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Design and Optimization of Conforming Lattice Structures

Jun Wu, Weiming Wang, Xifeng Gao

Abstract—Inspired by natural cellular materials such as trabecular bone, lattice structures have been developed as a new type of lightweight material. In this paper we present a novel method to design lattice structures that conform with both the principal stress directions and the boundary of the optimized shape. Our method consists of two major steps: the first optimizes concurrently the shape (including its topology) and the distribution of orthotropic lattice materials inside the shape to maximize stiffness under application-specific external loads; the second takes the optimized configuration (i.e. locally-defined orientation, porosity, and anisotropy) of lattice materials from the previous step, and extracts a globally consistent lattice structure by field-aligned parameterization. Our approach is robust and works for both 2D planar and 3D volumetric domains. Numerical results and physical verifications demonstrate remarkable structural properties of conforming lattice structures generated by our method.

Index Terms—Lattice structures, topology optimization, homogenization, 3D printing.



1 INTRODUCTION

The design of lightweight structures by optimization is a classical and still active topic in engineering. Stimulated by the increasingly high flexibility and resolution offered by 3D printing, there has been a growing interest in optimizing structures that are composed of delicate microstructures [1], [2]. These approaches assume that the microstructures are aligned with a prescribed regular grid. This simplifies modelling, simulation and optimization. It, however, limits the solution space, and thus the achievable structural performance. The microstructures are typically anisotropic (e.g. a hollowed cubic cell with uniform thickness is stiffer along its axes than along its diagonals). It is known that the orientation of anisotropic materials in stiffness-optimal structure coincides with the principal stress directions resulting from forces acting on these materials [3]. Furthermore, axis-aligned microstructures do not match the curved surfaces of 3D shapes, which may create problems in assembly of mechanical components.

To address the above issues, in this paper we propose an efficient and robust method to generate *conforming* lattice structures. A *lattice* is a connected array of struts. The lattice structure generated by our method is conforming in two aspects: the struts align with principal stress directions, maximizing structural stiffness; and, struts on the boundary capture the curved surface of the optimized shape. We note

that the shape, according to design options accessible to the user, is allowed to evolve together with the optimization of lattice distribution, i.e. the optimized shape is a subset of the design domain.

Our method has two major steps, in line with the homogenization-based optimization method proposed by Bendsøe and Kikuchi [4] and the post-treatment of the homogenization proposed by Pantz and Trabelsi [5] which was recently revisited [6], [7]. In the first step of our method, both the shape and the spatially-varying orientation of lattices inside the shape evolve simultaneously according to stress analysis and numerical optimization. Rather than relying on extremely high-resolution finite elements to capture the evolving lattice geometric details, we develop a homogenization-based topology optimization method which allows to efficiently simulate and optimize the lattice material distribution on a relatively coarse level. By introducing a novel parameterization of the unit cell, our method ensures a uniform thickness of struts while allowing a sufficient degree of lattice anisotropy. The second step, which we call *lattice compilation*, extracts a globally consistent lattice structure from the optimized, locally-defined lattice configuration, including orientation, porosity, and anisotropy. We address the challenging problem of extracting connected lattices across cells with spatially-varying orientation, by extending field-aligned meshing techniques. This extension allows a fast and robust lattice compilation where anisotropic geometric features are incorporated.

The specific contributions of our paper include:

- Jun Wu and Weiming Wang are with the Department of Design Engineering, Delft University of Technology, Delft, The Netherlands.
- Xifeng Gao is with the Department of Computer Science, Florida State University, US.
- Weiming Wang is also with the School of Mathematical Sciences, Dalian University of Technology, China.
- Corresponding Author: Jun Wu, E-mail: j.wu-1@tudelft.nl

- A novel workflow for designing, in both 2D and 3D, conforming lattice structures based on homogenization-based topology optimization and field-aligned parameterization.
- A simple and effective parameterization of the unit cell for allowing structural anisotropy while ensuring a uniform thickness of struts.

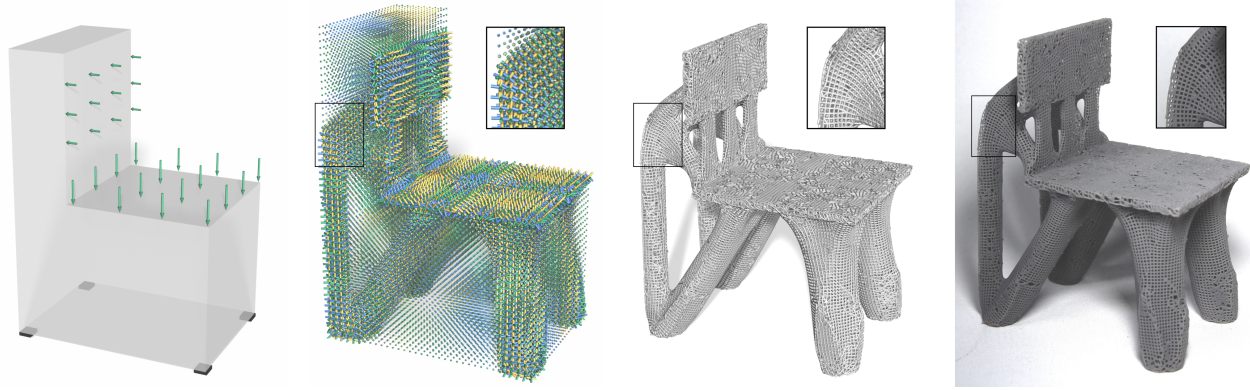


Fig. 1: From left to right: Given a design domain with specified external loads, our method optimizes the distribution of lattice materials for maximizing stiffness. From the optimized, locally-defined lattice configuration, a globally connected lattice structure is compiled, and fabricated by 3D printing.

- A new formulation to allow simultaneous optimization of the shape and the lattice distribution.
- A novel approach for extracting globally consistent lattice structures that accommodate anisotropy and heterogeneity.

Our method generates highly detailed lattice structures. The optimized lattice chair in Figure 1, for instance, consists of 178,291 struts, achieved on a simulation resolution of $140 \times 100 \times 200$.

The remainder of the paper is organized as follows. In the next section we review related work. In Section 3 we give an overview of the proposed method. In Sections 4 and 5 the two major steps of our method, lattice optimization and compilation, are presented. Results and analysis are presented in Section 6, before the conclusions are given in Section 7.

2 RELATED WORK

2.1 Structural Optimization for 3D Printing

In the era of 3D printing (and more broadly, digital fabrication), structural optimization becomes increasingly relevant in computational design [8]. Skin-frame structures [9], honeycomb-like Voronoi structures [10], tree-like supporting structures [11], and bone-inspired porous structures [12] have been optimized as lightweight infill for prescribed 3D shapes. Guided by outputs from structural optimization, Martinez et al. proposed to use graded orthotropic foams as a parameterized metamaterial to fill a prescribed shape [13], [14]. In contrast to design and optimize internal structures for prescribed shapes, our method optimizes concurrently the shape and its internal microstructures for application-specific loads. Different from two-scale structural optimizations (e.g. [15], [1]) which assume axis-aligned microstructures, our method optimizes the orientation of microstructures, in particular, to align it with spatially-varying stress directions. We restrict our design method to lightweight microstructures that are composed of struts, i.e. lattice structures. We note that lattice structures are less optimal than closed-walled shell structures regarding stiffness, yet they have potential benefits regarding, among others, structural stability and manufacturing [16].

Lattice structures are typically aligned with a regular grid [17], [18]. Rosen and his co-authors proposed a method

to design lattice structures that conform with the boundary surface of a prescribed 3D shape [19], [20]. Our method optimizes concurrently the shape and align the lattices with stress directions. The alignment of structures along principal stress directions improves structural performance [21], [3]. This principle has been applied to 2D planes (e.g. [22]) and curved surfaces [23], [24], [25]. The appealing 2D results are achieved by tracing stress directions or based on a ground structure approach [26], [27]. Due to their inherent challenges associated with the initialization of samples/nodes, a direct extension of these methods to 3D volumetric lattices is not applicable. Our method constructs stress-aligned 3D volumetric lattices, relying on homogenization-based topology optimization and field-aligned meshing.

Our method is among recent efforts on structural analysis and optimization for 3D printing. Stava et al. proposed a method to detect and correct structural defects [28]. Recent efforts include worst-case structural analysis [29], [30], and stochastic structural analysis [31]. Chen et al. proposed a solver for inverse elastic shape design [32]. Ulu et al. optimized structures under force location uncertainty [33]. Our method, targeting on stiffness maximization of lattice structures under certain static loads, is complementary to these efforts. Yet the integration goes beyond the scope of this paper.

2.2 Homogenization-based Topology Optimization

Topology optimization is an important design method for 3D printing, as it effectively leverages the fabrication flexibility to create structures with exceptional (mechanical) properties. Topology optimization transforms optimal shape design as a material distribution problem. In their seminal work, Bendsøe and Kikuchi proposed a homogenization method, which optimizes the distribution of square unit cells with variable rectangular holes [4]. Due to the lack of manufacturing means for such microstructures back then, the homogenization method was replaced by density-based approaches (e.g. SIMP [34]) which have since been widely used in industry and in many academic contributions (e.g. large scale optimization [35], [36], [37]).

In light of the capability of 3D printing to fabricate microstructures, the homogenization method was recently revisited to design structures with manufacturable mi-

crostructures [6], [7], [38], based on the rectangular hole model [4]. A challenge is to compile a continuous structure from hollowed cells that are defined on a regular grid, and that, after optimization, have different orientations. To this end, a projection approach proposed by Pantz and Trabelsi [5] was improved to connect the orthotropic microstructures [6], [39]. The output structure is represented by high-resolution pixels or voxels.

Our approach to design conforming lattice structures follows the homogenization approaches, but differs in three aspects. First, we propose a new parameterization of cells to ensure that the variable cells have a constant thickness, while allowing a large degree in anisotropy. Our intention of creating uniform thickness is to simplify all downstream operations including surface mesh creation, process planning, fabrication, surface treatment, inspection and qualification. Uniform thickness is a common practice in (metal) additive manufacturing of (axis-aligned) lattices [17], [18]. Such structures can also be fabricated by a direct extrusion in 3D [40], or by robotic fabrication [41], [42]. We note that variable thickness is not impossible with 3D printing. Second, our method simultaneously optimizes the lattice distribution and the shape, achieved using multiple design variables. Last but not least, while existing works exploited projection methods for generating high-resolution pixel or voxel models, we develop a novel approach based on field-aligned meshing to compile the lattice structure. The optimized structure is compactly represented by a graph. This direction shares a similar goal with the recent work by Arora *et al.* [43]. In contrast to the design approach [43], our method unlocks a large solution space by optimizing the porosity, anisotropy, and orientation of lattices.

2.3 Field-aligned Parameterization

We develop a lattice compilation method based on field-aligned parameterization which has been researched intensively in the past decade, especially for generating quadrilateral (quad-) mesh. We review briefly the more recent works on hexahedral (hex-) meshing, and for quad-meshing we refer an interested reader to the survey by Bommers *et al.* [44].

For a given 3D closed shape, field-aligned hex/hex-dominant meshing techniques typically consist of three steps [45], [46], [47], [48], [49], [50]. It starts by estimating the gradients of a volumetric parameterization using a directional field [51], [52], where the field is discretized per vertex or per tetrahedron and smoothly interpolated within the volume under a boundary alignment constraint. This is followed by computing a parameterization aligned with the estimated gradients by fitting. Finally it extracts the hex-mesh in the parametric space [53]. Robust hex-meshing remains a challenging problem. A promising direction is to topologically correct the directional fields [47], [48], [54], [55]. Lei *et al.* introduced a hex-mesh generation method based on surface foliation theory [56]. This approach, however, requires heavy topological pre-processing of the input.

The field-aligned parameterization pipeline is primarily used for generating semi-regular meshes. To ensure the validity of the mesh, complex geometric and topological computations are involved. In this paper we make use of

field-aligned parameterization to generate lattice structures. This new application differs from mesh generation, as lattices are encoded by graphs rather than meshes. This goal sidesteps the numerical stability issue and geometrical and topological complexities typically occurred during mesh extraction from the parameterization.

To efficiently and robustly extract consistent lattice structures, we extend the robust meshing approach that was proposed by Jacob *et al.* [57] and further developed by Gao *et al.* [50]. The per vertex local parameterization from [50] fits our purpose well since the local parameterization aligns exactly with the direction field by construction and permits fast and scalable computations. The extension proposed in this paper allows to incorporate anisotropy and heterogeneity.

The recent work by Arora *et al.* [43] shares a similar goal as ours, *i.e.* to extract field aligned struts from stress directions. Our approach takes the optimized stress fields as input, without a field smoothing operation that compromises the accuracy of input fields. During lattice compilation, while they extract the struts by tracing stress lines and simplifying the duplicated ones, our approach directly generates struts by simple and efficient graph operations. This makes our approach fast and scalable, taking a couple of minutes for an input with tens of millions of tetrahedral elements (see Table 2).

3 OVERVIEW

Given a design domain and application-specific loads, our method generates a lattice structure that maximizes structural stiffness. The struts in the optimized lattice structure conform with principal stress directions. Moreover, the struts on the boundary span a smooth surface faithfully approximating the optimized shape.

As illustrated in Figure 2 for 2D and Figure 1 for 3D, our approach consists of two steps. The first optimizes the shape (including its topology) and the distribution of lattice material within the shape. The input includes a design domain and boundary conditions (Figure 1 left and Figure 2a), as well as design specifications such as the target fraction of solid material. The design domain in 3D is represented by a closed triangle surface mesh. This mesh is voxelized, generating finite elements for simulation and optimization. The output is a set of fields, indicating, per element, the occupancy of lattice material, and the orientation and anisotropy of lattice material (Figure 1 second left and Figure 2b). A surface mesh is then reconstructed using Marching Cubes, representing the optimized shape, *i.e.* the interface between elements that are filled with lattice material and that are empty. The shape enclosed by this reconstructed surface mesh (or, the input surface mesh which defines the design domain, in case that the entire design domain is to be filled with lattices) is tetrahedralized. The optimized fields are then interpolated on the vertices of the tetrahedral model. The vertices, including their connectivity and their associated field values, serve as input for the second step, which compiles a globally connected conforming lattice structure composed of struts (Figure 1 second right and Figure 2c). The output lattice structure is encoded by a graph.

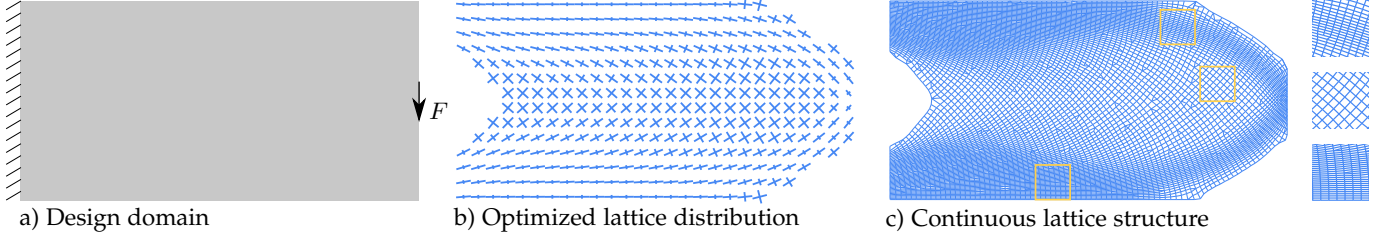


Fig. 2: A 2D example, illustrating the pipeline of our approach. Given the design specification (a), the first step optimizes the distribution of lattice materials (b). The second step extracts a continuous lattice structure corresponding to the optimized lattice configuration (c).

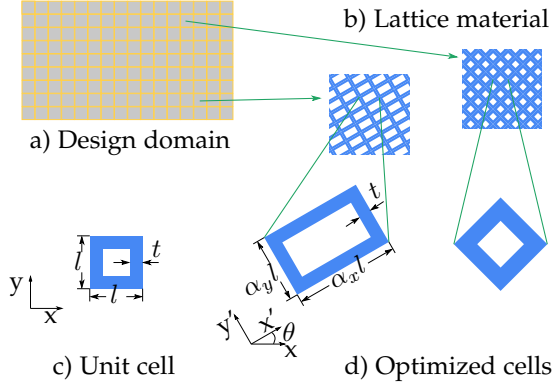


Fig. 3: The design domain (a) is discretized into bilinear quadrilateral elements. Each element is filled with lattice material (b), i.e. a block of periodic cells (d). The cells are adapted by scaling and rotating a unit cell (c).

4 LATTICE OPTIMIZATION

264

The goal of our optimization is to find the optimal distribution of lattice material that maximizes structural stiffness, subject to a number of design constraints. To this end, the design space is discretized into a regular grid of bilinear square elements in 2D or trilinear cubic elements in 3D. As illustrated in Figure 3 for a 2D rectangular design domain, each element is to be filled by repeating a unique, rectangular-shaped cell. The cells are adapted from a unit cell by scaling and rotation. The scaling factors and rotation matrices are to be optimized. The scaling factors for the cell in element e are denoted by α_e , and in 2D by $(\alpha_{e,x}, \alpha_{e,y})$ and in 3D by $(\alpha_{e,x}, \alpha_{e,y}, \alpha_{e,z})$. The rotation matrix for element e is denoted by R_e .

The unit cell in 2D is a hollowed square with a side length of l and a thickness of t , which are specified by the user. In 3D, the cubic unit cell consists of the union of all 12 edges with square cross section of thickness t . During scaling the side length of cells (l) is elongated, while the thickness (t) is kept constant. We note that this treatment differs from the standard scaling where the thickness is also scaled. This creates cells with gradation in the fraction of solid material (v_e),

$$v_e(\alpha_e) = 1 - \frac{(\alpha_{e,x}l - 2t)(\alpha_{e,y}l - 2t)}{\alpha_{e,x}\alpha_{e,y}l^2}. \quad (1)$$

This gradation allows the optimization to place adapted cells with a smaller fraction of solid material in regions where the stress is relatively small. Furthermore, per axis elongation allows to increase the mechanical anisotropy of

cells. This is beneficial since the stress tensors are typically anisotropic.

Besides a scaling factor per axis and a rotation matrix, each element is assigned a variable φ_e , to indicate whether the element is empty ($\varphi_e = 0$) or filled ($\varphi_e = 1$) with lattice material. The set of elements that are filled with lattice material defines the overall shape of the optimized structure. To allow for gradient-based numerical optimization, the variable φ_e is relaxed to take intermediate values, i.e. $\varphi_e \in [0, 1]$. This variable is akin to the density variable in classical density-based topology optimization, which in that context indicates the fraction of *solid* material. In the context of lattice optimization, this variable shall be interpreted as the fraction of *lattice* material. The fraction of solid material per element (ρ_e) depends on φ_e and the fraction of solid material within an adapted cell (v_e), i.e.

$$\rho_e(\varphi_e, \alpha_e) = \varphi_e v_e(\alpha_e). \quad (2)$$

As the design space is parameterized by the fraction of lattice material (φ), scaling factor (α), and orientation matrix (R), the optimization problem is given as

$$\min_{\varphi, \alpha, R} J = \mathbf{F}^T \mathbf{U}(\varphi, \alpha, R) \quad (3a)$$

$$s.t. : \sum_e \rho_e(\varphi_e, \alpha_e) \leq \bar{v}N \quad (3b)$$

$$\varphi_e \in [0.0, 1.0], \forall e \quad (3c)$$

$$\alpha_{e,k} \in [\underline{\alpha}_k, \bar{\alpha}_k], k \in \{x, y, z\}, \forall e. \quad (3d)$$

Here the objective is to minimize the work done by the external force, which is equivalent to minimize compliance (i.e. stiffness maximization). \mathbf{F} denotes the force vector that is applied to the design domain. The force vector is constant. \mathbf{U} denotes the displacement vector of the shape when it comes to its static equilibrium under the external force \mathbf{F} . The first constraint, Eq. 3b, restricts the amount of solid material, where \bar{v} is the fraction of available solid material, and N is the number of elements in the design domain. The second constraint, Eq. 3c, sets bounds for the fraction of lattice material (φ). The third constraint, Eq. 3d, sets bounds for the scaling factors ($\alpha_x, \alpha_y, \alpha_z$). The lower and upper bounds of the scaling factors are user-defined.

The novelty of this formulation is two-fold. First, by optimizing the scaling factors rather than the thickness of hollowed cells, it ensures that all struts in the optimized structure have the same thickness. As discussed in Section 2.2, this eases the control of the 3D printing process. Second, we assign an additional variable φ to indicate the occupation of lattice material. This makes the formulation more general. Prescribing $\varphi = 1$ leads to optimized lattices

328 that fill the entire design domain. This is useful as infill
329 for prescribed shapes. Allowing φ to be decided by the
330 optimization enables both the shape and the lattice to evolve
331 simultaneously, achieving a higher stiffness.

332 4.1 Stiffness Matrix for Lattices

333 The objective function, Eq. 3a, involves the displacement
334 vector (\mathbf{U}), which is related to the external force (\mathbf{F}). The
335 unknown \mathbf{U} is computed by solving the equilibrium equa-
336 tion with finite element analysis,

$$\mathbf{K}(\varphi, \alpha, \mathbf{R})\mathbf{U} = \mathbf{F}. \quad (4)$$

337 Here the stiffness matrix, \mathbf{K} , is assembled from per element
338 stiffness matrix, $\mathbf{K}_e(\varphi_e, \alpha_e, R_e)$.

339 In standard finite element analysis of solids [58], the
340 element stiffness matrix \mathbf{K}_e is computed by integrating over
341 the domain of an element, Ω_e ,

$$\mathbf{K}_e = \int_{\Omega_e} B^T D_e B dx, \quad (5)$$

342 where B is the element strain-displacement matrix for linear
343 basis functions [58]. D_e represents the fourth order elasticity
344 tensor, computed based on the Young's modulus and Pois-
345 son's ration of the solid material.

346 For analyzing elements that are filled with lattice ma-
347 terial, the elasticity tensor D_e is not constant but rather
348 depends on design variables α_e and R_e . Let us first consider
349 an element that is filled lattice with $\varphi_e = 1$. The stiffness
350 matrix for lattices is calculated by

$$\mathbf{K}_e(1, \alpha_e, R_e) = \int_{\Omega_e} B^T D_e(\alpha_e, R_e) B dx, \quad (6)$$

351 The elasticity tensor of a rotated lattice cell, $D_e(\alpha_e, R_e)$, is
352 computed by rotating the elasticity tensor of this cell in its
353 local coordinate system, $D_e(\alpha_e)$. In engineering notation,
354 D_e is represented as a 3×3 matrix for 2D problems or a
355 6×6 matrix for 3D. The rotation of tensor is realized by

$$D_e(\alpha_e, R_e) = \bar{R}_e(R_e) D_e(\alpha_e) \bar{R}_e^T(R_e), \quad (7)$$

356 where the tensor rotation matrix \bar{R} is given in the Appendix.

357 The effective elasticity tensor of an elongated cell,
358 $D_e(\alpha_e)$, is evaluated by numerical homogenization. We
359 make use of the Matlab code provided in [59] and [60]
360 for homogenization in 2D and 3D, respectively. Given the
361 scaling factors, the domain of the elongated unit cell is
362 discretized by square/cube finite elements with bilinear/tri-
363 linear basis functions. To avoid performing homogeniza-
364 tion for every α_e during the optimization process, we pre-
365 compute D_e for regularly sampled α values. In 2D, we fit a
366 surface for every non-zero entry in D over the 2D domain
367 of $[\underline{\alpha}_x, \bar{\alpha}_x] \times [\underline{\alpha}_y, \bar{\alpha}_y]$. In 3D we use trilinear interpolation.
368 The derivative of each non-zero entry in D with respect to
369 α is evaluated using the interpolation.

370 For elements with φ_e between 0 and 1, we use the power
371 law from density-based approaches [34] to interpolate,

$$\mathbf{K}_e(\varphi_e, \alpha_e, R_e) = \varphi_e^p \mathbf{K}_e(1, \alpha_e, R_e), \quad (8)$$

372 where the parameter p (typically $p = 3$) is introduced to
373 penalize intermediate values in φ_e , and consequently the
374 optimization steers φ_e towards either 0 or 1.

4.2 Solving

375 The optimization problem (Eq. 3) is solved in an iterative
376 process, as in [6]. In each iteration the following compu-
377 tational steps are performed, until the maximum change
378 in design variables is smaller than a threshold (or the
379 maximum number of iterations is reached). 380

381 First, the equilibrium equation (Eq. 4) is solved, obtain-
382 ing the displacement vector, \mathbf{U} . From the element displace-
383 ment vector (\mathbf{U}_e), strain ($\bar{\epsilon}_e$) and stress ($\bar{\sigma}_e$) per element,
384 in engineering notation, are calculated by $\bar{\epsilon}_e = B\mathbf{U}_e$ and
385 $\bar{\sigma}_e = D_e(\alpha_e, R_e)\epsilon_e$, respectively. 385

386 Second, design variables φ and α are updated using
387 a gradient-based solver. We make use of the method of
388 moving asymptotes (MMA) [61]. To avoid checkerboard
389 patterns, the design variables are regularized into $\tilde{\varphi}$ and
390 $\tilde{\alpha}$ using the so-called density filter. $\tilde{\varphi}$ is then projected into
391 $\bar{\varphi}$ by a (smoothed) Heaviside operation, to approach a 0-1
392 solution. The filter and Heaviside operator are widely used
393 in density-based approaches, e.g. in [62], [12]. 393

394 Third, the orientation of each element (R_e) is updated
395 based on the associated stress tensor (σ_e). The stress tensor
396 is symmetric positive-definite. By eigendecomposition we
397 obtain three mutually orthogonal principal stress direc-
398 tions (v_1, v_2, v_3). The eigenvectors are ordered by respective
399 eigenvalues in ascending order, i.e. $\gamma_1 \leq \gamma_2 \leq \gamma_3$. As shown
400 by Pedersen [3], the optimal orientation of an orthotropic
401 material coincides with the principal stress directions, hence
402 the element is rotated by $R_e = [v_1^T; v_2^T; v_3^T]$. 402

403 Fourth, the stiffness matrices of lattices, K_e , are re-
404 calculated based on the updated orientation (R_e) and reg-
405 ularized variables $\bar{\varphi}$ and $\bar{\alpha}$, according to Section 4.1. 405

4.3 Example

406 The output of our optimization is a set of fields defined
407 on the design domain. Figure 4 visualizes these fields for a
408 rectangular domain, which is discretized by a grid of 80×40
409 elements. The unit cell has $l = 10t$. The maximum fraction
410 of solid material is 0.15. Figure 4a shows the optimized
411 lattice fraction field. The field contains values very close 0
412 or 1 (cf. the colorbar). Even with a fraction of solid material
413 as small as 0.15, the lattice covers a large portion of the
414 design space. This is due to the fact that the unit cell has
415 a small fraction of solid material (i.e. 36%, with $l = 10t$).
416 Figure 4b visualizes the orientation of optimized cells. Here
417 the rotated frame is elongated according to the respective
418 scaling factor per axis. For clarity the frames are shown
419 for regularly-spaced samples. On the right, the frame field
420 is visualized for elements which have a fraction of lattice
421 material (φ_e) that is larger than a threshold (0.5). We note
422 that since the optimized field φ contains values close to 0
423 or 1, Figure 4c is independent (almost) of the value of the
424 threshold. 425

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426 Up to this step, we have equipped with a design domain
427 with a set of fields including fraction of lattice infill, orien-
428 tation, and scaling, that are optimized for the prescribed
429 external loads (cf Figure 4). Since a region with a low
430 fraction implies that little material is required, we extract
431

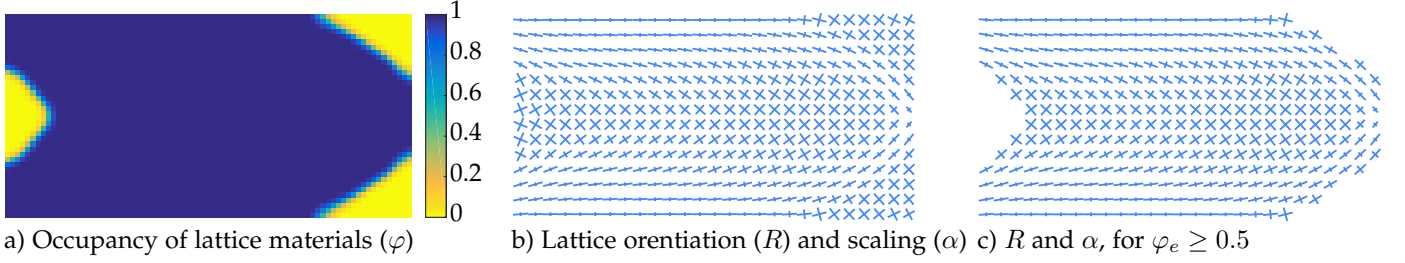


Fig. 4: Visualization of the optimized fields. In (b) and (c) the frame is elongated according to the respective scaling per axis (α), and then rotated according to the orientation (R).

a sub-area (volume) from the design space by thresholding (≥ 0.5) out low infill regions. With the actual shape being extracted, we now focus on generating a lattice structure that conforms to both the boundary of the shape and the directional and scaling fields.

Our problem setting differs from the typical meshing problem in that both of our input and output are quite relaxed from the conditions of being a mesh. For the input, we put no constraints on its geometrical quality (i.e. angles, edge ratios, etc) nor its topological correctness (i.e. permitting non-manifold features, holes, and intersections). This maximizes the scope of the problem but poses a great challenge to the design of a robust solution. For the lattice output, it does not require face (solid) elements, making complex topological operations in most of the meshing methods unnecessary for our purpose. Moreover, considering that it is not a hard requirement for our lattice structure to be all-hex cells for the designed structure to function, we choose the parameterization optimization in [50] that can be easily adapted to handle graphs and propose a simple extraction strategy to generate a lattice structure. The produced lattice structure contains mostly quad (hex)-like connections while allowing certain irregularity to adapt for rapid changes in the directions and scales.

Our method takes a graph with vertices of the optimized shape as the input: $G = (V, E)$, where every vertex $\mathbf{v} \in V$ has a position $\mathbf{x} \in \mathbb{R}^k$ (k is 2 for 2D and 3 for 3D), an orientation matrix $\mathbf{R} \in \mathbb{R}^{k \times k}$ encoding the cross directions and also denoting a local coordinate system, and a scaling vector $\alpha \in \mathbb{R}^k$ composed of scales for the k axes of the local coordinate system. Our goal is to extract a lattice structure, which is another graph $G' = (V', E')$ that (1) reproduces the input direction and anisotropy as much as possible, and (2) has a resolution that can be flexibly controlled by a target edge length h .

In the following, we first describe the parameterization optimization that incorporates anisotropic orientations, and then present the lattice structure extraction.

5.1 Parameterization

Given an orientation field \mathbf{O} that includes the cross directions for all the vertices, we want to compute a parameterization \mathbf{P} with the gradient aligned to \mathbf{O} . Methods that compute a global parameterization with the gradient aligning to the orientation field in a least-square sense (e.g. [63], [64], [43] and [5], [7], [6], [39]), rely on non-linear optimization solvers which are not scalable to large datasets. We instead compute a parameterization for the input graph by representing it with a set of local parameterizations and

minimizing an energy between the local parameterizations of adjacent vertices [57], [50]. The local nature of the parameterization makes it easily parallelizable and scalable to large inputs.

As illustrated in Figure 5a, the local parameterization of a vertex in 2D plane (or 3D volume) can be uniquely determined by its origin \mathbf{p} , the orientation matrix \mathbf{R} , and unit lengths $h \cdot \alpha$, where h is the user-defined global target edge length. The unit lengths are fixed through the entire process. Unlike the previous approaches [50], [57] that treat directions as a 4 rotational symmetric field in 2D or a 24 rotational symmetric field in 3D, since the unit length varies for different axis, our coordinate system is mutable only by flipping the signs of each axis. The origin with a random initialization, is the variable we need to optimize.

Given the above setting, the optimization energy of the parameterization \mathbf{P} is defined as the summation of all the squared differences of local parameterizations for each edge:

$$E(\mathbf{P}) = \sum_{i \in V} \sum_{j \in N(i)} \|\mathbf{p}_i - (\mathbf{M}_{ij} \mathbf{t}_{ij} + \mathbf{p}_j)\|^2, \quad (9)$$

where $N(i)$ is a set of all the vertices sharing an edge with vertex i , \mathbf{M}_{ij} is an interpolation of \mathbf{M}_i and \mathbf{M}_j where $\mathbf{M} = \mathbf{R}\mathbf{S}$ and \mathbf{S} is the scaling matrix converted from $h \cdot \alpha$, and $\mathbf{t}_{ij} \in \mathbb{Z}^k$ encodes the integer translations of \mathbf{p}_j . $\mathbf{M}_{ij} \mathbf{t}_{ij} + \mathbf{p}_j$ translates \mathbf{p}_j by integer moves to the nearest position to \mathbf{p}_i , effectively avoiding the integer jumps between the two local parameterization and only the difference of their fractional parts is measured. The computation of \mathbf{M}_{ij} requires interpolating the directions and scales separately,

$$\mathbf{M}_{ij} = \text{ortho}(\mathbf{R}_i + \mathbf{R}_j r(\mathbf{R}_i, \mathbf{R}_j)) \cdot \frac{(\mathbf{S}_i + \mathbf{S}_j)}{2}, \quad (10)$$

where ortho denotes the orthogonalization of a matrix, and $r(a, b)$ is the closest matching that gives the smallest difference between two coordinate systems which can be computed efficiently by enumerating all the cases. Note that, there is no scaling involved when computing the matching between orientations, which is similar to the one in [50]. The only difference is that in [50], there are 8 cases to compare for 2D and 48 cases for 3D, while we need to consider 4 and 8 cases in 2D and 3D respectively.

The integer translation between two connecting vertices in the parameterization space, \mathbf{t}_{ij} , is computed by a rounding operation,

$$\mathbf{t}_{ij} = \text{round}[\mathbf{M}_{ij}^{-1}(\mathbf{p}_i - \mathbf{p}_j)]. \quad (11)$$

By doing so, the energy between the two vertices will be minimized.

521 We minimize $E(\mathbf{P})$ in a Gauss-Seidel style by iteratively
 522 visiting every vertex and smoothing the origin of each
 523 vertex by computing an average of all its neighbors. The
 524 pseudo code of the optimization is provided in Algorithm 1.

Algorithm 1 Optimize-Parameterizations (\mathbf{P})

```

1: for  $i = 1, \dots, n$  do
2:    $\mathbf{p}'_i \leftarrow \mathbf{p}_i, d \leftarrow 0$ 
3:   for all  $j \in \mathcal{N}(i)$  do
4:      $\mathbf{p}'_i \leftarrow d\mathbf{p}'_i + \mathbf{p}_j + \mathbf{M}_{ij}\mathbf{t}_{ij}$ 
5:      $d \leftarrow d + 1$ 
6:    $\mathbf{p}'_i \leftarrow \mathbf{p}'_i/d$ 
7:   end for
8:    $\mathbf{p}_i \leftarrow \mathbf{p}'_i + \mathbf{M}_i \text{round}[\mathbf{M}_i^{-1}(\mathbf{x}_i - \mathbf{p}'_i)]$ 
9: end for
  
```

525

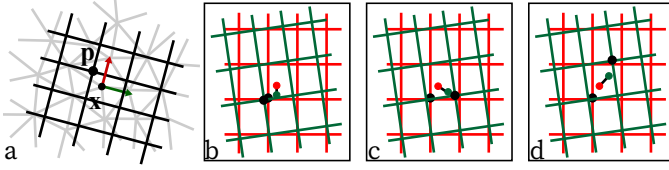


Fig. 5: (a) The local parameterization of a vertex in 2D plane. (b-d) Two local parameterizations of two close vertices in the input graph where they are not necessarily close in the parameterization space.

526 The last step in line 8 rounds each origin of a local
 527 parameterization \mathbf{p}_i to the integer position closest to the
 528 vertex position \mathbf{x}_i . Consequently, each component of \mathbf{t}_{ij}
 529 becomes $-1, 0$, or 1 . For example, as illustrated in Fig.
 530 5, $\mathbf{t}_{ij} = (0, 0)$ for Fig. 5b, $\mathbf{t}_{ij} = (\pm 1, 0)$ or $(0, \pm 1)$ for
 531 Fig. 5c, and $\mathbf{t}_{ij} = (\pm 1, \pm 1)$ for Fig. 5d. We note that this
 532 approach requires an input graph where the length of edges
 533 is much smaller than the desired length of the lattice struts,
 534 otherwise the integer translations after rounding could be
 535 larger than $|\pm 1|$.

536 After the rounding step at line 8 of Algorithm 1, if
 537 vertex i is on the input boundary, then \mathbf{p}_i is projected
 538 onto the tangent plane of i . This projection step ensures the
 539 conformity of the finally extracted lattice structure.

540 To speed up the optimization, similar to [50], we con-
 541 struct a hierarchical structure of the input graph by halving
 542 the number of vertices for each level and perform the
 543 optimization on each level of the hierarchy by 50 iterations
 544 for 2D and 200 iterations for 3D.

545 5.2 Graph Extraction

546 In the input graph G , each vertex \mathbf{v} has a smoothed local
 547 parameterization. The origin \mathbf{p} of \mathbf{v} provides a guidance
 548 for the vertex position in the output graph $G' = (V', E')$.
 549 Besides, the integer translation associated with each edge
 550 $(\mathbf{v}_i, \mathbf{v}_j)$ of G , $\mathbf{t}_{ij} \in \mathbb{Z}^k (k = 2, 3)$, categorizes this edge as a
 551 specific element in G' , depending on the L_0 norm of \mathbf{t}_{ij}
 552 which is the number of ± 1 s in \mathbf{t}_{ij} . In 3D ($k = 3$), this
 553 number can be

- 554 • 0 (i.e. $\mathbf{t}_{ij} = (0, 0, 0)$), indicating that the two vertices
 555 are very close in the parameterization space, and thus
 556 will be collapsed into a point in G' ,

- 557 • 1 ($\mathbf{t}_{ij} = (\pm 1, 0, 0), (0, \pm 1, 0),$ or $(0, 0, \pm 1)$), meaning
 558 that the edge is parallel to one of the stress directions,
 559 and thus will be kept in G' ,
- 560 • 2 ($\mathbf{t}_{ij} = (\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1),$ or $(0, \pm 1, \pm 1)$) or 3
 561 ($\mathbf{t}_{ij} = (\pm 1, \pm 1, \pm 1)$), respectively corresponding to a
 562 rectangular or cuboid diagonal, which deviates from
 563 the stress directions and thus shall not appear in G' .

564 For example, black edges and green dashed edges in Fig-
 565 ure 6 left correspond to $\|\mathbf{t}_{ij}\|_0 = 1$ and 2, respectively.

566 By utilizing the positional guidance of \mathbf{p} and the indica-
 567 tion of \mathbf{t}_{ij} , the graph extraction is straightforward: collapse
 568 the edges with $\|\mathbf{t}_{ij}\|_0 = 0$ (dots in Figure 6 represent
 569 the averaged positions of collapsed edges), keep the edges
 570 with $\|\mathbf{t}_{ij}\|_0 = 1$, and remove the diagonal edges (i.e.
 571 $\|\mathbf{t}_{ij}\|_0 = 2$ or 3).

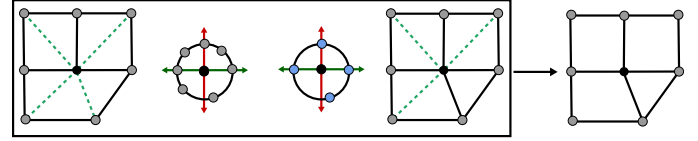


Fig. 6: Left: for a vertex in the graph, the nearest diagonal edges from its rotational directions will be relabelled to be maintained in the final graph if there is no edge representing the corresponding directions. Right: after the relabeling, our final graph is generated by discarding all the diagonal edges.

572 While the above procedure generates a graph with
 573 mostly right angles, we notice T-junctions in the final graph
 574 with near flat angles that are suboptimal for the stiffness of
 575 the lattice structure. Figure 6 left illustrates a vertex with
 576 T-junction in 2D. This can be attributed to the fact that the
 577 removal of the diagonal edges is aggressive. The T-junctions
 578 appear near singularities of the parameterization (similar to
 579 the positional singularities in [50]) which result in elements
 580 with non-right angles, for example, triangles and pentagons
 581 in 2D, and prisms and general polyhedra in 3D.

582 To address this issue, we propose to keep some diagonal
 583 edges in the final graph. Specifically, right after collapsing
 584 edges with $\|\mathbf{t}_{ij}\|_0 = 0$, we check the configuration of every
 585 vertex in the graph and identify critical diagonals. As illus-
 586 trated in Figure 6, for a vertex in black, the process is done
 587 by first normalizing all of its adjacent edge vectors onto
 588 a unit circle (sphere in 3D), then computing their nearest
 589 directions over 4 rotational-symmetric ones in 2D (6 in 3D),
 590 e.g. red and dark green arrows, and finally relabelling a
 591 diagonal edge to be an edge with $\|\mathbf{t}_{ij}\|_0 = 1$ such that each
 592 of the 4 (6 in 3D) stress directions is represented by an edge
 593 that is close to the direction (Figure 6 right).

594 In summary, the process to extract the graph G' , i.e. a
 595 lattice structure, consists of the following steps.

- 596 1) Categorize the edges in G based on $\|\mathbf{t}_{ij}\|_0$.
- 597 2) Group vertices in G according to $\|\mathbf{t}_{ij}\|_0$ such that groups
 598 are connected by edges with $\|\mathbf{t}_{ij}\|_0 \neq 0$. Note that a
 599 group might contain only a single vertex.
- 600 3) Generate the initial G' . For each group, a new vertex is
 601 positioned at the average of the origins of vertices in G .
 602 This vertex inherits the edges to new vertices that are
 603 converted from neighbouring groups.

- 604 4) Categorize the edges in G' based on $\|t_{ij}\|_0$.
 605 5) Identify and relabel critical diagonals in G' to avoid T-
 606 junctions, and remove other diagonal edges.

607 6 RESULTS

608 6.1 Examples

609 Our method works for both 2D and 3D. Figure 7 shows three
 610 optimized 2D lattice structures. In (a), the lattice distributes
 611 across the prescribed curved shape, with spatial variations
 612 in orientation, porosity, and anisotropy. In (b) and (c), the
 613 optimized lattices cover a subset of a rectangular design do-
 614 main, with variations in orientation in (c) and additionally
 615 variations in porosity and anisotropy in (b). The unit cell in
 616 2D is specified with $l = 10t$, $\underline{\alpha} = 1$, and $\bar{\alpha} = 4$.

617 Figure 8 shows 3D lattice structures optimized by our
 618 method. Our method is also applicable to design lattices
 619 that spread over a prescribed 3D curved shape, as shown in
 620 Figure 9. 3D examples are optimized with a unit cell using
 621 $l = 4t$, $\underline{\alpha} = 1$, and $\bar{\alpha} = 2$. Figure 10 shows fabricated
 622 bridge and cantilevers. The models are 3D printed with a
 623 Formlab Form 2 printer which uses stereolithography (SLA).
 624 The dimension of models and the thickness of struts are
 625 scaled to comply with the volume and feature size of the
 626 printer. The printed femur (Figure 11) has a dimension of
 627 $112.4 \times 77.9 \times 133.1 \text{ mm}^3$, with a thickness of 0.5 mm . The
 628 chair (Figure 1) is $110.8 \times 76.6 \times 142.1 \text{ mm}^3$, with a thickness
 629 of 0.6 mm .

630 6.2 Evaluation

631 6.2.1 Design options

632 We evaluate the influence of various design options on
 633 the resulting lattice structures using the 2D cantilever (
 634 Section 4.3, Figure 4), with the fraction of solid material
 635 $\bar{v} = 0.15$, and bounds for the scaling factors $\underline{\alpha} = 1$ and
 636 $\bar{\alpha} = 4$. The optimized fields and compiled lattice structures
 637 are shown in Figures 12 and 13, respectively.

638 In the first row of Figure 12, the fraction of lattice is
 639 fixed, $\varphi = 1$. Consequently the lattice distributes across the
 640 entire rectangular design domain. In (a) the scaling is also
 641 fixed, while in (b) the optimization of scaling is enabled.
 642 The enlarged solution space leads a decrease in compliance
 643 (i.e. improved stiffness), 418.33 (a) vs. 282.62 (b). In (c),
 644 the scaling factors along individual axes are decoupled,
 645 resulting in a further decrease in compliance to 239.97.

646 In the second row of Figure 12, the fraction of lattice
 647 is optimized. Consequently, a shape evolves from the opti-
 648 mization, corresponding to $\varphi_e \geq 0.5$. Similar to the trend of
 649 compliance in the first row, it decreases from (d), to (e), and
 650 to (f), along with the increased flexibility in optimization.
 651 (f) has the smallest compliance among the six cases. It
 652 decreases from (a) by 44.39%. This study, in agreement with
 653 similar numerical comparisons for various optimization op-
 654 tions [7], confirms the significance of adaptive porosity and
 655 anisotropic features for stiffness maximization [65].

656 As a reference, an axis-aligned uniform lattice struc-
 657 ture covering the entire domain (i.e. corresponding to the
 658 initialization of Figure 12) is evaluated. Its compliance is
 659 852.30, which is more than twice larger than the design in
 660 Figure 12a, and 3.66 times larger than the design in

Figure 12f. This comparison confirms the importance of
 aligning anisotropic microstructures with internal stress di-
 rections for stiffness maximization.

664 6.2.2 Accuracy

665 To evaluate the accuracy of our lattice compilation method,
 666 we perform a comparison of the compliance predicted by
 667 homogenization with the compliance of lattice structures
 668 by a full-resolution finite element analysis. To this end,
 669 the computational domain of the six lattice structures in
 670 Figure 13 is discretized by a finite element resolution of
 671 4096×2048 , and analyzed using a geometric multigrid elas-
 672 ticity solver [66]. The voids among lattices are approximated
 673 by a weak ersatz material with a (relative) Young's modulus
 674 of 10^{-2} . The comparison is summarized in Table 1. The
 675 difference in compliance is between 2.89% and 6.46%. This
 676 demonstrates that our lattice compilation introduces little
 677 error to the predicted performance from homogenization-
 678 based optimization. We note that homogenization theory
 679 assumes infinite periodicity of the cells, while for fabrication
 680 the compiled lattice has a finite physical size. This effect has
 681 been evaluated and reported in e.g. [6], [67]. Furthermore,
 682 we notice that the compiled lattices exhibit a small number
 683 of triangles. This also partially explains the discrepancy
 684 since pure rectangles are assumed in homogenization.

TABLE 1: The difference in compliance predicted by ho-
 mogenization and a full-resolution analysis, for the lattice
 structures shown in Fig. 13.

	a	b	c	d	e	f
Homo.	418.33	282.62	239.97	332.81	277.27	232.64
Full res.	444.78	300.15	255.48	323.18	292.66	241.94
Diff.	6.32%	6.20%	6.46%	2.89%	5.55%	4.00%

685 6.2.3 Computational performance

686 Table 2 presents statistics of 3D model complexity and
 687 computational performance. The experiments were run on a
 688 standard desktop PC equipped with an Intel Xeon E5-1650
 689 v3 processor (12 cores) running at 3.5 GHz, 64 GB RAM, and
 690 an Nvidia GTX1080 graphics card with 8 GB memory. The
 691 optimization and compilation together take less than 1 hour
 692 even for complex models such as the chair and femur.

693 The group of columns 2-8 is related to lattice optimiza-
 694 tion. From the cantilever and bridge examples, it can be
 695 observed that with increasing design flexibility the compli-
 696 ance (J_{com}) decreases. This agrees with the 2D tests in
 697 Figure 12. The increased design flexibility is also reflected
 698 by an increase of time associated with updating stiffness
 699 matrices, which is counted in T_{FEA} . The optimization time
 700 of the gradient-based solver for φ and α , T_{Opt} , increases
 701 accordingly as well.

702 The resolution of optimized fields is refined by a regular
 703 subdivision (1 element \rightarrow 2^3 elements), followed by tri-
 704 linear interpolation of the fields. While our lattice compi-
 705 lation algorithm takes a general graph as the input, in our
 706 implementation, we use triangle meshes and tetrahedral
 707 meshes which are purely for the convenience of computing
 708 vertex normal. This step costs 45~70 seconds (cantilevers,
 709 Figure 8) to 4 minutes and 26 seconds (chair, Figure 1). The
 710 refinement generates a large number of vertices (#vertex)

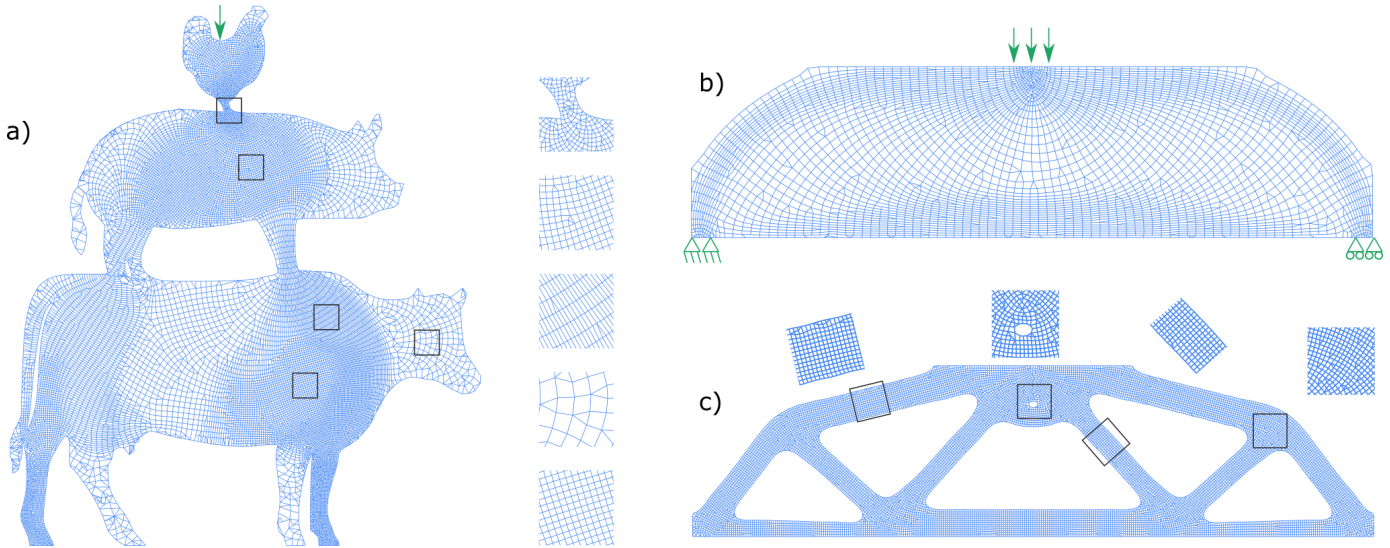


Fig. 7: Optimized 2D lattice structures for a prescribed freeform shape (a) and a rectangular design domain (b and c). The optimized lattice structures possess spatial variations in orientation, porosity, and anisotropy.

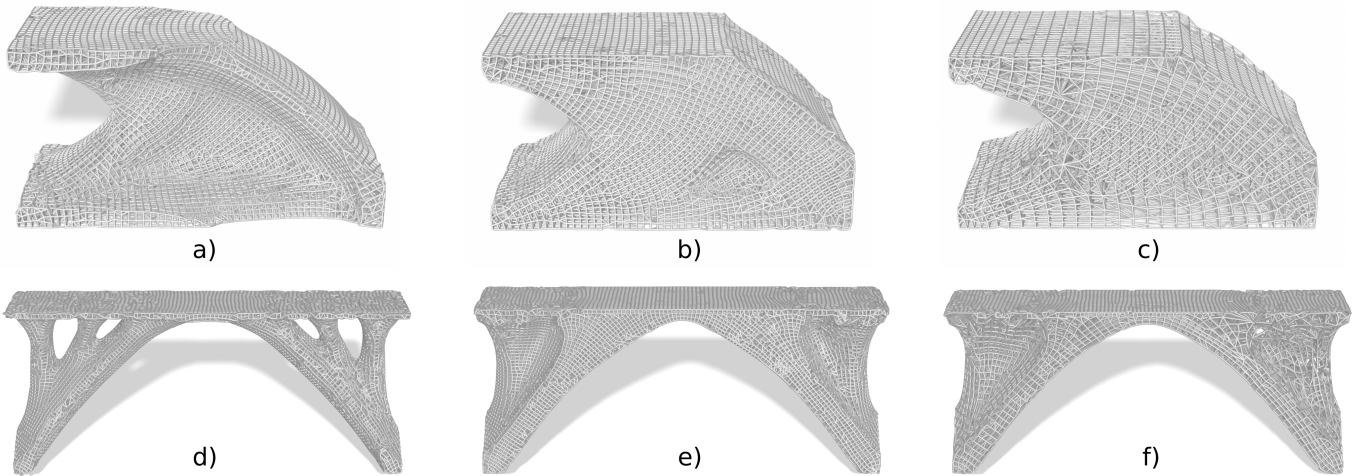


Fig. 8: 3D lattice structures optimized from cuboid design domains, showing spatial variations in orientation, porosity and anisotropy. The design options are: (left) fixed $\alpha = 1$ with design variables R and φ , (middle) design variables R , φ , and α with $\alpha_x = \alpha_y = \alpha_z$, (right) full flexibility. With the increased design flexibility, the compliance reduces from left to right: 110.84 \rightarrow 96.03 \rightarrow 85.85 (cantilever), 230.52 \rightarrow 177.86 \rightarrow 149.96 (bridge).

711 organized as tetrahedral elements (#tet), supplied to the
 712 lattice compilation. The compiled lattice has as many as
 713 305k struts, for the femur model. Timings for pre-processing,
 714 i.e. building data structures (T_{pre}), local parameterization
 715 (T_{posy}), and graph extraction (T_{extr}) are reported.

716 In the last two rows, the optimized fields are refined
 717 twice (1 element \rightarrow 4^3 elements). This creates highly de-
 718 tailed lattice structures as shown in Figure 14.

719 6.3 Comparison and Discussion

720 **Comparison with solid structures from density-based**
 721 **topology optimization [34]** A 2D simply supported beam
 722 is optimized using our method and the classic density-
 723 based approach – Solid Isotropic Material with Penaliza-
 724 tion (SIMP). The lattice and solid structure generated by
 725 our method and SIMP, are shown in Figure 15 a) and b),
 726 respectively. The physical sizes are $294.8 \times 74.2 \times 60 \text{ mm}^3$,
 727 and the struts have an in-plane thickness of 0.8 mm , which

728 is twice the nozzle size. They were fabricated by a Ultimaker
 729 S5 printer using flexible TPU (thermoplastic polyurethane)
 730 material. While the digital models were designed using the
 731 same fraction of solid material, with 3D printing the lattice
 732 structure is heavier (52 grams vs. 46 grams) due to the
 733 delicate tool-path.

734 The load condition of the 3D printed specimen is shown
 735 in Figure 15c. It is supported at the two ends on the bottom,
 736 while a downward force is applied on the top middle. To
 737 avoid out-of-plane buckling of these thin specimens, two
 738 wooden plates (with open square windows for observation)
 739 are placed to clamp the specimen (Figure 15d). The clamp-
 740 ing plates are placed with a gap of 62 mm , slightly larger
 741 than the thickness of the specimen.

742 The force-displacement plots for multiple tests are
 743 shown in Figure 15e. The forces on the solid structure
 744 increase more steeply than on the porous structure, meaning
 745 that the solid structure from SIMP has a higher stiffness.
 746 However, the forces on the solid structure turn down after

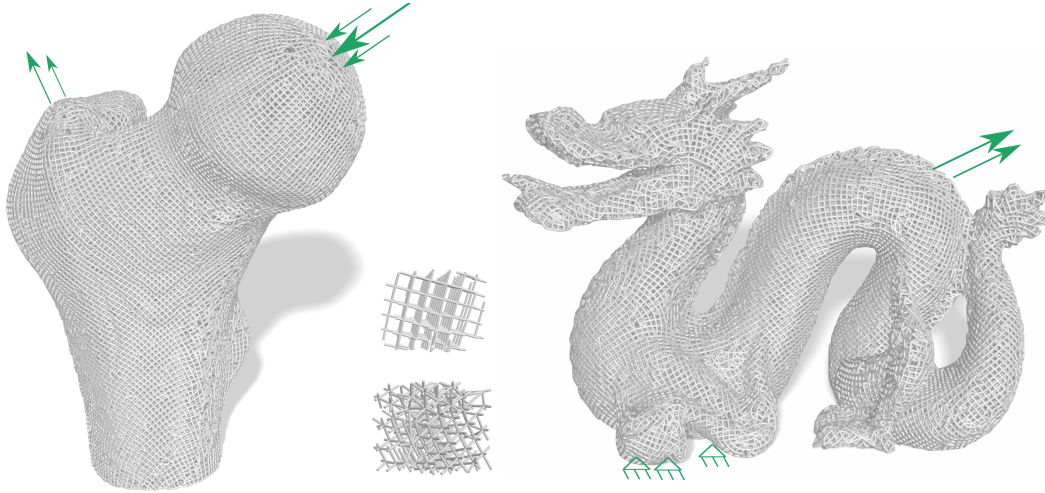


Fig. 9: Optimized 3D lattice structures for prescribed curved shapes. The optimized lattice structures possess spatial variations in orientation. The two samples are taken from inside the femur.

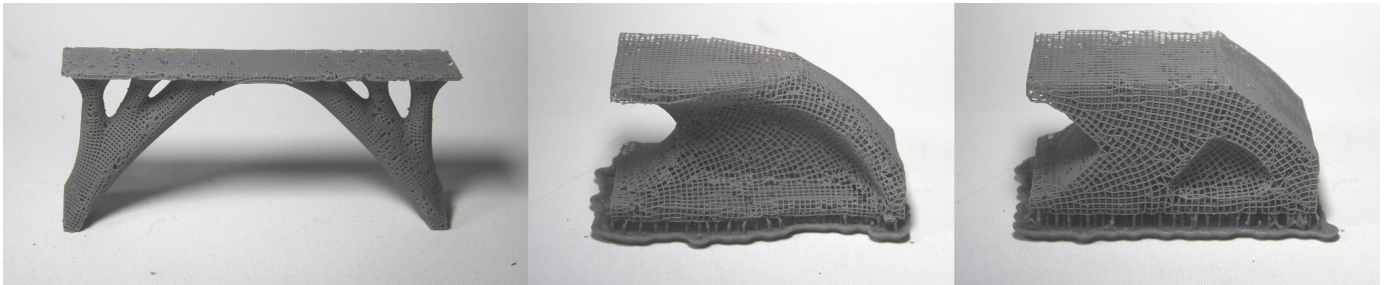


Fig. 10: Optimized lattice structures fabricated by 3D printing.

TABLE 2: Statistics of 3D model complexity and computational performance. The timing is reported in minutes.

Model	Resolution	#Ele.	Vol.	#It.	J_{com}	T_{FEA}	T_{Opt}	#vertex	#tet	#strut	T_{pre}	T_{posy}	T_{extr}	T_{Total}
3D cantilever (Fig. 8a)	100 × 50 × 50	250k	0.2	60	110.84	3.11	0.85	0.89m	5.0m	48k	1.80	5.28	0.33	11.36
3D cantilever (Fig. 8b)	100 × 50 × 50	250k	0.2	60	96.03	3.97	1.62	1.65m	9.62m	87k	1.75	9.98	0.70	18.03
3D cantilever (Fig. 8c)	100 × 50 × 50	250k	0.2	60	85.85	5.83	2.65	1.62m	9.41m	25k	1.83	6.50	0.57	17.38
Bridge (Fig. 8d)	200 × 38 × 88	644k	0.1	60	230.52	15.13	1.88	1.18m	6.54m	63k	1.41	7.09	0.47	25.97
Bridge (Fig. 8e)	200 × 38 × 88	644k	0.1	60	177.86	16.76	3.80	2.02m	11.57m	111k	2.84	13.58	1.14	38.12
Bridge (Fig. 8f)	200 × 38 × 88	644k	0.1	60	149.96	21.10	6.29	1.89m	10.77m	35k	2.40	8.29	0.72	38.79
chair (Fig. 1)	140 × 100 × 200	1.8m	0.1	60	193.5	30.92	5.03	3.32m	18.60m	178k	4.15	18.66	1.87	60.63
femur (Fig. 9)	140 × 93 × 182	696k	0.5	6	163.4	0.99	0	5.86m	14.26m	305k	12.36	35.50	5.94	54.79
dragon (Fig. 9)	200 × 90 × 143	461k	0.5	6	99.4	1.12	0	4.09m	23.31m	200k	5.09	24.84	2.88	33.92
3D cantilever (Fig. 14)	100 × 50 × 50	250k	0.2	60	110.84	3.11	0.85	6.65m	38.50m	351k	7.19	33.44	6.25	50.84
Bridge (Fig. 14)	200 × 38 × 88	644k	0.1	60	230.52	15.13	1.88	8.64m	49.63m	462k	12.50	56.56	14.35	100.42

747 they reach a peak of about 62 N. This is due to the (in-
 748 plane) buckling of the compressed bars. In contrast, the
 749 lattice structure can support a maximum force that is twice
 750 larger before it buckles. This is due to the increased effective
 751 cross-section area of the substructures. This test, in agree-
 752 ment with previous physical tests on 3D printed isotropic
 753 infill [68], confirms the significance of lattice structures for
 754 buckling stability. We note that directly accounting for buck-
 755 ling stability in topology optimization is more expensive
 756 than just compliance minimization, due to the less intu-
 757 itive definition of the buckling mechanism and demanding
 758 eigenvalue problems [69]. Lattice structures, although not
 759 directly optimized for maximal buckling load, have a very
 760 good behaviour against buckling.

Comparison with bone-like porous structures [12] Wu
 et al. proposed a density-based approach to design bone-
 like porous structures using constraints on local material
 volume [12]. Figure 16 compares the porous structure and
 the lattice structure, generated with the same boundary con-
 ditions (see Fig. 2a) and the same fraction of solid material.
 The porous structure was optimized with a local volume
 fraction of 0.36, leading to a total volume fraction of 0.288.
 We then run lattice optimization with this total volume
 fraction, with the design options of rotation and scaling.
 The bone-inspired infill was optimized with a finite element
 resolution of 400×200 , while the lattice was obtained with
 a simulation resolution of 80×40 .

The convergence in compliance is plotted in Figure 16

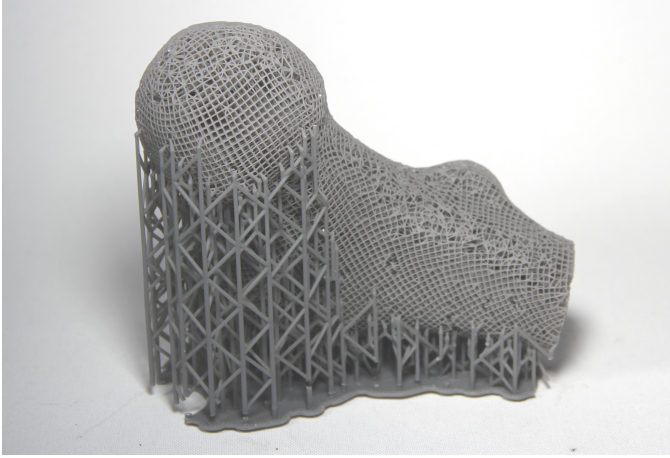


Fig. 11: 3D printed femur with supports.

775 right. The compliance of bone-like infill and conforming
 776 lattice is 184.64 and 177.29, respectively, meaning that the
 777 lattice structure is stiffer. Lattice optimization converges
 778 much faster, and since it runs on a lower resolution, this
 779 leads to a significant speed up. The optimization of lattice
 780 took 1 minute and 7 seconds (60 iterations), while the opti-
 781 mization of bone-like infill cost 40 minutes (1000 iterations).
 782 Both 2D tests are performed in Matlab.

783 **Discussion of Arora et al. [43]** Arora et al. proposed a
 784 method to construct a graph with its edges aligned with
 785 stress directions from simulation of the solid shape [43].
 786 This approach does not explicitly optimize the porosity nor
 787 anisotropy. It relates to the option in our method with fixed
 788 φ and α (cf. Figures 12a and 13a). The result in Figure 12f has
 789 demonstrated that with optimized porosity and anisotropy,
 790 the compliance reduces by 44.39%. We note that under
 791 the option of fixed φ and α , after aligning the lattice, our
 792 method re-calculates stress directions and updates the lattice
 793 orientation. This leads a minor but noticeable decrease in
 794 compliance than aligning the lattice with stress directions
 795 from the solid shape (420.58 \rightarrow 418.90).

796 Our lattice compilation approach is scalable, for exam-
 797 ple, the number of struts is more than two orders of mag-
 798 nitude larger compared with examples shown in [43]. This
 799 allows to design highly detailed lattice structures. Figure 14
 800 shows optimized lattice structures with 462k struts (bridge)
 801 and 351k struts (cantilever).

802 7 CONCLUSIONS

803 In this paper we have presented a novel method to design
 804 conforming lattice structures by homogenization-based
 805 topology optimization and field-aligned parameterization.
 806 It can compute not only an optimized lattice structure that
 807 occupies certain subregions of regular design domains but
 808 also lattices that spread over prescribed (curved) shapes.
 809 The optimized lattice structures conform with principal
 810 stress directions and the boundary of the (optimized) shape.
 811 Our method is scalable and allows to optimize highly de-
 812 tailed lattice structures, which can be fabricated by 3D print-
 813 ing. Numerical analysis on different design options confirms
 814 the importance of aligning anisotropic lattice with internal
 815 stress directions and the importance of lattice gradation in
 816 porosity and anisotropy. The compiled lattice structure, by

a full-resolution finite element analysis, has a compliance 817
 very close to the compliance predicted by homogenization- 818
 based optimization. By physical tests we demonstrate that 819
 the optimized lattice structure can support a buckling load 820
 twice as large as topology optimized solid structures, at the 821
 price of a slight decrease in stiffness. Besides quantified 822
 structural performance, the optimized conforming lattice 823
 structures look remarkably appealing. 824

Future work Our method generates lattice structures par- 825
 ticularly optimized for mechanical properties. It provides 826
 options to steer the optimization by configuring the design 827
 variables, and to adapt the output graph resolution in lattice 828
 compilation. For designs with lattice spreading across the 829
 entire design domain, it is found that the generated lattice, 830
 in certain areas where the stress is small, is less regular, e.g. 831
 around top right and bottom right corner in Fig. 13 (c), and 832
 near the boundary at the back of the pig in Fig. 7. This is 833
 attributed to the fact that principal stress directions, and 834
 consequently the optimal orientation field, do not coincide 835
 with the boundary (see the visualization of the orientation 836
 field in Fig. 12c). A potential solution to this problem is 837
 to incorporate explicit constraints on the orientation field, 838
 i.e. enforcing an alignment of the orientation field with the 839
 prescribed boundary in stress minimal regions. 840

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849 APPENDIX

Denoting a 3×3 rotation matrix by 850

$$R = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \quad (12)$$

the 6×6 rotation matrix for the elasticity tensor in engineer- 851
 ing notation is written as 852

$$\bar{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (13)$$

with 853

$$A = \begin{pmatrix} l_1^2 & m_1^2 & n_1^2 \\ l_2^2 & m_2^2 & n_2^2 \\ l_3^2 & m_3^2 & n_3^2 \end{pmatrix}, \quad (14)$$

$$B = \begin{pmatrix} 2m_1n_1 & 2n_1l_1 & 2l_1m_1 \\ 2m_2n_2 & 2n_2l_2 & 2l_2m_2 \\ 2m_3n_3 & 2n_3l_3 & 2l_3m_3 \end{pmatrix}, \quad (15)$$

$$C = \begin{pmatrix} l_2l_3 & m_2m_3 & n_2n_3 \\ l_3l_1 & m_3m_1 & n_3n_1 \\ l_1l_2 & m_1m_2 & n_1n_2 \end{pmatrix}, \quad (16)$$

and 856

$$D = \begin{pmatrix} m_2n_3 + m_3n_2 & n_2l_3 + n_3l_2 & m_2l_3 + m_3l_2 \\ m_3n_1 + m_1n_3 & n_3l_1 + n_1l_3 & m_3l_1 + m_1l_3 \\ m_1n_2 + m_2n_1 & n_1l_2 + n_2l_1 & m_1l_2 + m_2l_1 \end{pmatrix}. \quad (17)$$

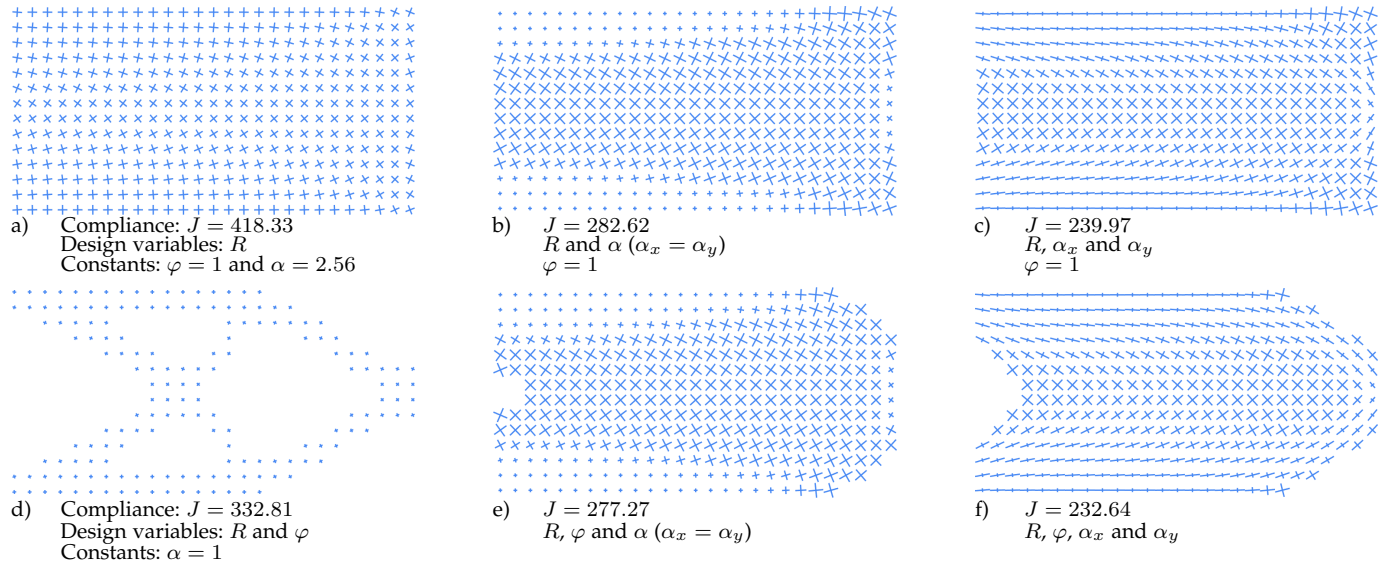


Fig. 12: Visualization of optimized 2D fields corresponding to different design options.

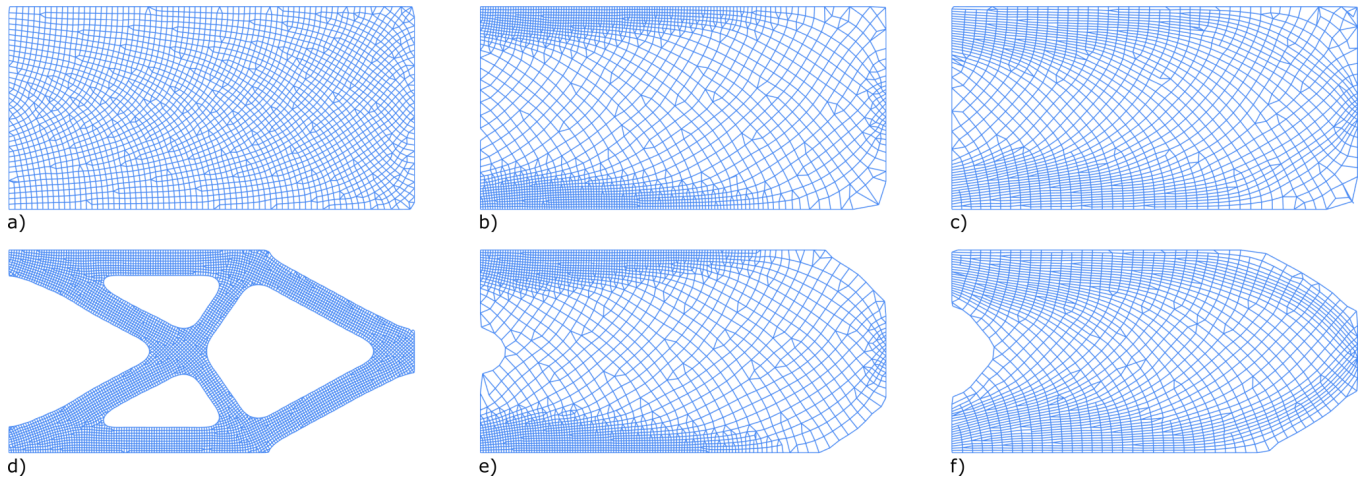


Fig. 13: Compiled lattice structures from the optimized, locally-defined lattice configuration (cf Fig. 12).



Fig. 14: Optimized lattice structures composed of 462k struts (bridge) and 351k struts (cantilever).

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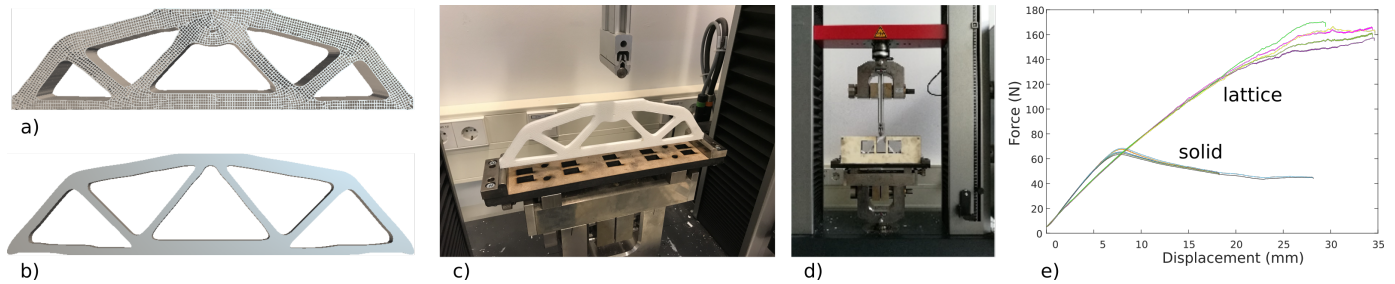


Fig. 15: Physical test comparing an optimized lattice structure (a) with an optimized solid structure (b). (c) and (d) show the experimental setup. From the force-displacement plots for multiple tests (e), it can be observed that, while the lattice structure is slightly less stiff, it supports a compressive force twice larger than the peak force supported by the solid counterpart.

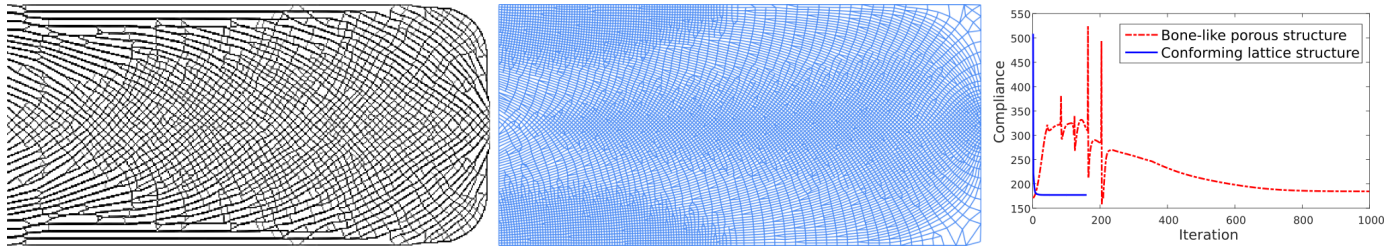
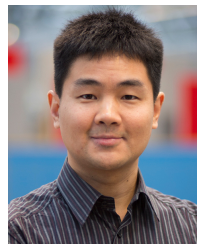


Fig. 16: Left: A bone-like porous structure generated by local volume constraints [12]. Middle: A conforming lattice structure generated by the proposed method. The lattice is stiffer, with a compliance of 177.29, compared to 184.64 of the porous structure. Right: Convergence in compliance for the optimizations.

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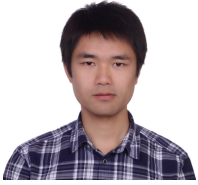
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topology optimization.

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