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De Borsuk-Ulam Stelling in de Combinatoriek

(Engelse titel: The Borsuk-Ulam Theorem in Combinatorics)

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Abstract

We examine and prove the Borsuk-Ulam theorem and its combinatorial equivalent Fan's lemma. The theory of simplicial complexes and triangulations plays an important role in this. The Borsuk-Ulam theorem and Fan's lemma will be used to provide proofs for the Brouwer fixed point theorem and the combinatorial Sperner's lemma. Lastly the Borsuk-Ulam theorem will be applied in determining the chromatic number of Kneser graphs.

Contents

Introduction

One of the versions of the Borsuk-Ulam theorem states that for every continuous mapping $f : S^n \to \mathbb{R}^n$, where $S^n := \boldsymbol{x} \in \mathbb{R}^{n+1} : \|\boldsymbol{x}\| = 1$ is the n-sphere, there exists a point $\boldsymbol{x} \in S^n$ such that $f(\boldsymbol{x}) = f(-\boldsymbol{x})$. For $n = 2$ this can be illustrated by an inflatible ball that first is inflated and then, when the air is pushed out, is laid flat without tearing or cutting it. The Borsuk-Ulam theorem then says that there are two points on the surface of the ball that first were diametrically opposite (antipodal) and later, when the ball is laid flat, are lying on top of each other.

We will examine and proof this theorem and are particularly interested in the connections it has in the field of combinatorics. We will encounter a combinatorial equivalent of the Borsuk-Ulam theorem: Fan's lemma. The combinatorial proof given for this lemma provides an alternative proof for the Borsuk-Ulam theorem using the equivalence of Fan's lemma and the Borsuk-Ulam theorem.

The theory of simplicial complexes and triangulations will play an important role throughout this report and will be studied first.

Later the Borsuk-Ulam theorem and Fan's lemma will be used to prove the Brouwer fixed point theorem and the combinatorial Sperner's lemma.

Finally, a version of the Borsuk-Ulam theorem will be applied in determining the chromatic number of Kneser graphs, one of the earliest applications of topological methods in combinatorics.

As part of this bachelor project a proof is given for a number of lemmas, many of which were given as exercises in [Mat08]. These proofs will be marked with a *.

Chapter 1

Simplicial Complexes

1.1 Geometric Simplicial Complexes

Simplicial complexes play an important role in the connection between topology and combinatorics. In this chapter we will provide an introducion to simplices and simplicial complexes based on the first chapter of [Mat08]. The theory presented here will be of great importance throughout the rest of this report.

An important notion in the theory of simplices is that of affine indepence.

Definition 1.1. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be points in $\mathbb{R}^n$. We call them *affinely dependent* if there are real numbers $\alpha_1, \ldots, \alpha_k$, not all of them 0, such that $\sum_{i=1}^k \alpha_i \boldsymbol{v}_i = \mathbf{0}$ and $\sum_{i=1}^k \alpha_i = 0$. Otherwise $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are called *affinely independent.*

For two points affine independence means that $\boldsymbol{v}_1 \neq \boldsymbol{v}_2$; for three points it means that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ do not lie on a common line; for four points it means that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ do not lie on a common plane; and so on. Two other useful characterizations of affine independence are given in the next lemma.

Lemma 1.2. *Both of the following conditions are equivalent to affine independence of points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ in $\mathbb{R}^n$:*

- *The $k-1$ vectors $\boldsymbol{v}_1 - \boldsymbol{v}_k, \ldots, \boldsymbol{v}_{k-1} - \boldsymbol{v}_k$ are linearly independent.*
- *The $(n+1)$-dimensional vectors $(\boldsymbol{v}_1, 1), \ldots, (\boldsymbol{v}_k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.*

Proof.* We start by showing the equivalence for the first condition. Assuming $\boldsymbol{v}_1 - \boldsymbol{v}_k, \ldots, \boldsymbol{v}_{k-1} - \boldsymbol{v}_k$ are linearly dependent, we can find real numbers $\alpha_1, \ldots, \alpha_{k-1}$ not all of them zero such that $\sum_{i=1}^{k-1} \alpha_i (\boldsymbol{v}_i - \boldsymbol{v}_k) = \mathbf{0}$. Now define $\alpha_k = -\sum_{i=1}^{k-1} \alpha_i$, then $\sum_{i=1}^k \alpha_i = 0$, $\alpha_1, \ldots, \alpha_k$ are not all zero and

$$\begin{aligned}
\sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i &= \sum_{i=1}^{k-1} \alpha_i \boldsymbol{v}_i + \alpha_k \boldsymbol{v}_k \\
&= \sum_{i=1}^{k-1} \alpha_i \boldsymbol{v}_i - \sum_{i=1}^{k-1} \alpha_i \boldsymbol{v}_k \\
&= \sum_{i=1}^{k-1} \alpha_i (\boldsymbol{v}_i - \boldsymbol{v}_k) = \mathbf{0},
\end{aligned}$$

so $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are affinely dependent.

For the other implication, assume that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are affinely dependent, which means we can find real numbers $\alpha_1, \ldots, \alpha_k$ not all of them zero, such that $\sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i = \mathbf{0}$ and $\sum_{i=1}^{k} \alpha_i = 0$. Then it follows that $\alpha_k = -\sum_{i=1}^{k-1} \alpha_i$, so that

$$\begin{aligned}\sum_{i=1}^{k-1} \alpha_i(\boldsymbol{v}_i - \boldsymbol{v}_k) &= \sum_{i=1}^{k-1} \alpha_i \boldsymbol{v}_i - \sum_{i=1}^{k-1} \alpha_i \boldsymbol{v}_k \\ &= \sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i = \mathbf{0}.\end{aligned}$$

This means that $\boldsymbol{v}_1 - \boldsymbol{v}_k, \ldots, \boldsymbol{v}_{k-1} - \boldsymbol{v}_k$ are linearly dependent.

Now we show the equivalence of the second condition. Start by assuming that the vectors $(\boldsymbol{v}_1, 1), \ldots, (\boldsymbol{v}_k, 1)$ are linearly dependent. This means we can find real numbers $\alpha_1, \ldots, \alpha_k$, not all of them zero, such that $\sum_{i=1}^{k} \alpha_i(\boldsymbol{v}_i, 1) = \mathbf{0}$. But then $\sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i, = \mathbf{0}$ and $\sum_{i=1}^{k} \alpha_i = 0$, which means $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are affinely dependent.

For the other implication we assume that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are affinely dependent. Therefore we can find real numbers $\alpha_1, \ldots, \alpha_k$, not all of them zero such that $\sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i = \mathbf{0}$ and $\sum_{i=1}^{k} \alpha_i = 0$. Then

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i(\boldsymbol{v}_i, 1) &= (\sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i, \sum_{i=1}^{k} \alpha_i) \\ &= (\mathbf{0}, 0) = \mathbf{0},\end{aligned}$$

so $(\boldsymbol{v}_1, 1), \ldots, (\boldsymbol{v}_k, 1)$ are linearly dependent.

□

Note that $n + 1$ is the largest size of an affinely independent set of points in $\mathbb{R}^n$. Now we are ready to give some definitions concerning simplices and simplicial complexes.

Definition 1.3. A *simplex* σ is the convex hull of a finite affinely independent set A in $\mathbb{R}^n$. The points of A are called the *vertices* of σ. The *dimension* of σ is $\dim \sigma := |A| - 1$. Thus every k-dimensional simplex, or k-simplex, has $k + 1$ vertices.

Definition 1.4. The convex hull of an arbitrary subset, possibly empty, of vertices of a simplex σ is a *face* of σ. Thus every face is itself a simplex. Also, every simplex has $\emptyset$ as a face. A face of σ of dimension $\dim \sigma - 1$ is called a *facet* of σ.

The *relative interior* of a simplex σ arises from σ by removing all faces of dimension smaller than $\dim \sigma$.

Lemma 1.5. *Let σ be a simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Then any point $\boldsymbol{x} \in \sigma$ can be uniquely written as a convex combination of the vertices of σ:*

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i \text{ with } \alpha_1, \ldots, \alpha_n \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1.$$

Proof.* The existence of such a convex combination follows from the fact that σ is the convex hull of its vertices. Now suppose we have two distinct convex combinations equal to $\boldsymbol{x}$, say $\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i$ and $\boldsymbol{x} = \sum_{i=1}^{n} \beta_i \boldsymbol{v}_i$. Then by subtracting them we get $\sum_{i=1}^{n} (\alpha_i - \beta_i) \boldsymbol{v}_i = \mathbf{0}$ with

$(\alpha_1 - \beta_1), \dots, (\alpha_n - \beta_n)$ not all zero and $\sum_{i=1}^{n}(\alpha_i - \beta_i) = \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i = 0$, which is in contradiction with the affine independence of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Therefore, the convex combination is unique.

□

Definition 1.6. A nonempty family Δ of simplices is a *simplicial complex* if the following two conditions hold:

1. Each face of any simplex $\sigma \in \Delta$ is also a simplex of Δ.
2. The intersection $\sigma_1 \cap \sigma_2$ of any two simplices $\sigma_1, \sigma_2 \in \Delta$ is a face of both σ_1 and σ_2.

The union of all simplices in a simplicial complex Δ is called the *polyhedron* of Δ and is denoted by $\|\Delta\|$.

The *dimension* of a simplicial complex Δ is defined as the largest dimension of a simplex in Δ: $\dim \Delta := \max\{\dim \sigma : \sigma \in \Delta\}$.

The *vertex set* of Δ, denoted by $V(\Delta)$, is the union of the vertex sets of all simplices of Δ.

It is intuitively clear that the set of all faces of a simplex forms a simplicial complex. A proof can be found in [Mat08].

A simplicial complex consisting of all faces of an arbitrary n-dimensional simplex, including the simplex itself, will be denoted by σ^n. Note that $\|\sigma^n\|$ is a geometric n-simplex.

All simplicial complexes we will encounter in this report will be *finite*, meaning that they contain only finitely many simplices. Note that the polyhedron of a finite simplicial complex is always a compact space.

Definition 1.7. The relative interiors of all simplices of a simplicial complex Δ form a partition of the polyhedron $\|\Delta\|$: For each point $\boldsymbol{x} \in \|\Delta\|$ there exists exactly one simplex $\sigma \in \Delta$ containing $\boldsymbol{x}$ in its relative interior. This simplex is denoted by $\operatorname{supp}(\boldsymbol{x})$ and is called the *support* of $\boldsymbol{x}$. When $\boldsymbol{x}$ is written as a convex combination of the vertices of its support all coefficients α are greater than zero.

Definition 1.8. If $f : V(\Delta) \to \mathbb{R}^n$ is a mapping defined on the vertex set of a simplicial complex Δ, we define the function

$$\|f\| : \|\Delta\| \to \mathbb{R}^n,$$

the *affine extension* of f, by extending f *affinely* to the relative interiors of the simplices of Δ as follows: If $\sigma \in \Delta$, with vertices $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$, is the support of a point $\boldsymbol{x} \in \|\Delta\|$ then, according to Lemma 1.5, $\boldsymbol{x}$ can be written uniquely as $\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i$ with $\alpha_1, \dots, \alpha_k \geq 0$ and $\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i = 1$. Then we put

$$\|f\|(\boldsymbol{x}) = \sum_{i=1}^{k} \alpha_i f(\boldsymbol{v}_i).$$

From the construction it is clear that the resulting function is continuous on $\|\Delta\|$.

Definition 1.9. A *subcomplex* of a simplicial complex Δ is a subset of Δ that is itself a simplicial complex.

An important example of a subcomplex is the k-skeleton of a simplicial complex Δ. It consists of all simplices of Δ of dimension at most k.

1.2 Triangulations

Let X be a topological space. A simplicial complex Δ such that $X \cong \|\Delta\|$, if one exists, is called a *triangulation* of X.

Next we will give some examples of triangulations that we will encounter again later in this report.

Definition 1.10. The n-dimensional *cross-polytope* is the convex hull

$$\operatorname{conv}(\boldsymbol{e}_1, -\boldsymbol{e}_1, \dots, \boldsymbol{e}_n, -\boldsymbol{e}_n)$$

of the vectors of the standard orthonormal basis and their negatives. It can also be described by $\{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$.

For example, the 3-dimensional cross-polytope is a regular octahedron.

The boundary of the n-dimensional cross-polytope is homeomorphic to S^{n-1}, as can be seen using a central projection. Hence, the natural triangulation of the boundary of the n-dimensional cross-polytope provides a triangulation of the sphere S^{n-1}.

Lemma 1.11. *Let σ be a simplex with vertices $\boldsymbol{v}_1, \dots \boldsymbol{v}_n$, and let $P = \sigma \times [0,1]$ be the n-dimensional "prism above σ". Let the vertices of P be $\boldsymbol{v}'_1, \dots \boldsymbol{v}'_n, \boldsymbol{v}''_1, \dots \boldsymbol{v}''_n$, where each $\boldsymbol{v}'_i = (\boldsymbol{v}_i, 0)$ is a bottom vertex and $\boldsymbol{v}''_i = (\boldsymbol{v}_i, 1)$ is the top vertex above it. For $i = 1, \dots, n$ define $\sigma_i = \operatorname{conv}(\boldsymbol{v}'_1, \dots \boldsymbol{v}'_i, \boldsymbol{v}''_i, \dots \boldsymbol{v}''_n)$. Then $\sigma_1, \dots, \sigma_n$ triangulate P.*

Proof.* We first establish that σ_i is an n-dimensional simplex for $i = 1, \dots, n$ by showing that $\boldsymbol{v}'_1, \dots \boldsymbol{v}'_i, \boldsymbol{v}''_i, \dots \boldsymbol{v}''_n$ are affinely independent. By Lemma 1.2 this is equivalent to showing that $\boldsymbol{v}'_1 - \boldsymbol{v}'_i, \dots \boldsymbol{v}'_{i-1} - \boldsymbol{v}'_i, \boldsymbol{v}''_i - \boldsymbol{v}'_i, \boldsymbol{v}''_{i+1} - \boldsymbol{v}'_i, \dots, \boldsymbol{v}''_n - \boldsymbol{v}'_i$ are linearly independent. By taking these vectors as the columns of a matrix we get

$$\begin{pmatrix} \boldsymbol{v}_1 - \boldsymbol{v}_i & \dots & \boldsymbol{v}_{i-1} - \boldsymbol{v}_i & \boldsymbol{v}_i - \boldsymbol{v}_i & \boldsymbol{v}_{i+1} - \boldsymbol{v}_i & \dots & \boldsymbol{v}_n - \boldsymbol{v}_i \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{pmatrix},$$

which is similar to

$$\begin{pmatrix} \boldsymbol{v}_1 - \boldsymbol{v}_i & \dots & \boldsymbol{v}_{i-1} - \boldsymbol{v}_i & \boldsymbol{0} & \boldsymbol{v}_{i+1} - \boldsymbol{v}_i & \dots & \boldsymbol{v}_n - \boldsymbol{v}_i \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

From the affine independence of $\boldsymbol{v}_1, \dots \boldsymbol{v}_n$ and Lemma 1.2 it follows that the $n-1$ vectors $\boldsymbol{v}_1 - \boldsymbol{v}_i, \dots, \boldsymbol{v}_{i-1} - \boldsymbol{v}_i, \boldsymbol{v}_{i+1} - \boldsymbol{v}_i, \dots, \boldsymbol{v}_n - \boldsymbol{v}_i$ are linearly dependent and the vector $(\boldsymbol{0}, 1)$ is clearly linearly independent from the rest of the columns of this second matrix. This establishes the linear independence of $\boldsymbol{v}'_1 - \boldsymbol{v}'_i, \dots \boldsymbol{v}'_{i-1} - \boldsymbol{v}'_i, \boldsymbol{v}''_i - \boldsymbol{v}'_i, \boldsymbol{v}''_{i+1} - \boldsymbol{v}'_i, \dots, \boldsymbol{v}''_n - \boldsymbol{v}'_i$, so we can conclude that the σ_i are indeed n-dimensional simplices.

Next we show that $\sigma_1, \dots, \sigma_n$ cover P. To this end, take $(\boldsymbol{x}, h) \in P$ with $\boldsymbol{x} \in \sigma$ and $h \in [0,1]$. Then we can write $\boldsymbol{x}$ uniquely as $\boldsymbol{x} = \sum_{k=1}^n \alpha_k \boldsymbol{v}_k$ with $\sum_{k=1}^n \alpha_k = 1$ and $\alpha_1, \dots, \alpha_n \geq 0$. Now take $1 \leq i \leq n$ such that $\sum_{k=i}^n \alpha_k \geq h$ and $\sum_{k=i+1}^n \alpha_k \leq h$. Then, defining $t := h - \sum_{k=i+1}^n \alpha_k$, we can write

$$\begin{aligned}(\boldsymbol{x}, h) = {} & \alpha_1 \boldsymbol{v}'_1 + \dots + \alpha_{i-1} \boldsymbol{v}'_{i-1} \\ & + \alpha_i \left((1-t)\boldsymbol{v}'_i + t\boldsymbol{v}''_i\right) + \alpha_{i+1} \boldsymbol{v}''_{i+1} + \dots + \alpha_n \boldsymbol{v}''_n,\end{aligned}$$

which shows that $(\boldsymbol{x}, h)$ is a convex combination of the vertices of σ_i, so in particular $(\boldsymbol{x}, h) \in \sigma_i$.

The last thing we need to show, is that if two simplices of $\sigma_1, \dots, \sigma_n$ share a point, then this point has to lie in a common face of the two simplices. Take a point $(\boldsymbol{x}, h) \in P$ that lies in both σ_i and σ_j, with $i < j$. Then from the discussion above, we get that $\sum_{k=i}^n \alpha_k \geq h$, $\sum_{k=i+1}^n \alpha_k \leq h$, $\sum_{k=j}^n \alpha_k \geq h$ and $\sum_{k=j+1}^n \alpha_k \leq h$, from which the second and third condition can only be met simultaneously when $\sum_{k=j}^n \alpha_k = h$ and $\alpha_k = 0$ for $i < k < j$. Therefore $(\boldsymbol{x}, h)$ must lie in the common face $\operatorname{conv}(\boldsymbol{v}_1', \dots, \boldsymbol{v}_i', \boldsymbol{v}_j'', \dots, \boldsymbol{v}_n'')$.

Now we can conclude that $\sigma_1, \dots, \sigma_n$ together with all of their faces form a simplicial complex that triangulates P.

□

1.3 The Barycentric Subdivision

In this section we will present a way to refine an existing triangulation. This means that simplices in the triangulation are subdivided into simplices of smaller diameter. The next lemma gives a useful characterization of the diameter of a simplex. Throughout this report we will use the Euclidean norm and distance, unless stated otherwise.

Lemma 1.12. *The diameter of an arbitrary simplex σ is equal to the distance between some two vertices of σ.*

Proof.* Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be the vertices of σ. The diameter of σ is defined in the usual way as $\operatorname{diam}(\sigma) = \max\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{x}, \boldsymbol{y} \in \sigma\}$. Now suppose that for $\boldsymbol{x}, \boldsymbol{y} \in \sigma$, with $\boldsymbol{y} \neq \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, we have $\|\boldsymbol{x} - \boldsymbol{y}\| = \operatorname{diam}(\sigma)$. Then, writing $\boldsymbol{y} = \sum_{i=1}^n \alpha_i \boldsymbol{v}_i$ with $\alpha_1, \dots, \alpha_n \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$, we get

$$\begin{aligned}
\|\boldsymbol{x} - \boldsymbol{y}\| &= \|\sum_{i=1}^n \alpha_i \boldsymbol{x} - \sum_{i=1}^n \alpha_i \boldsymbol{v}_i\| \\
&= \|\sum_{i=1}^n \alpha_i (\boldsymbol{x} - \boldsymbol{v}_i)\| \\
&\leq \sum_{i=1}^n \alpha_i \|\boldsymbol{x} - \boldsymbol{v}_i\| \\
&\leq \sum_{i=1}^n \alpha_i \max\{\|\boldsymbol{x} - \boldsymbol{v}_j\| : \boldsymbol{v}_j \in \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}\} \\
&= \max\{\|\boldsymbol{x} - \boldsymbol{v}_j\| : \boldsymbol{v}_j \in \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}\},
\end{aligned}$$

where the first inequality follows from the triangle inequality.

Thus, we have shown that if $\boldsymbol{y} \neq \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ there is a vertex $\boldsymbol{v}_j$ of σ such that $\|\boldsymbol{x} - \boldsymbol{y}\| \leq \|\boldsymbol{x} - \boldsymbol{v}_j\|$. Repeating the same argument for $\boldsymbol{x}$, we get that

$$\operatorname{diam}(\sigma) = \max\{\|\boldsymbol{v}_i - \boldsymbol{v}_j\| : \boldsymbol{v}_i, \boldsymbol{v}_j \in \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}\},$$

which shows the diameter of σ is equal to the distance between some two vertices of σ.

□

Definition 1.13. Let Δ be a simplicial complex. The *(first) barycentric subdivision* of Δ, denoted by $\operatorname{sd}(\Delta)$, is the simplicial complex constructed as follows:

- For each simplex $\sigma \in \Delta$, $\mathrm{sd}(\Delta)$ has a vertex at the gravitational center, or barycenter, of σ. The gravitational center is the mean of all vertices of σ.
- If $\sigma_1 \subset \ldots \subset \sigma_n$ is a chain of simplices in Δ, then the simplex spanned by the corresponding vertices in $\mathrm{sd}(\Delta)$ is a simplex in $\mathrm{sd}(\Delta)$.

Note that this definition ensures that if σ is a simplex in $\mathrm{sd}(\Delta)$, then all faces of σ are simplices of $\mathrm{sd}(\Delta)$ as well.

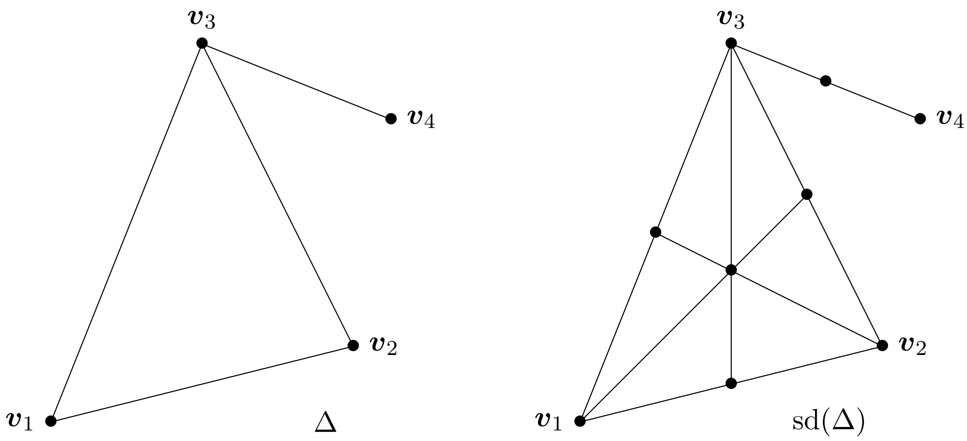

Figure 1.1: A simplicial complex Δ and its first barycentric subdivision $\mathrm{sd}(\Delta)$.

To confirm that the barycentric subdivision $\mathrm{sd}(\Delta)$ of a simplicial complex Δ is indeed a simplicial complex and that it triangulates $\|\Delta\|$, it suffices to show this for the case in which Δ is the simplicial complex σ^n.

Lemma 1.14. *The barycentric subdivision* $\mathrm{sd}(\sigma^n)$ *is a triangulation of* $\|\sigma^n\|$.

Proof.* First we check that the n-simplices in $\mathrm{sd}(\sigma^n)$ cover $\|\sigma^n\|$. To this end take an arbitrary $\boldsymbol{x} \in \|\sigma^n\|$. Number the vertices of σ^n in such a way that we get

$$\boldsymbol{x} = \sum_{i=1}^{n+1} \alpha_i \boldsymbol{v}_i \text{ with } \alpha_1 \geq \ldots \geq \alpha_{n+1} \geq 0 \text{ and } \sum_{i=1}^{n+1} \alpha_i = 1. \tag{1.1}$$

Now define

$$\overline{\boldsymbol{v}}_k := \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{v}_i \text{ for } k = 1, \ldots n+1,$$

then $\mathrm{conv}(\overline{\boldsymbol{v}}_1, \ldots, \overline{\boldsymbol{v}}_{n+1})$ is an n-simplex in $\mathrm{sd}(\sigma^n)$ since it corresponds to a chain of simplices in σ^n. Further, define

$$\beta_k := k(\alpha_k - \alpha_{k+1}) \text{ for } k = 1, \ldots, n \text{ and } \beta_{n+1} := (n+1)\alpha_{n+1},$$

then we get $\beta_1, \ldots, \beta_{n+1} \geq 0$ since $\alpha_i \geq \alpha_{i+1} \geq 0$ for $i = 1, \ldots, n$ and also

$$\begin{aligned} \sum_{i=1}^{n+1} \beta_i &= (\alpha_1 - \alpha_2) + 2(\alpha_2 - \alpha_3) + \ldots + n(\alpha_n - \alpha_{n+1}) + (n+1)\alpha_{n+1} \\ &= \sum_{i=1}^{n+1} \alpha_i = 1. \end{aligned}$$

Now observe that

$$\begin{aligned}\sum_{i=1}^{n+1} \beta_i \overline{\boldsymbol{v}}_i &= \alpha_{n+1}(\boldsymbol{v}_1 + \ldots + \boldsymbol{v}_{n+1}) + (\alpha_n - \alpha_{n+1})(\boldsymbol{v}_1 + \ldots + \boldsymbol{v}_n) + \ldots \\ &\quad + (\alpha_2 - \alpha_3)(\boldsymbol{v}_1 + \boldsymbol{v}_2) + (\alpha_1 - \alpha_2)\boldsymbol{v}_1 \\ &= \sum_{i=1}^{n+1} \alpha_i \boldsymbol{v}_i \\ &= \boldsymbol{x},\end{aligned} \tag{1.2}$$

which means that $\boldsymbol{x} \in \operatorname{conv}(\overline{\boldsymbol{v}}_1, \ldots, \overline{\boldsymbol{v}}_{n+1})$.

Notice that if some of the α_i are equal or zero then one or more of the $\overline{\boldsymbol{v}}_k$ do not contribute to $\boldsymbol{x}$. More precisely, $\boldsymbol{x}$ lies in the relative interior of the face of $\operatorname{conv}(\overline{\boldsymbol{v}}_1, \ldots, \overline{\boldsymbol{v}}_{n+1})$ spanned by the $\boldsymbol{v}_k$ for which $\beta_k \neq 0$. This face is also a simplex in $\operatorname{sd}(\sigma^n)$, as it corresponds to a subchain of the chain corresponding to $\operatorname{conv}(\overline{\boldsymbol{v}}_1, \ldots, \overline{\boldsymbol{v}}_{n+1})$. This simplex is uniquely determined since the convex combination 1.1 is unique and it is clear from 1.2 that it does not matter in which order the $\boldsymbol{v}_i$ for which the α_i are equal are numbered. Thus, this simplex is the support of $\boldsymbol{x}$ in $\operatorname{sd}(\sigma^n)$.

Each face of a simplex σ in $\operatorname{sd}(\sigma^n)$ is also a simplex of $\operatorname{sd}(\sigma^n)$, because it corresponds to a subchain of the chain of simplices in σ^n corresponding to σ.

Further, if σ and τ are simplices of $\operatorname{sd}(\sigma^n)$ then $\sigma \cap \tau$ is the convex hull of their common vertices, which is a face of both σ and τ and therefore also a simplex in $\operatorname{sd}(\sigma_n)$. This can be shown as follows: Suppose $\sigma \cap \tau$ is not the convex hull of their common vertices. Then there is an $\boldsymbol{x} \in \sigma \cap \tau$ that does not lie in the convex hull of their common vertices. However, $\operatorname{supp}(\boldsymbol{x})$ is a face of both σ and τ and therefore also a face of the convex hull of their common vertices, so $\boldsymbol{x}$ lies in the convex hull of the common vertices of σ and τ after all.

This confirms that $\operatorname{sd}(\sigma^n)$ is a simplicial complex. Thus, we have confirmed that $\operatorname{sd}(\sigma^n)$ is a triangulation of $\|\sigma^n\|$.

□

Now that we know that the barycentric subdivision of a simplicial complex Δ is a triangulation of $\|\Delta\|$, we will show that, using the barycentric subdivision, we can construct arbitrarily fine triangulations. We will do this using the following lemmas found in [Mun84].

Lemma 1.15. *Let σ be an n-dimensional simplex with barycenter $\hat{\sigma} := \frac{1}{n+1}\sum_{i=1}^{n+1} \boldsymbol{v}_i$, where $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1}$ are the vertices of σ. Then for all $\boldsymbol{x} \in \sigma$*

$$\|\hat{\sigma} - \boldsymbol{x}\| \leq \frac{n}{n+1}\operatorname{diam}(\sigma).$$

Proof. For any vertex $\boldsymbol{v}_j$ of σ we get

$$\begin{aligned}
\|\hat{\sigma} - \boldsymbol{v}_j\| &= \|\frac{1}{n+1}\sum_{i=1}^{n+1} \boldsymbol{v}_i - \boldsymbol{v}_j\| \\
&= \|\frac{1}{n+1}\sum_{i=1}^{n+1} (\boldsymbol{v}_i - \boldsymbol{v}_j)\| \\
&\leq \sum_{i=1}^{n+1} \|\frac{1}{n+1}(\boldsymbol{v}_j - \boldsymbol{v}_i)\| \\
&\leq \frac{n}{n+1} \max\{\|\boldsymbol{v}_j - \boldsymbol{v}_i\|\} \\
&\leq \frac{n}{n+1}\mathrm{diam}(\sigma),
\end{aligned}$$

where the first inequality follows from the triangle inequality, the second from the fact that the term in the summation for $i = j$ is zero and the last one from Lemma 1.12. Since this holds for any vertex $\boldsymbol{v}_j$ of σ we get that all vertices of σ lie in the closed ball with radius $\frac{n}{n+1}\mathrm{diam}(\sigma)$ centered at $\hat{\sigma}$. From the convexity of σ it follows that this closed ball contains σ, which concludes our proof. □

This result will be used in the proof of the next lemma.

Lemma 1.16. *Let σ be an n-dimensional simplex and let τ be a simplex in the first barycentric subdivision of σ, then*

$$\mathrm{diam}(\tau) \leq \frac{n}{n+1}\mathrm{diam}(\sigma).$$

Proof. The proof uses induction on the dimension n. For $n = 0$ the result is trivial. Now suppose it is true in dimensions less than n. By lemma 1.12 and the definition of the barycentric subdivision it suffices to show that if s and s' are faces of σ such that $'s \subset s$, then

$$\|\hat{s} - \hat{s}'\| \leq \frac{n}{n+1}\mathrm{diam}(\sigma).$$

If s equals σ itself, this inequality follows from lemma 1.15. If s is a proper face of σ of dimension $m < n$ we have

$$\begin{aligned}
\|\hat{s} - \hat{s}'\| &\leq \frac{m}{m+1}\mathrm{diam}(s) \\
&\leq \frac{n}{n+1}\mathrm{diam}(\sigma).
\end{aligned}$$

The first inequality follows by the induction hypothesis and the second from the fact that $f(x) = \frac{x}{x+1}$ is increasing for $x > 0$. □

From this lemma the next corollary easily follows.

Corollary 1.17. *Let Δ be a simplicial complex. For every $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that the diameter of all simplices in $\mathrm{sd}^k(\Delta)$ (barycentric subdivision iterated k times) is at most ε.*

Proof.* Let n be the dimension of Δ and d the maximum diameter of any simplex in Δ. Take $k \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^k \leq \frac{\varepsilon}{d}$. Then, using Lemma 1.16, we get that the diameter of any simplex in $\mathrm{sd}^k(\Delta)$ is at most $\left(\frac{n}{n+1}\right)^k d \leq \frac{\varepsilon}{d}d = \varepsilon$. □

Chapter 2

The Borsuk-Ulam Theorem

2.1 Borsuk-Ulam Equivalents

The Borsuk-Ulam theorem has many equivalent versions. In the next theorem we will state four equivalent statements that are all known as the Borsuk-Ulam theorem. We will also verify that they are indeed equivalent.

Theorem 2.1 (Borsuk-Ulam theorem)**.** *The following statements are equivalent.*

1. *For every continuous mapping $f : S^n \to \mathbb{R}^n$ there exists a point $\boldsymbol{x} \in S^n$ with $f(\boldsymbol{x}) = f(-\boldsymbol{x})$.*
2. *For every continuous* antipodal mapping *$f : S^n \to \mathbb{R}^n$, that is, $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S^n$, there exists a point $\boldsymbol{x} \in S^n$ satisfying $f(\boldsymbol{x}) = \mathbf{0}$.*
3. *There is no continuous antipodal mapping $f : S^n \to S^{n-1}$.*
4. *There is no continuous mapping $f : B^n \to S^{n-1}$ that is antipodal on the boundary, that is, satisfies $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S^{n-1} = \partial B^n$.*

Proof. The equivalence of the four statements follows from the following implications, which are based on a proof given in [Mat08].

1 $\Longrightarrow$ 2 If $f : S^n \to \mathbb{R}^n$ is a continuous antipodal mapping, according to statement 1 there exists a point $\boldsymbol{x} \in S^n$ with $f(\boldsymbol{x}) = f(-\boldsymbol{x})$. But since f is antipodal, we also get that $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$, which means that $f(\boldsymbol{x}) = \mathbf{0}$.

2 $\Longrightarrow$ 1 Let $f : S^n \to \mathbb{R}^n$ be a continuous mapping. Then $g(\boldsymbol{x}) := f(\boldsymbol{x}) - f(-\boldsymbol{x})$ is a continuous antipodal mapping. Statement 2 says that there is a point $\boldsymbol{x} \in S^n$ with $g(\boldsymbol{x}) = \mathbf{0}$ and so $f(\boldsymbol{x}) = f(-\boldsymbol{x})$.

2 $\Longrightarrow$ 3 Suppose, for contradiction, that there is a continuous antipodal mapping $f : S^n \to S^{n-1}$. Then, since $S^{n-1} \subset \mathbb{R}^n$, f would be a continuous antipodal mapping $S^n \to \mathbb{R}^n$ without a zero, which is in contradiction with statement 2. This means that there is no continuous antipodal mapping $f : S^n \to S^{n-1}$.

3 $\Longrightarrow$ 2 Assume, for contradiciton, that $f : S^n \to \mathbb{R}^n$ is a continuous nowhere zero antipodal mapping. Then the mapping $g : S^n \to S^{n-1}$ given by $g(\boldsymbol{x}) := f(\boldsymbol{x})/\|f(\boldsymbol{x})\|$ is a continuous antipodal mapping $S^n \to S^{n-1}$, so it contradicts statement 3. Thus every continuous antipodal mapping $S^n \to \mathbb{R}^n$ has a zero.

3 $\implies$ 4 First note that the orthogonal projection $\pi : (x_1, \dots, x_{n+1}) \to (x_1, \dots, x_n)$ is a homeomorphism of the upper hemisphere $U = \{(x_1, \dots, x_{n+1}) \in S^n : x_{n+1} \geq 0\}$ of S^n to B^n. For a continuous mapping $g : B^n \to S^{n-1}$ that is antipodal on ∂B^n we can define a mapping $f : S^n \to S^{n-1}$ by $f(\boldsymbol{x}) = g(\pi(\boldsymbol{x}))$ and $f(-\boldsymbol{x}) = -g(\pi(\boldsymbol{x}))$ for $\boldsymbol{x}$ in the upper hemisphere U. This specifies f on the whole of S^n and since g is antipodal on ∂B^n, which can be seen as the equator of S^n, it is consistent. Further f is continuous since it is continuous on both of the closed hemispheres. Thus f is a continuous antipodal mapping $S^n \to S^{n-1}$ which contradicts statement 3, so there is no continuous mapping $f : B^n \to S^{n-1}$ that is antipodal on the boundary.

4 $\implies$ 3 From a continuous antipodal mapping $f : S^n \to S^{n-1}$ we can construct a continuous mapping $g : B^n \to S^{n-1}$ that is antipodal on the boundary ∂B^n by $g(\boldsymbol{x}) = f(\pi^{-1}(\boldsymbol{x}))$, where π^{-1} is the inverse of the projection π above. This contradicts statement 4, so there is no continuous antipodal mapping $f : S^n \to S^{n-1}$.

□

Another well know equivalent of the Borsuk-Ulam theorem is stated next.

Theorem 2.2 (Lyusternik-Schnirel'man theorem). *For any cover $U_1, \dots, U_{n+1}$ of the sphere S^n by $n+1$ sets, such that each of the first n sets of $U_1, \dots, U_{n+1}$ is either open or closed, at least one of the $n+1$ sets contains a pair of antipodal points.*

The case when all $n+1$ sets are closed is also known as the Lyusternik-Schnirel'man theorem, just as the case when all $n+1$ sets are open. All versions are equivalent, but since Theorem 2.2 is the most general, we will only verify the equivalence of this version and the Borsuk-Ulam theorem.

Theorem 2.3. *The Borsuk-Ulam theorem (Theorem 2.1) and the Lyusternik-Schnirel'man theorem (Theorem 2.2) are equivalent.*

Proof. This proof is based on proofs given in [AZ14] and [Mat08]. We first show that the first statement of the Borsuk-Ulam theorem implies the Lyusternik-Schnirel'man theorem.
Let $U_1, \dots, U_{n+1}$ be a cover of the sphere S^n, such that $U_1, \dots, U_n$ are either open or closed and assume that no U_i contains a pair of antipodal points. We define a map $f : S^n \to \mathbb{R}^n$ by

$$f(\boldsymbol{x}) := (\delta(\boldsymbol{x}, U_1), \dots, \delta(\boldsymbol{x}, U_{n+1})),$$

where $\delta(\boldsymbol{x}, U_i)$ denotes the distance from $\boldsymbol{x}$ to U_i: $\delta(x, U_i) := \inf\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{y} \in U_i\}$. From the continuity of δ it follows that f is also continuous. From the first statement of Theorem 2.1 we get that there is a point $\boldsymbol{x} \in S^n$ with $f(\boldsymbol{x}) = f(-\boldsymbol{x})$. Because U_{n+1} does not contain a pair of antipodal points, at least one of the points $\boldsymbol{x}$ and $-\boldsymbol{x}$ must lie in one of the sets $U_1, \dots, U_n$. After exchanging $\boldsymbol{x}$ with$-\boldsymbol{x}$, if necessary, we may assume that $\boldsymbol{x} \in U_k$ for some $1 \leq k \leq n$. This yields $\delta(\boldsymbol{x}, U_k) = 0$ and since $f(\boldsymbol{x}) = f(-\boldsymbol{x})$ we get that $\delta(-\boldsymbol{x}, U_k) = 0$ as well.

If U_k is closed, then $\delta(-\boldsymbol{x}, U_k) = 0$ implies that $-\boldsymbol{x} \in U_k$. But then both $\boldsymbol{x}$ and $-\boldsymbol{x}$ are in U_k, which contradicts the assumption that no U_i contains a pair of antipodal points.

If U_k is open, then $\delta(-\boldsymbol{x}, U_k) = 0$ means that $-\boldsymbol{x}$ lies in the closure $\overline{U_k}$ of U_k. This $\overline{U_k}$ is contained in $S^n \backslash (-U_k)$, since this is a closed subset of S^n that contains U_k (because U_k does not contain a pair of antipodal points $U_k \cap -U_k = \emptyset$). This means that $-\boldsymbol{x}$ lies in $S^n \backslash (-U_k)$, so it cannot lie in $-U_k$, and therefore $\boldsymbol{x}$ cannot lie in U_k. Again, we have reached a contradiction.

Since the assumption that no U_i contains a pair of antipodal points leads to a contradiction in both cases, our assumption must be false, which means that one of the U_i does contain a pair of antipodal points.

Next we show that the Lyusternik-Schirel'man theorem implies the third statement of the Borsuk-Ulam theorem. First note that if we take an n-simplex in $\mathbb{R}^n$ containing $\mathbf{0}$ in its interior and we project the facets centrally from $\mathbf{0}$ on S^{n-1}, then the projection of a facet falls within an open hemisphere of S^{n-1}. This hemisphere is defined by the $(n-1)$-dimensional hyperplane through $\mathbf{0}$ parallel to the facet. Now let $F_1, \dots, F_{n+1}$ be the projections of the $n+1$ facets of the n-simplex on S^{n-1}, then no F_i contains a pair of antipodal points.

Now, assume that $f : S^n \to S^{n-1}$ is a continuous antipodal mapping, so $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$, then the sets $f^{-1}(F_1), \dots, f^{-1}(F_{n+1})$ are $n+1$ closed sets (since the F_i are closed and f is continuous) containing no antipodal points (since f is antipodal) that cover S^n. This is in contradiction with Theorem 2.2, so there can not be a continuous antipodal mapping $f : S^n \to S^{n-1}$. $\square$

2.2 A Geometric Proof

In this section we present a geometric proof of the Borsuk-Ulam theorem, based on a proof given in [Mat08]. For this proof we need a number of lemmas which will be stated and proved here first.

Lemma 2.4. *Let $p(x_1, x_2, \dots, x_n) = p(\mathbf{x})$ be a nonzero polynomial in n variables. Then the zero set $Z(p) := \{\boldsymbol{x} \in \mathbb{R}^n : p(\boldsymbol{x}) = 0\}$ is nowhere dense, meaning that any open ball $B \subseteq \mathbb{R}^n$ contains an open ball B' with $B' \cap Z(p) = \emptyset$.*

Proof.* Suppose there exists an open ball B with center $\mathbf{a}$ for which $p(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in B$. Take the polynomial $q(\boldsymbol{x}) = p(\boldsymbol{x} + \mathbf{a})$ which is zero on an open ball around the origin and therefore all partial derivatives of $q(\boldsymbol{x})$ are zero in $\mathbf{0}$. Let m be the maximum degree of any term of q. Then, writing

$$q(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \{0,1,\dots,m\}^n} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \text{ where } \boldsymbol{x}^{\boldsymbol{\alpha}} := \prod_{i=1}^{n} x_i^{\alpha_i},$$

we get

$$\prod_{i=1}^{n} \frac{\partial^{\alpha_i}}{\partial x^{\alpha_i}} q(\boldsymbol{x}) = \sum_{i=1}^{n} \prod_{k=1}^{\alpha_i} k c_{\boldsymbol{\alpha}} + r(\boldsymbol{x}),$$

where $\sum_{i=1}^{n} \prod_{k=1}^{\alpha_i} k > 0$ for $\boldsymbol{\alpha} \neq 0$ and $r(\boldsymbol{x})$ is a polynomial without a constant term. This is because taking $\boldsymbol{\alpha}$-repeated partial derivatives of a monomial $\boldsymbol{x}^{\boldsymbol{\beta}}$ can only result in the following:

1. A nonzero constant if $\boldsymbol{\beta} = \boldsymbol{\alpha}$.
2. A non-constant monomial if $\beta_i > \alpha_i$ for some $i \in [n] = \{1, \dots, n\}$ and $\beta_i < \alpha_i$ for no $i \in [n]$.
3. Zero if $\beta_i < \alpha_i$ for some $i \in [n]$.

Thus, for each $\boldsymbol{\alpha} \in \{0, 1, \dots, m\}^n$, plugging in $\mathbf{0}$ in the corresponding partial derivative (for $\boldsymbol{\alpha} = \mathbf{0}$ this is just $q(\boldsymbol{x})$) then yields that all coefficients $c_{\boldsymbol{\alpha}}$ of $q(\boldsymbol{x})$ must be zero and therefore $q(\boldsymbol{x})$ is the zero polynomial. But this implies that $p(\boldsymbol{x})$ must also be the zero polynomial and this is in contradiction with our assumptions. Therefore every open ball B must contain a point on which $p(\boldsymbol{x}) \neq 0$.

Now suppose, for contradiction, that $Z(p)$ is not nowhere dense, this means that there is an open ball B so that for every open ball $B' \subseteq B$ there is an $\boldsymbol{x} \in B'$ with $p(\boldsymbol{x}) = 0$. Take B as above and take a point $\mathbf{a} \in B$ with $p(\mathbf{a}) \neq 0$. Now for every $n \in \mathbb{N}$ fix a point $\boldsymbol{x}_n \in B(\mathbf{a}, \frac{1}{n})$ so that $p(\boldsymbol{x}_n) = 0$. From the construction it is clear that $\boldsymbol{x}_n \to \mathbf{a}$. The continuity of $p(\boldsymbol{x})$ implies that $p(\boldsymbol{x}_n) \to p(\mathbf{a})$. However $p(\boldsymbol{x}_n) = 0$ for all $n \in \mathbb{N}$ and $p(\mathbf{a}) \neq 0$. This contradiction implies that $Z(p)$ is nowhere dense.

□

Lemma 2.5. *Let $A_1, A_2, \ldots, A_n$ be a collection of nowhere dense sets. Then the union $\bigcup_{i=1}^n A_i$ is also nowhere dense.*

Proof.* Suppose, as our induction hypothesis, that $\bigcup_{i=1}^k A_i$ is nowhere dense. We show that $\bigcup_{i=1}^{k+1} A_i$ is nowhere dense. Take an open ball B, then there is an open ball $B_k \subseteq B$ with $B_k \cap \bigcup_{i=1}^k A_i = \emptyset$. Since A_{k+1} is nowhere dense, we can find an open ball $B_{k+1} \subseteq B_k$ with $B_{k+1} \cap A_{k+1} = \emptyset$. This implies that $B_{k+1} \subseteq B$ and $B_{k+1} \cap \bigcup_{i=1}^{k+1} A_i = \emptyset$, so $\bigcup_{i=1}^{k+1} A_i$ is nowhere dense. Since all the A_i are nowhere dense, our induction hypothesis holds trivially for $k = 1$, so by induction it follows that $\bigcup_{i=1}^n A_i$ is nowhere dense.

□

Let $\sigma := \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+2})$ be an $n+1$ dimensional simplex in $\mathbb{R}^{n+1}$ and $h : \sigma \to \mathbb{R}^n$ be an affine map, that is, a map of the form $\boldsymbol{x} \mapsto A\boldsymbol{x} + \mathbf{b}$, where A is an $n \times (n+1)$ matrix and $\mathbf{b} \in \mathbb{R}^n$. We call h *generic* if $h^{-1}(\mathbf{0})$ intersects no face of σ of dimension smaller than n. This means that $h^{-1}(\mathbf{0})$ is either empty or a segment lying in the interior of σ with endpoints lying in the interior of two distinct n-dimensional faces of σ. If h is nongeneric then there must be a point $\boldsymbol{x}$ lying in an $(n-1)$-dimensional face of σ that gets mapped to $\mathbf{0}$. Assume the vertices of this face are $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. Then $\boldsymbol{x}$ is a convex combination of these n vertices:

$$\boldsymbol{x} = \sum_{i=1}^n \alpha_i \boldsymbol{v}_i \text{ with } \alpha_1, \ldots, \alpha_n \geq 0 \text{ and } \sum_{i=1}^n \alpha_i = 1.$$

Since $\boldsymbol{x} \in h^{-1}(\mathbf{0})$ we get

$$\begin{aligned}
\mathbf{0} = h(\boldsymbol{x}) &= A\boldsymbol{x} + \mathbf{b} \\
&= A(\sum_{i=1}^n \alpha_i \boldsymbol{v}_i) + \mathbf{b} \\
&= \sum_{i=1}^n \alpha_i A\boldsymbol{v}_i + \sum_{i=1}^n (\alpha_i \mathbf{b}) \\
&= \sum_{i=1}^n \alpha_i (A\boldsymbol{v}_i + \mathbf{b}) \\
&= \sum_{i=1}^n \alpha_i h(\boldsymbol{v}_i)
\end{aligned}$$

As the α_i can not all be zero, we get that $h(\boldsymbol{v}_1), \ldots, h(\boldsymbol{v}_n)$ are linearly dependent.

Lemma 2.6. *Let X be an $(n+1)$-dimensional subset of $\mathbb{R}^{n+2}$ that has a triangulation T Let $H : X \to \mathbb{R}^n$ be a function that is affine on each simplex of T. Then there exists a* generic *map G, a map that is generic on each full dimensional simplex of T, arbitrarily close to H, that is, for all $\varepsilon > 0$ there is a generic map G such that $\|H(\boldsymbol{x}) - G(\boldsymbol{x})\| < \varepsilon$ for all $\boldsymbol{x} \in X$.*

*Proof**. Let $N = V(\mathsf{T})$ be the number of vertices in T. Then a function $F : X \to \mathbb{R}^n$ that is affine on each simplex of T is fully determined by the N values in $\mathbb{R}^n$ it takes on the vertices of T and therefore each possible F can be associated with a vector in $\mathbb{R}^{nN}$.

Let Σ be the set of all $(n-1)$-dimensional simplices in T and for each $\sigma \in \Sigma$ let A_σ be the $n \times n$ matrix with the images of the vertices of σ as columns. For each $\sigma \in \Sigma$ the determinant $\det(A_\sigma)$ can be written as a polynomial p_σ in the nN coefficients of F. We know that if the n vertices of an $(n-1)$-dimensional simplex in T are mapped to n linearly independent vectors, then this simplex does not contain a zero. So the vectors in $\mathbb{R}^{nN}$ belonging to the maps F that lead to a zero in a particular $\sigma \in \Sigma$ lie in the zero set $Z(p_\sigma)$. Now, according to Lemma 2.4, the zero set $Z(p_\sigma)$ is nowhere dense in $\mathbb{R}^{nN}$ for all $\sigma \in \Sigma$ and by Lemma 2.5 it follows that the union $\bigcup_{\sigma\in\Sigma} Z(p_\sigma)$ is also nowhere dense in $\mathbb{R}^{nN}$. Therefore the set of all vectors in $\mathbb{R}^{nN}$ for which F is nongeneric is nowhere dense in $\mathbb{R}^{nN}$ and this means that there is a generic map G arbitrarily close to H.

□

Before we begin our proof of the Borsuk-Ulam theorem, we will first state the exact version of the theorem that we will prove.

Theorem 2.7. *Let $f : S^n \to \mathbb{R}^n$ be a continuous antipodal map. Then there exists a point $\boldsymbol{x} \in S^n$ satisfying $f(\boldsymbol{x}) = \boldsymbol{0}$. This is statement 2 of Theorem 2.1.*

Proof. Let $\hat{S}^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : \sum_{i=1}^n |x_i| = 1\}$ be the boundary of a cross-polytope. Since $\hat{S}^n$ is homeomorphic to S^n we can use $\hat{S}^n$ instead of S^n and we will do so in the rest of the proof. Because of the symmetry of $\hat{S}^n$ we can make sure that f is antipodal on $\hat{S}^n$ just like on S^n. Suppose, for contradiction, that $f : \hat{S}^n \to \mathbb{R}^n$ has no zeros. Since $\hat{S}^n$ is compact, there is an $\varepsilon > 0$ such that $\|f(\boldsymbol{x})\| > \varepsilon$ for all $\boldsymbol{x} \in \hat{S}^n$. For this ε we can find $\delta > 0$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in \hat{S}^n$ if $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$ then $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| < \varepsilon$. We now construct a triangulation T_S of $\hat{S}^n$ by iterated barycentric subdivision of the natural triangulation of $\hat{S}^n$ such that all simplices in T_S have diameter smaller than this δ. The number of iterated barycentric subdivisions needed, denoted by k, will be used later.

Let $\bar{f} : \hat{S}^n \to \mathbb{R}^n$ be the map that agrees with f on the vertex set $V(\mathsf{T}_S)$ of T_S and is affine on each simplex of T_S. Note that $\bar{f}$ is antipodal since f is antipodal and the triangulation T_S is symmetric. For all $\boldsymbol{y} \in \hat{S}^n$ we can write $\boldsymbol{y} = \sum_{i=1}^{n+1} \alpha_i \boldsymbol{v}_i$ with $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n+1} \in V(\mathsf{T}_S)$, $\alpha_1, \dots, \alpha_{n+1} \geq 0$ and $\sum_{i=1}^{n+1} \alpha_i = 1$. Therefore we get

$$\begin{aligned} \|f(\boldsymbol{y}) - \bar{f}(\boldsymbol{y})\| &= \|\sum_{i=1}^{n+1} \alpha_i (f(\boldsymbol{y}) - f(\boldsymbol{v}_i))\| \\ &\leq \sum_{i=1}^{n+1} \alpha_i \|f(\boldsymbol{y}) - f(\boldsymbol{v}_i)\| \\ &< \sum_{i=1}^{n+1} \alpha_i \varepsilon \\ &= \varepsilon, \end{aligned}$$

so $\bar{f}$ has no zeros in $\hat{S}^n$. The inequality above follows from the triangle inequality and the fact that $\boldsymbol{y}$ lies in the simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1}$, so the distance from $\boldsymbol{y}$ to either of these vertices is smaller than δ.

Let g be an orthogonal projection from $\hat{S}^n$ to $\mathbb{R}^n$ in a "generic" direction such that the only two zeros $\mathbf{z}_1$ and $\mathbf{z}_2$ lie in the interior of n-dimensional simplices of the triangulation T_S. Note that g is affine and antipodal, so in particular $\mathbf{z}_1 = -\mathbf{z}_2$.

Now we construct a space $X := \hat{S}^n \times [0,1]$ which can be seen as a "hollow cylinder" placed in $\mathbb{R}^{n+2}$ with two copies of $\hat{S}^n$ at the top and bottom. We will refer to $\hat{S}^n \times \{0\}$ as the *bottom sphere* and to $\hat{S}^n \times \{1\}$ as the *top sphere* in X. In order to construct a triangulation T of X we first triangulate the simplicial prisms $\sigma \times [0,1]$ for all n-dimensional simplices σ in the natural triangulation of $\hat{S}^n$ according to Lemma 1.11 in such a way that the triangulations of adjacent prisms are mirror images of each other. This is possible since an even number of n-simplices of the natural triangulation of $\hat{S}^n$ meets in each vertex, due to the symmetry of $\hat{S}^n$. Next we refine the triangulation by k iterated subdivisions. Note that then the triangulation T is the same as T_S on the top and bottom sphere.

Next we define the function $F : X \to \mathbb{R}^n$ by $F(\boldsymbol{x}, t) = (1-t)g(\boldsymbol{x}) + t\bar{f}(\boldsymbol{x})$. Let ν be the map on X with $\nu(\boldsymbol{x}, t) = (-\boldsymbol{x}, t)$ and call it the *antipodality on* X. Since both g an $\bar{f}$ are antipodal it follows that $F(\nu(\boldsymbol{x}, t)) = -F(\boldsymbol{x}, t)$. By construction it is clear that F is affine and it has the following two properties.

(2.1) $\qquad F$ has no zeros on the top sphere,

and

(2.2) $\qquad F$ has exactly two zeros on the bottom sphere, lying in the interiors of n-dimensional antipodal simplices.

The following step is to introduce a perturbation map $P_0 : V(\mathsf{T}) \to \mathbb{R}^n$ satisfying $P_0(\nu(\boldsymbol{v})) = -P_0(\boldsymbol{v})$ for each $\boldsymbol{v} \in V(\mathsf{T})$, which will be specified later. P_0 is extended affinely on each simplex of T to a map $P : X \to \mathbb{R}^n$ and then we set $\tilde{F} = F + P$. If the values of P_0 lie sufficiently close to $\mathbf{0}$, then the perturbed map $\tilde{F}$ still has properties 2.1 and 2.2. Since F has no zero on the top sphere, then if the perturbation is small enough $\tilde{F}$ has no zero there either. Further, if σ is a simplex on the bottom sphere containing one of the two zeros of F, then F maps this simplex to some n-dimensional simplex τ in $\mathbb{R}^n$ containing the origin in its interior. Again, if the perturbation, which causes a small movement of the vertices of τ is small enough, $\tilde{F}$ has a zero in the interior of the same simplex σ. Similarly, a simplex that does not contain a zero retains this property after a small enough perturbation. If T has $2N$ vertices, then the space of all possible antipodal perturbation maps P_0 on $V(\mathsf{T})$ has dimension nN, since the value can be chosen freely on a set of N vertices containing no two antipodal vertices. Then by Lemma 2.6 there exists a generic perturbed map $\tilde{F}$ arbitrarily close to F. Here we use the fact that no simplex in T contains any two antipodal vertices. Otherwise all perturbed mappings would have a zero on the middle of the edge connecting them and would be nongeneric for that reason. This means that an arbitrarily small perturbation P_0 exists, such that $\tilde{F}$ is generic.

Thus, there is a sufficiently small perturbation P_0 such that the perturbed mapping $\tilde{F}$ is generic and still has no zeros on the top sphere and exactly two zeros on the bottom sphere, lying in the interiors of n-dimensional antipodal simplices. For this $\tilde{F}$ the zero set $\tilde{F}^{-1}(\mathbf{0})$ is a locally polygonal path consisting of segments with no branchings. This is because each n-simplex $\tau \in \mathsf{T}$ is the face of exactly two $(n+1)$-simplices, unless τ lies in the top or bottom sphere, in which case it is a face of only one $(n+1)$-simplex in T. Therefore the components of $\tilde{F}^{-1}(\mathbf{0})$ are zero or more closed polygonal cycles and a polygonal path γ connecting $\mathbf{z}_1$ and $\mathbf{z}_2$.

This γ consists of finitely many segments and is symmetric under ν. This means that, depending on whether the number of segments in γ is even or odd, either the middle vertex on γ has to be its own antipode or the middle segment has to connect two antipodal vertices of T. However X does not contain a point antipodal to itself and there are no segments in T connecting two antipodal vertices since T uses the natural triangulation of the boundary of the cross-polytope. We have reached a contradiction, so we can conclude that f does have a zero, which concludes our proof.

□

Chapter 3

Fan's Lemma

3.1 Fan's $n + 1$ Lemma

In this section, based on [NS13] we will present Fan's $n + 1$ lemma, which is a combinatorial equivalent of the Borsuk-Ulam theorem. As we shall see in the next chapter, there is a direct proof that Fan's $n+1$ lemma implies Sperner's lemma, a combinatorial equivalent to the Brouwer fixed point theorem.

In order to state Fan's $n + 1$ lemma, we need to introduce some terminology.

A triangulation T of S^n is *symmetric* if when a simplex σ is in T, then $-\sigma$ is also in T. Define an *m-labeling* of a triangulation T to be a function ℓ that assigns to each vertex $\boldsymbol{v}$ of T one of $2m$ possible integers: $\{\pm 1, \ldots, \pm m\}$. We will call a labeling of a symmetric triangulation *anti-symmetric* if the labels of each pair of antipodal vertices sum to zero. Further, a labeling has a *complementary edge* if two adjacent vertices have labels that sum to zero.

We call an n-simplex in an m-labeled triangulation *alternating* if its vertex labels are distinct in magnitude and alternate in sign when arranged in order of increasing absolute value, that is, the labels have the form

$$\{k_0, -k_1, k_2, \ldots, (-1)^n k_n\} \text{ or } \{-k_0, k_1, -k_2, \ldots, (-1)^{n+1} k_n\},$$

where $1 \leq k_0 < k_1 < \ldots < k_n \leq m$. The *sign* of an alternating simplex is the sign of k_0, that is, the sign of the smallest label in absolute value. Depending on the sign we will call them positive alternating simplices or negative alternating simplices.

Theorem 3.1 (Fan's $n+1$ lemma). *Let T be a symmetric triangulation of S^n that is a refinement of the natural triangulation of $\hat{S}^n$ with an $(n + 1)$-labeling that is anti-symmetric and has no complementary edge. Then T has a positive alternating n-simplex.*

We will show now that Fan's $n + 1$ lemma is equivalent to the Borsuk-Ulam theorem. We will identify S^n again with the boundary of a concentric cross-polytope $\hat{S}^n$ through a projection from the origin, which is a homeomorphism.

Theorem 3.2. *The Borsuk-Ulam theorem and Fan's $n + 1$ lemma are equivalent.*

Proof. We first show that the Borsuk-Ulam theorem implies Fan's $n+1$ lemma. Let T be a symmetric triangulation of $\hat{S}^n$ with an anti-symmetric $(n+1)$-labeling L that has no complementary edges. Let $\boldsymbol{w}_i \in \mathbb{R}^{n+1}$ be the point with ith coordinate n and other coordinates -1:

$$\boldsymbol{w}_i = (-1, \ldots, -1, n, -1, \ldots, -1).$$

Let $W_+ = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n+1}\}$ and $W_- = \{-\boldsymbol{w}_1, \ldots, -\boldsymbol{w}_{n+1}\}$ and define $\boldsymbol{w}_{-i} = -\boldsymbol{w}_i$. Then the $2n+2$ points in $W = W_+ \cup W_-$ lie on the n-dimensional hyperplane $H = \{(x_1, \ldots, x_{n+1}) : \sum_{i=1}^{n+1} x_i = 0\}$.

We now define a continuous map $h : \hat{S}^n \to H$ as follows. For each $\boldsymbol{v} \in V(\mathsf{T})$, let

$$h(\boldsymbol{v}) = \begin{cases} \boldsymbol{w}_{L(\boldsymbol{v})} & \text{if } L(\boldsymbol{v}) \text{ is odd} \\ -\boldsymbol{w}_{L(\boldsymbol{v})} & \text{if } L(\boldsymbol{v}) \text{ is even.} \end{cases} \tag{3.1}$$

Then h is affinely extended on each simplex of T. From the anti-symmetry of L it follow that $h(-\boldsymbol{x}) = -h(\boldsymbol{x})$ for all $\boldsymbol{x} \in \hat{S}^n$. Therefore, by the Borsuk-Ulam theorem (the second statement of Theorem 2.1) and our homeomorphism of S^n with $\hat{S}^n$, we get that there is a $\boldsymbol{z} \in \hat{S}^n$ such that $h(\boldsymbol{z}) = \boldsymbol{0}$.

Since the n-simplices in T cover $\hat{S}^n$, $\boldsymbol{z}$ is in some n-simplex σ for which $h(\sigma)$ contains the origin. The images of the vertices of this σ form a set K, which is a subset of W of size $n+1$ (or smaller, in case L assigns the same label to more than one vertex of σ). Since there are no complementary edges in T, we know that K contains no pair $\{\boldsymbol{w}_j, -\boldsymbol{w}_j\}$. This means that we can write $K = \{\boldsymbol{w}_j\}_{j\in B} \cup \{-\boldsymbol{w}_j\}_{j\in B'}$, where B and B' are disjoint subsets of $\{1, \ldots, n+1\}$.

Now define the vector $\boldsymbol{v}$ as the sum of all vectors in K:

$$\boldsymbol{v} = \sum_{j\in B} \boldsymbol{w}_j - \sum_{j\in B'} \boldsymbol{w}_j,$$

Note that for the dot product between two vectors in W_+ we get $\boldsymbol{w}_i \cdot \boldsymbol{w}_i = n(n+1)$ for all $i \in \{1, \ldots, n+1\}$ and $\boldsymbol{w}_i \cdot \boldsymbol{w}_j = -(n+1)$ for all $j \neq i$. Then, for $i \in B$, we get

$$\boldsymbol{w}_i \cdot \boldsymbol{v} = n(n+1) - (|B|-1)(n+1) + |B'|(n+1) = (n+1)(n+1-|B|+|B'|),$$

which is positive unless $|B| = n+1$ and $|B'| = 0$, meaning that $K = W_+$. For $i \in B'$ we get

$$-\boldsymbol{w}_i \cdot \boldsymbol{v} = |B|(n+1) - n(n+1) - (|B'|-1)(n+1) = (n+1)(|B|-|B'|+n+1),$$

which is positive unless $|B'| = n+1$ and $|B| = 0$, meaning that $K = W_-$. Since the convex hull of K contains the origin, it cannot be the case that all vectors in K have a positive dot product with $\boldsymbol{v}$. This means that either $K = W_+$ or $K = W_-$.

If $K = W_+$, then from (3.1) it follows that a vector with image $\boldsymbol{w}_i$ must have label i if i is odd and label $-i$ if i is even. This means that σ has labels $1, -2, \ldots, (-1)^n(n+1)$. If $K = W_-$, then from (3.1) and the anti-symmetry of L it follows that a vector with image $-\boldsymbol{w}_i$ must have label i if i is odd and label $-i$ if i is even. Again we get that σ has labels $1, -2, \ldots, (-1)^n(n+1)$, so in all cases we find a positive alternating n-simplex.

Next we show that Fan's $n+1$ lemma implies the Borsuk-Ulam theorem. Let $h : S^n \to \mathbb{R}^n$ be a continuous antipodal function. Again we use $\hat{S}^n$ instead of S^n, where our homeomorphism of S^n and $\hat{S}^n$ ensures that $h : \hat{S}^n \to \mathbb{R}^n$ is antipodal as well, so $h(-\boldsymbol{x}) = -h(\boldsymbol{x})$ for all $\boldsymbol{x} \in \hat{S}^n$. Assume, for contradiction, that there is no point $\boldsymbol{z} \in \hat{S}^n$ such that $h(\boldsymbol{z}) = \boldsymbol{0}$. If $h(\boldsymbol{x}) = (x'_1, \ldots, x'_n)$, let $\hat{h} : \hat{S}^N \to \mathbb{R}^{n+1}$ be the function defined by $\hat{h}(\boldsymbol{x}) = (x'_1, \ldots, x'_n, -\sum_{i=1}^n x'_i)$. Note that $\hat{h}$ maps $\hat{S}^n$ to the hyperplane H defined above and that it is continuous and antipodal, which follows from the continuity and antipodality of h. Also there is no point $\boldsymbol{z}$ so that $\hat{h}(\boldsymbol{z}) = 0$.

Let T_k be a symmetric triangulation of $\hat{S}^n$ such that the diameter of all simplices in T_k is smaller than $\frac{1}{k}$ and let the set W be as above. Our next aim is to construct a labeling L_k of the vertices of T_k that is antisymmetric.

Define $L_k(\boldsymbol{v})$ to be the index i of smallest absolute value such that for all $j \in \{\pm 1, \dots, \pm(n+1)\}$ we have $\|\boldsymbol{w}_i - \hat{h}(\boldsymbol{v})\| \leq \|\boldsymbol{w}_j - \hat{h}(\boldsymbol{v})\|$. To see that this is well defined, note that $\hat{h}(\boldsymbol{v})$ is never zero and that no other point than the origin can be equidistant from $\boldsymbol{w}_i$ and $\boldsymbol{w}_{-i}$. This last fact is not directly obvious, but can be explained by the following.

Since $\|\boldsymbol{w}_i\|$ is the same for all $i \in [n+1] = \{1, \dots, n+1\}$ and $\|\boldsymbol{w}_i - \boldsymbol{w}_j\|$ is also equal for all $i \neq j \in [n+1]$, we know that the points in W_+ are evenly distributed round the origin on the n-dimensional hyperplane $H = \{(x_1, \dots, x_{n+1}) : \sum_{i=1}^{n+1} x_i = 0\}$. And in the same way the points in W_- lie at their negative positions. Also note that for $i \in [n+1]$ we have that $\|\boldsymbol{w}_i - \boldsymbol{w}_{-j}\|$ is the same for all $j \in [n+1] \setminus i$. Now let P_{ij} be the hyperplane of points that have the same distance to $\boldsymbol{w}_i$ as to $\boldsymbol{w}_{-j}$. Then the collection of hyperplanes $\{P_{ij}\}$ with $i \neq j \in [n+1]$ divides H in $2n+2$ subsets where the points in each subset lie closest to some $\boldsymbol{w}_i$ or $\boldsymbol{w}_{-i}$. The only point that lies in all these subsets is the origin and because of the regular distribution of W_+ and W_- a point other than the origin cannot have equal distance to both $\boldsymbol{w}_i$ and $\boldsymbol{w}_{-i}$ without being closer to one of the other points in W_+ or W_-.

Further L_k is anti-symmetric, because $\hat{h}$ is anti-symmetric, so $\hat{h}(\boldsymbol{v})$ is closest to $\boldsymbol{w}_i$ if and only if $\hat{h}(-\boldsymbol{v})$ is closest to $\boldsymbol{w}_{-1}$.

Now for all $0 < k \in \mathbb{N}$ we get a triangulation T_k with an $(n+1)$-labeling L_k that is anti-symmetric. Then from Fan's $n+1$ lemma we get that there is either a complementary edge $(+i, -i)$ for some i, or an alternating n-simplex with labels $\{1, -2, \dots, (-1)^n(n+1)\}$ in each T_k. This gives an infinite sequence of complementary edges and alternating simplices which, by the compactness of $\hat{S}^n$, has at least one of the following subsequences:

1. A subsequence of complementary edges of decreasing length, involving the same index i, such that the two sequences formed by their vertices, for each distinct label one, both converge.

2. A subsequence of alternating n-simplices of decreasing diameter, such that the $n+1$ sequences formed by their vertices, for each distinct label from $\{1, -2, \dots, (-1)^n(n+1)\}$ one, all converge.

In the first case, let $\{\boldsymbol{y}_n\}_{n\in\mathbb{N}}$ be the sequence of vertices with label i with $\boldsymbol{y}_n \to \boldsymbol{y}$ and let $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}}$ be the sequence of vertices with label $-i$ with $\boldsymbol{z}_n \to \boldsymbol{z}$. Since $\hat{h}(\boldsymbol{y}_n) = \boldsymbol{w}_i$ and $\hat{h}(\boldsymbol{z}_n) = \boldsymbol{w}_{-i}$ for all $n \in \mathbb{N}$ we get, by the continuity of $\hat{h}$ that $\hat{h}(\boldsymbol{y}) = \boldsymbol{w}_i$ and $\hat{h}(\boldsymbol{z}) = \boldsymbol{w}_{-i}$, so $\hat{h}(\boldsymbol{y})$ lies closest to w_i and $\hat{h}(\boldsymbol{z})$ lies closest to w_{-i}. Now suppose that $\boldsymbol{y} \neq \boldsymbol{z}$, let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that for all $n \geq N$ we get that $\|\boldsymbol{y} - \boldsymbol{y}_n\| < \frac{\varepsilon}{3}$, $\|\boldsymbol{z} - \boldsymbol{z}_n\| < \frac{\varepsilon}{3}$ and $\|\boldsymbol{y}_n - \boldsymbol{z}_n\| < \frac{\varepsilon}{3}$. This is possible since $\boldsymbol{y}_n \to \boldsymbol{y}$, $\boldsymbol{z}_n \to \boldsymbol{z}$ and the length of the complementary edges goes to zero. Then, by the triangle inequality, we get for all $n \geq N$

$$\|\boldsymbol{y} - \boldsymbol{z}\| \leq \|\boldsymbol{y} - \boldsymbol{y}_n\| + \|\boldsymbol{y}_n - \boldsymbol{z}_n\| + \|\boldsymbol{z}_n - \boldsymbol{z}\| < \varepsilon$$

Therefore $\boldsymbol{y} = \boldsymbol{z}$ and we get a limit point $\hat{h}(\boldsymbol{y}) = \hat{h}(\boldsymbol{z})$ that lies closest to w_i and lies closest to w_{-i}, which means that it is equidistant from both $\boldsymbol{w}_i$ and $\boldsymbol{w}_{-i}$. Considering the discussion above, the only point satisfying this condition is $\mathbf{0}$.

In the second case, we get in a similar way that the $n+1$ sequences of vertices all converge to the same limit point $\boldsymbol{z}$, for which $\hat{h}(\boldsymbol{z})$, by the continuity of $\hat{h}$, is equidistant from all points in $\{\boldsymbol{w}_1, -\boldsymbol{w}_2, \dots, (-1)^n \boldsymbol{w}_{n+1}\}$, and again the only point satisfying this condition is $\mathbf{0}$.

This means that in either case, the limit point $\boldsymbol{z}$ must satisfy $\hat{h}(\boldsymbol{z}) = \mathbf{0}$ and therefore $h(\boldsymbol{z}) = \mathbf{0}$, contradicting our assumption. This means that h does have a zero.

□

Note that if an anti-symmetric m-labeling of a symmetric triangulation T of S^n has no complementary edge, then $m \geq n+1$ since alternating n-simplices must have $n+1$ labels of distinct magnitude. Therefore if we have an n-labeling there cannot be an alternating n-simplex. The contrapositive of Fan's $n+1$ lemma then yields that there must be a complementary edge. This result is known as Tucker's lemma.

Theorem 3.3 (Tucker's lemma)**.** *Let T be a symmetric triangulation of S^n that is a refinement of the natural triangulation of $\hat{S}^n$ with an n-labeling that is antisymmetric. Then T has a complementary edge.*

Tucker's lemma is also equivalent to the Borsuk-Ulam theorem (see e.g. [Mat08]), but there is no known proof that it directly implies Sperner's lemma.

3.2 A Combinatorial Proof of Fan's Lemma

In order to provide a combinatorial proof of the Borsuk-Ulam theorem, we will prove a more general version of Fan's n+1 lemma, to which we will refer to as Fan's lemma. We will closely follow the proof given in [PS05]. The Borsuk-Ulam theorem then follows from its equivalence with Fan's $n+1$ lemma. In order to state and prove Fan's lemma, we first need to introduce some terminology. If A is a set in S^n then $-A$ is the *antipodal set*, so if $\boldsymbol{x} \in A$ then $-\boldsymbol{x} \in -A$. A *flag of hemispheres* in S^n is a sequence $H_0 \subset \ldots \subset H_n$ where each H_d is homeomorphic to a d-dimensional ball, and for $1 \leq d \leq n$, $\partial H_d = \partial(-H_d) = H_d \cap -H_d = H_{d-1} \cup -H_{d-1} \cong S^{d-1}$, $H_n \cup -H_n = S^n$, and $\{H_0, -H_0\}$ are antipodal points.

We call a symmetric triangulation T *aligned with hemispheres* if we can find a flag of hemispheres such that H_d is the union of a number of d-simplices of the triangulation. The *carrier hemisphere* of a simplex $\sigma \in \mathsf{T}$ is the minimal H_d or $-H_d$ that contains σ. Note that the sign of a carrier hemisphere is either positive or negative.

We define a simplex to be *almost-alternating* if it is not alternating, but by deleting one of the vertices, the resulting simplex, which is a facet of the original simplex, is alternating. The *sign* of an almost-alternating simplex without a complementary edge is defined to be the sign of any of its alternating facets. To see that this is well defined, we look at the two possible cases in which such a simplex σ is not alternating.

1. Two vertices of σ have the same label.
2. When placed in order of increasing absolute value, two adjacent vertex labels of distinct magnitude of σ have the same sign.

Now, let σ be an almost-alternating simplex without a complementary edge.

In the first case, deleting either one of the vertices with the same label makes the resulting simplex alternating and its sign does not depend on which of the two vertices is deleted.

In the second case, deleting either one of the vertices with adjacent labels of the same sign, when placed in order of increasing absolute value, makes the resulting simplex alternating. Its sign does not depend on which of the two vertices is deleted because they both have the same sign.

Thus, in both cases the sign of the almost alternating simplex is well defined. Further, note that if an almost-alternating simplex does not have a complementary edge, it has exactly two facets that are alternating simplices. For example a simplex with labels $\{-1, 3, 5, -7\}$ has facets with labels $\{-1, 3, -7\}$ and $\{-1, 5, -7\}$ respectively, that are alternating simplices. Now we are ready to present a constructive proof of Fan's lemma, stated for more general triangulations than Fan's original version in [Fan52].

Theorem 3.4. *Let T be a symmetric triangulation of S^n aligned with hemispheres. Suppose that T has an anti-symmetric labeling by labels $\{\pm 1, \ldots, \pm m\}$ without a complementary edge. Then there are an odd number of positive alternating n-simplices and an equal number of negative alternating n-simplices in T. And, in particular, $m \geq n + 1$.*

Proof. Suppose that the triangulation T of S^n is aligned with the flag of hemispheres $H_0 \subset \ldots \subset H_n$. We call an alternating or almost-alternating simplex *agreeable* if its sign matches that of its carrier hemisphere. For example, the simplex with labels $\{-1, 3, 5, -7\}$ discussed above, is agreeable if its carrier hemisphere is $-H_d$ for some d. Next we construct a graph G. A simplex $\sigma \in \mathsf{T}$ with carrier H_d is a node of G if it is one of the following.

1. σ is an agreeable alternating $(d-1)$-simplex.
2. σ is an agreeable almost-alternating d-simplex.
3. σ is an alternating d-simplex.

Further, two nodes σ and τ are adjacent in G, meaning that there is an edge between them if all the following conditions are met.

a. σ and τ are not both of type 1.

b. σ is a facet of τ or τ is a facet of σ.

c. $\sigma \cap \tau$ is alternating.

d. The sign of the carrier hemisphere of $\sigma \cup \tau$ matches the sign of $\sigma \cap \tau$.

e. The difference between the carrier hemisphere dimensions of σ and τ is at most 1.

Conditions a and e were not mentioned in [PS05], but are necessary. We claim that all vertices of G have degree 1 or 2 and a vertex has degree 1 if and only if its corresponding vertex in T is carried by $\pm H_0$ or is an n-dimensional alternating simplex. We will verify this for all three different types of nodes in G.

1. Let σ be an agreeable alternating $(d-1)$-simplex with carrier $\pm H_d$.
   We first look at d-simplices that have σ as a facet. Because of condition a, only d-simplices of type 2 or 3 can be adjacent to σ. This means that only the two d-simplices in the same carrier as σ that have σ as a facet might qualify. For both types condition c and d are satisfied since σ is alternating and agreeable, so its sign matches that of $\pm H_d$, which is also the carrier of the type 2 or type 3 d-simplex.
   Next we look at $(d-2)$-simplices that are a facet of σ. Because of conditions a and c, $(d-2)$-simplices of type 1 and 2 cannot be adjacent to σ. A $(d-2)$-simplex of type 3 with carrier $\pm H_{d-2}$, cannot be adjacent to σ either, because of condition e.
   Thus, σ has degree 2 in G.

2. Let σ be an agreeable almost-alternating d-simplex σ with carrier $\pm H_d$.
   We start again by looking at simplices that have σ as a facet. Because of condition c, no $d+1$-simplices can be adjacent to σ since σ is not alternating.
   Next we look at the $(d-1)$-simplices that are a facet of σ. Because of condition c again, no $(d-1)$-simplices of type 2 are adjacent to σ. So the only facets of σ that can be adjacent to σ are its two alternating facets, for which condition c is satisfied. Each of these facets is either an agreeable alternating $(d-1)$-simplex with the same carrier as σ or an alternating $(d-1)$-simplex with carrier $\pm H_{d-1}$, so that condition e is met. Since σ is agreeable its sign matches that of its carrier hemisphere, and because an almost-alternating simplex has the same sign as its two alternating facets, condition d is also satisfied for both facets. So σ is only adjacent in G to its two alternating facets, each of which is either of type 1 or 3, which gives σ degree 2 in G.

3. Let σ be an alternating d-simplex σ carried by $\pm H_d$.
   Again, we first look at the simplices that have σ as a facet. Note that if $d = n$, the number of such simplices is zero, so assume $d < n$. Since σ is carried by $\pm H_d$ it lies on $\partial H_{d+1} = \partial(-H_{d+1})$, which means that it is a facet of exactly two $(d+1)$-simplices, one in H_{d+1} and one in $-H_{d+1}$. If σ is carried by H_d it is adjacent to the one in H_{d+1} and if its carrier is $-H_d$ it is adjacent to the one in $-H_{d+1}$. It cannot be adjacent to both, since only one of them satisfies condition d.
   Next we look at the facets of σ. Assume $d > 0$, since σ does not have any labeled facets if $d = 0$. Because of condition c an almost-alternating facets of σ cannot be adjacent to σ. Thus any facet of σ that is adjacent to σ must be alternating and has the same carrier as σ or is carried by $\pm H_{d-1}$. There are only two such facets of σ, which are obtained from σ by deleting either the highest labeled or the lowest labeled vertex, when arranged in order of increasing magnitude. Note that one of these is a positive alternating simplex and the other one is a negative alternating simplex. They both satisfy adjacency conditions a, b, c and e, but only one of them meets condition d and is therefore adjacent to σ. This is because the sign of only one of them agrees with the sign of the carrier hemisphere of σ.
   Thus σ has degree 2 in G, unless $d = 0$ or $d = n$. If $d = 0$, then σ is one of the points $\pm H_0$, so it has no labeled facets and therefore only has degree 1. If $d = n$, then σ can not be the facet of any other simplex and therefore it also has degree 1.

We have now confirmed our claim that every node in G has degree 2 with the exception of the points at $\pm H_0$ and all alternating n-simplices. This means that G consists of a collection of disjoint paths with endpoints at $\pm H_0$ or at alternating n-simplices.

Since T is symmetric and has an anti-symmetric labeling, the antipode of each path in G is also a path in G. Also, note that no path can have antipodal endpoints, since then the center edge or node of this path would be antipodal to itself. In the first case, two antipodal simplices would be adjacent in G, which is impossible because one cannot be a facet of the other. The second case is also impossible since a simplex in T can never be equal to its antipodal simplex. Therefore a path is never identical to its antipodal path, so all paths in G must come in pairs. This gives a multiple of four endpoints of paths in G. Since all endpoints correspond to $\pm H_0$ or an alternating n-simplex, we get that there is twice an odd number of alternating n-simplices. And, because every positive alternating simplex has a negative alternating simplex as its antipode, half of the alternating n-simplices are positive. Thus there is an odd number of positive alternating simplices and an equal number of negative alternating simplices.

Since an alternating n-simplex has $n+1$ labels of distinct magnitude, it follows that $m \geq n+1$.

□

This proof also provides a procedure to find an alternating n-simplex. A path that begins at H_0 cannot terminate at $-H_0$, since a path is never identical to its antipodal path. Therefore, the path starting at H_0 must terminate at an alternating n-simplex. So following this path we eventually find an alternating n-simplex. Note that the antipode of this simplex is an alternating n-simplex of the opposite sign.

It is an open question whether any symmetric triangulation of S^n can be aligned with a flag of hemispheres. If it is possible, the proof above would be valid for all symmetric triangulations of S^n. Also the procedure to find an alternating n-simplex would work for all symmetric triangulations of S^n.

In [Mat08] two proofs of Tucker's lemma are given, one of which somewhat resembles the proof of Fan's lemma given above, that through the equivalence of Tucker's lemma and the Borsuk-Ulam theory also provide combinatorial proofs of the Borsuk-Ulam theorem.

Chapter 4

The Brouwer Fixed Point Theorem and Sperner's Lemma

The Borsuk-Ulam theorem and Fan's lemma we have seen in the previous chapters all concern the n-sphere. In this chapter we will discuss related theorems that concern the n-ball $B^n = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$. We start by introducing the Brouwer fixed point theorem.

Theorem 4.1 (Brouwer fixed point theorem)**.** *For any continuous mapping $f : B^n \to B^n$, there exists a point $\boldsymbol{x} \in B^n$ such that $f(\boldsymbol{x}) = \boldsymbol{x}$.*

For the combinatorial equivalent of the Brouwer fixed point theorem we will consider the regular n-simplex Δ^n embedded in $\mathbb{R}^{n+1}$:

$$\Delta^n := \{(x_1, \ldots, x_{n+1}) : x_i \geq 0, \sum x_i = 1\},$$

which is homeomorphic to B^n. Next, for any $\boldsymbol{v} = (v_1, \ldots, v_{n+1}) \in \Delta^n$, let

$$Z(\boldsymbol{v}) = \{i : v_i \neq 0\}$$

be the set of indices of coordinates of $\boldsymbol{v}$ that are non-zero. Suppose T is a triangulation of Δ^n. We call an $(n+1)$-labeling ℓ of T a *Sperner labeling* if for each vertex $\boldsymbol{v}$ of T we have

$$\ell(\boldsymbol{v}) \in Z(\boldsymbol{v}).$$

This forces each main vertex of Δ^n to have a different label, which is the index of its only non-zero coordinate. Further, any vertex on a face of Δ^n must have one of the labels assigned to the main vertices that span that face. We call an n-simplex in T *fully-labeled* if its vertices all have distinct labels and therefore all labels $\{1, \ldots, n+1\}$. Now we are able to state Sperner's lemma.

Theorem 4.2 (Sperner's lemma)**.** *Any Sperner labeled triangulation T of Δ^n must have a fully-labeled n-simplex.*

A more general version of this lemma says that, in fact, the number of fully-labeled n-simplices is odd. This statement is easily proved by induction on the dimension n.

As we shall see in this chapter the Borsuk-Ulam theorem implies the Brouwer fixed point theorem. We will prove this by a direct construction. We also show that Fan's $n+1$ lemma directly implies Sperner's lemma. As mentioned before, through the equivalence of the Borsuk-Ulam theorem and Tucker's lemma and the equivalence of the Brouwer fixed point theorem and Sperner's lemma, Tucker's lemma also implies Sperner's lemma, but there is no direct proof known of this implication.

The original link between the Brouwer fixed point theorem and Sperner's lemma was provided by Knaster, Kuratowski and Mazurkiewicz and is known as the KKM lemma.

Theorem 4.3 (KKM lemma)**.** *Let $C_1, \dots, C_n$ be a collection of closed sets that cover Δ^n such that for each $I \subseteq [n+1]$, the face spanned by the set $\{\boldsymbol{e}_i : i \in I\}$ is covered by $\{C_i : i \in I\}$. Then $\bigcap_{i=1}^n C_i$ is non-empty.*

This lemma is a set-covering theorem equivalent to the topological Brouwer fixed point theorem and the combinatorial Sperner's lemma. Similarly, the Lyusternik-Schnirel'man theorem is a set-covering theorem equivalent to the topological Borsuk-Ulam theorem and the combinatorial Fan's $n+1$ lemma. This suggest that the Lyusternik-Schnirel'man theorem implies the KKM lemma. A direct proof of this implication, using the closed version of the Lyusternik-Schnirel'man theorem, can be found in [SS07].

4.1 Borsuk-Ulam implies Brouwer Fixed Point Theorem

In this section we will prove that the Brouwer fixed point theorem follows from the Borsuk-Ulam theorem.

Theorem 4.4. *The Borsuk-Ulam theorem implies the Brouwer fixed point theorem.*

Proof.* Let $f : B^n \to B^n$ be a continuous mapping. Assume, for contradiction, that f has no fixed point, so there is no $\boldsymbol{x} \in B^n$ for which $f(\boldsymbol{x}) = \boldsymbol{x}$. Now define a function $g : B^n \to S^n$ as follows: let $g(\boldsymbol{x})$ be the point in which the ray originating in $f(\boldsymbol{x})$ and going through $\boldsymbol{x}$ intersects $\partial B^n = S^{n-1}$. Since we assume that f has no fixed point, this is well defined. Next we will show that g is continuous.

Since f is continuous and has no fixed point and since B^n is compact, there exists some $\gamma > 0$ such that for all $\boldsymbol{x} \in B^n$ we have $\|f(\boldsymbol{x}) - \boldsymbol{x})\| > 2\gamma$.

Now let $\varepsilon > 0$ and set $\varepsilon_1 = \gamma \sin(\frac{\varepsilon}{2})$ Since f is continuous and B^n is compact, we get that f is uniformly continuous, so there is a δ_1 such that for all $\boldsymbol{x}, \boldsymbol{y} \in B^n$ for which $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta_1$ we have $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| < \varepsilon_1$. Fix such a δ_1 and then set

$$\delta = \min\{\delta_1, \varepsilon_1\}.$$

Then, from figure 4.1 with $BE = \delta$ and $AD = \varepsilon_1$ we get, using the similarity of the triangles $\triangle ASD$ and $\triangle BSE$ and the fact that $\delta \leq \varepsilon_1$, that $DS \geq DE/2$. Now if $\boldsymbol{x}$ is situated at E and $f(x)$ at D, we get that $DE \geq \gamma$ which yields $\sin(\angle ASD) = \frac{AD}{DS} \leq \frac{\varepsilon_1}{\gamma}$.

Further we have $\alpha := \angle FGH = \angle ASD$ since GH is parallel to AC and from the inscribed angle theorem we get that the arc length FH is equal to 2α (since the radius of B^n is 1 the arc length of a central angle is equal to the angle itself in radians).

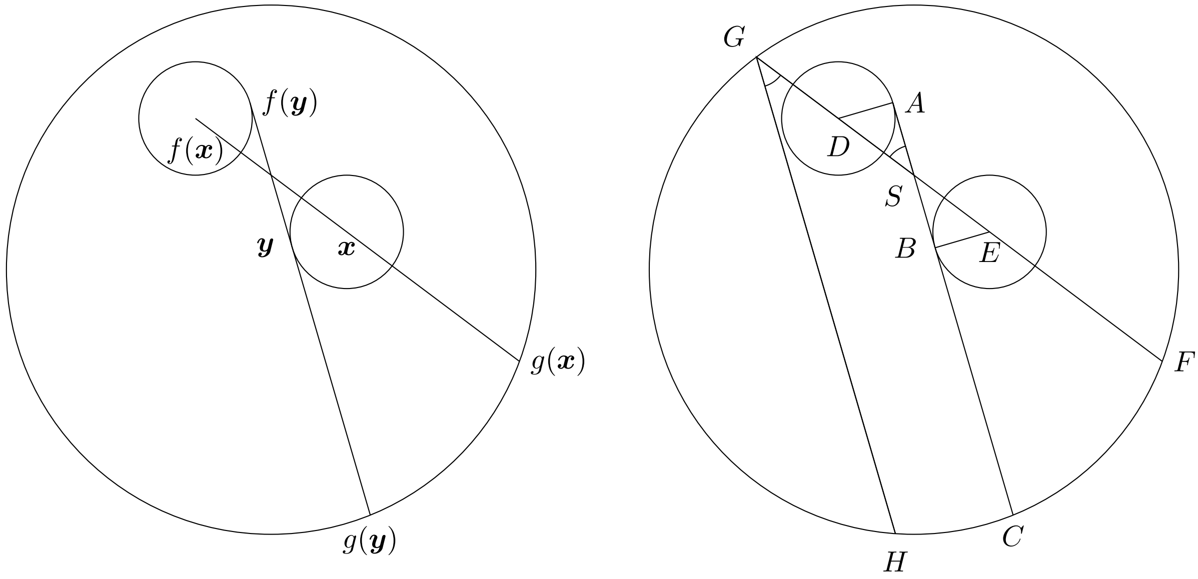

Figure 4.1: A cross section of B^n.

For $\|\boldsymbol{x} - \boldsymbol{y}| < \delta$ we get from figure 4.1 that $\|g(\boldsymbol{x}) - g(\boldsymbol{y})\|$ is always smaller than when $\boldsymbol{y}$ would lie at point B and $f(\boldsymbol{y})$ at point A, so whenever $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$ we get

$$\begin{aligned}
\|g(\boldsymbol{x}) - g(\boldsymbol{y})\| &< CF \\
&\leq \mathrm{arc}FH \\
&= 2\alpha \\
&= 2\sin^{-1}(\frac{AD}{DS}) \\
&\leq 2\sin^{-1}\left(\frac{\varepsilon_1}{\gamma}\right) \\
&= 2\sin^{-1}\left(\frac{\gamma\sin(\varepsilon/2)}{\gamma}\right) \\
&= 2\sin^{-1}(\sin(\varepsilon/2)) \\
&= \varepsilon,
\end{aligned}$$

where we use that the inverse sine is an increasing function. This proofs that g is (uniformly) continuous. But since $g(\boldsymbol{x}) = \boldsymbol{x}$ for every $\boldsymbol{x}$ on the boundary $\partial B^n = S^{n-1}$ of B^n, the function g contradicts statement 4 of theorem 2.1. This means our assumption is false so f does have a fixed point, proving the Brouwer fixed point theorem.

A different proof of this implication using a direct construction can be found in [Su97].

$\square$

4.2 Fan's $n+1$ Lemma implies Sperner's Lemma

The proof of the following theorem closely follows the proof given in [NS13].

Theorem 4.5. *Fan's $n+1$ lemma implies Sperner's lemma.*

Proof. Let S be a triangulation of Δ^n with Sperner labeling ℓ. Note that Δ^n is a facet of the $n+1$-dimensional cross-polytope. We extend S to a triangulation T of $\hat{S}^n$ the boundary of the cross-polytope (T is then also a triangulation of S^n since $\hat{S}^n$ and S^n are homeomorphic) by reflecting copies of S to the other orthants of $\hat{S}^n$. Let $G = \{\pm\}^{n+1}$ denote the group of symmetries of $\hat{S}^n$ generated by reflections that change the sign of selected coordinates. Then the action of $\boldsymbol{g} = (g_1, \ldots, g_{n+1}) \in G$ on $\boldsymbol{v} = (v_1, \ldots, v_{n+1}) \in \hat{S}^n$ produces $\boldsymbol{g}\boldsymbol{v} = (g_1 v_1, \ldots, g_{n+1} v_{n+1}) \in \hat{S}^n$. This means that $\boldsymbol{g}$ reflects $\boldsymbol{v}$ in all coordinates i for which $g_i = -1$.

For a simplex σ in S spanned by a set of vertices V, we define $\boldsymbol{g}\sigma$ to be the simplex spanned by the vertices in $\boldsymbol{g}V = \{\boldsymbol{g}\boldsymbol{v} : \boldsymbol{v} \in V\}$. Now let T be the collection of simplices $\{\boldsymbol{g}\sigma : \sigma \in \mathsf{S} \text{ and } g \in G\}$. Then T is a triangulation of $\hat{S}^n$ since S is a triangulation of Δ^n and because the reflection method ensures that also on the boundary of reflected copies of Δ^n all simplices in T meet face to face.

Next we extend the labeling ℓ of the vertices of S to a labeling L on the vertices of T. We define

$$L(\boldsymbol{g}\boldsymbol{v}) = g_{\ell(\boldsymbol{v})} \cdot (-1)^{\ell(\boldsymbol{v})+1} \cdot \ell(\boldsymbol{v})$$

for each vertex $\boldsymbol{v}$ of S. Note that a reflected vertex of a vertex $\boldsymbol{v}$ in S gets the same label as $\boldsymbol{v}$, but possibly with a change of sign. When $\boldsymbol{g} = (1, 1, \ldots, 1)$, this defines L on S, where the factor $(-1)^{\ell(\boldsymbol{v})+1}$ turns fully labeld n-simplices into positive alternating n-simplices. For other $\boldsymbol{g} \in G$ this defines L on the reflected copies of S in the other orthants of $\hat{S}^n$.

To see that L is well defined where orthants meet, note that orthants meet where $\boldsymbol{g}\boldsymbol{v} = \hat{\boldsymbol{g}}\hat{\boldsymbol{v}}$ for some $\boldsymbol{g}, \hat{\boldsymbol{g}} \in G$ and some $\boldsymbol{v}, \hat{\boldsymbol{v}} \in \mathsf{S}$. This means $g_i v_i = \hat{g}_i \hat{v}_i$ for each i and since $g_i, \hat{g}_i = \pm 1$, this implies that $v_i = \hat{v}_i$ for each i. Then $g_i = \hat{g}_i$ for all i for which $v_i \neq 0$, that is when $i \in Z(\boldsymbol{v})$. But since ℓ is a Sperner labeling, we have $\ell(\boldsymbol{v}) \in Z(\boldsymbol{v})$, which yields that $g_{\ell(\boldsymbol{v})} = \hat{g}_{\ell(\boldsymbol{v})}$. Then from the definition of L it follows that $L(\boldsymbol{g}\boldsymbol{v}) = L(\hat{\boldsymbol{g}}\boldsymbol{v})$, so L is well defined.

Next we show that the triangulation T of S^n with labeling L satisfies the conditions of Fan's $n+1$ lemma. Let $-\boldsymbol{v} = \overline{\boldsymbol{g}}\boldsymbol{v}$, where $\overline{\boldsymbol{g}} = (-1, -1, \ldots, -1)$, be the vertex antipodal to a vertex $\boldsymbol{v}$ of T. Then from the definition of L we get that $L(-\boldsymbol{v}) = -L(\boldsymbol{v})$, which means that L is anti-symmetric. Now we only need to show that L has no complementary edges. The labeling ℓ has no complementary edges (since all labels are positive) and every edges in T is a reflected copy of some edge in S via some $\boldsymbol{g} \in G$. Then from the definition of L we get that for any $\boldsymbol{g} \in G$, two vertices $\boldsymbol{v}, \boldsymbol{w} \in \mathsf{S}$ have identical ℓ-labels ($\ell(\boldsymbol{v}) = \ell(\boldsymbol{w})$) if and only if $\boldsymbol{g}\boldsymbol{v}$ and $\boldsymbol{g}\boldsymbol{w}$ have identical L-labels ($L(\boldsymbol{g}\boldsymbol{v}) = L(\boldsymbol{g}\boldsymbol{w})$). Then, since ℓ has no complementary edges, it follows that T has no complementary edges either.

Now from Fan's $n+1$ lemma it follows that there exists a positive alternating n-simplex in T. And since Δ^n is the only facet of $\hat{S}^n$ that contains the labels $\{1, -2, 3, \ldots, (-1)^n(n+1)\}$, there must be a fully-labeled n-simplex in S.

□

When we use Fan's lemma (Theorem 3.4) with $m = n+1$ instead of Fan's $n+1$ lemma in the proof above, we get the more general version of Sperner's lemma that says that there is an odd number of fully-labeled n-simplices in S.

Chapter 5

An Application of Borsuk-Ulam in Combinatorics

The Borsuk-Ulam theorem has a number of applications in combinatorics. In this chapter we will focus on the problem of determining the chromatic number of Kneser graphs, a problem posed in a slightly different form by the number theorist Martin Kneser in 1955. The combinatorial problem was first solved 23 years later by Lásló Lovász, surprisingly using the Borsuk-Ulam theorem from topology. The content of this chapter is based on [AZ14] and [Mat08]. For more applications of the Borsuk-Ulam theorem in combinatorics see [Mat08].

5.1 The Chromatic Number of Kneser Graphs

We start by giving the definition of a Kneser graph.

Definition 5.1. Let $n \geq k \geq 1$ in $\mathbb{N}$, then the Kneser graph $\mathrm{KG}_{n,k}$ is defined as follows:

1. The vertex set of $\mathrm{KG}_{n,k}$ is the family of all k-subsets of $[n] = \{1, \ldots, n\}$. Thus, the number of vertices of $\mathrm{KG}_{n,k}$ is $\binom{n}{k}$.
2. Two vertices are connected if their corresponding k-sets are disjoint.

In Kneser graphs for which $k = 1$ any two k-subsets are trivially disjoint, which means every two vertices are connected, so $\mathrm{KG}_{n,1}$ is the complete graph K_n. Another well known example is $\mathrm{KG}_{5,2}$, the famous Petersen graph (see figure 5.1). Notice that if $n < 2k$ any two k-sets intersect, which means that $\mathrm{KG}_{n,k}$ has no edges at all.

A proper m-coloring of a graph $G = (V, E)$ is a mapping $c : V(G) \to [m]$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$ is an edge. The *chromatic number* $\chi(G)$ is defined as the smallest m such that G has an m-coloring.

In figure 5.1 a 3-coloring of the Petersen Graph is shown. It is not possible to color it with less than three colors, so $\chi(\mathrm{KG}_{5,2}) = 3$.

The origninal proof of the following theorem was given by Lovász, but the proof given here is based on a shorter one presented in [Gre02].

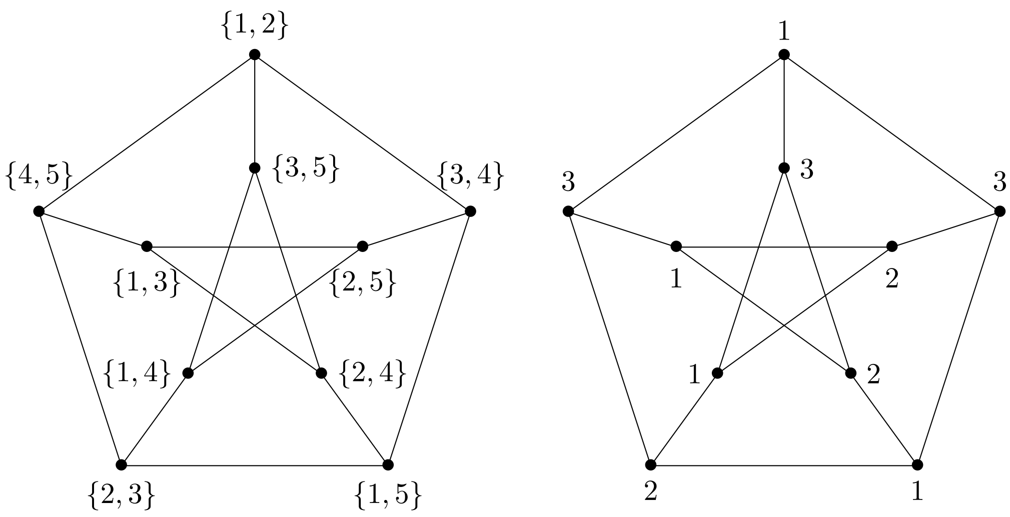

Figure 5.1: The Petersen graph and its 3-coloring.

Theorem 5.2 (Lovász-Kneser theorem)**.** *For all $k > 0$ and $n \geq 2k-1$, the chromatic number of the Kneser Graph $\mathrm{KG}_{n,k}$ is $\chi(\mathrm{KG}_{n,k}) = n - 2k + 2$.*

Proof. First we show that there is a coloring of $\mathrm{KG}_{n,k}$ with $n - 2k + 2$ colors. We color the vertices of the Kneser graph by

$$c(F) := \min\{\min(F), n - 2k + 2\}.$$

This assigns a color $c(F) \in \{1, 2, \ldots, n - 2k + 2\}$ to each k-subset F of $[n]$. If two k subsets F, F' get the same color $c(F) = c(F') = i < n - 2k + 2$, then they both contain the element i, so they cannot be disjoint. If the two k-subsets both get the color $n - 2k + 2$, then they are both contained in the set $\{n-2k+2, \ldots, n\}$, which has only $2k-1$ elements, which means they cannot be disjoint either.

Next we prove that there cannot be a proper coloring of $\mathrm{KG}_{n,k}$ with only $n - 2k + 1$ colors. To this end set $d := n-2k+1$ and take n points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^{d+1}$ lying on S^d in *general position*, meaning that no $d + 1$ of them lie on a common hyperplane through the origin.

Now suppose, for contradiction, that there is a proper coloring of $\mathrm{KG}_{n,k}$ by at most $n-2k+1 = d$ colors. We fix one such coloring and we define sets $A_1, \ldots, A_d \subseteq S^d$: For a point $\boldsymbol{x} \in S^d$, we have $\boldsymbol{x} \in A_i$ if there is at least one k-subset F of $[n]$ that has been given color i contained in the set of indices of the points from $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ lying in the open hemisphere $H(\boldsymbol{x})$ centered at $\boldsymbol{x}$ (formally, $H(\boldsymbol{x}) = \{\boldsymbol{y} \in S^d : \langle \boldsymbol{x}, \boldsymbol{y} \rangle > 0\}$). Finally, we put $A_{d+1} = S^d \setminus (A_1 \cup \ldots \cup A_d)$.

From the construction it is clear that $A_1, \ldots, A_d \subseteq S^d$ are open sets and A_{d+1} is closed, and together they cover S^d. Now, from the Lyusternik-Schnirel'man theorem (theorem 2.2) we get that there is at least one $i \in [d+1]$ such that A_i contains a pair of antipodal points $\boldsymbol{x}$ and $-\boldsymbol{x}$.

If $i \leq d$, we get two disjoint k-subsets of color i, one consisting of indices from points in $H(\boldsymbol{x})$ and the other consisting of indices from points in the opposite open hemisphere $H(-\boldsymbol{x})$. Which means that our fixed coloring is not a proper coloring of $\mathrm{KG}_{n,k}$.

In the other case, if $i = d+1$, $H(\boldsymbol{x})$ contains at most $k-1$ points of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$, and so does $H(-\boldsymbol{x})$. Therefore, since we took n points on S^d, the complement $S^d \setminus (H(\boldsymbol{x}) \cup H(-\boldsymbol{x}))$ has to contain at least $n-2k+2$ points of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$. But since $S^d \setminus (H(\boldsymbol{x}) \cup H(-\boldsymbol{x}))$ is an "equator" (the intersection of S^d with a hyperplane through the origin), this contradicts the fact that the n points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ were in general position.

In both cases we reached a contradiction, so we can conclude that there is no proper coloring of $\mathrm{KG}_{n,k}$ with only $n-2k+1$ colors, which proofs that $\chi(\mathrm{KG}_{n,k}) = n-2k+2$.

□

The fact that the Borsuk-Ulam theorem that lies at the heart of the proof (in this case the Lyusternik-Schirel'man theorem) has combinatorial equivalents, suggests that it may be possible to give a purely combinatorial proof of the Kneser-Lovász theorem. Indeed, in [Mat04] Jiří Matoušek derives a proof from Tucker's lemma (3.3) and even obtains a self contained purely combinatorial proof of the Kneser-Lovász theorem that avoids any mentioning of topology or triangulations.

Concluding Remarks

In this report we focused on the connections between the Borsuk-Ulam theorem and Combinatorics which offered a lot of interesting results. However, even within this confined area we did not cover everything. For further study [Mat08] gives a great survey of theory and applications concerning the Borsuk-Ulam theorem.

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