

## Quadratic systems with a symmetrical solution

Zegeling, A.; Kooij, Robert E.

**DOI**

[10.14232/ejqtde.2018.1.32](https://doi.org/10.14232/ejqtde.2018.1.32)

**Publication date**

2018

**Document Version**

Final published version

**Published in**

Electronic Journal of Qualitative Theory of Differential Equations

**Citation (APA)**

Zegeling, A., & Kooij, R. E. (2018). Quadratic systems with a symmetrical solution. *Electronic Journal of Qualitative Theory of Differential Equations*, 2018, 1-18. Article 32.  
<https://doi.org/10.14232/ejqtde.2018.1.32>

**Important note**

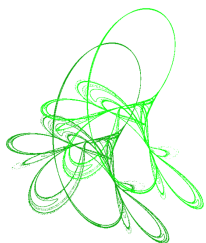
To cite this publication, please use the final published version (if applicable).  
Please check the document version above.

**Copyright**


Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

**Takedown policy**

Please contact us and provide details if you believe this document breaches copyrights.  
We will remove access to the work immediately and investigate your claim.



## Quadratic systems with a symmetrical solution

André Zegeling <sup>1,2</sup> and Robert E. Kooij<sup>3,4</sup>

<sup>1</sup>Shanghai Normal University, Department of Mathematics, Shanghai 200234, P. R. China

<sup>2</sup>Guilin University of Aerospace Technology, Jinji Road 2, Guilin, China

<sup>3</sup>Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, Network Architectures and Services, Mekelweg 4, 2628 CD Delft, the Netherlands

<sup>4</sup>Singapore University of Technology and Design, 8 Somapah Rd, Singapore 487372, Singapore

Received 12 September 2017, appeared 23 May 2018

Communicated by Gabriele Villari

**Abstract.** In this paper we study the existence and uniqueness of limit cycles for so-called quadratic systems with a symmetrical solution:

$$\begin{aligned}\frac{dx(t)}{dt} &= P_2(x, y) \equiv a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \frac{dy(t)}{dt} &= Q_2(x, y) \equiv b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2\end{aligned}$$

where  $(x, y) \in \mathbb{R}^2, t \in \mathbb{R}, a_{ij}, b_{ij} \in \mathbb{R}$ , i.e. a real planar system of autonomous ordinary differential equations with linear and quadratic terms in the two independent variables.

We prove that a quadratic system with a solution symmetrical with respect to a line can be of two types only. Either the solution is an algebraic curve of degree at most 3 or all solutions of the quadratic system are symmetrical with respect to this line.

For completeness we give a new proof of the uniqueness of limit cycles for quadratic systems with a cubic algebraic invariant, a result previously only available in Chinese literature. Together with known results about quadratic systems with algebraic invariants of degree 2 and lower, this implies the main result of this paper, i.e. that quadratic systems with a symmetrical solution have at most one limit cycle which if it exists is hyperbolic.

**Keywords:** ordinary differential equations, limit cycle, algebraic curve.


**2010 Mathematics Subject Classification:** 34C05, 34C07.

### 1 Introduction

We will study the existence and uniqueness of limit cycles for so-called quadratic systems with a symmetrical solution:

$$\begin{aligned}\frac{dx(t)}{dt} &= P_2(x, y) \equiv a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \frac{dy(t)}{dt} &= Q_2(x, y) \equiv b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2\end{aligned}\tag{1.1}$$

---

 Corresponding author. Email: zegela1@yahoo.com

where  $(x, y) \in \mathbb{R}^2, t \in \mathbb{R}, a_{ij}, b_{ij} \in \mathbb{R}$ , i.e. a real planar system of autonomous ordinary differential equations with linear and quadratic terms in the two independent variables.

For notational purposes we will write system (1.1) in the following form:

$$\begin{aligned}\frac{dx(t)}{dt} &= A_0(x) + A_1(x)y + a_{02}y^2 \\ \frac{dy(t)}{dt} &= B_0(x) + B_1(x)y + b_{02}y^2\end{aligned}\tag{1.2}$$

where

$$\begin{aligned}A_0(x) &\equiv a_{00} + a_{10}x + a_{20}x^2 \\ A_1(x) &\equiv a_{01} + a_{11}x \\ B_0(x) &\equiv b_{00} + b_{10}x + b_{20}x^2 \\ B_1(x) &\equiv b_{01} + b_{11}x.\end{aligned}$$

By a solution symmetrical with respect to a line we mean that the image of the curve after reflection in this line is a solution curve of the system as well.

After an affine transformation of the variables we may assume that the line of symmetry is at the  $x$ -axis  $y = 0$ . A solution is symmetrical with respect to this line when for a solution curve  $(x(t), y(t))$  the reflected curve  $(x(t), -y(t))$  is a solution as well or when the reflected curve with reversed time traversal is a solution  $(x(-t), -y(-t))$ . These two distinct types are covered by writing the system of differential equations without explicit time-parametrization:

$$\frac{dy}{dx} = \frac{B_0(x) + b_{02}y^2 + B_1(x)y}{A_0(x) + a_{02}y^2 + A_1(x)y}\tag{1.3}$$

In this notation a solution is symmetrical if it has the form  $y^2 = \phi(x)$ , defined for  $x$ -values where  $\phi(x) \geq 0$ . In principle there could be solutions which are symmetrical with respect to a line, where there more than two  $y$ -values for a given  $x$ . These cases will automatically be included in our study of the restricted form  $y^2 = \phi(x)$  if we just take one branch of this multivalued function and ignore possible other symmetrical branches. The results of the next sections will show that the multivalued case cannot occur because of the specific structure that the symmetrical solutions necessarily have.

The paper is organized as follows. First in the next section the possible forms for the symmetrical solutions of (1.1) are derived, basically stating that either all solutions are symmetrical or there is exactly one symmetrical solution which is necessarily an algebraic curve of degree at most 3. In the last sections it is shown that such systems can have at most one limit cycle by using a transformation to a Liénard system and by applying a well-known uniqueness theorem.

## 2 Quadratic systems with a symmetrical solution

We provide the main result about the structure of systems (1.2) with a symmetrical solution in two steps. The first step uses a simple argument to show that if (1.1) has an isolated symmetrical solution then it is necessarily algebraic of degree at most 4. The second non-trivial step consists of showing that if a symmetrical algebraic curve of degree 4 exists, then all solutions are symmetrical. It follows that isolated symmetrical solutions are algebraic and of degree at most 3. A simple example shows that this case can occur proving the sharp upper bound on the degree of the algebraic curve.

## 2.1 First estimates on the structure of possible symmetrical solutions

This section provides a estimate of the types of symmetrical solutions in a quadratic system. Similar results were communicated to us by [16].

A simple observation on the properties of a symmetrical solution will lead to a necessary but not sufficient restriction on the types of symmetrical solutions to a quadratic system.

**Definition 2.1.** An algebraic curve of order  $n$  is the set of points  $(x, y)$  satisfying the equation:

$$C(x, y) = \sum_{0 \leq i+j \leq n} c_{ij} x^i y^j = 0 \quad (2.1)$$

where  $c_{ij} \in \mathbb{R}$  and at least one  $c_{ij} \neq 0$ ,  $i + j = n$ .

With this definition we can formulate the following lemma.

**Lemma 2.2.** *If a quadratic system (1.3) contains a solution  $y^2 = \phi(x)$ , then either all solutions are symmetrical (and not necessarily algebraic) with respect to the line  $y = 0$  or the solution lies in a component of an algebraic curve of degree at most 4.*

*Proof.* For notational purposes we denote the two symmetrical branches of the solution by  $y = \pm u(x)$ , i.e.  $u^2(x) \equiv \phi(x)$ . Since each branch satisfies (1.3), we get two equations:

$$\begin{aligned} \frac{d(-u(x))}{dx} &= \frac{B_0(x) - B_1(x)u(x) + b_{02}u^2(x)}{A_0(x) - A_1(x)u(x) + a_{02}u^2(x)} \\ \frac{du(x)}{dx} &= \frac{B_0(x) + B_1(x)u(x) + b_{02}u^2(x)}{A_0(x) + A_1(x)u(x) + a_{02}u^2(x)}. \end{aligned} \quad (2.2)$$

Adding the two equations leads to the following identity:

$$\begin{aligned} (B_0(x) + b_{02}u^2(x))(A_0(x) + a_{02}u^2(x)) &= A_1(x)B_1(x)u^2(x) \\ \Leftrightarrow C_0u^4(x) + C_1(x)u^2(x) + C_2(x) &= 0 \end{aligned} \quad (2.3)$$

where  $C_0 = a_{02}b_{02}$ ,  $C_1(x) = b_{02}A_0(x) + a_{02}B_0(x) - A_1(x)B_1(x)$ ,  $C_2(x) = A_0(x)B_0(x)$ .

This equation represents an algebraic curve of degree 4 unless all terms vanish identically. Therefore to conclude the proof we need to show that if (2.3) vanishes identically, then all solutions to (1.3) are symmetrical. The conditions under which all coefficients vanish are:

$$\begin{aligned} B_0(x)A_0(x) &\equiv 0 \\ b_{02}A_0(x) + a_{02}B_0(x) &\equiv A_1(x)B_1(x) \\ b_{02}a_{02} &\equiv 0. \end{aligned}$$

We list all possible solution combinations satisfying these 3 conditions:

- (i)  $B_0(x) \equiv 0$ ,  $b_{02}A_0(x) \equiv A_1(x)B_1(x)$ ,  $a_{02} \equiv 0$
- (ii)  $B_0(x) \equiv 0$ ,  $A_1(x) \equiv 0$ ,  $b_{02} \equiv 0$
- (iii)  $B_0(x) \equiv 0$ ,  $B_1(x) \equiv 0$ ,  $b_{02} \equiv 0$
- (iv)  $A_0(x) \equiv 0$ ,  $A_1(x) \equiv 0$ ,  $a_{02} \equiv 0$
- (v)  $A_0(x) \equiv 0$ ,  $B_1(x) \equiv 0$ ,  $a_{02} \equiv 0$
- (vi)  $A_0(x) \equiv 0$ ,  $a_{02}B_0(x) \equiv A_1(x)B_1(x)$ ,  $b_{02} \equiv 0$ .

In these 6 cases all solutions are symmetrical with respect to the line  $y = 0$ , concluding the proof of the lemma:

Case (i)

$$\begin{aligned}\frac{dx(t)}{dt} &= A_1(x)(B_1(x) + b_{02}y) \\ \frac{dy(t)}{dt} &= b_{02}y(B_1(x) + b_{02}y)\end{aligned}$$

which is a degenerate system for which all solutions are symmetrical with respect to  $y = 0$ .

Case (ii)

$$\begin{aligned}\frac{dx(t)}{dt} &= A_0(x) + a_{02}y^2 + A_1(x)y \\ \frac{dy(t)}{dt} &= 0\end{aligned}$$

which is a degenerate system for which all solutions are symmetrical with respect to  $y = 0$ .

Case (iii)

$$\begin{aligned}\frac{dx(t)}{dt} &= A_0(x) + a_{02}y^2 \\ \frac{dy(t)}{dt} &= B_1(x)y\end{aligned}$$

which is a quadratic system for which all solutions are symmetrical with respect to  $y = 0$ . To this important class belongs the so-called reversible center case [26].

Case (iv)

$$\begin{aligned}\frac{dx(t)}{dt} &= A_1(x)y \\ \frac{dy(t)}{dt} &= B_0(x) + b_{02}y^2\end{aligned}$$

which is a quadratic system for which all solutions are symmetrical with respect to  $y = 0$ , taking a time-reversal into account.

Case (v)

$$\begin{aligned}\frac{dx(t)}{dt} &= 0 \\ \frac{dy(t)}{dt} &= B_0(x) + b_{02}y^2 + B_1(x)y\end{aligned}$$

which is a degenerate system for which all solutions are vertical lines in the phase plane, i.e.  $x = \text{constant}$ , and therefore symmetrical with respect to  $y = 0$ .

Case (vi)

$$\begin{aligned}\frac{dx(t)}{dt} &= a_{02}y(a_{02}y + A_1(x)) \\ \frac{dy(t)}{dt} &= B_1(x)(a_{02}y + A_1(x))\end{aligned}$$

which is a degenerate quadratic system for which all solutions are symmetrical with respect to  $y = 0$ . □

**Remark 2.3.** It is trivial to see that in the particular cases of quadratic systems where all solutions are symmetrical, no limit cycles can occur. For the existence of limit cycles, we therefore only need to consider the case where the symmetrical solution is isolated and according to the lemma is of a degree at most 4. The next section will show that the strict upper bound on this algebraic curve is 3, i.e. isolated symmetrical algebraic solutions of degree 4 cannot occur in quadratic systems.

## 2.2 A sharp upper bound on the symmetrical algebraic solution

The previous section showed that if (1.3) has an isolated symmetrical solution, then it is an algebraic curve of at most 4<sup>th</sup> degree. This section improves the result to the strict upper bound of degree 3.

First we introduce some other useful necessary conditions for a solution to be symmetrical with respect to a line.

An additional condition can be imposed on the structure by subtraction of the two differential equations in (2.2). This leads to the following expressions for the derivative of the function  $u(x)$ :

$$\frac{du(x)}{dx} = \frac{u(x)B_1(x)}{A_0(x) + a_{02}u^2(x)} = \frac{B_0(x) + b_{02}u^2(x)}{u(x)A_1(x)}. \quad (2.4)$$

Taking the derivative of the first condition (2.3) with respect to  $x$  using the derivative expression (2.4) we get a second necessary condition on the structure of a symmetrical solution:

$$D_0(x)u^4(x) + D_1(x)u^2(x) + D_2(x) = 0 \quad (2.5)$$

$$D_0(x) = 4b_{02}C_0,$$

$$D_1(x) = -2b_{02}(A_1(x)B_1(x) - (b_{02}A_0(x) + a_{02}B_0(x)) + 4b_{02}a_{02}B_0(x) - A_1(x)((A_1(x)B_1(x))' - b_{02}A_0'(x) - a_{02}B_0'(x))),$$

$$D_2(x) = -2B_0(x)(A_1(x)B_1(x) - (b_{02}A_0(x) + a_{02}B_0(x)) + A_1(x)(A_0(x)B_0(x))'.$$

A straightforward elimination of the  $u^4(x)$  term from conditions (2.3) and (2.5) leads to:

$$u^2(x) = \frac{W_1(x)}{W_2(x)} \quad (2.6)$$

where  $W_1(x) = -(D_2(x) - 4b_{02}C_2(x))$  and  $W_2(x) = -(D_1(x) - 4b_{02}C_1(x))$ .

Using this expression in condition (2.3) leads to an expression which necessarily needs to be satisfied by the coefficients of system (1.1) in order to have a symmetrical solution:

$$C_0W_1^2(x) + C_1(x)W_1(x)W_2(x) + C_2(x)W_2^2(x) \equiv 0. \quad (2.7)$$

The crucial point is that the coefficient  $C_0$  is a constant.

First we assume that  $C_0 \neq 0$  and consider in (2.7) three possibilities for the polynomial  $W_2(x)$ :

(i)  $W_2(x)$  is quadratic

(ii)  $W_2(x)$  is linear

(iii)  $W_2(x)$  is a constant

Case (i)

If  $W_2(x)$  is quadratic, then  $W_1(x)$  vanishes for each of the 2 (possibly complex) zeros of  $W_2(x)$ , i.e.  $W_1(x)$  and  $W_2(x)$  have a common quadratic factor. This means that the symmetrical solution given in (2.6) is an algebraic curve of at most degree 2, because the righthand-side is a (at most) quadratic polynomial in  $x$ .

Case (ii)

If  $W_2(x)$  is linear, then a similar reasoning shows that  $W_1(x)$  has a linear factor in common with  $W_2(x)$ . Then (2.6) implies that the symmetrical solution is an algebraic curve of at most degree 3.

Case (iii)

Suppose that  $W_2(x)$  is a constant.  $W_1(x)$  can be at most quartic in  $x$ . However, a straightforward substitution of a solution of the type  $y^2 = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$  into the quadratic system (1.1) shows that  $a_{02}$  has to be 0 which contradicts our assumption for this case. Intuitively this is logical since the faster than linear growth in  $x$  of the solution for large  $x$  implies that a singularity needs to exist at the end of the  $y$ -axis, i.e.  $a_{02} = 0$ .

Finally we need to consider the case  $C_0 = 0$ . Either  $a_{02} = 0$  or  $b_{02} = 0$ . First suppose that  $b_{02} = 0$ . In that case expression (2.4) becomes:

$$\frac{du(x)}{dx} = \frac{B_0(x)}{u(x)A_1(x)}$$

which in integral form becomes:

$$\frac{1}{2}u^2(x) = \int^x \frac{B_0(\bar{x})}{A_1(\bar{x})} d\bar{x}.$$

Since we have already established that this equation should represent an algebraic curve, the right-hand side should be a rational function of  $x$ . The two polynomial functions in the integrand are  $B_0(\bar{x})$  and  $A_1(\bar{x})$ , respectively of degree at most 2 and 1. If  $A_1(\bar{x})$  is constant then the integral becomes a polynomial of degree at most 3. If it is not a constant then the integral can only become a rational polynomial function in  $x$  if  $B_0(\bar{x})$  and  $A_1(\bar{x})$  have a common linear factor, implying that the integral becomes a quadratic polynomial. In both cases the algebraic solution is of degree at most 3 as we needed to establish.

Now suppose that  $a_{02} = 0$ .

Under the simplifying assumption of the lemma, the conditions as derived in the proof of Lemma 2.2 become:

$$C_1(x)u^2(x) + C_2(x) = 0 \tag{2.8}$$

where  $C_1(x) = b_{02}A_0(x) - A_1(x)B_1(x)$ ,  $C_2(x) = A_0(x)B_0(x)$ ,

$$\frac{du(x)}{dx} = \frac{u(x)B_1(x)}{A_0(x)} = \frac{B_0(x) + b_{02}u^2(x)}{u(x)A_1(x)}. \tag{2.9}$$

Differentiation of (2.8) with respect to  $x$  leads to an expression for the coefficients of the equation which we rearranged appropriately:

$$C_1(x)(2B_0(x)B_1(x) - (A_0(x)B_0(x))') = -A_0(x)B_0(x)C_1'(x).$$

Since  $C_1(x)$  has a degree higher in  $x$  than  $C_1'(x)$ , it follows that  $C_1(x)$  has a factor in common with  $A_0(x)B_0(x)$ . The numerator and denominator in the expression  $u^2(x) = \frac{-C_2(x)}{C_1(x)} =$

$\frac{-A_0(x)B_0(x)}{b_{02}A_0(x)-A_1(x)B_1(x)}$  have degrees at most 4 and 2 respectively. Since they have a factor in common, it follows that the whole expression represents an algebraic curve of at most order 3 which concludes the proof for the case  $a_{02} = 0$ .

The previous discussion showed that the actual degree of the symmetrical algebraic curve is at most 3, improving the result of Lemma 2.2.

**Theorem 2.4.** *If a quadratic system (1.1) contains a solution  $y^2 = \phi(x)$ , then either all solutions are symmetrical (and not necessarily algebraic) with respect to the line  $y = 0$  or the solution lies in a component of an algebraic curve of degree at most 3.*

**Remark 2.5.** In the next section an example will be given of a quadratic system with an isolated symmetrical cubic solution showing that the bound in the theorem is strict.

### 3 Uniqueness of limit cycles

#### 3.1 Limit cycles in quadratic systems with algebraic invariants

The relationship between invariant algebraic curves and the existence of limit cycles has been studied extensively in the literature (see [26]). The study of limit cycles is related to the second part of 16th Hilbert problem which asks for upper bounds on the number of limit cycles in polynomial systems.

For quadratic systems with algebraic invariant curves the following results regarding the number of limit cycles have been proved (for a complete discussion we recommend [6] where an excellent historical overview is presented):

- invariant line: at most one limit cycle (see [8,9,11,20]);
- 2 real invariant lines: no limit cycle (see [26]);
- 2 complex invariant lines: at most one limit cycle (see [26]);
- invariant hyperbola: no limit cycle (see [26]);
- invariant ellipse/circle: at most one limit cycle. The limit cycle in this case is algebraic and is formed by the ellipse/circle (see [26]);
- invariant parabola: at most one limit cycle (see [7] for existence and [27] for uniqueness);
- invariant cubic curve: at most one limit cycle (see [21,22]).

Therefore for algebraic curves of degree 3 or lower the limit cycle problem has been solved completely.

In the discussion about limit cycles in families of differential equations with an algebraic invariant curve, an important distinction needs to be made between limit cycles contained in the algebraic curve itself, i.e. algebraic limit cycles, and limit cycles which are not part of this algebraic invariant curve. A large part of the literature is devoted to the investigation of algebraic limit cycles, see [17,18,23]. An important result in this area was the proof by [19] that the limit cycle in the well-known van der Pol equation cannot be algebraic.

This paper focusses on the other type, because it is well-known that in a quadratic system with an algebraic cubic invariant curve, a limit cycle cannot be part of the algebraic curve itself, see [3,13–15]. In [6] and [25] it was shown that such systems can contain limit cycles which are



not part of the cubic invariant curve. Their uniqueness was proved in [21,22]. However, since these journals are not easily accessible in the west and their proofs are in Chinese, we provide here a complete proof using methods which have some similarity to the Chinese version but still use different transformations and Liénard theorems.

Specifically the aforementioned papers identify two families of quadratic systems with a cubic algebraic curve which contain cases with limit cycles. In [6] it is shown that the remaining families of quadratic systems with a cubic invariant cannot contain limit cycles. The two families of quadratic systems with an invariant cubic curve which we need to investigate are:

*Type I*

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{1}{2}\phi x(1-x) + \frac{1}{2}\mu(-1+2x+xy) - \frac{1}{2}\lambda x^2, \\ \frac{dy(t)}{dt} &= \phi(-1-y) - \mu y(1+y) + \lambda(1+xy),\end{aligned}\tag{3.1}$$

where  $\mu \geq 0$  with invariant  $yx^2 + 2x - 1 = 0$ .

The restriction on  $\mu$  can be obtained by observing that the system is invariant under the transformations  $t \rightarrow -t, \lambda \rightarrow -\lambda, \phi \rightarrow -\phi, \mu \rightarrow -\mu$ .

*Type II*

$$\begin{aligned}\frac{dx(t)}{dt} &= \phi x(x-1) + \mu xy + \lambda y, \\ \frac{dy(t)}{dt} &= \phi \left(-y + \frac{3}{2}xy\right) + \frac{1}{2}\mu(x^2 + 3y^2) + \frac{1}{2}\lambda(-2x + 3x^2),\end{aligned}\tag{3.2}$$

where  $\phi \geq 0, \lambda \geq 0$  with invariant  $y^2 + x^2 - x^3 = 0$ .

The restriction on  $\phi$  can be obtained by observing that under the transformations  $t \rightarrow -t, y \rightarrow -y, \phi \rightarrow -\phi$  the system is invariant. The restriction on  $\lambda$  can be obtained by observing that under the transformations  $y \rightarrow -y, \lambda \rightarrow -\lambda, \mu \rightarrow -\mu$  the system is invariant.

In these two families of quadratic systems limit cycles may occur.

**Remark 3.1.** Type I systems have a cubic invariant which is not symmetrical with respect to a line as discussed in this paper. Type II has a symmetrical solution and is therefore of direct relevance to the main conclusions of the paper. For completeness we also provide the proof of uniqueness of the limit cycle for the Type I case.

### 3.2 Quadratic systems with type I cubic algebraic invariant

This case is relatively easy and follows the same proof as for quadratic systems with an invariant parabola in [27]. To start we apply the following transformation to the type I system:

$$yx^2 + 2x - 1 = e^z.$$

The transformation moves the invariant curve to negative infinity (i.e.  $z = -\infty$ ). We get

$$\begin{aligned}\frac{dx(t)}{dt} &= \psi(z) - F(x), \\ \frac{dz(t)}{dt} &= -g(x)\end{aligned}\tag{3.3}$$

with

$$\begin{aligned}\psi(z) &= \mu e^z, \\ F(x) &= \phi x^2(x-1) + \lambda x^3 + \mu(2x-1)(1-x), \\ g(x) &= 2x(\phi x - \mu).\end{aligned}\tag{3.4}$$

This form falls under the structure of a system for which the uniqueness of limit cycles has been proved, see [27, Lemma 2.4].

**Lemma 3.2.** *Let  $g(x) = \frac{p(x)}{r(x)}$ ,  $f(x) = \frac{dF(x)}{dx} = \frac{q(x)}{r(x)}$  where  $p(x)$ ,  $q(x)$  are polynomials of degree 2 or less,  $r(x) \in C^\infty$ ,  $r(x) \neq 0$  on the open interval  $(r_1, r_2)$ , and let  $\frac{d\psi(z)}{dz} > 0$ . Then system (3.3) has at most one limit cycle in the strip  $r_1 < x < r_2$  which, if it exists, is hyperbolic.*

This lemma can be applied to our case to show the uniqueness of limit cycle. Existence of a limit cycle in this family was proved in [6] and [25].

**Theorem 3.3.** *Quadratic systems of type I can have at most one limit cycle. If it exists, it is hyperbolic.*

*Proof.* In our case  $f(x) \equiv \frac{dF(x)}{dx}$  and  $g(x)$  are both quadratic functions, and we can apply the lemma to system (3.3) by taking  $r(x) = 1$ . Since  $\mu \geq 0$  the condition  $\frac{d\psi(z)}{dz} > 0$  is satisfied. Note that for  $\mu = 0$  the system cannot have limit cycles because in that case  $\frac{dx(t)}{dt}$  in (3.1) does not depend on  $y$ .  $\square$

### 3.3 Quadratic systems with type II cubic algebraic invariant

This family is much harder to investigate than the type I family. A similar approach as for type I leads to a system where uniqueness of the limit cycle is hard to prove. The proof is split into two parts. In the first section we derive some properties of the type II family using the original system.

#### 3.3.1 Properties of quadratic systems with type II cubic algebraic invariant

The components of the cubic invariant for the type II system have an interesting structure. There are two components lying in the phase plane. One component is an isolated point at the origin, which is an antisaddle. It will be shown later in this section that this is not the singularity which can be surrounded by a limit cycle. The other component is a symmetrical parabola-like curve  $y = \pm x\sqrt{x-1}$ ,  $x \geq 1$ . For future reference we refer to  $y^2 < x^2(x-1)$ ,  $x \geq 1$  as the inside of the cubic invariant. We will show in the next section that limit cycles can only occur in that region.

**Lemma 3.4.** *Limit cycles in systems of type II can only lie in the region  $x > 1$ .*

*Proof.* First we observe that limit cycles cannot cross the line  $x = 1$ . This follows from the fact that on the line  $x = 1$  we have  $\frac{dx(t)}{dt} = (\mu + \lambda)y$ . It follows that any limit cycle crossing the line  $x = 1$  would have to cross it once for  $y > 0$  and once for  $y < 0$ . It cannot cross it at  $y = 0$  because a branch of the cubic invariant is already passing through the point  $x = 1, y = 0$ . If a limit cycle would cross the line  $x = 1$  for  $y > 0$  it would not be able to cross  $x = 1$  again because it cannot cross the branch of the cubic invariant  $y = x\sqrt{x-1}$ ,  $x \geq 1$ , preventing the

solution to return to  $x = 1$ . Therefore each periodic orbit (if it exists) will lie in a region  $x > 1$  or  $x < 1$ . We will show that the latter cannot occur. Consider the family of type II in the form

$$\begin{aligned}\frac{dx(t)}{dt} &= B(x, y)(\phi x(x-1) + \mu xy + \lambda y), \\ \frac{dy(t)}{dt} &= B(x, y)\left(\phi\left(-y + \frac{3}{2}xy\right) + \frac{1}{2}\mu(x^2 + 3y^2) + \frac{1}{2}\lambda(-2x + 3x^2)\right)\end{aligned}$$

with  $B(x, y) \equiv |y^2 + x^2 - x^3|^{-\frac{4}{3}}$ .

The divergence of this system is  $\frac{1}{6}B(x, y)^{-\frac{4}{3}}\phi(3x-4)$  and its sign is determined by a factor  $3x-4$ . This implies that periodic orbits of the system have to cross the line  $x = \frac{4}{3}$  according to Dulac's lemma (see [26]). It follows that periodic solutions can only lie in the region  $x > 1$ .  $\square$

The singularities of the quadratic system are distributed as follows. At the origin an antisaddle singularity resides which as said is part of the cubic invariant. On the parabola-like component of the cubic invariant  $y = \pm x\sqrt{x-1}$ ,  $x \geq 1$  two real or complex singularities lie, leaving only one other possible singularity outside the cubic invariant (because in a non-degenerate quadratic system at most 4 finite singularities can exist, see [26]). Its location and nature depend on the three parameters of the family.

Next we determine a necessary condition on the parameters for existence of limit cycles. The singularity at the origin cannot be surrounded by a limit cycle due to Lemma 3.4. A straightforward calculation shows that the other singularity not lying on the algebraic curve has  $x$ -coordinate  $x_g \equiv \frac{2\lambda}{3\lambda+\mu}$ . Limit cycles, if they exist, need to surround this singularity. From Lemma 3.4 it follows that  $x_g > 1$  is a necessary condition for the existence of limit cycles. In terms of the parameters of the system, we rewrite  $x_g - 1$  as

$$\frac{\frac{-\lambda}{\mu} - 1}{3\frac{\lambda}{\mu} + 1}.$$

This expression is only positive on the interval  $-1 < \frac{\lambda}{\mu} < -\frac{1}{3}$ . The two interval bounds represent  $x_g = 1$ ,  $x_g = +\infty$  for  $\frac{\lambda}{\mu} = -1$ ,  $\frac{\lambda}{\mu} = -\frac{1}{3}$  respectively. We get the following lemma.

**Lemma 3.5.** *A necessary condition for the existence of limit cycles in systems of type II is*

$$-1 < \frac{\lambda}{\mu} < -\frac{1}{3}.$$

### 3.3.2 Uniqueness of limit cycles in quadratic systems with type II cubic algebraic invariant

Several transformations exist for which the quadratic system (3.2) can be transformed into a Liénard system. It turned out that most transformations we tried did not lead to a form where we could apply a uniqueness theorem for limit cycles, except for the following transformation:

$$\begin{aligned}2x &= (z^2 + 1)(u + 1), \\ 2y &= z(z^2 + 1)(u + 1).\end{aligned}$$

Under this transformation the family of type II transforms into

$$\begin{aligned}\frac{dz(t)}{dt} &= u + 1 + \frac{8z_f}{3z_g K(z)}, \\ \frac{du(t)}{dt} &= \frac{2z_f(z - z_g)}{(z^2 + 1)z_g K(z)}(u^2 - 1)\end{aligned}\tag{3.5}$$

with

$$\begin{aligned} K(z) &\equiv z^2 - 2z_f z - 2\frac{z_f}{z_g} + 1, \\ z_g &\equiv \frac{\phi}{3\lambda}, \\ z_f &\equiv \frac{-\phi}{2\mu}, \end{aligned} \tag{3.6}$$

where

- the zeroes of  $K(z)$  represent the directions of the singularities at infinity of the original system;
- $z_g$  corresponds to the  $z$ -coordinate of the only singularity in the system which can be surrounded by limit cycles;
- $z_f$  corresponds to the  $z$ -coordinate of the only zero of the divergence of the transformed system (after an appropriate transformation of the  $u$ -variable, as indicated in the next sections).

With these definitions the structure of the system will be easier to understand.

### 3.3.3 Restrictions on the parameters and functions in the system of differential equations

Before we can apply theorems on Liénard systems, it is useful to investigate the parameters of the system and the functions defining the differential equations closer. This will lead to further necessary conditions for the existence of limit cycles.

The restriction on the parameters derived in Lemma 3.5 becomes in the new notation:

**Lemma 3.6.** *A necessary condition for the existence of limit cycles in (3.5) is that:*

$$\frac{1}{2} < \frac{z_f}{z_g} < \frac{3}{2}$$

and  $z_g > 0$ .

**Remark 3.7.**  $z_g > 0$  holds because of the restrictions on the original parameters in (3.2). It follows from the lemma that  $z_f > 0$  holds as well. The values defining the interval in the lemma are related to the condition in lemma 3.4:  $\frac{z_f}{z_g} = \frac{3}{2}$  corresponds to  $x = 1$  while  $\frac{z_f}{z_g} = \frac{1}{2}$  corresponds to  $x = +\infty$ .

The function  $K(z)$  plays an important role in the differential equation. It has some nice properties which can be used in the study of limit cycles:

**Lemma 3.8.** *The function  $K(z)$  has two real zeroes  $z_K^{(1)} < z_K^{(2)}$  for the parameter values where limit cycles can occur in system (3.5).*

*Proof.* For the quadratic function  $K(z)$  under the conditions of Lemma 3.6, we have  $(z_g^2 + 1)(1 - 2\frac{z_f}{z_g}) < 0$ , while  $\lim_{z \rightarrow \infty} K(z) = +\infty$ .  $\square$

It is easy to see that the vertical lines  $z = z_K^{(1)}$  and  $z = z_K^{(2)}$  are lines without contact. Limit cycles cannot cross these lines and therefore we can restrict the investigation of limit cycles to the region  $z_K^{(1)} < z < z_K^{(2)}$ . It follows that:

**Lemma 3.9.** *A necessary condition for the existence of limit cycles in (3.5) is that  $z_K^{(1)} < z < z_K^{(2)}$  and in this region  $K(z) < 0$ .*

A simple calculation show that the singularity at  $z = z_g$  is an antisaddle only if  $u^2 > 1$ . However, it follows from the previous lemmas and the fact that  $z_f > 0, z_g > 0$  that  $\frac{dz(t)}{dt} < 0$  if  $u < -1$ , i.e. no antisaddle can occur. Therefore the restriction  $u^2 > 1$  can be simplified to:

**Lemma 3.10.** *Limit cycles can only lie in the region  $u > 1$ .*

**Corollary 3.11.** *This implies that limit cycles in the original quadratic system (3.2) can only lie in the region inside the parabola-like component of the cubic invariant, i.e.  $y^2 < x^2(x - 1), x > 1$ .*

Next we will show that for  $\frac{1}{2} < \frac{z_f}{z_g} \leq 1$  the system cannot have limit cycles, and has at most one limit cycle for  $1 < \frac{z_f}{z_g} < \frac{3}{2}$  (equivalent to  $z_g < z_f < \frac{3}{2}z_g$ ).

First we transform (3.5) into a generalized Liénard form of the type (3.3), using the restriction on  $u$  in lemma 3.10. Applying the change of variables  $u = \frac{1+e^{2\bar{u}}}{1-e^{2\bar{u}}}$  to (3.5), we get

$$\begin{aligned}\frac{dz(t)}{dt} &= \psi(\bar{u}) - F(z), \\ \frac{d\bar{u}(t)}{dt} &= -g(z)\end{aligned}\tag{3.7}$$

with

$$\psi(\bar{u}) = \frac{1 + e^{2\bar{u}}}{1 - e^{2\bar{u}}} - u_g,\tag{3.8}$$

where  $u_g \equiv 1 + \frac{8z_f}{3z_g K(z_g)}$  and

$$F(z) \equiv \frac{8z_f}{3z_g K(z_g)} - \frac{8z_f}{3z_g K(z)} = \frac{8z_f}{3z_g K(z_g)} \frac{(z - z_g)(z - z^*)}{K(z)},\tag{3.9}$$

where  $z^* \equiv 2z_f - z_g$  and

$$g(z) = \frac{-2z_f(z - z_g)}{(z^2 + 1)z_g K(z)}.\tag{3.10}$$

The function  $F(z)$  has two real zeroes  $z = z_g$  and  $z = z^*$  with the property that if  $z_f > z_g$  then  $z^* > z_g$ .

The following property for the function  $K(z)$  holds true.

**Lemma 3.12.** *For the real zeros of  $K(z)$ , i.e.  $z_K^{(1)}$  and  $z_K^{(2)}$  we have  $z_K^{(1)} < z_g < z^* < z_K^{(2)}$ .*

*Proof.* This follows from Lemma 3.9 and the observation that

$$K(z_g) = K(z^*) = (z_g^2 + 1) \left(1 - 2\frac{z_f}{z_g}\right) < 0. \quad \square$$

We define:

**Definition 3.13.**  $f(z) \equiv \frac{dF(z)}{dz} = \frac{8z_f K'(z)}{3z_g K^2(z)} = \frac{16z_f(z - z_f)}{3z_g K^2(z)}$ .

The zero of  $f(z)$ , i.e.  $z_f$  corresponds to a zero of the divergence of the system but also corresponds to a local minimum of the function  $K(z)$ . This implies the following.

**Lemma 3.14.** *For  $z < z_f$  ( $z > z_f$ ) we have  $K'(z) < 0$  ( $K'(z) > 0$ ).*

### 3.3.4 Liénard-related properties of the functions in the system of differential equations

In order to apply theorems for Liénard systems we first derive an important property of the following system of equations. These equations are familiar expressions in Liénard systems and play an essential role in many theorems.

**Lemma 3.15.** *Consider the system of equations for the functions defined in (3.7)*

$$\begin{aligned} F(z_1) &= F(z_2), \\ \frac{f(z_1)}{g(z_1)} &= \frac{f(z_2)}{g(z_2)}, \end{aligned}$$

where  $f(z) \equiv \frac{dF(z)}{dz}$ . For  $z_g < z_f < \frac{3}{2}z_g$  the system has at most one non-trivial (i.e.  $z_1 < z_g < z^* < z_2$ ) pair of solutions ( $z_1 = z_1^* > 0, z_2 = z_2^* > 0$ ) For  $\frac{1}{2}z_g < z_f \leq z_g$  the system does not have a non-trivial (i.e.  $z_1 < z_g < z^* < z_2$ ) pair of solutions ( $z_1 = z_1^* > 0, z_2 = z_2^* > 0$ )

*Proof.* In our case the system of equations can be written in the form

$$\begin{aligned} \frac{(z_1 - z_g)(z_1 - z^*)}{K(z_1)} &= \frac{(z_2 - z_g)(z_2 - z^*)}{K(z_2)}, \\ \frac{(z_1 - z_f)(z_1^2 + 1)}{(z_1 - z_g)K(z_1)} &= \frac{(z_2 - z_f)(z_2^2 + 1)}{(z_2 - z_g)K(z_2)}. \end{aligned}$$

The first equation can easily be seen to be equivalent to

$$z_1 + z_2 = 2z_f. \quad (3.11)$$

With this relation the second equation can be shown to have the following two possible solutions:

$$z_1 z_2 = \frac{2z_f^2 z_g + z_g - z_f}{z_g + z_f} \quad (3.12)$$

or

$$z_1 z_2 = z_g z^*. \quad (3.13)$$

The second solution (3.13) does not lead to essential solutions: we are typically looking for solutions  $z_1 < z_g < z^* < z_2$  or  $z_1 < z^* < z_g < z_2$  while the second solution leads to a solution pair  $z_1 = z^*, z_2 = z_g$  or  $z_1 = z_g, z_2 = z^*$ .

Using the first solution (3.12) in combination with the solution form (3.11) we can deduce the statement of the lemma. Under the parameter restriction  $\frac{1}{2}z_g < z_f \leq z_g$  the straight line as defined by (3.11) does not intersect the branch of the hyperbola (3.12) in the first quadrant of the  $(z_1, z_2)$  plane. For  $z_g = z_f$  the line is tangent to the hyperbola and for  $z_g < z_f < \frac{3}{2}z_g$  the line intersects the hyperbola in two points  $(z_1^*, z_2^*)$  and  $(z_2^*, z_1^*)$  which is essentially one pair of solutions under the restriction  $z_1 < z_2$ .  $\square$

### 3.3.5 Non-existence of limit cycles

With lemma (3.15) we can apply a well-known theorem on non-existence of limit cycles for generalized Liénard systems (see [12, 26]) stating that, in our notation:

**Theorem 3.16.** *Suppose the system*

$$\begin{aligned}\frac{dz(t)}{dt} &= h(\bar{u}) - F(z), \\ \frac{d\bar{u}(t)}{dt} &= -g(z)\end{aligned}$$

*satisfies the following conditions:*

1.  $(z - z_g)g(z) > 0$  for  $z \in (d', d)$ ,  $z \neq z_g$ ;
2.  $h(\bar{u})$  is continuous and increasing and  $(\bar{u} - u_g)h(\bar{u}) > 0$ ,  $\bar{u} \neq u_g$ ;
3. *the system of equations:*

$$\begin{aligned}F(z_1) &= F(z_2), \\ \frac{f(z_1)}{g(z_1)} &= \frac{f(z_2)}{g(z_2)}\end{aligned}$$

*does not have a solution  $d' < z_1 < z_g, z_g < z_2 < d$ .*

*Then the system does not have limit cycles.*

In our case the conditions are easily seen to be satisfied under the parameter conditions of Lemma 3.15 and therefore we can conclude:

**Proposition 3.17.** *If  $\frac{1}{2}z_g < z_f \leq z_g$ , then system (3.7) does not have limit cycles.*

### 3.3.6 Uniqueness of limit cycles

It remains to be proved that for  $z_g < z_f < \frac{3}{2}z_g$  at most one limit cycle occurs. The critical part of the proof is to show monotonicity of the following quotient of functions:

**Lemma 3.18.** *For  $z_g < z_f < \frac{3}{2}z_g$  on the interval  $z_f < z < z_K^{(2)}$  we have  $\frac{d}{dz} \frac{f(z)}{g(z)} > 0$ .*

*Proof.* A straightforward calculation shows that

$$\frac{d}{dz} \frac{f(z)}{g(z)} \propto -(z^2 + 1)K(z)(z_f - z_g) - 2z(z - z_f)(z - z_g)K(z) + (z - z_f)(z - z_g)(z^2 + 1) \frac{K(z)}{dz}.$$

Since  $K(z) < 0$  (because  $z_K^{(1)} < z_f < z < z_K^{(2)}$ ),  $\frac{dK(z)}{dz} > 0$  (because  $z > z_f$  and  $\frac{dK(z)}{dz} = 2(z - z_f)$ ),  $z > z_f > 0$ ,  $z_f > z_g$  and  $z > z_g$ , the expression is negative on the interval  $z_f < z^* < z < z_K^{(2)}$ .  $\square$

**Remark 3.19.** It is important to note that the derivative  $\frac{d}{dz} \frac{f(z)}{g(z)}$  does not have fixed sign on the interval  $z_K^{(1)} < z < z_g$ . If it would have been fixed sign, we would have been able to apply a generalization of the famous Zhang–Zhifen theorem [26]. However, since the sign is not fixed, we need to apply a stronger theorem, which only requires monotonicity of the function  $\frac{f(z)}{g(z)}$  on one side of the singularity at  $z = z_g$ .

The uniqueness theorem we apply is a theorem introduced in [28] (Theorem 3, page 485), written in our notation as follows.

**Theorem 3.20** ([28]). *Suppose the system*

$$\begin{aligned}\frac{dz(t)}{dt} &= h(\bar{u}) - F(z), \\ \frac{d\bar{u}(t)}{dt} &= -g(z)\end{aligned}$$

*satisfies the following conditions:*

1.  $(z - z_g)g(z) > 0$  for  $z \in (d', d)$ ,  $z \neq -z_g$ ;
2.  $(z - z_f)f(z) < 0$ , for  $z \in (d', d)$ ,  $x \neq z_f$ ,  $z_f < z_g$ ;
3.  $h(\bar{u})$  is continuous and increasing and  $(\bar{u} - u_g)h(\bar{u}) > 0$ ,  $\bar{u} \neq u_g$ ;
4. the system of equations

$$\begin{aligned}F(z_1) &= F(z_2), \\ \frac{f(z_1)}{g(z_1)} &= \frac{f(z_2)}{g(z_2)}\end{aligned}$$

*has at most one solution*  $d' < z_1 < z^* < z_f$ ,  $z_g < z_2 < d$ , where  $F(z^*) = 0$ ;

5.  $\frac{f(z)}{g(z)}$  is nondecreasing in  $d' < z_1 < z^*$ .

*Then the system has at most one limit cycle, which is hyperbolic if it exists.*

In words the theorem proves uniqueness of limit cycles if the following conditions are satisfied.

1. There is a unique singularity at  $z_g$  around which the limit cycle can reside on some interval  $z_- < z_g < z_+$ .
2. There is a unique zero  $z_f$  of the divergence function  $f(z)$  which lies to the left of the singularity,  $z_f < z_g$ .
3. The function  $h(\bar{u})$  is monotonically increasing and changes sign at  $u_g$ .
4. The function  $\frac{f(z)}{g(z)}$  is monotonic on the interval  $z_- < z < z^* < z_f$
5. The system of equations

$$\begin{aligned}F(z_1) &= F(z_2), \\ \frac{f(z_1)}{g(z_1)} &= \frac{f(z_2)}{g(z_2)}\end{aligned}$$

*has a unique non-trivial solution.*

We have adapted the statement of the theorem in [28] to exclude an additional condition which applies to the case of a weak focus. Since it already follows from the previous section on non-existence of limit cycles that in the case of  $z_f = z_g$  no limit cycles occur, we do not have to take it into consideration here.

With this theorem we can prove the main result on limit cycles for families of type II.



**Theorem 3.21.** *Quadratic systems of type II have at most one limit cycle, which is hyperbolic if it exists.*

*Proof.* In order to apply Theorem 3.20 to our situation, first we need to switch the relative position of  $z_f$  and  $z_g$ . The following transformation leaves system (3.7) in Liénard form:  $z \rightarrow 2z_g - z$  and  $\bar{u} \rightarrow 2u_g - \bar{u}$ , with new functions  $F(z)$ ,  $g(z)$  and  $\psi(\bar{u})$  satisfying the first 3 conditions of Theorem 3.20. The last two conditions are satisfied because of Lemmas 3.15 and 3.18.  $\square$

Combining this with Theorem 3.3 for type I, we have proved the following result.

**Theorem 3.22.** *Any quadratic system with a cubic algebraic invariant has at most one limit cycle, which is hyperbolic if it exists.*

Since it is known that quadratic systems with an algebraic invariant of degree at most 2 have at most one limit cycle – see the above references – the next theorem follows.

**Theorem 3.23.** *Any quadratic system with a symmetrical solution has at most one limit cycle, which is hyperbolic if it exists.*

## 4 Acknowledgement

The main work for this paper was done during a visit to Shanghai Normal University, April 2017. The first author would like to thank Prof Han Maoan for useful discussions and his invitation to the Shanghai Normal University.

## References

- [1] J. S. BARDAJI, *On the algebraic limit cycles of quadratic systems*, PhD thesis, 2004, Departament de Matemàtiques, Universitat Àutònoma de Barcelona, Bellaterra.
- [2] A. BENDJEDDOU, R. CHEURFA, Cubic and quartic planar differential systems with exact algebraic limit cycles, *Electron. J. Differential Equations* **2011**, No. 15, 1–12. [MR2781050](#)
- [3] J. CHAVARRIGA, J. LLIBRE, On the algebraic limit cycles of quadratic systems, in: C. García et al. (Eds.), *Proceedings of the IV Catalan Days of Applied Mathematics (Tarragona, 1998)*, Univ. Rovira Virgili, Tarragona, 1998, pp. 17–24. [MR1613021](#)
- [4] J. CHAVARRIGA, H. GIACOMINI, J. LLIBRE, Uniqueness of algebraic limit cycles for quadratic systems, *J. Math. Anal. Appl.* **261**(2001), No. 1, 85–99. <https://doi.org/10.1006/jmaa.2001.7476>; [MR1850958](#)
- [5] J. CHAVARRIGA, J. LLIBRE, J. SOROLLA, Algebraic limit cycles of degree 4 for quadratic systems, *J. Differential Equations* **200**(2004), No. 2, 206–244. <https://doi.org/10.1016/j.jde.2004.01.003>; [MR2052614](#)
- [6] J. CHAVARRIGA, I. A. GARCÍA, Existence of limit cycles for real quadratic differential systems with an invariant cubic, *Pacific J. Math.* **223**(2006), No. 2, 201–218. <https://doi.org/10.2140/pjm.2006.223.201>; [MR2221024](#)

- [7] S. CHEN, Limit cycles of a real quadratic differential system having a parabola as an integral curve (in Chinese), *Kexue Tongbao* **30**(1985), 401–405.
- [8] L. A. CHERKAS, L. I. ZHILEVICH, Some tests for the absence or uniqueness of limit cycles (in Russian), *Differentsialnye Uravneniya* **6**(1970), 1170–1178, translated in *Differential Equations* **6**(1970), 891–897. [MR0271458](#)
- [9] L. A. CHERKAS, L. I. ZHILEVICH, The limit cycles of certain differential equations (in Russian), *Differentsialnye Uravneniya* **8**(1972), 1207–1213, translated in *Differential Equations* **8**(1972), 924–929. [MR0306605](#)
- [10] C. CHRISTOPHER, J. LLIBRE, G. ŚWIRSZCZ, Invariant algebraic curves of large degree for quadratic system, *J. Math. Anal. Appl.* **303**(2005), No. 2, 450–461. <https://doi.org/10.1016/j.jmaa.2004.08.042>; [MR2122228](#)
- [11] W. A. COPPEL, Some quadratic systems with at most one limit cycle, in: *Dynamics reported. Vol. 2*, Dynam. Report. Ser. Dynam. Systems Appl., Vol. 2, Wiley, Chichester, 1989, pp. 61–88. [MR1000976](#)
- [12] F. DUMORTIER, C. ROUSSEAU, Cubic Liénard equations with linear damping, *Nonlinearity* **3**(1990), 1015–1039. [MR1079280](#)
- [13] R. M. EVDOKIMENKO, Construction of algebraic trajectories and a qualitative investigation in the large of the behavior of the integral curves of a certain system of differential equations (in Russian), *Differentsialnye Uravneniya* **6**(1970), 1780–1791, translated in *Differential Equations* **6**(1970), 1349–1358. [MR0277810](#)
- [14] R. M. EVDOKIMENKO, Behavior of the integral curves of a certain dynamical system (in Russian), *Differentsialnye Uravneniya* **12**(1976), No. 9, 1557–1567, translated in *Differential Equations* **12**(1976), No. 9, 1095–1103. [MR0437852](#)
- [15] R. M. EVDOKIMENKO, Investigation in the large of a dynamical system in the presence of a given integral curve (in Russian), *Differentsialnye Uravneniya* **15**(1979), No. 2, 215–221, translated in *Differential Equations* **15**(1979), No. 2, 145–150. [MR527322](#)
- [16] A. GASULL, H. GIACOMINI, private communication.
- [17] J. LLIBRE, G. ŚWIRSZCZ, Classification of quadratic systems admitting the existence of an algebraic limit cycle, *Bull. Sci. Math.* **131**(2007), No. 5, 405–421. <https://doi.org/10.1016/j.bulsci.2006.03.014>; [MR2337734](#)
- [18] J. LLIBRE, Open problems on the algebraic limit cycles of planar polynomial vector fields, *Bul. Acad. Ştiinţe Repub. Mold. Mat.* **2008**, No. 1, 19–26. [MR2392677](#)
- [19] K. ODANI, The limit cycle of the van der Pol equation is not algebraic, *J. Differential Equations*, **115**(1995), 146–152. <https://doi.org/10.1006/jdeq.1995.1008>; [MR1308609](#)
- [20] G. S. RYCHKOV, The limit cycles of the equation  $u(x+1)du = (-x + ax^2 + bxu + cu + du^2)dx$  (in Russian), *Differentsialnye Uravneniya* **8**(1972), 2257–2259, translated in *Differential Equations Vol.8* **1972**, 1748–1750. [MR0316816](#)
- [21] S. SHUI, The uniqueness of limit cycle of a quadratic system with an invariant cubic curve (in Chinese), *J. Zhejiang Normal Univ. Natur. Sci.* **22**(1999), No. 1, 8–11.

- [22] S. SHUI, A planar quadratic system with an invariant cubic curve has at most one limit cycle (in Chinese), *Acta Math. Appl. Sinica* **24**(2001), No. 4, 590–595. [MR1894454](#)
- [23] G. ŚWIRSZCZ, An algorithm for finding invariant algebraic curves of a given degree for polynomial planar vector fields, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **15**(2005), No. 3, 1033–1044. <https://doi.org/10.1142/S0218127405012442>; [MR2136759](#)
- [24] X. XIE, The quadratic system with an invariant curve  $y^n = x^{n+1} - x^n$  (in Chinese), *J. Ningde Teachers College (Natural science)* **14**(2002), No. 4.
- [25] C. XU, Quadratic systems that have cubic curves as particular integrals, *Acta Math. Sci.* (in Chinese), **12**(1992), No. 1, 9–18. [MR1196976](#)
- [26] Y. YE, *Theory of limit cycles*, Translations of Mathematical Monographs, Vol. 66, American Mathematical Society, Providence, R.I., 1986. [MR854278](#)
- [27] A. ZEGELING, R. E. KOOIJ, Uniqueness of limit cycles in polynomial systems with algebraic invariants, *Bull Austral. Math. Soc.* **49**(1994), No. 1, 7–20. <https://doi.org/10.1017/S0004972700016026>; [MR1262668](#)
- [28] Y. ZHOU, C. WANG, D. BLACKMORE, The uniqueness of limit cycles for Liénard system, *J. Math. Anal. Appl.* **304**(2005), No. 2, 473–489. <https://doi.org/10.1016/j.jmaa.2004.09.037>; [MR2126544](#)