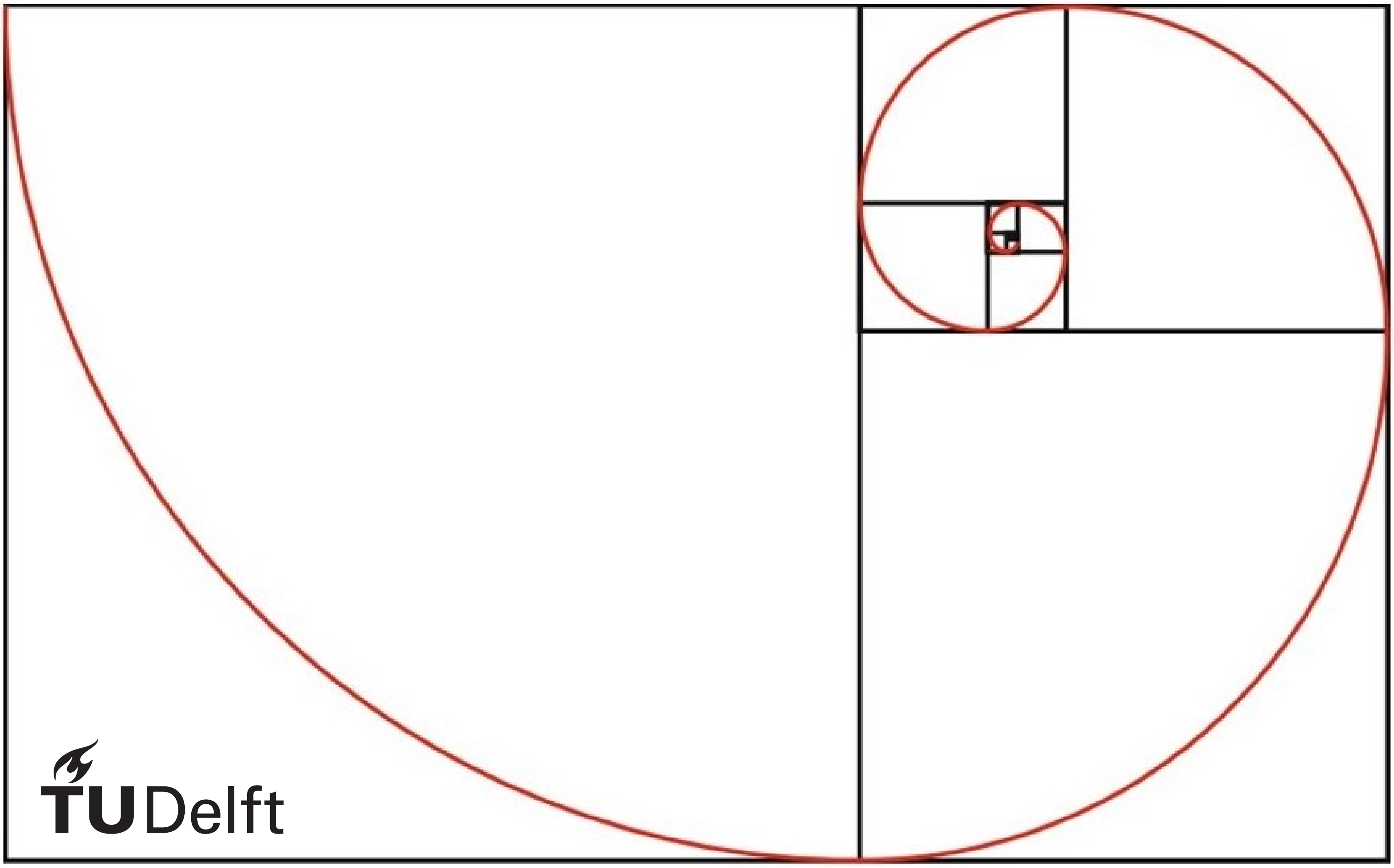
On Euler and Fibonacci Numbers

Why Pi is Bounded by Twice Phi

G. K. van der Wal



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by

G. K. van der Wal

to obtain the degree of Bachelor of Science
at Delft University of Technology,
Department of Applied Mathematics.

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Thesis committee:	R. Fokkink,	TU Delft, supervisor
	Y. van Gennip,	TU Delft

TU Delft

Preface

This is the report “On Euler and Fibonacci Numbers: Why Pi is Bounded by Twice Phi", which is based on the article from Alejandro H. Morales, Igor Pak & Greta Panova [1], where a link is made between the Fibonacci and Euler numbers. It is meant for mathematicians without the prior knowledge that is needed to understand this article.

This report has been written to obtain the degree of Bachelor of Science at Delft University of Technology, Department of Applied Mathematics. It has been written in the period from May 2020 to September 2020 at Delft University of Technology under the supervision of Robbert Fokkink. I would like to thank Robbert Fokkink for his supervision and support.

G. K. van der Wal
Delft, September 4, 2020

Abstract

In this report, we will look at the connection between the Fibonacci and Euler numbers. By using a combinatorial argument including the Fibonacci and Euler numbers, we will prove our main theorem:

$$F_n \cdot E_n \geq n!$$

From the main theorem and the asymptotics of these numbers, we will conclude that $\pi \leq 2\varphi$. We follow the proof in the article of Alejandro H. Morales, Igor Pak & Greta Panova [1], but we will give a more detailed proof and some extra facts about the Golden Ratio, φ, and the Fibonacci and Euler numbers. Finally, the article discusses the number of linear extensions of certain partially ordered sets, or *posets*. We see that there exist a two-dimensional poset $\mathscr{U}_n$ and complement poset $\overline{\mathscr{U}}_n$, both with n elements, such that the number of linear extensions are respectively E_n and F_n. We conclude that the Fibonacci and Euler numbers are related to each other.

Contents

1

The Golden ratio

1.1. Definition

Euclid of Alexandria, sometimes called “the founder of geometry" [2], was the first to give a clear definition of the Golden Ratio, around 300 B.C. Euclid defined the ratio as follows:

> “*a straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.*" [3]

Meaning: in Figure 1.1, the ratio of the length of the whole line, $a+b$, to that of the greater segment, a, is the same as the ratio of the length of the greater segment, a, to the lesser segment, b, i.e.,

$$\frac{a+b}{a} = \frac{a}{b}$$

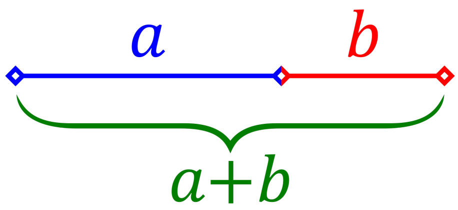

Figure 1.1: Line segments in the Golden Ratio, where $a+b$ is to a as a is to b. [4]

Using this definition, we can calculate what this ratio is.

$$\begin{aligned}
\frac{a}{b} &= \frac{a+b}{a} \\
\frac{a}{b} &= 1+\frac{b}{a} \\
\text{Let } x &= \frac{a}{b} \\
x &= 1+\frac{1}{x} \\
x^2 &= x+1 \\
x^2-x-1 &= 0 \\
x &= \frac{1\pm\sqrt{5}}{2}
\end{aligned} \tag{1.1}$$

The ratio of the line segments should be a positive number, therefore, we define the Golden Ratio as follows:

Definition 1.1.1. The Golden Ratio is the positive solution to $x^2-x-1=0$ (equation 1.1), which we denote $\varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}618034$.

We denote the negative solution to equation 1.1 by $\psi = \frac{1-\sqrt{5}}{2} \approx 0,618034$. ψ is sometimes called the little golden number and it can be expressed in terms of φ:

$$\begin{aligned} \varphi + \psi &= \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1 \\ \psi &= 1 - \varphi = -\frac{1}{\varphi} \end{aligned}$$

There also exists a *silver ratio*, δ_S. If we look at Figure 1.1, a and b are in the silver ratio if the ratio of the length of the greater segment, a, to the lesser segment, b, is the same as the ratio of the sum of the smaller segment plus twice the greater segment, $2a+b$, to that of the greater segment, a, i.e.,

$$\frac{2a+b}{a} = \frac{a}{b}$$

If we calculate this in the same way as in equation 1.1, we find that the silver ratio is the positive solution to the equation $x^2 - 2x - 1 = 0$, which we denote $\delta_S = 1 + \sqrt{2}$.

Euclid defined the Golden Ratio, but he named it “extreme and mean ratio" [3]. It was not called the Golden Ratio until 1835, when it was first used by Martin Ohm in the book *Die reine Elementar-Mathematik*. He named the division of a line AB at a point C such that $AB \cdot CB = AC^2$ the ‘golden section’ (or *goldener Schnitt* in German) [6]. In Figure 1.2 we have such a line.

Figure 1.2: Line AB divided at point C such that $AB \cdot CB = AC^2$.

If we rewrite its property, we get:

$$\begin{aligned} AB \cdot CB &= AC^2 \\ \frac{AB}{AC} &= \frac{AC}{CB} \end{aligned}$$

If we let the length of AC be a and the length of CB be b, we have:

$$\frac{a+b}{a} = \frac{a}{b}$$

Which is the same as Euclid’s definition.

1.1.1. φ as Continued Fraction and Square Root

We define the recursive sequence $(x_n)_{n\geq 1}$ by $x_n = 1 + \frac{1}{x_{n-1}}$ for all $n \geq 1$ and $x_0 = 1$. We get:

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 + \frac{1}{1} \\ x_2 &= 1 + \frac{1}{1+\frac{1}{1}} \\ x_3 &= 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}} \\ &\vdots \end{aligned}$$

If we let n go to infinity, we get a continued fraction. We call this expression x.

$$\lim_{n\to\infty} x_n = 1 + \frac{1}{1+\frac{1}{1+\dots}} = x$$

We can then see the same expression in the denominator of the fraction:

$$\begin{aligned} x &= 1+\frac{1}{1+\frac{1}{1+\dots}} \\ x &= 1+\frac{1}{x} \end{aligned}$$

This is equation 1.1 and x is a positive number, so the solution is:

$$\begin{aligned} \lim_{n\to\infty} x_n = x &= \varphi \\ \varphi &= 1+\frac{1}{1+\frac{1}{1+\dots}} \end{aligned} \tag{1.2}$$

This only holds under the assumption that the sequence converges to x. In section 2.1, we simplify the fractions x_1, x_2, x_3, etc. and we will see that they equal the ratio between two consecutive Fibonacci numbers. Then, we prove in Proposition 2.1.2 that these fractions do converge, so equation 1.2 is indeed well-defined.

We can do something similar with square roots. We define the recursive sequence $(y_n)_{n\geq 1}$ by $y_n = \sqrt{1+y_{n-1}}$ for all $n \geq 1$ and $y_0 = 1$. We get:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= \sqrt{1+1} \\ y_2 &= \sqrt{1+\sqrt{1+1}} \\ y_3 &= \sqrt{1+\sqrt{1+\sqrt{1+1}}} \\ &\vdots \end{aligned}$$

If we let n go to infinity, we get an infintely long square root. We call this expression y.

$$\lim_{n\to\infty} y_n = \sqrt{1+\sqrt{1+\sqrt{1+\dots}}} = y$$

Now, we square both sides:

$$\begin{aligned} y &= \sqrt{1+\sqrt{1+\sqrt{1+\dots}}} \\ y^2 &= 1+\sqrt{1+\sqrt{1+\sqrt{1+\dots}}} \end{aligned}$$

But then we have y again on the right side:

$$y^2 = y+1$$

This is again equation 1.1 and y is a positive number, so the solution is:

$$\begin{aligned} \lim_{n\to\infty} y_n = y &= \varphi \\ \varphi &= \sqrt{1+\sqrt{1+\sqrt{1+\dots}}} \end{aligned} \tag{1.3}$$

1.2. The Golden Ratio in Architecture, Art, the Human Body and Nature

Many people have tried finding this "perfect proportion" in objects such as art, architecture and nature. There has been suggested that the Great Pyramid of Giza and the Parthenon in Athens contain the Golden ratio in their dimensions [5]. But this seems to be untrue, as the measurements of these buildings vary from source to source, so one could easily choose the measurements that would get a result close to φ.

There also have been claims that the human body contains the Golden Ratio, such as the ratio of the height of a person to the height of their navel [6]. For example, in *De Divinia Proportione* by Luca Pacioli. In this book, Pacioli describes the application of the Golden Ratio in mathematics, art and architecture [7]. The book is illustrated by Leonardo da Vinci and contains one of da Vinci's most famous drawings: "The Vitruvian Man", see Figure 1.3. The ratio of the height of The Vitruvian Man to the height of his navel is indeed the Golden Ratio. But that is not the only occurence of the Golden Ratio in this drawing, see Figure 1.4.

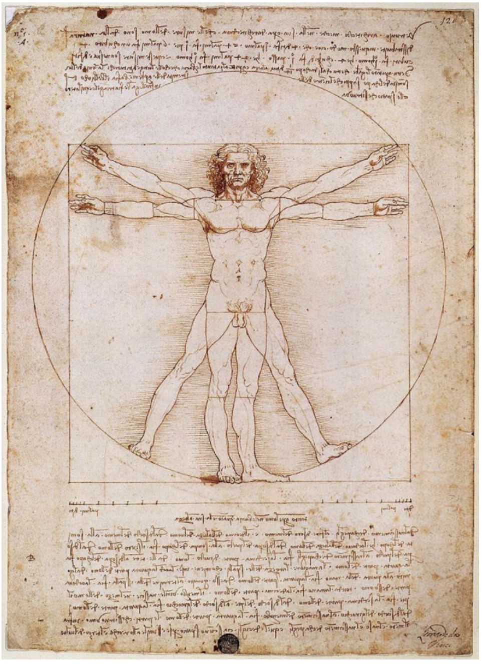

Figure 1.3: "The Vitruvian Man", drawn by Leonardo da Vinci. [7]

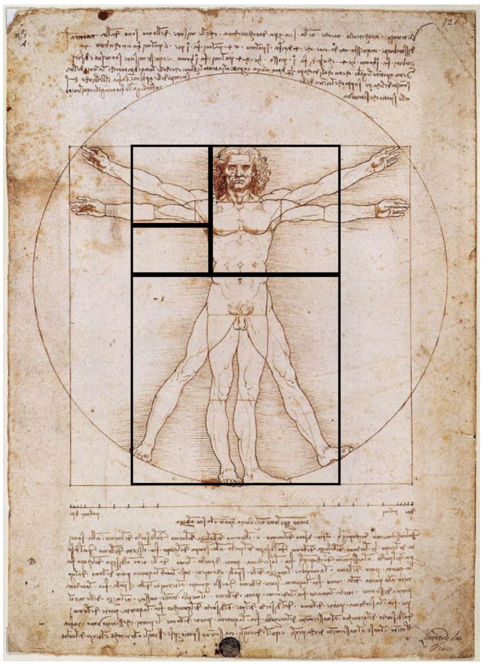

Figure 1.4: "The Vitruvian Man" with Golden Ratios drawn in. [7]

If you want to know, my 'height to navel ratio' is $\frac{170}{105} \approx 1,619$, so about 0,06% off. This seems very close, but obviously these measurements vary widely from person to person and my own measurements were not very precise either.

1.2.1. The Golden Spiral

Since φ is the solution to equation 1.1, we have that:

$$\varphi = 1 + \frac{1}{\varphi}$$

If we divide by φ we get:

$$\begin{aligned} 1 &= \frac{1}{\varphi} + \frac{1}{\varphi^2} \\ \frac{1}{\varphi} &= \frac{1}{\varphi^2} + \frac{1}{\varphi^3} \\ &\vdots \end{aligned}$$

From these equations, we will construct a Golden Spiral, see Figure 1.5.
First we make a rectangle of 1 by φ. Since $\varphi > 1$, we can place a 1 by 1 one square inside this rectangle, and we will be left with a 1 by $\varphi - 1 = \frac{1}{\varphi}$ rectangle.Since $\frac{1}{\varphi} < 1$, we can place a $\frac{1}{\varphi}$ by $\frac{1}{\varphi}$ square inside this rectangle, and we will be left with a $1 - \frac{1}{\varphi} = \frac{1}{\varphi^2}$ by $\frac{1}{\varphi}$ rectangle. Once again we can place a $\frac{1}{\varphi^2}$ by $\frac{1}{\varphi^2}$ square inside and are left with a $\frac{1}{\varphi^2}$ by $\frac{1}{\varphi^3}$ rectangle. If we do this infinitely many times and we draw a quarter circle inside every square, we are left with a Golden Spiral.

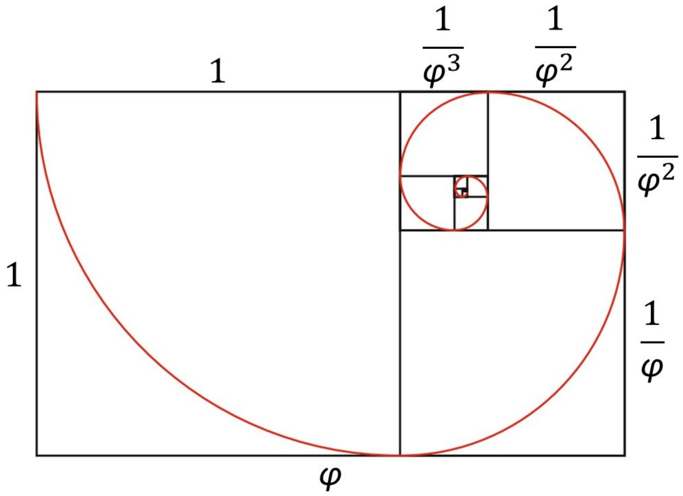

Figure 1.5: Golden Spiral [8]

Figure 1.6: Nautilus shell [8]

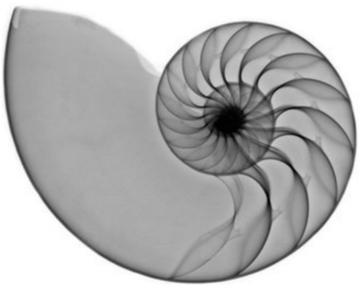

Figure 1.7: X-ray image of a Nautilus shell [8]

The pearly, or chambered, nautilus (Nautilus) is a genus of cephalopod mollusks. This animal has a "smooth, coiled external shell about 25 cm in diameter, consisting of about 36 separate chambers, the outermost of which it lives in" [9]. This shell has a spiral pattern and there have been claims that this spiral resembles a Golden Spiral. To see if this true, we compare the expansion coefficients of both spirals.
The Golden Spiral is a logarithmic spiral. We write down the polar equation for a logarithmic spiral:

$$r = ae^{k\theta}$$

where r is the new radius or distance to the origin of the spiral, a the initial radius, k the expansion coefficient, and θ the angle from the positive x-axis [10] [11]. If we look at the Golden Spiral, every quarter turn (so an angle of $\frac{\pi}{2}$) the spiral grows with a factor φ. We get:

$$\begin{aligned} \varphi r &= re^{k\frac{\pi}{2}} \\ e^{k\frac{\pi}{2}} &= \varphi \\ k\frac{\pi}{2} &= \ln(\varphi) \\ k &= \frac{2}{\pi}\ln(\varphi) \approx 0,30635 \end{aligned}$$

[12]

But the expansion coefficient of a Nautilus shell lies around $k = 0,177$, which is not even close [8].

So it seems that there are some doubtful claims about the Golden Ratio in architecture, the human body and nature. However, spirals that have a close relation to the Golden Ratio can be found in the leaf arrangement of plants, as we will see in paragraph 2.1.2.

1.2.2. The Golden Angle

Not all claims about the Golden Ratio are incorrect. Let us have a look at the Golden Angle. We take a circle with circumference $a+b$ and divide it into a greater segment, a, and a lesser segment, b, such that:

$$\frac{a+b}{a} = \frac{a}{b}$$

The Golden Angle is the angle subtended by the arc of length b, see Figure 1.8.

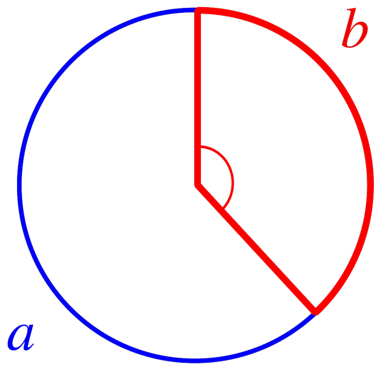

Figure 1.8: Circle divided into two arcs of length a and b, such that $a+b$ is to a as a is to b. [13]

We know that

$$\frac{a}{b} = \varphi, \text{ so } \frac{a+b}{b} = \frac{a}{b} + 1 = \varphi + 1$$

Let f be the smaller fraction of the circumference.

$$f = \frac{b}{a+b} = \frac{1}{\varphi+1}$$

From equation 1.1 we know that $\varphi^2 = \varphi + 1$, thus

$$f = \frac{1}{\varphi^2}$$

The whole circle equals 2π radians, so the Golden Angle equals:

$$\begin{aligned} 2\pi f &= \frac{2\pi}{\varphi^2} = \frac{2\pi}{\left(\frac{1+\sqrt{5}}{2}\right)^2} \\ &= \frac{2\pi}{\left(\frac{6+2\sqrt{5}}{4}\right)} = \frac{8\pi}{6+2\sqrt{5}} \\ &= \frac{4\pi}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{4\pi(3-\sqrt{5})}{4} = \pi(3-\sqrt{5}) \end{aligned}$$

Or $180(3-\sqrt{5}) \approx 137,5°$. This Golden Angle can be found in nature. Mathematicians have found that the angle between each sunflower floret and its neighbor, equals approximately 137°, the Golden Angle [14]. We will see more about the sunflower in paragraph 2.1.2.

1.3. The Golden Ratio in Geometry

The Golden Ratio can also be found in geometry. For example in a pentagram.

First we take a look at the isosceles triangle $\triangle ABC$ in Figure 1.9, where $\angle ABC = 36°$ and $\angle BAC = \angle BCA = 72°$. We make the triangle such that $|AB| = |BC| = 1$ and $|AC| = x$. If we divide $\angle BAC$ in two, and we call the point where this dividing line and BC intersects D, then we have two other isosceles triangles: $\triangle CAD$ and $\triangle BDA$. $\triangle ABC$ and $\triangle CAD$ are similar triangles.

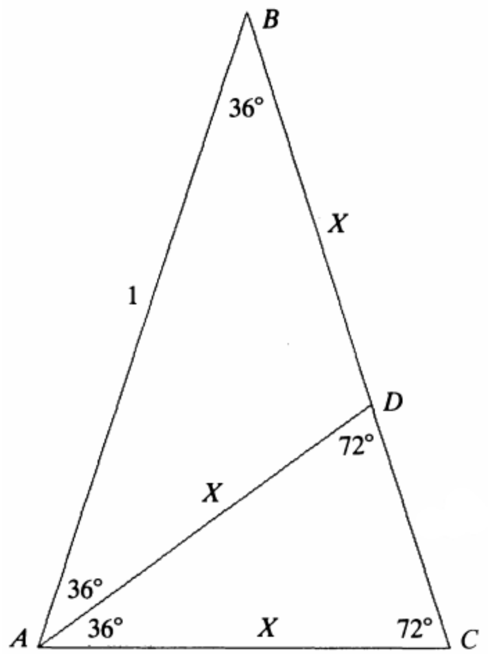

Figure 1.9: Isosceles triangle $\triangle ABC$ with $\angle ABC = 36°$, $\angle BAC = \angle BCA = 72°$, $|AB| = |BC| = 1$ and $|AC| = x$. [6]

We now have:

1. $\angle BAC = \angle BCA = \angle ACD = \angle ADC = 72°$
2. $\angle ABC = \angle BAD = \angle CAD = 36°$
3. $|AC| = |AD| = |BD| = x$
4. $|AB| = |BC| = 1$
5. $\triangle ABC \sim \triangle CAD$ (because of 1 and 2)

We get:

$$\begin{aligned} \frac{|AB|}{|AC|} &= \frac{|CA|}{|CD|} \\ \frac{1}{x} &= \frac{x}{|CD|} \\ |CD| &= x^2 \end{aligned}$$

But also:

$$\begin{aligned} |BC| &= |BD| + |CD| \\ 1 &= x + |CD| \\ |CD| &= 1 - x \end{aligned}$$

Thus

$$\begin{aligned} x^2 &= 1-x \\ x^2+x-1 &= 0 \\ x &= \frac{-1\pm\sqrt{5}}{2} \\ x &> 0,\ \text{so:} \\ x &= \frac{-1+\sqrt{5}}{2} = \varphi - 1 = \frac{1}{\varphi} \end{aligned}$$

We now have the following lengths:

$$|AB| = |BC| = 1,\ |AC| = |AD| = |BD| = \frac{1}{\varphi} \text{ and } |CD| = \frac{1}{\varphi^2}$$

This gives us the following ratios in Figure 1.9:

$$\frac{|AB|}{|AD|} = \frac{|AC|}{|CD|} = \varphi$$

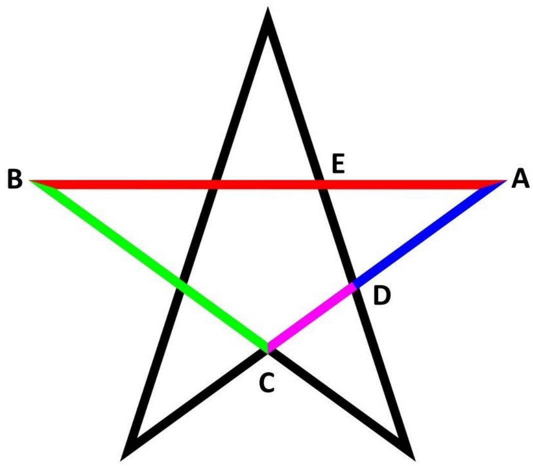

Figure 1.10: A pentagram colored to distinguish its line segments of different lengths. The four lengths are in Golden Ratio to one another. [15]

If we look at Figure 1.10, we see that the point of a pentagram, $\triangle DAE$, also is an isosceles triangle. Because all five points of a pentagram are similar isosceles triangles, we get that $\angle DAE = \frac{180}{5} = 36°$. Therefore $\angle ADE = \angle AED = 72°$. This gives us the following ratios in Figure 1.10:

$$\frac{|AB|}{|BC|} = \frac{|AD|}{|CD|} = \varphi$$

We see that

$$\begin{aligned} |BC| = |AC| &= |AD| + |CD| \\ &= |AD| + \frac{|AD|}{\varphi} = \left(1+\frac{1}{\varphi}\right)\cdot|AD| = \varphi\cdot|AD| \\ \frac{|BC|}{|AD|} &= \varphi \end{aligned}$$

So all four colored line segment in Figure 1.10 are in Golden Ratio to one another.

$$\frac{|AB|}{|BC|} = \frac{|BC|}{|AD|} = \frac{|AD|}{|CD|} = \varphi$$

2
The Fibonacci Numbers

The Fibonacci sequence is a sequence of integers, where the sum of every two consecutive numbers in the sequence equals the next number. We call these numbers the Fibonacci numbers, named after Leonardo of Pisa, who was also known as Fibonacci. Even though the sequence is named after him, it was not Leonardo of Pisa who came up with this sequence. The sequence was already known in India hundreds of years before Fibonacci, but he introduced the sequence to Western mathematics in 1202 [16].

Definition 2.0.1. We define the Fibonacci numbers recursively by $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$ starting from $F_0 = F_1 = 1$.

This gives us the following sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987,\ \ldots$$

[17]

In section 1.1.1, we already noted that the Fibonacci numbers are actually strongly related to the Golden Ratio. We will verify this in this chapter.

2.1. Properties of the Fibonacci Numbers

Let us start with the continued fraction in equation 1.2 and approximate its value:

$$\begin{aligned}
1 \qquad &= 1{,}000 \\
1+\frac{1}{1} = \frac{2}{1} &= 2{,}000 \\
1+\frac{1}{1+1} = \frac{3}{2} &= 1{,}500 \\
1+\frac{1}{1+\frac{1}{1+1}} = \frac{5}{3} &= 1{,}666 \\
1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} = \frac{8}{5} &= 1{,}600 \\
1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}} = \frac{13}{8} &= 1{,}625
\end{aligned}$$

We see that the fractions equal the ratio between the $n+1$ and nth Fibonacci number. This is true for all n, because if we look at a fraction, we see that it equals 1 plus (1 divided by the fraction that came before), which was the ratio between the $n-1$ and nth Fibonacci number:

$$1+\frac{1}{\left(\frac{F_n}{F_{n-1}}\right)} = 1+\frac{F_{n-1}}{F_n} = \frac{F_n}{F_n}+\frac{F_{n-1}}{F_n} = \frac{F_n+F_{n-1}}{F_n} = \frac{F_{n+1}}{F_n}$$

We also see that the fractions get closer to $\varphi \approx 1{,}618$. In fact, Johannes Kepler first discovered that the golden ratio, φ, is the limit of the ratios of consecutive Fibonacci numbers [18]. We will prove this in Proposition 2.1.2. But in order to prove this, we first need Binet's Formula.

2.1.1. Binet's Formula

We can write the nth Fibonacci number as a closed-form expression, using the positive and negative solution to equation 1.1, φ and ψ. This is known as Binet's Formula:

Lemma 2.1.1. *Binet's Formula*

$$F_n = \frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}$$

For $n = 0, 1, 2, \ldots$

Proof. φ and ψ were the solutions to the equation $x^2 - x - 1 = 0$, so

$$\begin{aligned}
x^2 &= x + 1 \\
x^3 &= x^2 + x = 2x + 1 \\
x^4 &= 2x^2 + x = 3x + 2 \\
x^5 &= 3x^2 + 2x = 5x + 3 \\
x^6 &= 5x^2 + 3x = 8x + 5 \\
&\vdots
\end{aligned}$$

We will prove by induction that

$$x^n = F_{n-1}x + F_{n-2} \tag{2.1}$$

For all $n \geq 2$.

Proof correctness equation 2.1 by induction
First we prove this to be true for $n = 2$.
$x^2 - x - 1 = 0$, so:

$$x^2 = x + 1 = F_1 x + F_0$$

since $F_0 = F_1 = 1$. Thus the equation is true for $n = 2$.
Now assume equation 2.1 is true for n, we will prove that it is also true for $n + 1$.

$$\begin{aligned}
x^n &= F_{n-1}x + F_{n-2} \\
x^{n+1} &= x \cdot x^n = F_{n-1}x^2 + F_{n-2}x \\
&= F_{n-1}(x+1) + F_{n-2}x \\
&= x(F_{n-1} + F_{n-2}) + F_{n-1} \\
&= F_n x + F_{n-1}
\end{aligned}$$

This is in accordance with equation 2.1, so the equation also holds for $n + 1$.
By induction, equation 2.1 holds for all $n \geq 2$. □

Since we have used $x^2 = x + 1$ to obtain equation 2.1, and the solutions to this equation are φ and ψ, we fill in $x = \varphi$ and $x = \psi$:

$$\begin{aligned}
\varphi^{n+1} &= F_n\varphi + F_{n-1} \\
\psi^{n+1} &= F_n\psi + F_{n-1} \\
\varphi^{n+1} - \psi^{n+1} &= F_n(\varphi - \psi) + F_{n-1} - F_{n-1} \\
\frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} &= F_n
\end{aligned}$$

□

Now that we have a closed-form expression for the nth Fibonacci number, we can easily prove that the Golden Ratio, φ, is the limit of the ratios of two consecutive Fibonacci numbers.

Proposition 2.1.2.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

Proof. We use Lemma 2.1.1, Binet's formula:

$$\begin{aligned}
\frac{F_{n+1}}{F_n} &= \frac{\left(\frac{\varphi^{n+2}-\psi^{n+2}}{\sqrt{5}}\right)}{\left(\frac{\varphi^{n+1}-\psi^{n+1}}{\sqrt{5}}\right)} = \frac{\varphi^{n+2}-\psi^{n+2}}{\varphi^{n+1}-\psi^{n+1}} \\
&= \frac{\varphi^{n+2}}{\varphi^{n+1}-\psi^{n+1}} - \frac{\psi^{n+2}}{\varphi^{n+1}-\psi^{n+1}} \\
&= \frac{\varphi}{1-\left(\frac{\psi}{\varphi}\right)^{n+1}} - \frac{\psi}{\left(\frac{\varphi}{\psi}\right)^{n+1}-1} \\
\frac{\psi}{\varphi} &= \frac{\left(\frac{1-\sqrt{5}}{2}\right)}{\left(\frac{1+\sqrt{5}}{2}\right)} = \frac{1-\sqrt{5}}{1+\sqrt{5}} \approx -0,382 \\
\left|\frac{\psi}{\varphi}\right| &< 1, \text{ thus } \lim_{n\to\infty}\left(\frac{\psi}{\varphi}\right)^{n+1} = 0 \\
\text{Thus } & \lim_{n\to\infty}\left(\frac{\varphi}{1-\left(\frac{\psi}{\varphi}\right)^{n+1}}\right) = \frac{\varphi}{1-0} = \varphi \\
\frac{\varphi}{\psi} &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)}{\left(\frac{1-\sqrt{5}}{2}\right)} = \frac{1+\sqrt{5}}{1-\sqrt{5}} \approx -2,618 \\
\left|\frac{\varphi}{\psi}\right| &> 1, \text{ thus } \lim_{n\to\infty}\left|\frac{\varphi}{\psi}\right|^{n+1} = \infty \\
\text{Thus } & \lim_{n\to\infty}\left(\frac{\psi}{\left(\frac{\varphi}{\psi}\right)^{n+1}-1}\right) = 0 \\
\lim_{n\to\infty}\frac{F_{n+1}}{F_n} &= \lim_{n\to\infty}\left(\frac{\varphi}{1-\left(\frac{\psi}{\varphi}\right)^{n+1}} - \frac{\psi}{\left(\frac{\varphi}{\psi}\right)^{n+1}-1}\right) = \varphi - 0 \\
&= \varphi
\end{aligned}$$

□

We can also define the *Pell numbers* recursively by $P_{n+1} = 2P_n + P_{n-1}$ for all $n \geq 1$ starting from $P_0 = 0$ and $P_1 = 1$. We see that this is similar to the Fibonacci recurrence, except adding the previous numbers twice. This gives the following sequence:

$$0,\ 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378,\ 5741,\ \ldots$$

[19]

We can write the nth Pell number as a closed-form expression, using the positive and negative solution to the equation $x^2 - 2x - 1 = 0$, $\delta_S = 1 + \sqrt{2}$ and $\delta_T = 1 - \sqrt{2}$. If we do the exact same prove of Binet's formula for the Pell numbers, we get:

$$P_n = \frac{{\delta_S}^n - {\delta_T}^n}{2\sqrt{2}}$$

In the same way as Proposition 2.1.2, we can prove that the silver ratio, δ_S, is the limit of the ratios of two consecutive Pell numbers:

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \delta_S = 1 + \sqrt{2}$$

We now make a rectangle of the nth Fibonacci number by the $(n+1)$th number. We can then place a F_n by F_n square inside and are left with a F_n by $F_{n+1} - F_n = F_{n-1}$ rectangle. If we continue doing this the same way we made a Golden Spiral we are left with a Fibonacci Spiral, see Figure 2.1.

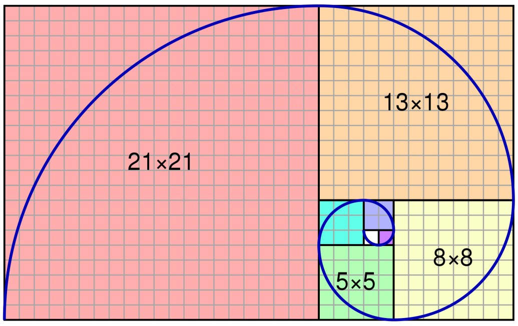

Figure 2.1: Fibonacci Spiral [20]

Since we proved in Proposition 2.1.2 that the ratio of two consecutive Fibonacci numbers goes to φ, this Fibonacci spiral approximates the Golden Spiral.

2.1.2. Fibonacci numbers in Nature

We have been looking for the Golden Ratio in nature, but now that we know that the Golden Ratio and the Fibonacci numbers are closely related, we can also look for these numbers instead.

Leaves that grow on a branch or a twig do often not grow directly above each other, as they will block each other from sunlight, air and rain. Instead, they grow around a branch or twig in a spiral. The arrangement of leaves around a branch is an example of *phyllotaxis*, which means “leaf arrangement" in Greek. It was named by Charles Bonnet in 1754. The amount of turns ik takes to get from one leaf to the next is called the *phyllotactic ratio*. Basswood leaves grow on opposite sides of a stem, so it takes half a turn to get from one leaf to the next. This is a $1/2$ phyllotactic ratio. A beech, blackberry and hazel have a phyllotactic ratio of $1/3$. An apple, apricot and coast live oak have a phylotactic ratio of $2/5$, and a pear and weaping willow have a phylactic ratio of $3/8$ [3]. Note that all these ratios are made of Fibonacci numbers.

Let us look at the florets in a sunflower again. The florets are arranged in clockwise and counter-clockwise spirals. The amount of clockwise and counter-clockwise spirals depends on the size of the sunflower. In Figure 2.2 we see a sunflower with 34 counter-clockwise (from the middle) spirals and 21 clockwise spirals. Most commonly there are 55 spirals going one way and 34 the other, but there have been seen sunflowers with ratios of numbers of spirals of 89/55 and 144/89 as well. [21]. If we look at Figure 2.3, we see a yellow chamomile with 21 clockwise (from the middle) and 13 counter-clockwise spirals . All these ratios are ratios of two consecutive Fibonacci numbers, which approximate the Golden Ratio. This relation to the Fibonacci numbers was noted as early as 1917 by D'Arcy Wentworth Thompson in *On Growth and Form*.

Figure 2.2: Head of a sunflower (*Helianthus annuus*) displaying spiral patterns [22].

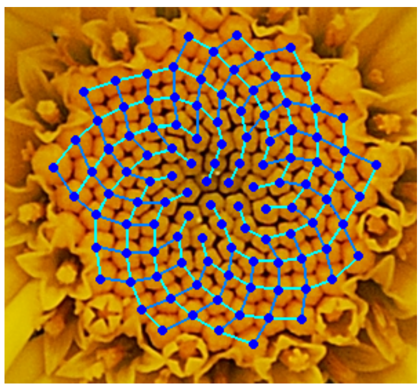

Figure 2.3: “ Disk florets of a yellow chamomile (*Anthemis tinctoria*) with spirals indicating the arrangement drawn in." [23]

These spirals are a very efficient way of packing a lot of florets in the head of a flower, without them blocking the sun for one another too much [14]. These spirals also appear in pineapples and pinecones [24]. For instance, out of 505 cones of the Norway spruce, 467 cones had five spirals in one direction and eight the other. This is about 92,4%. Out of these 467 cones, 224 cones had the eight spirals clockwise and 243 cones had the eight spirals counter-clockwise. A 48:52 ratio [21].

We cannot only find Fibonacci numbers in plants, but also in the family tree of honeybees. Honeybees can be either male or female. Females can either be workers, which are females that do not lay eggs, or queens. Queen honeybees can store the sperm of a male, which allows the queen to control the fertilization of her eggs. Thus queens can lay eggs that are either unfertilized or fertilized. Unfertilized eggs develop into males, whereas fertilized eggs develop into females [25]. Now, we look at the ancestors of a male honeybee. We represent males by '0' and females by '1'. Since a male hatches from an unfertilized egg, it only has a female as parent. Since a female hatches from a fertilized egg, it has a male and female as parents. We now have:

$$\begin{aligned} 0 &\rightarrow 1 \\ 1 &\rightarrow 10 \end{aligned}$$

This specific substitution is called the *Fibonacci substitution*.
If we start with one male honeybee, a zero, we get:

n		# ancestors at n	# 1's or female ancestors at n	# 0's, or male ancestors at n
0	0	0	0	0
1	$0 \rightarrow 1$	1	1	0
2	$1 \rightarrow 10$	2	1	1
3	$10 \rightarrow 101$	3	2	1
4	$101 \rightarrow 10110$	5	3	2
5	$10110 \rightarrow 10110101$	8	5	3
6	$10110101 \rightarrow 1011010110110$	13	8	5

Figure 2.4: Ancestry of a male honeybee, where ones represent females and zeros represent males.

We can see that a male honeybee has 1 parent, 2 grandparents, 3 great-grandparents, 5 great-great-grandparents, and so on. The Fibonacci sequence again! We can also see that if a honeybee has F_n great-great-...-grandparents, that they consist of F_{n-1} females and F_{n-2} males. This is, however, under the assumption that no ancestors are the same, which is highly unrealistic.
If we look at the sequence in the second column of Figure 2.4, we see that it always starts with the sequence that came before and ends with the sequence that came before that. Just like the Fibonacci numbers, this sequence is the addition of the two sequences that came before. If we do this infinitely many times, we get the following sequence:

$$1\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,1\,0\ldots$$

[26]

We saw that for the sequence of length F_n, the ratio of ones to zeros was $\frac{F_{n-1}}{F_{n-2}}$. So for the infinite sequence, the ratio of ones to zeros equals the limit of $\frac{F_{n-1}}{F_{n-2}}$ in n. So the ratio of ones to zeros is the Golden Ratio, φ, see Proposition 2.1.2. Therefore, this sequence is sometimes called the *Golden Sequence* [3], but it is also known as the *Fibonacci word*. Sometimes males are represented by '1' and females by '0', so the zeros and ones are switched.

2.1.3. The Generating Function of the Fibonacci Numbers

The generating function of the Fibonacci numbers can be used to find the asymptotics of said numbers.

Proposition 2.1.3.

$$\mathscr{F}(t) = \sum_{n=0}^{\infty} F_n t^n = \frac{1}{1-t-t^2}$$

For $0 \le t < 1$.

Proof.

$$\begin{aligned}
(1-t-t^2)\mathscr{F}(t) &= (1-t-t^2)\sum_{n=0}^{\infty} F_n t^n \\
&= \sum_{n=0}^{\infty} F_n t^n - t\sum_{n=0}^{\infty} F_n t^n - t^2\sum_{n=0}^{\infty} F_n t^n \\
&= \sum_{n=0}^{\infty} F_n t^n - \sum_{n=0}^{\infty} F_n t^{n+1} - \sum_{n=0}^{\infty} F_n t^{n+2} \\
&= \sum_{n=0}^{\infty} F_n t^n - \sum_{n=1}^{\infty} F_{n-1} t^n - \sum_{n=2}^{\infty} F_{n-2} t^n \\
&= F_0 + F_1 t + \sum_{n=2}^{\infty} F_n t^n - F_0 t - \sum_{n=2}^{\infty} F_{n-1} t^n - \sum_{n=2}^{\infty} F_{n-2} t^n \\
&= F_0 + t(F_1 - F_0) + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) t^n \\
\text{Since } F_0 = F_1 = 1 \quad &\text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for all } n \ge 2\text{, we get} \\
&= 1 + (1-1)t + \sum_{n=2}^{\infty} (F_n - F_n) t^n \\
&= 1 \\
(1-t-t^2)\mathscr{F}(t) &= 1, \quad \text{thus } \mathscr{F}(t) = \frac{1}{1-t-t^2}
\end{aligned}$$

□

2.1.4. Asymptotics of the Fibonacci Numbers

We get the following asymptotics of the Fibonacci numbers:

Lemma 2.1.4.

$$F_n \sim \frac{1}{\sqrt{5}}\varphi^{n+1}$$

Where $a_n \sim b_n$ means $a_n / b_n \to 1$ for $n \to \infty$.

Proof. We use Proposition 2.1.3:

$$\mathscr{F}(t) = \sum_{n=0}^{\infty} F_n t^n = \frac{1}{1-t-t^2}$$

Associate with a power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ the number

$$\frac{1}{R} = \limsup \sqrt[n]{|c_n|} \tag{2.2}$$

Where R is the radius of convergence of the series [27].
In this case $c_n = F_n$ and $z_0 = 0$. Now we need to find R.

$\mathscr{F}(t)$ has singularities where there occurs a division by 0, so at $1-t-t^2=0$.

$$\begin{aligned}
1-t-t^2 &= 0\\
t_{1,2} &= \frac{1\pm\sqrt{5}}{-2}=\frac{-1\pm\sqrt{5}}{2}\\
t_1 &= \frac{-1+\sqrt{5}}{2}=-\psi\\
t_2 &= \frac{-1-\sqrt{5}}{2}=-\varphi\\
\frac{1}{1-t-t^2} &= \frac{1}{-(t+\psi)(t+\varphi)}
\end{aligned}$$

We see that $\mathscr{F}(t)$ has singularities at $t=-\psi$ and $t=-\varphi$. Both poles of order 1.

$$\begin{aligned}
-\psi=\varphi-1 &\approx 0,618\\
-\varphi &\approx -1,618
\end{aligned}$$

So in absolute value, $-\psi$ is our smallest pole, which means our radius of convergence R of the series $\mathscr{F}(t)$ is $|-\psi|$.

$$\begin{aligned}
R &= |-\psi|=\varphi-1\\
\varphi &= 1+\frac{1}{\varphi} \qquad \text{(equation 1.1)}\\
R &= \varphi-1=\frac{1}{\varphi}
\end{aligned}$$

From equation 2.2 we get:

$$\limsup \sqrt[n]{F_n}=\frac{1}{R}=\varphi$$

This means that for all $\varepsilon>0$ there is an n_0 such that for all $n\geq n_0$

$$\sqrt[n]{F_n} < \varphi+\varepsilon$$

Let $\varepsilon>0$ be arbitrary, for n large enough we get:

$$\begin{aligned}
\sqrt[n]{F_n} &< \varphi+\varepsilon\\
F_n &< (\varphi+\varepsilon)^n
\end{aligned}$$

Since we have a closed form expression for the Fibonacci number, we can actually calculate the asymptotics exactly. Use Binet's Formula, Lemma 2.1.1:

$$\begin{aligned}
\frac{F_n}{\left(\frac{\varphi^{n+1}-\psi^{n+1}}{\sqrt{5}}\right)} &= 1\\
\lim_{n\to\infty}\frac{F_n\sqrt{5}}{\varphi^{n+1}-\psi^{n+1}} &= 1\\
\text{Since } |\psi| &= \left|\frac{1-\sqrt{5}}{2}\right|\approx|-0,618|<1\\
\text{We get } \lim_{n\to\infty}\psi^{n+1} &= 0\\
\lim_{n\to\infty}\frac{F_n\sqrt{5}}{\varphi^{n+1}-\psi^{n+1}} &= \lim_{n\to\infty}\frac{F_n\sqrt{5}}{\varphi^{n+1}}=1
\end{aligned}$$

Thus

$$F_n\sim\frac{1}{\sqrt{5}}\varphi^{n+1}$$

□

2.2. $\mathscr{B}_n$

Now we will consider sequences consisting of the symbols {⊂, ⊃, ♢}, where each open bracket "⊂" must be followed by a closed bracket "⊃", and a closed bracket is always preceded by an open bracket. The brackets only come in pairs. Let $\mathscr{B}_n$ bet the set of such sequences of length n. For example:

$$\mathscr{B}_4 = \{\lozenge\lozenge\lozenge\lozenge,\ \subset\supset\lozenge\lozenge,\ \lozenge\subset\supset\lozenge,\ \lozenge\lozenge\subset\supset,\ \subset\supset\subset\supset\}$$

We see that $|\mathscr{B}_4| = 5 = F_4$. This is actually true for all n.

Proposition 2.2.1.

$$|\mathscr{B}_n| = F_n \qquad \textit{for all } n \geq 1$$

Proof. We can clearly see that Proposition 2.2.1 holds for small n.

$\mathscr{B}_1$	=	{♢}	$\lvert\mathscr{B}_1\rvert$	=	1
$\mathscr{B}_2$	=	{♢♢, ⊂⊃}	$\lvert\mathscr{B}_2\rvert$	=	2
$\mathscr{B}_3$	=	{♢♢♢, ⊂⊃ ♢, ♢ ⊂⊃}	$\lvert\mathscr{B}_3\rvert$	=	3
$\mathscr{B}_4$	=	{♢♢♢♢, ⊂⊃ ♢♢, ♢ ⊂⊃ ♢, ♢♢ ⊂⊃, ⊂⊃⊂⊃}	$\lvert\mathscr{B}_4\rvert$	=	5
$\mathscr{B}_5$	=	{♢♢♢♢♢, ⊂⊃ ♢♢♢, ♢ ⊂⊃ ♢♢, ♢♢ ⊂⊃ ♢, ♢♢♢ ⊂⊃, ⊂⊃⊂⊃ ♢, ⊂⊃ ♢ ⊂⊃, ♢ ⊂⊃⊂⊃}	$\lvert\mathscr{B}_5\rvert$	=	8

Figure 2.5: $\mathscr{B}_n$ for $n = 1,\dots,5$.

Now we will prove that Proposition 2.2.1 holds for all n, by using induction.
Assume the equation holds for all $m = 1,\dots,n-1$, we prove that it also holds for n.
We want to know $|\mathscr{B}_n|$, so we split $\mathscr{B}_n$ up in two sets.
Let U_n be the set of all sequences of length n that end with "⊂⊃".
Let V_n be the set of all sequences of length n that end with "♢".
Clearly, we have

$$|\mathscr{B}_n| = |U_n| + |V_n| \tag{2.3}$$

All elements in U_n are of length n and end with "⊂⊃", so the sequence preceding the pair of brackets must be of length $n-2$. Thus, all elements in U_n can be created by taking an element of $\mathscr{B}_{n-2}$ and putting "⊂⊃" behind it. This gives us:

$$|U_n| = |\mathscr{B}_{n-2}| \tag{2.4}$$

All elements in V_n are of length n and end with "♢", so the sequence preceding the pair of brackets must be of length $n-1$. Thus, all elements in V_n can be created by taking an element of $\mathscr{B}_{n-1}$ and putting "♢" behind it. This gives us:

$$|V_n| = |\mathscr{B}_{n-1}| \tag{2.5}$$

Equations 2.3, 2.4 and 2.5 together with the assumption that $|\mathscr{B}_m| = F_m$ for all $m = 1,\dots,n-1$, give:

$$|\mathscr{B}_n| = |U_n| + |V_n| = |\mathscr{B}_{n-2}| + |\mathscr{B}_{n-1}| = F_{n-2} + F_{n-1} = F_n$$

□

2.3. Extra Facts About the Fibonacci Numbers

Kepler also observed that the square of any Fibonacci number minus the product of the two adjacent numbers alternates between + and -1 [28].

$$F_n{}^2 - F_{n-1} \cdot F_{n+1} = (-1)^n \quad \text{for all } n \geq 1 \tag{2.6}$$

We will not prove this statement, as it is not important for our main theorem, but you can prove it yourself by using induction.
Some other fun facts about the Fibonacci numbers you can try proving by yourselves:

- The sum of ten consecutive Fibonacci numbers equals 11 times the seventh number in the row, i.e.,

  $$F_n + F_{n+1} + \cdots + F_{n+9} = 11 \cdot F_{n+6}$$

  Hint: Use $F_{n+1} = F_n + F_{n-1}$ and repeatedly substitute this into the sum.

- The nth Fibonacci number squared equals the sum from $k = 0$ to $n-1$ of $F_k \cdot F_{k+1}$, if n is odd. If n is even, it equals that same sum +1.

  $$F_n{}^2 = \sum_{k=0}^{n-1} F_k F_{k+1} \quad \text{if } n \text{ is odd}$$
  $$F_n{}^2 = 1 + \sum_{k=0}^{n-1} F_k F_{k+1} \quad \text{if } n \text{ is even}$$

  Hint: Use equation 2.6 and $F_{n+1} = F_n + F_{n-1}$ and repeatedly substitute this into the sum.

- The sum of the first n Fibonacci numbers equals the $(n+2)$th number minus 1.

  $$F_{n+2} - 1 = \sum_{k=0}^{n} F_n$$

  Hint: Start with F_{n+2} and repeatedly use $F_n = F_{n-1} + F_{n-2}$.

3

The Euler Numbers

3.1. Alternating Permutations

A permutation σ of a set is a possible way its elements can be ordered or arranged. For example, if we have the set $\{1,2,3\}$, we can arrange its elements in six ways:

$$\{1,2,3\},\ \{1,3,2\},\ \{2,1,3\},\ \{2,3,1\},\ \{3,1,2\},\ \{3,2,1\}$$

Thus, we have six permutations of the set $\{1,2,3\}$.
A permutation σ of a set S with $|S| = n$ is a bijection from $\{1,2,\ldots,n\}$ to S. $\sigma : \{1,2,\ldots,n\} \to S$. For example, the permutation $\{2,1,3\}$ is given by the function σ, defined as follows: $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$.

Let S_n be the set of all permutations of $\{1,2,\ldots,n\}$. Clearly, we can choose n numbers for the first position, after we picked that we have $n-1$ numbers left for the second position, $n-2$ for the third position, etc. So

$$|S_n| = n! \tag{3.1}$$

We also see that it does not matter what elements we choose. We can arrange $\{a,b,c\}$ in equallly many ways as $\{1,2,3\}$. As long as no elements are the same, it only matters how many elements we have.

Definition 3.1.1. A permutation $\sigma \in S_n$ is called an alternating permutation if:

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$$

Let $\mathscr{A}_n$ be the set of alternating permutations in S_n.

An alternating permutation in $\mathscr{A}_n$ is made with integers 1 up to n. But if we fix any n integers $x_1, x_2, \ldots, x_n$, then we can make exactly as many alternating permutations with them as with 1 up to n, as long as no integers are the same.

Definition 3.1.2. We define the sequence of Euler numbers, also called the zigzag numbers or up-down numbers [29], as a sequence of numbers E_n, where E_n is the number of alternating permutations from the set S_n.

$$E_0 = 1 \qquad \text{and}$$

$$E_n = |\mathscr{A}_n| \qquad \text{for all } n \geq 1$$

This gives us the following sequence:

$$1,\ 1,\ 1,\ 2,\ 5,\ 16,\ 61,\ 272,\ 1385,\ 7936,\ 50521,\ 353792,\ \ldots$$

[29]

3.2. Properties of the Euler Numbers

3.2.1. The Seidel-Entringer Triangle

We can also get the Euler numbers via the Seidel-Entringer triangle. This triangle is constructed as follows:

- The first row will contain 1 number, the next row 2 numbers, etc.
- Begin at the top, where we place $E_0 = 1$.
- We alternate the direction we go in. In the second row we go from left to right, in the third from right to left, in the fourth from left to right, etc.
- Each row starts with 0, each new number is equal to the previous number plus the number above.
- The last number in the row is E_n.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 0 & \rightarrow & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & \mathbf{2} & & & \\
 & & \mathbf{5} & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & \mathbf{16} & \\
\mathbf{61} & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 & \leftarrow & 0
\end{array}
$$

Figure 3.1: The Seidel-Entringer triangle, where the red numbers equal the Euler numbers.

If we take the numbers from the triangle in Figure 3.1 and leave out the zeros and the first 1, we get the following sequence:

$$1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 4,\ 5,\ 5,\ 5,\ 10,\ 14,\ 16,\ 16,\ 16,\ 32,\ 46,\ 56,\ 61,\ 61,\ \ldots$$

[30]

These are called the *Entringer numbers*.

Definition 3.2.1. We define the Entringer numbers, $E_{n,k}$ by $E_{n,k} = |\mathscr{A}_{n,k}|$ for $n = 1,2,\ldots$ and $k = 1,\ldots,n$, where $\mathscr{A}_{n,k} = \{\sigma \in \mathscr{A}_n,\ \sigma(1) = k\}$, i.e., the set of alternating permutations $\sigma \in S_n$ starting with k.

	Alternating permutations		Alternating permutations
$E_{1,1} = 1$	(1)	$E_{5,1} = 5$	(1 3 2 5 4), (1 4 2 5 3), (1 4 3 5 2),
$E_{2,1} = 1$	(1 2)		(1 5 3 4 2), (1 5 2 4 3)
$E_{2,2} = 0$	-	$E_{5,2} = 5$	(2 3 1 5 4), (2 4 1 5 3), (2 4 3 5 1),
$E_{3,1} = 1$	(1 3 2)		(2 5 1 4 3), (2 5 3 4 1)
$E_{3,2} = 1$	(2 3 1)	$E_{5,3} = 4$	(3 4 1 5 2), (3 4 2 5 1)
$E_{3,3} = 0$	-		(3 5 1 4 2), (3 5 2 4 1)
$E_{4,1} = 2$	(1 3 2 4), (1 4 2 3)	$E_{5,4} = 2$	(4 5 1 3 2), (4 5 2 3 1)
$E_{4,2} = 2$	(2 3 1 4), (2 4 1 3)	$E_{5,5} = 0$	-
$E_{4,3} = 1$	(3 4 1 2)		
$E_{4,4} = 0$	-		

Figure 3.2: Entringer numbers $E_{n,k}$ for $n = 1,\ldots,5$ with corresponding alternating permutations starting with k.

We can see that these numbers $E_{n,k}$ from Figure 3.2 fit the Seidel-Entringer triangle.

$$\begin{array}{ccccccccc} & & & & E_{1,1} & & & & \\ & & & E_{2,2} & \rightarrow & E_{2,1} & & & \\ & & E_{3,1} & \leftarrow & E_{3,2} & \leftarrow & E_{3,3} & & \\ & E_{4,4} & \rightarrow & E_{4,3} & \rightarrow & E_{4,2} & \rightarrow & E_{4,1} & \\ E_{5,1} & \leftarrow & E_{5,2} & \leftarrow & E_{5,3} & \leftarrow & E_{5,4} & \leftarrow & E_{5,5} \end{array}$$

Figure 3.3: The Seidel-Entringer triangle with the Entringer numbers filled in.

If we replace $E_{n,k}$ in Figure 3.3 with the corresponding numbers from Figure 3.2, we do indeed get the triangle from Figure 3.1. So these numbers seem to fit for $n = 1,\dots,5$. Since each number is equal to the previous number plus the number above, we get the following formula:

Proposition 3.2.2.

$$E_{n,k} = E_{n,k+1} + E_{n-1,n-k}$$

With

$$\begin{aligned} E_{1,1} &= 1 \\ E_{n,n} &= 0 \quad \text{for all } n \geq 2 \end{aligned}$$

We can see that Proposition 3.2.2 holds for $n = 1,\dots,5$. Now we will prove that this holds for all $n \geq 2$.

Proof. First we prove that $E_{n,n} = 0$ for all $n \geq 2$.

$$E_{n,n} = |\mathscr{A}_{n,n}| = \#(\text{alternating permutations } \sigma \in S_n \text{ with } \sigma(1) = n)$$

But since n is the biggest number in $\sigma \in S_n$, we cannot find a bigger integer such that $n = \sigma(1) < \sigma(2)$.
Thus there are no alternating permutations $\sigma \in S_n$ starting with n.

Now assume Proposition 3.2.2 holds for $n = 1,\dots,m-1$, we prove that it then also holds for m. Thus by induction it will hold for all $m \geq 2$.

$$E_{m,k} = |\mathscr{A}_{m,k}| = \#(\text{alternating permutations } \sigma \in S_m \text{ with } \sigma(1) = k)$$

We can see that

$$|\mathscr{A}_{m,k+1}| = \#(\text{alternating permutations } \sigma \in S_m \text{ with } \sigma(1) = k+1)$$

comes somewhat close to this. Since there are no integers in between k and $k+1$, they have the same relations to the rest of the the integers in the permutation. If $a < k < b$, then also $a < k+1 < b$, with
$a, b \in \{1,\dots k-1, k+2,\dots,m\}$. So we take the alternating permutations $\sigma \in S_m$ with $\sigma(1) = k$ and we switch the numbers k and $k+1$. This will still be an alternating permutation when k and $k+1$ are not next to each other. If they are next to each other and they switch place, we get $\sigma(1) = k+1 > \sigma(2) = k$ and we will not have an alternating permutation anymore. We conclude that:

$$|\mathscr{A}_{m,k}| = |\mathscr{A}_{m,k+1}| + \#(\text{permutations } \sigma \in S_m \text{ with } \sigma(1) = k,\ \sigma(2) = k+1) \tag{3.2}$$

A permutation $\sigma \in S_n$ with $\sigma(1) = k$ and $\sigma(2) = k+1$ looks like:

$$k < k+1 > \sigma(3) < \sigma(4) > \dots$$

So if we take k and $k+1$ away, we get another alternating permutation $\sigma(3) < \sigma(4) > \dots$ of $m-2$ numbers, with $\sigma(3) < k$. Since no integers are the same, the number of alternating permutations made with $m-2$ integers is $|\mathscr{A}_{m-2}|$, so the number of alternating permutations made from $\{1,\dots,k-1,k+2,\dots,m\}$ starting with $i < k$, equals $|\mathscr{A}_{m-2,i}|$. We sum over all i:

$$\begin{aligned} \#(\text{permutations } \sigma \in S_m \text{ with } \sigma(1) = k,\ \sigma(2) = k+1) &= \sum_{i=1}^{k-1} |\mathscr{A}_{m-2,i}| \\ &= \sum_{i=1}^{k-1} E_{m-2,i} \end{aligned} \tag{3.3}$$

If

$$\sum_{i=1}^{k-1} E_{m-2,i} = |\mathscr{A}_{m-1,m-k}| \tag{3.4}$$

Then equation 3.2 and 3.3 together give:

$$|\mathscr{A}_{m,k}| = |\mathscr{A}_{m,k+1}| + |\mathscr{A}_{m-1,m-k}|$$

In that case, Proposition 3.2.2 holds.
We know that $E_{n,k} = E_{n,k+1} + E_{n-1,n-k}$ for $n = 1,\ldots,m-1$. We write this formula for $E_{m-1,m-k}$, then for $E_{m-1,m-k+1}$, etc.:

$$\begin{aligned}
E_{m-1,m-k} &= E_{m-1,m-k+1} + E_{m-2,k-1}\\
E_{m-1,m-k+1} &= E_{m-1,m-k+2} + E_{m-2,k-2}\\
E_{m-1,m-k+2} &= E_{m-1,m-k+3} + E_{m-2,k-3}\\
&\vdots\\
E_{m-1,m-2} &= E_{m-1,m-1} + E_{m-2,1}\\
E_{m-1,m-1} &= 0
\end{aligned}$$

Now we substitute all these fomulas into $E_{m-1,m-k}$:

$$\begin{aligned}
E_{m-1,m-k} &= E_{m-2,k-1} + E_{m-2,k-2} + E_{m-2,k-3} + \cdots + E_{m-2,1} + E_{m-1,m-1}\\
&= \sum_{i=1}^{k-1} E_{m-2,i} = |\mathscr{A}_{m-1,m-k}|
\end{aligned}$$

Thus equation 3.4 holds, thus Proposition 3.2.2 holds.

□

3.2.2. Generating Function of the Euler Numbers

Lemma 3.2.3.

$$\mathscr{E}(t) = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n = \frac{1+\sin(t)}{\cos(t)}$$

Proof. We will follow the proof from [31], Theorem 1. But three other proofs can be found in [32], Theorem 1.1. We will prove that

$$\cos(t) \cdot \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n = 1 + \sin(t) \tag{3.5}$$

holds. Use the Maclaurin series of $\cos(t)$:

$$\begin{aligned}
\cos(t) \cdot \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n &= \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{2n!}(-1)^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{E_n}{n!} t^n\right)\\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \cdot \left(E_0 + E_1 \frac{x}{1!} + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + \cdots\right)\\
&= E_0 + x\frac{E_1}{1!} + x^2\left(\frac{E_2}{2!} - \frac{E_0}{2!}\right) + x^3\left(\frac{E_3}{3!} - \frac{E_1}{1!\cdot 2!}\right)\\
&+ \quad x^4\left(\frac{E_4}{4!} - \frac{E_2}{2!\cdot 2!} + \frac{E_0}{4!}\right) + \cdots
\end{aligned}$$

We see that we get t^n if we do

$$\begin{aligned}
& 1\cdot E_n\frac{t^n}{n!} \\
\text{or}\quad & -\frac{t^2}{2!}\cdot E_{n-2}\frac{t^{n-2}}{(n-2)!} = -E_{n-2}\frac{t^n}{2!(n-2)!} \\
\text{or}\quad & \frac{t^4}{4!}\cdot E_{n-4}\frac{t^{n-4}}{(n-4)!} = E_{n-4}\frac{t^n}{4!(n-4)!} \\
\vdots \\
\text{or}\quad & (-1)^n\frac{t^n}{n!}\cdot E_0 \text{ if } n \text{ is even} \\
& (-1)^{n-1}\frac{t^{n-1}}{(n-1)!}\cdot E_1\frac{t}{1!} = (-1)^{n-1}E_1\frac{t^n}{1!(n-1)!} \text{ if } n \text{ is odd}
\end{aligned}$$

Thus the coefficient of t^n is the sum of all coefficients above:

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\frac{1}{(n-2k)!(2k)!}$$

So the coefficient of $t^n/n!$ is:

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\frac{n!}{(n-2k)!(2k)!} = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\binom{n}{2k}$$

We get

$$\cos(t)\cdot\sum_{n=0}^{\infty}\frac{E_n}{n!}t^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\binom{n}{2k}$$

If $n = 0$ we get

$$\frac{t^0}{0!}\sum_{k=0}^{0}(-1)^k E_{n-2k}\binom{n}{2k} = 1$$

Thus

$$\cos(t)\cdot\sum_{n=0}^{\infty}\frac{E_n}{n!}t^n = 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\binom{n}{2k}$$

We have proved that equation 3.5 holds, if

$$\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\binom{n}{2k} = \sin(t) \tag{3.6}$$

We know the Maclaurin series expansions of the sine:

$$\sin(t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}$$

The coefficient of $\frac{t^n}{n!}$ is 0 if n is even, and $(-1)^{(n-1)/2}$ if n is odd. So we will have to show that:

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k E_{n-2k}\binom{n}{2k} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} & \text{if } n \text{ is odd}\end{cases} \tag{3.7}$$

Let $D_n = \{1,2,\dots,n\}$ and let B_n be a subset of D_n of even size. Place the elements of B_n in increasing order to the left of a bar. Now let C_n be the set of remaning integers , i.e., $D_n \setminus B_n$. Place all the elements of C_n like an alternating permutation to right of the bar. Let A_n be the set of objects formed in this way. For example, an element of A_9 is: 2 4 5 7 8 9 | 3 6 1. Define for $a \in A_n$:

$$sign(a) = (-1)^{\#(\text{integers to the left of the bar in } a)/2}$$

Now let $k \in \{0, 1, \ldots \lfloor \frac{n}{2} \rfloor\}$ and let $2k$ be the number of integers to the left of the bar, then $|B_n| = 2k$. We can choose these numbers from D_n in $\binom{n}{2k}$ ways, since the order we pick them in does not matter. We then place them in increasing order, which can only be done in one way. Thus if we have $2k$ integers to the left of the bar, we can choose them in $\binom{n}{2k}$ ways. Now we have $n-2k$ integers remaining to put on the right of the bar. Since no integers are the same, the number of alternating permutations made from $n-2k$ integers is $|\mathscr{A}_{n-2k}|$. By definition 3.1.2, this equals E_{n-2k}. Let a_{2k} be an element in A_n such that there are $2k$ integers to the left of the bar. Then, taking the amount of possibilities from the left and the right of the bar together, we get that $\binom{n}{2k} E_{n-2k}$ is the number of possible a_{2k}. And

$$sign(a_{2k}) = (-1)^{\#(\text{integers to the left of the bar in } a_{2k})/2} = (-1)^{\frac{2k}{2}} = (-1)^k$$

$$\begin{aligned} \text{if } k \text{ is even} \sum_{a_{2k} \in A_n} sign(a_{2k}) &= \binom{n}{2k} E_{n-2k} = (-1)^k \binom{n}{2k} E_{n-2k} \\ \text{if } k \text{ is odd} \sum_{a_{2k} \in A_n} sign(a_{2k}) &= -\binom{n}{2k} E_{n-2k} = (-1)^k \binom{n}{2k} E_{n-2k} \end{aligned}$$

Sum over all k

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{a_{2k} \in A_n} sign(a_{2k}) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} E_{n-2k} \\ \sum_{a \in A_n} sign(a) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} E_{n-2k} \qquad (3.8) \end{aligned}$$

Suppose that we have an element $a = \sigma_1\ \sigma_2\ \ldots\ \sigma_n \in A_n$, but the bar is missing. Can we find where the bar was supposed to be?

- If n is odd and $\sigma_1, \sigma_2, \ldots, \sigma_n$ is increasing, then the bar could only have been between σ_{n-1} and σ_n. Then we have an increasing sequence of $n-1$ integers before the bar, which is even, and one integer after the bar, which is correct. If the bar is placed anywhere before σ_{n-2}, then after the bar we get $\ldots \sigma_{n-2}\ \sigma_{n-1} \sigma_n$ which cannot be part of alternating permutation, since it is increasing. If we place the bar right before σ_{n-1}, then there are $n-2$ numbers before the bar, which cannot be, since n was an odd number. If we place the bar after σ_n, then there are n numbers before the bar, which also cannot be, since n was an odd number.

- If n is even and $\sigma_1, \sigma_2, \ldots, \sigma_n$ is increasing, then the bar could only have been between σ_{n-2} and σ_{n-1} or after σ_n. If we place the bar between σ_{n-2} and σ_{n-1}, we get an increasing sequence of $n-2$ integers before the bar and $\sigma_{n-1}\ \sigma_n$ after the bar, which is alternating because $\sigma_{n-1} < \sigma_n$. If we place the bar after σ_n, we get an increasing sequence of n integers before the bar and nothing after, which also is correct. If the bar is placed anywhere before σ_{n-2} , then after the bar we get $\ldots \sigma_{n-2}\ \sigma_{n-1} \sigma_n$ which cannot be part of alternating permutation, since it is increasing. If we place the bar between σ_{n-1} and σ_n, we get $n-1$ integers before the bar, whoch connot be, since n was an even number.

- If $\sigma_1, \sigma_2 \ldots, \sigma_n$ is not increasing, suppose i is the first integer such that $\sigma_i > \sigma_{i+1}$. The sequence will then look like: $\sigma_1 < \sigma_2 < \cdots < \sigma_{i-2} < \sigma_{i-1} < \sigma_i > \sigma_{i+1} < \sigma_{i+2} > \ldots \sigma_n$. Then, the bar could have been in two places: directly before σ_{i-1} or directly after σ_i. The part after σ_i is definitely part of the alternating permutation, because $\sigma_i > \sigma_{i+1}$, so it cannot be part of the increasing sequence. Clearly, placing the bar directly after σ_i gives us an increasing sequence before the bar and an alternating permutation after. Placing the bar directly before σ_{i-1} als gives a correct permutation, since $\sigma_1, \sigma_2, \ldots, \sigma_{i-2}$ is increasing and $\sigma_{i-1} < \sigma_i > \sigma_{i+1} < \ldots$ is an alternating permutation. $\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_i$ is increasing, so if we place the bar anywhere before σ_{i-2}, then we get $\ldots \sigma_{i-2}\ \sigma_{i-1}\ \sigma_i \ldots$ after the bar which cannot be part of alternating permutation, since it is increasing. If we place the bar between σ_{i-1} and σ_i then the alternating permutation starts with $\sigma_i\ \sigma_{i+1}\ \ldots$, which cannot be because $\sigma_i > \sigma_{i+1}$ and an alternating permutation should start with an integer smaller than the second integer. Therefore, the bar must be placed between σ_{i-2} and σ_{i-1} or between σ_i and σ_{i+1}, creating a sequence before the bar of recpectively $i-2$ or i integers. This means that i must be even. If i is odd, we cannot create a correct element of A_n.

Let $a \in A_n$. If n is even or a is not increasing, define $I(a)$ as the element in A_n found by moving the bar to its other possible location. For example, if

$$a = 2\,4\,5\,7\,8\,9\,|\,3\,6\,1\text{ , then } I(a) = 2\,4\,5\,7\,|\,8\,9\,3\,6\,1$$

If n is odd and a is an increasing sequence, define $I(a) = a$, because the bar can only be in one place.
This mapping is an involution, i.e., it's own inverse: $I(I(a)) = a$. Since, in the case where n is even or a not increasing, the bar can only be directly before σ_{i-1} or directly after σ_i, applying I to a moves the bar with + or −2. So:

$$\begin{aligned} sign(I(a)) &= (-1)^{\#(\text{integers to the left of the bar in } I(a))/2} \\ &= (-1)^{(\#(\text{integers to the left of the bar in } a)\pm 2)/2} \\ &= (-1)^{(\#(\text{integers to the left of the bar in } a))/2} \cdot (-1)^{\pm 2/2} = -sign(a) \end{aligned}$$

In equation 3.8 we sum over all $a \in A_n$. So if n is even, all $sign(a)$ cancel each other out, because $sign(I(a)) = -sign(a)$. If n is odd, the only term that does not get cancelled out is when $I(a) = a$. This only happens when a is increasing. As we have seen before, the number of integers to the left of the bar equals $n-1$ in that case, so:

$$\begin{aligned} sign(a) &= (-1)^{\#(\text{integers to the left of the bar in } a)/2} \\ &= (-1)^{n-1}/2 \end{aligned}$$

Thus

$$\sum_{a \in A_n} sign(a) = \sum_{\substack{a \in A_n \\ I(a)=a}} sign(a) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases} \tag{3.9}$$

These were the coefficients we looked for in equation 3.7.
Substituting equation 3.9 in equation 3.6, by using the equality in equation 3.8, gives us:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k E_{n-2k} \binom{n}{2k} &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\substack{a \in A_n \\ I(a)=a}} sign(a) \\ &= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (-1)^{(2n+1-1)/2} \\ &= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (-1)^n = \sin(t) \end{aligned}$$

□

Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$, Lemma 3.2.3 can also be written as

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} t^n = \frac{1+\sin(t)}{\cos(t)} = \tan(t) + \sec(t) \tag{3.10}$$

Note that $\sec(x)$ is an even function and $\tan(x)$ is an odd function:

$$\begin{aligned} \sec(-x) &= \frac{1}{\cos(-x)} = \frac{1}{\cos(x)} = \sec(x) \\ \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x) \end{aligned}$$

It follows from equation 3.10 that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} t^{2n} &= \sec(t) \\ \sum_{n=0}^{\infty} \frac{E_{2n+1}}{(2n+1)!} t^{2n+1} &= \tan(t) \end{aligned}$$

This is why E_{2n} is called a secant number, and E_{2n+1} is called a tangent number [32].

3.2.3. Asymptotics of the Euler Numbers

From the classical generating function, Lemma 3.2.3, we get the following asymptotics of the Euler numbers:

Lemma 3.2.4. *For all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$:*

$$\frac{E_n}{n!} < \left(\frac{2+\varepsilon}{\pi}\right)^n$$

Proof. To prove that Lemma 3.2.4 holds, we use equation 2.2 again. In this case $c_n = \frac{E_n}{n!}$ and $z_0 = 0$. Now we need to find R. Therefore, we have to find the poles of $\frac{1+\sin(z)}{\cos(z)}$. $\mathscr{E}(t)$ has singularities where there occurs a division by 0, so at $\cos(z) = 0$. $\cos(z) = 0$ for $z = \frac{\pi}{2} + k\pi$, with $k \in \mathbb{Z}$. These are all zeros of order 1.

1. at $z = -\frac{\pi}{2} + 2k\pi$, with $k \in \mathbb{Z}$, we have $1 + \sin(z) = 1 + \sin\left(-\frac{\pi}{2} + 2k\pi\right) = 1 - 1 = 0$. Thus here we have a removable singularity.

2. at $z = \frac{\pi}{2} + 2k\pi$, with $k \in \mathbb{Z}$, we have $1 + \sin(z) = 1 + \sin\left(\frac{\pi}{2} + 2k\pi\right) = 1 + 1 = 2 \neq 0$. Thus here we have poles of order 1.

Thus we have poles at $\frac{\pi}{2} + 2k\pi$ with $k \in \mathbb{Z}$. So at $\frac{\pi}{2}$, then $-\frac{3\pi}{2}$, then $\frac{5\pi}{2}$, then $-\frac{7\pi}{2}$, etc.. The smallest pole in absolute value is $\frac{\pi}{2}$, this is our radius of convergence R.
From equation 2.2 we get:

$$\limsup \sqrt[n]{\frac{E_n}{n!}} = \frac{1}{R} = \frac{2}{\pi}$$

This means that for all $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$

$$\sqrt[n]{\frac{E_n}{n!}} < \frac{2+\varepsilon}{\pi}$$

Let $\varepsilon > 0$ be arbitrary, for n large enough, we get:

$$\begin{aligned} \sqrt[n]{\frac{E_n}{n!}} &< \frac{2+\varepsilon}{\pi} \\ \frac{E_n}{n!} &< \left(\frac{2+\varepsilon}{\pi}\right)^n \end{aligned}$$

□

4

Combining the Fibonacci and Euler numbers

In this chapter, we will combine the Fibonacci and Euler numbers to prove that $\pi \leq 2\varphi$. To do that, we first have to prove our main theorem, Theorem 4.0.1.

Theorem 4.0.1. *Main Theorem*

$$E_n \cdot F_n \geq n!$$

For example, $F_3 \cdot E_3 = 3 \cdot 2 = 3!$, $F_4 \cdot E_4 = 5 \cdot 5 = 25 > 24 = 4!$ and $F_5 \cdot E_5 = 8 \cdot 16 = 128 > 120 = 5!$.

4.1. Combinatorial Proof of the Main Theorem

From Proposition 2.2.1, equation 3.1 and Definition 3.1.2, we have that $|\mathscr{B}_n| = F_n$, $|S_n| = n!$ and $|\mathscr{A}_n| = E_n$. Therefore we can rewrite Theorem 4.0.1 as:

$$|\mathscr{A}_n| \cdot |\mathscr{B}_n| \geq |S_n| \quad \text{for } n \geq 1 \tag{4.1}$$

Consider the map $\phi : \mathscr{A}_n \times \mathscr{B}_n \to S_n$ defined as follows: $\phi(\sigma,\ w) = \omega$, where $\omega \in S_n$ is a permutation obtained by swapping the numbers from the alternating permutation $\sigma \in \mathscr{A}_n$ when they are in the same positions of a pair of consecutive brackets "⊂⊃" in $w \in \mathscr{B}_n$. The numbers that are in the same position as a diamond "◊", stay in the same position. For example:

$$\phi((3\ 6\ 2\ 5\ 4\ 7\ 1\ 8), (\Diamond\Diamond \subset\supset \Diamond \subset\supset \Diamond)) = (3\ 6\ 5\ 2\ 4\ 1\ 7\ 8)$$

We saw that $F_3 \cdot E_3 = 3 \cdot 2 = 3!$. So in this case, every permutation can be made by exactly one combination of elements from $\mathscr{A}_3$ and $\mathscr{B}_3$ (see Figure 4.1).

$\sigma \in \mathscr{A}_3$	$w \in \mathscr{B}_3$	$\phi(\sigma, w) = \omega \in S_3$
(1 3 2)	(◊ ⊂⊃)	(1 2 3)
(1 3 2)	(◊◊◊)	(1 3 2)
(2 3 1)	(◊ ⊂⊃)	(2 1 3)
(2 3 1)	(◊◊◊)	(2 3 1)
(2 3 1)	(⊂⊃ ◊)	(3 2 1)
(1 3 2)	(⊂⊃ ◊)	(3 1 2)

Figure 4.1: All possible combinations of elements from $\mathscr{A}_3$ and $\mathscr{B}_3$ create all possible permutations from S_3.

$F_4 \cdot E_4 = 5 \cdot 5 = 25 > 24 = 4!$, so here we have exactly one element that can be made by two combinations of elements from $\mathscr{A}_4$ and $\mathscr{B}_4$:

$$\phi((3\ 4\ 1\ 2), (\Diamond \subset\supset \Diamond)) = (3\ 1\ 4\ 2)$$
$$\phi((1\ 3\ 2\ 4), (\subset\supset\subset\supset)) = (3\ 1\ 4\ 2)$$

(See Figure 2.5 and Figure 3.2 for more elements from respectively $\mathscr{B}_n$ and $\mathscr{A}_n$.)

Lemma 4.1.1. *The map $\phi : \mathscr{A}_n \times \mathscr{B}_n \to S_n$ is a surjection.*

Proof. We need to show that for every permutation $\omega \in S_n$, there exist $\sigma \in \mathscr{A}_n$ and $w \in \mathscr{B}_n$ such that $\phi(\sigma, w) = \omega$. Let $J = \{\omega(2), \omega(4), \dots\}$ be the set of entries from ω in even positions and let $b = \omega(i)$ be the smallest element in J. Locally, the permutation ω looks like this:

$$\omega = (\dots, x, a, b, c, y, \dots)$$

Remember that an alternating permutation looked like this:

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) \dots$$

So the elements in even positions are bigger than their adjacent elements.

- Therefore if $b > a, c$, we do nothing. Since b was the smallest element in J and $x, y \in J$ we have $x, y > b > a, c$. We get the desired inequalities: $x > a < b > c < y$. Let w have a diamond $\diamondsuit$ at the corresponding position, indicating that b does not move.
  Repeat the procedure by induction for subpermutations $\sigma_1 = (\dots, x, a)$ and $\sigma_2 = (c, y, \dots)$.

- If $b < \max\{a, c\}$, swap b with $\max\{a, c\}$.
  If $\max\{a, c\} = a$, we swap a with b. We get the desired inequalities again: $x > b < a > c$. Let w have a pair of brackets $\subset\supset$ at the corresponding positions, indicating that a and b have been swapped.
  Repeat the procedure by induction for subpermutations $\sigma_1 = (\dots, x)$ and $\sigma_2 = (c, y, \dots)$.

- If $\max\{a, c\} = c$, we swap b with c. We get the desired inequalities again: $a < c > b < y$. Let w have a pair of brackets $\subset\supset$ at the corresponding positions, indicating that b and c have been swapped.
  Repeat the procedure by induction for subpermutations $\sigma_1 = (\dots, x, a)$ and $\sigma_2 = (b, y, \dots)$.

Let σ denote the resulting permutation at the end of the process.

Note that elements in even positions will only swap with an adjacent element if that number is bigger, so elements at even positions can only increase and elements at odd positions can only decrease. This is also true for the subpermutations σ_1 and σ_2, which have been chosen in a way such that the even positions in σ stay in even positions in the subpermutations.

If we swap a with b, we repeat the procedure for $\sigma_1 = (\dots, x)$ and $\sigma_2 = (c, y, \dots)$. These subpermutations do not include a and b, thus a and b will not move again. If we swap b with c, we repeat the procedure for $\sigma_1 = (\dots, x, a)$ and $\sigma_2 = (b, y, \dots)$. Thus c will not move again. Because b is in an odd position, it can only decrease, y can only increase and $b < y$, so b will also not move again. We see that every element moves at most once, so the sequence w is well-defined.

By induction, σ_1 and σ_2 both have alternating inequalities. Since we always got the "desired inequalities", the last element of σ_1, the elements in the middle of σ_1 and σ_2 and the first element of σ_2 all put together give the correct alternating inequalities. Thus σ is alternating, as desired. Finally, note that $\phi(\sigma, w) = \omega$, by construction. □

Example: (3 6 5 2 4 1 7 8)
$J = \{6, 2, 1, 8\}$, so $b = 1$. $\omega = (\dots, 2, 4, 1, 7, 8)$. $b = 1 < \max\{a, c\} = c = 7$. Swap 1 and 7. (3 6 5 2 4 7 1 8) and $w = (\dots\dots \subset\supset .)$. Repeat for $\sigma_1 = (\dots, x, a) =$ (3 6 5 2 4) and $\sigma_2 = (b, y, \dots) =$ (1 8).
$\sigma_2 =$ (1 8): $J = \{8\}$, so $b = 8$. $b = 8 > 1$. Do nothing. $w = (\dots\dots \subset\supset \diamondsuit)$.
$\sigma_1 =$ (3 6 5 2 4): $J = \{6, 2\}$, so $b = 2$. $\sigma_2 = (\dots, 6, 5, 2, 4)$ $b = 2 < \max\{a, c\} = a = 5$. Swap 2 and 5. (3 6 2 5 4 7 1 8) and $w = (..\subset\supset . \subset\supset \diamondsuit)$. Repeat for $\sigma_{1_1} = (\dots, x) =$ (3 6) and $\sigma_{1_2} = (c, y, \dots) =$ (4).
$\sigma_{1_2} =$ (4): $J = \emptyset$. Do nothing. $w = (..\subset\supset \diamondsuit \subset\supset \diamondsuit)$.
$\sigma_{1_1} =$ (3 6): $J = \{6\}$, so $b = 6$. $b = 6 > a = 3$. Do nothing. $w = (\diamondsuit\diamondsuit \subset\supset \diamondsuit \subset\supset \diamondsuit)$.
Finally, we are left with $\sigma =$ (3 6 2 5 4 7 1 8) and $w = (\diamondsuit\diamondsuit \subset\supset \diamondsuit \subset\supset \diamondsuit)$. Indeed, $\phi(\sigma, w) =$ (3 6 5 2 4 1 7 8).

The fact that the map $\phi : \mathscr{A}_n \times \mathscr{B}_n \to S_n$ is a surjection, Lemma 4.1.1, proves equation 4.1. The main theorem, Theorem 4.0.1 is hereby proven.

4.2. Combining the Asymptotics of the Fibonacci and Euler Numbers to prove that $\pi \leq 2\varphi$

We will now prove that $\pi \leq 2\varphi$ by combining the asymptotics of the Fibonacci and Euler numbers, Lemma 2.1.4 and Lemma 3.2.4, and the main theorem, Theorem 4.0.1.

Theorem 4.2.1.

$$\pi \leq 2\varphi$$

Proof.

$$\begin{aligned} n! &\leq F_n \cdot E_n \\ 1 &\leq \frac{F_n \cdot E_n}{n!} \quad \text{for all } n \geq 1 \end{aligned} \tag{4.2}$$

From the asymptotics of the Fibonacci and Euler numbers, Lemma 2.1.4 and Lemma 3.2.4, we get:
Let $\varepsilon > 0$ be arbitrary, then for n large enough:

$$\begin{aligned} 1 &\leq \frac{F_n \cdot E_n}{n!} < \left(\frac{2+\varepsilon}{\pi}\right)^n \frac{\varphi^{n+1}}{\sqrt{5}} \\ 1 &< \frac{\phi}{\sqrt{5}} \left(\frac{(2+\varepsilon)\varphi}{\pi}\right)^n \end{aligned}$$

This also holds for $n \to \infty$.

$$\begin{aligned} 1 &< \lim_{n\to\infty} \frac{\phi}{\sqrt{5}} \left(\frac{(2+\varepsilon)\varphi}{\pi}\right)^n \\ \frac{\sqrt{5}}{\varphi} &< \lim_{n\to\infty} \left(\frac{(2+\varepsilon)\varphi}{\pi}\right)^n \\ \frac{\sqrt{5}}{\varphi} &> 0 \text{ and } \neq 1 \\ \text{If } \frac{(2+\varepsilon)\varphi}{\pi} &< 1: \\ \lim_{n\to\infty} \left(\frac{(2+\varepsilon)\varphi}{\pi}\right)^n &= 0 \end{aligned}$$

Contradiction. Note that $\varepsilon > 0$ is arbitrary, so $\frac{(2+\varepsilon)\varphi}{\pi} \neq 1$. Therefore,

$$\begin{aligned} 1 &< \frac{(2+\varepsilon)\varphi}{\pi} \\ \pi &< (2+\varepsilon)\varphi \end{aligned}$$

Let $\varepsilon \downarrow 0$:

$$\pi \leq 2\varphi$$

□

4.3. Linear Extensions of Partially Ordered Sets

However, there is a deeper connection between the Fibonacci and Euler numbers than just $\pi \leq 2\varphi$. Our main theorem, Theorem 4.0.1, turns out to be a special case of a more abstract theorem, as we will see in this section.

Definition 4.3.1. We denote a partially ordered set, or *poset* for short, by $\mathscr{P}$. A poset consists of a set X of $n = |X|$ elements, and an order relation $\preceq$. The order relation indicates that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering, i.e., the elements are comparable. Let x, y be elements of a poset $\mathscr{P}$, then x and y are comparable if $x \preceq y$ or $y \preceq x$. A poset is called partially ordered, because not every pair of elements is necessarily comparable. If every pair of elements is comparable, we are speaking of a *totally ordered* set.

Definition 4.3.2. We define a linear extension of $\mathscr{P}$ as a bijection $f : X \to \{1,\dots,n\}$ such that $f(u) < f(v)$ for all $u, v \in X$ with $u \prec v$. Let $e(\mathscr{P})$ be the number of linear extensions of $\mathscr{P}$.

For example, if we look at a poset $\mathscr{P}$ consisting of a set $X = \{x_1,\dots,x_n\}$ with $x_i \in \mathbb{R}$ for $i = 1,\dots,n$ and $x_i \neq x_j$ if $i \neq j$, together with the order relation $<$, then every two elements are comparable. $\mathscr{P}$ is a totally ordered set. Therefore, we only have one bijection f such that $f(x_i) < f(x_j)$ for all $x_i, x_j \in X$ with $x_i \prec x_j$. We have $e(\mathscr{P}) = 1$. This is an example of an n*-chain*: all n elements are comparable. However, if no two elements are comparable, we have $e(\mathscr{P}) = n!$. This is called an n*-antichain*.

Definition 4.3.3. For a poset $\mathscr{P}$ on a set S, denote the comparability graph of $\mathscr{P}$ by $C(\mathscr{P})$, i.e., the graph with vertices $\mathscr{P}$ and edges $\{x, y\}$ if x and y are comparable in the poset. The complement of this comparability graph, $\overline{C(\mathscr{P})}$, is a graph on the same vertices, such that two vertices in $\overline{C(\mathscr{P})}$ are adjacent if and only if they are not adjacent in $C(\mathscr{P})$. A poset $\overline{\mathscr{P}}$ on S is called a *complement* if its comparability graph $C(\overline{\mathscr{P}})$ is a complement of $C(\mathscr{P})$, i.e., if $C(\overline{\mathscr{P}}) = \overline{C(\mathscr{P})}$. So if two elements are comparable in $\mathscr{P}$, they are not in $\overline{\mathscr{P}}$, and if two elements are not comparable in $\mathscr{P}$, they are comparable in $\overline{\mathscr{P}}$.

Note that if x and y are not comparable in $\mathscr{P}$, they are comparable in $\overline{\mathscr{P}}$, but we don't know whether $x \preceq y$ or $y \preceq x$. Therefore, a poset can have more than one complement.

Let $S \subset \mathbb{R}^2$ be a finite set of points. Define an ordering $(x_1, y_1) \preceq (x_2, y_2)$ when $x_1 \leq x_2$ and $y_1 \leq y_2$. The resulting poset $\mathscr{P}_S$ is called a *two-dimensional* poset. For example, let us have a look at the poset $\mathscr{H}_{p,q}$. $\mathscr{H}_{p,q}$ has $p+q+1$ elements forming a hook. One chain of p elements, one chain of q elements and one extra minimal element, Every pair of elements consisting of an element of the first chain and an element of the second chain, is incomparable. This means that in Figure 4.2, we have the following ordering:
$e \prec d \prec c \prec b \prec a$ and $e \prec f \prec g \prec h \prec i \prec j$.

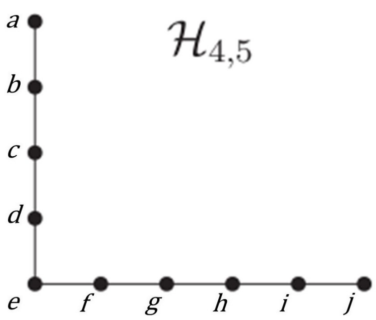

Figure 4.2: $\mathscr{H}_{p,q}$ with $p = 4$ and $q = 5$ [1].

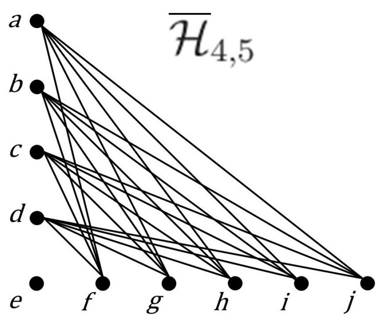

Figure 4.3: $\overline{\mathscr{H}}_{p,q}$ with $p = 4$ and $q = 5$.

If we want to make a linear extension of $\mathscr{H}_{p,q}$, the minimal element on the bottom left must be mapped to 1. If we now look at the chain with p elements, we can choose p numbers from $\{2,\dots,p+q+1\}$. Thus, we can choose from $p+q$ numbers, so the number of possibilities is $\binom{p+q}{p}$. Since all elements in the chain are comparable, we can order this in only one way. Now there are q numbers left for the chain with q elements, so this can also be ordered in exactly one way. We get a total of $\binom{p+q}{p}$ linear extensions.

$$e(\mathscr{H}_{p,q}) = \binom{p+q}{p} \tag{4.3}$$

In Figure 4.3, we see the complement of $\mathscr{H}_{4,5}$. Every pair of elements from one chain is now incomparable. Every pair of elements x, y from different chains is now compairable, with either $x \prec y$ or $y \prec x$. We have two possibilities for the relations: either $a, b, c, d \prec f, g, h, i, j$ or $f, g, h, i, j \prec a, b, c, d$. e is incomparable. There are no other posibilities, since if we take a and b from one chain and f from the other and let $a \prec f$ and $f \prec b$, we get $a \prec f \prec b$. By transitivity, $a \prec b$, but a and b were incomparable. This is a contradiction, so those are the only two posibilities.

A possible complement poset $\overline{\mathcal{H}}_{p,q}$ is described as follows:

1. If i and j were part of the same chain, i and j are now incomparable. $i \npreceq j$ and $j \npreceq i$
2. If i was part of the p-chain and j was part of the q-chain, i and j are now comparable. $j \preceq i$
3. If i was part of neither chains, i.e., i is the bottom left element, then i cannot be compared to any other element anymore.

If we want to make a linear extension of $\overline{\mathcal{H}}_{p,q}$, we see that the bottom left element can be mapped to any number from $1,\ldots,p+q+1$. For any element i from the p-chain and any element j from the q-chain, we have $j \preceq i$, so for the linear extension f we should have $f(j) \leq f(i)$. Therefore we can only map the elements from the p-chain to the p highest numbers left in $\{1,\ldots,p+q+1\}$, minus the number already chosen for the bottom left element. So we cannot choose these numbers, there is only one possible way. Since no elements from the p-chain are comparable, it does not matter which elements maps to which number. Therefore, this can be done in $p!$ ways. For the elements in the q-chain, we then have q numbers left to map to. Since no elements from the q-chain are comparable, it does not matter which elements maps to which number. Therefore, this can be done in $q!$ ways. Take everything together and we get:

$$e(\overline{\mathcal{H}}_{p,q}) = (p+q+1)p!q! \tag{4.4}$$

Now we look at the poset $\mathcal{U}_n$, forming a zigzag pattern with n points, see Figure 4.4. We have the following relations: $a \succ b \prec c \succ d \prec e \succ f \prec g$. Which means that our linear extension v should be defined such that $v(a) > v(b) < v(c) > v(d) < v(e) > v(f) < v(g)$. This is a *reverse alternating permutation*.
So the number of linear extensions of $\mathcal{U}_n$ is equal to the number of reverse alternating permutations.

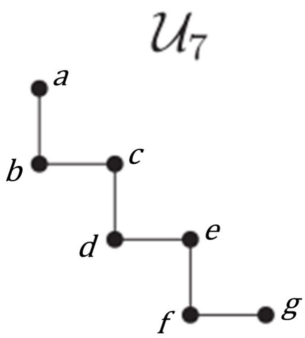

Figure 4.4: $\mathcal{U}_n$ with $n = 7$ [1].

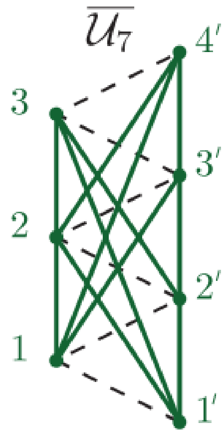

Figure 4.5: $\overline{\mathcal{U}}_n$ with $n = 7$ [1].

Lemma 4.3.4. *The number of reverse alternating permutations of length n equals the number of alternating permutations of length n.*

Proof. Let ρ be a reverse alternating permutation of length n made from the numbers $\{1,\ldots,n\}$.

$$\rho(1) > \rho(2) < \rho(3) > \ldots$$

Let $\mathcal{R}_n$ be the set of reverse alternating permutation of length n. We define a map $f : \mathcal{R}_n \to \mathcal{A}_n$ such that $1 \mapsto n,\ 2 \mapsto n-1,\ \ldots,\ n \mapsto 1$. Then we get an alternating permutation $\sigma \in \mathcal{A}_n$ and $f(\rho) = \sigma$.
A map $f : X \to Y$ is called *invertible* if $\exists g : Y \to X$ such that $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$. In our case, such a g exist, namely f^{-1}. The inverse of the map is defined as $f^{-1} : \mathcal{A}_n \to \mathcal{R}_n$ such that $n \mapsto 1,\ n-1 \mapsto 2,\ \ldots,\ 1 \mapsto n$. We then get the reverse alternating permutation $\rho \in \mathcal{R}_n$ again: $f^{-1}(\sigma) = \rho$. Thus, f in invertible. A map is invertibile if and only if it is bijective. Thus, f is a bijection. So $|\mathcal{A}_n| = |\mathcal{R}_n|$. □

We saw that the number of linear extensions of $\mathcal{U}_n$ is equal to the number of reverse alternating permutations. By Lemma 4.3.4:

$$e(\mathcal{U}_n) = |\mathcal{A}_n| = E_n \tag{4.5}$$

A possible complement poset $\overline{\mathscr{U}}_n$ is described as follows: we have n elements from $X = X_1 \cup X_2$, where $X_1 = \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and $X_2 = \left\{1', \dots, \lceil \frac{n}{2} \rceil'\right\}$, and we have the following relations:

1. $i \prec j$ and $i' \prec j'$ if $i < j$
2. $i \prec j'$ if $j - i > 1$
3. $i' \prec j$ if $j - i > 0$

See Figure 4.5 for an example.

Next, we willl prove that the number of linear extensions of $\overline{\mathscr{U}}_n$ equals the nth Fibonacci number.

Lemma 4.3.5.

$$e(\overline{\mathscr{U}}_n) = F_n$$

Proof. Note that $\overline{\mathscr{U}}_0 = 1$, since $X = \emptyset$. We have no elements, and there is only one way to order this. $e(\overline{\mathscr{U}}_1) = 1$, since $X_1 = \emptyset$ and $X_2 = \{1'\}$. We only have one element, and there is only one way to order this. Thus $e(\overline{\mathscr{U}}_0) = F_0 = 1$ and $e(\overline{\mathscr{U}}_1) = F_1 = 1$. Now we will prove by induction that $e(\overline{\mathscr{U}}_{n+1} = F_{n+1}$ for $n \geq 1$.
For $n \geq 1$ we have $n+1 \geq 2$, so X_1 is at least $\{1\}$ and X_2 is at least $\{1'\}$. The minimal elements of $\overline{\mathscr{U}}_{n+1}$ are 1 and $1'$. We can't compare 1 and $2'$, either $i = 1$ and $j = 2$ gives $2 - 1 = 1 \not> 1$ (relation 2), or $i = 2$ and $j = 1$ gives $1 - 2 = -1 \not> 0$ (relation 3). Therefore, the minimal elements of $\overline{\mathscr{U}}_{n+1} - \{1'\}$ are 1 and $2'$. The minimal element of $\overline{\mathscr{U}}_{n+1} - \{1\}$ is $1'$, because we can compare $1'$ and 2: $i = 1$ and $j = 2$ gives $j - i = 2 - 1 = 1 > 0$, so $1' \prec 2$ (relation 3). Thus, the linear extensions of $\overline{\mathscr{U}}_{n+1}$ either start with $1'$ or with both $11'$. Thus

$$e(\overline{\mathscr{U}}_{n+1}) = e\left(\overline{\mathscr{U}}_{n+1} - \{1'\}\right) + e\left(\overline{\mathscr{U}}_{n+1} - \{1, 1'\}\right) \tag{4.6}$$

First we show that $\overline{\mathscr{U}}_{n+1} - \{1'\}$ is isomorphic to $\overline{\mathscr{U}}_n$.
Let $f : \overline{\mathscr{U}}_{n+1} - \{1'\} \to \overline{\mathscr{U}}_n$ be such that

$$\begin{cases} f(i) = i' & \forall i \in \{1, \dots, \lfloor \frac{n+1}{2} \rfloor\} \\ f(i') = i - 1 & \forall i \in \left\{2', \dots, \lceil \frac{n+1}{2} \rceil'\right\} \end{cases}$$

f is clearly injective. We also get

$$\begin{aligned} f(i) &\in \left\{1', \dots, \left\lfloor \frac{n+1}{2} \right\rfloor'\right\} = \left\{1', \dots, \left\lceil \frac{n}{2} \right\rceil'\right\} \\ f(i') &\in \left\{1, \dots, \left\lceil \frac{n+1}{2} \right\rceil - 1\right\} = \left\{1, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \end{aligned}$$

Thus f is surjective. Therefore, f is bijective.

- If $i < j$, then $i \prec j$ and $i' \prec j'$. Also $f(i') \prec f(j')$ and $f(i) \prec f(j)$
- If $j - i > 1$, then $i \prec j'$. Also $(j - 1) - i > 0$, by relation (3) $i' \prec j - 1$, which equals $f(i) \prec f(j')$
- If $j - i > 0$, then $i' \prec j$. Also $j - (i - 1) > 1$, by relation (2) $i - 1 \prec j'$, which equals $f(i') \prec f(j)$

Thus $u \prec v$ if and only if $f(u) \prec f(v)$ for all $u, v \in X$. This together with the fact that f is bijective, means that f is an *order isomorphism*. Thus, $\overline{\mathscr{U}}_{n+1} - \{1'\}$ and $\overline{\mathscr{U}}_n$ are ismorphic. Therefore, they have the same number of linear extensions.

$$e\left(\overline{\mathscr{U}}_{n+1} - \{1'\}\right) = e\left(\overline{\mathscr{U}}_n\right) \tag{4.7}$$

Now we show that $\overline{\mathscr{U}}_{n+1} - \{1, 1'\}$ is isomorphic to $\overline{\mathscr{U}}_{n-1}$.
Let $f : \overline{\mathscr{U}}_{n+1} - \{1, 1'\} \to \overline{\mathscr{U}}_{n-1}$ be such that

$$\begin{cases} f(i) = i - 1 & \forall i \in \{2, \dots, \lfloor \frac{n+1}{2} \rfloor\} \\ f(i') = (i - 1)' & \forall i \in \left\{2', \dots, \lceil \frac{n+1}{2} \rceil'\right\} \end{cases}$$

f is clearly injective. We also get

$$\begin{aligned} f(i) &\in \left\{1,\ldots,\left\lfloor\frac{n+1}{2}\right\rfloor-1\right\}=\left\{1,\ldots,\left\lfloor\frac{n-1}{2}\right\rfloor\right\} \\ f(i') &\in \left\{1',\ldots,\left(\left\lceil\frac{n+1}{2}\right\rceil-1\right)'\right\}=\left\{1,\ldots,\left\lceil\frac{n-1}{2}\right\rceil'\right\} \end{aligned}$$

Thus f is surjective. Therefore, f is bijective.

- If $i<j$, then $i \prec j$ and $i' \prec j'$. Also $i-1<j-1$, by relation (1) $f(i) \prec f(j)$ and $f(i') \prec f(j')$
- If $j-i>1$, then $i \prec j'$. Also $(j-1)-(i-1)>1$, by relation (2) $i-1 \prec (j-1)'$, which equals $f(i) \prec f(j')$
- If $j-i>0$, then $i' \prec j$. Also $(j-1)-(i-1)>0$, by relation (3) $(i-1)' \prec j-1$, which equals $f(i') \prec f(j)$

Thus $u \prec v$ if and only if $f(u) \prec f(v)$ for all $u, v \in X$. This together with the fact that f is bijective, means that f is an *order isomorphism*. Thus, $\overline{\mathscr{U}}_{n+1}-\{1,1'\}$ and $\overline{\mathscr{U}}_{n_1}$ are ismorphic. Therefore, they have the same number of linear extensions.

$$e\left(\overline{\mathscr{U}}_{n+1}-\{1,1'\}\right)=e\left(\overline{\mathscr{U}}_{n_1}\right) \tag{4.8}$$

We get from equations 4.6, 4.7 and 4.8 and the assumptions that $e(\overline{\mathscr{U}}_n)=F_n$ and $e(\overline{\mathscr{U}}_{n-1})=F_{n-1}$:

$$e(\overline{\mathscr{U}}_{n+1})=e(\overline{\mathscr{U}}_n)+e(\overline{\mathscr{U}}_{n-1})=F_n+F_{n-1}=F_{n+1}$$

□

For examples on the isomorphisms, see Figure 4.6, 4.7, 4.8 and 4.9.

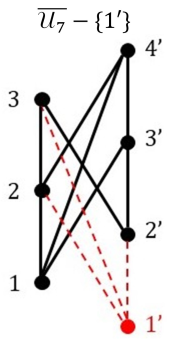

Figure 4.6: $\overline{\mathscr{U}}_7-\{1'\}$.

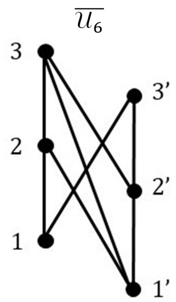

Figure 4.7: $\overline{\mathscr{U}}_6$.

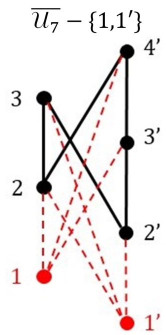

Figure 4.8: $\overline{\mathscr{U}}_7-\{1,1'\}$.

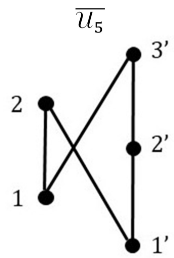

Figure 4.9: $\overline{\mathscr{U}}_5$.

Theorem 4.3.6. *Sidorenko*
Let $\mathscr{P}$ be a two-dimensional poset with n elements, and let $\overline{\mathscr{P}}$ be a complement of $\mathscr{P}$. We have

$$e(\mathscr{P})\cdot e(\overline{\mathscr{P}})\geq n!$$

[33]

The proof of this theorem uses "Stanley's interpretation of $e(\mathscr{P})$ as volumes of certain polytopes" [1], which is beyond the scope of this report.

If $\mathscr{P}$ is an n-chain, every element is comparable. Therefore, in $\overline{\mathscr{P}}$, no element is comparable. $\overline{\mathscr{P}}$ is an n-antichain. We saw that the number of linear extensions equals 1 and $n!$, respectively.

$$e(\mathscr{P})\cdot e(\overline{\mathscr{P}})=1\cdot n!=n!$$

So in this case the inequality in Theorem 4.3.6 is tight.

Let us have a look at $\mathscr{H}_{p,q}$ again. In this case $n = |\mathscr{H}_{p,q}| = p+q+1$. From equation 4.3 and 4.4, we get:

$$\begin{aligned} e(\mathscr{H}_{p,q}) \cdot e(\overline{\mathscr{H}}_{p,q}) &= \binom{p+q}{p} \cdot (p+q+1)\,p!q! \\ &= \frac{(p+q)!}{p!q!} \cdot (p+q+1)\,p!q! \\ &= (p+q)! \cdot (p+q+1) = (p+q+1)! \end{aligned}$$

So the inequality in Theorem 4.3.6 is tight again.

If we look at $\mathscr{U}_n$ we see that the main theorem, Theorem 4.0.1, follows immediately from Sidorenko's Theorem 4.3.6.

$$\begin{aligned} e(\mathscr{U}_n) \cdot e(\overline{\mathscr{U}}_n) &\geq n! \\ E_n \cdot F_n &\geq n! \end{aligned}$$

Thus, the Fibonacci and Euler numbers are connected.

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