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Janssens, Bas; Niestijl, Milan

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Generalized Positive Energy Representations of the Group of Compactly Supported Diffeomorphisms

Bas Janssens[1], **Milan Niestijl**[2]

[1] Delft Institute of Applied Mathematics, Technische Universiteit Delft, Mekelweg 4, 2628 CD Delft, The Netherlands.
E-mail: b.janssens@tudelft.nl
[2] Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany.
E-mail: niestijl@math.uni-paderborn.de

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Abstract: Motivated by asymptotic symmetry groups in general relativity, we consider projective unitary representations $\overline{\rho}$ of the Lie group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms of a smooth manifold M that satisfy a so-called generalized positive energy condition. In particular, this captures representations that are in a suitable sense compatible with a KMS state on the von Neumann algebra generated by $\overline{\rho}$. We show that if M is connected and $\dim(M) > 1$, then any such representation is necessarily trivial on the identity component $\mathrm{Diff}_c(M)_0$. As an intermediate step towards this result, we determine the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ of the Lie algebra of compactly supported vector fields. This is subtly different from Gelfand–Fuks cohomology in view of the compact support condition.

1. Introduction

The mathematical results in this paper are motivated by asymptotic symmetry groups in general relativity, as in the seminal work of and Bondi, van der Burg, Metzner and Sachs [1–3]. These groups can typically be described as follows. First, one carefully selects a set of boundary conditions for the gravitational fields, usually in terms of fall-off conditions at null or spacelike infinity [4–6]. This gives rise to a *weak asymptotic symmetry group* $G \subseteq \mathrm{Diff}(M)$, whose action on the gravitational fields preserves the specified boundary conditions. In the context of classical field theory, infinitesimal symmetries yield conserved currents by Noether's theorem [7]. If one selects a normal subgroup $N \subseteq G$ for which the corresponding currents are trivial (the group of *trivial gauge transformations*), then the quotient G/N can be interpreted as an *asymptotic symmetry group*. For asymptotically flat space–times, this is the Bondi–Metzner–Sachs group (or BMS group for short), which is a semidirect product of the Lorentz group with an infinite dimensional abelian Lie group [8,9]. However, in general the precise form of G/N depends quite sensitively on the choice of boundary conditions [10].

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In the present paper, we take a complementary approach. For a given group $G \subseteq \mathrm{Diff}(M)$ of 'weak asymptotic symmetries', we expect a putative quantum theory of gravity on a space–time manifold M to come with a Hilbert space $\mathcal{H}$ of states, and with a projective unitary representation $(\overline{\rho}, \mathcal{H})$ of G. We investigate representations of G that are subject to a (generalized) positive energy condition, whose precise meaning will be given shortly. For M connected with $\dim(M) > 1$, our main result implies that the common kernel N of all such representations contains the identity component $\mathrm{Diff}_c(M)_0$ of the group of compactly supported diffeomorphisms, assuming of course that G contains $\mathrm{Diff}_c(M)_0$ in the first place. We therefore arrive at the conclusion that the connected group $(G/N)_0$ is localized at infinity from representation theory alone, without any reference to classical field theory.

In future work, we intend to isolate groups G that admit non-trivial (generalized) positive energy representations, and subsequently to classify these representations. For now, let us remark that the universal central extension of the BMS group in three dimensions exhibits coadjoint orbits with energy bounded from below [11,12], hinting at the existence of induced unitary representations that are of positive energy. We refer to [13–18] for unitary representations of the BMS group in four dimensions, and to [19] for an approach to positive energy representations of gauge groups in the setting of Yang-Mills theory that is similar to the one in the present paper.

Generalized positive energy representations. Let us describe in more detail the type of representations that we wish to consider in the present paper. Let υ be a complete vector field on M. Then the action of $\mathbb{R}$ on $\mathrm{Diff}_c(M)$ by conjugation with the flow Φ^{υ}_t of υ gives rise to the semidirect product $\mathrm{Diff}_c(M) \rtimes_{\upsilon} \mathbb{R}$, which is a locally convex Lie group by [20]. If G is a group of diffeomorphisms that contains both $\mathrm{Diff}_c(M)$ and $\{\Phi^{\upsilon}_t;\ t \in \mathbb{R}\}$, then every projective unitary representation of G pulls back to a projective unitary representation of $\mathrm{Diff}_c(M) \rtimes_{\upsilon} \mathbb{R}$. In order to show that the kernel of the G-representation contains $\mathrm{Diff}_c(M)_0$, we can therefore restrict attention to the group $\mathrm{Diff}_c(M) \rtimes_{\upsilon} \mathbb{R}$, and we will do so from now on.

We consider projective unitary representations $\overline{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$ of the group $G = \mathrm{Diff}_c(M) \rtimes_{\upsilon} \mathbb{R}$ that are *smooth*, in the sense that they possess a dense set $\mathcal{H}^{\infty} \subseteq \mathcal{H}$ of smooth vectors. Such a representation is said to be of *positive energy* at υ if the strongly continuous one-parameter group $[U_t] := \overline{\rho}(\mathrm{id}_M, t)$ of projective unitary operators has a generator $H := -i \left.\frac{d}{dt}\right|_{t=0} U_t$ (defined up to an additive constant) whose spectrum is bounded from below.

If υ admits an interpretation as a future timelike vector field, then H is the corresponding Hamilton operator, and the positive energy condition is quite natural from a physical perspective. Positive energy representations of possibly infinite-dimensional Lie groups have consequently been the subject of a great deal of research [19,21–29]. For $\mathrm{Diff}(S^1)$, they were considered in e.g. [21,22,30,31].

In order to describe systems at positive temperature, the positive energy condition can be replaced by the *KMS condition*. KMS states (for Kubo–Martin–Schwinger) on operator algebras were introduced by Hugenholtz, Haag and Winnink [32] in order to describe quantum statistical systems at positive temperature. In the context of projective unitary representations, we say that $\overline{\rho}$ is *KMS at υ relative to* $\mathrm{Diff}_c(M)$ if there exists a normal state ϕ on the von Neumann algebra $\mathcal{N} := \rho(\mathrm{Diff}_c(M))''$ that satisfies the KMS condition w.r.t. the automorphism group $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{N}),\ \ t \mapsto \mathrm{Ad}(\overline{\rho}(\mathrm{id}_M, t))$. If for such a state ϕ, the canonical cyclic vector Ω_ϕ in the corresponding GNS Hilbert space $\mathcal{H}_\phi$ defines a smooth ray for the associated projective unitary representation

$\overline{\rho}_\phi : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H}_\phi)$, we say that $\overline{\rho}$ is *smoothly-KMS* (cf. Definition 8 and [33, Lem. 5.8]).

The notion of a KMS state on a von Neumann algebra is closely related to Tomita–Takesaki modular theory [34, Ch. VIII]. It plays an important role in the operator algebraic formulation of quantum statistical physics [32], [35, Ch. 5.3], and in algebraic quantum field theory [36–38], motivating our desire to consider representations that are suitably compatible with a KMS state.

In order to handle positive energy representations and KMS representations at the same time, we study *generalized positive energy representations*, a notion that is sufficiently flexible to capture positive energy representations as well as a large class of KMS representations [33,39]. We say that $(\overline{\rho}, \mathcal{H})$ is of *generalized positive energy* if there exists a dense subspace $\mathcal{D} \subseteq \mathcal{H}^\infty$ of smooth vectors such that for every $\psi \in \mathcal{D}$, the expected energy

$$\mu : \mathrm{P}(\mathcal{H}^\infty) \to \mathbb{R}, \qquad \mu([\psi]) := \frac{1}{\|\psi\|^2}\langle \psi, H\psi\rangle$$

is bounded below on the $\mathrm{Diff}_c(M)_0$-orbit $\mathcal{O}_{[\psi]} \subseteq \mathrm{P}(\mathcal{H})$ (cf. Definition 6).

It is important to mention that the (generalized) positive energy condition is invariant under the adjoint action of $\mathrm{Diff}_c(M)$ on $\mathcal{X}_c(M) \rtimes_\upsilon \mathbb{R}$, in the sense that $\overline{\rho}$ is of (generalized) positive energy at υ if and only if it is of (generalized) positive energy at $\mathrm{Ad}_f(\upsilon) := T(f) \circ \upsilon \circ f^{-1}$ for all $f \in \mathrm{Diff}_c(M)$. More generally, if $\overline{\rho}$ extends to a projective representation of a Lie group G that contains both $\mathrm{Diff}_c(M)$ and the flow of υ, then the choice of υ is only significant up to the adjoint action of G. We expect to find interesting (generalized) positive energy representations only when the adjoint orbit of G through υ generates a convex cone that is pointed. This is the case, for example, in the context of the BMS group for a suitable choice of υ.

The main result and its consequences. The following is our main result.

Theorem A. *Suppose that M is connected and that* $\dim(M) > 1$. *Let* $\upsilon \in \mathcal{X}(M)\backslash\{0\}$ *be a complete vector field on M. Let* $\overline{\rho} : \mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ *be a smooth projective unitary representation that is of generalized positive energy at υ. Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

This has at least three noteworthy consequences. First of all, it follows immediately that $\mathrm{Diff}_c(M)_0$ is in the kernel of every *positive energy representation*. Secondly, $\mathrm{Diff}_c(M)_0$ is in the kernel of every *smoothly-KMS representation* whose image generates a factor in $\mathrm{B}(\mathcal{H})$ (Corollary 7). And finally, $\mathrm{Diff}_c(M)_0$ is in the kernel of every projective unitary representation $(\overline{\rho}, \mathcal{H})$ of $\mathrm{Diff}_c(M)$ that is *bounded*, i.e. continuous in the norm topology (Corollary 8).

Compactly supported Lie algebra cohomology. From a technical point of view, a key step towards Theorem A is determining the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ of the Lie algebra $\mathcal{X}_c(M)$ of compactly supported vector fields, equipped with its natural locally convex LF-topology. Indeed, any smooth projective unitary representation $(\overline{\rho}, \mathcal{H})$ of $\mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R}$ gives rise to a canonical class in $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ that controls the corresponding central extension at the infinitesimal level [40]. If $\overline{\rho}$ is of generalized positive energy, then this cohomology class carries substantial information about the kernel of $\overline{\rho}$ (Proposition 1). Our main result in this regard is the following theorem:

Theorem B. *Let M be a smooth manifold.*

1. *If* $\dim(M) > 1$*, then* $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \{0\}$.
2. *If* $\dim(M) = 1$*, then* $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) \cong H^0_{\mathrm{dR}}(M)$ *is the de Rham cohomology of M in degree 0.*

The continuous Lie algebra cohomology $H^n_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is closely related to the *Gelfand–Fuks cohomology* $H^n_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$, the continuous Lie algebra cohomology of the Fréchet–Lie algebra $\mathcal{X}(M)$ of all vector fields. However, these two notions do *not* in general coincide [41], and many of the tools that are used in Gelfand–Fuks cohomology (such as Bott's homotopy operators [42]) break down in the compactly supported case.

The study of Gelfand–Fuks cohomology was initiated by Gelfand and Fuks [43–46], and carried further by Bott, G. Segal, Haefliger and many others [47–51], see also [42,52] and the recent exposition [53]. For compact manifolds M with $\dim(M) > 1$, it is well known that the second Gelfand–Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ vanishes ([53, Thm. 4.13 and Cor. 4.25]), covering part 1 of Theorem B in the compact case.

Nevertheless, Theorem B appears to be new if M is a non-compact manifold, which is of course the case of primary interest in the context of asymptotic symmetry groups. The proof is very much inspired by [54], and by joint work in progress [55] with Cornelia Vizman and Leonid Ryvkin on the second Lie algebra cohomology of the Lie algebra of exact volume preserving vector fields. We are grateful for their kind permission to use ideas from this unpublished work in the current setting.

Outline and further references. The paper is organized as follows. In Sect. 2 we recall various preliminary definitions and observations. In Sect. 3 we proceed with the proof of Theorem A and its consequences, subject to the assumption that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = 0$ if $\dim(M) > 1$. Finally, in Sect. 4, we justify this assumption by determining the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ for arbitrary manifolds M, culminating in Theorem B.

KMS representations of infinite dimensional Lie groups were studied by Strătilă and Voiculescu for $\mathrm{U}(\infty)$ [56], see [57,58] for related results in the context of Hilbert–Lie groups. For finite dimensional Lie groups, KMS representations that generate a factor of type I were fully classified in [59]. Projective unitary KMS representations were constructed for the loop group $C^\infty(S^1, \mathrm{U}(N))$ in [60,61], and for certain Sobolev maps from $\mathbb{R}$ to $\mathrm{U}(N)$ in [62]. Various other examples of KMS representations are given in [33, Sec. 5.2.2].

2. Preliminaries

We briefly discuss projective unitary representations for locally convex Lie groups, and recall some properties of generalized positive energy representations that we will need in Sect. 3.

2.1. Projective unitary representations. Let G be a locally convex Lie group, in the sense of Bastiani [63–65], with Lie algebra $\mathfrak{g}$.

Let $\mathcal{D}$ be a complex pre-Hilbert space with Hilbert space completion $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of linear operators $\mathcal{D} \to \mathcal{D}$. Define the algebra

$$\mathcal{L}^\dagger(\mathcal{D}) := \left\{ X \in \mathcal{L}(\mathcal{D}) \,:\, \exists X^\dagger \in \mathcal{L}(\mathcal{D}) \,:\, \forall \psi, \eta \in \mathcal{D} : \langle X^\dagger \psi, \eta \rangle = \langle \psi, X\eta \rangle \right\}.$$

It carries a natural involution $T \mapsto T^\dagger$. Let I denote the identity on $\mathcal{D}$ and define the Lie algebra

$$\mathfrak{u}(\mathcal{D}) := \left\{ X \in \mathcal{L}^\dagger(\mathcal{D}) \,:\, X^\dagger + X = 0 \right\}.$$

Define also the Lie algebra $\mathfrak{pu}(\mathcal{D}) := \mathfrak{u}(\mathcal{D})/i\mathbb{R}I$.

Definition 1. We define smooth projective unitary representations as follows:

- A unitary representation $(\rho, \mathcal{H})$ of G is *continuous* if $g \mapsto \rho(g)\psi$ is continuous for every $\psi \in \mathcal{H}$. Similarly, a projective unitary representation $(\overline{\rho}, \mathcal{H})$ is continuous if the orbit map $g \mapsto \overline{\rho}(g)[\psi]$ is continuous for every $\psi \in \mathcal{H} \setminus \{0\}$.
- If $(\rho, \mathcal{H})$ is a unitary representation of G, then a vector $\psi \in \mathcal{H}$ is called *smooth* if the orbit map $G \to \mathcal{H}$, $g \mapsto \rho(g)\psi$ is smooth. We denote by $\mathcal{H}^\infty$ the set of smooth vectors in $\mathcal{H}$, and we call ρ smooth if $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.
- Similarly, if $(\overline{\rho}, \mathcal{H})$ is a projective unitary representation of G, then a ray $[\psi] \in \mathrm{P}(\mathcal{H})$ is said to be *smooth* if the orbit map $G \to \mathrm{P}(\mathcal{H})$, $g \mapsto \overline{\rho}(g)[\psi]$ is smooth. We denote by $\mathrm{P}(\mathcal{H})^\infty$ the set of smooth rays, and we say that $\overline{\rho}$ is smooth if $\mathrm{P}(\mathcal{H})^\infty$ is dense in $\mathrm{P}(\mathcal{H})$.
- A unitary representation of a locally convex Lie algebra $\mathfrak{g}$ on $\mathcal{D}$ is a homomorphism $\pi : \mathfrak{g} \to \mathfrak{u}(\mathcal{D})$ of Lie algebras. It is called continuous if the map $\xi \mapsto \pi(\xi)\psi$ is continuous for every $\psi \in \mathcal{D}$. A projective unitary representation of $\mathfrak{g}$ on $\mathcal{D}$ is a Lie algebra homomorphism $\overline{\pi} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D})$.

Remark 1. A smooth unitary representation $(\rho, \mathcal{H})$ of G defines a unitary $\mathfrak{g}$-representation $d\rho : \mathfrak{g} \to \mathfrak{u}(\mathcal{H}^\infty)$ on $\mathcal{H}^\infty$ by $d\rho(\xi)\psi := \frac{d}{dt}\big|_{t=0} \rho(\gamma_t)\psi$, where $\gamma : \mathbb{R} \to G$ is a smooth curve satisfying $T_0(\gamma) = \xi$. If G is finite-dimensional, then $\mathcal{H}^\infty$ is dense in $\mathcal{H}$ for any continuous unitary representation ρ of G, by a result of Gårding [66] (cf. [67, Prop. 4.4.1.1]). The analogous statement is generally false for infinite-dimensional Lie groups [68].

Definition 2. A *central extension* of G by the circle group $\mathbb{T}$ is an exact sequence

$$1 \to \mathbb{T} \to \mathring{G} \to G \to 1 \tag{1}$$

of groups for which the image of $\mathbb{T}$ in $\mathring{G}$ is central. It is a central extension of *topological groups* if $\mathring{G}$ is a topological group, and the group homomorphisms in (1) are continuous. It is a central extension of *locally convex Lie groups* if $\mathring{G}$ is a locally convex Lie group, the group homomorphisms in (1) are smooth, and $\mathring{G} \to G$ is a locally trivial smooth principal $\mathbb{T}$-bundle. An *isomorphism* $\mathring{G} \to \mathring{G}'$ of central extensions is an isomorphism of groups (topological groups, Lie groups) that induces the identity on G and $\mathbb{T}$.

Definition 3. A *central extension of Lie algebras* is an exact sequence

$$0 \to \mathbb{R} \to \mathring{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{2}$$

of Lie algebras for which the image of $\mathbb{R}$ in $\mathring{\mathfrak{g}}$ is central. It is a *continuous* central extension if $\mathring{\mathfrak{g}}$ is a locally convex Lie algebra, and the Lie algebra homomorphisms in (2) are continuous.

Any central $\mathbb{T}$-extension of locally convex Lie groups determines a corresponding continuous central $\mathbb{R}$-extension of Lie algebras.

If $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H})$ is a continuous projective unitary representation of G, then the pullback

$$\mathring{G} := \left\{(g, U) \in G \times \mathrm{U}(\mathcal{H})\,;\; \overline{\rho}(g) = [U]\right\} \tag{3}$$

is a central extension $\mathring{G} \to G$ of topological groups, and $\overline{\rho}$ lifts to a continuous unitary representation $\rho\colon \mathring{G} \to \mathrm{U}(\mathcal{H})$ that satisfies $\rho(z) = zI$ for all $z \in \mathbb{T}$. We call ρ the *lift* of $\overline{\rho}$.

If, moreover, the projective unitary representation $\overline{\rho}$ of G is smooth, then $\mathring{G}$ is a locally convex Lie group, and $\mathring{G} \to G$ is a central extension of locally convex Lie groups [40, Thm. 4.3]. The lift ρ is smooth [40, Cor. 4.5], and $\mathrm{P}(\mathcal{H})^{\infty} = \mathrm{P}(\mathcal{H}^{\infty})$ [40, Thm. 4.3]. Suppose that $(\overline{\rho}_1, \mathcal{H}_1)$ and $(\overline{\rho}_2, \mathcal{H}_2)$ are two smooth projective unitary representations with lifts $\rho_1 : \mathring{G}_1 \to \mathrm{U}(\mathcal{H}_1)$ and $\rho_2 : \mathring{G}_2 \to \mathrm{U}(\mathcal{H}_2)$, respectively. Then $\overline{\rho}_1$ and $\overline{\rho}_2$ are unitarily equivalent if and only if there exists an isomorphism $\Phi : \mathring{G}_1 \to \mathring{G}_2$ of central extensions and a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\rho_2(\Phi(x)) = U\rho_1(x)U^{-1}$ for all $x \in \mathring{G}_1$ [40, Thm. 7.3].

Analogously, any projective unitary $\mathfrak{g}$-representation $\overline{\pi}$ on $\mathcal{D}$ can be lifted to a unitary representation $\pi : \mathring{\mathfrak{g}} \to \mathfrak{u}(\mathcal{D})$ of some central $\mathbb{R}$-extension $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$, by considering the pull-back of $\mathfrak{u}(\mathcal{D}) \to \mathfrak{pu}(\mathcal{D})$ along $\overline{\pi}$.

Definition 4. We call a projective unitary representation $\overline{\pi} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D})$ *continuous* if its lift $\pi : \mathring{\mathfrak{g}} \to \mathfrak{u}(\mathcal{D})$ is continuous.

2.2. Cohomology of Lie algebras and Lie groups. Smooth projective unitary representations of G give rise to central $\mathbb{T}$-extensions of locally convex Lie groups, and these in turn determine continuous central $\mathbb{R}$-extensions of the Lie algebra $\mathfrak{g}$. The latter can be described in terms of continuous Lie algebra cohomology.

Definition 5 *(Lie algebra cohomology)*. Let E be a module over a Lie algebra $\mathfrak{g}$.

- The Lie algebra cohomology $H^{\bullet}(\mathfrak{g}, E)$ of $\mathfrak{g}$ with values in E is the cohomology of the complex $C^{\bullet}(\mathfrak{g}, E)$, where $C^{q}(\mathfrak{g}, E)$ consists of alternating multilinear maps $\mathfrak{g}^{q} \to E$ for $q \geq 0$, and it is zero for $q < 0$. The differential $d_{\mathfrak{g}} : C^{\bullet}(\mathfrak{g}, E) \to C^{\bullet+1}(\mathfrak{g}, E)$ is given by

$$\begin{aligned} d_{\mathfrak{g}}\omega(\xi_0, \ldots, \xi_q) := &\sum_{j=0}^{q} (-1)^{j} \xi_j \cdot \omega(\xi_0, \ldots, \widehat{\xi_j}, \ldots, \xi_q) \\ &+ \sum_{0 \leq i < j \leq q} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_q). \end{aligned} \tag{4}$$

  As usual, the arguments in (4) with a caret are to be omitted. Unless mentioned otherwise, the vector space $\mathbb{R}$ is considered as a trivial $\mathfrak{g}$-module.
- If $\mathfrak{g}$ is a locally convex Lie algebra and E a topological $\mathfrak{g}$-module, then the continuous Lie algebra cohomology $H^{\bullet}_{\mathrm{ct}}(\mathfrak{g}, E)$ is the cohomology of the subcomplex $C^{\bullet}_{\mathrm{ct}}(\mathfrak{g}, E)$ of continuous alternating multilinear maps.

The continuous central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are classified up to isomorphism by $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$, the continuous second Lie algebra cohomology with trivial coefficients [40, Prop. 6.3]. In order to study smooth projective unitary representations of $G = \mathrm{Diff}_c(M)$, it is sensible to determine $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ for the Lie algebra $\mathfrak{g} = \mathcal{X}_c(M)$. This is done in Sect. 4.

The interpretation of the second Lie algebra cohomology in terms of central extensions is already implicit in the work of Schur [69,70], and its use in the projective representation theory of Lie groups was pioneered by E. Wigner [71] and V. Bargmann [72], see [73] for an exposition and further references.

To some extent, Lie algebra cohomology functions as an infinitesimal counterpart of Lie group cohomology, with their relation typically given by a van Est-type spectral sequence [74]. Although group cohomology for discrete groups admits a good description in terms of commutative algebra [75], the appropriate cohomology theory in the context of Lie groups requires a bit more care. There are in fact many different flavours of group cohomology for Lie groups, grounded either in Čech cohomology or in explicit cocycle models [76–78]. With the notable exception of bounded cohomology [79], they mostly agree on the domain for which they are intended, see [80] for an overview and comparison, as well as for further references.

2.3. Generalized positive energy (GPE) representations. In the following, G denotes a locally convex Lie group which is regular in the sense of Milnor [64, Def. 7.6] (cf. [65, Def. II.5.2]). We denote by $\mathfrak{g}$ the Lie algebra of G.

This paper is concerned with projective unitary representations $\overline{\rho}$ of G that satisfy a so-called generalized positive energy condition. This class of representations was introduced in [33, Sec. 4]. It includes representations that satisfy a positive energy condition, and also representations that are in a suitable sense compatible with a KMS state on the von Neumann algebra generated by $\overline{\rho}(G)$.

We first introduce the precise definitions, and then review the restrictions that the cohomology class $[\omega] \in H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ associated to such a representation imposes on its kernel. These restrictions play a crucial role in the proof of Theorem A.

Definition 6. Let $\mathcal{D}$ be a complex pre-Hilbert space with Hilbert space completion $\mathcal{H}$. Let $\mathfrak{h}$ be a locally convex topological Lie algebra.

- Let $\pi : \mathfrak{h} \to \mathfrak{u}(\mathcal{D})$ be a continuous unitary representation of $\mathfrak{h}$ on $\mathcal{D}$. We say that π is of *positive energy* at $\xi \in \mathfrak{h}$ if

$$\inf_{\psi \in \mathcal{D}} \langle \psi, -i\pi(\xi)\psi \rangle \geq 0.$$

We say that $\pi : \mathfrak{h} \to \mathfrak{u}(\mathcal{D})$ is of *generalized positive energy* (GPE) at $\xi \in \mathfrak{h}$ if there exists a 1-connected regular Lie group H with Lie algebra $\mathfrak{h}$ and a dense linear subspace $\mathcal{D}_\xi \subseteq \mathcal{D}$ such that

$$\forall \psi \in \mathcal{D}_\xi \ : \ \inf_{h \in H} \langle \psi, -i\pi(\mathrm{Ad}_h(\xi))\psi \rangle > -\infty. \tag{5}$$

- Let $\overline{\pi}$ be a continuous projective unitary representation of $\mathfrak{h}$ with lift $\pi : \mathring{\mathfrak{h}} \to \mathfrak{u}(\mathcal{D})$. We say that $\overline{\pi}$ is of (generalized) positive energy at $\xi \in \mathfrak{h}$ if π is so at some $\mathring{\xi} \in \mathring{\mathfrak{h}}$ covering ξ.

- A smooth unitary representation $(\rho, \mathcal{H})$ of G is of (generalized) positive energy at $\xi \in \mathfrak{g}$ if the derived representation $d\rho$ of $\mathfrak{g}$ on $\mathcal{H}^\infty$ is so.
- Let $\overline{\rho}$ be a smooth projective unitary representation of G on $\mathcal{H}$ with lift $\rho\colon \mathring{G} \to \mathrm{U}(\mathcal{H})$. We say that $\overline{\rho}$ is of (generalized) positive energy at $\xi \in \mathfrak{g}$ if there exists an element $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ such that ρ is of (generalized) positive energy at $\mathring{\xi}$.

Remark 2. Suppose that π is a continuous unitary representation of $\mathfrak{g}$ on $\mathcal{D}$ that is of GPE at some element in $\mathfrak{g}$. Then the group H in (5) is the simply connected cover $\widetilde{G}_0$ of the identity component of G, because two regular 1-connected Lie groups are isomorphic if their Lie algebras are so [64, Cor. 8.2], and $\widetilde{G}_0$ is regular whenever G is so [65, Thm. V.1.8].

Remark 3. If a unitary representation $(\rho, \mathcal{H})$ of G is of positive energy at $\xi \in \mathfrak{g}$, then it is also of generalized positive energy at ξ. Indeed, since

$$\langle \psi, d\rho(\mathrm{Ad}_g^{-1}(\xi))\psi \rangle = \langle \rho(g)\psi, d\rho(\xi)\rho(g)\psi \rangle \qquad \text{for all } g \in G,\ \xi \in \mathfrak{g} \text{ and } \psi \in \mathcal{H}^\infty,$$

Remark 2 implies that ρ is of generalized positive energy at $\xi \in \mathfrak{g}$ if and only if

$$I(\xi, \psi) := \inf_{g_0 \in G_0} \langle \rho(g_0)\psi, -id\rho(\xi)\rho(g_0)\psi \rangle > -\infty \tag{6}$$

for all ψ in some linear subspace $\mathcal{D}_\xi \subseteq \mathcal{H}^\infty$ that is dense in $\mathcal{H}$. If ρ is of positive energy at ξ, then the left hand side of (6) is nonnegative for any $\psi \in \mathcal{H}^\infty$, since $\mathcal{H}^\infty$ is G-invariant.

Remark 4. Let $(\rho, \mathcal{H})$ be a smooth unitary representation of G.

- The *generalized positive energy cone*

$$\mathcal{C}(\rho) := \{\, \xi \in \mathfrak{g} \ :\ \rho \text{ is of GPE at } \xi \,\}$$

  is Ad_G-invariant for the (not necessarily connected) Lie group G. Indeed, if $\xi \in \mathcal{C}(\rho)$ and $g \in G$, then (6) for ξ and $\psi \in \mathcal{D}_\xi$ implies the corresponding inequality for $\xi' = \mathrm{Ad}_g(\xi)$ and $\psi' \in \mathcal{D}_{\xi'}$ with $\mathcal{D}_{\xi'} := \rho(g)\mathcal{D}_\xi$.
- If $\xi \in \mathcal{C}(\rho)$, then $\mathcal{C}(\rho)$ also contains the Ad_{G_0}-invariant convex cone generated by ξ. To see this, suppose that (6) is satisfied for $\psi \in \mathcal{D}_\xi \subseteq \mathcal{H}^\infty$. Let $n \in \mathbb{N}$, $g_k \in G_0$ and $c_k \geq 0$ for $k \in \{1, \dots, n\}$, and define $C := \sum_{k=1}^n c_k$. Then for $\xi' := \sum_{k=1}^n c_k \mathrm{Ad}_{g_k}(\xi)$ we have that $I(\xi', \psi) \geq C \cdot I(\xi, \psi)$, so that $\xi' \in \mathcal{C}(\rho)$.
- If, moreover, ρ is of positive energy at $\xi \in \mathfrak{g}$, then it is of positive energy at every element of the Ad_G-invariant closed convex cone $\mathcal{C}_\xi \subseteq \mathfrak{g}$ generated by ξ. It follows that $\ker(d\rho)$ contains the Ad_G-invariant closed ideal $\mathcal{C}_\xi \cap -\mathcal{C}_\xi$ of $\mathfrak{g}$. If $\mathcal{C}_\xi \cap -\mathcal{C}_\xi = \mathfrak{g}$, then [40, Prop. 3.4] implies that $G_0 \subseteq \ker \rho$.
- In particular, if $\mathfrak{g}$ admits no non-zero proper Ad_G-invariant closed ideals, then G_0 can only act non-trivially in a smooth unitary representation of G that is of positive energy at ξ if the cone $\mathcal{C}_\xi$ is pointed. This is the situation for $G = \mathrm{Diff}_c(M)$ if M is connected, as we shall see in Corollary 5.

Remark 5. A smooth projective unitary G-representation $\overline{\rho}$ with lift ρ is of GPE at $\xi \in \mathfrak{g}$ if and only if for some (and hence any) $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ, the function

$$\mu : \mathrm{P}(\mathcal{H}^\infty) \to \mathbb{R},\ \ \mu([\psi]) := \frac{1}{\|\psi\|^2}\langle \psi, -id\rho(\mathring{\xi})\psi \rangle$$

is bounded below on the G_0-orbit $\mathcal{O}_{[\psi]} \subseteq \mathrm{P}(\mathcal{H}^\infty)$ for all ψ in some dense linear subspace $\mathcal{D}_\xi \subseteq \mathcal{H}^\infty$.

The following observation plays a crucial role in Sect. 3. It shows that the cohomology class in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ associated to a projective GPE-representation carries substantial information about the kernel.

Proposition 1 ([33, Prop. 4.4]). *Let $\overline{\pi} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D})$ be a projective unitary representation of $\mathfrak{g}$ that is of generalized positive energy at $\xi \in \mathfrak{g}$. Let $\omega \in C^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ be a 2-cocycle whose class $[\omega]$ in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ corresponds to the central extension $\mathring{\mathfrak{g}} \to \mathfrak{g}$. Suppose that $\eta \in \mathfrak{g}$ satisfies $[[\xi, \eta], \eta] = 0$. Then $\omega([\xi, \eta], \eta) \geq 0$ and*

$$\omega([\xi, \eta], \eta) = 0 \iff \overline{\pi}([\xi, \eta]) = 0.$$

In particular, if $[\omega] = 0$ in $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$, then

$$[[\xi, \eta], \eta] = 0 \implies \overline{\pi}([\xi, \eta]) = 0.$$

Before defining the notion of a KMS-representation, we briefly recall the definition of a KMS state on a von Neumann algebra (cf. [35, Ch. 5.3], [34, Ch. VIII]). Let $\mathrm{St} := \{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1 \}$.

Definition 7. Let ϕ be a normal state on a von Neumann algebra $\mathcal{N}$, and let $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{N})$ be a one-parameter group.

- We say that ϕ satisfies the *modular condition* for σ if:
  1. $\phi = \phi \circ \sigma_t$ for every $t \in \mathbb{R}$.
  2. For every pair $x, y \in \mathcal{N}$, there is a bounded continuous function $F_{x,y} : \overline{\mathrm{St}} \to \mathbb{C}$, holomorphic on St, so that for every $t \in \mathbb{R}$ we have:

$$\begin{aligned} F_{x,y}(t) &= \phi(\sigma_t(x)y), \\ F_{x,y}(t+i) &= \phi(y\sigma_t(x)). \end{aligned}$$

- We say that ϕ is *σ-KMS* if it satisfies the modular condition for the automorphism group $t \mapsto \sigma_{-t}$.

Definition 8. Let $\xi \in \mathfrak{g}$, and let $N \subseteq G$ be an embedded Lie subgroup such that $e^{t\xi} N e^{-t\xi} \subseteq N$ for all $t \in \mathbb{R}$.

1. A continuous unitary representation $(\rho, \mathcal{H})$ of G is *KMS at ξ relative to N* if the von Neumann algebra $\mathcal{N} := \rho(N)''$ admits a normal state ϕ that is σ-KMS for $\sigma_t(x) := \rho(e^{t\xi})x\rho(e^{-t\xi})$. It is *smoothly KMS* if additionally $n \mapsto \phi(\rho(n))$ is a smooth function $N \to \mathbb{C}$.
2. Let $\overline{\rho}: G \to \mathrm{PU}(\mathcal{H})$ be a smooth projective unitary representation of G, with lift $\rho: \mathring{G} \to \mathrm{U}(\mathcal{H})$. Let $\mathring{\mathfrak{g}}$ be the Lie algebra of $\mathring{G}$ and let $\mathring{N} \subseteq \mathring{G}$ be the Lie subgroup covering N. We say that $\overline{\rho}$ is smoothly-KMS at $\xi \in \mathfrak{g}$ relative to N if there exists $\mathring{\xi} \in \mathring{\mathfrak{g}}$ covering ξ such that ρ is smoothly-KMS at $\mathring{\xi}$ relative to $\mathring{N}$.

Various examples of KMS-representation are considered in [33, Sec. 5.2.2].

Suppose that $N \subseteq G$ and $\xi \in \mathfrak{g}$ are as in Definition 8. Let $\mathfrak{n}$ be the Lie algebra of N, ρ a unitary representation of G, and let $\mathcal{N} := \rho(N)''$ be the von Neumann algebra generated by $\rho(N)$. Define $\alpha : \mathbb{R} \to \mathrm{Aut}(N)$ by $\alpha_t(n) = e^{t\xi} n e^{-t\xi}$ and $D \in \mathrm{der}(\mathfrak{n})$ by $D\eta := [\xi, \eta]$. Suppose that the normal state ϕ is KMS w.r.t. the automorphism group $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{N})$, $\sigma_t(x) := \rho(e^{t\xi})x\rho(e^{-t\xi})$. The GNS-construction (for Gelfand–Naimark–Segal) provides a $*$-representation π_ϕ of the von Neumann algebra $\mathcal{N}$ on the GNS-Hilbert space $\mathcal{H}_\phi$. We therefore obtain a unitary representation

$\rho_\phi := \pi_\phi \circ \rho$ of N on $\mathcal{H}_\phi$. Letting Δ_ϕ denote the modular operator associated to ϕ (cf. [81, Ch. 2.5]), the representation ρ_ϕ extends to $N \rtimes_\alpha \mathbb{R}$ by setting $\rho_\phi(n,t) := \rho_\phi(n)\Delta_\phi^{-it}$. This representation is smooth if $n \mapsto \phi(\rho(n))$ is a smooth map $N \to \mathbb{C}$ [33, Lem. 5.10].

The following relates KMS-representations to the generalized positive energy condition:

Theorem 2 ([33, Thm. 5.13]). *Let ρ be a unitary representation of G that is smoothly-KMS at $\xi \in \mathfrak{g}$ relative to $N \subseteq G$, and let $\phi\colon \mathcal{N} \to \mathbb{C}$ be as in Definition 8. Then the associated unitary representation ρ_ϕ of $N \rtimes_\alpha \mathbb{R}$ on the GNS-Hilbert space $\mathcal{H}_\phi$ is smooth and of generalized positive energy at* $(0,1) \in \mathfrak{n} \rtimes_D \mathbb{R}$.

3. GPE Representations of $\mathrm{Diff}_c(M) \rtimes_v \mathbb{R}$

Let M be a smooth manifold of dimension $\dim(M) > 1$. If $v \in \mathcal{X}(M)$ is a complete vector field on M with flow $\Phi^v : \mathbb{R} \to \mathrm{Diff}(M)$, we write $\mathrm{Diff}_c(M) \rtimes_v \mathbb{R}$ for the semidirect product of $\mathrm{Diff}_c(M)$ and $\mathbb{R}$ relative to the smooth $\mathbb{R}$-action on $\mathrm{Diff}_c(M)$ defined by $\alpha_s(f) = \Phi^v_s \circ f \circ \Phi^v_{s^{-1}}$ for $s \in \mathbb{R}$ and $f \in \mathrm{Diff}_c(M)$. The corresponding Lie algebra is $\mathcal{X}_c(M) \rtimes \mathbb{R}v$, where v acts on $\mathcal{X}_c(M)$ by the derivation $Dw := [v, w]$.

In Sect. 4, we will see that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is trivial for $\dim(M) > 1$. This puts severe restrictions on the class of projective unitary representations of $\mathcal{X}_c(M) \rtimes \mathbb{R}v$ that are of generalized positive energy at v. The following result on Lie algebra representations is the crux of the matter.

Theorem 3. *Suppose that* $\dim(M) > 1$. *Let $\overline{\pi} : \mathcal{X}_c(M) \to \mathfrak{pu}(\mathcal{D})$ be a continuous projective unitary representation of $\mathcal{X}_c(M)$ on the complex pre-Hilbert space $\mathcal{D}$. Let $\mathcal{C} \subseteq \mathcal{X}(M)$ be a cone of complete vector fields on M, and define the open set*

$$U := \bigcup_{v\in\mathcal{C}} \{\, p \in M \,:\, v(p) \neq 0 \,\}\,.$$

Suppose that for every $v \in \mathcal{C}$ the representation $\overline{\pi}$ extends to a continuous projective unitary representation of $\mathcal{X}_c(M) \rtimes \mathbb{R}v$ that is of generalized positive energy at v. Then $\mathcal{X}_c(U) \subseteq \ker\overline{\pi}$.

In Sect. 3.1, we classify invariant ideals in $\mathcal{X}_c(M)$. This is an intermediate step towards the proof of Theorem 3, which is given in Sect. 3.2. Lastly, in Sect. 3.3 we use this result to derive Theorem A, which is a group-level analogue of Theorem 3.

3.1. Ideals of the Lie algebra $\mathcal{X}_c(M)$.

Let M be a smooth manifold. For any $x \in M$, let $I_x \subseteq \mathcal{X}_c(M)$ denote the closed ideal of vector fields that are flat at x. So $v \in I_x \iff j^\infty_x(v) = 0$ for $v \in \mathcal{X}_c(M)$. The proof of Theorem 3 uses Proposition 4 below.

Definition 9. If $J \subseteq \mathcal{X}_c(M)$ is an ideal, define its *hull* by

$$h(J) := \{\, x \in M \,:\, v(x) = 0 \text{ for all } v \in J \,\}\,.$$

Remark 6. The set of maximal ideals in $\mathcal{X}_c(M)$ is given by $\{I_x : x \in M\}$ [82, Thm. 1] (cf. [83, prop. 7.2.2] or [84, Prop. 1]). Moreover, if $x \in M$ and $J \subseteq \mathcal{X}_c(M)$ is an ideal, then $x \in h(J)$ if and only if $j^\infty_x(v) = 0$ for all $v \in J$. Indeed, if $x \in h(J)$, then for any $w_1, \ldots, w_m \in \mathcal{X}_c(M)$ and $v \in J$ we have $\mathcal{L}_{w_1}\cdots\mathcal{L}_{w_m}v \in J$, as J is an ideal, and so $\big(\mathcal{L}_{w_1}\cdots\mathcal{L}_{w_m}v\big)(x) = 0$. Consequently $j^\infty_x(v) = 0$. We thus see that $h(J) = \{x \in M : J \subseteq I_x\}$ corresponds to the set of maximal ideals of $\mathcal{X}_c(M)$ containing J.

Proposition 4. *Let $J \subseteq \mathcal{X}_c(M)$ be an ideal and let $x \in M$. Then either $x \in h(J)$, or there is an open neighborhood $U \subseteq M$ of x such that $\mathcal{X}_c(U) \subseteq [J, \mathcal{X}_c(M)] \subseteq J$.*

Proof. This is immediate from the proof of [84, Lem. 2.1], which does not require the ideal $J \subseteq \mathcal{X}_c(M)$ to be maximal. □

Although $\mathcal{X}_c(M)$ is not simple, the following related result does hold true:

Corollary 5. *Assume that M is connected. Suppose that $J \subseteq \mathcal{X}_c(M)$ is an ideal that is stable, in the sense that $\mathrm{Ad}_g(J) \subseteq J$ for all $g \in \mathrm{Diff}_c(M)_0$. Then either $J = \mathcal{X}_c(M)$ or $J = \{0\}$.*

Proof. That J is stable implies that its hull $h(J) \subseteq M$ is $\mathrm{Diff}_c(M)_0$-invariant. Since M is connected, $\mathrm{Diff}_c(M)_0$ acts transitively on M (cf. [85, p. 22]). It follows that either $h(J) = \emptyset$ or $h(J) = M$. Using a partition of unity argument, Proposition 4 implies that either $J = \mathcal{X}_c(M)$ or $J = \{0\}$. □

Remark 7. Suppose that M is connected. Let $\rho : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H})$ be a smooth projective unitary representation. Let $\overline{d\rho} : \mathcal{X}_c(M) \to \mathfrak{pu}(\mathcal{H}^\infty)$ be its derived representation. Its kernel $J := \ker \overline{d\rho}$ is a closed ideal in $\mathcal{X}_c(M)$ satisfying $\mathrm{Ad}_g(J) \subseteq J$ for all $g \in \mathrm{Diff}_c(M)$. So $\overline{d\rho}$ is either trivial or injective by Corollary 5.

3.2. The proof of Theorem 3. We now proceed with the proof of Theorem 3. Let $n := \dim(M) > 1$. We start with a lemma that concerns the local situation near a regular point of a vector field $v \in \mathcal{C}$. We thus consider the following setting:

Let $I \subseteq \mathbb{R}$ be an open interval containing zero. Let $U_0 \subseteq \mathbb{R}^{n-1}$ be an open subset that is diffeomorphic to $\mathbb{R}^{n-1}$. Define $U := I \times U_0$, which is then diffeomorphic to $\mathbb{R}^n$. We consider the locally convex Lie algebra $\mathcal{X}_c(U)$ of compactly supported smooth vector fields on U. We write $(t, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n$ for the coordinates on $\mathbb{R}^n$, and $(\partial_t, \partial_{x_1}, \ldots, \partial_{x_{n-1}})$ for the corresponding basis of $\mathcal{X}(\mathbb{R}^n)$ over $C^\infty(\mathbb{R}^n)$. Notice that the derivation $D := [\partial_t, -]$ on $\mathcal{X}_c(U)$ does not necessarily integrate to a 1-parameter group of automorphisms of $\mathcal{X}_c(U)$, because the open set U need not be invariant under the flow of ∂_t.

Lemma 6. *Let $\overline{\pi} : \mathcal{X}_c(U) \rtimes \mathbb{R}\partial_t \to \mathfrak{pu}(\mathcal{D})$ be a continuous projective unitary representation on the pre-Hilbert space $\mathcal{D}$. Assume that*

$$[v, Dv] = 0 \implies \overline{\pi}(Dv) = 0, \qquad \forall v \in \mathcal{X}_c(U). \tag{7}$$

Then $\mathcal{X}_c(U) \subseteq \ker \overline{\pi}$.

Proof. Let $p_0 = (t_0, x_0) \in U = I \times U_0$ be arbitrary. Let $f \in C_c^\infty(I)$ and $w \in \mathcal{X}_c(U_0)$ be s.t. $f'(t_0) \neq 0$ and $w(x_0) \neq 0$. Define $v \in \mathcal{X}_c(U)$ by $v(t, x) := f(t)w(x)$ for $t \in I$ and $x \in U_0$. Observe that $Dv(t, x) = f'(t)w(x)$. In particular, $Dv(p_0) \neq 0$ and $[v, Dv](t, x) = f(t)f'(t)[w, w](x) = 0$. It follows using (7) that $Dv \in \ker \overline{\pi}$. Let $J \subseteq \mathcal{X}_c(U)$ be the ideal generated by Dv. Then $J \subseteq \ker \overline{\pi}$. As $Dv(p_0) \neq 0$, it follows using Proposition 4 that $\mathcal{X}_c(V) \subseteq J$ for some open neighborhood $V \subseteq U$ of p_0. So we have $\mathcal{X}_c(V) \subseteq \ker \overline{\pi}$. We have thus shown that any $p \in U$ has a neighborhood $V \subseteq U$ for which $\mathcal{X}_c(V) \subseteq \ker \overline{\pi}$. Consequently, if $K \subseteq U$ is a compact subset, we can find a finite open cover $\{U_1, \ldots, U_m\}$ of K with $\mathcal{X}_c(U_k) \subseteq \ker \overline{d\rho}$ for all $k \in \{1, \ldots, m\}$. Using a partition of unity argument, it follows that $\mathcal{X}_K(U) \subseteq \ker \overline{d\rho}$ for any compact set $K \subseteq M$, so that $\mathcal{X}_c(U) \subseteq \ker \overline{d\rho}$. □

We now return to the global setting, and prove Theorem 3.

Proof (of Theorem 3). Let $p \in U$ and let $\upsilon \in \mathcal{C}$ satisfy $\upsilon(p) \neq 0$. By assumption, $\overline{\pi}$ extends to a continuous projective unitary representation of $\mathcal{X}_c(M) \rtimes \mathbb{R}\upsilon$ that is of generalized positive energy at υ, again denoted $\overline{\pi}$. Since $\upsilon(p) \neq 0$, we can find an open neighborhood $U_p \subseteq M$ of p, an open interval $I \subseteq \mathbb{R}$ containing zero, an open subset $U_0 \subseteq \mathbb{R}^{n-1}$ that is diffeomorphic to $\mathbb{R}^{n-1}$, and a diffeomorphism $\phi : I \times U_0 \to U_p$ such that $\phi_*([\partial_t, w]) = [\upsilon, \phi_*(w)]$ for all $w \in \mathcal{X}_c(I \times U_0)$ [86, Thm. 9.22]. So ϕ_* defines an isomorphism

$$\phi_* : \mathcal{X}_c(I \times U_0) \rtimes \mathbb{R}\partial_t \to \mathcal{X}_c(U_p) \rtimes \mathbb{R}\upsilon.$$

In view of Theorem B, we know that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = 0$. As $\overline{\pi}$ is of generalized positive energy at υ, it follows using Proposition 1 that $[w, Dw] = 0$ implies $\overline{\pi}(Dw) = 0$ for any $w \in \mathcal{X}_c(M)$. As a consequence, the pull-back of $\overline{\pi}$ along the composition

$$\mathcal{X}_c(I \times U_0) \rtimes \mathbb{R}\partial_t \xrightarrow{\phi_*} \mathcal{X}_c(U_p) \rtimes \mathbb{R}\upsilon \hookrightarrow \mathcal{X}_c(M) \rtimes \mathbb{R}\upsilon$$

satisfies the conditions of Lemma 6, from which it then follows that $\mathcal{X}_c(U_p) \subseteq \ker \overline{\pi}$. So any $p \in U$ has an open neighborhood $U_p \subseteq M$ satisfying $\mathcal{X}_c(U_p) \subseteq \ker \overline{\pi}$. This implies that $\mathcal{X}_c(U) \subseteq \ker \overline{\pi}$.

3.3. The proof of Theorem A. In this section, we derive Theorem A as a group-level consequence of Theorem 3. As a special case, we obtain similar results for KMS and bounded representations.

Proof (of Theorem A). Since the derived representation $\overline{d\rho} : \mathcal{X}_c(M) \rtimes \mathbb{R}\upsilon \to \mathfrak{pu}(\mathcal{H}^\infty)$ is of generalized positive energy at υ, it is so at every υ' in the cone $\mathcal{C}$ generated by the adjoint orbit of υ in $\mathcal{X}_c(M) \rtimes \mathbb{R}\upsilon$. Since υ is non-zero, there exists some open subset $U_0 \subseteq M$ on which υ is non-vanishing. Then $\mathrm{Ad}_f(\upsilon)$ is non-zero on $f(U_0)$ for every $f \in \mathrm{Diff}_c(M)$. Since M is connected, $\mathrm{Diff}_c(M)$ acts transitively on M (because all orbit are open and therefore also closed, cf. [85, p. 22]). Hence $\bigcup_{\upsilon' \in \mathcal{C}} \{\, p \in M \,:\, \upsilon'(p) \neq 0 \,\} = M$. We obtain using Theorem 3 that $\mathcal{X}_c(M) \subseteq \ker \overline{d\rho}$. Corollary 21 now implies that $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

Corollary 7. *Suppose that M is connected and that* $\dim(M) > 1$. *Let $\upsilon \in \mathcal{X}(M)\backslash\{0\}$ be a complete vector field on M. Let $\overline{\rho} : \mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ be a smooth projective unitary representation that is smoothly-KMS at υ relative to* $\mathrm{Diff}_c(M)$. *Assume that the von Neumann algebra $\rho(\mathrm{Diff}_c(M))''$ is a factor. Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

Proof. Let $\rho : G \to \mathrm{U}(\mathcal{H})$ be the lift of $\overline{\rho}$, where the Lie group G is a central $\mathbb{T}$-extension of $\mathrm{Diff}_c(M) \rtimes_\upsilon \mathbb{R}$. Let $H \subseteq G$ be the Lie subgroup covering $\mathrm{Diff}_c(M)$. Let $\mathfrak{h}$ and $\mathfrak{g}$ denote the Lie algebras of H and G, respectively. Let $\mathcal{N} := \rho(H)''$ be the von Neumann algebra generated by $\rho(H)$. As $\overline{\rho}$ is smoothly-KMS at υ relative to $\mathrm{Diff}_c(M)$, there is some $\xi \in \mathfrak{g}$ covering υ such that ρ is smoothly-KMS at $\xi \in \mathfrak{g}$ relative to H. Let ϕ be a normal state on $\mathcal{N}$ for which the function $H \to \mathbb{C}, h \mapsto \phi(\rho(h))$ is smooth, and that is σ-KMS for $\sigma_t(x) = \rho(e^{t\xi})x\rho(e^{-t\xi})$ with $t \in \mathbb{R}$ and $x \in \mathcal{N}$. Let $\rho_\phi : H \rtimes \mathbb{R} \to \mathrm{U}(\mathcal{H}_\phi)$ be the associated unitary representation of $H \rtimes \mathbb{R}$ on the GNS-Hilbert space $\mathcal{H}_\phi$. According to Theorem 2, the representation ρ_ϕ on $\mathcal{H}_\phi$ is smooth and of generalized positive energy at $(0, 1) \in \mathfrak{h} \rtimes \mathbb{R}$. It follows from Theorem A that $\rho_\phi(H_0) \subseteq \mathbb{T}\mathrm{id}_{\mathcal{H}_\phi}$, where H_0 denotes the identity component of H. Because the von

Neumann algebra $\mathcal{N}$ is a factor, the GNS-representation $\mathcal{N} \to \mathcal{B}(\mathcal{H}_\phi)$ is injective (see e.g. [33, Rem. 5.3 items 1 and 3]). It follows that $\rho(H_0) \subseteq \mathbb{T}\mathrm{id}_{\mathcal{H}}$. Since H_0 covers $\mathrm{Diff}_c(M)_0$, this implies that $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$. □

Corollary 8. *Suppose that* $\dim(M) > 1$. *Let* $\overline{\rho} : \mathrm{Diff}_c(M) \to \mathrm{PU}(\mathcal{H})$ *be a smooth projective unitary representation that is bounded, i.e., continuous w.r.t. the norm topology on* $\mathrm{PU}(\mathcal{H})$. *Then* $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$.

Proof. Let $\rho : G \to \mathrm{U}(\mathcal{H})$ be the lift of $\overline{\rho}$, where G is a central $\mathbb{T}$-extension of $\mathrm{Diff}_c(M)$ with Lie algebra $\mathfrak{g}$. Let $p \in M$. Take $v \in \mathcal{X}_c(M)$ with $v(p) \neq 0$ and let $\xi \in \mathfrak{g}$ cover v. Since ρ is continuous w.r.t. the norm-topology on $\mathrm{U}(\mathcal{H})$, the self-adjoint operator $-i \left.\frac{d}{dt}\right|_{t=0} \rho(\exp_G(t\xi))$ is bounded. It follows that $\overline{\rho}$ is of (generalized) positive energy at $v \in \mathcal{X}_c(M)$. Using Theorem A, this implies that $v' \in \ker \overline{d\rho}$ for any $v' \in \mathcal{X}_c(M)$ for which $\mathrm{supp}(v')$ is contained in the connected component of p in M. As $p \in M$ was arbitrary, we find that $\mathcal{X}_c(M) \subseteq \ker \overline{d\rho}$. We conclude using Corollary 21 that $\mathrm{Diff}_c(M)_0 \subseteq \ker \overline{\rho}$. □

4. Continuous Second Lie Algebra Cohomology

In this section we prove Theorem B, that is, we determine the continuous second Lie algebra cohomology $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$. This is a crucial ingredient for the results of Sect. 3.

In Sect. 4.1 we consider the proof of Theorem B for the case $\dim(M) > 1$, and in Sect. 4.2 we consider the case $\dim(M) = 1$. The cohomology $H^\bullet_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is generally different from the Gelfand–Fuks cohomology $H^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ (cf. also [41]). In Sect. 4.3 we therefore clarify the relationship between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the second Gelfand–Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$. They both vanish when $\dim(M) > 1$, but differ for noncompact manifolds of dimension 1.

We first make some general observations that will be useful for the proof of Theorem B.

Definition 10. A 2-cochain $\psi : \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ is called *diagonal* if

$$\mathrm{supp}(v) \cap \mathrm{supp}(w) = \emptyset \quad \Longrightarrow \quad \psi(v, w) = 0, \qquad \forall v, w \in \mathcal{X}_c(M).$$

Lemma 9. *Every* 2*-cocycle on* $\mathcal{X}_c(M)$ *is diagonal.*

Proof. Let $v, w \in \mathcal{X}_c(M)$ have disjoint support. Then we can find open subsets $U_1, U_2 \subseteq M$ with $U_1 \cap U_2 = \emptyset$ such that $\mathrm{supp}(v) \subseteq U_1$ and $\mathrm{supp}(w) \subseteq U_2$. Since the Lie algebra $\mathcal{X}_c(U_1)$ is perfect ([83, Thm. 1.4.3] or [84, Cor. 1]), there exist $v_1^i, v_2^i \in \mathcal{X}_c(U_1)$ s.t. $v = \sum_{i=1}^N [v_1^i, v_2^i]$. Since $\psi : \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ satisfies the cocycle identity

$$\psi([u, v], w) = \psi([u, w], v) + \psi(u, [v, w]), \tag{8}$$

and since w has support disjoint from that of v_1^i and v_2^i, we have

$$\psi(v, w) = \sum_{i=1}^N \psi([v_1^i, v_2^i], w) = \sum_{i=1}^N \psi([v_1^i, w], v_2^i) + \psi(v_1^i, [v_2^i, w]) = 0,$$

as required. □

Let $\mathfrak{g}$ be a locally convex Lie algebra, and let $\mathfrak{g}'$ be its continuous dual. We consider $\mathfrak{g}'$ as a $\mathfrak{g}$-module with the coadjoint action, defined by $(\xi \cdot \alpha)(\eta) := -\alpha([\xi, \eta])$ for $\alpha \in \mathfrak{g}'$. For a continuous cochain $\psi \in C^q_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ with values in $\mathbb{R}$, we define a (not necessarily continuous) cochain $\hat{\psi} \in C^{q-1}(\mathfrak{g}, \mathfrak{g}')$ by

$$\hat{\psi}(\xi_1, \dots, \xi_{q-1})(\eta) := \psi(\xi_1, \dots, \xi_{q-1}, \eta). \tag{9}$$

Let $C^{\bullet-1}(\mathfrak{g}, \mathfrak{g}')$ denote the shifted complex with $C^{q-1}(\mathfrak{g}, \mathfrak{g}')$ in degree q, and with differential from the q^{th} to the $(q+1)^{\text{th}}$ degree given by $d_{\mathfrak{g}} \colon C^{q-1}(\mathfrak{g}, \mathfrak{g}') \to C^q(\mathfrak{g}, \mathfrak{g}')$ as in (4). We note the following:

Proposition 10. *The assignment $\psi \mapsto \hat{\psi}$ defines a morphism $C^\bullet_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R}) \to C^{\bullet-1}(\mathfrak{g}, \mathfrak{g}')$ of cochain complexes. Moreover, the induced map $H^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R}) \to H^1(\mathfrak{g}, \mathfrak{g}')$ is injective.*

Proof. That $\psi \mapsto \hat{\psi}$ defines a morphism $C^\bullet_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R}) \to C^{\bullet-1}(\mathfrak{g}, \mathfrak{g}')$ of cochain complexes follows from a straightforward computation using (4). For the final assertion, note that $\psi \mapsto \hat{\psi}$ is injective and restricts to an isomorphism $C^1_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R}) \to C^0(\mathfrak{g}, \mathfrak{g}')$. This implies that a 2-cocycle $\psi \in C^2_{\mathrm{ct}}(\mathfrak{g}, \mathbb{R})$ is a coboundary if and only if $\hat{\psi} \in C^1(\mathfrak{g}, \mathfrak{g}')$ is a coboundary □

So if $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is a continuous 2-cocycle on $\mathcal{X}_c(M)$ with trivial coefficients, then $\hat{\psi} \in C^1(\mathcal{X}_c(M), \mathcal{X}_c(M)')$ is a 1-cocycle with values in the continuous dual space $\mathcal{X}_c(M)'$, and

$$\hat{\psi}([v, w]) = v \cdot \hat{\psi}(w) - w \cdot \hat{\psi}(v) \tag{10}$$

for all $v, w \in \mathcal{X}_c(M)$.

Lemma 11. *Let $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M))$ be a 2-cocycle. Then $\hat{\psi} \in C^1(\mathcal{X}_c(M), \mathcal{X}_c(M)')$ extends to a 1-cocycle $\hat{\psi} \in C^1(\mathcal{X}(M), \mathcal{X}_c(M)')$.*

Proof. Let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion of M by compact subsets. For $v \in \mathcal{X}(M)$ and $i \in \mathbb{N}$, we define $\hat{\psi}_i(v) \in \mathcal{X}_{K_i}(M)'$ by $\hat{\psi}_i(v)(w) := \psi(f_{K_i} v, w)$ for an arbitrary $f_{K_i} \in C^\infty_c(M)$ that satisfies $f_{K_i}(x) = 1$ for all x in some open neighborhood U_i of K_i. This is independent of f_{K_i} by Lemma 9. The various $\hat{\psi}_i(v)$ define an element $\hat{\psi}(v)$ of $\mathcal{X}_c(M)'$. Indeed, $\mathcal{X}_c(M)$ is the locally convex inductive limit $\mathcal{X}_c(M) = \varinjlim_i X_{K_i}(M)$, and the functionals $\hat{\psi}_i(v)$ are compatible in the sense that the restriction of $\psi_j(v)$ to $\mathcal{X}_{K_i}(M) \subseteq \mathcal{X}_{K_j}(M)$ coincides with $\psi_i(v)$ if $K_i \subseteq K_j$. The linear map $\hat{\psi} : \mathcal{X}(M) \to \mathcal{X}_c(M)'$ obtained in this way clearly extends the original map $\hat{\psi} : \mathcal{X}_c(M) \to \mathcal{X}_c(M)'$ and because ψ is diagonal, it satisfies the cocycle identity

$$\hat{\psi}([v, w]) = v \cdot \hat{\psi}(w) - w \cdot \hat{\psi}(v) \tag{11}$$

for the action of $\mathcal{X}(M)$ on $\mathcal{X}_c(M)'$ by $(v \cdot \phi)(u) = \phi([-v, u])$. □

Remark 8. Because the continuous 2-cocycle ψ is diagonal (Lemma 9), the map $\hat{\psi}$: $\mathcal{X}(M) \to \mathcal{X}_c(M)'$ from Lemma 11 is *support decreasing*, in the sense that $\mathrm{supp}(\hat{\psi}(v)) \subseteq \mathrm{supp}(v)$ for any $v \in \mathcal{X}(M)$. By Peetre's Theorem, we conclude that the linear map $\hat{\psi} : \mathcal{X}(M) \to \mathcal{X}_c(M)'$ is a differential operator of locally finite degree. This follows from [87, Thm. 1]; the locally finite set of discontinuous points for $\hat{\psi} \colon \mathcal{X}(M) \to \mathcal{X}_c(M)'$ is empty because $\psi \colon \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ is continuous.

4.1. Manifolds M of dimension $\dim(M) > 1$. We now proceed with the proof of Theorem B for manifolds of dimension $\dim(M) > 1$. Note that we will occasionally use Einstein summation convention, so repeated indices imply a sum.

4.1.1. The local setting We begin with the case where $M = \mathbb{R}^n$. In Lemma 13 below, we will show that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}^n), \mathbb{R}) = \{0\}$ for $n > 1$. This should be regarded as the local analog of Theorem B for $\dim(M) > 1$. The analogous statement in Gelfand–Fuks cohomology follows e.g. from [53, Thm. 3.12].

The following is an adaptation of a result in [55], and we thank Cornelia Vizman and Leonid Ryvkin for illuminating discussions on this topic.

We first make some preliminary observations. The Lie algebra $W_n \subseteq \mathcal{X}(\mathbb{R}^n)$ of vector fields with polynomial coefficients is $\mathbb{Z}$-graded, with W_n^k consisting of vector fields with homogeneous polynomial coefficients of degree $k+1$ for $k \geq -1$, and $W_n^k = \{0\}$ for $k < -1$. Since $[W_n^k, W_n^l] \subseteq W_n^{k+l}$, the constant vector fields W_n^{-1} decrease the degree by 1. Also, every W_n^k is a representation of the Lie algebra W_n^0 of linear vector fields, which we identify with $\mathfrak{gl}(n, \mathbb{R})$ via the isomorphism $\mathfrak{gl}(n, \mathbb{R}) \to W_n^0$ that maps $(A^\mu_\nu)^n_{\mu,\nu=1}$ to the linear vector field $a^\mu_\nu x^\nu \partial_\mu$ with constant coefficients $a^\mu_\nu = -A^\mu_\nu$. Under this identification, we have $W_n^k \cong S^{k+1}(\mathbb{R}^d)^* \otimes \mathbb{R}^d$ as $\mathfrak{gl}(n, \mathbb{R})$-representation for every $k \in \mathbb{N}_{\geq 0}$, where $S^{k+1}(\mathbb{R}^d)^*$ denotes the space of homogeneous polynomials on $\mathbb{R}^n$ of degree $k+1$. The Euler vector field $E = x^\mu \partial_\mu$ acts on $v \in W_n^k$ by $[E, v] = kv$.

Lemma 12. *The space $\mathcal{X}_c(\mathbb{R}^n)'$ of translation invariant elements is equivalent to the space $(\mathbb{R}^n)^* \otimes \wedge^n(\mathbb{R}^n)^*$ as a $\mathfrak{gl}(n, \mathbb{R})$-representation. In particular, the Euler vector field $E \in W_n^0$ acts on a translation-invariant $\phi \in \mathcal{X}_c(\mathbb{R}^n)'$ by*

$$E \cdot \phi = (n+1)\phi. \tag{12}$$

Proof. The linear vector field $a^\mu_\nu x^\nu \partial_\mu$ corresponding to $A = (A^\mu_\nu)^n_{\mu,\nu=1}$ acts on $\phi \in \mathcal{X}_c(\mathbb{R}^n)'$ according to

$$\begin{aligned}(a^\mu_\nu x^\nu \partial_\mu \cdot \phi)(u^\sigma \partial_\sigma) &= \phi(-a^\mu_\nu [x^\nu \partial_\mu, u^\sigma \partial_\sigma]) \\ &= -\phi(a^\mu_\nu x^\nu (\partial_\mu u^\sigma)\partial_\sigma) + \phi(a^\mu_\sigma u^\sigma \partial_\mu) \\ &= -\mathrm{tr}(A)\phi(u^\sigma \partial_\sigma) + \phi(a^\mu_\sigma u^\sigma \partial_\mu) + (\partial_\mu \cdot \phi)(a^\mu_\nu x^\nu u^\sigma \partial_\sigma),\end{aligned} \tag{13}$$

where the last equality uses $(\partial_\mu \cdot \phi)(a^\mu_\nu x^\nu u^\sigma \partial_\sigma) = -\phi(a^\mu_\nu u^\sigma \partial_\sigma)\delta^\nu_\mu - \phi(a^\mu_\nu x^\nu (\partial_\mu u^\sigma)\partial_\sigma)$. Assume now that ϕ is translation-invariant. Then $(\partial_\mu \cdot \phi)(a^\mu_\nu x^\nu u^\sigma \partial_\sigma) = 0$, and $\phi(u^\sigma \partial_\sigma) = b_\sigma I(u^\sigma)$ for some vector $b = (b_\sigma)^n_{\sigma=1} \in \mathbb{R}^n$, where $I(f) := \int_{\mathbb{R}^n} f dx$ for $f \in C^\infty_c(\mathbb{R}^n)$. It follows using (13) that

$$(a^\mu_\nu x^\nu \partial_\mu \cdot \phi)(u^\sigma \partial_\sigma) = \big(-\mathrm{tr}(A)b_\sigma + a^\mu_\sigma b_\mu\big) I(u^\sigma) = b'_\sigma I(u^\sigma),$$

where $b' := -\mathrm{tr}(A^T)b - A^T b$. This corresponds to the natural action of $\mathfrak{gl}(n, \mathbb{R})$ on $(\mathbb{R}^n)^* \otimes \wedge^n(\mathbb{R}^n)^*$ under the isomorphism $\mathfrak{gl}(n, \mathbb{R}) \cong W_n^0$ specified above, so the assertion follows. □

Lemma 13. *Let $n > 1$ be an integer. Then* $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}^n), \mathbb{R}) = \{0\}$.

Proof. Let ψ be a continuous 2-cocycle on $\mathcal{X}_c(\mathbb{R}^n)$, and let $\hat{\psi}$ be the corresponding 1-cocycle in $C^1(\mathcal{X}(\mathbb{R}^n), \mathcal{X}_c(\mathbb{R}^n)')$, obtained using Lemma 11. By Remark 8, we can expand $\hat{\psi}$ into a locally finite sum as

$$\hat{\psi}(v) = \sum_{\vec{\sigma} \in \mathbb{N}^n_{\geq 0}} \left(\frac{\partial^{|\vec{\sigma}|}}{\partial x^{\vec{\sigma}}} v^\mu \right) \phi^{\vec{\sigma}}_\mu, \tag{14}$$

where $\phi^{\vec{\sigma}}_\mu \in \mathcal{X}_c(\mathbb{R}^n)'$. Here $\frac{\partial^{|\vec{\sigma}|}}{\partial x^{\vec{\sigma}}} := (\frac{\partial}{\partial x^1})^{\sigma_1} \cdots (\frac{\partial}{\partial x^n})^{\sigma_n}$ is a higher order partial derivative. We show for any integer $k \geq -1$ that $\hat{\psi}$ is cohomologous to a 1-cocycle that vanishes on the subspace $W_n^{\leq k}$ of vector fields with polynomial coefficients of degree at most $k+1$.

The case $k = -1$. The cocycle identity (11) for constant vector fields $v = \partial_\mu$ and $w = \partial_\nu$ yields

$$\partial_\mu \cdot \phi^{\vec{0}}_\nu - \partial_\nu \cdot \phi^{\vec{0}}_\mu = 0. \tag{15}$$

We identify $\mathcal{X}_c(\mathbb{R}^n)' \simeq \mathcal{D}'(\mathbb{R}^n) \otimes (\mathbb{R}^n)^*$ with n copies of the distributions $\mathcal{D}'(\mathbb{R}^n)$ by setting

$$(\zeta_\sigma \otimes dx^\sigma)(v^\mu \partial_\mu) := \zeta_\sigma(v^\sigma), \qquad \text{for } \zeta_\sigma \in \mathcal{D}'(\mathbb{R}^n) \text{ and } v^\mu \in C^\infty_c(\mathbb{R}^n).$$

The action of ∂_μ on $\mathcal{X}_c(\mathbb{R}^n)'$ is then simply given by differentiating the components in $\mathcal{D}'(\mathbb{R}^n)$, so that for $\phi^{\vec{0}}_\nu = \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma$ we have $\partial_\mu \cdot \phi^{\vec{0}}_\nu = \partial_\mu \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma$. Indeed, we compute that

$$\begin{aligned}(\partial_\mu \cdot \phi^{\vec{0}}_\nu)(X^\tau \partial_\tau) &= \phi^{\vec{0}}_\nu(-[\partial_\mu, X^\tau \partial_\tau]) = \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma(-(\partial_\mu X^\tau)\partial_\tau) \\ &= \phi^{\vec{0}}_{\nu\sigma}(-(\partial_\mu X^\sigma)) = (\partial_\mu \phi^{\vec{0}}_{\nu\sigma})(X^\sigma) = (\partial_\mu \phi^{\vec{0}}_{\nu\sigma} \otimes dx^\sigma)(X^\tau \partial_\tau).\end{aligned}$$

Equation (15) therefore yields for each σ that $\partial_\mu \cdot \phi^{\vec{0}}_{\nu\sigma} - \partial_\nu \cdot \phi^{\vec{0}}_{\mu\sigma} = 0$ for all integers μ, ν ranging from 1 to n. With respect to the differential $d : \Omega^{n-p}_c(\mathbb{R}^n)' \to \Omega^{n-(p+1)}_c(\mathbb{R}^n)'$, $\langle dT, \alpha \rangle := (-1)^{p+1} \langle T, d\alpha \rangle$ on the space of currents (cf. [88, III§11]), this means precisely that the current $c_\sigma := \phi^{\vec{0}}_{\mu\sigma} dx^\mu \in \Omega^{n-1}_c(\mathbb{R}^n)'$ is closed, because

$$dc_\sigma = \sum_{1 \leq \mu < \nu \leq n} (\partial_\mu \cdot \phi^{\vec{0}}_{\nu\sigma} - \partial_\nu \cdot \phi^{\vec{0}}_{\mu\sigma}) dx^\mu \wedge dx^\nu = 0.$$

(Identifying $\mathcal{D}'(\mathbb{R}^n) \cong \Omega^n_c(\mathbb{R}^n)'$ using the volume form $dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$, for any $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha \in \Omega^p(\mathbb{R}^n)$, we interpret $T\alpha$ as element of $\Omega^{n-p}_c(\mathbb{R}^n)'$ via the pairing $\langle T\alpha, \beta \rangle := T(\alpha \wedge \beta)$ for $\beta \in \Omega^{n-p}_c(\mathbb{R}^n)$, cf. [88, p. 36].) By the Poincaré Lemma for currents [88, IV§19], it follows that there exist distributions $\eta_\sigma \in \mathcal{D}'(\mathbb{R}^n)$ with $\partial_\mu \eta_\sigma = \phi^{\vec{0}}_{\mu\sigma}$ for all integers $1 \leq \mu, \sigma \leq n$. The 1-coboundary $d_{\mathfrak{g}}(\eta_\sigma \otimes dx^\sigma)$ in $C^1(\mathcal{X}(\mathbb{R}^n), \mathcal{X}_c(\mathbb{R}^n)')$ thus agrees with $\hat{\psi}$ on $\partial_\mu \in W_n^{-1}$:

$$d_{\mathfrak{g}}(\eta_\sigma \otimes dx^\sigma)(\partial_\mu) = \partial_\mu \cdot (\eta_\sigma \otimes dx^\sigma) = \partial_\mu \eta_\sigma \otimes dx^\sigma = \phi^{\vec{0}}_{\mu\sigma} \otimes dx^\sigma = \phi^{\vec{0}}_\mu = \hat{\psi}(\partial_\mu)$$

Replacing $\hat{\psi}$ by the 1-cocycle $\hat{\psi} - d_{\mathfrak{g}}(\eta_\sigma \otimes dx^\sigma)$, we assume from now on that $\hat{\psi}$ vanishes on W_n^{-1}.

The case $0 \leq k \leq n$. Suppose that $\hat{\psi}$ vanishes on $W_n^{\leq(k-1)}$. Let $v \in W_n^k$. Since $[\partial_\mu, v] \in W_n^{k-1}$, the cocycle identity (11) yields $\partial_\mu \cdot \hat{\psi}(v) = 0$ for all μ, so that $\hat{\psi}(v)$ is translation invariant. So $E \cdot \hat{\psi}(v) = (n+1)\hat{\psi}(v)$, in view of Lemma 12. From the cocycle identity $\hat{\psi}([E, v]) = E \cdot \hat{\psi}(v) - v \cdot \hat{\psi}(E)$, we find for any $v \in W_n^k$ that

$$(n + 1 - k)\hat{\psi}(v) = v \cdot \hat{\psi}(E). \tag{16}$$

We consider separately the cases $k = 0$ and $0 < k \leq n$. Suppose that $k = 0$. The preceding then shows that $\hat{\psi}(v) = \frac{1}{n+1} v \cdot \hat{\psi}(E)$ for all $v \in W_n^0$. The 0-cochain $\eta = \frac{1}{n+1}\hat{\psi}(E)$ therefore satisfies $(d_{\mathfrak{g}}\eta)(v) = \hat{\psi}(v)$ for any $v \in W_n^0$. Since $E \in W_n^0$, we know that $\hat{\psi}(E)$ is translation-invariant, so we also have $(d_{\mathfrak{g}}\eta)(v) = \frac{1}{n+1} v \cdot \hat{\psi}(E) = 0$ for $v \in W_n^{-1}$. Replacing $\hat{\psi}$ by the cohomologous cocycle $\hat{\psi} - d_{\mathfrak{g}}\eta$ if necessary, we may assume that $\hat{\psi}$ vanishes on $W_n^{\leq 0}$. Suppose next that $0 < k \leq n$. Then $E \in W_n^{\leq(k-1)}$, so $\hat{\psi}(E) = 0$. Consequently, (16) implies that $\hat{\psi}(v) = 0$ for any $v \in W_n^k$ and hence $\hat{\psi}$ vanishes on $W_n^{\leq k}$. Inductively, we thus find that $\hat{\psi}$ vanishes on $W_n^{\leq n}$, and that $\hat{\psi}(v)$ is translation invariant for any $v \in W_n^{n+1}$.

The case $k = n + 1$. The cocycle identity (11) for $A \in W_n^0$ and $v \in W_n^{n+1}$ reads $\hat{\psi}([A, v]) = A \cdot \hat{\psi}(v)$, because $\hat{\psi}(A) = 0$. Since $\hat{\psi}(v)$ is translation invariant for any $v \in W_n^{n+1}$, we conclude using Lemma 12 that the linear map

$$\hat{\psi}\Big|_{W_n^{n+1}} : W_n^{n+1} \to (\mathbb{R}^n)^* \otimes \wedge^n(\mathbb{R}^n)^* \subseteq \mathcal{X}_c(\mathbb{R}^n)'$$

is an intertwiner of $\mathfrak{gl}(n, \mathbb{R})$-representations. The action of $\mathfrak{sl}(n, \mathbb{R})$ on $\wedge^n(\mathbb{R}^n)^*$ is trivial, and we notice that

$$\mathrm{Hom}_{\mathfrak{sl}(n,\mathbb{R})}\left(W_n^{n+1}, (\mathbb{R}^n)^*\right) \cong \mathrm{Hom}_{\mathfrak{sl}(n,\mathbb{R})}\left(S^{n+2}(\mathbb{R}^n)^*, (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*\right) = 0,$$

because $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \cong S^2(\mathbb{R}^n)^* \oplus \bigwedge^2(\mathbb{R}^n)^*$ does not contain the irreducible $\mathfrak{sl}(n, \mathbb{R})$-representation on $S^{n+2}(\mathbb{R}^n)^*$ (cf. [89, Prop. 15.15]). So $\hat{\psi}(v) = 0$ for any $v \in W_n^{n+1}$.

The case $k > n + 1$. Suppose that $\hat{\psi}$ vanishes on $W_n^{\leq(k-1)}$ for $k > n + 1$. Then (16) implies that $\hat{\psi}(v) = 0$ for any $v \in W_n^k$, so $\hat{\psi}$ vanishes on $W_n^{\leq k}$. Inductively, we thus find that $\hat{\psi}$ vanishes on $W_n^{\leq k}$ for any integer $k \geq -1$. This implies that all the coefficients $\phi_\mu^{\vec{\sigma}}$ in equation (14) are zero, so $\hat{\psi} = 0$ and hence $\psi = 0$. □

4.1.2. A local-to-global argument Having established that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}^n), \mathbb{R}) = \{0\}$, we now show that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ vanishes for general manifolds M of dimension greater than 1, using a local-to-global argument. This completes the proof of the first part of Theorem B.

We let $\mathcal{X}_c'$ denote the presheaf defined by $U \mapsto \mathcal{X}_c(U)'$ and the natural restriction maps. This is in fact an acyclic sheaf by Proposition 19.

Lemma 14. *Assume that* $\dim(M) > 1$. *Then* $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \{0\}$.

Proof. Let $n := \dim(M)$. The continuous Chevalley–Eilenberg cochains define a presheaf $U \mapsto C^m_{\mathrm{ct}}(\mathcal{X}_c(U), \mathbb{R})$ for any $m \in \mathbb{N}$, that we denote by $C^m_{\mathrm{ct}}(\mathcal{X}_c)$. We denote by $Z^m_{\mathrm{ct}}(\mathcal{X}_c) \subseteq C^m_{\mathrm{ct}}(\mathcal{X}_c)$ its sub-presheaf consisting of cocycles. Let $\mathcal{U} = \{U_i \,:\, i \in S\}$ be an open cover of M such that every U_i is diffeomorphic to $\mathbb{R}^n$, and consider the (augmented) double complex $\check{C}^\bullet(\mathcal{U}, C^\bullet_{\mathrm{ct}}(\mathcal{X}_c))$ for the Čech-cohomology with coefficients in the presheaf $C^\bullet_{\mathrm{ct}}(\mathcal{X}_c)$. Restricted to cocycles in Chevalley–Eilenberg degree 2, the left lower portion looks as follows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Z^2_{\mathrm{ct}}(\mathcal{X}_c(M)) & \xrightarrow{\check{\delta}} & \prod_{i\in S} Z^2_{\mathrm{ct}}(\mathcal{X}_c(U_i)) & \xrightarrow{\check{\delta}} & \prod_{i,j\in S} Z^2_{\mathrm{ct}}(\mathcal{X}_c(U_i \cap U_j)) \\
 & & d_{\mathfrak{g}}\uparrow & & d_{\mathfrak{g}}\uparrow & & d_{\mathfrak{g}}\uparrow \\
0 & \longrightarrow & C^1_{\mathrm{ct}}(\mathcal{X}_c(M)) & \xrightarrow{\check{\delta}} & \prod_{i\in S} C^1_{\mathrm{ct}}(\mathcal{X}_c(U_i)) & \xrightarrow{\check{\delta}} & \prod_{i,j\in S} C^1_{\mathrm{ct}}(\mathcal{X}_c(U_i \cap U_j)) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0.
\end{array}$$

The middle column is exact by Lemma 13, as every $U_i \in \mathcal{U}$ is diffeomorphic to $\mathbb{R}^n$, and the column on the right is exact at $\prod_{i,j\in S} C^1_{\mathrm{ct}}(\mathcal{X}_c(U_i \cap U_j))$ as $\mathcal{X}_c(U_i \cap U_j)$ is perfect for any $i, j \in S$ [83, Thm. 1.4.3]. The bottom row is exact because $C^1_{\mathrm{ct}}(\mathcal{X}_c) = \mathcal{X}'_c$ is an acyclic sheaf, by Proposition 19. Lemma 9 further guarantees that the map $\check{\delta}\colon Z^2_{\mathrm{ct}}(\mathcal{X}_c(M)) \to \prod_{i\in S} Z^2_{\mathrm{ct}}(\mathcal{X}_c(U_i))$ is injective. Indeed, suppose that $\psi(\mathcal{X}_c(U_i), \mathcal{X}_c(U_i)) = \{0\}$ for all $i \in S$. Then $\psi(X_c(U_i), \mathcal{X}_c(M)) = \{0\}$ for any $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M))$ and $i \in S$, because ψ is diagonal. So $\psi = 0$, by a partition of unity argument. A straightforward diagram chase now shows that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ vanishes. □

4.2. Manifolds M of dimension one. Having proven Theorem B for manifolds of dimension greater than 1, we proceed with the remaining case, and determine $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ for manifolds of dimension 1. In the connected case, M must be diffeomorphic to either $\mathbb{R}$ or S^1. It is well-known that $H^2_{\mathrm{ct}}(\mathcal{X}(S^1), \mathbb{R}) = \mathbb{R}$ is spanned by the class of the Virasoro cocycle (cf. [90, Prop. 2.3]):

$$\psi_{\mathrm{vir}}(f\partial_\theta, g\partial_\theta) = \int_{S^1} f'''(\theta)g(\theta)d\theta, \qquad f, g \in C^\infty(S^1).$$

A slight adaptation of the proof of Lemma 13 allows us to prove the analogous result on the real line:

Lemma 15. *The second Lie algebra cohomology* $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R})$ *is* 1*-dimensional. It is spanned by the class of the Virasoro cocycle*

$$\psi_{\mathrm{vir}}(f\partial_x, g\partial_x) = \int_{\mathbb{R}} f'''(x)g(x)dx, \qquad f, g \in C^\infty_c(\mathbb{R}). \tag{17}$$

Proof. Let us first observe that the cocycle

$$\psi_{\mathrm{vir}}(f\partial_x, g\partial_x) = \int_{\mathbb{R}} f'''(x)g(x)dx$$

is not a coboundary. Indeed, if $\eta \in C^1_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R})$ is a 1-cochain, then the map $\widehat{d_{\mathfrak{g}}\eta} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}_c(\mathbb{R})'$ obtained using Lemma 11 is the first-order differential operator $\widehat{d_{\mathfrak{g}}\eta}(f\partial_x) = f(\partial_x \cdot \eta) + 2f'\eta$. Indeed, this follows from the calculation

$$\begin{aligned}\widehat{d_{\mathfrak{g}}\eta}(f\partial_x)(g\partial_x) &= -\eta([f\partial_x, g\partial_x]) = \eta(f'g\partial_x - fg'\partial_x)\\ &= 2\eta(f'g\partial_x) - \eta((fg)'\partial_x) = (2f'\eta + f(\partial_x \cdot \eta))(g\partial_x)\end{aligned}$$

for $f \in C^\infty(\mathbb{R})$ and $g \in C^\infty_c(\mathbb{R})$. On the other hand, $\hat{\psi}_{\mathrm{vir}}$ is the third-order differential operator $\hat{\psi}_{\mathrm{vir}}(f\partial_x) = f'''I$, where $I \in \mathcal{X}_c(\mathbb{R})'$ is defined by $I(f\partial_x) = \int_{\mathbb{R}} f(x)dx$. So ψ_{vir} can not be a coboundary.

Let ψ be a continuous 2-cocycle. Let $\hat{\psi} \in C^1(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ be the corresponding 1-cocycle obtained using Lemma 11. We show that $\hat{\psi}$ is cohomologous to a 1-cocycle in $C^1(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ that vanishes on the subspace $W_1^{-1} = \mathbb{R}\partial_x$. Choose $\chi \in C^\infty_c(\mathbb{R})$ with $\int_{\mathbb{R}} \chi(x)dx = 1$. For any $f \in C^\infty_c(\mathbb{R})$, the function

$$P(f)(x) := \int_{-\infty}^{x} f(s) - I(f\partial_x)\chi(s)ds$$

is smooth and compactly supported. We moreover have $P(f') = f$, because $I(f'\partial_x) = 0$. Observe that the 0-cochain $\eta \in \mathcal{X}_c(\mathbb{R})' = C^0(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ defined by $\eta(f\partial_x) := \hat{\psi}(\partial_x)(P(f)\partial_x)$ satisfies $\hat{\psi}(\partial_x) + (d_{\mathfrak{g}}\eta)(\partial_x) = 0$, because

$$(d_{\mathfrak{g}}\eta)(\partial_x)(f\partial_x) = -\eta(f'\partial_x) = -\hat{\psi}(\partial_x)(P(f')\partial_x) = -\hat{\psi}(\partial_x)(f\partial_x), \quad \forall f \in C^\infty_c(\mathbb{R}).$$

Replacing $\hat{\psi}$ by $\hat{\psi} + d_{\mathfrak{g}}\eta$, we assume from now on that $\hat{\psi}$ vanishes on $W_1^{-1} = \mathbb{R}\partial_x$. Following the case $0 \leq k \leq n$ in the proof of Lemma 13, we may then further assume that $\hat{\psi}$ vanishes on $W_1^{\leq 1}$ and that $\hat{\psi}(x^3\partial_x) \in \mathcal{X}_c(\mathbb{R})'$ is translation invariant. The latter implies that $\hat{\psi}(x^3\partial_x) = cI$ for some constant $c \in \mathbb{R}$. It follows that the 1-cocycle $\hat{\psi} - c\hat{\psi}_{\mathrm{vir}} \in C^1(\mathcal{X}(\mathbb{R}), \mathcal{X}_c(\mathbb{R})')$ vanishes on $W_1^{\leq 2}$. Following the case $k > n+1$ in the proof of Lemma 13, this implies that $\hat{\psi} - c\hat{\psi}_{\mathrm{vir}}$ vanishes on $W_1^{\leq k}$ for any integer $k \geq -1$ and therefore that $\hat{\psi} = c\hat{\psi}_{\mathrm{vir}}$. Hence $\psi = c\psi_{\mathrm{vir}}$. □

We have thus shown that $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) \cong \mathbb{R}$ for any connected 1-dimensional manifold. Combined with Lemma 14, the following now completes the proof of Theorem B.

Lemma 16. *Let M be a smooth manifold of dimension* 1. *Then*

$$H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = H^0_{\mathrm{dR}}(M).$$

Proof. Let $\{M_\alpha\}_{\alpha \in \mathcal{I}}$ be the set of connected components of M, where $\mathcal{I}$ is some countable indexing set. (Here we used that M is second-countable.) As the support of a compactly supported vector field on M intersects only finitely many M_α non-trivially, $\mathcal{X}_c(M)$ is isomorphic to the locally convex direct sum $\mathcal{X}_c(M) \cong \bigoplus_{\alpha \in \mathcal{I}} \mathcal{X}_c(M_\alpha)$. Hence $\mathcal{X}_c(M)' \cong \prod_{\alpha \in \mathcal{I}} \mathcal{X}_c(M_\alpha)'$. Furthermore, any 2-cocycle $\psi : \mathcal{X}_c(M) \times \mathcal{X}_c(M) \to \mathbb{R}$ is diagonal by Lemma 9, and therefore decomposes as $\psi = \sum_\alpha \psi_\alpha$ for some 2-cochains ψ_α on $\mathcal{X}_c(M_\alpha)$. Moreover, ψ is a cocycle (or a coboundary) if and only if every ψ_α is so. It follows that

$$H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = \prod_{\alpha \in \mathcal{I}} H^2_{\mathrm{ct}}(\mathcal{X}_c(M_\alpha), \mathbb{R}) = \prod_{\alpha \in \mathcal{I}} \mathbb{R} \cong H^0_{\mathrm{dR}}(M).$$

□

4.3. Relation with Gelfand–Fuks cohomology. Finally, let us consider the relationship between $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ and the Gelfand–Fuks cohomology $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$. The continuous injection $\mathcal{X}_c(M) \hookrightarrow \mathcal{X}(M)$ induces a natural morphism $C^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to C^\bullet_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ of cochain complexes, which descends to a linear map $H^\bullet_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^\bullet_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ on cohomology. We show that this map is injective in degree 2. We also show that this map is in general not surjective, so that the continuous cohomology of the compactly supported vector fields is different from Gelfand–Fuks cohomology.

If $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is a diagonal 2-cochain, its *support* $\mathrm{supp}(\psi)$ is the set of points $x \in M$ with the property that for any neighborhood U of x, there exist $v, w \in \mathcal{X}_c(U)$ with $\psi(v, w) \neq 0$. If $x \notin \mathrm{supp}(\psi)$ and U is a neighborhood of x with $\psi(\mathcal{X}_c(U), \mathcal{X}_c(U)) = \{0\}$, then $\psi(\mathcal{X}_c(U), \mathcal{X}_c(M)) = \{0\}$, because ψ is diagonal. The following is a straightforward adaptation of [54, Lem. 4.19] to the present setting:

Proposition 17. *Let M be a smooth manifold.*

1. *A continuous 2-cocycle $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ extends to a continuous 2-cocycle on $\mathcal{X}(M)$ if and only if it has compact support.*
2. *Assume that the 2-cocycle $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ has compact support and satisfies $\psi = d_{\mathfrak{g}}\eta$ for some $\eta \in \mathcal{X}_c(M)'$. Then $\mathrm{supp}(\eta) = \mathrm{supp}(\psi)$ and η extends to a continuous linear map $\mathcal{X}(M) \to \mathbb{R}$.*
3. *The canonical linear map $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is injective.*

Proof. 1. Assume that ψ has compact support, say $\mathrm{supp}(\psi) = K$. Consider the 1-cocycle $\hat{\psi} \in C^1(\mathcal{X}(M), \mathcal{X}_c(M)')$ obtained from Lemma 11. Let $\chi \in C^\infty_c(M)$ satisfy $\chi|_U = 1$ for some open neighborhood U of K. Define a bilinear map $\widetilde{\psi} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathbb{R}$ extending ψ by setting $\widetilde{\psi}(v, w) := \hat{\psi}(v)(\chi w) = \psi(\chi v, \chi w)$. This is independent of the choice of χ because ψ has support K. It is moreover continuous, in view of the continuity of both ψ and the map $\mathcal{X}(M) \to \mathcal{X}_c(M)$, $w \mapsto \chi w$. We next show that $\widetilde{\psi}$ is a 2-cocycle. Observe that $\hat{\psi}(u)(\chi[v, w]) = \hat{\psi}(u)([v, \chi w])$ for any $u, v, w \in \mathcal{X}(M)$, because $\mathcal{L}_v(\chi)w \in \mathcal{X}_c(M)$ vanishes on a neighborhood of K, so that $\hat{\psi}(u)(\mathcal{L}_v(\chi)w) = 0$. Using (11), we therefore have

$$
\begin{aligned}
\widetilde{\psi}([u, v], w) + \widetilde{\psi}(v, [u, w]) &= \hat{\psi}([u, v])(\chi w) + \hat{\psi}(v)(\chi[u, w]) \\
&= \hat{\psi}(u)([v, \chi w]) - \hat{\psi}(v)([u, \chi w]) + \hat{\psi}(v)(\chi[u, w]) \\
&= \hat{\psi}(u)(\chi[v, w]) \\
&= \widetilde{\psi}(u, [v, w]).
\end{aligned}
$$

Conversely, assume that ψ extends to a continuous 2-cocycle on $\mathcal{X}(M)$, again denoted ψ. Suppose that $K := \mathrm{supp}(\psi)$ is not compact. Then we can find a countably infinite sequence $(x_i)_{i\in\mathbb{N}}$ in K of distinct points which has no convergent subsequence. Let $\{U_i\}_{i\in\mathbb{N}}$ be a collection of pairwise disjoint open subsets of M so that $x_i \in U_i$ for all $i \in \mathbb{N}$. Since $x_i \in K$, there exist for every $i \in \mathbb{N}$ some $v_i, w_i \in \mathcal{X}_c(U_i)$ satisfying $\psi(v_i, w_i) = 1$. Notice that $v := \sum_{i=1}^\infty v_i$ and $w := \sum_{i=1}^\infty w_i$ are well-defined smooth vector fields on M, because the open sets U_i are pairwise disjoint. Since $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ is diagonal and continuous, we obtain the evident contradiction that

$$
\lim_{N\to\infty} N = \lim_{N\to\infty} \sum_{i=1}^{N} \psi(v_i, w_i) = \lim_{N\to\infty} \psi\left(\sum_{i=1}^{N} v_i, \sum_{i=1}^{N} w_i\right) = \psi(v, w) < \infty.
$$

So $\mathrm{supp}(\psi)$ must be compact.

2. Let $x \notin \mathrm{supp}(\psi)$. Then there exists an open neighborhood U of x such that $\psi(\mathcal{X}_c(U), \mathcal{X}_c(U)) = \{0\}$. Let $u \in \mathcal{X}_c(U)$. Since $\mathcal{X}_c(U)$ is perfect ([83, Thm. 1.4.3]), there exist $N \in \mathbb{N}$ and $v_i, w_i \in \mathcal{X}_c(U)$ for $i \in \{1, \ldots, N\}$ s.t. $u = \sum_{i=1}^N [v_i, w_i]$. Then $\eta(u) = \sum_{i=1}^N \eta([v_i, w_i]) = -\sum_{i=1}^N \psi(v_i, w_i) = 0$. Thus η vanishes on $\mathcal{X}_c(U)$, and $x \notin \mathrm{supp}(\eta)$. It follows that $\mathrm{supp}(\eta) \subseteq \mathrm{supp}(\psi)$. Conversely, suppose that $x \notin \mathrm{supp}(\eta)$. Then there exists an open neighborhood U of x such that $\eta(\mathcal{X}_c(U)) = \{0\}$. Then $\psi = d_{\mathfrak{g}}\eta$ implies that $\psi(\mathcal{X}_c(U), \mathcal{X}_c(U)) = \{0\}$. Thus $x \notin \mathrm{supp}(\psi)$. Hence $\mathrm{supp}(\eta) = \mathrm{supp}(\psi) =: K$. As η has compact support K, it admits a continuous linear extension $\widetilde{\eta}$ to $\mathcal{X}(M)$ by setting $\widetilde{\eta}(v) := \eta(\chi v)$ for any $\chi \in C_c^\infty(M)$ satisfying $\chi|_V = 1$ for some open neighborhood V of K. Notice that $\widetilde{\eta}$ is indeed well-defined and continuous.
3. Let $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$ be a 2-cocycle and assume that $\eta \in \mathcal{X}_c(M)'$ satisfies $\psi(v, w) = -\eta([v, w])$ for all $v, w \in \mathcal{X}_c(M)$. The previous items ensure that η extends to a continuous functional on $\mathcal{X}(M)$. As $\mathcal{X}_c(M)$ is dense in $\mathcal{X}(M)$ and ψ is continuous on $\mathcal{X}(M) \times \mathcal{X}(M)$, it follows that $\psi(v, w) = -\eta([v, w])$ for all $v, w \in \mathcal{X}(M)$. So $\psi = d_{\mathfrak{g}}\eta$. Hence $[\psi] = 0$ in $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R})$. □

Proposition 17, Lemma 14 and Lemma 15 have the following consequence for Gelfand–Fuks cohomology:

Corollary 18. *Let M be a smooth manifold.*

1. *If* $\dim(M) > 1$, *then* $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = 0$.
2. *If* $\dim(M) = 1$, *then* $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = H^0_c(M)$ *is the compactly supported de Rham cohomology of M in degree 0. In particular,* $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R}) = 0$.

Proof. 1. Assume that $\dim(M) > 1$. Then $H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R}) = 0$ by Theorem B. We know using Proposition 17 that the linear map $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) \to H^2_{\mathrm{ct}}(\mathcal{X}_c(M), \mathbb{R})$ is injective. It follows that $H^2_{\mathrm{ct}}(\mathcal{X}(M), \mathbb{R}) = 0$.

2. By reasoning similar to that in the proof of Lemma 16, it suffices to consider the case where M is connected, so that M is either S^1 or $\mathbb{R}$. Since $H^2_{\mathrm{ct}}(\mathcal{X}(S^1), \mathbb{R}) \cong \mathbb{R}$ [90, Prop. 2.3], it remains to show that $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R}) = 0$. By Lemma 15 we know that $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R}) \cong \mathbb{R}$, which by Proposition 17 implies that $H^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R})$ is at most one-dimensional. The non-trivial class in $H^2_{\mathrm{ct}}(\mathcal{X}_c(\mathbb{R}), \mathbb{R})$ is spanned by the cocycle ψ_{vir}, defined by (17). Assume that $\psi \in C^2_{\mathrm{ct}}(\mathcal{X}(\mathbb{R}), \mathbb{R})$ is a 2-cocycle on $\mathcal{X}(\mathbb{R})$ whose restriction $r(\psi)$ to $\mathcal{X}_c(M) \times \mathcal{X}_c(M)$ is cohomologous to ψ_{vir}. Then $r(\psi) = \psi_{\mathrm{vir}} + d_{\mathfrak{g}}\eta$ for some $\eta \in \mathcal{X}_c(\mathbb{R})'$. By Proposition 17, we know that $r(\psi)$ has compact support. Consider the associated map $\widehat{r(\psi)} : \mathcal{X}(\mathbb{R}) \to \mathcal{X}_c(\mathbb{R})'$. We saw in the proof of Lemma 15 that $\widehat{d_{\mathfrak{g}}\eta}(f\partial_x) = f(\partial_x \cdot \eta) + 2f'\eta$, and that $\hat{\psi}_{\mathrm{vir}}(f\partial_x) = f'''I$, where $I(f\partial_x) := \int_{\mathbb{R}} f(x)dx$. So $\widehat{r(\psi)}$ is the differential operator given by

$$\widehat{r(\psi)}(f\partial_x) = f'''I + f(\partial_x \cdot \eta) + 2f'\eta. \tag{18}$$

Since $\widehat{r(\psi)}(f\partial_x) \in \mathcal{X}_c(\mathbb{R})'$ has compact support for any $f \in C^\infty(\mathbb{R})$, we obtain by taking $f = 1$ in (18) that $\partial_x \cdot \eta$ has compact support. Choosing subsequently $f(x) = x$ in (18), it follows that η has compact support, and hence so does $\hat{\psi}_{\mathrm{vir}} = \widehat{r(\psi)} - \widehat{d_{\mathfrak{g}}\eta}$. But the support of $\hat{\psi}_{\mathrm{vir}}$ is all of $\mathbb{R}$, which is not compact, a clear contradiction. □

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Declarations

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Appendix A: Sheaves of Distributions

Let $E \to M$ be a smooth vector bundle over the smooth manifold M. If $U \subseteq M$ is an open subset, we denote by $\Gamma_c(U, E)$ the locally convex vector space of smooth compactly supported sections of $E|_U \to U$, equipped with the natural LF-topology. Let $\Gamma_c(U, E)'$ denote its continuous dual space. It is clear that the assignment $U \mapsto \Gamma_c(U, E)'$ defines a presheaf Γ_c' w.r.t. the natural restriction maps. In the following, we show that Γ_c' actually defines an acyclic sheaf. Since we are considering the continuous dual space, we have to slightly extend the usual sheaf-theoretic arguments (such as [91, §V.1 Prop. 1.6 and 1.10]).

Proposition 19. *Γ_c' is an acyclic sheaf.*

Proof. Let $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ be a collection of open subsets of M and define $U := \bigcup_{\alpha\in\mathcal{I}} U_\alpha$. Let $\{\chi_\alpha\}_{\alpha\in\mathcal{I}}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of U [86, Thm. 2.23]. Notice for $s \in \Gamma_c(U, E)$ that $\chi_\alpha s$ is non-zero for only finitely many $\alpha \in \mathcal{I}$, because $\{\operatorname{supp} \chi_\alpha\}_{\alpha\in\mathcal{I}}$ is locally finite. To see that Γ_c' satisfies the locality axiom, suppose that $\lambda \in \Gamma_c(U, E)'$ satisfies $\lambda_\alpha := \lambda|_{\Gamma_c(U_\alpha, E)} = 0$ for all $\alpha \in \mathcal{I}$. Then $\lambda(s) = \sum_{\alpha\in\mathcal{I}} \lambda_\alpha(\chi_\alpha s) = 0$ for any $s \in \Gamma_c(U, E)$, so $\lambda = 0$. For the gluing axiom, take $\lambda_\alpha \in \Gamma_c(U_\alpha, E)'$ for all $\alpha \in \mathcal{I}$ and suppose for any $\alpha, \beta \in \mathcal{I}$ that the restrictions of λ_α and λ_β to $\Gamma_c(U_\alpha \cap U_\beta, E)$ coincide whenever $U_\alpha \cap U_\beta \neq 0$. Define $\lambda \in \Gamma_c(U, E)'$ by $\lambda(s) := \sum_{\alpha\in\mathcal{I}} \lambda_\alpha(\chi_\alpha s)$ for $s \in \Gamma_c(U, E)$. Notice that λ does indeed define a continuous functional on the LF-space $\Gamma_c(U, E)$ because $\{\operatorname{supp} \chi_\alpha\}_{\alpha\in\mathcal{I}}$ is locally finite. If $s \in \Gamma_c(U_\beta, E)$ for some $\beta \in \mathcal{I}$, then $\chi_\alpha s \in \Gamma_c(U_\alpha \cap U_\beta, E)$ and consequently $\lambda_\alpha(\chi_\alpha s) = \lambda_\beta(\chi_\alpha s)$ for any $\alpha \in \mathcal{I}$. Hence $\lambda(s) = \sum_\alpha \lambda_\alpha(\chi_\alpha s) = \sum_\alpha \lambda_\beta(\chi_\alpha s) = \lambda_\beta(s)$. So $\lambda|_{\Gamma_c(U_\beta, E)} = \lambda_\beta$ for any $\beta \in \mathcal{I}$. It follows that Γ_c' is a sheaf. We show next that it is fine (cf. [92, Def. II.3.3]). Assume henceforth that $U = M$. Define for any open set $V \subseteq M$ and $\alpha \in \mathcal{I}$ the linear map $\eta_\alpha : \Gamma_c(V, E)' \to \Gamma_c(V, E)'$ by $\eta_\alpha(\lambda)(s) := \lambda(\chi_\alpha s)$. This defines a morphism $\eta_\alpha : \Gamma_c' \to \Gamma_c'$ of sheaves. Since $\sum_\alpha \eta_\alpha(\lambda)(s) = \sum_\alpha \lambda(\chi_\alpha s) = \lambda(s)$ for any

$s \in \Gamma_c(V, E)'$, the sum being finite, we have $\sum_\alpha \eta_\alpha = 1$. Additionally, η_α vanishes on the stalk of the sheaf Γ_c' at x for any x in the open neighborhood $M \setminus \operatorname{supp} \chi_\alpha$ of $M \setminus U_\alpha$. So Γ_c' is fine and therefore acyclic [92, II. Prop. 3.5 and Thm. 3.11]. □

Appendix B: Unitary Equivalence of Projective Representations

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$.

Definition 11. Suppose for $k \in \{1, 2\}$ that $\mathcal{D}_k$ is a complex pre-Hilbert space, and let $\overline{\pi}_k : \mathfrak{g} \to \mathfrak{pu}(\mathcal{D}_k)$ be a projective unitary representation of $\mathfrak{g}$ on $\mathcal{D}_k$. We say that $\overline{\pi}_1$ and $\overline{\pi}_2$ are unitarily equivalent if there exists a unitary operator $U : \mathcal{D}_1 \to \mathcal{D}_2$ such that $\overline{\pi}_2(\xi) = \overline{U}\,\overline{\pi}_1(\xi)\,\overline{U}^{-1}$ for all $\xi \in \mathfrak{g}$, where $\overline{U} : \mathrm{P}(\mathcal{D}_1) \to \mathrm{P}(\mathcal{D}_2)$ is the descent of U to the projective spaces. In this case, we write $\overline{\pi}_1 \cong \overline{\pi}_2$.

The following is the projective analogue of [40, Prop. 3.4]:

Proposition 20. *Assume that G is connected. For $k \in \{1, 2\}$, let $(\overline{\rho}_k, \mathcal{H}_k)$ be a smooth projective unitary representation of G with derived representation $\overline{d\rho_k} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{H}_k^\infty)$ on $\mathcal{H}_k^\infty$. Then*

$$\overline{\rho}_1 \cong \overline{\rho}_2 \quad \Longleftrightarrow \quad \overline{d\rho}_1 \cong \overline{d\rho}_2.$$

Proof. Passing to the universal cover of G, which is a Lie group by [65, Cor. II.2.4], we may and do assume that G is 1-connected. Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary map, and let $\overline{U} : \mathrm{P}(\mathcal{H}_1) \to \mathrm{P}(\mathcal{H}_2)$ be its descent to the projective spaces. Assume first that $\overline{U}\overline{\rho}_1(g)\overline{U}^{-1} = \overline{\rho}_2(g)$ for all $g \in G$. Then by [40, Cor. 4.5], we know that $\overline{\rho}_1$ and $\overline{\rho}_2$ correspond to the same central $\mathbb{T}$-extension $\mathring{G}$ of G, up to isomorphism of central extensions. Let $\mathring{\mathfrak{g}}$ denote the Lie algebra of $\mathring{G}$. Let the smooth unitary $\mathring{G}$-representations ρ_1 and ρ_2 be lifts of $\overline{\rho}_1$ and $\overline{\rho}_2$ respectively. Then there exists a smooth character $\zeta : \mathring{G} \to \mathrm{U}(1)$ such that $\rho_2(\mathring{g}) = \zeta(\mathring{g})U\rho_1(\mathring{g})U^{-1}$ for all $\mathring{g} \in \mathring{G}$. This implies in particular that U maps $\mathcal{H}_1^\infty$ onto $\mathcal{H}_2^\infty$. Differentiating the preceding equation at the identity of $\mathring{G}$, it also follows that $d\rho_2(\mathring{\xi}) = Ud\rho_1(\mathring{\xi})U^{-1} + d\zeta(\mathring{\xi})I$ for all $\mathring{\xi} \in \mathring{\mathfrak{g}}$, where I denotes the identity on $\mathcal{H}_2^\infty$. Hence $\overline{d\rho}_2(\xi) = \overline{U}\,\overline{d\rho}_1(\xi)\overline{U}^{-1}$ for all $\xi \in \mathfrak{g}$. So $\overline{d\rho}_1 \cong \overline{d\rho}_2$.

Assume conversely that U maps $\mathcal{H}_1^\infty$ onto $\mathcal{H}_2^\infty$ and that $\overline{U}\,\overline{d\rho}_1(\xi)\,\overline{U}^{-1} = \overline{d\rho}_2(\xi)$ for all $\xi \in \mathfrak{g}$. This implies that $\overline{d\rho}_1$ and $\overline{d\rho}_2$ induce isomorphic central $\mathbb{R}$-extension of $\mathfrak{g}$, up to isomorphism. Since G is 1-connected, it follows using [93, Cor. 7.15(i)] that $\overline{\rho}_1$ and $\overline{\rho}_2$ induce the same central $\mathbb{T}$-extension $\mathring{G}$ of G, up to isomorphism. Let the smooth unitary $\mathring{G}$-representations ρ_1 and ρ_2 once again be lifts of $\overline{\rho}_1$ and $\overline{\rho}_2$, respectively. There exists a continuous linear map $\lambda : \mathring{\mathfrak{g}} \to \mathbb{R}$ such that

$$d\rho_2(\mathring{\xi}) = Ud\rho_1(\mathring{\xi})U^{-1} + i\lambda(\mathring{\xi})I, \qquad \forall \mathring{\xi} \in \mathring{\mathfrak{g}}, \tag{B1}$$

where I denotes the identity on $\mathcal{H}_2^\infty$. Let $\psi \in \mathcal{H}_1^\infty$ and $\chi \perp \psi$. Let $(\chi_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}_1^\infty$ such that $\chi_k \to \chi$ in $\mathcal{H}_1$. Let $\gamma : \mathbb{R} \to \mathring{G}$ be a smooth path in $\mathring{G}$ with $\gamma_0 = 1$ being the identity of $\mathring{G}$, and let $\overline{\gamma} : \mathbb{R} \to G$ be its projection to G. Define the functions

$$f(t) := \langle U\chi, \rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi\rangle,$$
$$f_k(t) := \langle U\chi_k, \rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi\rangle, \qquad t \in \mathbb{R}.$$

Then $f_k \to f$ pointwise and f_k is smooth for every $k \in \mathbb{N}$, as $U\chi_k \in \mathcal{H}_2^\infty$. Let $\gamma' : \mathbb{R} \to \mathring{\mathfrak{g}}$ be the left-logarithmic derivative of γ, defined by $\gamma'_s := \frac{d}{dt}\big|_{t=s} \gamma_s^{-1}\gamma_t$ for $s \in \mathbb{R}$. Let $k \in \mathbb{N}$. Observe using equation (B1) that the derivative f'_k of f_k satisfies

$$\begin{aligned} f'_k(s) &= \langle U\chi_k, \rho_2(\gamma_s) d\rho_2(\gamma'_s) U\rho_1(\gamma_s)^{-1}\psi\rangle - \langle U\chi_k, \rho_2(\gamma_s) U d\rho_1(\gamma'_s)\rho_1(\gamma_s)^{-1}\psi\rangle \\ &= i\lambda(\gamma'_s)\langle U\chi_k, \rho_2(\gamma_s)U\rho_1(\gamma_s)^{-1}\psi\rangle \\ &= i\lambda(\gamma'_s) f_k(s), \qquad \forall s \in \mathbb{R}. \end{aligned}$$

We see that f_k satisfies the ordinary differential equation $f'_k(s) = i\lambda(\gamma'_s) f_k(s)$ with initial condition $f_k(0) = \langle \chi_k, \psi\rangle$. It follows that $f_k(t) = \langle \chi_k, \psi\rangle g(t)$, where $g(t) = e^{\int_0^t i\lambda(\gamma'_s)\,\mathrm{d}s}$. Consequently, $f(t) = \lim_k f_k(t) = \langle \chi, \psi\rangle g(t) = 0$ for all $t \in \mathbb{R}$. Hence $\langle U\chi, \rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi\rangle = 0$ for every $\chi \perp \psi$ and $t \in \mathbb{R}$. Thus $[\rho_2(\gamma_t)U\rho_1(\gamma_t)^{-1}\psi] = [U\psi]$ for all $t \in \mathbb{R}$ and $\psi \in \mathcal{H}_1^\infty$, which implies that $\overline{U}\,\overline{\rho}_1(\overline{\gamma}_t^{-1}) = \overline{\rho}_2(\overline{\gamma}_t^{-1})\,\overline{U}$ for all $t \in \mathbb{R}$. Since $\mathring{G}$ is a path-connected principal $\mathbb{T}$-bundle over G, it follows that $\overline{U}\,\overline{\rho}_1(g) = \overline{\rho}_2(g)\,\overline{U}$ for all $g \in G$. Thus $\overline{\rho}_1 \cong \overline{\rho}_2$. □

Corollary 21. *Let $\overline{\rho} : G \to \mathrm{PU}(\mathcal{H})$ be a smooth projective unitary representation with derived representation $\overline{d\rho} : \mathfrak{g} \to \mathfrak{pu}(\mathcal{H}^\infty)$. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$, and let $f : H \to G$ be smooth homomorphism of Lie groups. If $T_e(f)(\mathfrak{h}) \subseteq \ker \overline{d\rho}$, then $f(H) \subseteq \ker \overline{\rho}$.*

Proof. By considering the pull-back of $\overline{\rho}$ along f, it suffices to consider the case where $H = G$ and $f = \mathrm{id}_G$. Thus, assume that G is connected and that $\mathfrak{g} \subseteq \ker \overline{d\rho}$. Then Proposition 20 implies that $G \subseteq \ker \overline{\rho}$. □

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