The Structure of Hecke Operator Algebras

Klisse, M.

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Mario Klisse

Master of Science in Mathematics
Westfälische Wilhelms-Universität Münster, Germany
geboren te Essen, Duitsland.

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promotor: Prof. dr. J.M.A.M. van Neerven
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Dedicated to my sister Miriam.
* 1989 – †2021

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Introduction

A core theme of mathematical research, and of science in general, is the building of bridges between seemingly unrelated fields. The idea occurs in Galois theory which provides a link between groups and field extensions, it helped in proving Fermat's infamous last theorem by connecting modular forms with Galois representations and it appears in K-theory where (in its classical form) groups are associated with compact topological spaces. The power of translating a given mathematical concept into another stems from the new set of tools often provided by adopting a new perspective. Conversely, such a translation often leads to interesting examples and new structures, one of which are group operator algebras associated with a given group and crossed product operator algebras associated with a given dynamical system.

Originating in Heisenberg's formalism of matrix mechanics, the theory of operator algebras was introduced as an abstraction of algebras of physical observables appearing in quantum theory. Its rigorous foundation was built by Murray and von Neumann who realized that the modeling of infinite systems of particles requires the study of certain infinite-dimensional algebras. These ideas led to the theory of C^*-algebras and von Neumann algebras which, somewhat detached from its origins in quantum physics, over time developed a vibrant life on its own. A C^*-algebra is an algebra over the complex numbers (i.e. a vector space with a multiplication), equipped with a submultiplicative norm with respect to which it is complete, and an anti-linear involution (the "star") whose properties resemble the properties of conjugation of complex numbers. The prescribed compatibility of the norm and the involution implies a strong connection between the C^*-algebra's algebraic and its analytic properties. This has a number of intriguing implications, one of which is a fundamental result of Gelfand and Naimark, that states that every (abstract) C^*-algebra can be viewed as a norm-closed involutive subalgebra of the bounded linear operators $\mathscr{B}(\mathscr{H})$ on some complex Hilbert space $\mathscr{H}$.

Von Neumann algebras build a subclass of C*-algebras. They can be concretely defined as those norm-closed involutive subalgebras of the bounded linear operators $\mathscr{B}(\mathscr{H})$ on some complex Hilbert space $\mathscr{H}$ that are closed in the strong operator topology (that is, the topology of pointwise convergence). Despite their evident proximity, the study of von Neumann algebras has a very different flavor compared to that of C*-algebras: whereas C*-algebra theory is often referred to as non-commutative topology, von Neumann algebra theory is called non-commutative measure theory.

Two well-studied classes of C*-algebras and von Neumann algebras, that already appear in the early works of Murray and von Neumann and that provide interesting examples of operator algebras, are those associated with groups and, more generally, with dynamical systems. Groups constitute some of the most fundamental objects in mathematics. A given (discrete) group G induces its complex group (convolution) algebra $\mathbb{C}[G]$ which can be completed to the (reduced) group C*-algebra $C_r^*(G)$ and the group von Neumann algebra $\mathscr{L}(G)$. Very much in the spirit of the first paragraph above, the procedure provides a way to encode group theoretical data in the language of operator algebras. Similarly, a (discrete) group G that acts on a compact Hausdorff space X gives rise to the reduced crossed product C*-algebra $C(X) \rtimes_r G$ and the corresponding crossed product von Neumann algebra $L^\infty(X)\overline{\rtimes}G$, where in the case of the one-point set $X = \{\bullet\}$ simply $C(X) \rtimes_r G = C_r^*(G)$ and $C(X)\overline{\rtimes}G = \mathscr{L}(G)$. The study of group algebras and crossed products turned out to be of use both for the investigation of internal structures as well as for applications. It further motivated other fascinating concepts such as free products and graph products of operator algebras.

The aim of this thesis is the study of a somewhat "deformed" setting, namely that of operator algebras associated with so-called Iwahori-Hecke algebras. In our attempt to unravel the structure of these algebras, we encounter and study several other concepts such as (Khintchine inequalities of) graph products of operator algebras, topological dynamics associated with boundaries and compactifications of groups, the approach by Kalantar and Kennedy to the C*-simplicity problem of discrete groups, the relative Haagerup property of general unital inclusions of von Neumann algebras, approximation properties of operator algebras, and rigidity theory of von Neumann algebras.

IWAHORI-HECKE ALGEBRAS

The concept of deformation is ubiquitous in mathematics and can be traced back to Euler's work on q-analogues of the natural logarithm. The underlying idea is to generalize a mathematical statement, a formula, or an expression by introducing a deformation parameter q for which, if q approaches 1, the original statement, formula or expression is recovered. Examples of this appear in combinatorics (see e.g. q-factorials, q-binomial coefficients), analysis (see e.g. q-derivatives, q-difference polynomials), in the study of quantum groups, and the idea is the core theme of deformation quantization where, roughly speaking, classical physical systems are deformed into non-classical (non-commutative) ones. Also, Iwahori-Hecke algebras, the operator algebraic completions of which we will be concerned with in this dissertation, fall into this category.

A Coxeter group is a group universally generated by a (possibly infinite) generating set S with respect to relations of the form $(st)^{m_{st}} = e$ where $s, t \in S$, $m_{st} \in \mathbb{N} \cup \{\infty\}$ and where $m_{ss} = 2$, $m_{st} = m_{ts}$ and $m_{st} \geq 2$ if $s \neq t$. The class of Coxeter groups, first studied by Coxeter in [62] and [63], was introduced as a natural generalization of finite reflection groups, i.e. finite groups generated by a set of reflections on a finite-dimensional Euclidean space. It is very accessible but also provides a rich source of interesting structures. The multiplication of the group algebra $\mathbb{C}[W]$ of a given Coxeter group W generated by a set S admits a natural deformation depending on a deformation parameter q. It gives rise to an algebra, the Iwahori-Hecke algebra $\mathbb{C}_q[W]$ of the Coxeter group with respect to the parameter q, which can be endowed with an involution and which in the case where q equals 1 coincides with the group algebra $\mathbb{C}[W]$.

Definition ([67, Proposition 19.1.1])**.** Let W be a Coxeter group generated by a set S, let $q = (q_s)_{s\in S} \in \mathbb{R}^S_{>0}$ be a multi-parameter for which $q_s = q_t$ if s and t are conjugate to each other (i.e. if there exists $\mathbf{w} \in W$ with $s = \mathbf{w}^{-1} t \mathbf{w}$) and let $(\widetilde{T}_{\mathbf{w}})_{\mathbf{w}\in W}$ be the canonical basis of the free $\mathbb{C}$-module $\mathbb{C}^{(W)}$ on W. Then there exists a unique $*$-algebra structure on $\mathbb{C}^{(W)}$ such that for all $\mathbf{w} \in W$, $s \in S$,

$$\widetilde{T}_s \widetilde{T}_{\mathbf{w}} = \begin{cases} \widetilde{T}_{s\mathbf{w}} & \text{, if } |s\mathbf{w}| > |\mathbf{w}| \\ q_s \widetilde{T}_{s\mathbf{w}} + (1 - q_s)\widetilde{T}_{\mathbf{w}} & \text{, if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}$$

and

$$(\widetilde{T}_{\mathbf{w}})^* = \widetilde{T}_{\mathbf{w}^{-1}}.$$

Here $|\cdot|$ denotes the word length with respect to the generating set S. The induced $*$-algebra $\mathbb{C}_q[W]$ is called the Iwahori-Hecke algebra of the pair (W, S) and the multi-parameter q.

Iwahori-Hecke algebras very naturally appear as abstractions of certain endomorphism rings that play a role in the representation theory of Lie groups. Their history is vast and reaches into diverse fields such as knot theory (see [118]), combinatorics (see [107], [106]), the theory of buildings (see [87, Section 6.2], [160, Section 2.4]). They are linked to quantum groups and non-commutative geometry, and due to their relation with reductive groups they occur in the local Langlands program. The notion can be traced back to Iwahori in [112] who studied (double coset) algebras associated with certain inclusions of groups.

The Iwahori-Hecke algebras $\mathbb{C}_q[W]$, $q \in \mathbb{R}^S_{>0}$ can conveniently be viewed as $*$-subalgebras of the bounded linear operators $\mathcal{B}(\ell^2(W))$ on the Hilbert space $\ell^2(W)$ of all square-summable functions on the (discrete) group W and thus norm complete to C*-algebras and strongly complete to von Neumann algebras. These are

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respectively called the corresponding Hecke C*-algebras $C^*_{r,q}(W)$ and the corresponding Hecke-von Neumann algebras $\mathcal{N}_q(W)$ of the Coxeter group W. It is natural to ask how much the structure of these algebras resembles that of $C^*_r(W)$ and $\mathcal{L}(W)$ and to what extent it depends on the deformation parameter q. In this context, as we will see, the class of Hecke operator algebras constitutes a natural setting to look for extensions of ideas and methods developed in the study of group operator algebras.

Probably the first source discussing topological operator algebraic completions of Iwahori-Hecke algebras is Matsumoto's book [133]. In the eighties, Baum, Higson, and Plymen studied them in relation to the important Baum-Connes conjecture (see [15]), which links the K-theory of reduced group C*-algebras with the (topological) K-homology of certain spaces and which has been proved to hold for lots of classes of groups, one of which are Coxeter groups. Later Opdam [146] considered Hecke C*-algebras in the setting of harmonic analysis which inspired several other results, see [69], [147], [167], [168]. In all these instances the study of Hecke operator algebras mostly restricted to a certain tractable class of Coxeter groups, the so-called affine ones. The study of (possibly non-affine) Hecke-von Neumann algebras was initiated by Davis, Dymara and Januszkiewicz (see [76], [66], [67]) in an attempt to investigate a certain q-analogue to the classical ℓ^2-cohomology of buildings. This "weighted ℓ^2-cohomology" is tied to the central decomposition of Hecke-von Neumann algebras. Motivated by this, in his book on the geometry and topology of Coxeter groups [67] Davis asked for a classification of Hecke-von Neumann algebras up to isomorphism. This question was picked up by Garncarek in [86] who calculated the center of single-parameter Hecke-von Neumann algebras of a certain class of Coxeter groups, namely the right-angled ones, and who demonstrated that although the centers of these von Neumann algebras can be non-trivial, they contribute nothing to the decomposition of the weighted cohomology of W. His results are however of independent interest and initiated the study of Hecke operator algebras on their own rights. In [160] Raum and Skalski extended Garncarek's results to the multi-parameter setting.

Complementing his results on the center of Hecke-von Neumann algebras, in [86, Section 6] Garncarek discussed a curious connection to the interpolated free group factors which occur in the context of the infamous free factor problem. The free factor problem, going back to a question raised by Kadison in the 1960s, is one of the major open questions in the theory of von Neumann algebras. It asks whether the group von Neumann algebras, the "free group factors", of groups freely generated by a certain finite set of generators can be distinguished from each other, i.e. do there exist natural numbers $m \neq n$ bigger than 1 such that $\mathcal{L}(\mathbb{F}_n) \cong \mathcal{L}(\mathbb{F}_m)$ where $\mathbb{F}_n$ denotes the free group on n generators and where $\mathbb{F}_m$ denotes the free group on m generators? Despite its straightforward formulation, the free factor problem is notoriously hard to answer and led to the creation of whole new fields such as Voiculescu's free probability theory. By introducing "interpolated free group factors" (independently discovered by Radulescu in [159]), Dykema demonstrated in [74] that the $\mathcal{L}(\mathbb{F}_n)$, $n \in \mathbb{N}$ are either all isomorphic to each other, or none of them are. Garncarek's discussion in [86, Section 6] implies that the interpolated free group factors arise as Hecke-von Neumann algebras of certain Coxeter groups, that is Hecke-von Neumann algebras can be viewed as a natural generalization of the interpolated free group factors. The classification of these operator algebras is hence tied to, and thus motivated by, one of the major questions in the theory of operator algebras.

After providing the background on C^*-algebras and von Neumann algebras (see Section 2.2), dynamical systems and crossed products (see Section 2.3), partially ordered sets (see Section 2.4), graphs and trees (see Section 2.5), amalgamated free products of groups (see Section 2.6) and Coxeter groups (see Section 2.7) in Chapter 2, in Chapter 3 we make an effort to gently introduce Iwahori-Hecke algebras and their operator algebraic counterparts. After that, we study isomorphism properties and decompositions of Hecke algebras and their generating elements, and we discuss our results with respect to the free factor problem. The corresponding results will be crucial for the later chapters.

Graph product Khintchine inequalities

Creating new objects out of simpler building blocks often leads to interesting new structures. Instances of this common idea are CW-complexes in topology, free products, and graph products of groups and inductive limits. The approach is also employed in the theory of operator algebras where, after their introduction as a part of Voiculescu's non-commutative probability theory, (reduced) free products nowadays play an important role. Voiculescu's construction starts with a family of C^*-algebras (or von Neumann algebras) endowed with states, and spits out a new C^*-algebra (or von Neumann algebra) into which the family canonically embeds and for which operators coming from different members of the family are in a certain sense "free" with respect to each other. It can be viewed as an operator algebraic analogue to free products of groups.

In the group setting free products can be generalized in terms of Green's graph products of groups. Her construction starts from a simplicial graph with a discrete group attached to each vertex and results in a new discrete group that suitably contains its building blocks and for which the corresponding commutation relations resemble the structure of the underlying graph. What makes the construction so interesting is that it interpolates between free products and Cartesian products, it covers important examples such as right-angled Coxeter groups and right-angled Artin groups, and - other than the more general construction of amalgamated free products - it preserves many important group theoretical properties. It is a natural question to ask whether graph products of groups admit an operator algebraic counterpart. This question was answered by Caspers and Fima in [44] in the affirmative. They came up with a construction that associates with a simplicial graph with C^*-algebras (or von Neumann algebras) endowed with states attached to each vertex, a new C^*-algebra (or von Neumann algebra) into which the vertex operator algebras canonically embed and for which the commutation relations

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resemble the structure of the underlying graph.

Theorem ([44, Proposition 2.12])**.** Let $K = (V, E)$ be a finite, undirected, simplicial graph and let $(A_v, \varphi_v)_{v \in V}$ be a family of unital C*-algebras equipped with GNS-faithful states. Then there exists a (up to isomorphism) unique C*-algebra A, the (reduced) graph product C*-algebra, such that:

(1) There exists a GNS-faithful state φ on A;

(2) There exist unital inclusions $A_v \subseteq A$ such that the union of all $A_v, v \in V$ generates A and such that for all $(v, v') \in E$ the elements in A_v commute with the elements in $A_{v'}$;

(3) $\varphi|_{A_v} = \varphi_v$ for every $v \in V$;

(4) Freeness: For every reduced word $\mathbf{v} = v_1 ... v_n$ with $v_1, ..., v_n \in V$ and $a_1 \in A_{v_1}^\circ$, ..., $a_n \in A_{v_n}^\circ$ one has $\varphi(a_1 ... a_n) = 0$.

Similar to their group theoretical counterparts, graph products of operator algebras generalize Voiculescu's free products as well as tensor products and they admit useful preservation properties. What makes the construction so interesting in the context of this dissertation is that Hecke operator algebras coming from right-angled Coxeter groups decompose as graph products over very simple (2-dimensional) building blocks. This observation provides useful tools to study them, some of which have been employed in [46] to study rigidity properties of right-angled Hecke-von Neumann algebras. On the other hand, this demonstrates that operator algebraic graph products lead to interesting structures to examine.

Since their invention in the 1980s, various structural aspects of free products of operator algebras have been studied. Results in particular concern the ideal structure and approximation properties (see the later sections for an explanation of what that means), the construction of free product maps between operator algebras and a detailed study of free products of finite-dimensional operator algebras. In [163] Ricard and Xu, in an attempt to deduce (the preservation of) approximation properties of free product C*-algebras, proved a "Khintchine type inequality". Inequalities of this type estimate the operator norm of an operator of a given length with the norm of certain Haagerup tensor products of column and row Hilbert spaces. In the case of group C*-algebras $C_r^*(\mathbb{F}_n)$, $n \in \mathbb{N}$ of free groups and operators of length 1, their approach goes back to Haagerup's fundamental paper [93]. In Chapter 4 we walk a similar path by proving a Khintchine type inequality for general graph products of C*-algebras.

Theorem (Graph product Khintchine inequality)**.** Let $K = (V, E)$ be a finite, undirected, simplicial graph and let $(A_v, \varphi_v)_{v \in V}$ be a family of unital C*-algebras equipped with GNS-faithful states. Denote by A the corresponding graph product C*-algebra and by $\chi_n : A \to A$ the word length projection of length n. Then for every $n \in \mathbb{N}_{\geq 1}$ there exists some operator space X_n and maps

$$j_n : \chi_n(A) \to X_n, \qquad \pi_n : \mathrm{dom}(\pi_n) \subseteq X_n \to \chi_n(A),$$

with $\mathrm{dom}(\pi_n) = j_n(\chi_n(A))$ such that the following statements hold:

(1) X_n is a direct sum of Haagerup tensor products of column and row Hilbert spaces;

(2) $\pi_n \circ j_n$ is the identity on $\chi_n(A)$;

(3) The map π_n is completely bounded with $\|\pi_n\|_{cb} \leq Cn$ for some (explicit) constant C, depending only on the graph K.

The proof of this theorem employs an intertwining technique between graph products and free products as well as a careful decomposition of graph product operators of a given length into sums of creation, diagonal and annihilation operators. Our main objective will be to apply the theorem in the setting of right-angled Hecke C*-algebras where, due to the graph product decomposition of the latter into very simple building blocks, the Khintchine inequality conveniently simplifies. This will allow us to derive a "Haagerup type inequality" for Hecke C*-algebras of right-angled Coxeter groups. Such a Haagerup type inequality states that the norm of an operator of length n can be estimated with its 2-norm up to a polynomial bound depending on n. It is a generalization of Haagerup's inequality for free group C*-algebras $C_r^*(\mathbb{F}_n)$, $n \in \mathbb{N}$ in [93] which entails that there exists a constant C such that for every $x \in C_r^*(\mathbb{F}_n) \subseteq \mathcal{B}(\ell^2(\mathbb{F}_n))$ supported on group elements of length n one has $\|x\| \leq Cn\|x\delta_e\|_2$. In particular, in this case the polynomial can be chosen to be n, that is we have a linear estimate in the length n.

Theorem (Haagerup inequality for right-angled Hecke C*-algebras)**.** Let W be a finitely generated right-angled Coxeter group and let q be a multi-parameter. Then for each $n \in \mathbb{N}$ and $x \in \chi_n(C_{r,q}^*(W))$ we have that

$$\|x\| \leq Cn\|x\delta_e\|_2$$

for some (explicit) constant depending only on q and the group W.

Haagerup and Khintchine inequalities have found a wide range of applications in operator theory. We will employ them in the context of the trace-uniqueness problem of Hecke C*-algebras in Section 6.3.

Topological boundaries and compactifications of graphs and Coxeter groups

The basic idea of geometric group theory is to explore the connections between abstract algebraic properties of a group and geometric properties of a space on which this group acts nicely. Many of the concepts in the field are heavily influenced by the work of Gromov who introduced the notion of word hyperbolic groups and (more generally) hyperbolic graphs. Hyperbolic graphs are graphs that satisfy a certain negative curvature condition. Intuitively, a hyperbolic graph's large-scale geometry looks similar to that of a tree; in particular, every tree is hyperbolic. The

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notion can be formally introduced in several equivalent ways, one of which is to declare a (undirected and simplicial) graph K with graph metric d_K to be hyperbolic, if there exists a constant $\delta > 0$ such that for any four vertices $w, x, y, z \in K$ the inequality

$$d_K(w,x) + d_K(y,z) \leq \max\{d_K(w,y) + d_K(x,z), d_K(w,z) + d_K(x,y)\} + \delta$$

holds. A group G generated by a finite set S is word hyperbolic, if its Cayley graph $\mathrm{Cay}(G,S)$ with respect to S, consisting of the vertex set G and the edge set $\{(g,h) \in G \times G \mid g^{-1}h \in S \cup S^{-1}\}$, is hyperbolic. It might not be immediately clear, but the class of hyperbolic groups is very accessible and the notion of hyperbolicity is quite rigid. For instance, it does not depend on the generating set S.

An important tool in the study of hyperbolic graphs (and in particular word hyperbolic groups) is its "space at infinity", the Gromov boundary $\partial_h K$. One can think of it as a set of (equivalence classes of) infinite geodesic rays. It can be equipped with a topology that turns it into a compact Hausdorff space. This topological space has a rich structure which provides an excellent tool to study the underlying graph as well as groups acting on it. This led to a number of breakthroughs in the fields of geometric and combinatorial group theory.

Following Gromov's ideas, many similar constructions assigning topological spaces to graphs and groups have been presented. In Chapter 5 we will follow an analogous path by defining certain topological spaces associated with (countable, undirected, and simplicial) connected graphs with a designated root vertex. Given such a graph K and a root $o \in K$ we consider (equivalence classes of) sequences in K that in a certain sense "converge to infinity". The set of all such elements can be endowed with a natural topology that turns it into a compact Hausdorff space and that we denote by $\partial(K,o)$. Similarly, the set $\overline{(K,o)} := K \cup \partial(K,o)$ can be turned into a compact Hausdorff space that naturally contains the vertex set of K as a dense subset; we call it the compactification of (K,o). Our construction covers several interesting examples. It reflects combinatorial and order theoretical properties and (for hyperbolic graphs) nicely relates to Gromov's construction.

Theorem. Let (K,o) be a hyperbolic connected rooted graph. Then there exists a canonical continuous surjection $\phi : \partial(K,o) \to \partial_h K$. If the graph is locally finite, then ϕ extends to a continuous surjection $\widetilde{\phi} : \overline{(K,o)} \to K \cup \partial_h K$ with $\widetilde{\phi}|_K = \mathrm{id}_K$.

In general, the structure of the spaces $\overline{(K,o)}$ and $\partial(K,o)$ is much less tractable than that of hyperbolic compactifications and boundaries; for instance, an isometric bijection of a graph does not necessarily extend to a homeomorphism of $\overline{(K,o)}$. However, for certain classes of graphs, this can be different. This is the case for Cayley graphs of Coxeter groups with respect to their canonical generating set. For a given Coxeter system (W,S) we denote the corresponding compactification and boundary associated with its Cayley graph $\mathrm{Cay}(W,S)$ by $\overline{(W,S)}$ and $\partial(W,S)$. One can conveniently think of elements in $\partial(W,S)$ as infinite reduced words with letters in S. As it turns out, the canonical action of W on itself via left multiplication induces a continuous action of W on $\overline{(W,S)}$ and $\partial(W,S)$; one can thus build the corresponding crossed product C*-algebras $C(\overline{(W,S)}) \rtimes_r W$ and $C(\partial(W,S)) \rtimes_r W$. These crossed products are strongly connected to the Hecke C*-algebras of W.

Theorem. Let W be a Coxeter group generated by S with $\#S < \infty$ and let q be a multi-parameter. Then the Hecke C*-algebra $C^*_{r,q}(W)$ canonically embeds into the crossed product C*-algebra $C(\overline{(W,S)}) \rtimes_r W$. If q lies outside a certain set $\mathscr{R}'(W,S)$ of parameters associated with the region of convergence of the multi-variate growth series $\sum_{\mathbf{w} \in W} z_{\mathbf{w}}$ of W, then the Hecke C*-algebra $C^*_{r,q}(W)$ canonically embeds into the crossed product C*-algebra $C(\partial(W,S)) \rtimes_r W$.

The theorem provides a direct link between the topological spaces $\overline{(W,S)}$ and $\partial(W,S)$ (and the action of W on them) and the Hecke operator algebras of the system. This motivates the usage of geometrical ideas to deduce information about Hecke C*-algebras and Hecke-von Neumann algebras. For this reason, in Section 5.2 we study dynamical properties of $W \curvearrowright \overline{(W,S)}$ and $W \curvearrowright \partial(W,S)$, such as amenability, smallness at infinity, boundary actions and probability measures on $\overline{(W,S)}$ and $\partial(W,S)$. These properties all have interesting operator algebraic implications that we will pick up in the later chapters. We further identify the spaces $\overline{(W,S)}$ and $\partial(W,S)$ with a construction by Caprace and Lécureux [37] that was given in the setting of buildings.

Ideal structure and unique trace property of Hecke C*-algebras

A von Neumann algebra is called a factor if its center (i.e. the set of elements that commute with every other element of the von Neumann algebra) is trivial. By a classical theorem of von Neumann every von Neumann algebra admits a decomposition into factorial building blocks via direct integrals which can be thought of as a generalization of the concept of direct sums. The power of such a decomposition follows from the fact that it often allows passing in considerations about general von Neumann algebras to the factorial case.

As mentioned before, motivated by the study of the weighted ℓ^2-cohomology of buildings, in [67] Davis asked for a classification of Hecke-von Neumann algebras up to isomorphism. This question was picked up by Garncarek in [86] who calculated the center of right-angled single-parameter Hecke-von Neumann algebras. His characterization was extended to the right-angled multi-parameter case by Raum and Skalski by using a combinatorial approach.

Theorem ([160, Theorem A])**.** Let (W,S) be a right-angled, irreducible Coxeter system with $3 \leq \#S < \infty$ and let q be a multi-parameter with $0 < q_s \leq 1$ for all $s \in S$. Then the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ decomposes as a direct sum

$$\mathcal{N}_q(W) \cong \mathcal{M} \oplus \bigoplus_{\epsilon} \mathbb{C}$$

where $\mathscr{M}$ is a factor and where the direct summands $\mathbb{C}$ correspond to the (1-dimensional) central projections of $\mathcal{N}_q(W)$. The central projections can be characterized explicitly.

The characterization of the factoriality for broader classes of Hecke-von Neumann algebras is still open.

Unfortunately, there exists no C*-algebraic version of von Neumann's central decomposition theorem. The most suitable analogue to factoriality in the C*-algebraic setting is the notion of simplicity. A C*-algebra is called simple if it contains no non-trivial closed two-sided ideal. Since Powers' short paper [157] in 1975, especially the characterization of (discrete) groups whose reduced group C*-algebras are simple (such groups are called C*-simple) was a major open problem. By using a clever averaging argument he proved that for every $n \geq 2$ the C*-algebra $C_r^*(\mathbb{F}_n)$ is simple and that it carries a unique tracial state. Influenced by Powers' approach, large classes of C*-simple groups that appear naturally in geometry have been identified, including non-elementary word hyperbolic groups and lattices in semisimple groups.

In [120] Kalantar and Kennedy found a new dynamical approach to the question for C*-simplicity by considering the action of a given discrete group G on its "Furstenberg boundary". The continuous action of a group G on a compact Hausdorff space X is called minimal if the orbit of every element under the action is dense in X. It is further strongly proximal if the weak-$*$ closure of the orbit of every probability measure on X contains a point mass. The space X is called a G-boundary if $G \curvearrowright X$ is both minimal and strongly proximal. In [85] Furstenberg demonstrated that every group admits a unique G-boundary $\partial_F G$, its Furstenberg boundary, which is universal in the sense that every other G-boundary is a continuous G-equivariant image of $\partial_F G$. By identifying the continuous functions on $\partial_F G$ with the minimal C*-subalgebra of $\ell^\infty(G)$ that arises as the image of a unital positive G-equivariant projection (this is the so-called G-injective envelope of $\mathbb{C}$), Kalantar and Kennedy were able to deduce several interesting implications, one of which is the characterization of C*-simplicity in terms of boundary actions.

Theorem ([120, Theorem 6.2])**.** A discrete group G is C*-simple if and only if its action on some G-boundary is topologically free.

The approach by Kalantar and Kennedy implied further results on the uniqueness of trace and tightness of nuclear embeddings of group C*-algebras and it inspired various generalizations.

Motivated by the results on the factoriality of Hecke-von Neumann algebras, the aim of Chapter 6 is to study the ideal structure (and in particular simplicity) and the trace-uniqueness of Hecke C*-algebras. It is known that the reduced group C*-algebra (i.e. the Hecke C*-algebra with parameter 1) of an irreducible Coxeter system is simple if and only if the corresponding Coxeter system is non-affine (see [80], [101], [61]). By demonstrating that the 1-dimensional central projections that lie inside certain Hecke-von Neumann algebras induce unital, linear, multiplicative functionals, we deduce non-simplicity results that in particular imply that Hecke C*-algebras of irreducible Coxeter groups that are not non-affine are never simple.

Theorem. Let (W, S) be an irreducible Coxeter system and q some multi-parameter. If the multi-parameter $(q_s^{\epsilon_s})_{s\in S}$ with

$$\epsilon_s := \begin{cases} +1, & \text{if } q_s \leq 1 \\ -1, & \text{if } q_s > 1 \end{cases}$$

lies in the closure of the region of convergence of the growth series $\sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ of W, then $C^*_{r,q}(W)$ is not simple and does not have a unique tracial state.

Corollary. Let (W, S) be an irreducible Coxeter system that is not non-affine and q some multi-parameter. Then $C^*_{r,q}(W)$ is not simple and does not have unique tracial state.

By using the Haagerup inequality from Chapter 4 we further prove a decomposition of right-angled Hecke C*-algebras which is analogous to the one for right-angled Hecke-von Neumann algebras and we prove that, if the parameter q is close enough to 1, the Hecke C*-algebra of a right-angled Coxeter group W generated by S with $3 \leq \#S < \infty$ carries a unique tracial state. The trace-uniqueness result is inspired by Powers' classical averaging argument in [157].

Theorem. Let (W, S) be a right-angled Coxeter system with $3 \leq \#S < \infty$ and let q be a multi-parameter. Then the corresponding Hecke C*-algebra $C^*_{r,q}(W)$ decomposes as a direct sum

$$C^*_{r,q}(W) \cong A \oplus \bigoplus_{\epsilon} \mathbb{C},$$

where A is a C*-algebra and where the direct summands $\mathbb{C}$ correspond to the (1-dimensional) central projections of $\mathcal{N}_q(W)$.

Theorem. Let (W, S) be an irreducible, right-angled Coxeter system with $3 \leq \#S < \infty$. Then there exists an open neighborhood $\mathcal{U}$ of 1 such that for all multi-parameters $q \in \mathcal{U}$ the Hecke C*-algebra $C^*_{r,q}(W)$ has unique tracial state.

The proof of the chapter's main result is inspired by Kalantar-Kennedy's boundary methods and the approach in [95]. By using the construction from Chapter 5 and by viewing certain Hecke C*-algebras as C*-subalgebras of the crossed product $C(\partial(W, S)) \rtimes_r W$, we obtain a complete characterization of right-angled simple Hecke C*-algebras.

Theorem. Let (W, S) be an irreducible, right-angled Coxeter system with $\#S < \infty$ and let q be a multi-parameter. Then the Hecke C*-algebra $C^*_{r,q}(W)$ is simple if and only if q lies outside the closure of a set $\mathcal{R}'(W, S)$ of parameters associated with the region of convergence of the multi-variate growth series $\sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ of W

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Corollary. Let (W,S) be an irreducible, right-angled Coxeter system with $\#S = \infty$ and let q be a multi-parameter. Then the Hecke C*-algebra $C^*_{r,q}(W)$ is simple if and only if there exists a finite subset $T \subseteq S$ such that the Hecke C*-algebra $C^*_{r,q_T}(W_T)$ with $q_T := (q_t)_{t\in T}$ is simple.

THE RELATIVE HAAGERUP PROPERTY

Chapter 7 is somewhat isolated from the rest of this dissertation as it does not a priori relate to the study of Hecke operator algebras. In the broad context of this thesis its relevance becomes apparent as soon as one invokes the results of Section 7.7 in the context of Hecke-von Neumann algebras of Coxeter groups that conveniently decompose as amalgamated free products, as we will do in Chapter 8. The aim of Chapter 7 is the introduction and study of a suitable notion of the relative Haagerup property for arbitrary inclusions of (σ-finite) von Neumann algebras.

The story of the (group theoretic) Haagerup property began with Haagerup's celebrated article [93] in which he noted that the free group $\mathbb{F}_n$, $n \in \mathbb{N}$ admits a sequence of positive definite functions that are in a certain sense small and that pointwise converge to a constant function equal to 1. An important notion both in group theory as well as in the study of operator algebras associated with groups is that of amenability. A group G is amenable if there exists a sequence of finitely supported positive definite functions on G that converges to 1. Amenability has a number of strong implications. Haagerup's result indicates that, even though the free group $\mathbb{F}_n$ is not amenable for $n \geq 2$, it satisfies a property that is close to amenability and that can be viewed as a natural weakening of this notion. This "Haagerup property" holds for many groups and is a subject of intense study, in particular because it is related to deep conjectures such as the Baum-Connes conjecture and the associated Novikov conjecture. It can be encoded operator algebraically in terms of the group von Neumann algebra of the group. Motivated by this, in [55] Choda introduced a definition of the Haagerup property for a von Neumann algebra $\mathcal{M}$ equipped with a (faithful normal) tracial state in terms of the existence of certain approximating maps on $\mathcal{M}$, that behave well with respect to the trace in question. Later Jolissaint proved that the property does not depend on the choice of the trace; it is thus an intrinsic invariant of $\mathcal{M}$.

In several group theoretic and operator algebraic contexts it is important to consider also relative properties; such properties are for instance the key to showing that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ does not have the Haagerup property, which in turn has several von Neumann algebraic consequences. In the context of von Neumann algebras that admit a tracial state the relative Haagerup property, first defined by Boca [23], was used in [155] as part of Popa's deformation/rigidity theory as a key tool to obtain deep structural results about von Neumann algebras containing a certain type of large commutative subalgebras. Such "Cartan subalgebras" are deeply related to von Neumann algebras of equivalence relations (which generalize certain crossed product von Neumann algebras). In this setting the first analogue to the

Haagerup property outside the tracial case was proposed in [176], [6].

Also in the non-relative case for several years considerations focused on finite von Neumann algebras, mainly as the motivating examples came from discrete groups. This changed when a suitable analogue for quantum groups and their von Neumann algebras was investigated. Soon after that Okayasu and Tomatsu on one hand, and Caspers and Skalski on another, gave a definition of the Haagerup property for an arbitrary von Neumann algebra equipped with a (faithful normal semifinite) weight and proved that in fact the notion does not depend on the choice of the weight in question. In all the cases above the Haagerup property should be thought of as a natural weakening of important approximation properties (namely injectivity/amenability), that permits applying approximation ideas and techniques beyond the class of amenable groups or algebras. Further, the class of discrete groups enjoying the Haagerup property has good permanence properties, one of which is the preservation under taking free products amalgamated over finite subgroups.

In Chapter 7 we extend the ideas from above by undertaking a systematic study of a von Neumann algebraic relative Haagerup property for a unital inclusion $\mathscr{N} \subseteq \mathscr{M}$ of von Neumann algebras equipped with a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}} : \mathscr{M} \to \mathscr{N}$. We define it in terms of a fixed (faithful normal) state (preserved by $\mathbb{E}_{\mathscr{N}}$) but then quickly show that it depends only on the conditional expectation in question. Essentially, for a triple $(\mathscr{M}, \mathscr{N}, \mathbb{E}_{\mathscr{N}})$ to have the relative Haagerup property we require the existence of certain $\mathscr{N}$-bimodular maps on $\mathscr{M}$ that preserve $\mathbb{E}_{\mathscr{N}}$, that in a certain sense decrease rapidly and that converge to the identity function. Allowing non-trivial inclusions allows us to significantly broaden the class of examples fitting into our framework and yields certain facts that are new even in the context of the non-relative tracial Haagerup property.

If the smaller von Neumann algebra admits a tracial state, we prove that the relative Haagerup property is an intrinsic invariant of the inclusion $\mathscr{N} \subseteq \mathscr{M}$.

Theorem. Suppose that $\mathscr{N} \subseteq \mathscr{M}$ is a unital, expected inclusion of von Neumann algebras and assume that $\mathscr{N}$ admits a faithful normal tracial state. Then the relative Haagerup property of the triple $(\mathscr{M}, \mathscr{N}, \mathbb{E}_{\mathscr{N}})$ does not depend on the choice of the conditional expectation $\mathbb{E}_{\mathscr{N}}$.

We further prove that, similar to the non-relative setting, certain conditions of the definition can be relaxed upon. The key idea to the proofs of these statements is to use modular theory to pass to a tracial setting. However, the relative context makes the technical details quite demanding.

Other main results of Chapter 7 are the following.

Theorem. Suppose that $\mathscr{N} \subseteq \mathscr{M}$ is a unital, expected inclusion of von Neumann algebras and assume that $\mathscr{N}$ is finite-dimensional. Then the relative Haagerup property of the triple $(\mathscr{M}, \mathscr{N}, \mathbb{E}_{\mathscr{N}})$ is equivalent to the non-relative Haagerup property of $\mathscr{M}$.

Theorem. The relative Haagerup property is preserved under taking amalgamated free products over finite-dimensional subalgebras.

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We illustrate our results with examples coming from discrete quantum groups. Of particular interest is further the elementary case of $\mathscr{M} = \mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. It provides us with both triples $(\mathscr{B}(\mathscr{H}), \mathscr{N}, \mathbb{E}_{\mathscr{N}})$ that do and do not have the relative Haagerup property. This leads to rather surprising examples.

APPROXIMATION PROPERTIES OF HECKE OPERATOR ALGEBRAS

The idea of approximating complicated structures by simpler building blocks appears all over mathematics. One such instance is the Stone-Weierstrass theorem which, in its simplest form, states that continuous functions on an interval can be approximated by polynomials. Another example is property (AP) in Banach space theory which demands that every compact operator is the norm limit of finite rank operators. Also in the theory of C^*-algebras and von Neumann algebras approximation properties have huge importance. Over the years some of the most profound developments of the field, such as Murray-von Neumann's uniqueness result for the hyperfinite II_1-factor or Elliott's classification program for C^*-algebras, would not have been possible without the consideration of suitable approximation properties.

A famous question going back to von Neumann (and which of course relates to the free factor problem) asks if a discrete group can be recovered from its group von Neumann algebra. Treating the question – hence contributing to the classification theory of von Neumann algebras – is highly ambitious. An approach that has been effective so far is the usage of finite-dimensional approximations. An approximation property of a C^*-algebra or von Neumann algebra is a way of approximating the algebra by finite-dimensional (matrix) algebras. Nowadays there exists a whole zoo of approximation properties, one of which is the notion of nuclearity. A C^*-algebra A is called nuclear if there exists a sequence of contractive completely positive maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ and $\psi_n : M_{k_n}(\mathbb{C}) \to A$ with $k_n \in \mathbb{N}$ such that $\psi_n \circ \varphi_n \to \mathrm{id}_A$ pointwise, so roughly speaking A is nuclear if the identity map on A approximately factors through matrix algebras. One can think of nuclearity as a non-commutative analogue to the existence of a partition of unity. One of the reasons why this property is so useful is that lots of interesting C^*-algebras are known to be nuclear. It can further be characterized in a seemingly completely different way: a C^*-algebra A is nuclear if and only if for every other C^*-algebra B there is a unique C^*-algebra completion of the algebraic tensor product $A \odot B$. Other approximation properties are the (relative) Haagerup property that we discussed before in the general setting as well as exactness. A C^*-algebra A is called exact if it can be embedded into some space $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$ such that the embedding approximately factors through matrix algebras. Just as nuclearity, this property can be characterized qualitatively: a C^*-algebra A is exact if and only if (spatial) tensoring with A preserves short exact sequences.

The examples above indicate that the approximation theory of operator algebras lives on the edge between quantitative and qualitative considerations. It combines hard and soft analysis.

The aim of Chapter 8 is to study approximation properties of Hecke operator algebras. By using the results of the previous chapters we will prove that Hecke C^*-algebras are exact, we characterize their nuclearity and injectivity and we consider classes of Hecke-von Neumann algebras that satisfy the Haagerup property.

Theorem. Let W be a Coxeter system and let q be a multi-parameter. Then $C^*_{r,q}(W)$ is an exact C^*-algebra.

Theorem. Let W be a Coxeter group and q a multi-parameter. Then the following statements are equivalent:

(1) W is not non-affine;

(2) The Hecke C^*-algebra $C^*_{r,q}(W)$ is nuclear;

(3) The Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is injective.

Theorem. Let W be a virtually free Coxeter group and q a multi-parameter. Then the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ has the Haagerup property.

Fundamental properties of Hecke operator algebras

As mentioned earlier, a question going back to von Neumann asks if a discrete group can be recovered from its group von Neumann algebra. That is, is every (discrete) group G W^*-superrigid in the sense that if $\mathscr{L}(G) \cong \mathscr{L}(H)$ for some other (discrete) group H, then $G \cong H$? Similarly one can ask for whether an action $G \curvearrowright X$ can be recovered from its crossed product von Neumann algebra $L^\infty(X) \overline{\rtimes} G$. These questions were famously answered in the negative by Connes who proved that for every countable amenable group G that only has infinite non-trivial conjugacy classes, the von Neumann algebra $\mathscr{L}(G)$ is isomorphic to the unique hyperfinite II_1-factor. A similar result holds in the crossed product setting as well. Connes' result implies that group von Neumann algebras and crossed product von Neumann algebras associated with certain amenable groups admit a striking lack of rigidity; almost all the group theoretic information is lost as soon as one passes to the von Neumann algebra setting. On the other hand, in 1980 Connes proved that if G is a (discrete) group with infinite conjugacy classes that admits "property (T)", then any automorphism of the group von Neumann algebra $\mathscr{L}(G)$ which is in a certain sense close to the identity must be implemented by conjugation with unitary elements from $\mathscr{L}(G)$. In other words, the identity map on $\mathscr{L}(G)$ can not be "deformed" by automorphisms all of which are outer.

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Initiated by the seminal work of Popa starting in 2001 the classification of group and crossed product von Neumann algebras made enormous progress. Popa's deformation/rigidity approach, consisting of studying classes of II_1-factors (i.e. indecomposable infinite-dimensional von Neumann algebras that carry a tracial state) that admit both a deformation property as well as a rigidity property, led to several important rigidity results, one of which states that for $m \neq n$ the crossed product von Neumann algebra $L^\infty(X)\overline{\rtimes}\mathbb{F}_n$ associated with a certain type of action $\mathbb{F}_n \curvearrowright X$ can be distinguished from that of $L^\infty(Y)\overline{\rtimes}\mathbb{F}_m$ for a certain type of other action $\mathbb{F}_m \curvearrowright Y$. This statement should be compared with the free factor problem.

In the setting of the rigidity theory of (group) von Neumann algebras the consideration of boundaries has turned out to be very useful. By using properties of the canonical action of word hyperbolic groups on their Gromov boundary, Higson and Guentner deduced in [105] that for every such group G the algebraic tensor product map

$$C_r^*(G) \odot JC_r^*(G)J \to \mathscr{B}(\ell^2(W))/\mathscr{K}(\ell^2(G)), x \otimes y \mapsto xy + \mathscr{K}(\ell^2(G))$$

continuously extends with respect to the minimal tensor norm on $C_r^*(G) \odot JC_r^*(G)J$. Here J is the anti-linear operator $\delta_g \mapsto \delta_{g^{-1}}$ and $\mathscr{K}(\ell^2(G))$ denotes the compact operators on $\ell^2(G)$. The same statement had earlier been shown by Akemann and Ostrand for free groups by a different method, see [1]. This "Akemann-Ostrand property" (property $(\mathscr{AO})$) was formally introduced and famously applied by Ozawa in [148] to rigidity questions of von Neumann algebras. In particular, he proved that certain von Neumann algebras that admit a dense C*-subalgebra satisfying (a variant of) property $(\mathscr{AO})$, are prime in the sense that they can not be decomposed as a tensor product of infinite-dimensional von Neumann algebras. Using similar notions and ideas, Ozawa and Popa were able to find classes of von Neumann algebras that can not be written as certain crossed product von Neumann algebras (or more generally von Neumann algebras induced by equivalence relations).

By using a method similar to that of Higson and Guentner in combination with our study of the canonical action of a Coxeter group W on its compactification $\overline{(W,S)}$ and boundary $\partial(W,S)$ in Chapter 5, in Chapter 9 we prove that Hecke-von Neumann algebras of Coxeter groups whose reflection centralizers are all finite satisfy a variant of the Akemann-Ostrand property.

Theorem. Let W be a Coxeter group generated by S with $\#S < \infty$ and $\#\{\mathbf{w} \in W \mid s\mathbf{w} = \mathbf{w}s\} < \infty$ for all $s \in S$. Let further q be a multi-parameter. Then the Hecke-von Neumann algebra $\mathscr{N}_q(W) \subseteq \mathscr{B}(\ell^2(W))$ satisfies the Akemann-Ostrand property.

As an immediate consequence, we deduce that Dykema's interpolated free group factors share this property. The statement is known to experts.

Corollary. For every $t \in \mathbb{R}_{>1}$ the interpolated free group factors $\mathscr{L}(\mathbb{F}_t)$ satisfy the Akemann-Ostrand property.

We further exploit some related implications of property $(\mathscr{AO})$ that are related to Connes' notion of fullness.

2

Preliminaries and notation

In the following, we present the background required for the later parts of this thesis. The content and the notation is as it appears in [49], [126], [127], [128] and [50].

2.1. General notation

We will write $\mathbb{N} := \{0, 1, 2, ...\}$ and $\mathbb{N}_{\geq 1} := \{1, 2, ...\}$ for the natural numbers. The disjoint union of a family $(A_i)_{i \in I}$ of sets is denoted by $\bigsqcup_{i \in I} A_i$. For a unital ring R, $R^\times$ is the set of units. Hilbert spaces and von Neumann algebras are usually denoted by calligraphic letters where scalar products of Hilbert spaces are always assumed to be linear in the first variable. We denote the bounded operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ and the corresponding unitary group is denoted by $\mathcal{U}(\mathcal{H})$. For a discrete group G write $(\delta_g)_{g \in G} \subseteq \ell^2(G)$ for the canonical orthonormal basis in $\ell^2(G)$. The symbol $\odot$ denotes the algebraic tensor product of $*$-algebras. The neutral element of a group is always denoted by e and for a set S we write $\#S$ for the number of elements in S and χ_S for the characteristic function on S.

2.2. C^*-algebras and von Neumann algebras

Originating in Heisenberg's formalism of matrix mechanics, the theory of operator algebras was introduced as an abstraction of algebras of physical observables appearing in quantum theory. Its rigorous foundation was built by Murray and von Neumann in a series of publications on rings of operators, see [136], [137], [140], [138].

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A C^**-algebra* is a complex Banach algebra A endowed with an involution $x \mapsto x^*$ which is compatible with the norm on A in the sense that $\|x^* x\| = \|x\|^2$ for all $x \in A$. Though it is not immediately clear, this inconspicuous C^**-condition* implies an intimate relationship between the algebraic and the analytic structure of the C^*-algebra A. Basic examples of C^*-algebras are the commutative algebras $C_0(X)$ of functions on X which *vanish at infinity* where X is a locally compact Hausdorff space, where the algebra is endowed with the supremum norm and where the involution is given by complex conjugation $f \mapsto \overline{f}$. One of the first structural results on C^*-algebras is Gelfand's theorem which asserts that (up to $*$-isomorphism) every commutative C^*-algebra arises in such a way. A second result by Gelfand and Naimark states that every (abstract) C^*-algebra can be represented concretely as a norm-closed $*$-subalgebra of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on some complex Hilbert space $\mathscr{H}$. Gelfand's theorem and the Gelfand-Naimark theorem (and their constructive proofs) form the foundation of the theory of operator algebras.

We shall assume throughout this dissertation that the reader is familiar with the basic concepts of C^*-algebras, including *Gelfand's theorem*, *positive elements*, (*continuous*) *functional calculus*, *inductive limits* of C^*-algebras, the *GNS-construction*, spatial *tensor products* of C^*-algebras (which we will denote by $\otimes$) and *completely positive maps*. There are a number of standard textbooks on C^*-algebras, some of which are [70], [171], [172], [135], [65], [20], [33].

Though introduced earlier by Murray and von Neumann, von Neumann algebras build a subclass of C^*-algebras. A *von Neumann algebra* is a unital $*$-subalgebra of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on some complex Hilbert space $\mathscr{H}$, which is closed in the strong operator topology. Von Neumann algebras can be described more algebraically as unital $*$-subalgebras of $\mathscr{B}(\mathscr{H})$ which coincide with their bicommutant. There exists a von Neumann algebraic analogue to Gelfand's theorem which identifies a commutative von Neumann algebra $\mathscr{N}$ with $L^\infty(X,\mu) \subseteq \mathscr{B}(L^2(X,\mu))$ where (X,μ) is a locally finite measure space. Consequently, the study of von Neumann algebras is sometimes referred to as *non-commutative measure theory*. In many applications, it suffices to study *factorial* von Neumann algebras (that is von Neumann algebras whose center is trivial), the reason being that every von Neumann algebra decomposes as a *direct integral* of factorial von Neumann algebras. These von Neumann algebras can hence be viewed as the building blocks of more general von Neumann algebras. Factorial von Neumann algebras fall into one of three classes: *type I* von Neumann algebras contain a *minimal projection* (i.e. a projection $p \neq 0$ such that for all other projections q with $q \leq p$ either $q = 0$ or $q = p$), *type II* von Neumann algebras contain no minimal projection but a *finite* one (i.e. a projection p such that no other projection $q \lneq p$ is equivalent to p in the sense that $vv^* = p$, $v^* v = q$ for some partial isometry v), and *type III* von Neumann algebras are those von Neumann algebras whose non-zero projections are *infinite*.

Again, we shall assume throughout this dissertation that the reader is familiar with the basic concepts of von Neumann algebra theory, including *type classifica-*

tion, *direct integral decompositions*, the different *topologies on von Neumann algebras*, *Borel functional calculus*, *tensor products* of von Neumann algebras (which we will denote by $\overline{\otimes}$), *standard forms*, *Tomita-Takesaki modular theory* and the (*Takai-*)*Takesaki duality*. Standard references are [169], [171], [172], [20], [5].

2

2.3. DYNAMICAL SYSTEMS AND CROSSED PRODUCTS

The study of dynamical systems is an important subject on its own which has a long history. It can be linked to the theory of operator algebras which leads to the concept of C*-dynamical and W*-dynamical systems and their crossed products. This link, going back to von Neumann's work, provides interesting examples of operator algebras and turned out to be of use both for the investigation of internal structures as well as for applications. Standard references are [172], [182] and [33].

A *group action* of a group G on a set X is a map $G \times X \to X$, $(g,x) \mapsto g.x$ which is compatible with the group structure of G in the sense that $e.x = x$ and $g.(h.x) = (gh).x$ for all $g,h \in G$, $x \in X$. A subset $Y \subseteq X$ is called *G-invariant* if $g.y \in Y$ for all $g \in G$, $y \in Y$. If X is a topological space and G is a topological group acting on it, then the action is called *continuous* if the map $(g,x) \mapsto g.x$ is continuous. In this case X is called a (*left*) *G-space*, the pair (G,X) is called a *transformation group* and we often write $G \curvearrowright X$ for the (continuous) group action. Without further mention, we will usually assume that the space X is a locally compact Hausdorff space on which the locally compact group G acts continuously.

The C*-algebraic analogue to these topological dynamics are C*-dynamical systems.

Definition 2.3.1. A *C*-dynamical system* is a triple (A,G,α) consisting of a C*-algebra A, a locally compact group G and a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ of G into the *automorphism group* of A which is continuous with respect to the point-norm topology on $\mathrm{Aut}(A)$. We usually just say that the group G acts (C*-continuously) on A and write $G \overset{\alpha}{\curvearrowright} A$ or $G \curvearrowright A$. Further, we usually write $g.x := \alpha_g(x)$ for $g \in G$, $x \in A$. A subset $X \subseteq A$ is called *G-invariant* if $g.x \in X$ for all $g \in G$, $x \in X$.

The link between the notion of C*-dynamical systems and classical dynamics becomes apparent in the following way. A continuous action $G \curvearrowright X$ of a locally compact group G on a locally compact Hausdorff space X induces a C*-dynamical system $\alpha : G \to \mathrm{Aut}(C_0(X))$, $g \mapsto \alpha_g$ via $\alpha_g(f)(x) := f(g^{-1}.x)$ where $f \in C_0(X)$ and $x \in X$. Conversely, if G acts on $C_0(X)$ this induces a unique continuous group action $G \curvearrowright X$ with $(g.f)(x) = f(g^{-1}.x)$ for all $g \in G$, $f \in C_0(X)$, $x \in X$, see [182, Proposition 2.7].

Every action $G \curvearrowright A$ of a locally compact group G on a C*-algebra A induces a natural continuous action of G on the *state space* $\mathscr{S}(A)$ of A equipped with the weak-$*$ topology via $(g.\phi)(x) := \phi(g^{-1}.x)$ for $g \in G$, $x \in A$. Similarly, for a locally compact Hausdorff space X denote by $\mathrm{Prob}(X)$ the space of all *probability measures* on X equipped with the weak-$*$ topology. Then a continuous action $G \curvearrowright X$ of G on X induces a continuous action $G \curvearrowright \mathrm{Prob}(X)$ via $(g.\mu)(x) := \mu(g^{-1}.x)$.

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Definition 2.3.2. Let G be a locally compact group which acts on a C^*-algebra A. A covariant representation (π, U) of the C^*-dynamical system (A, G, α) consists of a Hilbert space $\mathscr{H}$, a $*$-representation $\pi : A \to \mathscr{B}(\mathscr{H})$ and a unitary representation $U : G \to \mathscr{U}(\mathscr{H})$, $g \mapsto U_g$ such that $U_g \pi(x) U_g^* = \pi(g.x)$ for all $g \in G$, $x \in A$.

In the following let μ be a fixed left Haar measure on G.

Every covariant representation (π, U) of the C^*-dynamical system (A, G, α) induces a $*$-representation $\pi \rtimes U : C_c(G, A) \to \mathscr{B}(\mathscr{H})$, the *induced representation*, of the *convolution algebra* $C_c(G, A)$ of compactly supported continuous functions $f : G \to A$ on $\mathscr{B}(\mathscr{H})$ via

$$\pi \rtimes U(f) := \int_G \pi(f(g)) U_g \, d\mu(g).$$

Covariant representations always exist. Indeed, assume that $A \subseteq \mathscr{B}(\mathscr{H})$ is a C^*-subalgebra for some Hilbert space $\mathscr{H}$ and define $\pi : A \to \mathscr{B}(L^2(G, \mathscr{H}))$ by

$$(\pi(x)\xi)(g) := (g^{-1}.x)(\xi(g))$$

for $g \in G$, $\xi \in L^2(G, \mathscr{H})$ and $\lambda : G \to \mathscr{U}(\mathscr{H})$ by

$$\lambda_g(\xi)(h) := \xi(g^{-1}h)$$

for $g, h \in G$, $\xi \in L^2(G, \mathscr{H})$. Then (π, λ) is a covariant representation, called the *(left) regular covariant representation* of the system $G \curvearrowright A$. It can be proved that the corresponding induced representation $\pi \rtimes \lambda : C_c(G, A) \to \mathscr{B}(L^2(G, \mathscr{H}))$ is faithful, that for every $f \in C_c(G, A)$ the operator norm $\|f\|_r := \|(\pi \rtimes \lambda)(f)\|$ is finite and that it does not depend on the choice of the embedding $A \subseteq \mathscr{B}(\mathscr{H})$. This motivates the following definition.

Definition 2.3.3. Let G be a locally compact group which acts on a C^*-algebra $A \subseteq \mathscr{B}(\mathscr{H})$. The corresponding *reduced crossed product C^*-algebra*, denoted by $A \rtimes_{\alpha,r} G$, is the completion of $C_c(G, A)$ with respect to the C^*-norm $\|\cdot\|_r$.

Similarly, the universal crossed product C^*-algebra, denoted by $A \rtimes_{\alpha,u} G$, is the completion of $C_c(G, A)$ with respect to the (well-defined) C^*-norm

$$\|f\|_u := \left\{ \|\pi \rtimes U(f)\| \mid (\pi, U) \text{ is a covariant representation of } (A, G, \alpha) \right\}.$$

If the action $\alpha : G \to \mathrm{Aut}(A)$ is clear, we usually just write $A \rtimes_r G$, instead of $A \rtimes_{\alpha,r} G$ and $A \rtimes_u G$, instead of $A \rtimes_{\alpha,u} G$.

From the definition it is clear that the identity map on $C_c(G, A)$ extends to a surjective $*$-homomorphism $A \rtimes_{\alpha,u} G \twoheadrightarrow A \rtimes_{\alpha,r} G$. By construction, we can further view $A \rtimes_{\alpha,r} G$ as a C^*-subalgebra of $\mathscr{B}(L^2(G, \mathscr{H}))$.

Following [33], for notational convenience we denote for C^*-dynamical systems (A, G, α) with discrete G the elements of $C_c(G, A) \subseteq A \rtimes_{\alpha,u} G$ by finite sums of the form $f = \sum_{g \in G} f_g g$ where $f_g \in A$. Similarly, we can view the elements of

$C_c(G,A) \subseteq A \rtimes_{\alpha,r} G$ as finite sums of the form $f = \sum_{g \in G} f_g \lambda_g$ where $f_g \in A$. In both cases A canonically embeds into the corresponding crossed product via $x \mapsto xe$ and $x \mapsto x\lambda_e$. The map $C_c(G,A) \to A$ given by $\sum_{g\in G} f_g\lambda_g \mapsto f_e$ continuously extends to a *conditional expectation* $E : A \rtimes_{\alpha,r} G \to A$ (i.e. a contractive completely positive A-bimodule map with $E|_A = \mathrm{id}_A$) which is *faithful* (i.e. $E(x^*x) \neq 0$ for all $0 \neq x \in A \rtimes_{\alpha,r} G$), see [33, Proposition 4.1.9].

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Particularly important is the case of trivial dynamical systems. Note that $\mathbb{C}$ only provides trivial $*$-automorphisms. For a locally compact group G we denote by $C_r^*(G) := \mathbb{C} \rtimes_{\alpha,r} G$ the *reduced group* C^**-algebra* of G and by $C_u^*(G) := \mathbb{C} \rtimes_{\alpha,u} G$ the *universal group* C^**-algebra* of G. The corresponding unitary representation $\lambda : G \to \mathcal{U}(L^2(G))$, $g \mapsto \lambda_g$ is called the *left regular representation.* If the group G is discrete, $C_r^*(G)$ carries a canonical tracial state $\tau : C_r^*(G) \to \mathbb{C}$ given by

$$\tau(\lambda_g) := \langle \lambda_g \delta_e, \delta_e \rangle = \begin{cases} 1 & , \text{if } g = e \\ 0 & , \text{if } g \neq e \end{cases}.$$

There is also a von Neumann algebraic analogue of crossed products and group algebras that will play a role in Chapter 7.

Definition 2.3.4. A W^**-dynamical system* is a triple $(\mathcal{N}, G, \alpha)$ consisting of a von Neumann algebra $\mathcal{N}$, a locally compact group G and a group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{N})$ of G into the *automorphism group* of $\mathcal{N}$ which is continuous with respect to the point-strong topology on $\mathrm{Aut}(\mathcal{N})$. We usually just say that the group G acts (W*-continuously) on $\mathcal{N}$ and write $G \overset{\alpha}{\curvearrowright} \mathcal{N}$ or $G \curvearrowright \mathcal{N}$. Further, we usually write $g.x := \alpha_g(x)$ for $g \in G$, $x \in \mathcal{N}$. A subset $X \subseteq \mathcal{N}$ is called *G-invariant* if $g.x \in X$ for all $g \in G$, $x \in X$.

Note that a W*-continuous action $G \overset{\alpha}{\curvearrowright} \mathcal{N}$ does not necessarily define a C*-dynamical system, i.e. the map α is not in general point-norm continuous. If the group G is discrete this is however the case.

Similar to before, for a W*-dynamical system $(\mathcal{N}, G, \alpha)$ one can define a W*-crossed product of $\mathcal{N}$ by G. For this, assume that $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann subalgebra for some Hilbert space $\mathcal{H}$ and define $\pi : \mathcal{N} \to \mathcal{B}(L^2(G,\mathcal{H}))$ by

$$(\pi(x)\xi)(g) := (g^{-1}.x)(\xi(g))$$

for $g \in G$, $\xi \in L^2(G,\mathcal{H})$ and $\lambda : G \to \mathcal{U}(\mathcal{H})$ by

$$\lambda_g(\xi)(h) := \xi(g^{-1}h)$$

for $g,h \in G$, $\xi \in L^2(G,\mathcal{H})$. Then (π,λ) is a covariant representation of the system $G \curvearrowright \mathcal{N}$. The corresponding induced representation $\pi \rtimes \lambda : C_c(G,A) \to \mathcal{B}(L^2(G,\mathcal{H}))$ is faithful.

Definition 2.3.5. Let G be a locally compact group which acts on a von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$. The corresponding *crossed product von Neumann algebra*, denoted by $A \bar{\rtimes}_\alpha G$, is defined as

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$$\mathcal{N} \bar{\rtimes}_\alpha G := ((\pi \rtimes \lambda)(C_c(G, \mathcal{N})))'' \subseteq \mathcal{B}(L^2(G, \mathcal{H})).$$

If the action $\alpha : G \to \mathrm{Aut}(\mathcal{N})$ is clear, we usually just write $\mathcal{N} \bar{\rtimes} G$, instead of $\mathcal{N} \bar{\rtimes}_\alpha G$.

The crossed product von Neumann algebra of a given W*-dynamical system does, up to isomorphism, not depend on the way $\mathcal{N}$ is represented on $\mathcal{H}$. Again, we can view $C_c(G, \mathcal{N}))$ as a $*$-subalgebra of $\mathcal{N} \bar{\rtimes}_\alpha G$ whose elements are - in the case where G is discrete - finite sums of the form $f = \sum_{g \in G} f_g \lambda_g$ where $f_g \in \mathcal{N}$ and $\mathcal{N}$ canonically embeds into $\mathcal{N} \bar{\rtimes}_\alpha G$ via $x \mapsto x\lambda_e$. In this setting the map $C_c(G, \mathcal{N}) \to \mathcal{N}$ given by $\sum_{g \in G} f_g \lambda_g \mapsto f_e$ continuously extends to a faithful normal conditional expectation $E : \mathcal{N} \bar{\rtimes}_\alpha G \to A$.

For a locally compact group G we denote by $\mathcal{L}(G) := \mathbb{C} \bar{\rtimes} G$ the *group von Neumann algebra* of G. Again, the corresponding representation $\lambda : G \to \mathcal{U}(L^2(G))$, $g \mapsto \lambda_g$ is called the *left regular representation* and if the group G is discrete, $\mathcal{L}(G)$ carries a canonical normal tracial state $\tau : \mathcal{L}(G) \to \mathbb{C}$ given by

$$\tau(\lambda_g) := \langle \lambda_g \delta_e, \delta_e \rangle = \begin{cases} 1 & \text{, if } g = e \\ 0 & \text{, if } g \neq e \end{cases}.$$

2.3.1. AMENABILITY

Crossed products of dynamical systems provide a way to encode the information of a given group action by an operator algebra. It is natural to ask if one can read off properties of the action from the corresponding crossed product C*-algebra (resp. crossed product von Neumann algebra) and vice versa. In the following, we will consider certain dynamical notions and their operator algebraic counterparts that will play a role in the later chapters. We will mostly restrict to discrete groups and C*-dynamical systems.

Definition 2.3.6 ([33, Definition 4.3.1])**.** Let G be a discrete group acting on a unital C*-algebra A. The action $G \curvearrowright A$ is called *amenable* if there exists a sequence $(T_i)_{i \in \mathbb{N}} \subseteq C_c(G, A)$ such that the following conditions hold:

- For every $g \in G$, $i \in \mathbb{N}$ the element $T_i(g) \in A$ is positive and contained in the center of A;
- $\sum_{g \in G} (T_i(g))^2 = 1_A$ for every $i \in \mathbb{N}$;
- $\sum_{g \in G} \left| h.(T_i(g)) - T_i(hg) \right|^2 \to 0$ for every $h \in G$.

Crossed product C*-algebras associated with amenable group actions satisfy several strong properties. For instance, the amenability of the action $G \curvearrowright A$ implies that the canonical surjection $A \rtimes_u G \twoheadrightarrow A \rtimes_r G$ is an isomorphism (see [33, Theorem

4.3.4]). Further, the nuclearity (resp. exactness) of A implies the nuclearity (resp. exactness) of $A \rtimes_r G$. For an excellent introduction to this topic (and discrete crossed products in general), see [33].

The commutative analogue of Definition 2.3.6 is the following.

Definition 2.3.7 ([33, Definition 4.3.5]). Let G be a discrete group acting on a compact Hausdorff space X. The action $G \curvearrowright X$ is called (*topologically*) *amenable* if there exists a sequence $(m_i : X \to \mathrm{Prob}(G), x \mapsto m_i^x)_{i\in\mathbb{N}}$ of continuous functions such that for every $g \in G$,

$$\lim_{i\to\infty} \sup_{x\in X} \left(\sum_{h\in G} \left| m_i^x(g^{-1}h) - m_i^{g.x}(h) \right| \right) = 0.$$

It can be shown that an action $G \curvearrowright X$ of a discrete group G on a compact Hausdorff space X is amenable if and only if the induced action $G \curvearrowright C(X)$ is amenable, see [33, Theorem 4.3.7]. In this sense Definition 2.3.6 and Definition 2.3.7 are compatible with each other. The reader should be aware that there is an analogue to topological amenability in the ergodic theoretic setting which is sometimes called *Zimmer amenability*. Both notions should not be confused.

A discrete group G is called *amenable* if the trivial action of G on the one-point set is amenable. Every group action $G \curvearrowright X$ of an amenable group on a compact Hausdorff space is amenable. Further, amenability of groups can be characterized in several equivalent ways, some of which are listed below.

Theorem 2.3.8 ([33, Theorem 2.6.8]). *Let G be a discrete group. Then the following statements are equivalent:*

- *G is amenable;*
- *$C_u^*(G) \cong C_r^*(G)$ via $g \mapsto \lambda_g$;*
- *$C_r^*(G)$ has a character (i.e. a one-dimensional representation);*
- *$C_r^*(G)$ is nuclear.*

2.3.2. SMALLNESS AT INFINITY

Let G be a discrete group. Recall that $\ell^\infty(G)$ can be viewed as a commutative C*-subalgebra of $\mathcal{B}(\ell^2(G))$ via $f\delta_g := f(g)\delta_g$ and that G acts on $\ell^\infty(G)$ via $(g.f)(h) := f(g^{-1}h)$ for $f \in \ell^\infty(G)$, $g, h \in G$. Following [33, Chapter 5.3], a compact Hausdorff space $\overline{G}$ which contains G as a dense open subset, is called a *compactification* of G. It is (*left*) *equivariant* if the canonical left translation action of G on itself extends to a continuous action on $\overline{G}$. From Gelfand duality it follows that (left equivariant) compactifications bijectively correspond to (G-invariant) C*-subalgebras A with $C_0(G) \subseteq A \subseteq \ell^\infty(G)$.

A notion that (often implicitly) appears in the rigidity theory of group von Neumann algebras (see e.g. [119], [104] and [105]) is that of compactifications that are small at infinity.

Definition 2.3.9 ([33, Definition 5.3.16])**.** Let G be a discrete group and $\overline{G}$ an equivariant compactification of G. Then $\overline{G}$ is said to be *small at infinity* if for every net $(g_i)_{i\in I} \subseteq G$ with $g_i \to z$ for some boundary point $z \in \overline{G} \setminus G$, one has $g_i h \to z$ for all $h \in G$.

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We will see in Chapter 9 that the study of a certain equivariant compactification which is small at infinity also implies rigidity properties of Hecke-von Neumann algebras.

Equivariant compactifications that are small at infinity can be characterized in the following way. For $f \in \ell^\infty(G)$ and $g \in G$ define $f^g \in \ell^\infty(G)$ by $f^g(h) := f(hg^{-1})$. Then the spectrum of a G-invariant C*-algebra A with $C_0(G) \subseteq A \subseteq \ell^\infty(G)$ is small at infinity if and only if $f^g - f \in C_0(G)$ for all $f \in A$, $g \in G$.

2.3.3. BOUNDARY ACTIONS

Topological and measurable boundary actions have been introduced by Furstenberg in [84], [85] in the context of rigidity of Lie groups. Compared to its measurable counterpart, the notion of topological boundary actions initially received much less attention. That changed after Kalantar and Kennedy established in [120] a connection between the dynamical properties of the Furstenberg boundary of a given discrete group and the question for simplicity, uniqueness of trace and tightness of nuclear embedding of the corresponding reduced group C*-algebra. A series of breakthroughs and generalizations followed (see e.g. [95], [30], [125] and [16], [102], [121], ...).

Definition 2.3.10. Let G be a discrete group continuously acting on a compact Hausdorff space X.

- The action $G \curvearrowright X$ is called *minimal* if for every $x \in X$ the G-orbit $G.x$ is dense in X.
- The action $G \curvearrowright X$ is called *strongly proximal* if for every probability measure $\nu \in \text{Prob}(X)$ the weak-$*$ closure of the G-orbit $G.\nu$ contains a point mass $\delta_x \in \text{Prob}(X)$ for some $x \in X$.
- The action $G \curvearrowright X$ is *topologically free* if for every $g \in G \setminus \{e\}$ the set X^g of points fixed by g has no inner points.

X is called a *G-boundary* if the action of G on X is both minimal and strongly proximal. In that case the action is called a *boundary action*.

Furstenberg proved in [85, Proposition 4.6] that every discrete group G admits a unique G-boundary $\partial_F G$ that is universal in the sense that every other G-boundary is a continuous G-equivariant image of $\partial_F G$. It is called the *Furstenberg boundary* of the group G. In [120] Kalantar and Kennedy demonstrated that the algebra $C(\partial_F G)$ of continuous functions on $\partial_F G$ identifies with Hamana's *G-injective envelope* (see [97], [98], [99], [100]) of the complex numbers $\mathbb{C}$, i.e. the minimal C*- subalgebra of $\ell^\infty(G)$ that arises as the image of a unital positive G-equivariant projection. This connection implies a dynamical characterization of the C*-simplicity of discrete groups: the reduced group C*-algebra of a discrete group G is *simple* (that is it contains no non-trivial two-sided closed ideals) if and only if the action $G \curvearrowright \partial_F G$ is topologically free, see [120, Theorem 6.2]. The proof of this statement requires a result by Archbold and Spielberg [9] which states that for a discrete group G which continuously acts on a compact Hausdorff space X, the corresponding reduced crossed product C*-algebra $C(X) \rtimes_r G$ is simple if and only if the action is minimal, topologically free and *regular* in the sense that the kernel of the canonical surjection $A \rtimes_u G \twoheadrightarrow A \rtimes_r G$ is trivial. We will also make use of this fact in Section 6.2.

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2.4. PARTIALLY ORDERED SETS

The following definitions are as they appear in [19]. Let $\mathscr{S}$ be a set. A *partial order* on $\mathscr{S}$ is a binary relation $\leq$ which is *reflexive*, *antisymmetric* and *transitive*. A set endowed with a partial order is called a *partially ordered set* (*poset*). If existent, the *join* of a subset $T \subseteq \mathscr{S}$, denoted by $\vee T$, is the least upper bound of T, meaning that $\vee T \geq y$ for every $y \in T$ and $\vee T \leq x$ for every $x \in \mathscr{S}$ with $x \geq y$ for every $y \in T$. In the same manner, the *meet* of T, denoted by $\wedge T$, is the greatest lower bound of T, meaning that $\wedge T \leq y$ for every $y \in T$ and $\wedge T \geq x$ for every $x \in \mathscr{S}$ with $x \leq y$ for every $y \in T$.

In general, the join and the meet of a subset of a partially ordered set do not necessarily exist. A poset $\mathscr{S}$ in which all pairs $\{x, y\}$, $x, y \in \mathscr{S}$ of elements have a join is called a *join-semilattice*. If every non-empty subset has a join, it is called a *complete join-semilattice*. Dually, a poset in which all pairs of elements have a meet is called a *meet-semilattice*. If every non-empty subset has a meet, it is called a *complete meet-semilattice*.

The following lemma is standard. We include a proof (which also appears in [127, Lemma 2.2]) for the convenience of the reader.

Lemma 2.4.1. *Let $\mathscr{S}$ be a complete meet-semilattice and $T \subseteq \mathscr{S}$ a set. If T has an upper bound (i.e. an element $x \in \mathscr{S}$ with $x \geq y$ for all $y \in T$), then the join $\vee T$ exists.*

Dually, if $\mathscr{S}$ is a complete join-semilattice and $T \subseteq \mathscr{S}$ a set with a lower bound (i.e. an element $x \in \mathscr{S}$ with $x \leq y$ for all $y \in T$), then the meet $\wedge T$ exists.

Proof. Let $\mathscr{S}$ be a complete meet-semilattice and $T \subseteq \mathscr{S}$ a set having an upper bound. Then the set $T' := \{x \in \mathscr{S} \mid y \leq x \text{ for all } y \in T\}$ is non-empty. Because $\mathscr{S}$ is a complete meet-semilattice, the meet $x := \wedge T'$ exists. It satisfies $y \leq x$ for all $y \in T$ and $x' \geq x$ for all $x' \in T'$, i.e. x is the join of the set S. The second statement follows analogously. □

2.5. GRAPHS AND TREES

2.5.1. BASIC NOTIONS

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A *graph* K is a pair $K = (V, E)$ consisting of a *vertex set* V and an *edge set* $E \subseteq V \times V$. In the case where the vertex and edge set of the graph K are not designated, we will often write $x \in K$, meaning that x is a vertex of the graph K. A graph is called *finite* if V is finite, it is called *countable* if V is countable, it is called *undirected* if for every element $(x, y) \in E$ also $(y, x) \in E$ and it is called *simplicial* if $(x, x) \notin E$ for every $x \in V$. We will always assume that the graphs appearing in this thesis are countable, undirected and simplicial. We call a second graph $K_0 = (V_0, E_0)$ a *subgraph* of K if $V_0 \subseteq V$ and $E_0 \subseteq E$. In this case we write $K_0 \subseteq K$.

Two vertices $x, y \in K$ are *adjacent* if $(x, y) \in E$. A path $\alpha = (\alpha_i)_i$ of length $n \in \mathbb{N} \cup \{\infty\}$ is a sequence $\alpha_0 ... \alpha_n$ of n vertices for which $(\alpha_i, \alpha_{i+1}) \in E$ for every $0 \le i < n$. We call K *connected* if every two vertices of K can be connected by a path. This induces a natural metric d_K on K via

$$d_K(x, y) := \min\{n \mid \text{there is a path of length } n \text{ that connects } x \text{ and } y\}.$$

We call this the *graph metric* on K. A path α is called *geodesic* if $d_K(\alpha_i, \alpha_j) = |i - j|$ for all i, j. Without further comments we will often extend a finite geodesic path $\alpha_0 ... \alpha_n$ to an infinite path via $\alpha_0 ... \alpha_n \alpha_n \alpha_n ...$ and still call it (finite) geodesic. Further, we say that K is a *tree* if it is connected and if there is no finite path $\alpha = (\alpha_i)_{i=0,...,n}$ with $\alpha_0 = \alpha_n$ for which the vertices $\alpha_1, ..., \alpha_n$ are pairwise distinct.

A vertex of a graph is said to have *finite degree* if the number of vertices that are adjacent to it is finite. A graph whose vertices all have finite degree is called *locally finite*. If there is a uniform bound on the degree of vertices, we say that the graph is *uniformly locally finite*.

For a vertex x in the graph $K = (V, E)$ we call

$$\operatorname{Link}(x) := \{y \in K \mid (x, y) \in E\}$$

the *link* of x and denote the *common link* of a subset $X \subseteq K$ by

$$\operatorname{Link}(X) := \bigcap_{x \in X} \operatorname{Link}(x),$$

where by convention $\operatorname{Link}(\emptyset) := V$. Note that we may view $\operatorname{Link}(X)$ as a subgraph of K by declaring two vertices $x, y \in \operatorname{Link}(X)$ to be connected if $(x, y) \in E$. We further define $\operatorname{Star}(x) := \operatorname{Link}(x) \cup \{x\}$.

A *clique* in the graph K is a subgraph $K_0 \subseteq K$ in which every two vertices share an edge. We write $\operatorname{Cliq}(K)$ for the set of cliques in K and $\operatorname{Cliq}(K, l)$ for the set of cliques with l vertices. By convention we will assume that $\emptyset \in \operatorname{Cliq}(K)$. For a subgraph $K_0 \subseteq K$ set

$$\operatorname{Comm}(K_0) := \{(K_1, K_2) \subseteq \operatorname{Link}(K_0) \times \operatorname{Link}(K_0) \mid K_1, K_2 \in \operatorname{Cliq}(K) \text{ and } K_1 \cap K_2 = \emptyset\}.$$

The following lemmas are standard (see for instance [33, Lemma 5.2.3 and Lemma E.2]).

Lemma 2.5.1. *Let K be a connected graph, $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ an infinite geodesic path in K and $x \in K$. Then there exists an infinite geodesic path $\beta = (\beta_i)_{i\in\mathbb{N}}$ with $x = \beta_0$ which eventually flows into α in the sense that there exists $k \in \mathbb{Z}$ such that $\beta_i = \alpha_{i+k}$ for all large enough i.*

Lemma 2.5.2. *Let T be a tree and $x, y \in T$. Then there exists a unqiue geodesic path connecting x and y.*

We will usually denote the unique geodesic path appearing in Lemma 2.5.2 by $[x, y]$.

The most important class of graphs that we consider in this dissertation are Cayley graphs of finitely generated groups.

Example 2.5.3. Let G be a group generated by a set S where $e \notin S$. Set $S^{-1} := \{g^{-1} \mid g \in S\}$. The *Cayley graph* $\mathrm{Cay}(G, S)$ of G with respect to S is the graph defined by the vertex set G and the edge set $\{(g, h) \in G \times G \mid g^{-1}h \in S \cup S^{-1}\}$. If S is finite, the corresponding Cayley graph is countable, undirected, simplicial and uniformly locally finite.

Given a family $(K_i)_{i\in I}$ of graphs $K_i := (V_i, E_i)$ one can build the *(Cartesian) product* $K := \prod_{i\in I} K_i$. Its vertex set is given by the product $\prod_{i\in I} V_i$ and two vertices $\mathbf{x} = (x_i)_{i\in I}$ and $\mathbf{y} = (y_i)_{i\in I}$ are defined to be adjacent to each other if and only if there exists $i_0 \in I$ with $x_i = y_i$ for all $i \in I \setminus \{i_0\}$ and $(x_{i_0}, y_{i_0}) \in E_{i_0}$. This gives K the structure of a graph. The corresponding graph metric d_K is given by the *ℓ^1-metric* on the product $\prod_{i\in I} K_i$, meaning that $d_K(\mathbf{x},\mathbf{y}) = \sum_{i\in I} d_{K_i}(x_i, y_i)$ for $\mathbf{x} = (x_i)_{i\in I}$, $\mathbf{y} = (y_i)_{i\in I} \in K$.

2.5.2. HYPERBOLIC GRAPHS AND COMPACTIFICATIONS

Hyperbolic graphs are graphs that satisfy a certain negative curvature condition. Intuitively, a hyperbolic graph is a graph whose large-scale geometry looks similar to that of a tree. The concept goes back to Gromov, see [90]. The results and definitions we present here are as they appear in [33, Chapter 5.3] and [122].

Let K be a connected graph. A *geodesic triangle* Δ consists of three points $x, y, z \in K$ and three geodesic paths connecting them. If there exists a number $\delta > 0$ for which each of the paths is contained in the open δ-tubular neighborhood of the union of the other two paths, such a triangle is called *δ-slim*. We say that the graph K is *hyperbolic* if there exists $\delta > 0$ such that every geodesic triangle is δ-slim. Note that trees are always hyperbolic with $\delta = 1$.

The *Gromov product* of a graph K with base point $o \in K$ is defined by

$$\langle x, y\rangle_o := \frac{1}{2}(d_K(o, x) + d_K(o, y) - d_K(x, y)).$$

Every hyperbolic graph K admits a topological "space at infinity" consisting of equivalence classes of certain sequences. Define an *equivalence relation* $\sim_h$ on the set of all sequences $\mathbf{x} := (x_i)_{i\in\mathbb{N}} \subseteq K$ which *converge to infinity* in the sense that $\liminf_{i,j}\langle x_i, x_j\rangle_o = \infty$ by declaring two such sequences $\mathbf{x}$ and $\mathbf{y}$ to be equivalent if and only if $\liminf_{i,j}\langle x_i, y_j\rangle_o = \infty$. This definition does not depend on choice of the base point o. We write $[\mathbf{x}]_h$ for the equivalence class of $\mathbf{x}$. The *hyperbolic boundary* (or *Gromov boundary*) $\partial_h K$ of K is the set of all equivalence classes of sequences in K which converge to infinity. The union $K \cup \partial_h K$ is called the *hyperbolic bordification* (or *Gromov bordification*). It is easy to see that for a locally finite graph every element in $\partial_h K$ can be represented by an infinite geodesic path starting in the base point o (see [122, Proposition 2.10]).

For $z \in \partial_h K$, $R > 0$ define the sets

$$U(z,R) := \{z' \in \partial_h K \quad | \quad \text{there are sequences } \mathbf{x},\mathbf{y} \text{ converging to infinity with } z = [\mathbf{x}]_h, z' = [\mathbf{y}]_h \text{ and } \liminf_{i,j\to\infty} \langle x_i, y_j\rangle_o > R\}$$

and

$$U'(z,R) := U(z,R) \cup \{y \in K \quad | \quad \text{there is a sequence } \mathbf{x} \text{ converging to infinity with } z = [\mathbf{x}]_h \text{ and } \liminf_{i\to\infty} \langle x_i, y\rangle_o > R\}.$$

One can topologize $\partial_h K$ by declaring the basis of neighborhoods for a point $z \in \partial_h K$ to be $\{U(z,R) \mid R > 0\}$. One can further topologize $K \cup \partial_h K$ by endowing K with the metric (i.e. discrete) topology and by declaring the basis of neighborhoods for a point $z \in \partial_h K$ to be $\{U'(z,R) \mid R > 0\}$. Again, these topologies are independent of the choice of the base point o. If we assume K to be locally finite, then $K \cup \partial_h K$ is a compact space (for a proof see e.g. [33, Proposition 2.14]) that contains K as a dense open subset. In that context, we also speak of $K \cup \partial_h K$ as the *hyperbolic compactification* of K. Further, every automorphism (i.e. isometric bijection) of K uniquely extends to a homeomorphism of $K \cup \partial_h K$, [33, Theorem 5.3.14].

Let us now turn our attention to Cayley graphs of finitely generated groups.

Definition 2.5.4. A group G generated by a finite set S whose Cayley graph $\mathrm{Cay}(G,S)$ is hyperbolic, is called *word hyperbolic*.

As it turns out, both the hyperbolicity of $\mathrm{Cay}(G,S)$ and the hyperbolic boundary $\partial_h G := \partial_h \mathrm{Cay}(G,S)$, do not depend on the choice of the finite generating set S. Further, since the left multiplication of G on itself induces an action by automorphism of G on its Cayley graph, it extends to continuous actions $G \curvearrowright G \cup \partial_h G$ and $G \curvearrowright \partial_h G$. These actions have some desirable properties.

Theorem 2.5.5 ([33, Theorem 5.3.15])**.** *Let G be a word hyperbolic group. Then the actions $G \curvearrowright G \cup \partial_h G$ and $G \curvearrowright \partial_h G$ are amenable.*

Proposition 2.5.6 ([33, Proposition 5.3.18])**.** *Let G be a word hyperbolic group. Then the hyperbolic compactification $G \cup \partial_h G$ is small at infinity.*

Theorem 2.5.7 ([88, Chapitre 8] and [120, Remark 5.6])**.** *Let G be a non-amenable word hyperbolic group. Then $\partial_h G$ is a G-boundary.*

2.6. AMALGAMATED FREE PRODUCTS AND GRAPH PRODUCTS OF GROUPS

The notion of amalgamated free products of groups generalizes free products of groups. Let $(G_i)_{i\in I}$ be a family of discrete groups indexed by a set I and assume that there exists a group H such that for every $i \in I$ there is a monomorphism $\iota_i : H \hookrightarrow G_i$. Then the *amalgamated free product* of $(G_i)_{i\in I}$ over H, denoted by $\star_H G_i$, is the quotient of the free product group $\star_{i\in I} G_i$ by the smallest normal subgroup of $\star_{i\in I} G_i$ generated by

$$\{\iota_i(h)^{-1}\iota_j(h) \mid i, j \in I, h \in H\}.$$

One can hence view the amalgamated free product as the free product of the family $(G_i)_{i\in I}$ where the embedded copies of H are identified with each other in a way that is compatible with the embeddings ι_i, $i \in I$. Note that the notation $\star_H G_i$ is slightly ambiguous.

For every $i \in I$ the group G_i canonically embeds into $\star_H G_i$. Further, note that in the case where H is trivial, $\star_H G_i$ is the ordinary free product $\star_{i\in I} G_i$. In the case of finite index sets $I = \{1, ..., n\}$ we sometimes write $\star_H G_i = G_1 \star_H G_2 \star_H ... \star_H G_n$, if convenient.

Graph products of groups were introduced by Green in her thesis [89] as a well-behaved generalization of free products and Cartesian products of groups. The construction associates with a simplicial graph with groups attached to each vertex a new group by taking the free product of the vertex groups, with added relations depending on the graph. To make this precise, let $K = (V, E)$ be a finite, undirected, simplicial graph and let $(G_v)_{v\in K}$ be a family of groups. Then the *graph product group* $\star_{v,K} G_v$ is the quotient of the free product group $\star_{v\in V} G_v$ by the smallest normal subgroup of $\star_{i\in I} G_i$ generated by

$$\{sts^{-1}t^{-1} \mid s \in G_v, t \in G_w \text{ with } (v, w) \in E\}.$$

Special (extremal) cases of graph products are free products (induced by graphs with no edges) and Cartesian products (induced by complete graphs). For every $v \in V$ the group G_v canonically embeds into $\star_{v,k} G_v$.

Green's construction preserves many important group theoretical properties (see [7], [8], [43], [54], [57], [56], [89], [109], [103]) and covers interesting examples such as right-angled Coxeter groups (see Subsection 2.7.4) and right-angled Artin groups. Further, any graph product of groups decomposes as iterated amalgamated free products of certain subgroups, as the following proposition demonstrates.

Proposition 2.6.1 ([89, Lemma 3.20]). *Let $K = (V, E)$ be a finite, undirected, simplicial graph, let $(G_v)_{v\in V}$ be a family of groups and let $v_0 \in V$. Define subgraphs $K_1 := Star(v_0)$, $K_2 := K \setminus \{v_0\}$ of K and set $H_0 := \star_{v,Link(v_0)} G_v$, $H_1 := \star_{v,K_1} G_v$ and $H_2 := \star_{v,K_2} G_v$. Then $\star_{v,K} G_v$ decomposes as $H_1 \star_{H_0} H_2$.*

2.7. COXETER GROUPS

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Coxeter groups, first systematically studied by Coxeter in [62] and [63], can be viewed as abstractions of (finite) reflection groups. Formally introduced by Tits (see [173]), they naturally appear in several branches of mathematics including algebra, geometry and combinatorics. Standard references on Coxeter groups are [26], [110], [19] and [67]. For a historical overview see [26] as well as [62].

2.7.1. BASIC NOTIONS

Let S be a (possibly infinite) set and let $M := (m_{st})_{s,t\in S}$ be a symmetric matrix with $m_{ss} = 1$ for all $s \in S$ and with $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for all $s,t \in S$, $s \neq t$. A matrix of this form is called a *Coxeter matrix* over S. The associated *Coxeter group* W is the discrete group freely generated by S subject to the relations $(st)^{m_{st}} = e$, i.e.

$$W = \langle S \mid (st)^{m_{st}} = e \text{ for all } s,t \in S \rangle .$$

Here the condition $m_{st} = \infty$ means that s and t are free with respect to each other, i.e. no relation of the form $(st)^m = e$ with $m \in \mathbb{N}$ is imposed. The pair (W,S) is called the *Coxeter system* associated with M. We will often assume that the generating set S is finite. In that case we say that (W,S) has *finite rank* and $\#S < \infty$ is called the *rank* of the system. The set of *reflections* of the Coxeter group W is often denoted by T and consists of all elements of the form $\mathbf{w}^{-1}s\mathbf{w}$ where $s \in S$, $\mathbf{w} \in W$.

Example 2.7.1. *(i)* There exists exactly one Coxeter group of rank 1. It is isomorphic to $\mathbb{Z}_2$, the group consisting of two elements.

(ii) For Coxeter systems (W,S), (W',S') associated with matrices $M := (m_{st})_{s,t\in S}$, $M' := (m'_{st})_{s,t\in S'}$ define coefficients $\tilde{m}_{st}$ via $\tilde{m}_{st} = m_{st}$ for $s,t \in S$, $\tilde{m}_{st} = m'_{st}$ for $s,t \in S'$ and $\tilde{m}_{st} = 2$ else. Then the Coxeter group associated with $(\tilde{m}_{st})_{s,t\in S\sqcup S'}$ is isomorphic to the direct product $W \times W'$ and we write $(W \times W', S \sqcup S') \cong (W,S) \times (W',S')$. In particular, every group of the form $\mathbb{Z}_2^k$, $k \in \mathbb{N}$ can be realized as a Coxeter group.

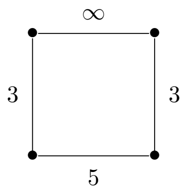
(iii) Similar to *(ii)*, for Coxeter systems (W,S), (W',S') associated with matrices $M := (m_{st})_{s,t\in S}$, $M' := (m'_{st})_{s,t\in S'}$ define coefficients $\tilde{m}_{st}$ via $\tilde{m}_{st} = m_{st}$ for $s,t \in S$, $\tilde{m}_{st} = m'_{st}$ for $s,t \in S'$ and $\tilde{m}_{st} = \infty$ else. Then the Coxeter group associated with $(\tilde{m}_{st})_{s,t\in S\sqcup S'}$ is isomorphic to the free product $W \star W'$ and we write $(W \star W', S \sqcup S') \cong (W,S) \star (W',S')$. In particular, every group of the form $\mathbb{Z}_2^{k_1} \star \ldots \star \mathbb{Z}_2^{k_l}$ with $l \in \mathbb{N}_{\geq 1}$ and $k_1,\ldots,k_l \in \mathbb{N}$ can be realized as a Coxeter group.

The data of (W,S) can be encoded in its *Coxeter diagram* which is an undirected simplicial labeled graph with vertex set S and edge set $\{(s,t) \in S \times S \mid m_{st} \geq 3\}$ where every edge connecting vertices $s,t \in S$ is labeled by the corresponding exponent $m_{st} = m_{ts}$.

Example 2.7.2. The diagram of the Coxeter group generated by the Coxeter matrix

$$\begin{pmatrix} 1 & 3 & 2 & \infty \\ 3 & 1 & 5 & 2 \\ 2 & 5 & 1 & 3 \\ \infty & 2 & 3 & 1 \end{pmatrix}$$

is given by



The *odd Coxeter diagram* is obtained from the Coxeter diagram by removing all edges whose labels are even or infinite. It encodes information about the conjugacy of generators.

Lemma 2.7.3 ([67, Lemma 3.3.3]). *Let (W,S) be a Coxeter system and $s,t \in S$. Then s and t are conjugate to each other in the sense that there exists $\mathbf{w} \in W$ such that $t = \mathbf{w}^{-1} s\mathbf{w}$, if and only if s and t lie in the same connected component of the odd Coxeter diagram of (W,S).*

For a subset $T \subseteq S$ the subgroup $W_T := \langle T\rangle$ of W generated by T is called a *special subgroup*. It is also a Coxeter group with the same exponents as W (see [67, Theorem 4.1.6]), i.e.

$$W_T \cong \left\langle T \mid (st)^{m_{st}} = e \text{ for all } s,t \in T\right\rangle$$

canonically. We briefly write $W_s = W_{\{s\}}$ for $s \in S$ and note that by the relation $s^2 = e$, $W_s \cong \mathbb{Z}_2$.

The system (W,S) is called *irreducible* if its Coxeter diagram is connected. This is equivalent to W not having a non-trivial decomposition into a direct product of special subgroups (compare with Example 2.7.1).

2.7.2. WORDS IN COXETER GROUPS

Let (W,S) be a Coxeter system. Because S generates the group W, every element $\mathbf{w} \in W$ can be written as a product $\mathbf{w} = s_1 \dots s_n$ with generators $s_1,\dots,s_n \in S$. We call

$$|\mathbf{w}| := \min\left\{n \mid \text{There exist } s_1,\dots,s_n \in S \text{ with } \mathbf{w} = s_1 \dots s_n\right\}$$

the *word length* of $\mathbf{w}$. An expression $\mathbf{w} = s_1 \dots s_n$ is called *reduced* if $n = |\mathbf{w}|$. The set of letters appearing in a reduced expression is independent of the choice of the reduced expression (see [67, Proposition 4.1.1]). For $\mathbf{v},\mathbf{w} \in W$ with $\left|\mathbf{v}^{-1}\mathbf{w}\right| = |\mathbf{w}| - |\mathbf{v}|$ (resp. $\left|\mathbf{w}\mathbf{v}^{-1}\right| = |\mathbf{w}| - |\mathbf{v}|$) we say that $\mathbf{w}$ *starts* (resp. *ends*) *with* $\mathbf{v}$ and write $\mathbf{v} \leq_R \mathbf{w}$ (resp. $\mathbf{v} \leq_L \mathbf{w}$). This defines a partial order which is called the *weak right* (resp. *weak left*) *Bruhat order*. For convenience, we will often write $\leq$ instead of $\leq_R$. The weak Bruhat orders have the important property that they define complete meet-semilattices on W.

Proposition 2.7.4 ([19, Proposition 3.2.1]). *Let (W,S) be a Coxeter system. Then the weak right Bruhat order and the weak left Bruhat order on W define complete meet-semilattices.*

Another useful property is the following.

Lemma 2.7.5 ([19, Proposition 3.1.2 (vi)]). *Let (W,S) be a Coxeter system, $\mathbf{v},\mathbf{w} \in W$ and $s \in S$. Then the following statements hold:*

- *Assume that $s \leq_R \mathbf{v}$, $s \leq_R \mathbf{w}$. Then, $\mathbf{v} \leq_R \mathbf{w}$ if and only if $s\mathbf{v} \leq_R s\mathbf{w}$.*
- *Assume that $s \leq_L \mathbf{v}$, $s \leq_L \mathbf{w}$. Then $\mathbf{v} \leq_L \mathbf{w}$ if and only if $\mathbf{v}s \leq_L \mathbf{w}s$.*

Coxeter groups can be characterized as those groups generated by a set of elements of order two which satisfy the following three equivalent cancellation conditions, see [67, Theorem 3.2.16 and Theorem 3.3.4]. We use the convention that $\widehat{s}$ means that s is removed from an expression.

Theorem 2.7.6 ([67, Theorem 3.2.16 and Theorem 3.3.4]). *Let (W,S) be a Coxeter system, $\mathbf{w} = s_1 ... s_n$ an expression for an element $\mathbf{w} \in W$ and $s,t \in S$. Then the following conditions hold:*

- Deletion condition*: If $s_1 \dots s_n$ is not a reduced expression for $\mathbf{w}$, then there exist $i < j$ such that $s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_n$ is also an expression for $\mathbf{w}$.*
- Exchange condition*: If $\mathbf{w} = s_1 \dots s_n$ is reduced, then either $|s\mathbf{w}| = n+1$ or there exists $1 \leq i \leq n$ with $s\mathbf{w} = s_1 \dots \widehat{s_i} \dots s_n$.*
- Folding condition*: If $|s\mathbf{w}| = n+1$ and $|\mathbf{w}t| = n+1$, then either $|s\mathbf{w}t| = n+2$ or $|s\mathbf{w}t| = n$.*

Following [19, Section 3.3], we call the deletion of a factor of the form ss inside a word $s_1 ... s_n$ in S a *nil-move*. Similarly, we call the replacement of a factor $stst...$ of length m_{st} by a factor $tsts...$ of length m_{st} inside $s_1 ... s_n$ a *braid-move*.

Theorem 2.7.7 ([19, Theorem 3.3.1]). *Let (W,S) be a Coxeter system and $\mathbf{w} \in W$. Then the following statements hold:*

- *Any expression $s_1 ... s_n$ with $s_1, ..., s_n \in S$ for $\mathbf{w}$ can be transformed into a reduced expression for $\mathbf{w}$ by a sequence of nil-moves and braid-moves.*
- *Every two reduced expressions for $\mathbf{w}$ can be transformed into each other by a sequence of braid-moves.*

2.7.3. CLASSES OF COXETER GROUPS

Coxeter groups are intimately related to reflection groups. In [62] Coxeter proved that every reflection group is a Coxeter group and demonstrated in [63] that conversely all finite Coxeter groups can be realized in this way. Further, the class of

Coxeter groups which can be realized as reflection groups has been entirely classified by their Coxeter diagrams (see [63] and [38]). Motivated by this one distinguishes three classes of (irreducible) Coxeter systems (compare with [67, Theorem 12.3.5]).

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Definition 2.7.8. Let (W,S) be an irreducible Coxeter system.

- It is of *spherical type* if it is locally finite, i.e. every finitely generated subgroup of W is finite.
- It is of *affine type* if it is infinite, virtually abelian and has finite rank.
- It is of *non-affine type* if it is neither spherical nor affine.

Both spherical type and affine type Coxeter systems are entirely classified by their Coxeter diagrams. These are called $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(D_n)_{n\geq 4}$, $(E_n)_{6\leq n\leq 8}$, F_4, G_2, $(H_n)_{2\leq n\leq 4}$, $(I_n)_{n\geq 3}$, A_∞, A'_∞, B_∞, D_∞ and $(\tilde{A}_n)_{n\geq 2}$, $(\tilde{B}_n)_{n\geq 3}$, $(\tilde{C}_n)_{n\geq 2}$, $(\tilde{D}_n)_{n\geq 4}$, $(\tilde{E}_n)_{6\leq n\leq 8}$, $\tilde{F}_4$, $\tilde{G}_2$, $\tilde{I}_1$, see the list in the appendix. The simplest Coxeter group of affine type is the *infinite dihedral group* $\mathbf{D}_\infty := \langle s,t \mid s^2 = t^2 = e\rangle \cong \mathbb{Z}_2 \star \mathbb{Z}_2$ which is generated by two elements.

Every finite rank spherical type (i.e. finite) Coxeter group contains a unique element of maximal length, see [67, Lemma 4.6.1]. We usually denote this element by $\mathbf{w}_0$.

By [67, Theorem 14.1.2 and Proposition 17.2.1] a Coxeter group W is amenable if and only if it decomposes as a direct product of spherical type and affine type Coxeter groups. By the work of several authors (see e.g. [80], [101], [61]) it has further been characterized when a Coxeter group W is C*-simple and when $C_r^*(W)$ carries the canonical tracial state τ (see Section 2.3) as its unique tracial state.

Theorem 2.7.9. *Let (W,S) be a Coxeter system. Then the following statements are equivalent:*

- *W decomposes as a direct product of non-affine type Coxeter groups;*
- *W is C^*-simple;*
- *$C_r^*(W)$ carries a unique tracial state.*

Let us expand on the well-known fact that irreducible affine Coxeter groups arise as subgroups generated by (affine) reflections associated with crystallographic root systems. Following [110, Chapter 1 and Chapter 4], let V be a finite-dimensional real Euclidean vector space with canonical inner product $\langle\cdot,\cdot\rangle$. For $\alpha \in V\setminus\{0\}$ write $\alpha^\vee := \frac{2}{\langle\alpha,\alpha\rangle}\alpha \in V$ and denote by s_α the *reflection*

$$s_\alpha : V \to V,\ x \mapsto x - \langle x,\alpha\rangle\,\alpha^\vee$$

2

in V mapping α to $-\alpha$ while fixing pointwise the *hyperplane* $H_{\alpha,0} := \{x \in V \mid \langle x, \alpha\rangle = 0\}$ orthogonal to α. Similarly, for $i \in \mathbb{Z}$ define the *affine hyperplane* $H_{\alpha,i} := \{x \in V \mid \langle x, \alpha\rangle = i\}$ and write $s_{\alpha,i}$ for the *affine* reflection

$$s_{\alpha,i} : V \to V,\ s_{\alpha,i}(x) := x - (\langle x, \alpha\rangle - i)\alpha^\vee$$

in V which fixes $H_{\alpha,i}$ pointwise and maps 0 to $i\alpha^\vee$. A *translation* in V is a map of the form $t_v : V \to V,\ x \mapsto x + v$ for some $v \in V$. Note that $s_{\alpha,1}s_{\alpha,0} = t_{\alpha^\vee}$.

Definition 2.7.10. Let V be a finite-dimensional real Euclidean vector space with canonical inner product $\langle\cdot,\cdot\rangle$. A *root system* in V is a finite subset $\Phi \subseteq V$ with $0 \notin \Phi$ satisfying the following conditions:

- $\mathrm{Span}(\Phi) = V$;
- $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$ for all $\alpha \in \Phi$ where $\mathbb{R}\alpha := \{\lambda\alpha \mid \lambda \in \mathbb{R}\}$;
- $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

A root system Φ is called *crystallographic* if $\frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. It is called *irreducible* if Φ cannot be written as a disjoint union of two subsets $\Phi_1, \Phi_2 \subseteq \Phi$ with $\langle\alpha, \beta\rangle = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$.

Similar to spherical and affine type irreducible Coxeter systems, irreducible crystallographic root systems have been completely classified by Cartan using Dynkin diagrams.

If V is a finite-dimensional real Euclidean vector space and Φ is a crystallographic root system, then the set $\{s_\alpha \mid \alpha \in \Phi\}$ generates a finite subgroup $W(\Phi)$ of the orthogonal group $O(V)$; this is the *Weyl group* which arises as a spherical type Coxeter group. Indeed, call a basis Δ of $V = \mathrm{Span}(\Phi)$ with $\Delta \subseteq \Phi$ a *simple system* if every $\alpha \in \Phi$ can be expressed as a linear combination of elements in Δ with coefficients all of the same sign, i.e. all coefficients are either positive or all coefficients are negative. Such a system always exists and its elements are called *simple (positive) roots*.

Theorem 2.7.11 ([110, Chapter 2])**.** *Let V be a finite-dimensional real Euclidean vector space with canonical inner product $\langle\cdot,\cdot\rangle$ and let Φ be a crystallographic root system. Then $W(\Phi)$ is generated by $S := \{s_\alpha \mid \alpha \in \Delta\} \subseteq W(\Phi)$ and $(W(\Phi), S)$ is a Coxeter system. Further, the system is irreducible if and only if Φ is irreducible.*

The Coxeter diagrams corresponding to Weyl groups are $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(D_n)_{n\geq 3}$, $(E_n)_{6\leq n\leq 8}$, F_4 and G_2.

Given a simple system Δ of the crystallographic root system Φ, define the *set of positive roots*

$$\Phi^+ := \left\{\alpha \in \Phi \mid \alpha = \sum_{\beta\in\Delta} a_\beta\beta \text{ with } a_\beta \geq 0 \text{ for all } \beta \in \Delta\right\}$$

and the *set of negative roots*

$$\Phi^- := \left\{ \alpha \in \Phi \mid \alpha = \sum_{\beta \in \Delta} a_\beta \beta \text{ with } a_\beta \leq 0 \text{ for all } \beta \in \Delta \right\}.$$

If Φ is irreducible, there exists a unique element $\tilde{\alpha} = \sum_{\beta \in \Delta} a_\beta \beta$ in Φ^+ such that $\sum_{\beta \in \Delta} a_\beta$ is maximal; we call it the *highest root*.

Similar to before, the set $T := \{s_{\alpha,i} \mid \alpha \in \Phi, i \in \mathbb{Z}\}$ generates a subgroup $W_{\text{aff}}(\Phi)$ of $O(V)$; the corresponding *affine Weyl group*. It is infinite, its set of reflections coincides with T and $W(\Phi) \subseteq W_{\text{aff}}(\Phi)$ is a subgroup. Note that, since Φ is assumed to be crystallographic, the set $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\} \subseteq V$ of co-roots also defines a crystallographic root system and write

$$L(\Phi^\vee) := \{n_1\alpha_1 + \ldots + n_l\alpha_l \mid \alpha_1, \ldots, \alpha_l \in \Phi^\vee, n_1, \ldots, n_l \in \mathbb{Z}\}$$

for the $\mathbb{Z}$-span of $\Phi^\vee$. One can show that the group of translations t_v, $v \in L(\Phi^\vee)$ is normal in $W_{\text{aff}}(\Phi)$ and that $W_{\text{aff}}(\Phi)$ decomposes as a semidirect product of $W(\Phi)$ and $\{t_v \mid v \in L(\Phi^\vee)\}$ (for details see [110, Section 4.2]). This decomposition will play an important role in Section 3.3 where Iwahori-Hecke algebras of affine type Coxeter systems are considered.

Theorem 2.7.12 ([110, Chapter 4]). *Let V be a finite-dimensional real Euclidean vector space with canonical inner product $\langle \cdot, \cdot \rangle$, let Φ be an irreducible crystallographic root system and let $\Delta \subseteq \Phi$ be a simple system. Then $W_{aff}(\Phi)$ is generated by*

$$S_{aff} := \{s_{\alpha,0} \mid \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\} \subseteq W_{aff}(\Phi)$$

where $\tilde{\alpha} \in \Phi$ denotes the highest root and $(W_{aff}(\Phi), S_{aff})$ is a Coxeter system. Further, every irreducible affine Coxeter group arises in such a way.

Example 2.7.13. Let $V := \mathbb{R}e$ be the 1-dimensional real Euclidean vector space with basis vector e and set $\Phi := \{e, -e\} \subseteq V$. Φ is an irreducible crystallographic root system, $\Delta := \{e\}$ defines a simple system and $e \in \Phi$ is the corresponding highest root. The induced Coxeter group $W_{\text{aff}}(\Phi)$ then identifies with the infinite dihedral group $\mathbf{D}_\infty$. We further have that $W(\Phi) \cong \mathbb{Z}_2$ and by the above discussion $\mathbf{D}_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}_2$.

In the context of Theorem 2.7.12 the word length $|\mathbf{w}|$ of an element $\mathbf{w} \in W_{\text{aff}}(\Phi)$ (with respect to the generating set S_{aff}) has a convenient geometrical interpretation. Write $V^\circ := V \setminus (\bigcup_{\alpha \in \Phi, i \in \mathbb{Z}} H_{\alpha,i})$ and call the connected components of V° *alcoves*. Two such alcoves are *separated by an affine hyperplane $H_{\alpha,i}$* if they lie in different half-spaces relative to $H_{\alpha,i}$. One checks that the *standard alcove* $A_\circ := \{x \in V \mid 0 < \langle x, \alpha \rangle < 1 \text{ for all } \alpha \in \Phi^+\}$ is indeed an alcove. Further, the natural action of $W_{\text{aff}}(\Phi)$ on V induces an action on the set of all alcoves. Then, for an element $\mathbf{w} \in W$ we have

$$|\mathbf{w}| = \#\{H_{\alpha,i} \mid H_{\alpha,i} \text{ separates } A_\circ \text{ and } \mathbf{w}.A_\circ\}.$$

2

2.7.4. RIGHT-ANGLED COXETER GROUPS

The combinatorial structure of general Coxeter groups can be very wild. An important subclass of Coxeter groups whose combinatorial structure is still rich but much more accessible is the class of right-angled Coxeter groups.

Definition 2.7.14. Let (W,S) be a Coxeter system. It is called *right-angled* if $m_{st} \in \{2,\infty\}$ for all $s \neq t$.

Right-angled Coxeter systems interpolate between free products and Cartesian products of $\mathbb{Z}_2$. In the right-angled case, if we have cancellation of the form $s_1...s_n = s_1...\widehat{s_i}...\widehat{s_j}...s_n$ for $s_1,...,s_n \in S$, then $s_i = s_j$ and s_i commutes with every letter in the reduced expression for $s_{i+1}...s_{j-1}$. Indeed, if $s_1...s_n = s_1...\widehat{s_i}...\widehat{s_j}...s_n$ then we have $s_j = (s_{i+1}...s_{j-1})^{-1}s_i(s_{i+1}...s_{j-1})$, i.e. s_i and s_j are conjugate to each other. Because the connected components of the odd Coxeter diagram of a right-angled Coxeter group are one-point sets, we deduce with Lemma 2.7.3 that $s_i = s_j$ and hence $(s_{i+1}...s_{j-1})s_j = s_i(s_{i+1}...s_{j-1})$. The statement then follows from Theorem 2.7.7.

The following lemma, that appears in [46], will play a role in Subsection 6.1.1.

Lemma 2.7.15 ([46, Lemma 4.4])**.** *Let (W,S) be a right-angled, finite rank Coxeter system and let $\mathbf{w} \in W$. For $l \in \mathbb{N}$ define*

$$\kappa_{\mathbf{w}}(l) := \#\{\mathbf{v} \in W \mid \mathbf{v} \leq \mathbf{w} \text{ and } |\mathbf{v}| = l\}.$$

Then there exists a constant $C > 0$ (which can be chosen uniformly in $\mathbf{w}$) such that $\kappa_{\mathbf{w}}(l) \leq Cl^{\#S-2}$ for all $l \in \mathbb{N}$.

Right-angled Coxeter groups decompose as graph products of $\mathbb{Z}_2$. Indeed, let $K := (V,E)$ be the complement of the (unlabeled) Coxeter diagram of (W,S), i.e. $V := S$ and $E := \{(s,t) \mid m_{st} = 2\}$. Then we find by $W_s = \langle s\rangle \cong \mathbb{Z}_2$ that

$$W \cong \star_{s,K} W_s \cong \star_{s,K} \mathbb{Z}_2$$

canonically. This follows from the fact that the defining (universal) properties of a right-angled Coxeter group and the corresponding graph product over the special subgroups $W_s \cong \mathbb{Z}_2$ are the same.

2.7.5. AMALGAMATED FREE PRODUCT DECOMPOSITIONS

Similar to the decomposition of right-angled Coxeter groups in terms of graph products of $\mathbb{Z}_2$, general Coxeter groups can be decomposed inductively as amalgamated free products of certain special subgroups (compare also with Proposition 2.6.1).

Following [67, Chapter 8.8], fix a finite rank Coxeter system (W,S). A subset $T \subseteq S$ is called *spherical* if the special subgroup $W_T \subseteq W$ generated by T is finite. Recall that a (*abstract*) *simplicial complex* consists of a set V (the *vertex set*) and a collection $\mathscr{S}$ of finite subsets of V (the *simplices*) such that $\{v\} \in \mathscr{S}$ for every $v \in V$

and such that $T \in \mathscr{S}$, $T' \subseteq T$ implies $T' \in \mathscr{S}$. One such simplicial complex is the *nerve of the system*, denoted by $\mathscr{N}(W,S)$, which is given by the vertex set S and the set $\{T \subseteq S \mid S \text{ spherical}\}$ of simplices. Let $\mathscr{N}_0$ be a *full subcomplex* of $\mathscr{N} := \mathscr{N}(W,S)$, meaning that every simplex of $\mathscr{N}$ whose vertices are contained in $\mathscr{N}_0$ is already a simplex in $\mathscr{N}$. Further assume that $\mathscr{N} \setminus \mathscr{N}_0$ can be written as a disjoint union of two subcomplexes. Denote the union of the corresponding vertex sets with the vertex set S_0 of N_0 by S_1 and S_2. Then, as explained in [67, Chapter 8.8], W decomposes as the amalgamated free product $W = W_1 \star_{W_0} W_2$ with $W_0 := W_{S_0}$, $W_1 := W_{S_1}$, $W_2 := W_{S_2}$ where W_0 embeds into W_1 and W_2 canonically. By iterating this procedure, every Coxeter group can be decomposed as an iterated amalgamated free product of 0- or 1-ended special subgroups over spherical special subgroups, see [67, Chapter 8.6 and Proposition 8.8.2] for details and the notion of ends of groups.

Conversely, if W is an arbitrary group which decomposes as an amalgamated free product $W = W_1 \star_{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $W_0 = W_1 \cap W_2$ and where $S_0 := S_1 \cap S_2$ generates W_0, then $(W, S_1 \cup S_2)$ is a Coxeter system as well.

Denote by $\mathscr{G}$ the smallest class of Coxeter groups which contains all finite rank spherical type Coxeter groups and which is closed under taking amalgamated free products over spherical special subgroups. Then, by [67, Proposition 8.8.5], $\mathscr{G}$ coincides with the class of Coxeter groups which are *virtually free* in the sense that they contain a finite index free subgroup. The class $\mathscr{G}$ will play a role in Chapter 8.

2.7.6. WORD HYPERBOLIC COXETER GROUPS

In [134] Moussong characterized word hyperbolic Coxeter groups.

Theorem 2.7.16 ([134, Theorem 17.1]). *For every finite rank Coxeter system (W,S) the following statements are equivalent:*

- *W is word hyperbolic;*
- *W contains no subgroup which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$;*
- *There is no subset $T \subseteq S$ such that (W_T, T) is an affine type Coxeter system of rank ≥ 3 or that there exists a pair of disjoint subsets $T_1, T_2 \subseteq T$ with $(W_T, T) \cong (W_{T_1} \times W_{T_2}, T_1 \cup T_2)$ where W_{T_1}, W_{T_2} are infinite.*

In the right-angled case this leads to a convenient characterization of word hyperbolicity in terms of the complement of the (unlabeled) Coxeter diagram. Let (W,S) be a right-angled Coxeter system and set $K := (V,E)$ with $V := S$ and $E := \{(s,t) \mid m_{st} = 2\}$. Then $W \cong \star_{s,K} \mathbb{Z}_2$ is word hyperbolic if and only if K contains no square as an induced subgraph, i.e. there are no elements $s,t,u,v \in E$ with $m_{st} = m_{tu} = m_{uv} = m_{vs} = 2$ and $m_{su} = m_{tv} = \infty$.

2.7.7. GROWTH SERIES OF COXETER GROUPS

Let (W,S) be a finite rank Coxeter system and let $\mathbb{C}^{(W,S)}$ be the set of tuples $z := (z_s)_{s \in S}$ in $\mathbb{C}^S$ with the property that $z_s = z_t$ whenever s and t are conjugate in W.

Further set $\mathbb{R}_{>0}^{(W,S)} := \mathbb{C}^{(W,S)} \cap \mathbb{R}_{>0}^{S}$ and $\{-1,1\}^{(W,S)} := \mathbb{C}^{(W,S)} \cap \{-1,1\}^{S}$. From Theorem 2.7.7 it follows that for every $z := (z_s)_{s\in S} \in \mathbb{C}^{(W,S)}$ and $\mathbf{w} \in W$ with reduced expression $\mathbf{w} = s_1 ... s_n$ the complex number $z_{\mathbf{w}} := z_{s_1} ... z_{s_n} \in \mathbb{C}$ does not depend on the choice of the reduced expression. Following [19] and [67], for a subset $X \subseteq W$ we can hence define the (*multivariate*) *growth series* (or *Poincaré series*) $X(z) := \sum_{\mathbf{w}\in A} z_{\mathbf{w}}$ of X.

As we will see in the later sections, several properties of Hecke operator algebras are related to the regions of convergence of the corresponding growth series of Coxeter groups. For the Coxeter system (W,S) write

$$\mathscr{R}(W,S) := \left\{ z \in \mathbb{C}^{(W,S)} \mid W(z) \text{ converges} \right\}$$

for the *region of convergence* of $W(z) = \sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ and define

$$\mathscr{R}'(W,S) := \left\{ \left(q_s^{\epsilon_s}\right)_{s\in S} \mid q \in \mathscr{R}(W,S) \cap \mathbb{R}_{>0}^{(W,S)}, \epsilon \in \{-1,1\}^{(W,S)} \right\}.$$

Further denote the closure of $\mathscr{R}'(W,S)$ in $\mathbb{R}_{>0}^{(W,S)}$ by $\overline{\mathscr{R}'(W,S)}$.

One of the first results about growth series of Coxeter groups is their rationality which inductively follows from the following lemma.

Lemma 2.7.17 ([67, Corollary 17.1.5] and [19, Corollary 7.1.4]). *Let (W,S) be a finite rank Coxeter system.*

- *If W is finite and $\mathbf{w}_0 \in W$ is the (unique) longest element in W, then for all $z \in \mathbb{C}^{(W,S)}$ the equality*

$$\sum_{T\subseteq S} \frac{(-1)^{\#T}}{W_T(z)} = \frac{z_{\mathbf{w}_0}}{W(z)}$$

  holds.

- *If W is infinite, then for all $z \in \mathbb{C}^{(W,S)}$ for which $\sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ absolutely converges, the equality*

$$\sum_{T\subseteq S} \frac{(-1)^{\#T}}{W_T(z)} = 0$$

  holds.

As the following proposition demonstrates, the growth series of any infinite Coxeter system can be computed conveniently in a non-recursive way by only considering finite Coxeter subgroups. It can be deduced from Lemma 2.7.17.

Proposition 2.7.18 ([67, Corollary 17.1.10] and [19, Proposition 7.1.7]). *Let (W,S) be a finite rank Coxeter system for which W is infinite and denote by $\mathcal{N} := \mathcal{N}(W,S)$ the nerve of (W,S). Consider $\mathcal{N} \cup \{S\}$ with respect to the partial order induced by the inclusion of sets and inductively define the corresponding* Möbius function *$\mu_{\mathcal{N}}$ by*

$$\mu_{\mathcal{N}}(T,T') := \begin{cases} 1 & , \text{if } T = T' \\ -\sum_{T\subseteq T''\subsetneq T'} \mu_{\mathcal{N}}(T,T'') & , \text{if } T \subsetneq T' \end{cases},$$

for $T, T' \in \mathcal{N} \cup \{S\}$ *with* $T \subseteq T'$. *Then for all* $z \in \mathbb{C}^{(W,S)}$ *for which* $\sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ *absolutely converges,*

$$\frac{1}{W(z)} = -\sum_{T\in\mathcal{N}} \frac{\mu_{\mathcal{N}}(T,S)}{W_T(z)}.$$

Example 2.7.19. Let (W,S) be a Coxeter system of the form $W = \mathbb{Z}_2^{k_1} \star \cdots \star \mathbb{Z}_2^{k_l}$ with $l, k_1, \ldots, k_l \in \mathbb{N}$ (compare with Example 2.7.1). Denote by $s_1^{(m)}, \ldots, s_{k_m}^{(m)}$ the mutually commuting generators corresponding of the component $\mathbb{Z}_2^{k_m}$ of W and set $S_m := \{s_1^{(m)}, \ldots, s_{k_m}^{(m)}\}$, so that in particular $S = \bigcup_{m=1}^{l} S_m$. The corresponding nerve of (W,S) is given by

$$\mathcal{N}(W,S) = \{\emptyset \neq T \subseteq S_m \mid m = 1, ..., l\}.$$

Using Lemma 2.7.17, an easy computation implies that for all $z \in \mathbb{C}^{(W,S)}$ for which $\sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ absolutely converges,

$$W(z) = \left(\sum_{m=1}^{l} \prod_{i=1}^{k_m} (1 + z_{s_i^{(m)}})^{-1} - (l-1) \right)^{-1}.$$

As we have seen before, irreducible amenable Coxeter groups are exactly those Coxeter groups that are of spherical or affine type. The consideration of the growth series of Coxeter groups leads to a number of other characterizations of this important property.

Proposition 2.7.20 ([67, Corollary 17.1.2])**.** *Let* (W,S) *be a finite rank Coxeter system. Then the following conditions are equivalent:*

- W *is amenable;*
- W *does not contain the free group* $\mathbb{F}_2$ *generated by two elements;*
- W *is not* large, *i.e.* W *does not contain a finite index subgroup which maps onto* $\mathbb{F}_2$*;*
- W *is virtually abelian;*
- W *decomposes as a direct product of spherical type and affine type Coxeter groups;*
- *the radius of convergence of the (single-parameter) growth series* $\sum_{\mathbf{w}\in W} z^{|\mathbf{w}|}$ *is 1;*
- W *has* subexponential growth, *i.e. there exists no* $\Lambda > 1$ *such that* $\#\{\mathbf{w} \in W \mid |\mathbf{w}| \leq l\} \geq \Lambda^l$ *for all* $l \in \mathbb{N}$.

3

HECKE OPERATOR ALGEBRAS

In this chapter, we introduce Iwahori-Hecke algebras and study their operator-algebraic counterparts. The first three sections, which are mostly not original but appear in [67, Chapter 19], [133, Chapter 2], [39, Chapter 10] and [35, Section 16] are concerned with the introduction of general Iwahori-Hecke algebras, the characterization of these algebras in the setting of affine type Coxeter systems and the introduction of their operator algebraic counterparts. In Section 3.5 we study isomorphisms of Hecke (operator) algebras and Section 3.6 treats amalgamated free product decompositions of Hecke operator algebras associated with virtually free Coxeter groups.

The content of these sections is based on parts of the articles

- M. Caspers, M. Klisse, N.S. Larsen, *Graph product Khintchine inequalities and Hecke C^*-algebras: Haagerup inequalities, (non)simplicity, nuclearity and exactness*, J. Funct. Anal. 280 (2021), no. 1, Paper No. 108795, 41 pp. no. 1, 108795;
- M. Klisse, *Topological boundaries of connected graphs and Coxeter groups*, to appear in the Journal of Operator Theory;
- M. Klisse, *Simplicity of right-angled Hecke C^*-algebras*, to appear in Int. Math. Res. Not. IMRN;
- M. Caspers, M. Klisse, A. Skalski, G. Vos, M. Wasilewski, *Relative Haagerup property for arbitrary von Neumann algebras*, arXiv preprint arXiv:2110.15078 (2021).

3

3.1. IWAHORI-HECKE ALGEBRAS

Iwahori-Hecke algebras can be viewed as deformations of the group algebra of Coxeter groups depending on a deformation (multi-)parameter q. They abstract certain endomorphism rings appearing in the representation theory of Lie groups and provide an important tool in the study of the representation theory of reflection groups and Weyl groups.

The history of Iwahori-Hecke algebras is vast and its detailed description is beyond the scope of this dissertation. They can be traced back to Iwahori [112] who discovered that double coset Hecke algebras associated with a Chevalley group acting on the cosets of a Borel subgroup define Iwahori-Hecke algebras. His result was later extended to the setting of arbitrary groups with a BN-pair by Matsumoto [132]. A lot of relevant insights, notably [113], [133], [123], [18], [124], [131], followed. Today Iwahori-Hecke algebras are ubiquitous, e.g. they occur in Jones' construction of an invariant for oriented knots and links [118], they play a role in combinatorics [107], [106] and the theory of buildings and Kac-Moody groups acting on them [87, Section 6.2] (see also [160, Section 2.4]). They are linked to quantum groups and non-commutative geometry, and due to their relation with reductive groups they occur in the local Langlands program.

In the following, let R be a commutative unital ring, let (W,S) be a Coxeter system, write $R^{(W)}$ for the free R-module on W with basis $(\widetilde{T}_{\mathbf{w}})_{\mathbf{w}\in W}$ and denote by $R^{(W,S)}$ the set of tuples $q := (q_s)_{s\in S}$ in R^S with the property that $z_s = z_t$ whenever s and t are conjugate in W.

Theorem 3.1.1 ([67, Proposition 19.1.1])**.** *For $a = (a_s)_{s\in S}, b = (b_s)_{s\in S} \in R^{(W,S)}$ there exists a unique ring structure on $R^{(W)}$ such that for all $\mathbf{w}\in W$, $s \in S$*

$$\widetilde{T}_s\widetilde{T}_{\mathbf{w}} = \begin{cases} \widetilde{T}_{s\mathbf{w}} & \text{, if } |s\mathbf{w}| > |\mathbf{w}| \\ a_s\widetilde{T}_{s\mathbf{w}} + b_s\widetilde{T}_{\mathbf{w}} & \text{, if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}. \tag{3.1.1}$$

Proof. For $s \in S$ define R-linear endomorphisms $\lambda_s : R^{(W)} \to R^{(W)}$ and $\rho_s : R^{(W)} \to R^{(W)}$ by

$$\lambda_s(\widetilde{T}_{\mathbf{w}}) := \begin{cases} \widetilde{T}_{s\mathbf{w}} & \text{, if } |s\mathbf{w}| > |\mathbf{w}| \\ a_s\widetilde{T}_{s\mathbf{w}} + b_s\widetilde{T}_{\mathbf{w}} & \text{, if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}$$

and

$$\rho_s(\widetilde{T}_{\mathbf{w}}) := \begin{cases} \widetilde{T}_{\mathbf{w}s} & \text{, if } |\mathbf{w}s| > |\mathbf{w}| \\ a_s\widetilde{T}_{\mathbf{w}s} + b_s\widetilde{T}_{\mathbf{w}} & \text{, if } |\mathbf{w}s| < |\mathbf{w}| \end{cases}.$$

By using Theorem 2.7.6 one checks that for all $s,t \in S$ the maps λ_s and ρ_t commute in the sense that $\lambda_s \circ \rho_t(\widetilde{T}_{\mathbf{w}}) = \rho_t \circ \lambda_s(\widetilde{T}_{\mathbf{w}})$ for all $\mathbf{w} \in W$. Denote by $\mathscr{H}$ the subalgebra of $\mathrm{End}_R(R^{(W)})$ generated by the set $\{\lambda_s \mid s \in S\}$. For $\mathbf{w} \in W$ choose a reduced expression $\mathbf{w} = s_1 \dots s_n$ and define

$$\lambda_{\mathbf{w}} := \lambda_{s_1} \circ \dots \circ \lambda_{s_n} \in \mathscr{H} \qquad \text{and} \qquad \rho_{\mathbf{w}} := \rho_{s_1} \circ \dots \circ \rho_{s_n}.$$

By the above, $\lambda_{\mathbf{v}} \circ \rho_{\mathbf{w}} = \rho_{\mathbf{w}} \circ \lambda_{\mathbf{v}}$ for all $\mathbf{v}, \mathbf{w} \in W$. For $s \in S$ and $\mathbf{v}, \mathbf{w} \in W$ with $|s\mathbf{w}| > |\mathbf{w}|$ we hence have

$$\begin{aligned}(\lambda_s \circ \lambda_{\mathbf{w}} - \lambda_{s\mathbf{w}})(\widetilde{T}_{\mathbf{v}}) &= (\lambda_s \circ \lambda_{\mathbf{w}} - \lambda_{s\mathbf{w}})(\rho_{\mathbf{v}}(\widetilde{T}_e)) \\ &= \rho_{\mathbf{v}}((\lambda_s \circ \lambda_{\mathbf{w}} - \lambda_{s\mathbf{w}})(\widetilde{T}_e)) \\ &= \rho_{\mathbf{v}}(0) \\ &= 0\end{aligned}$$

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and for $s \in S$ and $\mathbf{v}, \mathbf{w} \in W$ with $|s\mathbf{w}| < |\mathbf{w}|$,

$$\begin{aligned}(\lambda_s \circ \lambda_{\mathbf{w}} - a_s\lambda_{s\mathbf{w}} - b_s\lambda_{\mathbf{w}})(\widetilde{T}_{\mathbf{v}}) &= (\lambda_s \circ \lambda_{\mathbf{w}} - a_s\lambda_{s\mathbf{w}} - b_s\lambda_{\mathbf{w}})(\rho_{\mathbf{v}}(\widetilde{T}_e)) \\ &= \rho_{\mathbf{v}}((\lambda_s \circ \lambda_{\mathbf{w}} - a_s\lambda_{s\mathbf{w}} - b_s\lambda_{\mathbf{w}})(\widetilde{T}_e)) \\ &= \rho_{\mathbf{v}}(0) \\ &= 0.\end{aligned}$$

This implies

$$\lambda_s \circ \lambda_{\mathbf{w}} = \begin{cases} \lambda_{s\mathbf{w}} & \text{, if } |s\mathbf{w}| > |\mathbf{w}| \\ a_s\lambda_{s\mathbf{w}} + b_s\lambda_{\mathbf{w}} & \text{, if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}.$$

By the same argument as before, the map $\Phi : \mathscr{H} \to R^{(W)}$ given by $\lambda_{\mathbf{w}} \mapsto \lambda_{\mathbf{w}}(\widetilde{T}_e)$ for $\mathbf{w} \in W$ is an R-linear isomorphism. It induces the desired ring structure on $R^{(W)}$ via

$$\widetilde{T}_{\mathbf{v}}\widetilde{T}_{\mathbf{w}} := \Phi(\Phi^{-1}(\widetilde{T}_{\mathbf{v}}) \circ \Phi^{-1}(\widetilde{T}_{\mathbf{w}}))$$

for $\mathbf{v}, \mathbf{w} \in W$. This finishes the proof. □

The algebra $R_{a,b}[W]$ induced by Theorem 3.1.1 is called the *Iwahori-Hecke algebra* of the system (W, S) and the multi-parameters $a, b \in R^{(W,S)}$. Its unit is given by $1 := T_e$. In the following we will only consider the case where $a = q$ for some $q = (q_s)_{s\in S} \in R^{(W,S)}$ and where $b = (1 - q_s)_{s\in S} \in R^{(W,S)}$. In that case we denote the corresponding Iwahori-Hecke algebra by $R_q[W]$. For $q \in R$ the Iwahori-Hecke algebra associated with $(q)_{s\in S} \in R^{(W,S)}$ is called a *single-parameter Iwahori-Hecke algebra* which we still denote by $R_q[W]$. Note that $R_1[W]$ is the usual *group algebra* $R[W]$ of W. We can hence view $R_q[W]$ as a "deformation" of $R[W]$. To keep track of the deformation parameter, instead of $\widetilde{T}_{\mathbf{w}}$ we will usually write $\widetilde{T}_{\mathbf{w}}^{(q)}$ for the canonical basis elements of $R_q[W]$.

Remark 3.1.2. *(a)* Let (W, S) be a Coxeter system and let $a = (a_s)_{s\in S}, b = (b_s)_{s\in S} \in \mathbb{C}^{(W,S)}$ be multi-parameters for which $a_s \neq 0$ or $b_s \neq 0$ for every $s \in S$. If for every $s \in S$, $\lambda_s \in \mathbb{C}$ denotes the non-zero root of the polynomial $\lambda^2 - b_s\lambda - a_s$ and if $q := (a_s\lambda_s^{-2})_{s\in S} \in \mathbb{C}^{(W,S)}$, then the defining properties of the corresponding Iwahori-Hecke algebras induce an algebra isomorphism $R_{a,b}[W] \cong R_q[W]$ via $\widetilde{T}_{\mathbf{w}} \mapsto \lambda_{\mathbf{w}}\widetilde{T}_{\mathbf{w}}^{(q)}$ for $\mathbf{w} \in W$. Indeed, for $s \in S$ and $\mathbf{w} \in W$ with $|s\mathbf{w}| > |\mathbf{w}|$ one has

$$(\lambda_s\widetilde{T}_s^{(q)})(\lambda_{\mathbf{w}}\widetilde{T}_{\mathbf{w}}^{(q)}) = \lambda_{s\mathbf{w}}\widetilde{T}_{s\mathbf{w}}^{(q)}$$

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and for $s \in S$ and $\mathbf{w} \in W$ with $|s\mathbf{w}| < |\mathbf{w}|$ one has

$$(\lambda_s \widetilde{T}_s^{(q)})(\lambda_{\mathbf{w}} \widetilde{T}_{\mathbf{w}}^{(q)}) = \lambda_s \lambda_{\mathbf{w}} (q_s \widetilde{T}_{s\mathbf{w}}^{(q)} + (1-q_s)\widetilde{T}_{\mathbf{w}}^{(q)}) = a_s(\lambda_{s\mathbf{w}} \widetilde{T}_{s\mathbf{w}}^{(q)}) + b_s(\lambda_{\mathbf{w}} \widetilde{T}_{\mathbf{w}}^{(q)}).$$

The existence of the isomorphism then follows from Theorem 3.1.1. The observation demonstrates that in this case the consideration of Iwahori-Hecke algebras of the form $\mathbb{C}_q[W]$, $q \in \mathbb{C}^{(W,S)}$ is not really restrictive.
(b) Let R be a commutative unital ring, let (W,S) be a Coxeter system and let $q = (q_s)_{s\in S} \in R^{(W,S)}$. By the equality (3.1.1), for $s \in S$ with $q_s \in R^\times$, the element $\widetilde{T}_s^{(q)}$ is invertible with $(\widetilde{T}_s^{(q)})^{-1} = q_s^{-1}\widetilde{T}_s^{(q)} + (1-q_s^{-1})$.

Since this dissertation is concerned with the study of operator algebras associated with Iwahori-Hecke algebras, from Section 3.4 on we will only consider the case where $R = \mathbb{C}$ and $q \in \mathbb{R}_{>0}^{(W,S)}$. Following the notation in [86] (see also [46], [45], [49], [160], [126], [127], [161], [128], [50]), in this case we set $T_{\mathbf{w}}^{(q)} := q_{\mathbf{w}}^{-\frac{1}{2}} \widetilde{T}_{\mathbf{w}}^{(q)}$ for $\mathbf{w} \in W$. Then $(T_{\mathbf{w}}^{(q)})_{\mathbf{w}\in W} \subseteq \mathbb{C}_q[W]$ is a basis of $\mathbb{C}_q[W]$ which satisfies an analogue of (3.1.1),

$$T_s^{(q)} T_{\mathbf{w}}^{(q)} = \begin{cases} T_{s\mathbf{w}}^{(q)} & \text{, if } |s\mathbf{w}| > |\mathbf{w}| \\ T_{s\mathbf{w}}^{(q)} + p_s(q) T_{\mathbf{w}}^{(q)} & \text{, if } |s\mathbf{w}| < |\mathbf{w}| \end{cases}$$

for $s \in S$, $\mathbf{w} \in W$ where $p_s(q) := q_s^{-\frac{1}{2}}(q_s - 1)$ and we have $(T_s^{(q)})^{-1} = T_s^{(q)} - p_q$.

The following statements are easy to check and appear in [67, Chapter 19]. We will constantly make use of them in the later sections without further mention.

Lemma 3.1.3 ([67, Lemma 19.1.2]). *Let R be a commutative unital ring, let (W,S) be a Coxeter system and let $q = (q_s)_{s\in S} \in R^{(W,S)}$. Then the following statements hold:*

- *For all $\mathbf{v}, \mathbf{w} \in W$ with $|\mathbf{v}\mathbf{w}| = |\mathbf{v}| + |\mathbf{w}|$ the equality $\widetilde{T}_{\mathbf{v}}^{(q)} \widetilde{T}_{\mathbf{w}}^{(q)} = \widetilde{T}_{\mathbf{v}\mathbf{w}}^{(q)}$ holds;*
- *For all $s \in S$ the equality $(\widetilde{T}_s^{(q)})^2 = q_s + (1-q_s)\widetilde{T}_s^{(q)}$ holds;*
- *For all $s,t \in S$ with $m_{st} \neq \infty$, $\widetilde{T}_s^{(q)} \widetilde{T}_t^{(q)} \ldots = \widetilde{T}_t^{(q)} \widetilde{T}_s^{(q)} \ldots$ where the number of factors on the left-hand side and right-hand side equals m_{st}.*

In this context (as well as in other instances) the relation $\widetilde{T}_s^{(q)} \widetilde{T}_t^{(q)} \ldots = \widetilde{T}_t^{(q)} \widetilde{T}_s^{(q)} \ldots$ in Lemma 3.1.3 is often refered to as a *Braid relation*.

Iwahori-Hecke algebras also admit a right-handed version of (3.1.1).

Lemma 3.1.4 ([67, Proposition 19.1.1]). *Let R be a commutative unital ring, let (W,S) be a Coxeter system and let $q = (q_s)_{s\in S} \in R^{(W,S)}$. Then for all $s \in S$, $\mathbf{w} \in W$,*

$$\widetilde{T}_{\mathbf{w}} \widetilde{T}_s = \begin{cases} \widetilde{T}_{\mathbf{w}s} & \text{, if } |\mathbf{w}s| > |\mathbf{w}| \\ a_s \widetilde{T}_{\mathbf{w}s} + b_s \widetilde{T}_{\mathbf{w}} & \text{, if } |\mathbf{w}s| < |\mathbf{w}| \end{cases}.$$

In the case where $R = \mathbb{C}$ and $q \in \mathbb{R}_{>0}^{(W,S)}$, analogues of Lemma 3.1.3 and Lemma 3.1.4 also hold true for $T_{\mathbf{w}}^{(q)}$, $\mathbf{w} \in W$.

3.2. Iwahori-Hecke algebras of finite Coxeter groups

Let (W, S) be a Coxeter system for which W is finite. In the case where the commutative unital ring R coincides with the complex numbers, the group ring $R[W]$ is *semi-simple* in the sense that for every $0 \neq x \in R[W]$ there exists an irreducible representation π with $\pi(x) \neq 0$. From abstract algebraic arguments, it can be deduced that for most choices of the parameter $q \in R^{(W,S)}$ the representation theory of the corresponding Iwahori-Hecke algebra $R_q[W]$ is relatively easy. More concretely, it follows from *Tits' deformation theorem* (see e.g. [39, Theorem 10.11.2]) that the Iwahori-Hecke algebra $R_q[W]$ is isomorphic to the group algebra $R[W]$ if and only if $R_q[W]$ is semi-simple. The underlying reasoning is based on the classification of finite-dimensional semi-simple algebras and the isomorphism is not explicit. Similar arguments apply to more general choices of R as well.

In the case where $R_q[W]$ is not semi-simple, it contains a *nilpotent ideal* (i.e. an ideal I for which $I^k = 0$ for some $k \in \mathbb{N}$) and can look quite different from $R[W]$. It is therefore of interest to know for which choice of $q = (q_s)_{s \in S} \in R^{(W,S)}$ the algebra $R_q[W]$ is semi-simple. It can be shown that this happens if for every $s \in S$ the coefficient q_s is not a proper root of unity, so in particular $R_q[W] \cong R[W]$ for all $q \in \mathbb{R}_{>0}^{(W,S)}$.

3.3. Iwahori-Hecke algebras of affine type

Even though in recent years Iwahori-Hecke algebras of non-affine type Coxeter systems gained attention, traditionally the study of Iwahori-Hecke algebras was mainly focused on spherical and (extended) affine type Coxeter systems. The reason being that, similar to the characterization of affine Coxeter groups in terms of reflections of Euclidean vector spaces (see Subsection 2.7.3), these algebras admit a convenient characterization in terms of crystallographic root systems.

An important tool in the representation theory of Iwahori-Hecke algebras of affine type Coxeter systems is *Bernstein's* (unpublished) *presentation* (see e.g. [131], [96], [35]). It implies that these algebras are finitely generated over their centers which will play a role in Chapter 8. Following [35] (see also [131], [96]), we will present some details.

Let V be a finite-dimensional real Euclidean vector space with canonical inner product $\langle \cdot, \cdot \rangle$ and let Φ be an irreducible crystallographic root system. Recall that by Theorem 2.7.12 the subgroup $W_{\text{aff}} := W_{\text{aff}}(\Phi)$ of $O(V)$ generated by the affine reflections $\{s_{\alpha,i} \mid \alpha \in \Phi, i \in \mathbb{Z}\}$ is a Coxeter group with respect to the generating set $S_{\text{aff}} := \{s_{\alpha,0} \mid \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}$ where Δ is a simple system and where $\tilde{\alpha}$ denotes the corresponding highest root. Further recall that every irreducible affine type Coxeter group arises in that fashion and that W_{aff} decomposes as a semidirect product $W_{\text{aff}} \cong T \rtimes W$. Here T is the group of translations t_v, $v \in L(\Phi^\vee)$ with $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\} \subseteq V$, $L(\Phi^\vee) = \{n_1\alpha_1 + ... + n_l\alpha_l \mid \alpha_1, ..., \alpha_l \in \Phi^\vee, n_1, ..., n_l \in \mathbb{Z}\}$ and W is the irreducible finite Coxeter group with generating set $S = \{s_\alpha \mid \alpha \in \Delta\}$.

Let R be a commutative unital ring, let $q = (q_s)_{s \in S_{\text{aff}}} \in R^{(W_{\text{aff}}, S_{\text{aff}})}$ with $q_s \in R^\times$ for all $s \in S_{\text{aff}}$ and consider the corresponding Iwahori-Hecke algebra $R_q[W_{\text{aff}}]$. As before, denote by $|\cdot|$ the word length with respect to the generating set S_{aff}. Note that in general $|t_v t_w| \neq |t_v| + |t_w|$ for $v, w \in L(\Phi^\vee)$. In particular, usually $\widetilde{T}^{(q)}_{t_v t_w} \neq \widetilde{T}^{(q)}_{t_v} \widetilde{T}^{(q)}_{t_w}$. To fix this issue, introduce the set

$$L_{\text{dom}}(\Phi^\vee) := \left\{ x \in L(\Phi^\vee) \mid \langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta \right\} \subseteq V$$

3

of *dominant weights* in the co-root lattice. From the characterization of $|\cdot|$ in terms of the number of hyperplanes which separate a certain alcove and the fundamental alcove (see Subsection 2.7.3) one deduces $|t_v t_w| = |t_v| + |t_w|$ for all $v, w \in L_{\text{dom}}(\Phi^\vee)$ (see [35, Lemma 17] for details) and hence that $\widetilde{T}^{(q)}_{t_v t_w} = \widetilde{T}^{(q)}_{t_v} \widetilde{T}^{(q)}_{t_w}$. Now, a general weight $x \in L(\Phi^\vee)$ decomposes as $x = v - w$ where $v, w \in L_{\text{dom}}(\Phi^\vee)$. Define

$$\theta^{(q)}_x := \widetilde{T}^{(q)}_{t_v} (\widetilde{T}^{(q)}_{t_w})^{-1} \in (R_q[W])^\times$$

and note that $\theta^{(q)}_x$ does not depend on the choice of $v, w \in L_{\text{dom}}(\Phi^\vee)$. Indeed, if $v - w = v' - w'$ for $v, w \in L_{\text{dom}}(\Phi^\vee)$, then

$$\widetilde{T}^{(q)}_{t_{w'}} \widetilde{T}^{(q)}_{t_v} = \widetilde{T}^{(q)}_{t_{w'} t_v} = \widetilde{T}^{(q)}_{t_{v'} t_w} = \widetilde{T}^{(q)}_{t_{v'}} \widetilde{T}^{(q)}_{t_w}$$

and hence

$$\begin{aligned}
\widetilde{T}^{(q)}_{t_v} (T^{(q)}_{t_w})^{-1} &= (\widetilde{T}^{(q)}_{t_{w'}})^{-1} \widetilde{T}^{(q)}_{t_{v'}} \\
&= (\widetilde{T}^{(q)}_{t_{w'}})^{-1} \widetilde{T}^{(q)}_{t_{v'}} \widetilde{T}^{(q)}_{t_{w'}} (\widetilde{T}^{(q)}_{t_{w'}})^{-1} \\
&= (\widetilde{T}^{(q)}_{t_{w'}})^{-1} \widetilde{T}^{(q)}_{t_{w'}} \widetilde{T}^{(q)}_{t_{v'}} (\widetilde{T}^{(q)}_{t_{w'}})^{-1} \\
&= \widetilde{T}^{(q)}_{t_{v'}} (\widetilde{T}^{(q)}_{t_{w'}})^{-1}.
\end{aligned}$$

By a similar calculation we also have that $\theta^{(q)}_x \theta^{(q)}_y = \theta^{(q)}_{x+y}$ for all $x, y \in L(\Phi^\vee)$, so in particular the subalgebra $\mathscr{A}$ of $R_q[W]$ generated by all elements $\theta^{(q)}_x$, $x \in L(\Phi^\vee)$ is commutative. It is canonically isomorphic to the group algebra $R[T]$.

The group W naturally acts on $\mathscr{A}$ via $\mathbf{w}.\theta_x := \theta_{\mathbf{w}(x)}$ for $\mathbf{w} \in W$, $x \in L(\Phi^\vee)$. It is a remarkable (unpublished) result by Bernstein that the center $\mathcal{Z}(R_q[W])$ of $R_q[W]$ coincides with the set $\mathscr{A}^W$ of W-invariant elements in $\mathscr{A}$ and that it is large in the sense that $R_q[W]$ is finitely generated over $\mathcal{Z}(R_q[W])$. Further, the set of elements of the form $\theta^{(q)}_x \widetilde{T}^{(q)}_{\mathbf{w}}$ where $x \in L_{\text{dom}}(\Phi^\vee)$, $\mathbf{w} \in W$ forms an R-basis of $R_q[W]$. For a proof (in a slightly different setting) see [131], [96] and [35].

Theorem 3.3.1 (Bernstein). *Let R be a commutative unital ring, let (W, S) be an irreducible Coxeter system of affine type and let $q = (q_s)_{s \in S} \in R^{(W,S)}$ with $q_s \in R^\times$ for all $s \in S$. Then there exists a subset $T \subseteq S$ such that the special subgroup W_T is finite and such that every element in the Iwahori-Hecke algebra $R_q[W]$ is a sum of the form $\sum_{\mathbf{w} \in W_T} z_{\mathbf{w}} \widetilde{T}^{(q)}_{\mathbf{w}}$ with coefficients $z_{\mathbf{w}} \in \mathcal{Z}(R_q[W])$, i.e. $R_q[W]$ is finitely generated over its center.*

3.4. HECKE OPERATOR ALGEBRAS

We are finally in the position to introduce Hecke operator algebras which are the central focus of study of this dissertation. In the case of spherical and (extended) affine Coxeter systems these types of completions of Iwahori-Hecke algebras have been considered early by Matsumoto in [133]. In the eighties Baum, Higson and Plymen looked at C*-completions of affine Iwahori-Hecke algebras in relation to the Baum-Connes conjecture (an account of which can be found in [15]). Further, an extensive treatment of them appears in the article [146] which inspired several other results, some of which are [69], [147], [167], [168].

In the (possibly) non-affine setting Hecke operator algebras first occurred in the context of the ℓ^2-cohomology of buildings (see [76], [66] and also [67]). Motivated by this, Davis formulated in [67, Chapter 19] the question for a classification of factorial Hecke-von Neumann algebras. In the right-angled single-parameter setting such a classification was obtained by Garncarek (see [86]) who further observed that Hecke-von Neumann algebras of right-angled Coxeter groups are closely related to Dykema's interpolated free group factors, which play an important role in the treatment of the infamous free factor problem (see [73], [159], [86]). A number of other results followed, see [45], [46], [160], [24], [49], [161].

Let (W,S) be a Coxeter system and $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$. Write $\ell^2(W)$ for the Hilbert space of square-summable functions on W and denote its canonical orthonormal basis by $(\delta_{\mathbf{w}})_{\mathbf{w}\in W}\subseteq\ell^2(W)$. For every $s\in S$ the element $T_s^{(q)}=q_s^{-\frac{1}{2}}\widetilde{T}_s^{(q)}\in\mathbb{C}_q[W]$ acts boundedly on $\ell^2(W)$ via

$$T_s^{(q)}\delta_{\mathbf{w}}:=\begin{cases}\delta_{s\mathbf{w}} & \text{, if } |s\mathbf{w}|>|\mathbf{w}|\\ \delta_{s\mathbf{w}}+p_s(q)\delta_{\mathbf{w}} & \text{, if } |s\mathbf{w}|<|\mathbf{w}|\end{cases}\tag{3.4.1}$$

where as before $p_s(q):=q_s^{-\frac{1}{2}}(q_s-1)$. The induced map $S\to\mathcal{B}(\ell^2(W))$ extends to a faithful algebra representation $\mathbb{C}_q[W]\hookrightarrow\mathcal{B}(\ell^2(W))$ (compare with the proof of Theorem 3.1.1). We will identify $\mathbb{C}_q[W]$ with its image under this representation. With the involution coming from $\mathcal{B}(\ell^2(W))$, the Iwahori-Hecke algebra $\mathbb{C}_q[W]$ carries a $*$-algebra structure satisfying $(T_{\mathbf{w}}^{(q)})^*=T_{\mathbf{w}^{-1}}^{(q)}$ for $\mathbf{w}\in W$.

Remark 3.4.1. Let (W,S) be a Coxeter system and $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$. Note that for $s\in S$ the element $T_s^{(q)}$ (viewed as an operator in $\mathcal{B}(\ell^2(W))$) is self-adjoint. The equality $(T_s^{(q)})^2=1+p_s(q)T_s^{(q)}$ implies that every point λ in the spectrum of $T_s^{(q)}$ satisfies $\lambda^2=1+p_s(q)\lambda$. One further checks that the vectors

$$\xi_1:=q_s^{\frac{1}{2}}(q_s+1)^{-1}\delta_s+(q_s+1)^{-1}\delta_e$$

and

$$\xi_2:=-q_s^{\frac{1}{2}}(q_s+1)^{-1}\delta_s+q_s(q_s+1)^{-1}\delta_e$$

are eigenvectors of $T_s^{(q)}$ with $T_s^{(q)}\xi_1=q_s^{\frac{1}{2}}\xi_1$ and $T_s^{(q)}\xi_2=-q_s^{-\frac{1}{2}}\xi_2$. Hence the spectrum of $T_s^{(q)}$ is given by $\{-q_s^{-\frac{1}{2}},q_s^{\frac{1}{2}}\}$. In particular, $\|T_s^{(q)}\|=\max\{q_s^{\frac{1}{2}},q_s^{-\frac{1}{2}}\}$.

Consider the anti-linear isometric isomorphism J on $\ell^2(W)$ defined by $\delta_{\mathbf{v}} \mapsto \delta_{\mathbf{v}^{-1}}$ and define operators $T_{\mathbf{w}}^{(q),r} := JT_{\mathbf{w}}^{(q)}J \in \mathcal{B}(\ell^2(W))$ for $\mathbf{w} \in W$. One checks that every such operator satisfies a right-handed (i.e. with the order of s and $\mathbf{w}$ reversed) analogue of (3.4.1) and that it commutes with the elements in $\mathbb{C}_q[W] \subseteq \mathcal{B}(\ell^2(W))$. Hence, the *right-handed Iwahori-Hecke algebra*

$$\mathbb{C}_q^r[W] := *\text{-Alg}(\{T_{\mathbf{w}}^{(q),r} \mid \mathbf{w} \in W\}) \subseteq \mathcal{B}(\ell^2(W))$$

3 of the system (W,S) and the multi-parameter $q \in \mathbb{R}_{>0}^{(W,S)}$ is contained in the commutant $(\mathbb{C}_q[W])' \subseteq \mathcal{B}(\ell^2(W))$ of $\mathbb{C}_q[W]$ and conversely $\mathbb{C}_q[W]$ is contained in the commutant $(\mathbb{C}_q^r[W])' \subseteq \mathcal{B}(\ell^2(W))$ of $\mathbb{C}_q^r[W]$. A short calculation implies that the orthonormal basis vector $\delta_e \in \ell^2(W)$ associated with the neutral element e of the group is cyclic and separating for both $\mathbb{C}_q[W]$ and $\mathbb{C}_q^r[W]$.

We define the (*reduced*) *Hecke C^*-algebra* $C_{r,q}^*(W)$ of (W,S) and $q \in \mathbb{R}_{>0}^{(W,S)}$ as the norm closure of $\mathbb{C}_q[W]$ in $\mathcal{B}(\ell^2(W))$ and the corresponding *Hecke-von Neumann algebra* as $\mathcal{N}_q(W) := (\mathbb{C}_q[W])'' \subseteq \mathcal{B}(\ell^2(W))$. Similarly, the *right-handed* (*reduced*) *Hecke C^*-algebra* $C_{r,q}^{*,r}(W)$ is the norm closure of $\mathbb{C}_q^r[W]$ in $\mathcal{B}(\ell^2(W))$ and $\mathcal{N}_q^r(W) := (\mathbb{C}_q^r[W])'' \subseteq \mathcal{B}(\ell^2(W))$ is the *right-handed Hecke-von Neumann algebra*. For $q \in \mathbb{R}_{>0}$ we denote the *single-parameter Hecke operator algebras* associated with $(q)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ by $C_{r,q}^*(W)$, $\mathcal{N}_q(W)$, $C_{r,q}^{*,r}(W)$ and $\mathcal{N}_q^r(W)$ as well.

Note that for $q = 1 := (1)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ one has that $\mathbb{C}_q[W] = \mathbb{C}[W]$, $C_{r,q}^*(W) = C_r^*(W)$ and $\mathcal{N}_q(W) = \mathcal{L}(W)$ are respectively the group algebra, the reduced group C*-algebra and the group von Neumann algebra of W and that $T_{\mathbf{w}}^{(1)} \in \mathcal{B}(\ell^2(W))$, $\mathbf{w} \in W$ is nothing but the left regular representation operator $\lambda_{\mathbf{w}}$ (see Section 2.3). Hecke (C*- and von Neumann) algebras can hence be viewed as q-deformations of group (C*- and von Neumann) algebras of Coxeter groups where the deformation (in principle) depends on the parameter q.

Remark 3.4.2. *(a)* For completeness it should be mentioned that in [160] Raum and Skalski introduced ℓ^r-convolution algebra analogues to Hecke-von Neumann algebras which generalize the construction above. These analogues will however not occur as a part of this thesis.
(b) Similar to the group C*-algebraic setting (see Section 2.3) one can introduce a *universal* analogue to reduced Hecke C*-algebras which, to the author's best knowledge does not yet occur in the literature. For this, let (W,S) be a Coxeter system, let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ and define $C_{u,q}^*(W)$ to be the completion of the Iwahori-Hecke algebra $\mathbb{C}_q[W]$ (equipped with the $*$-algebra structure from before) with respect to the norm

$$\|x\|_u := \sup\left\{\|\pi(x)\| \mid \pi \text{ is a (cyclic) } *\text{-representation of } \mathbb{C}_q[W]\right\}.$$

Similar to the reasoning in Remark 3.4.1 it can be deduced that the supremum is indeed finite for every $x \in \mathbb{C}_q[W]$, i.e. the C*-algebra $C_{u,q}^*(W)$ is well-defined. It identifies with the universal C*-algebra generated by a family of self-adjoint elements $(R_s)_{s\in S}$ with respect to the relations $R_s^2 = 1 + p_s(q)R_s$ for $s \in S$ and $R_sR_tR_s... =$

$R_tR_sR_t...$ for $s,t \in S$ with $m_{st} \neq \infty$ where the number of factors on the left-hand side and the right-hand side equals m_{st}. Further, the identity map on $\mathbb{C}_q[W]$ uniquely extends to a surjective $*$-homomorphism $C^*_{u,q}(W) \twoheadrightarrow C^*_{u,r}(W)$.

In the following sections, we will be mostly concerned with reduced Hecke C*-algebras. The q-deformed setting however leads to a couple of curious phenomena of universal Hecke C*-algebras which will be discussed in Section 3.5.

Let us proceed by collecting some basic facts about Hecke operator algebras.

Lemma 3.4.3. *Let (W,S) be a Coxeter system and $q \in \mathbb{R}^{(W,S)}_{>0}$. Then the vector state*

$$\tau : \mathcal{B}(\ell^2(W)) \to \mathbb{C}, \tau(x) := \langle x\delta_e, \delta_e \rangle$$

restricts to a normal tracial state τ_q on $\mathcal{N}_q(W)$ with $\tau_q(T^{(q)}_{\mathbf{w}}) = 0$ for all $\mathbf{w} \in W \setminus \{e\}$.

Proof. It is clear that the restriction τ_q of τ to $\mathcal{N}_q(W)$ is normal and satisfies $\tau_q(T^{(q)}_{\mathbf{w}}) = 0$ for $\mathbf{w} \in W \setminus \{e\}$. The traciality follows from

$$\tau_q(T^{(q)}_{\mathbf{v}} T^{(q)}_{\mathbf{w}}) = \left\langle T^{(q)}_{\mathbf{v}} T^{(q)}_{\mathbf{w}} \delta_e, \delta_e \right\rangle = \left\langle \delta_{\mathbf{w}}, \delta_{\mathbf{v}^{-1}} \right\rangle = \begin{cases} 0 & \text{, if } \mathbf{w} \neq \mathbf{v}^{-1} \\ 1 & \text{, if } \mathbf{w} = \mathbf{v}^{-1} \end{cases}$$

for $\mathbf{v}, \mathbf{w} \in W$. Finally, the faithfulness of τ_q can be deduced from the fact that $\delta_e \in \ell^2(W)$ is separating for $\mathcal{N}_q(W)$. □

One easily checks that an analaogue of Lemma 3.4.3 also holds for $\mathcal{N}^r_q(W)$.

Lemma 3.4.4 ([76, Proposition 2.1]). *Let (W,S) be a Coxeter system and $q \in \mathbb{R}^{(W,S)}_{>0}$. Then $\mathcal{N}^r_q(W) = (\mathcal{N}_q(W))'$ and $\mathcal{N}_q(W) = (\mathcal{N}^r_q(W))'$, i.e. $\mathcal{N}_q(W)$ is the commutant of $\mathcal{N}^r_q(W)$ and vice versa.*

Proof. As mentioned before, $\mathbb{C}_q[W] \subseteq (\mathbb{C}^r_q[W])'$ and $\mathbb{C}^r_q[W] \subseteq (\mathbb{C}_q[W])'$ which implies that

$$\mathcal{N}_q(W) = (\mathbb{C}_q[W])'' \subseteq (\mathbb{C}^r_q[W])''' = (\mathcal{N}^r_q(W))'$$

and similarly $\mathcal{N}^r_q(W) \subseteq (\mathcal{N}_q(W))'$. Thus, by the well-known properties of the modular conjugation operator J,

$$\mathcal{N}_q(W) \subseteq (\mathcal{N}^r_q(W))' = J\mathcal{N}^r_q(W)J \subseteq J(\mathcal{N}_q(W))'J = \mathcal{N}_q(W),$$

i.e. $\mathcal{N}_q(W) = (\mathcal{N}^r_q(W))'$ and $\mathcal{N}^r_q(W) = (\mathcal{N}_q(W))'$. □

Proposition 3.4.5 ([67, Proposition 19.2.2]). *Let (W,S) be a Coxeter system, $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$, $T \subseteq S$ and set $q_T := (q_t)_{t\in T} \in \mathbb{R}^{(W_T,T)}_{>0}$. Then the canonical embedding $\mathbb{C}_{q_T}[W_T] \hookrightarrow \mathbb{C}_q[W]$, $T^{(q_T)}_{\mathbf{w}} \mapsto T^{(q)}_{\mathbf{w}}$ uniquely extends to $*$-embeddings $C^*_{r,q_T}(W_T) \hookrightarrow C^*_{r,q}(W)$ and $\mathcal{N}_{q_T}(W_T) \hookrightarrow \mathcal{N}_q(W)$.*

3

Proof. Denote the map $\mathbb{C}_{q_T}[W_T] \hookrightarrow \mathbb{C}_q[W]$ given by $T_{\mathbf{w}}^{(q_T)} \mapsto T_{\mathbf{w}}^{(q)}$ for $\mathbf{w} \in W$ by ι. It obviously preserves the $*$-structure of $\mathbb{C}_{q_T}[W_T] \subseteq \mathscr{B}(\ell^2(W_T))$. For $\mathbf{w} \in W$ write

$$\ell^2(W_T\mathbf{w}) := \overline{\text{Span}\{\delta_{\mathbf{v}} \mid \mathbf{v} \in W_T\mathbf{w}\}} \subseteq \ell^2(W)$$

and note that these subspaces give rise to a direct sum decomposition $\ell^2(W) \cong \bigoplus_{W_T\mathbf{w}} \ell^2(W_T\mathbf{w})$ where the sum runs over all right cosets $W_T\mathbf{w}$, $\mathbf{w} \in W$. In this picture, for every $x \in \mathbb{C}_{q_T}[W_T]$ the image $\iota(x)$ preserves the summands and acts on each of them in the same way in the sense that $U^*\iota(x)U = x$ where $U : \ell^2(W_T) \to \ell^2(W_T\mathbf{w})$ is the natural unitary. Hence $\iota(x)$ is a multiple of the operator x where the multiplicity coincides with the number of cosets. This implies the claim. □

It is often useful not to consider Hecke operator algebras of a given Coxeter system independently from each other, but to embrace that they are deformations continuously depending on the parameter q. This perspective will play a major role in the later chapters of this thesis. The following lemma is a multi-parameter version of [46, Lemma 2.7] and roughly states that the canonical basis elements (viewed as operators on $\ell^2(W)$) of a given right-angled Iwahori-Hecke algebra conveniently decompose as sums of products of group algebra elements and certain projections. It should be mentioned that decompositions of this flavour also exist outside of the right-angled setting. However, these are often much harder to formulate explicitly due to the (potentially) more complicated combinatorial structure of the underlying Coxeter group.

Proposition 3.4.6 ([46, Lemma 2.7])**.** *Let (W,S) be a right-angled Coxeter system. Denote the set of subsets of S whose elements pairwise commute (including the empty set) by Cliq, let $P_s \in \ell^\infty(W) \subseteq \mathscr{B}(\ell^2(W))$ be the orthogonal projection onto*

$$\overline{\mathit{Span}\{\delta_{\mathbf{v}} \mid \mathbf{v} \in W \text{ with } s \leq \mathbf{v}\}} \subseteq \ell^2(W)$$

and write $P_\Gamma := \prod_{s\in\Gamma} P_s$ for $\Gamma \in$ Cliq. Further, for $\mathbf{w} \in W$ let $A_{\mathbf{w}}$ be the set of triples $(\mathbf{w}', \Gamma, \mathbf{w}'')$ with $\mathbf{w}', \mathbf{w}'' \in W$ and $\Gamma \in$ Cliq such that $\mathbf{w} = \mathbf{w}'\left(\prod_{s\in\Gamma} s\right)\mathbf{w}''$, $|\mathbf{w}| = |\mathbf{w}'| + |\prod_{s\in\Gamma} s| + |\mathbf{w}''|$ and $|\mathbf{w}'t| > |\mathbf{w}'|$ for all $t \in S$ with $m_{st} = 2$ for all $s \in \Gamma$. Then the operator $T_{\mathbf{w}}^{(q)}$ decomposes as

$$T_{\mathbf{w}}^{(q)} = \sum_{(\mathbf{w}',\Gamma,\mathbf{w}'')\in A_{\mathbf{w}}} \left(\prod_{s\in\Gamma} p_s(q)\right) T_{\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)}. \tag{3.4.2}$$

Proof. The proof proceeds by induction over the word length of $\mathbf{w}$. Note that in the case where $|\mathbf{w}| = 1$ it follows from (3.4.1) that $T_{\mathbf{w}}^{(q)} = T_{\mathbf{w}}^{(1)} + p_{\mathbf{w}}(q)P_{\mathbf{w}}$.

Now suppose that (3.4.2) holds for $\mathbf{w} \in W$ and let $t \in S$ with $|t\mathbf{w}| = |\mathbf{w}| + 1$. One gets that

$$\begin{aligned} T_{t\mathbf{w}}^{(q)} &= T_t^{(q)} T_{\mathbf{w}}^{(q)} \\ &= \left(T_t^{(1)} + p_t(q)P_t\right)\left(\sum_{(\mathbf{w}',\Gamma,\mathbf{w}'')\in A_{\mathbf{w}}} \left(\prod_{s\in\Gamma} p_s(q)\right) T_{\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)}\right) \end{aligned}$$

$$= \sum_{(\mathbf{w}',\Gamma,\mathbf{w}'')\in A_{\mathbf{w}}} \left(\prod_{s\in\Gamma} p_s(q)\right)\left(T_t^{(1)} T_{\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)} + p_t(q) P_t T_{\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)}\right). \quad (3.4.3)$$

For a triple $(\mathbf{w}',\Gamma,\mathbf{w}'') \in A_{\mathbf{w}}$ distinguish the following two cases.

(1): If $t\mathbf{w}' = \mathbf{w}'t$, by the assumptions $|t\mathbf{w}| = |\mathbf{w}|+1$ and $|\mathbf{w}| = |\mathbf{w}'| + |\prod_{s\in\Gamma} s| + |\mathbf{w}''|$ we have that $t \not\leq \mathbf{w}'$ in the weak right Bruhat order (see Subsection 2.7.2). It follows that $P_t T_{\mathbf{w}'}^{(1)} = T_{\mathbf{w}'}^{(1)} P_t$ and hence $p_t(q) P_t T_{\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)} = p_t(q) T_{\mathbf{w}'}^{(1)} P_t P_\Gamma T_{\mathbf{w}''}^{(1)}$. One further has $P_t P_\Gamma = P_{\{t\}\cup\Gamma}$ if $\{t\}\cup\Gamma \in \mathrm{Cliq}$ and $P_t P_\Gamma = 0$ otherwise.

(2): Now assume that $t\mathbf{w}' \neq \mathbf{w}'t$. For every $\mathbf{u} \in W$ with $P_\Gamma \delta_{\mathbf{u}} \neq 0$ there exists a decomposition $\mathbf{u} = \mathbf{v}\mathbf{v}'$ with $|\mathbf{u}| = |\mathbf{v}| + |\mathbf{v}'|$ where $\mathbf{v} := \prod_{s\in\Gamma} s$ and $\mathbf{v}' \in W$. Let $\mathbf{w}' = s_1 ... s_n$ be a reduced expression for $\mathbf{w}'$. By the properties of $(\mathbf{w}',\Gamma,\mathbf{w}'')$ one has $|s_n\mathbf{v}\mathbf{v}'| = |\mathbf{v}|+|\mathbf{v}'|+1$. Indeed, if $|s_n\mathbf{v}\mathbf{v}'| = |\mathbf{v}|+|\mathbf{v}'|-1$ then s_n either cancels a letter in the expression $\prod_{s\in\Gamma} s$ for $\mathbf{v}$ or s commutes with $\mathbf{v}$ (see Subsection 2.7.4). Both options contradict the assumptions on the triple $(\mathbf{w}',\Gamma,\mathbf{w}'') \in A_{\mathbf{w}}$ that $|\mathbf{w}| = |\mathbf{w}'| + |\mathbf{v}| + |\mathbf{w}''|$ and $|\mathbf{w}'t'| > |\mathbf{w}'|$ for all $t' \in S$ with $m_{st'} = 2$ for all $s \in \Gamma$. By the same argument, $|s_{n-1}s_n\mathbf{v}\mathbf{v}'| = |\mathbf{v}| + |\mathbf{v}'| + 2$. Proceeding like this we get that $|\mathbf{w}'\mathbf{v}\mathbf{v}'| = |\mathbf{w}'| + |\mathbf{v}| + |\mathbf{v}'|$. Now, by $|t\mathbf{w}| = |\mathbf{w}|+1$ we in particular have that $t \not\leq \mathbf{w}'$ and since $t\mathbf{w}' \neq \mathbf{w}'t$ we deduce that $P_t T_{\mathbf{w}'}^{(1)} P_\Gamma \delta_{\mathbf{u}} = P_t \delta_{\mathbf{w}'\mathbf{v}\mathbf{v}'} = 0$ (again, see Subsection 2.7.4). It follows that $P_t T_{\mathbf{w}'}^{(1)} P_\Gamma = 0$.

The considerations above imply in combination with (3.4.3) that

$$\begin{aligned} T_{t\mathbf{w}}^{(q)} &= \sum_{(\mathbf{w}',\Gamma,\mathbf{w}'')\in A_{\mathbf{w}}} \left(\prod_{s\in\Gamma} p_s(q)\right) T_{t\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)} \\ &\quad + \sum_{(\mathbf{w}',\Gamma,\mathbf{w}'')\in A_{\mathbf{w}}: t\mathbf{w}'=\mathbf{w}'t, \{t\}\cup\Gamma\in\mathrm{Cliq}} \left(\prod_{s\in\{t\}\cup\Gamma} p_s(q)\right) T_{\mathbf{w}'}^{(1)} P_{\{t\}\cup\Gamma} T_{\mathbf{w}''}^{(1)} \\ &= \sum_{(\mathbf{w}',\Gamma,\mathbf{w}'')\in A_{t\mathbf{w}}} \left(\prod_{s\in\Gamma} p_s(q)\right) T_{\mathbf{w}'}^{(1)} P_\Gamma T_{\mathbf{w}''}^{(1)} \end{aligned}$$

from which the claim follows. □

We finish this subsection with a useful inequality.

Lemma 3.4.7. *Let (W,S) be a Coxeter system, $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ and $\mathbf{w} \in W$. Let further $\mathbf{w} = s_1 ... s_n$ with $s_1,...,s_n \in S$ be a reduced expression for $\mathbf{w}$. Then,*

$$\prod_{i=1}^n \min\{q_{s_i}, q_s^{-1}\} \leq (T_{\mathbf{w}}^{(q)})^* T_{\mathbf{w}}^{(q)} \leq \prod_{i=1}^n \max\{q_{s_i}, q_s^{-1}\}. \quad (3.4.4)$$

Proof. By Remark 3.4.1 for every $1 \leq i \leq n$ the spectrum of $(T_{s_i}^{(q)})^2$ is given by $\{q_{s_i}, q_{s_i}^{-1}\}$ and hence $\min\{q_{s_i}, q_{s_i}^{-1}\} \leq (T_{s_i}^{(q)})^2 \leq \max\{q_{s_i}, q_{s_i}^{-1}\}$. It inductively follows that

$$(T_{\mathbf{w}}^{(q)})^* T_{\mathbf{w}}^{(q)} = T_{s_n}^{(q)} ... T_{s_1}^{(q)} T_{s_1}^{(q)} ... T_{s_n}^{(q)}$$

$$\begin{aligned} &\geq \min\{q_{s_1}, q_{s_1}^{-1}\} T_{s_n}^{(q)} \dots T_{s_2}^{(q)} T_{s_2}^{(q)} \dots T_{s_n}^{(q)} \\ &\geq \dots \\ &\geq \prod_{i=1}^{n} \min\{q_{s_i}, q_{s_i}^{-1}\} \end{aligned}$$

and similarly

$$\begin{aligned} (T_{\mathbf{w}}^{(q)})^* T_{\mathbf{w}}^{(q)} &= T_{s_n}^{(q)} \dots T_{s_1}^{(q)} T_{s_1}^{(q)} \dots T_{s_n}^{(q)} \\ &\leq \max\{q_{s_1}, q_{s_1}^{-1}\} T_{s_n}^{(q)} \dots T_{s_2}^{(q)} T_{s_2}^{(q)} \dots T_{s_n}^{(q)} \\ &\leq \dots \\ &\leq \prod_{i=1}^{n} \max\{q_{s_i}, q_{s}^{-1}\}. \end{aligned}$$

This proves the inequality (3.4.4). □

3.5. ISOMORPHISMS OF HECKE ALGEBRAS

This section aims to discuss isomorphism properties of Hecke algebras and their implications for the associated operator algebras, in particular concerning the dependence on the multi-parameter. Recall that these properties are well understood in the case of finite Coxeter systems, see Section 3.2.

The following proposition is well-known on the algebraic level, see for instance [133, Chapter 2], [67, Chapter 19], [145, Section 9]. The proof of its operator algebraic counterparts makes use of the following general fact which is standard and shall be used several times in the course of this thesis.

Lemma 3.5.1. *Let A and B be unital C^*-algebras with respective faithful tracial states τ_A and τ_B. Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be dense $*$-subalgebras of A and B and let $\pi : A_0 \to B_0$ be a $*$-isomorphism such that $\tau_B \circ \pi = \tau_A$. Then π extends to $*$-isomorphisms $A \to B$ and $\pi_{\tau_A}(A)'' \to \pi_{\tau_B}(B)''$ where π_{τ_A} and π_{τ_B} denote the GNS-representations.*

Proof. The statement is essentially [135, Theorem 5.1.4]. Without loss of generality we may assume that A and B are represented on their GNS-Hilbert spaces $L^2(A,\tau_A)$ and $L^2(B,\tau_B)$ with cyclic vectors Ω_A and Ω_B. Then the map $U : L^2(A,\tau_A) \to L^2(B,\tau_B)$ induced by $U(a\Omega_A) := \pi(a)\Omega_B$ for $a \in A_0$ is a unitary with $\pi(a) = UaU^*$. Therefore π extends to a map $A \to B$ as well as a map $A'' \to B''$. □

Proposition 3.5.2. *Let (W,S) be a Coxeter system and $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$, $\epsilon = (\epsilon_s)_{s\in S} \in \{-1,1\}^{(W,S)}$. Then $C^*_{r,q}(W) \cong C^*_{r,q'}(W)$ and $\mathcal{N}_q(W) \cong \mathcal{N}_{q'}(W)$ via $T_s^{(q)} \mapsto \epsilon_s T_s^{(q')}$ where $q' := (q_s^{\epsilon_s})_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$.*

Proof. First note that $\epsilon_s p_s(q') = p_s(q)$. From the defining properties of the Iwahori-Hecke algebras $\mathbb{C}_q[W]$ and $\mathbb{C}_{q'}[W]$ (see Theorem 3.1.1) it follows that $T_s^{(q)} \mapsto \epsilon_s T_s^{(q')}$ for $s \in S$ determines a $*$-isomorphism $\pi_{q',q} : \mathbb{C}_q[W] \to \mathbb{C}_{q'}[W]$. Moreover,

$$\tau_{q'} \circ \pi_{q',q}(T_{\mathbf{w}}^{(q)}) = \epsilon_{\mathbf{w}} \tau_{q'}(T_{\mathbf{w}}^{(q')}) = \tau_q(T_{\mathbf{w}}^{(q)})$$

because $\tau_{q'}(T_{\mathbf{w}}^{(q')}) = \tau_q(T_{\mathbf{w}}^{(q)}) = 0$ for all $\mathbf{w} \in W \setminus \{e\}$. We get that $\pi_{q',q}$ is trace-preserving and hence, by Lemma 3.5.1, $\pi_{q',q}$ extends to $*$-isomorphisms $C_{r,q}^*(W) \cong C_{r,q'}^*(W)$ and $\mathcal{N}_q(W) \cong \mathcal{N}_{q'}(W)$. □

The following proposition easily follows from Proposition 3.4.5 in combination with the universal property of the inductive limit construction.

Proposition 3.5.3. *Let (W,S) be a Coxeter system, $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$, $T_0 \subseteq S$ finite and $\mathcal{S} := \{T \subseteq S \mid T \text{ finite with } T_0 \subseteq T\}$. For $T \in \mathcal{S}$ set $q_T := (q_t)_{t\in T}$. Then*

$$\left\{(C_{r,q_T}^*(W_T), \phi_{T,T'}) \mid T, T' \in \mathcal{S} \text{ with } T \subseteq T'\right\}$$

with $\phi_{T,T'}(T_t^{(q_T)}) := T_t^{(q_{T'})}$ for $t \in T$ defines an inductive system with $C_{r,q}^*(W) \cong \varinjlim C_{r,q_T}^*(W_T)$.

Theorem 3.5.4. *Let (W,S) be a spherical type Coxeter system and $q \in \mathbb{R}_{>0}^{(W,S)}$. Then the Hecke C^*-algebra $C_{r,q}^*(W)$ is $*$-isomorphic to the reduced group C^*-algebra $C_r^*(W)$. A similar statement holds for the corresponding Hecke-von Neumann algebras.*

Proof. First assume that $\#S < \infty$, that is W is a finite group and hence $\mathbb{C}_q[W] = C_{r,q}^*(W) = \mathcal{N}_q(W)$, $\mathbb{C}[W] = C_r^*(W) = \mathcal{L}(W)$. By the discussion in Section 3.2, the Iwahori-Hecke algebra $\mathbb{C}_q[W]$ is algebraically isomorphic to $\mathbb{C}_1[W]$. Recall that every finite-dimensional C^*-algebra is $*$-isomorphic to a direct sum of matrix algebras. There are different C^*-structures on such an algebra, but they are all equivalent under conjugation by suitable elements. It follows that finite-dimensional C^*-algebras are isomorphic as C^*-algebras if and only if they are isomorphic as algebras. We deduce that $C_{r,q}^*(W) \cong C_r^*(W)$ and $\mathcal{N}_q(W) \cong \mathcal{L}(W)$. In the case where $\#S = \infty$, the same statement follows from the finitely generated case in combination with Proposition 3.5.3. □

Proposition 3.5.5. *Let (W,S) be a Coxeter system that admits a non-trivial decomposition of the form $(W,S) = (W_T \times W_{T'}, T \sqcup T')$. For $q \in \mathbb{R}_{>0}^{(W,S)}$ set $q_T := (q_t)_{t\in T}$ and $q_{T'} := (q_t)_{t\in T'}$. Then the corresponding Hecke algebra decomposes as an algebraic tensor product $\mathbb{C}_q[W] \cong \mathbb{C}_{q_T}[W_T] \odot \mathbb{C}_{q_{T'}}[W_{T'}]$ via $T_{\mathbf{vw}}^{(q)} \mapsto T_{\mathbf{v}}^{(q_T)} \otimes T_{\mathbf{w}}^{(q_{T'})}$ for $\mathbf{v} \in W_T$, $\mathbf{w} \in W_{T'}$. This induces C^*-algebraic and von Neumann algebraic isomorphisms $C_{r,q}^*(W) \cong C_{r,q_T}^*(W_T) \otimes C_{r,q_{T'}}^*(W_{T'})$ and $\mathcal{N}_q(W) \cong \mathcal{N}_{q_T}(W_T)\overline{\otimes}\mathcal{N}_{q_{T'}}(W_{T'})$.*

Proof. Define a linear map $U : \ell^2(W) \to \ell^2(W_T) \otimes \ell^2(W_{T'})$ by $\delta_{\mathbf{vw}} \mapsto \delta_{\mathbf{v}} \otimes \delta_{\mathbf{w}}$ for $\mathbf{v} \in W_T$, $\mathbf{w} \in W_{T'}$. The operator U defines an isometric isomorphism with

$$UT_t^{(q)}U^*(\delta_{\mathbf{v}} \otimes \delta_{\mathbf{w}}) = \begin{cases} \delta_{t\mathbf{v}} \otimes \delta_{\mathbf{w}} & \text{, if } |t\mathbf{v}| > |\mathbf{v}| \\ \delta_{t\mathbf{v}} \otimes \delta_{\mathbf{w}} + p_t(q_T)\delta_{\mathbf{v}} \otimes \delta_{\mathbf{w}} & \text{, if } |t\mathbf{v}| < |\mathbf{v}| \end{cases}$$

for $t \in T$, $\mathbf{v} \in W_T, \mathbf{w} \in W_{T'}$ and

$$UT_t^{(q)}U^*(\delta_{\mathbf{v}} \otimes \delta_{\mathbf{w}}) = \begin{cases} \delta_{\mathbf{v}} \otimes \delta_{t\mathbf{w}} & \text{, if } |t\mathbf{w}| > |\mathbf{w}| \\ \delta_{\mathbf{v}} \otimes \delta_{t\mathbf{w}} + p_t(q_{T'})\delta_{\mathbf{v}} \otimes \delta_{t\mathbf{w}} & \text{, if } |t\mathbf{w}| < |\mathbf{w}| \end{cases}$$

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for $t \in T'$, $\mathbf{v} \in W_T, \mathbf{w} \in W_{T'}$. For $\mathbf{v} \in W_T$ with reduced expressions $\mathbf{v} = s_1 ... s_m$ where $s_1, ..., s_m \in T$ and $\mathbf{w} \in W_{T'}$ with reduced expression $\mathbf{w} = t_1 ... t_n$ where $t_1, ..., t_n \in T'$ it hence follows that

$$\begin{aligned} UT_{\mathbf{vw}}^{(q)}U^* &= (UT_{s_1}^{(q)}U^*)...(UT_{s_m}^{(q)}U^*)(UT_{t_1}^{(q)}U^*)...(UT_{t_n}^{(q)}U^*) \\ &= (T_{s_1}^{(q_T)} \otimes 1)...(T_{s_m}^{(q_T)} \otimes 1)(1 \otimes T_{t_1}^{(q_{T'})})...(1 \otimes T_{t_n}^{(q_{T'})}) \\ &= T_{\mathbf{v}}^{(q_T)} \otimes T_{\mathbf{w}}^{(q_{T'})}, \end{aligned}$$

i.e. the unitary operator U implements isomorphisms $\mathbb{C}_q[W] \cong \mathbb{C}_{q_T}[W_T] \odot \mathbb{C}_{q_{T'}}[W_{T'}]$, $C^*_{r,q}(W) \cong C^*_{r,q_T}(W_T) \otimes C^*_{r,q_{T'}}(W_{T'})$ and $\mathcal{N}_q(W) \cong \mathcal{N}_{q_T}(W_T) \overline{\otimes} \mathcal{N}_{q_{T'}}(W_{T'})$. ☐

Because many operator algebraic properties are preserved by taking inductive limits and tensor products, Proposition 3.5.2, Proposition 3.5.3 and Proposition 3.5.5 allow to restrict in the treatment of questions like the ones for simplicity of Hecke C*-algebras or factoriality of Hecke-von Neumann algebras to irreducible, finite rank Coxeter systems (W,S) and parameters $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $0 < q_s \leq 1$.

To the author's best knowledge other than in the spherical type case there exists no general statement for arbitrary Coxeter systems (W,S) on the dependence of the isomorphism class of the Hecke deformations on the multi-parameter $q \in \mathbb{R}_{>0}^{(W,S)}$. However, for right-angled (not necessarily spherical type) Coxeter systems one can still prove that all Iwahori-Hecke algebras are isomorphic; even with an explicit isomorphism. See e.g. [133, (2.1.13)], [145, Corollary 9.7] for this result which we present in an alternative way that is suited for the next sections.

Proposition 3.5.6. *Let (W,S) be a right-angled Coxeter system and $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$. Then the map $\pi_{q,1}\colon \mathbb{C}_1[W] \to \mathbb{C}_q[W]$ given by*

$$1 \mapsto 1 \quad \textit{and} \quad T_s^{(1)} \mapsto \frac{1-q_s}{1+q_s} + \frac{2\sqrt{q_s}}{1+q_s} T_s^{(q)} \tag{3.5.1}$$

for $s \in S$ defines an isomorphism of $*$-algebras.

Proof. Set $\alpha_s(q) := (1-q_s)(1+q_s)^{-1}$ and $\beta_s(q) := 2q_s^{\frac{1}{2}}(1+q_s)^{-1}$. Being its own inverse, for $s \in S$ the expression $\alpha_s(q) + \beta_s(q) T_s^{(q)} \in \mathbb{C}_q[W]$ is invertible. It induces a map $S \to \mathbb{C}_q[W]^\times$, $s \mapsto \alpha_s(q) + \beta_s(q) T_s^{(q)}$ that uniquely extends to a group homomorphism ϕ on the free group $F(S)$ in S. Because (W,S) is right-angled one easily checks that $\phi(st)^{m_{st}} = \left(\phi(s)\phi(t)\right)^{m_{st}}$ for all $s,t \in S$. This implies that ϕ induces a group homomorphism $\phi' : W \to \mathbb{C}_q[W]^\times$ with $(\phi'(\mathbf{w}))^* = \phi(\mathbf{w}^{-1})$ for every $\mathbf{w} \in W$. The universal property of the group algebra $\mathbb{C}_1[W]$ then implies the existence of the unital $*$-algebra homomorphism $\pi_{q,1}$. It is clearly surjective. The injectivity follows from the universal property of the Iwahori-Hecke algebra $\mathbb{C}_1[W]$, see Theorem 3.1.1. ☐

Remark 3.5.7. *(a)* The homomorphism prescribed by (3.5.1) does not necessarily exist if (W,S) is not right-angled. This already fails for the Coxeter group $W = \langle s,t \mid s^2 = t^2 = (st)^3 = e \rangle$ and points out an inaccuracy in [67, Note 19.2].

(b) Let (W,S) be a right-angled Coxeter system and $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$. By the universal properties of the universal group C*-algebra $C_u^*(W)$ and the universal Hecke C*-algebra $C_{u,q}^*(W)$ (see Remark 3.4.2), the isomorphism $\pi_{q,1}$ in Proposition 3.5.6 extends to an isomorphism $C_u^*(W)\cong C_{u,q}^*(W)$. In the case where the system (W,S) is of spherical type, for trivial reasons $\pi_{q,1}$ also extends to an isomorphism $C_r^*(W)\cong C_{r,q}^*(W)$ of the corresponding reduced operator algebras. However, if W is infinite this is not true anymore and already fails in the case where $W=\langle s,t\mid s^2=t^2=e\rangle$ is the infinite dihedral group which is the only irreducible right-angled Coxeter group of affine type. Indeed, let $C_{r,0}^*(0)\subseteq\mathcal{B}(\ell^2(W))$ be the unital C*-algebra generated by the orthogonal projections P_s and P_t onto the subspaces

$$\overline{\mathrm{Span}\left\{\delta_{\mathbf{w}}\mid \mathbf{w}\in W \text{ with } s\leq\mathbf{w}\right\}}$$

and

$$\overline{\mathrm{Span}\left\{\delta_{\mathbf{w}}\mid \mathbf{w}\in W \text{ with } s\leq\mathbf{w}\right\}}$$

of $\ell^2(W)$. Assume that for every $q\in\mathbb{R}_{>0}^{(W,S)}$ the isomorphism $\pi_{q,1}$ of Proposition 3.5.6 extends to a $*$-isomorphism . Then the map defined by $P_s\mapsto\frac{1-T_s^{(1)}}{2}$ and $P_t\mapsto\frac{1-T_t^{(1)}}{2}$ extends to a $*$-isomorphism $\pi_{0,1}:C_{r,0}^*(W)\cong C_r^*(W)$ as well. Indeed, for every finite sum $x:=\sum_{\mathbf{w}\in W}x(\mathbf{w})T_{\mathbf{w}}^{(1)}\in\mathbb{C}_1[W]$ with $x(\mathbf{w})\in\mathbb{C}$ one checks that as $q\downarrow 0$,

$$\pi_{q,1}^{-1}(x)\to\sum_{\mathbf{w}\in W}x(\mathbf{w})\pi_{0,1}^{-1}(T_{\mathbf{w}}^{(1)})=\pi_{0,1}^{-1}(x)$$

in $\mathcal{B}(\ell^2(W))$. That implies

$$\|\pi_{0,1}^{-1}(x)\|=\lim_{q\downarrow 0}\|\pi_{q,1}^{-1}(x)\|=\|x\|$$

as $\pi_{q,1}$ is isometric. Because the C*-algebra $C_{r,0}^*(W)$ is commutative, we have reached a contradiction. The example illustrates that in general $\pi_{q,1}$ does not extend to an isomorphism of (reduced) C*-algebras. This is somewhat unexpected because for amenable discrete groups the universal and the reduced group C*-algebra coincide (see Theorem 2.3.8). In the case of non-affine type Coxeter systems, it gets even worse. Indeed, if (W,S) is an irreducible right-angled Coxeter system of non-affine type, then by Theorem 2.7.9 the reduced group C*-algebra $C_r^*(W)$ carries a unique tracial state. For $q\neq 1$ the map $\pi_{q,1}$ is not trace-preserving with respect to the canonical traces τ_1 and τ_q, i.e. $\tau_q\circ\pi_{q,1}\neq\tau_1$. It can hence not extend to the reduced C*-algebraic level. To check that $\pi_{q,1}^{-1}$ does in general not extend, one can either use the same argument as before or apply Theorem 6.3.3 from Section 6.3. One should also compare these observations with the results in [161] where K-theoretic invariants of right-angled Hecke C*-algebras have been computed.

(c) In [86] Garncarek characterized the center of single-parameter right-angled Hecke-von Neumann algebras. He proved that for irreducible right-angled Coxeter systems (W,S) with $\#S\geq 3$ the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is a factor (necessarily of type II$_1$) if and only if $q\in[\rho,\rho^{-1}]$ where ρ is the radius of convergence of the growth series $\sum_{\mathbf{w}\in W}z^{|\mathbf{w}|}$. Moreover, for q outside this

interval, $\mathcal{N}_q(W)$ decomposes as a direct sum of a II_1-factor and $\mathbb{C}$. Garncarek's result was later extended by Raum and Skalski in [160] to the multi-parameter case. Similar to *(b)* this illustrates that the isomorphism of Proposition 3.5.6 does not necessarily extend to an isomorphism of the corresponding Hecke-von Neumann algebras. Moreover, for $q \in [\rho, \rho^{-1}]$ and $q' \notin [\rho, \rho^{-1}]$ there can be no isomorphism at all between $\mathcal{N}_q(W)$ and $\mathcal{N}_{q'}(W)$. But the situation is even more delicate. In combination with Garncarek's calculations in [86, Section 6], reference [73, Theorem 2.3 and Proposition 2.4] implies that for every Coxeter group $W = \mathbb{Z}_2^{*l}$, $l \geq 3$ (compare with Example 2.7.1) and $q \in [(l-1)^{-1}, 1]$ the II_1-factor $\mathcal{N}_q(W)$ is isomorphic to $\mathcal{L}(\mathbb{F}_{2lq(1+q)^{-2}})$ where $\mathcal{L}(\mathbb{F}_t)$, $t \in \mathbb{R}_{>1}$ denotes Dykema's and Radulescu's *interpolated free group factor*, cf. [73], [159]. It is known that the interpolated free group factors are either all isomorphic or they are all non-isomorphic. The problem which of the two in this dichotomy is true is known as the infamous *free factor problem*. Hence, solving the isomorphism question of $\mathcal{N}_q(W)$ for different $q \in [\frac{1}{l-1}, 1]$ is equivalent to the free factor problem. In Subsection 6.1.3 we shall show that for $q \in [\frac{1}{l-1}, 1]$ the C*-algebra $C^*_{r,q}(W)$ has a unique tracial state. Therefore, if any two $C^*_{r,q}(W)$ and $C^*_{r,q'}(W)$ with $q, q' \in [\frac{1}{l-1}, 1]$ are isomorphic, Lemma 3.5.1 implies that $\mathcal{N}_q(W) \cong \mathcal{N}_{q'}(W)$. Since solving the free factor problem using these C*-algebraic methods seems unrealistic (and solving it in the affirmative using C*-algebras seems even more unrealistic) we expect all Hecke C*-algebras $C^*_{r,q}(W)$ with $q \in [\frac{1}{l-1}, 1]$ to be non-isomorphic. In this context it should be mentioned that the reference [161] provides an explicit computation of the K-theory of right-angled Hecke C*-algebras, including concrete algebraic representants of K-theory classes. The results in particular allow to distinguish certain right-angled Hecke C*-algebras from each other.

3.6. AMALGAMATED FREE PRODUCT DECOMPOSITIONS

Building interesting new objects out of simpler building blocks is a common idea that appears all over mathematics. Being part of Voiculescu's groundbreaking noncommutative probability theory the (reduced) free product construction, which plays an important role in the theory of operator algebras, is an example of this. It can be viewed as a natural operator algebraic analogue to free products of groups where both constructions are in a certain sense compatible with each other (see Example 3.6.3 and Example 3.6.4).

In Section 2.6 we introduced the notion of amalgamated free products of groups. Similar to free products, this notion also admits an operator algebraic counterpart that will play a role in Subsection 6.1.3 and Chapter 8. It turns out to behave well with respect to amalgamated free product decompositions of Hecke operator algebras, see Subsection 3.6.2.

The aim of this section is to first introduce amalgamated free products of operator algebras in detail and to then study the concept in the setting of Hecke operator algebras. For further details see [179], [72], [180], [21] and [33].

3.6.1. AMALGAMATED FREE PRODUCTS OF OPERATOR ALGEBRAS

Following [33, Chapter 4.7], let $(A_i)_{i\in I}$ be a family of unital C*-algebras and let D be a unital C*-algebra such that for every $i \in I$ there exists a unital $*$-embedding $\iota_i : D \hookrightarrow A_i$. For convenience, we will often suppress the embedding ι_i and identify D with its image in A_i. Assume moreover that for every $i \in I$ there exists a faithful conditional expectation $\mathbb{E}_i : A_i \to D$.

For every $i \in I$ denote by $L^2(A_i,\mathbb{E}_i)$ the Hilbert D-module obtained from the pair $(A_i,\mathbb{E}_i)$, i.e. the completion of A with respect to the norm induced by the D-valued inner product $\langle a,b\rangle := \mathbb{E}_i(b^* a)$ for $a,b \in A_i$. Write $\Omega_{\mathbb{E}_i}$ for the element in $L^2(A_i,\mathbb{E}_i)$ corresponding to $1 \in A_i$ and denote by $\pi_{\mathbb{E}_i} : A_i \to \mathcal{B}(L^2(A_i,\mathbb{E}_i))$ the (faithful) $*$-representation induced by left multiplication by elements in A. In this context, the triple $(\pi_{\mathbb{E}_i}, L^2(A_i,\mathbb{E}_i),\Omega_{\mathbb{E}_i})$ is called the *GNS-representation* of $(A_i,\mathbb{E}_i)$. For brevity we write $\pi_i := \pi_{\mathbb{E}_i}$, $\mathcal{H}_i := L^2(A_i,\mathbb{E}_i)$ and $\Omega_i := \Omega_{\mathbb{E}_i}$.

Define the *free product Hilbert D-module* $(\mathcal{H},\Omega) := \star_{i\in I}(\mathcal{H}_i,\Omega_i)$ where Ω is the *vacuum vector* and where

$$\mathcal{H} := D\Omega \oplus \bigoplus_{n\geq 1} \bigoplus_{i_1\neq i_2, i_2\neq i_3,\dots,i_{n-1}\neq i_n} \mathcal{H}_{i_1}^\circ \otimes_D \dots \otimes_D \mathcal{H}_{i_n}^\circ.$$

Here $\mathcal{H}_i^\circ := \overline{(A_i \cap \ker(\mathbb{E}_i))\Omega_i} \subseteq \mathcal{H}_i$ denotes the closed subspace of $\mathcal{H}_i$ spanned by all elements $a\Omega_i$, $a \in A_i$ with $\mathbb{E}_i(a) = 0$ and $\otimes_D$ is the Hilbert D-module tensor product. For $i \in I$ define an isomorphism $U_i \in \mathcal{B}(\mathcal{H}_i \otimes_D \mathcal{H}(i),\mathcal{H})$ with $\mathcal{H}(i) := D\Omega \oplus \bigoplus_{n\geq 1}\bigoplus_{i\neq i_1, i_1\neq i_2, i_2\neq i_3,\dots,i_{n-1}\neq i_n} \mathcal{H}_{i_1}^\circ \otimes_D \dots \otimes_D \mathcal{H}_{i_n}^\circ$ by

$$U_i : \quad \begin{array}{c} D\Omega_i \otimes_D D\Omega \\ \mathcal{H}_i^\circ \otimes_D D\Omega \\ D\Omega_i \otimes_D (\mathcal{H}_{i_1}^\circ \otimes_D \dots \otimes_D \mathcal{H}_{i_n}^\circ) \\ \mathcal{H}_i^\circ \otimes_D (\mathcal{H}_{i_1}^\circ \otimes_D \dots \otimes_D \mathcal{H}_{i_n}^\circ) \end{array} \xrightarrow{\cong} \begin{array}{c} D\Omega \\ \mathcal{H}_i^\circ \\ \mathcal{H}_{i_1}^\circ \otimes_D \dots \otimes_D \mathcal{H}_{i_n}^\circ \\ \mathcal{H}_i^\circ \otimes_D \mathcal{H}_{i_1}^\circ \otimes_D \dots \otimes_D \mathcal{H}_{i_n}^\circ \end{array}.$$

where the actions are understood naturally. It gives rise to a faithful $*$-representation $\lambda_i : A_i \to \mathcal{B}(\mathcal{H})$ that satisfies $\lambda_i|_D = \lambda_j|_D$ for all $j \in I$ via $\lambda_i(x) := U_i(\pi_i(x)\otimes 1)U_i^*$.

Definition 3.6.1 ([33, Definition 4.7.1])**.** Let $(A_i)_{i\in I}$ be a family of unital C*-algebras and let D be a unital C*-algebra such that for every $i \in I$ there exists a unital $*$-embedding $\iota_i : D \hookrightarrow A_i$ and a faithful conditional expectation $\mathbb{E}_i : A_i \to D$. Then the *(reduced) amalgamated free product C*-algebra* $(A,\mathbb{E}_D) := \star_D(A_i,\mathbb{E}_i)$ is the C*-subalgebra A of $\mathcal{B}(\mathcal{H})$ (with $\mathcal{H}$ as above) generated by $\bigcup_{i\in I}\lambda_i(A_i)$ equipped with the (faithful) conditional expectation $\mathbb{E}_D : A \to D$, $x \mapsto \langle x\Omega,\Omega\rangle$.

Because the $*$-representations λ_i are faithful, we usually identify A_i, $i \in I$ with its image in A. The reduced amalgamated free product C*-algebra can be characterized in the following way.

Theorem 3.6.2 ([33, Theorem 4.7.2])**.** *Let $(A_i)_{i\in I}$ be a family of unital C*-algebras and let D be a unital C*-algebra such that for every $i \in I$ there exists a unital $*$-embedding $\iota_i : D \hookrightarrow A_i$ and a faithful conditional expectation $\mathbb{E}_i : A_i \to D$. Then the (reduced) amalgamated free product A is the (up to isomorphism) unique C*-algebra such that:*

(1) *There exists a unital inclusion* $D \subseteq A$ *and a faithful condition expectation* $\mathbb{E}_D : A \to D$*;*

(2) *There exist inclusions* $D \subseteq A_i \subseteq A$ *such that the union of all* $A_i, i \in I$ *generates* A*;*

(3) $(\mathbb{E}_D)|_{A_i} = \mathbb{E}_i$ *for every* $i \in I$*;*

(4) Freeness*: For all* $i_1, ..., i_n \in I$ *with* $i_1 \neq i_2$*,* $i_2 \neq i_3$*, ...,* $i_{n-1} \neq i_n$ *and* $a_1 \in A_{i_1} \cap \ker(\mathbb{E}_{i_1})$*, ...,* $a_n \in A_{i_n} \cap \mathbb{E}_{i_n}$ *one has* $\mathbb{E}_D(a_1 ... a_n) = 0$.

3

In the case where the common C*-subalgebra D of the A_i, $i \in I$ coincides with $\mathbb{C}$ and where the conditional expectations $\mathbb{E}_i$ are faithful states, the corresponding amalgamated free product C*-algebra is nothing but the free product $\star_{i \in I}(A_i, \mathbb{E}_i)$.

As mentioned before, the construction above can be viewed as a natural operator algebraic analogue to amalgamated free products of groups where both constructions are in a certain sense compatible with each other. We will make this precise in the following example.

Example 3.6.3. Let $G := \star_H G_i$ be the amalgamated free product of a family $(G_i)_{i \in I}$ of discrete groups over a common subgroup H (see Section 2.6). The inclusion induces expected inclusions $C_r^*(H) \subseteq C_r^*(G_i)$, $i \in I$ and $C_r^*(H) \subseteq C_r^*(G)$ of C*-algebras. Denote the corresponding faithful conditional expectations by $\mathbb{E}_{G_i}^H$ and $\mathbb{E}_G^H$. Then the reduced group C*-algebra $C_r^*(G)$ identifies with the amalgamated free product of the family $(C_r^*(G_i), \mathbb{E}_{G_i}^H)_{i \in I}$, i.e.

$$(C_r^*(G), \mathbb{E}_G^H) \cong \star_{C_r^*(H)}(C_r^*(G_i), \mathbb{E}_{G_i}^H).$$

There also exists a von Neumann algebraic analogue to amalgamated free products which has first been spelled out for arbitrary σ-finite von Neumann algebras in [175, Section 2] (compare with [179], [72]). In this context, the conditional expectations appearing in the construction above are assumed to be normal and the corresponding *amalgamated free product von Neumann algebra* is defined as the von Neumann algebra generated by the union of the images of the λ_i, $i \in I$. To distinguish between the C*-algebraic and the von Neumann algebraic case we denote amalgamated free products of von Neumann algebras by $\overline{\star}_D$. The construction satisfies a uniqueness property very similar to the one in Theorem 3.6.2, see [175, Proposition 2.5].

Example 3.6.4. Let $G := \star_H G_i$ be the amalgamated free product of a family $(G_i)_{i \in I}$ of discrete groups over a common subgroup H. The inclusion induces expected inclusions $\mathscr{L}(H) \subseteq \mathscr{L}(G_i)$, $i \in I$ and $\mathscr{L}(H) \subseteq \mathscr{L}(G)$ of von Neumann algebras. Denote the corresponding faithful normal conditional expectations by $\mathbb{E}_{G_i}^H$ and $\mathbb{E}_G^H$. Then the group von Neumann algebra $\mathscr{L}(G)$ identifies with the amalgamated free product of the family $(\mathscr{L}(G_i), \mathbb{E}_{G_i}^H)_{i \in I}$, i.e.

$$(\mathscr{L}(G), \mathbb{E}_G^H) \cong \overline{\star}_{\mathscr{L}(H)}(\mathscr{L}(G_i), \mathbb{E}_{G_i}^H).$$

3.6.2. AMALGAMATED FREE PRODUCTS OF HECKE OPERATOR ALGEBRAS

Similar to the decompositions in Example 3.6.3 and Example 3.6.4 Hecke operator algebras behave well with respect to amalgamated free product decompositions of the underlying Coxeter group (see Subsection 2.7.5). Recall that if W is an arbitrary group which decomposes as an amalgamated free product $W = W_1 \star_{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $W_0 = W_1 \cap W_2$ and where $S_0 := S_1 \cap S_2$ generates W_0, then $(W, S_1 \cup S_2)$ is a Coxeter system as well. In that case, for every multi-parameter $q := (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$, by Proposition 3.4.5 there are natural unital embeddings $C^*_{r,q_0}(W_0) \subseteq C^*_{r,q_1}(W_1) \subseteq C^*_{r,q}(W)$, $C^*_{r,q_0}(W_0) \subseteq C^*_{r,q_2}(W_2) \subseteq C^*_{r,q}(W)$ and $\mathcal{N}_{q_0}(W_0) \subseteq \mathcal{N}_{q_1}(W_1) \subseteq \mathcal{N}_q(W)$, $\mathcal{N}_{q_0}(W_0) \subseteq \mathcal{N}_{q_2}(W_2) \subseteq \mathcal{N}_q(W)$ where $q_0 := (q_s)_{s\in S_0}$, $q_1 := (q_s)_{s\in S_1}$, $q_2 := (q_s)_{s\in S_2}$. Denote by $\mathbb{E} : \mathcal{N}_q(W) \to \mathcal{N}_{q_0}(W_0)$ the unique faithful normal trace-preserving conditional expectation onto $\mathcal{N}_{q_0}(W_0)$ (see [33, Lemma 1.5.11]). Then, for $\mathbf{w} \in W$ the equality

$$\mathbb{E}(T_{\mathbf{w}}^{(q)}) = \begin{cases} T_{\mathbf{w}}^{(q_0)}, & \text{if } \mathbf{w} \in W_0 \\ 0, & \text{if } \mathbf{w} \notin W_0 \end{cases}$$

holds.

The following theorem (and its graph product analogue, see Chapter 4.3.1) will turn out to be very useful.

Theorem 3.6.5. *Let (W,S) be a finite rank Coxeter system that decomposes as $W = W_1 * _{W_0} W_2$ where (W_1, S_1), (W_2, S_2) are Coxeter systems with $S = S_1 \cup S_2$ and $W_0 = W_1 \cap W_2$ such that $S_0 := S_1 \cap S_2$ generates W_0. For a multi-parameter $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ decomposes as an amalgamated free product of the form*

$$\mathcal{N}_q(W) \cong (\mathcal{N}_{q_1}(W_1), \mathbb{E}_1)\overline{\star}_{\mathcal{N}_{q_0}(W_0)}(\mathcal{N}_{q_2}(W_2), \mathbb{E}_2),$$

where $q_0 := (q_s)_{s\in S_0}$, $q_1 := (q_s)_{s\in S_1}$, $q_2 := (q_s)_{s\in S_2}$ and where the decomposition is taken with respect to the restricted conditional expectations $\mathbb{E}_1 := \mathbb{E}|_{\mathcal{N}_{q_1}(W_1)}$ and $\mathbb{E}_2 := \mathbb{E}|_{\mathcal{N}_{q_2}(W_2)}$. Similarly,

$$C^*_{r,q}(W) \cong (C^*_{r,q_1}(W_1), \mathbb{E}_1) \star_{C^*_{r,q_0}(W_0)} (C^*_{r,q_2}(W_2), \mathbb{E}_2)$$

where $\mathbb{E}_1 := \mathbb{E}|_{C^*_{r,q_1}(W_1)}$ and $\mathbb{E}_2 := \mathbb{E}|_{C^*_{r,q_2}(W_2)}$.

Proof. We may restrict to the von Neumann algebraic case. By the uniqueness of the amalgamated free product construction (see [175, Proposition 2.5] and also Theorem 3.6.2) in combination with our previous discussion it suffices to show that $\mathbb{E}(x_1...x_n) = 0$ for all $x_1 \in \mathcal{N}_{q_{i_1}}(W_{i_1}) \cap \ker(\mathbb{E}_{i_1})$, ..., $x_k \in \mathcal{N}_{q_{i_k}}(W_{i_k}) \cap \ker(\mathbb{E}_{i_k})$ where $n \in \mathbb{N}$ and $i_1, ..., i_n \in \{1,2\}$ with $i_1 \neq i_2$, $i_2 \neq i_3$, ..., $i_{n-1} \neq i_n$. For $i \in \{1,2\}$ let $(\mathcal{N}_{q_i}(W_i))_1$ denote the unit ball of $\mathcal{N}_{q_i}(W_i)$ and write $\overline{\text{Span}}$ for the strong closure of the linear span. By Kaplansky's density theorem,

$$(\mathcal{N}_{q_1}(W_1))_1 \cap \ker(\mathbb{E}_1) = (\mathcal{N}_{q_1}(W_1))_1 \cap \overline{\text{Span}}\{T_{\mathbf{w}}^{(q_1)} \mid \mathbf{w} \in W_1 \setminus W_0\},$$

and

$$(\mathcal{N}_{q_2}(W_2))_1 \cap \ker(\mathbb{E}_2) = \mathcal{N}_{q_2}(W_2)_1 \cap \overline{\mathrm{Span}}\{T_{\mathbf{w}}^{(q_2)} \mid \mathbf{w} \in W_2 \setminus W_0\}.$$

The statement [135, Remark 4.3.1] hence implies that the element $x_1...x_n \in \mathcal{N}_q(W)$ can be approximated strongly by a bounded net of linear combinations of reduced expressions of the form $T_{\mathbf{w}_1}^{(q)}...T_{\mathbf{w}_n}^{(q)}$ with $\mathbf{w}_1 \in W_{i_1} \setminus W_0, ..., \mathbf{w}_n \in W_{i_n} \setminus W_0$. But this expression coincides with $T_{\mathbf{w}_1...\mathbf{w}_n}^{(q)}$ where $\mathbf{w}_1...\mathbf{w}_n \in W \setminus W_0$ is non-trivial, so $\mathbb{E}(x_1...x_n) = 0$ because $\mathbb{E}_{\mathcal{N}_q}$ is normal and hence weakly continuous on bounded sets. The claim follows. □

3

Remark 3.6.6. *(a)* In the case where the Coxeter subsystem is trivial, Theorem 3.6.5 implies that the Hecke operator algebras of free products of Coxeter groups decompose as (reduced) free products over the canonical tracial states. This will play a role in Subsection 6.1.3.
(b) Recall that the smallest class of Coxeter groups which contains all finite rank spherical type Coxeter groups and which is closed under taking amalgamated free products over spherical special subgroups coincides with the class of Coxeter groups which are virtually free. Combining this with Theorem 3.6.5 implies that Hecke-von Neumann algebras of finite rank virtually free Coxeter systems can be decomposed as iterated amalgamated free products over finite-dimensional von Neumann subalgebras. We will make use of this in fact in Chapter 8 where the Haagerup approximation property for Hecke-von Neumann algebras is studied by using very general results of Chapter 7.

4

Graph product Khintchine inequalities

Similar to amalgamated free products, Green's graph products of groups (see Section 2.6 and Section 3.6) admit an operator algebraic counterpart introduced and explored by Caspers and Fima in [44]. Generalizing results by Ricard and Xu on free products of C^*-algebras (see [163]), in this chapter we prove Khintchine type inequalities for general C^*-algebraic graph products and illustrate their relevance for the study of right-angled Hecke C^*-algebras by proving a Haagerup type inequality.

The chapter is structured as follows: we will first recall the necessary background material such as the notion of column and row Hilbert spaces and the construction of graph products of operator algebras. The corresponding references are [77], [152] and [44]. In Section 4.2 we will then prove the main theorem of this chapter, the graph product Khintchine type inequality for graph products of C^*-algebras. Section 4.3 is concerned with its connection to right-angled Hecke C^*-algebras. This link will later be picked up in Section 6.3 where the trace uniqueness of right-angled Hecke C^*-algebras will be studied.

The content of Section 4.2 and Section 4.3 is entirely based on the article

- M. Caspers, M. Klisse, N.S. Larsen, *Graph product Khintchine inequalities and Hecke C^*-algebras: Haagerup inequalities, (non)simplicity, nuclearity and exactness*, J. Funct. Anal. 280 (2021), no. 1, Paper No. 108795, 41 pp.

4.1. PRELIMINARIES

In the following we present the background required for Section 4.2 and Section 4.3. The content and the notation is as it appears in [49].

4.1.1. GENERAL NOTATION

For notational convenience, we write $\delta(P)$ for the function which equals 1 if a statement P is true and which equals 0 otherwise.

4.1.2. COLUMN AND ROW HILBERT SPACES

A (concrete) *operator space* on a Hilbert space $\mathscr{H}$ is a closed subspace of the bounded operators on $\mathscr{H}$. The concept, going back to Ruan's thesis, also admits an abstract characterization (see e.g. [152, Chapter 2.2]): let $\mathscr{K}(\ell^2(\mathbb{N}))$ denote the compact operators on the separable Hilbert space $\ell^2(\mathbb{N})$ and write $\mathscr{K}_{00}$ for the dense $*$-subalgebra of finite rank operators on $\ell^2(\mathbb{N})$. For a complex vector space E together with a sequence $(\|\cdot\|_{M_k(\mathbb{C})\otimes E})_{k\in\mathbb{N}}$ of norms on the spaces $M_k(\mathbb{C})\otimes E$ which are compatible with respect to the canonical embeddings $M_k(\mathbb{C})\otimes E \hookrightarrow M_{k+1}(\mathbb{C})\otimes E$ define a norm on $\mathscr{K}_{00}\otimes E$ via

$$\|x\|_{\mathscr{K}_{00}\otimes E} := \lim_{k\to\infty} \|x\|_{M_k(\mathbb{C})\otimes E}.$$

Then, for a suitable Hilbert space $\mathscr{H}$, there exists a linear embedding $\phi : E \to \mathscr{B}(\mathscr{H})$ such that for every $k \in \mathbb{N}$ the canonical map

$$\mathrm{id}_{M_k(\mathbb{C})} \otimes \phi : M_k(\mathbb{C})\otimes E \to M_k(\mathbb{C})\otimes \phi(E) \subseteq M_k(\mathbb{C})\otimes \mathscr{B}(\mathscr{H})$$

is isometric if and only if

$$\|\sum_{i=1}^n a_i x_i b_i\|_{\mathscr{K}_{00}\otimes E} \le \|\sum_{i=1}^n a_i a_i^*\|^{\frac{1}{2}} \|\sum_{i=1}^n b_i^* b_i\|^{\frac{1}{2}} \sup_{1\le i\le n} \|x_i\|_{\mathscr{K}_{00}\otimes E}$$

for all $n\in\mathbb{N}$ and $a_1, b_1, ..., a_n, b_n \in \mathscr{K}_{00}\otimes \mathbb{C}1$, $x_1, ..., x_n \in \mathscr{K}_{00}\otimes E$. This characterization implies a one-to-one correspondence between the class of (concrete) operator spaces and vector spaces E equipped with a sequence of norms as above. For more information on the theory of operator spaces we refer to [77] and [152].

A nice class of examples of operator spaces are column and row Hilbert spaces. For $n\in\mathbb{N}$ denote by $(f_i)_{1\le i\le n}$ an orthonormal basis of $\mathbb{C}^n$, write $E_{i,j}$, $1 \le i,j \le n$ for the matrix units with respect to this basis and set $E_i := E_{i,i}$ for the diagonal projections. The *column Hilbert operator space* C_n of dimension n is the operator space spanned by all matrix units $E_{i,1}$, $1\le i\le n$ in $M_n(\mathbb{C}) = \mathscr{B}(\mathbb{C}^n)$ with the operator space structure induced by the restriction of the operator space structure of $M_n(\mathbb{C})$ as a C*-algebra. Concretely, for $x_1, ..., x_n \in M_k(\mathbb{C})$ where $k\in\mathbb{N}$,

$$\|\sum_{i=1}^n x_i \otimes E_{i,0}\|_{M_k(\mathbb{C})\otimes \mathrm{C}_n} = \|\sum_{i=1}^n x_i^* x_i\|^{\frac{1}{2}}.$$

Similarly, the *row Hilbert operator space* R_n is spanned by all matrix units $E_{1,i}$, $1 \leq i \leq n$ in $M_n(\mathbb{C})$ with the operator space structure again induced by the restriction of the operator space structure of $M_k(\mathbb{C})$ as a C^*-algebra, i.e. for $x_1, ..., x_n \in M_k(\mathbb{C})$ where $k \in \mathbb{N}$,

$$\| \sum_{i=1}^{n} x_i \otimes E_{0,i} \|_{M_k(\mathbb{C}) \otimes \mathrm{C}_n} = \| \sum_{i=1}^{n} x_i x_i^* \|^{\frac{1}{2}}.$$

In the category of operator spaces there exists a natural concept of tensoring two such spaces, namely the Haagerup tensor product which we denote by $\otimes_h$ and for which we refer to [77, Section 9]. For two operator spaces $X \subseteq \mathcal{B}(\mathcal{H})$, $Y \subseteq \mathcal{B}(\mathcal{K})$ define for every $k \in \mathbb{N}$ the *Haagerup operator space tensor norm* $\|\cdot\|_{h, M_k(\mathbb{C}) \otimes (X \otimes Y)}$ on $M_k(\mathbb{C}) \otimes (X \otimes Y)$ via

$$\begin{aligned} \|x\|_{h, M_k(\mathbb{C}) \otimes (X \otimes Y)} := \inf\{ \|u\| \, \|v\| \quad | \quad & x = u \odot v, u \in M_{k,r}(\mathbb{C}) \otimes (X \otimes \mathbb{C}1), \\ & v \in M_{r,k}(\mathbb{C}) \otimes (\mathbb{C}1 \otimes Y), r \in \mathbb{N}\}, \end{aligned}$$

where $u \odot v$ is the canonical tensor matrix product induced by the identifications $M_{k,r}(\mathbb{C}) \otimes (X \otimes \mathbb{C}1) \cong M_{k,r}(X \otimes \mathbb{C}1)$, $M_{r,k}(\mathbb{C}) \otimes (\mathbb{C}1 \otimes Y) \cong M_{r,k}(\mathbb{C}1 \otimes Y)$ and where one checks that the set in the brackets is indeed non-empty. One further confirms that the sequence of norms $(\|\cdot\|_{h, M_k(\mathbb{C}) \otimes (X \otimes Y)})_{k \in \mathbb{N}}$ satisfies the conditions from above, hence after completion they define a (unique) operator space $X \otimes_h Y$ which we call the *Haagerup tensor product* of X and Y.

We shall mainly need the following completely isometric identifications

$$\mathrm{C}_m \otimes_h \mathrm{C}_n \cong \mathrm{C}_{m+n}, \qquad \mathrm{R}_m \otimes_h \mathrm{R}_n \cong \mathrm{R}_{m+n}, \qquad \mathrm{C}_m \otimes_h \mathrm{R}_n \cong M_{m,n}(\mathbb{C})$$

where $m, n \in \mathbb{N}$, see [77, Proposition 9.3.4 and 9.3.5].

4.1.3. WORDS IN GRAPHS

Let $K = (V, E)$ be an undirected and simplicial graph with vertex set V and edge set E (for the corresponding notions review Section 2.5). In this chapter we will always assume that K is finite. Similar to words in Coxeter groups (see Subsection 2.7.2) following [44] we introduce a suitable notion of words in graphs. A *word* in K is an expression $\mathbf{v} = v_1 v_2 \cdots v_n$ with $v_1, ..., v_n \in V$, i.e. a concatenation of elements in V which we call the *letters*. We say that two words are *shuffle equivalent* (also known as *II-equivalent*) if they are contained in the same equivalence class of the equivalence relation generated by

$$v_1 \cdots v_{i-1} v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_n \text{ if } (v_i, v_{i+1}) \in E.$$

We say that two words are *equivalent*, denoted by the symbol $\simeq$, if they are equivalent through shuffle equivalence and the additional relation:

$$v_1 \cdots v_i v_{i+1} v_{i+2} \cdots v_n \sim v_1 \cdots v_i v_{i+2} \cdots v_n \text{ if } v_i = v_{i+1}.$$

A word $v_1 \cdots v_n$ is called *reduced* if whenever $v_i = v_j$ for $i < j$ then there exists $i < k < j$ such that $(v_i, v_k), (v_j, v_k) \notin E$. If $\mathbf{v}$ and $\mathbf{w}$ are equivalent reduced words then necessarily $\mathbf{v}$ and $\mathbf{w}$ are shuffle equivalent. For a word $\mathbf{v}$ we define its *length* $|\mathbf{v}|$ as the number of letters in the shortest representative of $\mathbf{w}$ up to equivalence. We say that a word $\mathbf{v}$ *starts* with $v \in V$ if it is equivalent to a reduced word of the form $vv_1 \cdots v_n$ with $v_1,...,v_n \in V$. Similarly we say that a word $\mathbf{v}$ *ends* with $v \in V$ if it is equivalent to a reduced word $v_1 \cdots v_n v$ with $v_1,...,v_n \in V$.

4.1.4. Graph products of operator algebras

What makes Green's graph products of groups (see Section 2.6) so interesting is that the construction interpolates between free products and Cartesian products, it covers important examples (e.g. right-angled Coxeter groups and right-angled Artin groups, see Subsection 2.7.4) and it preserves many group theoretical properties (e.g. soficity [57], Haagerup property [7], residual finiteness [89], rapid decay [56] and linearity [109]). In [44] Caspers and Fima introduced a suitable operator algebraic analogue to graph products. Similar to the group case their construction generalizes both Voiculescu's free products and tensor products of operator algebras. It further covers interesting examples (such as right-angled Hecke operator algebras, see Subsection 4.3.1), it admits desirable stability properties (e.g. exactness, Haagerup property and II_1-factoriality, see [44]) and it is compatible with Green's construction (see Example 4.1.3 and Example 4.1.4).

In the following we present a slightly different viewpoint than in [44] by identifying Hilbert spaces up to shuffle equivalence. This makes the notation much shorter and yields the same construction. Compare the construction with the one in Subsection 3.6.1. As before let $K = (V, E)$ be a finite, undirected, simplicial graph. Further, for every $v \in V$ let A_v be a unital C*-algebra equipped with a *GNS-faithful* state φ_v. By this we mean that the GNS-representation of A_v on the GNS-Hilbert space $L^2(A_v, \varphi_v)$ is faithful. For notational convenience we will view A_v as a C*-subalgebra of the bounded operators on $L^2(A_v, \varphi_v)$. Write $A_v^\circ := \{a \in A_v \mid \varphi_v(a) = 0\}$ and set for $a \in A_v$

$$a^\circ := a - \varphi_v(a)1 \in A_v^\circ.$$

We further denote by $L^2(A_v^\circ, \varphi_v)$ the closure of A_v° viewed as a subspace of $L^2(A_v, \varphi_v)$ and define $\Omega_v \in L^2(A_v, \varphi_v)$ to be the vector corresponding to the unit of A_v. For a reduced word $\mathbf{v} = v_1 \cdots v_n$ with $v_1,...,v_n \in V$ set

$$\mathcal{H}_{\mathbf{v}} := L^2(A_{v_1}^\circ, \varphi_{v_1}) \otimes \cdots \otimes L^2(A_{v_n}^\circ, \varphi_{v_n}).$$

By convention we set $\mathcal{H}_\emptyset := \mathbb{C}\Omega$ where Ω is a unit vector called the *vacuum vector*. If $\mathbf{v} = v_1 \cdots v_n$ and $\mathbf{w} = w_1 \cdots w_n$ are reduced equivalent (hence shuffle equivalent) words then $\mathcal{H}_{\mathbf{v}} \cong \mathcal{H}_{\mathbf{w}}$ naturally by applying flip maps to the vectors dictated by the shuffle equivalence. More precisely, by [44, Lemma 1.3] if $\mathbf{v}$ and $\mathbf{w}$ are equivalent reduced words, there exists a unique permutation σ of the numbers $1,\ldots,n$ such that $v_{\sigma(i)} = w_i$ and such that if $i < j$ and $v_i = v_j$ then also $\sigma(i) < \sigma(j)$. There hence exists a unitary map $\mathcal{Q}_{\mathbf{v},\mathbf{w}} : \mathcal{H}_{\mathbf{v}} \to \mathcal{H}_{\mathbf{w}}$ which maps $\xi_1 \otimes \cdots \otimes \xi_n$ with $\xi_1 \in L^2(A_{v_1}^\circ, \varphi_{v_1})$, ..., $\xi_n \in L^2(A_{v_n}^\circ, \varphi_{v_n})$ to $\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$. From now on we will omit the unitary $\mathcal{Q}_{\mathbf{v},\mathbf{w}}$ in the notation and identify the spaces $\mathcal{H}_{\mathbf{v}}$ and $\mathcal{H}_{\mathbf{w}}$ through $\mathcal{Q}_{\mathbf{v},\mathbf{w}}$. Compared to

[44] this significantly simplifies our notation and it is directly verifyable that our constructions below agree with the ones in [44].

Let I be a set of *representatives* of all reduced words in K modulo shuffle equivalence and set $\mathcal{H} := \bigoplus_{\mathbf{v}\in I} \mathcal{H}_{\mathbf{v}}$. Each $A_v, v \in V$ can be canonically represented on $\mathcal{H}$ as follows: take $x \in A_v$ and $\xi_1 \otimes \cdots \otimes \xi_d \in \mathcal{H}_{\mathbf{w}}$ where $\mathbf{w} = w_1 \dots w_d$ with $w_1, ..., w_d \in V$ is a reduced word. If $\mathbf{w}$ does not start with v we set

$$x(\xi_1 \otimes \cdots \otimes \xi_d) := x^{\circ}\Omega_v \otimes \xi_1 \otimes \cdots \otimes \xi_d + \varphi_v(x)\xi_1 \otimes \cdots \otimes \xi_d.$$

If $\mathbf{w}$ starts with v we may assume (by shuffling the letters if necessary and by identifying the corresponding Hilbert spaces as described above) that $w_1 = v$ and we set

$$x(\xi_1 \otimes \cdots \otimes \xi_d) := (x\xi_1 - \langle x\xi_1, \Omega_v\rangle\Omega_v) \otimes \xi_2 \otimes \cdots \otimes \xi_d + \langle x\xi_1, \Omega_v\rangle\xi_2 \otimes \cdots \otimes \xi_d.$$

This defines a faithful $*$-representation $\lambda_v : A_v \hookrightarrow \mathcal{B}(\mathcal{H})$.

Definition 4.1.1 ([44, Section 2.2])**.** Let $K = (V, E)$ be a finite, undirected, simplicial graph and let $(A_v, \varphi_v)_{v\in V}$ be a family of unital C*-algebras equipped with GNS-faithful states. Then the (*reduced*) *graph product C*-algebra* $(A, \varphi) := \star_{v,K}(A_v, \varphi_v)$ is the C*-subalgebra A of $\mathcal{B}(\mathcal{H})$ (with $\mathcal{H}$ as above) generated by $\bigcup_{i\in I} \lambda_i(A_i)$ equipped with the (GNS-faithful) *graph product state* $\varphi : A \to \mathbb{C}$, $x \mapsto \langle x\Omega, \Omega\rangle$.

Because the $*$-representations λ_v are faithful, we usually identify the C*-algebra A_v, $v \in V$ with its image in A. Similar to Theorem 3.6.2 the reduced graph product can be characterized in the following way.

Theorem 4.1.2 ([44, Proposition 2.12])**.** *Let $K = (V, E)$ be a finite, undirected, simplicial graph and let $(A_v, \varphi_v)_{v\in V}$ be a family of unital C*-algebras equipped with GNS-faithful states. Then the (reduced) graph product A is the (up to isomorphism) unique C*-algebra such that:*

(1) There exists a GNS-faithful state φ;

(2) There exist unital inclusions $A_v \subseteq A$ such that the union of all $A_v, v \in V$ generates A and such that for all $(v, v') \in E$ the elements in A_v commute with the elements in $A_{v'}$;

(3) $\varphi|_{A_v} = \varphi_v$ for every $v \in V$;

(4) Freeness*: For every reduced word $\mathbf{v} = v_1 ... v_n$ with $v_1, ..., v_n \in V$ and $a_1 \in A^{\circ}_{v_1}, ..., a_n \in A^{\circ}_{v_n}$ one has $\varphi(a_1 ... a_n) = 0$.*

From Theorem 4.1.2 the following examples can be deduced easily.

Example 4.1.3. *(a)* Let $K = (V, E)$ be a finite, undirected, simplicial graph and let $(G_v)_{v\in K}$ be a family of groups. Then the reduced group C*-algebra $C^*_r(G)$ of the graph product group $G := \star_{v,K} G_v$ identifies with the (reduced) graph product C*-algebra of the family $(C^*_r(G_v), \tau_v)_{v\in I}$ where τ_v denotes for every $v \in V$ the canonical tracial state on $C^*_r(G_v)$, i.e.

$$(C^*_r(G), \tau) \cong \star_{v,K}(C^*_r(G_v), \tau_v).$$

(b) Let $K=(V,E)$ be a graph in which every two vertices are connected by an edge and let $(A_v,\varphi_v)_{v\in V}$ be a family of unital C*-algebras equipped with GNS-faithful states. Then the (reduced) graph product canonically identifies with the tensor product over all $A_v\subseteq\mathcal{B}(L^2(A_v,\varphi_v))$, $v\in V$, i.e. $\star_{v,K}(A_v,\varphi_v)\cong\bigotimes_{v\in V}(A_v,\varphi_v)$.
(c) Let $K=(V,E)$ be a graph with no edges and let $(A_v,\varphi_v)_{v\in V}$ be a family of unital C*-algebras equipped with GNS-faithful states. Then the (reduced) graph product canonically identifies with Voiculescu's (reduced) free product C*-algebra of the family $(A_v,\varphi_v)_{v\in V}$, i.e. $\star_{v,K}(A_v,\varphi_v)\cong\star_{v\in V}(A_v,\varphi_v)$.

The graph product construction above also admits a von Neumann algebraic counterpart which satisfies an analogue of Theorem 4.1.2 (see [44, Proposition 2.22]). If for every $v\in V$, $\mathcal{N}_v$ is a von Neumann algebra and φ_v is a faithful normal state on $\mathcal{N}_v$ then we define the *graph product von Neumann algebra* of $(\mathcal{N}_v,\varphi_v)_{v\in V}$ as

4

$$\overline{\star}_{v,K}(\mathcal{N}_v,\varphi_v):=(\star_{v,K}(\mathcal{N}_v,\varphi_v))''\subseteq\mathcal{B}(\mathcal{H})$$

and view $\mathcal{N}_v$, $v\in V$ as a von Neumann subalgebra of $\overline{\star}_{v,K}(\mathcal{N}_v,\varphi_v)$. We will usually write

$$L^2(\mathcal{N},\varphi):=\mathcal{H},$$

and call this the (*graph product*) *Fock space*. The notation is justified as in [44] it is shown that $\mathcal{H}$ is the Hilbert space of the standard form of the von Neumann algebraic graph product.

Example 4.1.4. *(a)* Let $K=(V,E)$ be a finite, undirected, simplicial graph and let $(G_v)_{v\in K}$ be a family of groups. Then the reduced group von Neumann algebra $\mathcal{L}(G)$ of the graph product group $G:=\star_{v,K}G_v$ identifies with the graph product von Neumann algebra of the family $(\mathcal{L}(G_v),\tau_v)_{v\in I}$ where τ_v denotes for every $v\in V$ the canonical tracial state on $\mathcal{L}(G_v)$, i.e.

$$(\mathcal{L}(G),\tau)\cong\overline{\star}_{v,K}(\mathcal{L}(G_i),\tau_v).$$

(b) Let $K=(V,E)$ be a graph in which every two vertices are connected by an edge and let $(\mathcal{N}_v,\varphi_v)_{v\in V}$ be a family of von Neumann algebras equipped with faithful normal states. Then the graph product canonically identifies with the (von Neumann algebraic) tensor product over all $\mathcal{N}_v\subseteq\mathcal{B}(L^2(\mathcal{N}_v,\varphi_v))$, $v\in V$, i.e. $\overline{\star}_{v,K}(\mathcal{N}_v,\varphi_v)\cong\overline{\bigotimes}_{v\in V}(\mathcal{N}_v,\varphi_v)$.
(c) Let $K=(V,E)$ be a graph with no edges and let $(\mathcal{N}_v,\varphi_v)_{v\in V}$ be a family of von Neumann algebras equipped with faithful normal states. Then the graph product canonically identifies with Voiculescu's free product von Neumann algebra of the family $(\mathcal{N}_v,\varphi_v)_{v\in V}$, i.e. $\overline{\star}_{v,K}(\mathcal{N}_v,\varphi_v)\cong\overline{\star}_{v\in V}(A_v,\varphi_v)$.

An operator $a_1\cdots a_n$ with $a_1\in A^\circ_{v_1}$, ..., $a_n\in A^\circ_{v_n}$ where $\mathbf{v}=v_1\cdots v_n$ with $\mathbf{v}_1,\dots,\mathbf{v}_n\in K$ is a reduced word is called a *reduced operator of type* $\mathbf{v}$. We refer to n as the *length* of the operator and define P_v to be the orthogonal projection of $\mathcal{H}$ onto $\bigoplus_{\mathbf{v}\in I_v}\mathcal{H}_\mathbf{v}$ where I_v is the set of representatives of all reduced words that start with v up to shuffle equivalence. For $n\in\mathbb{N}$ define

$$\chi_n: A\to A, a_1\cdots a_r\mapsto\delta(r=n)a_1\cdots a_r, \tag{4.1.1}$$

where $a_1 \cdots a_r \in A$ is a reduced operator, i.e. χ_n is the *word length projection* of length n.

The following proposition illustrates that, similar to the group case, graph products can be viewed as a well-behaved special case of amalgamated free products (compare with Proposition 2.6.1).

Proposition 4.1.5 ([44, Theorem 2.15]). *Let $K = (V,E)$ be a finite, undirected, simplicial graph, let $(A_v, \varphi_v)_{v \in V}$ be a family of unital C^*-algebras equipped with faithful states and let $v_0 \in V$. Define subgraphs $K_1 := Star(v_0)$, $K_2 := K \setminus \{v_0\}$ of K and set $(A_0, \varphi_0) := \star_{v,Link(v_0)}(A_v, \varphi_v)$, $(A_1, \varphi_1) := \star_{v,K_1}(A_v, \varphi_v)$ and $(A_2, \varphi_2) := \star_{v,K_2}(A_v, \varphi_v)$. Then we may view $A_0 \subseteq A_1$, $A_0 \subseteq A_2$ and there exist faithful conditional expectations $\mathbb{E}_1 : A_1 \to A_0$, $\mathbb{E}_2 : A_2 \to A_0$ such that $\star_{v,K}(A_v, \varphi_v)$ is canonically isomorphic to the (reduced) amalgamated free product C^*-algebra $(A_1, \mathbb{E}_1) \star_{A_0} (A_2, \mathbb{E}_2)$.*

Similarly, let $(\mathcal{N}_v, \varphi_v)_{v \in V}$ be a family of von Neumann algebras equipped with faithful normal states and set $(\mathcal{N}_0, \varphi_0) := \overline{\star}_{v,Link(v_0)}(\mathcal{N}_v, \varphi_v)$, $(\mathcal{N}_1, \varphi_1) := \overline{\star}_{v,K_1}(\mathcal{N}_v, \varphi_v)$ and $(\mathcal{N}_2, \varphi_2) := \overline{\star}_{v,K_2}(\mathcal{N}_v, \varphi_v)$. Then we may view $\mathcal{N}_0 \subseteq \mathcal{N}_1$, $\mathcal{N}_0 \subseteq \mathcal{N}_2$ and there exist faithful normal conditional expectations $\mathbb{E}_1 : \mathcal{N}_1 \to \mathcal{N}_0$, $\mathbb{E}_2 : \mathcal{N}_2 \to \mathcal{N}_0$ such that $\overline{\star}_{v,K}(A_v, \varphi_v)$ is canonically isomorphic to the amalgamated free product $(\mathcal{N}_1, \mathbb{E}_1) \overline{\star}_{\mathcal{N}_0} (\mathcal{N}_2, \mathbb{E}_2)$.

4.2. A GRAPH PRODUCT KHINTCHINE INEQUALITY

In this section we prove a *Khintchine inequality* for general C^*-algebraic graph products by introducing an intertwining technique between graph products and free products. Inequalities of this type estimate the operator norm of a reduced operator of a given length with the norm of certain Haagerup tensor products of column and row Hilbert spaces. In the case of free groups the concept goes back to Haagerup's fundamental paper [93]. For general free products and arbitrary length a Khintchine type inequality has been proved by Ricard and Xu in [163, Section 2] who applied it in the context of the exactness and the completely contractive approximation property of free products of C^*-algebras. Our estimate of norms holds up to a bound that is polynomial in n. We make this more precise in the current section.

Remark 4.2.1. If in the Ssection 4.3 we would only treat the case of Hecke C^*-algebras coming from right-angled Coxeter groups which decompose as free products of finite Coxeter subgroups (see Example 2.7.1) the results from [163] would be sufficient. Here however, we want a more general theorem. One of the problems that arise while proving such a theorem is that the analogue of [163, Lemma 2.3] in its form fails in a general graph product setting. We remedy this problem by using maps that intertwine graph products with free products.

Let us now prepare for the proof of the main theorem of this section. We fix notation for both a graph product and a free product. For the corresponding graph theoretical notions review Section 2.5. As before, let $K = (V,E)$ be a finite, undirected, simplicial graph and let I be a set of representatives of equivalence classes of reduced words with letters in V (see Subsection 4.1.3). Let $A_v, v \in V$ be unital

C^*-algebras with GNS-faithful states φ_v and let

$$(A,\varphi) := \star_{v,K}(A_v,\varphi_v)$$

be its graph product with vacuum vector Ω. We also set the free product (i.e. the graph product over K with all edges removed)

$$(A_f,\varphi_f) := \star_v(A_v,\varphi_v)$$

with vacuum vector Ω_f. For every $v \in V$ we shall view A_v as a C^*-subalgebra of both A and A_f. As before, define $P_v \in \mathcal{B}(L^2(A,\varphi))$ to be the orthogonal projection of $\mathcal{H}$ onto $\bigoplus_{\mathbf{v}\in I_v}\mathcal{H}_{\mathbf{v}}$ where I_v is the set of representatives of all reduced words that start with v up to shuffle equivalence and similarly let $P_v^f \in \mathcal{B}(L^2(A_f,\varphi_f))$ be the free product projection onto words which start with v. Let $K_0 \in \mathrm{Cliq}(K,l)$ and let $V(K_0) \in I$ be the word consisting of all letters in K_0. We write $P_{K_0}^f$ for the projection of $L^2(A_f,\varphi_f)$ onto the direct sum of all $\mathcal{H}_{\mathbf{w}}$ where $\mathbf{w}$ starts with $V(K_0)$. Recall that A_v° is the set of $a \in A_v$ with $\varphi_v(a) = 0$.

View A_v as a subalgebra of A, set $\Sigma_1 := \mathrm{Span}\{A_v^\circ \mid v \in V\}$ and define for $n \in \mathbb{N}$,

$$\Sigma_n := \{a_1 \otimes \cdots \otimes a_n \mid a_1 \in A_{v_1}^\circ, ..., a_n \in A_{v_n}^\circ \text{ where } v_1\cdots v_n \text{ reduced}\} \subseteq \Sigma_1^{\odot n},$$

where the latter is the n-fold algebraic tensor product.

Our first aim is to show that reduced operators of length d in A can be decomposed as sums of creation operators, annihilation operators, and diagonal operators in a sense to be made precise below. Crucial is that first the annihilation operators act, then the diagonal operators and then the creation operators. That is, we shall be looking for an analogue of the decomposition [163, Fact 2.6].

Remark 4.2.2. *(a)* Definition 4.2.3 below formally defines the following permutation. Let $\mathbf{v} = v_1\cdots v_n$ with $v_1,...,v_n \in V$ be a reduced word. Let $0 \le l \le n$, $0 \le k \le n-l$, $K_0 \in \mathrm{Cliq}(K,l)$ and $(K_1,K_2) \in \mathrm{Comm}(K_0)$. As above, define $V(K_0) \in I$ to be the word consisting of all letters in K_0, define $V(K_1) \in I$ to be reduced the word consisting of all letters in K_1 and define $V(K_2) \in I$ to be the word consisting of all letters in K_2. Then, if possible, we permute the letters of $\mathbf{v}$ through shuffle equivalence in the form

$$(v_{\sigma(1)}\cdots v_{\sigma(k)})(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})(v_{\sigma(k+l+1)}\cdots v_{\sigma(n)}) \simeq \overbrace{(*\cdots * V(K_1))}^{k \text{ letters}}\overbrace{(V(K_0))}^{l \text{ letters}}\overbrace{(V(K_2)\diamond\cdots\diamond)}^{n-k-l \text{ letters}}, \tag{4.2.1}$$

where $*$ and $\diamond$ are the remaining letters and each of the 3 respective terms in between brackets are shuffle equivalent themselves. This means that in between the first brackets there is a word of length k that ends on $V(K_1) \in I$, in between the second brackets there is the clique of length l given by $V(K_0)$, and at the end there is a word of length $n-k-l$ that starts with $V(K_2)$. Moreover, we want that $*\cdots*$ does not end on letters commuting with $V(K_0)$ and $V(K_1)$ and that $\diamond\cdots\diamond$ does not start with letters commuting with $V(K_0)$ and $V(K_2)$. This means that the cliques K_1

and K_2 are maximal for the property that a decomposition like (4.2.1) exists. If we demand that $v_{\sigma(1)}...v_{\sigma(k)}$, $v_{\sigma(k+1)}...v_{\sigma(k+l)}$ and $v_{\sigma(k+l+1)}...v_{\sigma(n)}$ are in I, then there can be at most one such permutation coming from shuffle equivalences.
(b) Of course not for every $n \in \mathbb{N}$, $0 \leq l \leq n$, $0 \leq k \leq n-l$, $K_0 \in \mathrm{Cliq}(K,l)$, $(K_1, K_2) \in \mathrm{Comm}(K_0)$ and reduced words $\mathbf{v} = v_1 \cdots v_n$ with $v_1, ..., v_n \in V$ a permutation σ as in *(a)* exists since $\mathbf{v}$ cannot always be written in the from (4.2.1).

Definition 4.2.3. Let $n \in \mathbb{N}$. Suppose that $0 \leq l \leq n$, $0 \leq k \leq n-l$, $K_0 = (V_0, E_0) \in \mathrm{Cliq}(K,l)$ and $(K_1, K_2) \in \mathrm{Comm}(K_0)$ where $K_1 = (V_1, E_1)$ and $K_2 = (V_2, E_2)$. So if $l = 0$ we have that K_0 is the empty clique, and Condition (2) below vanishes. Take a reduced word $\mathbf{v} = v_1 \cdots v_n$ where $v_1, ..., v_n \in V$. If existent, define $\sigma (= \sigma^{\mathbf{v}}_{l,k,K_0,K_1,K_2})$ as the permutation of indices $1, \ldots, n$ that satisfies:

(1) $v_1 \cdots v_n = v_{\sigma(1)} \cdots v_{\sigma(n)}$;

(2) $\{v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\} = V_0$;

(3) $|v_{\sigma(1)} \cdots v_{\sigma(k)} s| = k-1$ whenever $s \in V_1$;

(4) $|v_{\sigma(1)} \cdots v_{\sigma(k)} s| = k+1$ whenever $s \in \mathrm{Link}(K_0) \setminus V_1$;

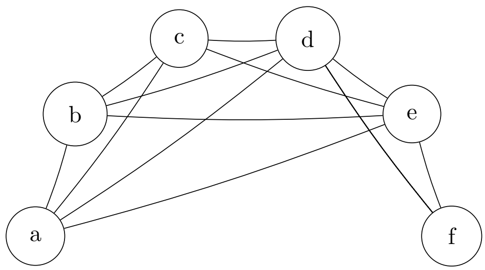
(5) $|s v_{\sigma(k+l+1)} \cdots v_{\sigma(n)}| = n-k-l-1$ whenever $s \in V_2$;

(6) $|s v_{\sigma(k+l+1)} \cdots v_{\sigma(n)}| = n-k-l+1$ whenever $s \in \mathrm{Link}(K_0) \setminus V_2$.

We shall moreover assume that $v_{\sigma(1)} \cdots v_{\sigma(k)}$, $v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}$ and $v_{\sigma(k+l+1)} \cdots v_{\sigma(n)}$ are in I (i.e. they are the representatives of their equivalence class) and that if $v_i = v_j$ for $i < j$ then $\sigma(i) < \sigma(j)$ so that σ comes from a shuffle equivalence. Then σ is unique if it exists.

The permutation σ of Definition 4.2.3 does not necessarily exist. All expressions below in which a non-existing σ occurs need to be interpreted as 0 and we shall recall this at the relevant places.

Example 4.2.4. Consider the following graph



This is the complete graph K_5 consisting of vertices a, b, c, d, e together with an extra vertex f that is connected only to d and e. Say that a word is in I (i.e. is a representative) if it is minimal in alphabetical order amongst all equivalent words. Now suppose that we have a reduced word $abcdef$.

- *Example 1.* Take $K_0 \in \mathrm{Cliq}(K,3)$ with vertex set $V_0 := \{a,b,c\}$ and $(K_1,K_2) \in \mathrm{Comm}(K_0)$ where K_1 has vertex set $V_1 := \{d,e\}$ and where $K_2 = \emptyset$. Set $l = 3, k = 2$. Then σ as in Definition 4.2.3 exists and it is the permutation moving the word $abcdef$ to $(de)(abc)f$ (since every letter occurs uniquely it is clear what the permutation is). Indeed, abc forms a clique, de ends on the vertices in K_1 and there is no other letter commuting with the vertices of K_1 at the end of de and f has no letters commuting with abc at the start.

- *Example 2.* Take $K_0 \in \mathrm{Cliq}(K,2)$ with vertex set $V_0 := \{a,b\}$ and take $(K_1,K_2) \in \mathrm{Comm}(K_0)$ where K_1 has vertex set $V_1 := \{d\}$ and K_2 has vertex set $V_2 := \{c\}$. Set $l = 2$, $k = 2$. Then σ as in Definition 4.2.3 does not exist. Indeed, by the choice of k,l and K_0 if σ exists there must be a word equivalent to $abcdef$ of the form ($*$ and $\diamond$ being undetermined letters) $**(ab)\diamond\diamond$. By the choice of K_1 we see that $**$ ends on d and there are no other letters in $\mathrm{Link}(K_0)$ at the end of $**$. So we must have $*d(ab)\diamond\diamond$. However, there is no choice for the letter $*$ (e is not allowed as it is in $\mathrm{Link}(K_0)$ and f cannot be moved past ab).

Now we find the following decomposition.

Lemma 4.2.5. *Let $a_1\cdots a_n$ be a reduced operator of type $\mathbf{v} = v_1\cdots v_n$ in the graph product (A,φ). Suppose that we have a non-zero expression of the form*

$$Q^{(1)}_{v_1} a_1 Q^{(2)}_{v_1} \cdots Q^{(1)}_{v_n} a_n Q^{(2)}_{v_n}, \tag{4.2.2}$$

in $\mathcal{B}(L^2(A,\varphi))$, where $Q^{(j)}_{v_i}$ equals either P_{v_i} or $P^{\perp}_{v_i} := 1 - P_{v_i}$. Then, for some permutation α of the indices $1,\ldots,n$ coming from a shuffle equivalence and for some $0 \le r \le n$ and $0 \le m \le n-r$ we have that (4.2.2) equals

$$\begin{aligned}&(P_{v_{\alpha(1)}} a_{\alpha(1)} P^{\perp}_{v_{\alpha(1)}})\cdots(P_{v_{\alpha(r)}} a_{\alpha(r)} P^{\perp}_{v_{\alpha(r)}})(P_{v_{\alpha(r+1)}} a_{\alpha(r+1)} P_{v_{\alpha(r+1)}})\cdots(P_{v_{\alpha(m)}} a_{\alpha(m)} P_{v_{\alpha(m)}})\\ &\times\quad (P^{\perp}_{v_{\alpha(m+1)}} a_{\alpha(m+1)} P_{v_{\alpha(m+1)}})\cdots(P^{\perp}_{v_{\alpha(n)}} a_{\alpha(n)} P_{v_{\alpha(n)}}).\end{aligned} \tag{4.2.3}$$

Further, for a non-zero expression of the form (4.2.3) we have that the induced subgraph of K with vertex set $\{v_{\alpha(r+1)},\ldots,v_{\alpha(m)}\}$ is in a clique.

Proof. We prove the statement of the lemma in a series of claims. To avoid cumbersome notation we shall not write the permutation of the shuffle equivalences in the proof.

Claim 1. Up to shuffle equivalence the expression (4.2.2) is equal to

$$Q^{(1)}_{v_1} a_1 Q^{(2)}_{v_1} \cdots Q^{(1)}_{v_m} a_m Q^{(2)}_{v_m} (P^{\perp}_{v_{m+1}} a_{m+1} P_{v_{m+1}})\cdots(P^{\perp}_{v_n} a_d P_{v_n}). \tag{4.2.4}$$

Moreover, the tail of annihilation operators is maximal in the sense that if for some $i \le m$ we have $Q^{(2)}_{v_i} = P_{v_i}$ then $Q^{(1)}_{v_i} = P_{v_i}$.

Proof of Claim 1. Suppose that we are given an expression as in (4.2.4). Suppose that for some $i < m$ we have $Q^{(1)}_{v_i} = P^{\perp}_{v_i}, Q^{(2)}_{v_i} = P_{v_i}$. Then we need to show that

v_i commutes with $v_{i+1}\cdots v_m$. To do so we may suppose the index i was chosen maximally. Suppose that v_i and $v_{i+1}\cdots v_m$ do not commute and let v_k be the first letter in $v_{i+1}\cdots v_m$ that does not commute with v_i. Our choice of i yields that $Q^{(1)}_{v_k} = P_{v_k}$. Indeed if $Q^{(1)}_{v_k}$ were to be $P^{\perp}_{v_k}$ then (4.2.4) is 0 in case $Q^{(2)}_{v_k} = P^{\perp}_{v_k}$ and in case $Q^{(2)}_{v_k} = P_{v_k}$ this would contradict maximality of i. But then (4.2.4) contains a factor $P_{v_i}P_{v_k} = 0$ which means that (4.2.4) would be zero which in turn is a contradiction.

Claim 2. The expression (4.2.2) is up to shuffle equivalence equal to:

$$\begin{aligned} &Q^{(1)}_{v_1} a_1 Q^{(2)}_{v_1} \cdots Q^{(1)}_{v_r} a_r Q^{(2)}_{v_r} (P_{v_{r+1}} a_{r+1} P_{v_{r+1}}) \cdots (P_{v_m} a_m P_{v_m}) \\ \times\quad &(P^{\perp}_{v_{m+1}} a_{m+1} P_{v_{m+1}}) \cdots (P^{\perp}_{v_n} a_n P_{v_n}). \end{aligned} \tag{4.2.5}$$

Moreover, the tail of annihilation and diagonal operators is maximal in the sense that if for some $i \le r$ we have $Q^{(1)}_{v_i} = P_{v_i}$ then $Q^{(2)}_{v_i} = P^{\perp}_{v_i}$.

Proof of Claim 2. Suppose that we are given a non-zero expression as in (4.2.5). Suppose that for some $i < r$ we have $Q^{(1)}_{v_i} = P_{v_i}, Q^{(2)}_{v_i} = P_{v_i}$. Then we need to show that v_i commutes with $v_{i+1}\cdots v_r$. To do so we may suppose the index $i < r$ was chosen maximally. Suppose that v_i and $v_{i+1}\cdots v_r$ do not commute and let v_k be the first letter in $v_{i+1}\cdots v_r$ that does not commute with v_i. We claim that our choice of i yields that $Q^{(2)}_{v_k} = P^{\perp}_{v_k}$. Indeed, suppose that $Q^{(2)}_{v_k} = P_{v_k}$. Then if $Q^{(1)}_{v_k} = P_{v_k}$ this contradicts maximality of i and if $Q^{(1)}_{v_k} = P^{\perp}_{v_k}$ it would contradict Claim 1. From $Q^{(2)}_{v_k} = P^{\perp}_{v_k}$ we find that $Q^{(1)}_{v_k} = P_{v_k}$ since if $Q^{(1)}_{v_k} = P^{\perp}_{v_k}$ then $P^{\perp}_{v_k} a_k P^{\perp}_{v_k} = 0$. But then (4.2.5) contains the factor $P_{v_i}P_{v_k} = 0$ with v_i and v_k non-commuting, which means that (4.2.5) would be zero. As this is a contradiction the claim follows.

Claim 3. The expression (4.2.2) is up to shuffle equivalence equal to:

$$\begin{aligned} &(P_{v_1} a_1 P^{\perp}_{v_1})\ldots(P_{v_r} a_r P^{\perp}_{v_r})(P_{v_{r+1}} a_{r+1} P_{v_{r+1}}) \cdots (P_{v_m} a_m P_{v_m}) \\ \times\quad &(P^{\perp}_{v_{m+1}} a_{m+1} P_{v_{m+1}})\ldots(P^{\perp}_{v_n} a_n P_{v_n}). \end{aligned}$$

Moreover $v_{r+1}\cdots v_m$ forms a clique.

Proof of Claim 3. This is obvious now from Claim 2 and the fact that $P^{\perp}_{v_i} a_{v_i} P^{\perp}_{v_i} = 0$. As $P_{v_i}P_{v_j}$ is non-zero only if v_i and v_j commute we must have that $v_{r+1}\cdots v_m$ forms a clique.

We may now directly conclude the lemma from Claim 3. The permutation α is then the composition of the shuffle equivalences coming from the Claims 1, 2 and 3. □

Proposition 4.2.6. *Let $a_1\cdots a_n \in A$ be a reduced operator of type $\mathbf{v} = v_1 \ldots v_n$ in the graph* *product* (A,φ). *We have the following equality of operators in* $\mathcal{B}(L^2(A,\varphi))$

$$
\begin{aligned}
a_1\cdots a_n = \sum_{l=0}^{n}\sum_{k=0}^{n-l}\sum_{K_0\in Cliq(K,l)}\sum_{(K_1,K_2)\in Comm(K_0)} & (P_{v_{\sigma(1)}}a_{\sigma(1)}P^{\perp}_{v_{\sigma(1)}})...(P_{v_{\sigma(k)}}a_{\sigma(k)}P^{\perp}_{v_{\sigma(k)}}) \\
& \times (P_{v_{\sigma(k+1)}}a_{\sigma(k+1)}P_{v_{\sigma(k+1)}})...(P_{v_{\sigma(k+l)}}a_{\sigma(k+l)}P_{v_{\sigma(k+l)}}) \\
& \times (P^{\perp}_{v_{\sigma(k+l+1)}}a_{\sigma(k+l+1)}P_{v_{\sigma(k+l+1)}})...(P^{\perp}_{v_{\sigma(n)}}a_{\sigma(n)}P_{v_{\sigma(n)}}),
\end{aligned}
\tag{4.2.6}
$$

where σ *(changing over the summation) is as in Definition 4.2.3. If such* σ *does not exist then the summand is understood as 0.*

Proof. First note that we may decompose,

4

$$a_1...a_n = (P_{v_1}+P^{\perp}_{v_1})a_1(P_{v_1}+P^{\perp}_{v_1})...(P_{v_n}+P^{\perp}_{v_n})a_n(P_{v_n}+P^{\perp}_{v_n}). \tag{4.2.7}$$

What we showed in Lemma 4.2.5 and (4.2.7) is that the product $a_1\cdots a_n$ decomposes as a sum of operators of the form (4.2.3) (where α depends on the summand). The proof is finished by arguing that the summation in (4.2.6) runs exactly over all these summands.

It is clear that each (non-zero) summand in (4.2.6) is an expression of the form (4.2.3) with $r=k$ and $l=m-r$. Conversely, take an expression of the form (4.2.3), then the letters $v_{\alpha(r+1)},...,v_{\alpha(m)}$ induce a clique $K_0=(V_0,E_0)$ by Lemma 4.2.5. Set $l:=\#V_0$ and $k:=r$. Now in $v_{\alpha(1)}...v_{\alpha(r)}$ there may be letters at the end that commute with the elements in V_0. Let $K_1=(V_1,E_1)$ be the clique of letters that appear at the end of $v_{\alpha(1)}...v_{\alpha(r)}$ that commute with the elements in V_0 that is maximal in the following sense: there are no letters s appearing at the end of $v_{\alpha(1)}\cdots v_{\alpha(r)}$ that commute with the elements in V_0 and the elements in V_1. Such a clique is unique since if both K_1 and K_1' would be such cliques, then so is $K_1\cup K_1'$ and hence $K_1=K_1'$ by the maximality. Similarly we may let $K_2:=(V_2,E_2)$ be a clique of letters that appear at the start of $v_{\alpha(m+1)}\cdots v_{\alpha(d)}$ that commute with the elements in V_0 and that is maximal. Then for this choice of K_0,K_1,K_2 we have that σ satisfying (1) - (6) of Definition 4.2.3 exists and it is moreover the only choice for which it exists. This shows that each non-zero expression (4.2.3) occurs exactly once in the summation (4.2.6). □

In order to prove our Khintchine inequality we introduce the necessary notation. We define as in [163, Section 2] the following subspaces of $\mathcal{B}(L^2(A_f,\varphi_f))$,

$$L_1 := \mathrm{Span}\{P^f_v a_v (P^f_v)^{\perp} \mid v\in V, a\in A_v\}, \qquad K_1 := L_1^*.$$

It is proved in [163, Lemma 2.3] that

$$L_1 \cong \left(\bigoplus_{v\in V} L^2(A_v^\circ,\varphi_v)\right)_{\mathrm{C}}, \qquad K_1 \cong \left(\bigoplus_{v\in V} L^2(A_v^\circ,\varphi_v)\right)_{\mathrm{R}} \tag{4.2.8}$$

completely isometrically, where the subscript C (resp. R) denotes the column (resp. row) Hilbert space structure. In particular, if each A_v° is one-dimensional (as is the case for right-angled Hecke algebras) we have that L_1 (resp. K_1) is completely isometrically isomorphic to the column Hilbert space $\mathrm{C}_{\#V}$ (resp. row Hilbert space $\mathrm{R}_{\#V}$). We set k-fold Haagerup tensor products,

$$L_k := L_1^{\otimes_h k}, \qquad K_k = K_1^{\otimes_h k}.$$

Fix $K_0 = (V_0, E_0) \in \mathrm{Cliq}(K, l)$. For a reduced word $\mathbf{v} = v_1 \cdots v_l$ in I consisting precisely of all letters of K_0 and $a_1 \in A_{v_1}$, ..., $a_l \in A_{v_l}$ we define an element of $\mathcal{B}(L^2(A_f, \varphi_f))$ by setting for $r > l$,

$$\mathrm{Diag}(a_1, ..., a_l) : b_1 \cdots b_l b_{l+1} \cdots b_r \Omega_f \mapsto (a_1 b_1)^\circ \cdots (a_l b_l)^\circ b_{l+1} \cdots b_r \Omega_f, \tag{4.2.9}$$

where $b_1 \in A_{w_1}^\circ$, ..., $b_r \in A_{w_r}^\circ$ with $w_1, ..., w_r \in V$ where $w_1 \neq w_2 \neq ... \neq w_r$ (that is $b_1 \cdots b_r$ is a reduced operator in the free product). If $r < l$ then the image in (4.2.9) is 0.

Lemma 4.2.7. *The operator defined in (4.2.9) is bounded.*

Proof. Let $r \in \mathbb{N}$ and fix a word $\mathbf{w} = w_1 \cdots w_r$ with $w_1, ..., w_r \in V$ and $w_1 \neq w_2 \neq ... \neq w_r$. Then

$$L^2(A_{w_1}^\circ, \varphi_{w_1}) \otimes \cdots \otimes L^2(A_{w_r}^\circ, \varphi_{w_r}) \subseteq L^2(A_f, \varphi_f)$$

is an invariant subspace for $\mathrm{Diag}(a_1, \ldots, a_l)$. Moreover, note that (4.2.9) is for $r \geq l$ just the tensor product operator

$$P_{w_1} a_1 P_{w_1} \otimes \cdots \otimes P_{w_l} a_l P_{w_l} \otimes 1^{\otimes r-l}, \tag{4.2.10}$$

where $P_{w_i} a_i P_{w_i}$ acts on $L^2(A_{w_i}^\circ, \varphi_{w_i})$. Clearly this operator is bounded. □

Set the diagonal space

$$A_{K_0} \subseteq \mathcal{B}(L^2(A_f, \varphi_f)) \tag{4.2.11}$$

to be the linear span of all operators of the form (4.2.9) where A_{K_0} inherits the operator space structure of A_f. Set for $n \in \mathbb{N}$

$$X_n := \bigoplus_{l=0}^{n} \bigoplus_{k=0}^{n-l} \bigoplus_{K_0 \in \mathrm{Cliq}(K,l)} \bigoplus_{(K_1,K_2) \in \mathrm{Comm}(K_0)} L_k \otimes_h A_{K_0} \otimes_h K_{n-k-l}. \tag{4.2.12}$$

Remark 4.2.8. In the case where the A_v°, $v \in V$ are all 1-dimensional, the space X_n can also be understood in terms of bounded operators on a Hilbert space. Indeed, the remarks after (4.2.8) and [22, Proposition 3.5] give the first two completely isometric isomorphisms of

$$\begin{aligned} L_k \otimes_h A_{K_0} \otimes_h K_{n-k-l} &\cong \mathrm{C}_k \otimes_h A_{K_0} \otimes_h \mathrm{R}_{n-k-l} \\ &\cong M_{k,n-k-l}(A_{K_0}) \cong M_{k,n-k-l}(\mathbb{C}) \otimes A_{K_0}. \end{aligned} \tag{4.2.13}$$

The third completely isometric isomorphism of (4.2.13) holds by the definition of the operator space structure on A_{K_0} as part of the C*-algebra A_f.

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Next consider the embedding $j_n : \Sigma_n \to X_n$, where the image of $a_1 \otimes \cdots \otimes a_n$ is given as follows: Consider a summand of X_n indexed by (l,k,K_0,K_1,K_2) with $0 \le l \le n, 0 \le k \le n-l$ and $K_0 \in \mathrm{Cliq}(K,l), (K_1,K_2) \in \mathrm{Comm}(K_0)$. Then the restriction of the image of j_n to this summand is given by

$$\begin{aligned} j_n(a_1\otimes\cdots\otimes a_n)|_{L_k\otimes_h A_{K_0}\otimes_h K_{n-k-l}} :=& (P^f_{v_{\sigma(1)}} a_{\sigma(1)} P^{f\perp}_{v_{\sigma(1)}}) \otimes\cdots\otimes (P^f_{v_{\sigma(k)}} a_{\sigma(k)} P^{f\perp}_{v_{\sigma(k)}}) \\ & \otimes \mathrm{Diag}(a_{\sigma(k+1)},\ldots,a_{\sigma(k+l)}) \\ & \otimes (P^{f\perp}_{v_{\sigma(k+l+1)}} a_{\sigma(k+l+1)} P^f_{v_{\sigma(k+l+1)}}) \otimes\cdots\otimes (P^{f\perp}_{v_{\sigma(n)}} a_{\sigma(n)} P^f_{v_{\sigma(n)}}), \end{aligned} \tag{4.2.14}$$

with σ given by Definition 4.2.3; if such σ is non-existent then the image of $j_n(a_1 \otimes \cdots \otimes a_n)$ in the summand of X_n corresponding to (l,k,K_0,K_1,K_2) is 0. Let $\pi^f_n : X_n \to \mathscr{B}(L^2(A_f,\varphi_f))$ be the direct sums of product maps. For a 5-tuple (l,k,K_0,K_1,K_2) as above, let

$$\pi^f_{n,l,k,K_0,K_1,K_2} : L_k \otimes_h A_{K_0} \otimes_h K_{n-k-l} \to \mathscr{B}(L^2(A_f),\varphi_f)$$

be the map π^f_n restricted to the corresponding summand of X_n. This map is completely bounded as follows from the very definition of the Haagerup tensor product. Consequently, $\pi^f_n : X_n \to \mathscr{B}(L^2(A_f,\varphi_f))$ is completely bounded by the number of summands of X_n, i.e. $\|\pi^f_n\|_{cb} \le (\#\mathrm{Cliq}(K))^3 n$.

Definition of two partial isometries. Given a 5-tuple (l,k,K_0,K_1,K_2) as in the previous paragraph, we define two partial isometries.

- We define a partial isometry

$$\mathscr{Q}_{l,k,K_0,K_1,K_2} : L^2(A,\varphi) \to L^2(A_f,\varphi_f)$$

  as follows. Consider a reduced operator $b_1 \cdots b_n \in A$ of type $\mathbf{v} = v_1 ... v_n$. We need to define a permutation $\sigma_{\mathscr{Q}} = \sigma^{\mathbf{v}}_{\mathscr{Q},l,k,K_0,K_1,K_2}$ coming from a shuffle equivalence satisfying (1) – (4) of Definition 4.2.3 and the additional relation that $|s v_{\sigma_{\mathscr{Q}}(k+l+1)} ... v_{\sigma_{\mathscr{Q}}(n)}| = n-k-l+1$ whenever s is a letter in the vertex set of K_2. Moreover we assume that this $\sigma_{\mathscr{Q}}$ is chosen such that each of the expressions $v_{\sigma_{\mathscr{Q}}(k)} ... v_{\sigma_{\mathscr{Q}}(1)}$, $v_{\sigma_{\mathscr{Q}}(k+1)} ... v_{\sigma_{\mathscr{Q}}(k+l)}$ and $v_{\sigma_{\mathscr{Q}}(k+l+1)} ... v_{\sigma_{\mathscr{Q}}(n)}$ are in I. If $\sigma_{\mathscr{Q}}$ exists it is unique and we set

$$\mathscr{Q}_{l,k,K_0,K_1,K_2}(b_1 \cdots b_n \Omega) := b_{\sigma_{\mathscr{Q}}(1)} \cdots b_{\sigma_{\mathscr{Q}}(n)} \Omega_f.$$

  If $\sigma_{\mathscr{Q}}$ does not exist we set $\mathscr{Q}_{l,k,K_0,K_1,K_2}(b_1 \cdots b_n \Omega) := 0$.

- We further define a partial isometry

$$\mathscr{R}_{l,k,K_0,K_1} : L^2(A,\varphi) \to L^2(A_f,\varphi_f), \tag{4.2.15}$$

as follows. Consider a reduced operator $b_1 \cdots b_n \in A$ of type $\mathbf{v} = v_1 ... v_n$. We need to define a permutation $\sigma_{\mathscr{R}} = \sigma^{\mathbf{v}}_{\mathscr{R},l,k,K_0,K_1}$ coming from a shuffle equivalence satisfying (1) – (4) of Definition 4.2.3. Moreover we assume that this $\sigma_{\mathscr{R}}$ is chosen such that each of the expressions

$$v_{\sigma_{\mathscr{R}}(1)} ... v_{\sigma_{\mathscr{R}}(k)},\ w_{\sigma_{\mathscr{R}}(k+1)} ... v_{\sigma_{\mathscr{R}}(k+l)},\ v_{\sigma_{\mathscr{R}}(k+l+1)} ... v_{\sigma_{\mathscr{R}}(n)}$$

is in I. If $\sigma_{\mathscr{R}}$ exists then it is unique and we set

$$\mathscr{R}_{l,k,K_0,K_1}(b_1 \cdots b_n \Omega) := b_{\sigma_{\mathscr{R}}(1)} \cdots b_{\sigma_{\mathscr{R}}(n)} \Omega_f.$$

If $\sigma_{\mathscr{R}}$ does not exist we set $\mathscr{R}_{l,k,K_0,K_1}(b_1 \cdots b_n \Omega) := 0$.

The maps $\mathscr{Q}_{l,k,K_0,K_1,K_2}$ and $\mathscr{R}_{l,k,K_0,K_1}$ preserve orthogonality and inner products and are therefore partial isometries.

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Proposition 4.2.9. *Let $x = a_1 \otimes \cdots \otimes a_n \in \Sigma_n$ be an operator of type $\mathbf{v} = v_1 ... v_n$ and let x_{n,l,k,K_0,K_1,K_2} with $0 \leq l \leq n$, $0 \leq k \leq n-l$, $K_0 = (V_0, E_0) \in Cliq(K,l)$ and $(K_1,K_2) \in Comm(K_0)$ be the corresponding summand of $j_n(x)$ in X_n as in (4.2.14). We have*

$$\begin{aligned}
&\mathscr{R}^*_{l,k,K_0,K_1} \pi^f_{n,l,k,K_0,K_1,K_2}(x_{n,l,k,K_0,K_1,K_2}) \mathscr{Q}_{l,n-l-k,K_0,K_1,K_2} \\
=&(P_{v_{\sigma(1)}} a_{\sigma(1)} P^{\perp}_{v_{\sigma(1)}}) \cdots (P_{v_{\sigma(k)}} a_{\sigma(k)} P^{\perp}_{v_{\sigma(k)}}) \\
&\times \quad (P_{v_{\sigma(k+1)}} a_{\sigma(k+1)} P_{v_{\sigma(k+1)}}) \cdots (P_{v_{\sigma(k+l)}} a_{\sigma(k+l)} P_{v_{\sigma(k+l)}}) \\
&\times \quad (P^{\perp}_{v_{\sigma(k+l+1)}} a_{\sigma(k+l+1)} P_{v_{\sigma(k+l+1)}}) \cdots (P^{\perp}_{v_{\sigma(n)}} a_{\sigma(n)} P_{v_{\sigma(n)}}),
\end{aligned} \tag{4.2.16}$$

where σ is defined as in (1) – (6) of Definition 4.2.3 and where the right-hand side should be understood as 0 if such σ does not exist.

Proof. Note that both sides of (4.2.16) equal 0 if a σ as in the statement of the proposition does not exist, c.f. the definition of j_n. So from now on we assume that σ exists and that the right-hand side of (4.2.16) is non-zero for some elementary tensor product $a_1 \otimes \cdots \otimes a_n \in \Sigma_n$ of type $\mathbf{v} = v_1 ... v_n$.

We first argue that without loss of generality we may assume that the permutation σ on the right-hand side of (4.2.16) is trivial. Indeed, if σ is non-trivial then we may replace x by the element $x' := a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \in \Sigma_n$. Then note that the left-hand sides of (4.2.16) coincide for x and x'. Similarly the right-hand side of (4.2.16) is the same for x and x'.

So assume that σ is trivial. Now take a reduced operator $b_1 ... b_r$ of type $\mathbf{w} = w_1 \cdots w_r$. We first prove the proposition in the case where the permutation $\sigma_{\mathscr{Q}} := \sigma^{\mathbf{w}}_{\mathscr{Q},l,n-l-k,K_0,K_1,K_2}$ exists. In that case set $w'_1 := w_{\sigma_{\mathscr{Q}}(1)}, ..., w'_r := w_{\sigma_{\mathscr{Q}}(r)}$ and

$$\mathscr{Q}_{l,n-l-k,K_0,K_1,K_2}(b_1 \cdots b_r \Omega) =: b'_1 ... b'_r \Omega_f$$

where $b'_1 := b_{\sigma_{\mathscr{Q}}(1)}, ..., b'_r := b_{\sigma_{\mathscr{Q}}(r)}$. As before, let $V(K_0) \in I$ be the word consisting of all letters in K_0 and let $V(K_2) \in I$ be the word consisting of all letters in K_2. Then

$$w'_1 ... w'_r \simeq \overbrace{(* \cdots * V(K_2))}^{r-l-k}(V(K_0))(\diamond \cdots \diamond),$$

where $(\diamond\cdots\diamond)$ has no letters in K_1 at the start. We have by definition of $\pi^f_{n,l,k,K_0,K_1,K_2}$,

$$\begin{aligned}&\pi^f_{n,l,k,K_0,K_1,K_2}(x_{n,l,k,K_0,K_1,K_2})\\ =&(P^f_{v_1}a_1P^{f\perp}_{v_1})\cdots(P^f_{v_k}a_kP^{f\perp}_{v_k})\mathrm{Diag}(a_{k+1},\cdots,a_{k+l})(P^{f\perp}_{v_{k+l+1}}a_{k+l+1}P^f_{v_{k+l+1}})\cdots(P^{f\perp}_{v_n}a_nP^f_{v_n}).\end{aligned}$$

Then, for the left-hand side of (4.2.16),

$$\begin{aligned}&\pi^f_{n,l,k,K_0,K_1,K_2}(x_{n,l,k,K_0,K_1,K_2})\mathscr{Q}_{l,n-l-k,K_0,K_1,K_2}(b_1\cdots b_r\Omega)\\ =&\langle a_nb'_1\Omega_f,\Omega_f\rangle...\langle a_{k+l+1}b'_{n-k-l}\Omega_f,\Omega_f\rangle a_1\cdots a_k(a_{k+1}b'_{n-k-l+1})^\circ...(a_{k+l}b'_{n-k})^\circ b'_{n-k+1}...b'_r\Omega_f.\end{aligned} \tag{4.2.17}$$

Now, for the right-hand side of (4.2.16) we consider an expression,

$$\begin{aligned}&(P_{v_1}a_1P^\perp_{v_1})\cdots(P_{v_k}a_kP^\perp_{v_k})(P_{v_{k+1}}a_{k+1}P_{v_{k+1}})\cdots(P_{v_{k+l}}a_{k+l}P_{v_{k+l}})\\ &\times(P^\perp_{v_{k+l+1}}a_{k+l+1}P_{v_{k+l+1}})\cdots(P^\perp_{v_n}a_nP_{v_n})(b_1\cdots b_r\Omega).\end{aligned} \tag{4.2.18}$$

The assumption that σ is trivial yields that $v_{k+l+1}\cdots v_n$ starts with $V(K_2)$, that the letters $v_k,...,v_{k+l}$ exhaust the vertex set of K_0 and that the letters at the end of $v_1...v_k$ that commute with K_0 are precisely given by the vertices in K_1. If (4.2.18) is non-zero then let us argue that there exists a word $w'_1...w'_r$ as defined above. Indeed, if (4.2.18) is non-zero, then we may shuffle $b_1...b_r$ into an operator $b'_1...b'_r$ of type $w'_1\cdots w'_r$ such that: $w'_1...w'_{n-l-k}$ equals $v_n...v_{k+l+1}$ and ends with $V(K_2)$; the letters $w'_{n-k-l+1},...,w'_{d-k}$ exhaust the vertex set of K_0; $w'_{n-k+1}\cdots w'_r$ does not have a letter of K_1 up front (because if that happens then applying $P^\perp_{v_i}$ with $i\le k$ will give zero). So we conclude that (4.2.18) can only be non-zero if there exist $w'_1,...,w'_r$ as defined above, in which case

$$\begin{aligned}(4.2.18)=&(P_{v_1}a_1P^\perp_{v_1})...(P_{v_k}a_kP^\perp_{v_k})(P_{v_{k+1}}a_{k+1}P_{v_{k+1}})...(P_{v_{k+l}}a_{k+l}P_{v_{k+l}})\\ &\times(P^\perp_{v_{k+l+1}}a_{k+l+1}P_{v_{k+l+1}})...(P^\perp_{v_n}a_nP_{v_n})b'_1...b'_r\Omega\\ =&\langle a_nb'_1\Omega,\Omega\rangle...\langle a_{k+l+1}b'_{n-k-l}\Omega,\Omega\rangle\\ &\times a_1...a_k(a_{k+1}b'_{n-k-l+1})^\circ...(a_{k+l}b'_{n-k})^\circ b'_{n-k+1}...b'_r\Omega.\end{aligned} \tag{4.2.19}$$

If one of the terms with a $(a_{k+1}b'_{n-k-l+1})^\circ,...,(a_{k+l}b'_{n-k})^\circ$ is zero, then also this term was zero in (4.2.17) and the proposition is proved. If none of these terms are zero, then the image of (4.2.19) under $\mathscr{R}_{l,k,K_0,K_1}$ equals (4.2.17) and (4.2.18) is in $\ker(\mathscr{R}_{l,k,K_0,K_1})^\perp$. This concludes the proposition in case $\sigma_{\mathscr{Q}}$ exists.

If $\sigma_{\mathscr{Q}}$ does not exist, then $\mathscr{Q}_{l,n-l-k,K_0,K_1,K_2}(b_1\cdots b_r\Omega)=0$. On the other hand we already noted that (4.2.19) can only be nonzero if a permutation $\sigma_{\mathscr{Q}}$ exists. So if $\sigma_{\mathscr{Q}}$ is non-existent then also (4.2.19) is zero, yielding the proposition. □

For $n\in\mathbb{N}$ set the product map

$$\rho_n:\Sigma_n\to\mathscr{B}(L^2(A,\varphi)),a_1\otimes\cdots\otimes a_n\mapsto a_1\cdots a_n.$$

and

$$\pi_n:j_n(\Sigma_n)\to\mathscr{B}(L^2(A,\varphi)),j_n(x)\mapsto\rho_n(x)$$

so that by definition $\pi_n \circ j_n = \rho_n$. Of course the crucial part is to show that the map π_n is well-defined and completely bounded with linear bound in n. This is where we use the announced intertwining argument between graph products and free products.

Now we are ready for the main theorem of this chapter. Recall that the word length projection χ_n was defined in (4.1.1).

Theorem 4.2.10 (Graph product Khintchine inequality)**.** *Let $K = (V, E)$ be a finite, simplicial graph and consider a graph product $(A, \varphi) = *_{v,K}(A_v, \varphi_v)$ of unital C*-algebras A_v with GNS-faithful states φ_v. Then for every $n \in \mathbb{N}$ there exist maps*

$$j_n : \chi_n(A) \to X_n, \qquad \pi_n : dom(\pi_n) \subseteq X_n \to \chi_n(A),$$

with $dom(\pi_n) = j_n(\chi_n(A))$ and where X_n is the operator space defined as in (4.2.12) such that:

(1) $\pi_n \circ j_n$ is the identity on $\chi_n(A)$;

(2) $\|\pi_n : dom(\pi_n) \to A\|_{cb} \leq (\#Cliq(K))^3 n$.

Proof. Part (1) follows from Proposition 4.2.6 and by identifying $\chi_n(A)$ with Σ_n canonically. From Proposition 4.2.9 we see that on the domain $j_n(\chi_n(A))$ the map π_n is given by the direct sum of the maps

$$\mathscr{R}^*_{l,k,K_0,K_1} \pi^f_{n,l,k,K_0,K_1,K_2}(\,\cdot\,)\mathscr{Q}_{l,n-l-k,K_0,K_1,K_2}.$$

In particular, π_n is well-defined. As each of these summands is completely contractive and there are at most $(\#\mathrm{Cliq}(K))^3 n$ summands, we see that π_n is completely bounded with the desired complete bound. ☐

4.3. APPLICATION TO RIGHT-ANGLED HECKE C*-ALGEBRAS

This section aims to apply the Khintchine inequality for arbitrary graph products from Section 4.2 to right-angled Hecke C*-algebras. As a consequence, we derive a Haagerup type inequality for right-angled Coxeter groups and their Hecke deformations which will turn out to be very useful in Section 6.3.

4.3.1. RIGHT-ANGLED HECKE OPERATOR ALGEBRAS AS GRAPH PRODUCTS

Similar to (operator algebras of) right-angled Coxeter groups (compare with Subsection 2.7.4, Example 4.1.3 and Example 4.1.4), Hecke C*-algebras and Hecke-von Neumann algebras of right-angled Coxeter groups admit a useful decomposition in terms of graph products. This decomposition makes available several tools, one of which is the inequality that we proved in Section 4.2.

In the single-parameter version, the following statement appears in [46] where it was used to deduce approximation properties (and subsequently the absence of Cartan subalgebras) for right-angled Hecke-von Neumann algebras.

Proposition 4.3.1 ([46, Corollary 3.4]). *Let (W,S) be a right-angled Coxeter system, let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter and define a graph $K = (V,E)$ by $V := S$ and $E := \{(s,t) \mid m_{st} = 2\}$. Then*

$$(C^*_{r,q}(W), \tau_q) \cong \star_{s,K}(C^*_{r,q_s}(W_s), \tau_{q_s})$$

via $T^{(q)}_{\mathbf{w}} \to T^{(q_{s_1})}_{s_1}..T^{(q_{s_n})}_{s_n}$ where $\mathbf{w} = s_1 \cdots s_n$ with $s_1,...,s_n \in S$ is a reduced expression. Similarly,

$$(\mathcal{N}_q(W), \tau_q) \cong \overline{\star}_{s,K}(\mathcal{N}_{q_s}(W_s), \tau_{q_s}).$$

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Proof. Let $\mathscr{A}$ be the $*$-subalgebra of $\star_{s,K}(C^*_{r,q_s}(W_s), \tau_{q_s})$ generated by all elements $T^{(q_s)}_s$, $s \in S$. From the defining properties of the Iwahori-Hecke algebra $\mathbb{C}_q[W]$ (see Theorem 3.1.1) and the defining commutation relations of the graph product C*-algebra it follows that there exists a $*$-isomorphism $\pi : \mathbb{C}_q[W] \to \mathscr{A}$ given by $T^{(q)}_s \mapsto T^{(q_s)}_s$. One easily checks that π intertwines the graph product state and τ_q. From Lemma 3.5.1 it then follows that π extends to $*$-isomorphisms $C^*_{r,q}(W) \cong \star_{s,K}(C^*_{r,q_s}(W_s), \tau_{q_s})$ and $(\mathcal{N}_q(W), \tau_q) \cong \overline{\star}_{s,K}(\mathcal{N}_{q_s}(W_s), \tau_{q_s})$. □

Remark 4.3.2. *(a)* Note that reduced words in the graph K in Proposition 2.6.1 correspond to reduced expressions of the corresponding Coxeter system. This will be needed in Subsection 4.3.2.
(b) In combination with Proposition 4.1.5 the Proposition 2.6.1 implies that Hecke operator algebras of right-angled Coxeter groups decompose as (iterated) amalgamated free products. This decomposition has for instance been used in [161] where Raum and Skalski investigated the K-theory of right-angled Hecke C*-algebras by using results by Fima and Germain [82].

4.3.2. KHINTCHINE AND HAAGERUP TYPE INEQUALITIES FOR RIGHT-ANGLED HECKE C*-ALGEBRAS

Let us make the Khintchine inequality for arbitrary graph products of C*-algebras from Section 4.2 explicit in the setting of Subsection 4.3.1. For this, let (W,S) be a finite rank right-angled Coxeter system and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter. Let $K = (V,E)$ be the graph associated to (W,S) defined in Proposition 4.3.1 and recall that we have a canonical isomorphism

$$(C^*_{r,q}(W), \tau_q) \cong \star_{s,K}(C^*_{r,q_s}(W_s), \tau_{q_s}).$$

We may specialize the reasoning in Section 4.2 to $\star_{s,K}(C^*_{r,q_s}(W_s), \tau_{q_s})$, so define the spaces A_{K_0} of diagonal operators (see (4.2.11)) and the operator spaces X_n, $n \in \mathbb{N}$ (see (4.2.12)) accordingly. We first observe that A_{K_0} simplifies. Recall that P_s, $s \in S$ and $P^f_{K_0}$, $K_0 \in \mathrm{Cliq}(K)$ denote orthogonal projections that were defined in the paragraph after Example 4.1.3 and in the paragraph after Remark 4.2.1.

Lemma 4.3.3. *Let (W,S) be a right-angled Coxeter system. For any $K_0 = (V_0, E_0) \in$ Cliq(K,l) with $l \in \mathbb{N}$ we have that $A_{K_0} = \mathbb{C}P^f_{K_0}$. Moreover, for a reduced word $\mathbf{v} = s_1 \cdots s_n$ in I where $\{s_1, \ldots, s_n\} = V_0$ we have that*

$$Diag\left(P_{s_1} T_{s_1}^{(q_{s_1})} P_{s_1}, ..., P_{s_n} T_{s_n}^{(q_{s_n})} P_{s_n}\right) = (\prod_{s \in V_0} p_s(q)) P^f_{K_0}.$$

Proof. For $s \in S$ one has $P_s T_s^{(q_s)} P_s = p_s(q) P_s$ and $C^*_{r,q_s}(W_s)$ is a two-dimensional C*-algebra spanned by the unit and $T_s^{(q_s)}$. For a reduced word $\mathbf{v} = s_1 \cdots s_n$ in I where $\{s_1, \ldots, s_n\} = V_0$ and operators $a_1 \in C^*_{r,q_{s_1}}(W_{s_1})$, ..., $a_l \in C^*_{r,q_{s_n}}(W_{s_n})$ we then have that $P_{s_i} a_i P_{s_i}$ is a scalar multiple of P_{s_i} so that Diag$(a_1, \ldots, a_l)$ is a scalar multiple of $P^f_{K_0}$, see (4.2.10). If $a_i = T_{s_i}^{(q_{s_i})}$ the scalar multiple is $\prod_{s \in V_0} p_s(q)$ which finishes the proof. □

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Lemma 4.3.3 shows that we may identify A_{K_0} with $\mathbb{C}$ completely isometrically. Then in (4.2.13) note that $M_{k,d-k-l}(\mathbb{C}) \otimes_h \mathbb{C} \cong M_{k,d-k-l}(\mathbb{C})$. So for a right-angled Coxeter system we get by Lemma 4.3.3, (4.2.12) and (4.2.13) that

$$X_n = \bigoplus_{l=0}^{n} \bigoplus_{k=0}^{n-l} \bigoplus_{K_0 \in \mathrm{Cliq}(K,l)} \bigoplus_{(K_1,K_2) \in \mathrm{Comm}(K_0)} M_{k,n-k-l}(\mathbb{C}). \tag{4.3.1}$$

Let p_{l,k,K_0,K_1,K_2} be the projection of X_n onto the summand $M_{k,n-k-l}(\mathbb{C})$ indexed by (l,k,K_0,K_1,K_2). We equip $M_{k,n-k-l}(\mathbb{C})$ with the inner product $\langle x,y\rangle_{\mathrm{Tr}} := (\mathrm{Tr}_{n-k-l})(y^*x)$, where Tr_{n-k-l} is the non-normalized trace that takes the value 1 on rank 1 projections. We further equip X_n with the direct sum of these inner products and for $x \in X_n$ we let $\|x\|_{2,\mathrm{Tr}} := \langle x,x\rangle_{\mathrm{Tr}}^{\frac{1}{2}}$. Then, as for any finite-dimensional type I von Neumann algebra, we have

$$\|x\| \le \|x\|_{2,\mathrm{Tr}}. \tag{4.3.2}$$

By Theorem 4.2.10 (and Proposition 4.3.1) we obtain maps

$$j_n : \chi_n(C^*_{r,q}(W)) \to X_n \qquad \text{and} \qquad \pi_n : \mathrm{dom}(\pi_n) \to \chi_n(C^*_{r,q}(W)),$$

with $\mathrm{dom}(\pi_n) = j_n(\chi_n(C^*_{r,q}(W))) \subseteq X_n$ such that $\pi_n \circ j_n$ is the identity on $C^*_{r,q}(W)$ and such that

$$\|\pi_n : \mathrm{dom}(\pi_n) \to X_n\|_{cb} \le (\#\mathrm{Cliq}(K))^3 n.$$

We now have the following orthogonality lemma.

Lemma 4.3.4. *Let $n \in \mathbb{N}$, $0 \le l \le n$, $0 \le k \le d-l$, $K_0 = (V_0,E_0) \in$ Cliq(K,l) and $(K_1,K_2) \in$ Comm(K_0). Let $\mathbf{v},\mathbf{w} \in W$ be elements with length $|\mathbf{v}| = |\mathbf{w}| = n$. If the permutation $\sigma^{\mathbf{v}}$ given in Definition 4.2.3 exists we have*

$$\langle p_{l,k,K_0,K_1,K_2} j_n(T_{\mathbf{v}}^{(q)}), j_n(T_{\mathbf{w}}^{(q)})\rangle_{\mathrm{Tr}} = \delta(\mathbf{v} \simeq \mathbf{w}) \prod_{s \in V_0} (p_s(q))^2. \tag{4.3.3}$$

If such $\sigma^{\mathbf{v}}$ does not exist then $p_{l,k,K_0,K_1,K_2} j_n(T_{\mathbf{v}}^{(q)}) = 0$.

Proof. The final claim of the statement follows from the definition of j_n. It hence remains to prove (4.3.3) and we assume that $\sigma^{\mathbf{v}}$ as in Definition 4.2.3 exists. Since the right-hand side of (4.3.3) contains the term $\delta(\mathbf{v} \simeq \mathbf{w})$ we may assume that also $\sigma^{\mathbf{w}}$ exists. Moreover, by shuffling the letters of $\mathbf{v}$ and $\mathbf{w}$ if necessary (which does not change the operators $T_{\mathbf{v}}^{(q)}$ and $T_{\mathbf{w}}^{(q)}$), we may assume that $\sigma^{\mathbf{v}}$ and $\sigma^{\mathbf{w}}$ are the identity permutation.

For $s \in S$ set

$$e_s := P_s^f T_s^{(q_s)} (P_s^f)^{\perp} \quad \text{and} \quad e_s' = (P_s^f)^{\perp} T_s^{(q_s)} P_s^f.$$

These elements form an orthonormal basis of the respective column Hilbert space L_1 and row Hilbert space K_1. Now write a reduced expression $\mathbf{v} = s_1 \cdots s_n$ with $s_1, ..., s_n \in S$. By the assumption that $\sigma^{\mathbf{v}}$ was trivial we see that the generators $s_{k+1}, ..., s_{k+l}$ commute and form a clique $K_0 = (V_0, E_0)$ in K. From Lemma 4.3.3 we deduce that

$$\mathrm{Diag}\left(P_{s_{k+1}} T_{s_{k+1}}^{(q_{s_{k+1}})} P_{s_{k+1}}, ..., P_{s_{k+l}} T_{s_{k+l}}^{(q_{s_{k+l}})} P_{s_{k+l}}\right) = (\prod_{s \in V_0} p_s(q)) P_{K_0}^f.$$

It now follows from the definition of j_n that

$$p_{l,k,K_0,K_1,K_2}\, j_n(T_{\mathbf{v}}^{(q)}) = (\prod_{s \in V_0} p_s(q))(e_{s_1} \otimes ... \otimes e_{s_k} \otimes e'_{s_{k+l+1}} \otimes ... \otimes e'_{s_n}).$$

From this we directly conclude (4.3.3) which finishes the proof. □

Theorem 4.3.5 (Khintchine inequality for right-angled Hecke C*-algebras)**.** *Let (W, S) be a right-angled finite rank Coxeter system and define a graph $K = (V, E)$ by $V := S$ and $E := \{(s,t) \mid m_{st} = 2\}$. Let $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter. Then for every $n \in \mathbb{N}_{\geq 1}$ there exist maps*

$$j_n : \chi_n(C_{r,q}^*(W)) \to X_n, \qquad \pi_n : dom(\pi_n) \subseteq X_n \to \chi_n(C_{r,q}^*(W))$$

with $j_n(\chi_n(C_q^*(W)) = dom(\pi_n)$ where X_n is the operator space defined as in (4.3.1) such that:

(1) $\pi_n \circ j_n$ is the identity on $\chi_n(C_{r,q}^*(W))$;

(2) $\|\pi_n : dom(\pi_n) \to C_{r,q}^*(W)\|_{cb} \leq (\#Cliq(K))^3 n$;

(3) j_n extends to a bounded map

$$L^2(\chi_n(C_{r,q}^*(W)), \tau_q) \to L^2(X_n, Tr),$$

with bound majorized by $\prod_{s \in S} p_s(q)$.

Proof. The statements (1) and (2) are immediate from Proposition 4.3.1 and Theorem 4.2.10. It thus remains to prove (3). Let $\mathbf{v} \in I$ have length n. Since $\|T_{\mathbf{v}}^{(q)}\Omega\|_2 = 1$ we find from (4.3.3) that

$$\|j_n : L^2(\chi_n(C_{r,q}^*(W)), \tau_q) \to L^2(X_n, \mathrm{Tr})\| \leq \sup_{\mathbf{v} \in W, |\mathbf{v}| = n} \|j_n(T_{\mathbf{v}}^{(q)})\|_{2,\mathrm{Tr}} \leq \prod_{s \in S} p_s(q)$$

which completes the proof. □

As a consequence of Theorem 4.3.5 we derive a Haagerup type inequality for Hecke C*-algebras of right-angled Coxeter groups. Such a *Haagerup type inequality* states that the norm of an operator of length n can be estimated with its 2-norm up to a polynomial bound depending on n. It is a generalization of Haagerup's inequality for free groups $\mathbb{F}_m$ (see [93] or also [152, Section 9.6]) which entails that there exists a constant C such that for every $x \in \mathbb{C}[\mathbb{F}_m] \subseteq \mathcal{B}(\ell^2(\mathbb{F}_m))$ supported on group elements of length n one has $\|x\| \leq Cn\|x\delta_e\|_2$. In particular, in this case the polynomial can be chosen to be n, that is we have a linear estimate in the length n. Haagerup and Khintchine inequalities have found a wide range of applications in operator theory. We will give further applications to the trace-uniqueness problem of Hecke C*-algebras in Section 6.3.

Theorem 4.3.6 (Haagerup inequality for right-angled Hecke C*-algebras)**.** *Let (W,S) be a right-angled finite rank Coxeter system and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter. Define a graph $K = (V,E)$ by $V := S$ and $E := \{(s,t) \mid m_{st} = 2\}$. Then for each $n \in \mathbb{N}_{\geq 1}$ and $x \in \chi_n(C^*_{r,q}(W))$ we have that*

$$\|x\| \leq n(\#Cliq(K))^3 (\prod_{s\in S} p_s(q)) \|x\delta_e\|_2.$$

Proof. By Theorem 4.3.5 and the inequality (4.3.2) we get for every $x \in \chi_n(C^*_{r,q}(W))$

$$\begin{aligned}\|x\| =& \|(\pi_n \circ j_n)(x)\| \leq \|\pi_n\| \|j_n(x)\| \leq \|\pi_n\| \|j_n(x)\|_{2,\mathrm{Tr}} \\ \leq& n(\#\mathrm{Cliq}(K))^3 \|j_n(x)\|_{2,\mathrm{Tr}} \leq n(\#\mathrm{Cliq}(K))^3 (\prod_{s\in S} p_s(q)) \|x\delta_e\|_2.\end{aligned}$$

This completes the proof. □

5

Topological boundaries and compactifications of graphs and Coxeter groups

The aim of this chapter is to introduce and study certain topological spaces associated with (countable) connected rooted graphs. These spaces reflect combinatorial and order theoretic properties of the underlying graph and relate in the case of hyperbolic graphs to Gromov's hyperbolic compactification (see Subsection 2.5.2). They are particularly tractable in the case of Cayley graphs of finite rank Coxeter groups. In that context, we speak of the compactification and the boundary of the group. As it turns out, the canonical action of the Coxeter group on its Cayley graph induces a natural action on the compactification and the boundary. From this, we deduce that in this case our construction coincides with spaces defined in [37] (see also [130] and [129]). We further prove the amenability of the action, we characterize when the compactification is small at infinity (see Subsection 2.3.2) and we study classes of Coxeter groups for which the action is a topological boundary action in the sense of Furstenberg (see Subsection 2.3.3). Section 5.3 then reveals an intimate relationship between our construction and Hecke operator algebras. This relationship will be crucial in the treatment of the characterization of the simplicity of right-angled Hecke C^*-algebras in Chapter 6 and has several other important applications (see Chapter 8 and Chapter 9).

The content of this chapter is based on parts of the articles

- M. Klisse, *Topological boundaries of connected graphs and Coxeter groups*, arXiv preprint arXiv:2010.03414v1 (2020).
- M. Klisse, *Topological boundaries of connected graphs and Coxeter groups*, to appear in the Journal of Operator Theory;
- M. Klisse, *Simplicity of right-angled Hecke C^*-algebras*, to appear in Int. Math. Res. Not. IMRN.

5.1. Boundaries and compactifications of graphs

5

Recall that hyperbolic graphs are graphs that satisfy a certain negative curvature condition (see Subsection 2.5.2). The hyperbolic boundary $\partial_h K$ and the hyperbolic compactification $K \cup \partial_h K$ of such a graph K have a rich structure which provides an excellent tool to study the underlying graph. Especially in the context of Cayley graphs of groups (see Definition 2.5.4) the notion of hyperbolicity allows exploring connections between algebraic properties of the group and geometric properties of certain topological spaces (see [122] for a survey). This led to a number of breakthroughs in the fields of geometric and combinatoric group theory.

Following Gromov's ideas, many similar constructions assigning topological spaces to graphs and groups have been presented. In the following, we will walk an analogous path by defining certain topological spaces associated with countable, undirected and simplicial connected rooted graphs. Our construction covers several interesting examples and, if the graph is hyperbolic, nicely relates to Gromov's hyperbolic compactification and boundary.

All graphs appearing in this chapter are assumed to be countable, undirected and simplicial. The reader may consult Section 2.5 again for the underlying graph theoretical notions.

5.1.1. Construction and basic properties

Definition 5.1.1. A *rooted graph* (K, o) is a graph K equipped with a root $o \in K$. If K is connected, we impose a partial order $\leq_o$ on K by declaring $x \leq_o y$ if and only if there exists a geodesic path starting in o and ending in y which passes x. If the root o is clear, we often just write $\leq$ instead of $\leq_o$. We call this the *graph order* on (K, o). Further, define relations $\geq_o$, $<_o$ and $>_o$ (resp. $\geq$, $<$ and $>$) in the natural way. If the *join* or *meet* (with respect to the partial order) of two elements $x, y \in K$ exists, we denote it by $x \vee_o y$ (resp. $x \vee y$) or $x \wedge_o y$ (resp. $x \wedge y$).

One easily checks that the graph order indeed defines a partial order. Based on it, we define a topological space associated with the connected rooted graph (K, o)

into which K naturally embeds as a dense subset. Let $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ be a sequence in K. We say that $\mathbf{x}$ *o-converges* if for every $x \in K$ one either has $x \le x_i$ for all large enough i or $x \not\le x_i$ for all large enough i. Write $x \le_o \mathbf{x}$ (resp. $x \le \mathbf{x}$) in the first case and $x \not\le_o \mathbf{x}$ (resp. $x \not\le \mathbf{x}$) in the second one. Note that constant sequences in K necessarily o-converge. We say that $\mathbf{x}$ *o-converges to infinity* if further $\sup_{x\le\mathbf{x}} d_K(x,o) = \infty$. One easily checks that infinite geodesic paths always o-converge to infinity. On the set of all sequences in K which o-converge we define an equivalence relation $\sim_o$ (resp. $\sim$) by declaring $\mathbf{x} \sim_o \mathbf{y}$ if and only if for every $x \in K$ the implications $x \le \mathbf{x} \Leftrightarrow x \le \mathbf{y}$ hold. Denote by $[\mathbf{x}]_o$ (resp. $[\mathbf{x}]$) the corresponding equivalence class of a sequence $\mathbf{x}$ and write $\partial(K,o)$ for the set of all equivalence classes of sequences which o-converge to infinity. We call this set the *boundary of* (K,o). Similarly, the *bordification* $\overline{(K,o)}$ is the set of all equivalence classes of sequences in K which o-converge. We will view K as a subset of its bordification by identifying the elements in K with equivalence classes of constant sequences.

The following lemma is easy to check.

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Lemma 5.1.2. *Let (K,o) be a connected rooted graph. Then the graph order on (K,o) extends to a partial order on $\overline{(K,o)}$ via*

$$[\mathbf{x}] \le_o [\mathbf{y}] :\Leftrightarrow x \le_o \mathbf{y} \text{ for every } x \in K \text{ with } x \le_o \mathbf{x}$$

for $[\mathbf{x}],[\mathbf{y}] \in \overline{(K,o)}$.

As before, we often just write $\le$ instead of $\le_o$ for the extended graph order. We equip $\overline{(K,o)}$ with the topology generated by the subbase of sets of the form

$$\mathcal{U}_x := \left\{z \in \overline{(K,o)} \mid x \le z\right\} \text{ and } \mathcal{U}_x^c := \left\{z \in \overline{(K,o)} \mid x \not\le z\right\}$$

where $x \in K$. In particular, $\mathcal{U}_x$ is *clopen* (closed and open) in $\overline{(K,o)}$. Further, we impose the subspace topology on $\partial(K,o)$.

Lemma 5.1.3. *Let (K,o) be a connected rooted graph. Then the following statements hold:*

- *$\overline{(K,o)}$ contains K as a dense subset;*
- *For $x \in K$ the one point set $\{x\}$ is clopen if x has finite degree;*
- *If the graph is locally finite, then K is a discrete subset of $\overline{(K,o)}$ and $\partial(K,o) = \overline{(K,o)} \setminus K$.*

Proof. The density of $K \subseteq \overline{(K,o)}$ is clear. If $x \in K$ has finite degree, then $\{x\}$ is open because either $\{x\} = \bigcap_{y\in K: x<y, d_K(x,y)=1}(\mathcal{U}_x \cap \mathcal{U}_y^c)$ or $\{x\} = \mathcal{U}_x$. In particular, if the graph is locally finite, K is a discrete subset of $\overline{(K,o)}$. It remains to show that $\partial(K,o) = \overline{(K,o)} \setminus K$. For this, let $z \in \overline{(K,o)}$ be a point represented by a sequence $\mathbf{x} = (x_i)_{i\in\mathbb{N}} \subseteq K$ which o-converges but which does not o-converge to infinity. Then $l := \sup\{d_K(y,o) \mid y \in K \text{ with } y \le z\}$ is finite. Because K is locally finite there exists $i_0 \in \mathbb{N}$ such that $x_i \notin \bigcup_{y\in K: d_K(y,o)=l+1} \mathcal{U}_y$ for all $i \ge i_0$. But then $d_K(x_i,o) \le l$ for all $i \ge i_0$. In particular, again by the local finiteness of K, there exists a subsequence of $\mathbf{x}$ which is constant. But $\mathbf{x}$ o-converges, so $z = y$ for some $y \in K$. This implies the claim. □

Remark 5.1.4. *(a)* It is in general not true that for a connected rooted graph (K,o) the set $K \subseteq \overline{(K,o)}$ is open. Indeed, if we consider the first graph in Figure 5.1 with the indicated sequence $(z_i)_{i\in\mathbb{N}}$ of boundary points represented by infinite geodesic paths, then $z_i \to z$. The example in particular demonstrates that the boundary $\partial(K,o)$ is not necessarily compact.
(b) Other than in the context of trees it is in general not true that for a connected rooted graph (K,o) and an element $x \in K$ the openness of the one point set $\{x\}$ implies that x has finite degree. Indeed, consider the second graph in Figure 5.1. Its vertex z does not have finite degree but the one point set $\{z\}$ is open since $\{z\} = \mathcal{U}_z \cap \mathcal{U}_{z'}^c$.
(c) Other than for the Gromov compactification of a hyperbolic graph, in general not every element of the bordification $\overline{(K,o)}$ is represented by a (possibly finite) geodesic path; not even in the locally finite case. Consider for instance the sequence $(x_i)_{i\in\mathbb{N}}$ indicated in the third graph of Figure 5.1. It o-converges but the corresponding equivalence class can not be represented by a geodesic path.

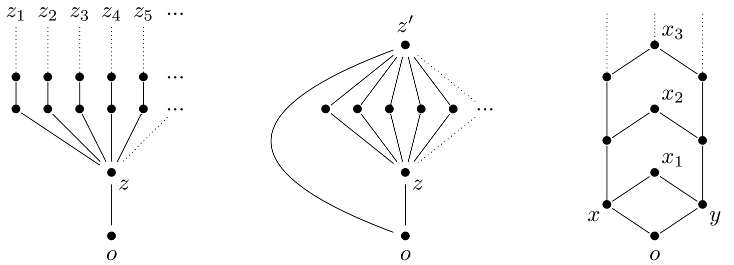

Figure 5.1: Example *(a)*, *(b)* and *(c)*

Definition 5.1.5. Let (K,o) be a connected rooted graph and let

$$\pi : \mathscr{B}(\ell^2(K)) \to \mathscr{B}(\ell^2(K))/\mathscr{K}(\ell^2(K))$$

be the quotient map where $\mathscr{K}(\ell^2(K))$ denotes the compact operators in $\mathscr{B}(\ell^2(K))$. For every element $x \in K$ let $P_x \in \ell^\infty(K) \subseteq \mathscr{B}(\ell^2(K))$ be the orthogonal projection onto the subspace

$$\overline{\mathrm{Span}\{\delta_y \mid y \in K \text{ with } y \leq x\}} \subseteq \ell^2(K).$$

Denote by $\mathscr{D}(K,o)$ the (commutative) unital C*-algebra generated by all P_x, $x \in K$. Note that $P_o = 1$.

Proposition 5.1.6. *Let (K,o) be a connected rooted graph. Then, Spec$(\mathscr{D}(K,o)) \cong \overline{(K,o)}$ where Spec$(\mathscr{D}(K,o))$ denotes the character spectrum of $\mathscr{D}(K,o)$. In particular, $\overline{(K,o)}$ is a compact Hausdorff space. Further, Spec$(\pi(\mathscr{D}(K,o))) \cong \overline{\partial(K,o)}$ where Spec$(\pi(\mathscr{D}(K,o)))$ denotes the character spectrum of $\pi(\mathscr{D}(K,o))$ and where $\overline{\partial(K,o)}$ is the closure of $\partial(K,o)$ in $\overline{(K,o)}$.*

Proof. Let $\mathbf{x}$ be a sequence in K which o-converges. It is clear that $\lim_i \langle(\cdot)\delta_{x_i},\delta_{x_i}\rangle \in \operatorname{Spec}(\mathscr{D}(K,o))$ is well-defined where the limit is taken in the weak-$*$ topology. Define a map $\psi : \overline{(K,o)} \to \operatorname{Spec}(\mathscr{D}(K,o))$ by $z \mapsto \lim \langle(\cdot)\delta_{x_i},\delta_{x_i}\rangle$ for $z \in \overline{(K,o)}$ where $\mathbf{x}$ is a sequence representing z. The image of z does not depend on the choice of the representing sequence for z. Indeed, let $\mathbf{x}$ and $\mathbf{y}$ be sequences in K which o-converge and are equivalent to each other. For all $x \in K$ one has

$$\lim_i \langle P_x\delta_{x_i},\delta_{x_i}\rangle = \begin{cases} 1 & \text{, if } x \leq \mathbf{x} \\ 0 & \text{, else} \end{cases} = \begin{cases} 1 & \text{, if } x \leq \mathbf{y} \\ 0 & \text{, else} \end{cases} = \lim_i \langle P_x\delta_{y_i},\delta_{y_i}\rangle,$$

implying that $\lim_i \langle(\cdot)\delta_{x_i},\delta_{x_i}\rangle$ and $\lim_i \langle(\cdot)\delta_{y_i},\delta_{y_i}\rangle$ coincide on $*\text{-Alg}\left(\{P_x \mid x \in K\}\right)$. Hence, ψ is well-defined.

We proceed by showing that ψ is continuous, injective, surjective and closed.

- *Continuity:* The continuity follows in the same way as the well-definedness above.
- *Injectivity:* Let $\mathbf{x}$ and $\mathbf{y}$ be sequences in K which o-converge and which are not equivalent to each other. Without loss of generality, we can assume that there exists $x \in K$ with $x \leq \mathbf{x}$ and $x \not\leq \mathbf{y}$. Then,

  $$\lim_i \langle P_x\delta_{x_i},\delta_{x_i}\rangle = 1 \text{ and } \lim_i \langle P_x\delta_{y_i},\delta_{y_i}\rangle = 0,$$

  which implies that $\psi([\mathbf{x}]) \neq \psi\left([\mathbf{y}]\right)$.
- *Surjectivity:* Let $\chi \in \operatorname{Spec}(\mathscr{D}(K,o))$ be a character on $\mathscr{D}(K,o)$. Define the set $\mathscr{S} := \{x \in K \mid \chi(P_x) = 1\}$ and choose an enumeration $y_1, y_2, \ldots$ of $\mathscr{S}$ (where we assume that the sequence becomes constant if $\mathscr{S}$ is finite) and an enumeration $y'_1, y'_2, \ldots$ of $K \setminus \mathscr{S}$ (where we assume that the sequence becomes constant if $K \setminus \mathscr{S}$ is finite). For every $i \in \mathbb{N}$ the intersection $\mathfrak{I}_i = K \cap (\mathscr{U}_{y_1} \cap \mathscr{U}^c_{y'_1}) \cap \ldots \cap (\mathscr{U}_{y_i} \cap \mathscr{U}^c_{y'_i})$ must be non-empty because otherwise $P_{y_1}(1-P_{y'_1})\ldots P_{y_i}(1-P_{y'_i}) = 0$ and hence

  $$1 = \chi(P_{y_1})\chi(1-P_{y_1})\ldots\chi(P_{y_i})\chi(1-P_{y'_i}) = \chi(P_{y_1}(1-P_{y'_1})\ldots P_{y_i}(1-P_{y'_i})) = 0.$$

  So choose for every $i \in \mathbb{N}$ an element $x_i \in \mathfrak{I}_i$ and consider the sequence $\mathbf{x} := (x_i)_{i\in\mathbb{N}}$ in K. By construction the sequence o-converges and for $z := [\mathbf{x}] \in \overline{(K,o)}$ we have $\psi(z) = \chi$. The surjectivity follows.
- *Closedness:* It suffices to show that for every $x \in K$ the sets $\psi(\mathscr{U}_x)$ and $\psi(\mathscr{U}^c_x)$ are closed in $\operatorname{Spec}(\mathscr{D}(K,o))$. Fix $x \in K$, let $(z^i)_{i\in I} \subseteq \mathscr{U}_x$ be a net and let $z \in \overline{(K,o)}$ with $\psi(z^i) \to \psi(z)$. We have $\left(\psi(z)\right)(P_x) = \lim\left(\psi(z^i)\right)(P_x) = 1$, so $z \in \mathscr{U}_x$. Hence, $\psi(\mathscr{U}_x)$ is closed in $\operatorname{Spec}(\mathscr{D}(K,o))$. The closedness of $\psi(\mathscr{U}^c_x)$ follows in the same way.

We have shown that ψ is a homeomorphism. The existence of a homeomorphism between $\operatorname{Spec}(\pi(\mathscr{D}(K,o)))$ and $\overline{\partial(K,o)}$ follows in a similar way. □

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Motivated by Proposition 5.1.6 we will often speak about $\overline{(K,o)}$ as the *compactification of the graph K*.

Remark 5.1.7. *(a)* The maps in Proposition 5.1.6 induce isomorphisms $\mathscr{D}(K,o) \cong C(\overline{(K,o)})$ via $P_x \mapsto \chi_{\mathscr{U}_x}$ and $\pi(\mathscr{D}(K,o)) \cong C(\overline{\partial(K,o)})$ via $\pi(P_x) \mapsto \chi_{\mathscr{U}_x \cap \overline{\partial(K,o)}}$, where $\chi_{\mathscr{U}_x}$ (resp. $\chi_{\mathscr{U}_x \cap \overline{\partial(K,o)}}$) denotes the characteristic function on $\mathscr{U}_x$ (resp. $\mathscr{U}_x \cap \overline{\partial(K,o)}$).
(b) The C*-algebra $\mathscr{D}(K,o)$ appearing in Proposition 5.1.6 is separable. This implies that the topological space $\overline{(K,o)}$ is metrizable. The same holds for $\overline{\partial(K,o)}$ and hence for $\partial(K,o)$.
(c) If (K,o) is a locally finite connected rooted graph, then Lemma 5.1.3 and Proposition 5.1.6 imply that $\partial(K,o) = \overline{\partial(K,o)} \cong \mathrm{Spec}(\pi(\mathscr{D}(K,o)))$ is a (metrizable) compact Hausdorff space.

As mentioned in Remark 5.1.4, in general not every element of the compactification $\overline{(K,o)}$ is represented by a (possibly finite) geodesic path. Proposition 5.1.9 characterizes when this is the case. Its proof requires the following simple lemma.

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Lemma 5.1.8. *Let (K,o) be a connected rooted graph and let $(x_i)_{i\in\mathbb{N}} \subseteq K$ be a sequence with $x_1 \le x_2 \le \ldots$. Then the sequence converges to a point in $\overline{(K,o)}$ which can be represented by a (possibly finite) geodesic path.*

Proof. Choose a geodesic path starting in o and ending in x_1 and denote it by $[o,x_1]$. Because $x_1 \le x_2$, there exists a geodesic path starting in o and ending in x_2 which passes x_1. Denote by $[x_1,x_2]$ its tail starting in x_1 and ending in x_2. Further, let $[o,x_1]\,[x_1,x_2]$ be the concatenation of $[o,x_1]$ and $[x_1,x_2]$. It is geodesic as well. Proceeding like this we get a geodesic path $\alpha := [o,x_1]\,[x_1,x_2]\,[x_2,x_3]\ldots$. If the path is finite, then the convergence of the sequence $(x_i)_{i\in\mathbb{N}}$ is clear, so assume that α is infinite. We claim that $x_i \to [\alpha]$. Indeed, if $y \le [\alpha]$, then $y \le \alpha_i$ for all large enough i and hence $y \le x_n$ for all large enough n. If $y \not\le [\alpha]$, then $y \not\le \alpha_i$ for all large enough i and then also $y \not\le x_n$ for all large enough n. Hence, $x_i \to [\alpha]$. □

Proposition 5.1.9. *Let (K,o) be a connected rooted graph. Then the following statements are equivalent:*

(1) *Every element in $\overline{(K,o)}$ is represented by a (possibly finite) geodesic path in K;*

(2) *For every clopen subset $S \subseteq \overline{(K,o)}$ the number of minimal elements in S is finite;*

(3) *For every $x,y \in K$ the number of minimal elements in $\mathscr{U}_x \cap \mathscr{U}_y$ is finite.*

In particular, if (K,o) satisfies one (and hence all) of the conditions above, then $\partial(K,o) = \overline{(K,o)} \setminus K$.

Proof. "(1) ⇒ (2)": Assume that every element in $\overline{(K,o)}$ is represented by a (possibly finite) geodesic path in K and that there exists a clopen subset $S \subseteq \overline{(K,o)}$ of K for which $\#T = \infty$, where $T := \{z \in S \mid z \text{ minimal element in } S\}$. By the first assumption we have $\overline{(K,o)} = K \cup \partial(K,o)$. Further, for every boundary point $z \in S \cap \partial(K,o)$ represented by an infinite geodesic path $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ with $\alpha_0 = o$ we find $i \in \mathbb{N}$ with $\alpha_i \in S$ (as S is open). But then $\alpha_i \le z$ and hence $z \notin T$. This implies that $T \subseteq K \cap S$, so $\bigcup_{x\in T}\mathscr{U}_x$ is an infinite open cover of S which has no finite subcover. But by the compactness of $\overline{(K,o)}$ the subset S must be compact as well. This leads to a contradiction.

"(2) ⇒ (3)": This implication is clear.

"(3) ⇒ (1)": Assume that for every $x, y \in K$ the number of minimal elements in $\mathscr{U}_x \cap \mathscr{U}_y$ is finite and let $z \in \overline{(K,o)}$. Define the subset $\mathscr{S} := \{x \in K \mid x \leq z\}$ of K and choose a (possibly finite) enumeration $y_1, y_2, \ldots$ of $\mathscr{S}$. We inductively define elements $x_1 \leq x_2 \leq \ldots$ in $\mathscr{S}$ with $y_i \leq x_{i+1} < z$ for all i. Set $x_1 := o \in \mathscr{S}$ and assume that for $i \in \mathbb{N}$ elements $x_1, \ldots, x_i \in \mathscr{S}$ with $o = x_1 \leq \ldots \leq x_i$ and $y_j \leq x_{j+1} < z$ for $j = 1, \ldots, i-1$ have been defined. The intersection $K \cap \mathscr{U}_{x_i} \cap \mathscr{U}_{y_i}$ must be non-empty and $\mathscr{U}_{x_i} \cap \mathscr{U}_{y_i} \subseteq \bigcup_{y \in \mathscr{U}_{x_i} \cap \mathscr{U}_{y_i} \text{ minimal}} \mathscr{U}_y$ where the union is finite by our assumption. We hence find $y \in K \cap \mathscr{U}_{x_i} \cap \mathscr{U}_{y_i}$ with $z \in \mathscr{U}_y$. Set $x_{i+1} := y$, then this element satisfies the condition $x_1 \leq \ldots \leq x_{i+1}$ and $y_i \leq x_{i+1} < z$. Now, Lemma 5.1.8 implies that the sequence $(x_i)_{i\in\mathbb{N}}$ converges to a boundary point z' which can be represented by an infinite geodesic path. By construction, for every $x \in \mathscr{S}$ one has $x \leq z'$ and for $x \in K \setminus \mathscr{S}$ one has $x \nleq z'$. Hence, $z' = z$.

We have shown the equivalence of the statements (1), (2) and (3). If $\overline{(K,o)}$ satisfies one (and hence all) of these conditions, it is clear that $\partial(K,o) = \overline{(K,o)} \setminus K$. The claim follows. □

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Remark 5.1.10. Let (K,o) be a connected rooted graph. One can show that the set of equivalence classes of infinite geodesic paths in K does not depend on the choice of the root $o \in K$. Indeed, let $o' \in K$ be a second root. Assume that α, β are infinite geodesic paths which are equivalent with respect to o. One finds $M \in \mathbb{N}$ such that for all $n \geq M$ there exist $k_n, l_n \in \mathbb{N}$ with $\alpha_n \leq_o \beta_{k_n} \leq_o \alpha_{l_n}$. That in particular implies that we find a geodesic path starting in α_n, passing β_{k_n} and ending in α_{l_n}. Denote this path by $[\alpha_n, \alpha_{l_n}]$. Now, there is a geodesic path α' starting in o' which eventually flows into α, i.e. there exist $N \in \mathbb{N}$, $i \in \mathbb{Z}$ such that $\alpha'_n = \alpha_{i+n}$ for all $n \geq N$ (see e.g. [33, Lemma E.2]). For $n \geq N - i$ write $[o', \alpha_n]$ for the corresponding head of this path starting in o' and ending in α_n. Then, for $n \geq \max\{M, N+i\}$ we have that the concatenation $[o', \alpha_n][\alpha_n, \alpha_{l_n}]$ is a geodesic path starting in o', passing α_n, passing β_{k_n} and ending in α_{l_n}. We get that $\alpha_n \leq_{o'} \beta_{k_n} \leq_{o'} \alpha_{l_n}$ for all $n \geq \max\{M, N+i\}$. It is then obvious that α and β are equivalent with respect to o'. However, even though the set of equivalence classes of infinite geodesic paths in K does not depend on the choice of the root $o \in K$, the topology of $\overline{(K,o)}$ can; even in the setting of Proposition 5.1.9. Consider for instance the third graph in Figure 5.1. Then the limit of the indicated sequence $(x_i)_{i\in\mathbb{N}} \subseteq K$ depends on whether one views it as a sequence in $\overline{(K,x)}$ or as a sequence in $\overline{(K,y)}$. Note that the connected rooted graphs (K,x) and (K,y) satisfy the equivalent conditions in Proposition 5.1.9 whereas (K,o) does not.

In general, the graph order of a connected rooted graph (K,o) does not necessarily define a (complete) meet-semilattice. However, the graphs that we are mainly interested in in this thesis satisfy this condition (see Example 5.1.11). The meet-semilattice property has the technical advantage that $\mathscr{U}_x \cap \mathscr{U}_y = \mathscr{U}_{x\vee y}$, $P_x P_y = P_{x\vee y}$ if $\{x,y\}$ has an upper bound and $\mathscr{U}_x \cap \mathscr{U}_y = \emptyset, P_x P_y = 0$ else. This in particular implies that

$$*\text{-Alg}(\{P_x \mid x \in K\}) = \text{Span}(\{P_x \mid x \in K\}).$$

Example 5.1.11. *(a)* The graph orders of connected rooted trees define complete meet-semilattices.
(b) Let (W,S) be a Coxeter group, let $\text{Cay}(W,S)$ be the Cayley graph of W with respect to the generating set S and identify elements in W with the corresponding vertices in the graph. It is easy to check that under this identification the graph order of the connected rooted graph $(\text{Cay}(W,S),e)$ coincides with the weak right Bruhat order of (W,S) (see Subsection 2.7.2). In particular, by Proposition 2.7.4, the graph order of $(\text{Cay}(W,S),e)$ defines a complete meet-semilattice.

In combination with the discussion above Proposition 5.1.9 immediately implies the following statement.

Corollary 5.1.12. *Let (K,o) be a connected rooted graph whose graph order defines a complete meet-semilattice. Then $\partial(K,o) = \overline{(K,o)} \setminus K$ and every element in $\partial(K,o)$ is represented by an infinite geodesic path in K.*

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Theorem 5.1.13. *Let (K,o) be a connected rooted graph for which the graph order defines a complete meet-semilattice. Then, $\mathscr{D}(K,o)$ is the universal C^*-algebra generated by projections $(P_x)_{x\in K}$ with $P_xP_y = P_{x\vee y}$ for all $x,y \in K$ where we assume that $P_{x\vee y} = 0$ if the join $x \vee y$ does not exist.*

Proof. Let $\mathscr{A}$ be the universal C*-algebra generated by projections $(\tilde{P}_x)_{x\in K}$ with $\tilde{P}_x\tilde{P}_y = \tilde{P}_{x\vee y}$ for all $x,y \in K$ and let χ be a character on $\mathscr{A}$. It suffices to show that the map $P_x \mapsto \chi(\tilde{P}_x)$ defines a character on $\mathscr{D}(K,o)$. As in the proof of Proposition 5.1.6, define the set $\mathscr{S} := \{x \in K \mid \chi(\tilde{P}_x) = 1\}$. For $i \in \mathbb{N}$ define $x_i := \bigvee_{x\in\mathscr{S}: d_K(x,o)\leq i} x$ and let $z \in \overline{(K,o)}$ be the point this sequence converges to (see Lemma 5.1.8). Further, let ψ be the homeomorphism appearing in the proof of Proposition 5.1.6. Then, $(\psi(z))(P_x) = 1 = \chi(\tilde{P}_x)$ if $x \in \mathscr{S}$ and $(\psi(z))(P_x) = 0 = \chi(\tilde{P}_x)$ if $x \notin \mathscr{S}$. The claim follows. □

We finish this subsection with three spaces that arise as special cases from our construction and which serve as additional motivation.

Example 5.1.14. *(a)* Let S be a countable (discrete) set and let $\{\bullet\}$ be the one-point set. Define a graph $K = (V,E)$ via $V := S \cup \{\bullet\}$ and $E := \{(\bullet,s) \mid s \in S\} \cup \{(s,\bullet) \mid s \in S\}$. Then, by Proposition 5.1.6, $\overline{(K,\bullet)}$ is a compact Hausdorff space. It is easy to check that it identifies with the one-point compactification of the discrete set S.
(b) Let $\mathbf{n} := (n_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$ be a sequence of natural numbers with $n_i > 1$, define $N_0 := 1$, $N_1 := n_1$, $N_2 := n_1n_2$, ... and consider the set $\mathscr{G}_\mathbf{n}$ consisting of all formal sums of the form $\mathbf{x} := \sum_{i=1}^{\infty} x_iN_{i-1}$ with $x_i \in \{0,1,...,n_i - 1\}$ where only finitely many of the coefficients x_i are non-zero. Set $0 := \sum_{i=1}^{\infty} 0N_{i-1}$. The set $\mathscr{G}_\mathbf{n}$ induces a locally finite graph $K = (V,E)$ via $V := \mathscr{G}_\mathbf{n}$ and where distinct vertices $\mathbf{x} = \sum_{i=1}^{\infty} x_iN_{i-1}$ and $\mathbf{y} = \sum_{i=1}^{\infty} y_iN_{i-1}$ are adjacent to each other if and only if there exists $L \in \mathbb{N}$ such that

$$x_1 = y_1, ..., x_L = y_L,\ x_{L+1} = 0 \neq y_{L+1} \text{ and } x_{L+2} = y_{L+2} = x_{L+3} = y_{L+3} = ... = 0$$

or

$$x_1 = y_1, \ldots, x_L = y_L, \; x_{L+1} \neq 0 = y_{L+1} \text{ and } x_{L+2} = y_{L+2} = x_{L+3} = y_{L+3} = \ldots = 0.$$

The graph order of $(K,0)$ gives rise to a complete meet-semilattice. Hence, by Proposition 5.1.6 the compactification $\overline{\mathcal{G}_{\mathbf{n}}} := \overline{(K,0)}$ is a metrizable compact space. It identifies with the Cantor set $\prod_{i=1}^{\infty} \mathbb{Z}/n_i\mathbb{Z}$ where the product is equipped with the product topology and where the $\mathbb{Z}/n_i\mathbb{Z}$ are viewed as discrete topological spaces. Note that the map $\mathcal{G}_{\mathbf{n}} \to \mathcal{G}_{\mathbf{n}}$, $\mathbf{x} \mapsto \mathbf{x} + 1N_0$ (*addition with carryover*) induces a homeomorphism of $\overline{\mathcal{G}_{\mathbf{n}}}$. The induced action of $\mathbb{Z}$ on $\overline{\mathcal{G}_{\mathbf{n}}}$ is the well-studied *odometer action* with respect to $\mathbf{n}$ and the corresponding crossed product $C(\overline{\mathcal{G}_{\mathbf{n}}}) \rtimes_r \mathbb{Z}$ is the *Bunce-Deddens algebra* with respect to $\mathbf{n}$ (see [36] and [65, Sections VIII.4 and V.3]). In the case where n_i is the i-th prime number, $\overline{\mathcal{G}_{\mathbf{n}}}$ identifies with the well-known *profinite completion* of the integers.

(c) The positive integers $\mathbb{N}_{\geq 1}$ induce a locally finite graph $K = (V,E)$ with $V := \mathbb{N}_{\geq 1}$ and

$$E := \{(m,n) \in \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1} \mid m = pn \text{ or } n = pm \text{ for a prime number } p\}.$$

In the graph order with respect to the root $o := 1$ one has $m \leq n$ for $m, n \in \mathbb{N}_{\geq 1}$ if and only if $m \mid n$. It hence defines a complete meet-semilattice where the meet of a subset $T \subseteq K$ is given by the greatest common divisor $\gcd(T)$. By Theorem 5.1.6 the corresponding compactification $\overline{(\mathbb{N}_{\geq 1}, 1)}$ is a metrizable compact space and by Proposition 5.1.9 every element in $\overline{(\mathbb{N}_{\geq 1}, 1)}$ can be represented by a (possibly infinite) geodesic path. The space can hence be viewed as the set of *supernatural numbers*. Further, the semigroup action of $\mathbb{N}_{\geq 1}$ on itself by multiplication extends to a continuous semigroup action on $\overline{(\mathbb{N}_{\geq 1}, 1)}$.

5.1.2. HYPERBOLIC GRAPHS AND TREES

In this subsection we will see that for a hyperbolic connected rooted graph (K,o) the topological spaces $\partial(K,o)$ and $\overline{(K,o)}$ behave well with respect to the hyperbolic (Gromov) boundary $\partial_h K$ and the corresponding compactification $K \cup \partial_h K$ of K. In the case where K is a tree, both spaces turn out to be homeomorphic to each other.

Theorem 5.1.15. *Let (K,o) be a hyperbolic connected rooted graph. Then the map $\phi : \partial(K,o) \to \partial_h K$ given by $\phi([\mathbf{x}]) = [\mathbf{x}]_h$ for a sequence $\mathbf{x}$ which o-converges to infinity is well-defined, continuous and surjective. If the graph is locally finite, then ϕ extends to a continuous surjection $\tilde{\phi} : \overline{(K,o)} \to K \cup \partial_h K$ with $\tilde{\phi}|_K = id_K$.*

Proof. Well-defined: Let $\mathbf{x}$, $\mathbf{y}$ be equivalent sequences which o-converge to infinity. These sequences converge to infinity in the sense that $\liminf_{i,j} \langle x_i, x_j \rangle_o = \infty$ and $\liminf_{i,j} \langle y_i, y_j \rangle_o = \infty$. Indeed, if we assume that $\liminf_{i,j} \langle x_i, x_j \rangle_o < \infty$, then there exist strictly increasing sequences $(m_i)_{i \in \mathbb{N}}, (n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\lim_{i \to \infty} \langle x_{m_i}, x_{n_i} \rangle_o < \infty$. Because $\mathbf{x}$ o-converges to infinity, for every $i \in \mathbb{N}$ there exists an element $z_i \in K$ such that $z_i \leq x_{m_i}$, $z_i \leq x_{n_i}$ and we can choose z_i in such a way that $d_K(z_i, o) \to \infty$. This implies

$$\langle x_{m_i}, x_{n_i} \rangle_o \quad = \quad \frac{1}{2}\left(d_K(x_{m_i}, o) + d_K(x_{n_i}, o) - d_K(x_{m_i}, x_{n_i})\right)$$

$$
\begin{aligned}
&\geq \frac{1}{2}\left(d_K(x_{m_i},o)+d_K(x_{n_i},o)-(d_K(z_i,x_{m_i})+d_K(z_i,x_{n_i}))\right)\\
&= \frac{1}{2}(d_K(x_{m_i},o)+d_K(x_{n_i},o)-(d_K(x_{m_i},o)\\
&\qquad -d_K(z_i,o)+d_K(x_{n_i},o)-d_K(z_i,o)))\\
&= d_K(z_i,o)\\
&\to \infty
\end{aligned}
$$

in contradiction to our assumption. Hence, $\mathbf{x}$ must converge to infinity. In a similar way one checks that the sequence $\mathbf{y}$ converges to infinity and that $\mathbf{x}\sim_h\mathbf{y}$. We get that ϕ is well-defined.

Continuity: Let $([\mathbf{x}^i])_{i\in I}\subseteq\partial(K,o)$ be a net of equivalence classes of sequences $\mathbf{x}^i$, $i\in I$ which o-converge to infinity that converges to a point $z\in\partial(K,o)$. Let $\mathbf{x}$ be a sequence which o-converges to infinity and which represents z. We claim that $[\mathbf{x}^i]_h\to\phi(z)=[\mathbf{x}]_h$. As the sets $\{U(\phi(z),R)\}_{R>0}$ with

$$
\begin{aligned}
U(\phi(z),R):=\{z'\in\partial_h K \quad | \quad &\text{there are sequences } \mathbf{y}^1,\mathbf{y}^2 \text{ converging to infinity}\\
&\text{with } \phi(z)=[\mathbf{y}^1]_h,\ z'=[\mathbf{y}^2]_h \text{ and } \liminf_{i,j\to\infty}\left\langle y^1_i,y^2_j\right\rangle_o>R\}
\end{aligned}
$$

define a neighborhood basis of $\phi(z)$, it suffices to show that for every $R>0$, $[\mathbf{x}^i]_h\in U(\phi(z),R)$ for i large enough. For $R>0$ we find $y_R\in K$ with $y_R\le z$ and $d_K(y_R,o)>R$. Further, as $[\mathbf{x}^i]\to z$, there exists $i_0(R)\in I$ such that $y_R\le[\mathbf{x}^i]$ for every $i\ge i_0(R)$. We claim that $[\mathbf{x}^i]_h\in U(\phi(z),R)$ for every $i\ge i_0(R)$. Assume that this is not the case. Then, as above, for fixed $i\ge i_0(R)$ we find strictly increasing sequences $(m_j)_{j\in\mathbb{N}},(n_j)_{j\in\mathbb{N}}\subseteq\mathbb{N}$ such that $\lim_{j\to\infty}\langle x_{m_j},x^i_{n_j}\rangle_o\le R$. Without loss of generality we can assume that $x_R\le x_{m_j},x^i_{n_j}$ for every $j\in\mathbb{N}$. Then,

$$
\begin{aligned}
\langle x_{m_j},x^i_{n_j}\rangle_o &= \frac{1}{2}\left(d_K(x_{m_j},o)+d_K(x^i_{n_j},o)-d_K(x_{m_j},x^i_{n_j})\right)\\
&\geq \frac{1}{2}\left(d_K(x_{m_j},o)+d_K(x^i_{n_j},o)-(d_K(y_R,x_{m_j})+d_K(y_R,x^i_{n_j}))\right)\\
&= d_K(y_R,o)\\
&> R
\end{aligned}
$$

in contradiction to our assumption. This implies that $[x^i]_h\in U(\phi(z),R)$ for $i\ge i_0(R)$, so $[x^i]_h\to\phi(z)=[\mathbf{x}]_h$.

Surjectivity: That is clear.

We have shown that the map ϕ is well-defined, continuous and surjective. If the graph is locally finite, K is an open subset of $\overline{(K,o)}$. Using this, one checks in the same way as above that the identity map on K continuously extends to a surjection $\tilde{\phi}:\overline{(K,o)}\to K\cup\partial_h K$ with $\tilde{\phi}|_{\partial(K,o)}=\phi$. □

In the case of a tree Theorem 5.1.15 can be strengthened.

Corollary 5.1.16. *Let $(\mathcal{T},o)$ be a connected rooted tree. Then the identity on $\mathcal{T}$ extends to a homeomorphism $\partial(\mathcal{T},o)\cong\partial_h\mathcal{T}$.*

Proof. It suffices to show that the map $[\mathbf{x}] \mapsto [\mathbf{x}]_h$ is injective. By Proposition 5.1.9 it further suffices to consider equivalence classes of infinite geodesic paths. So let α, β be infinite geodesic paths with $[\alpha]_h = [\beta]_h$, i.e. $\sup_i d_{\mathcal{T}}(\alpha_i, \beta_i) < \infty$. Then, since $\mathcal{T}$ is a tree, α and β must eventually flow together which implies that $[\alpha] = [\beta]$. □

Besides from Corollary 5.1.16 the compactification of trees has another useful property.

Lemma 5.1.17. *Let $(\mathcal{T}, o)$ be a rooted tree. Then, every element $z \in \partial(\mathcal{T}, o)$ is* maximal *in the sense that if $z' \in \partial(\mathcal{T}, o)$ is another element with $z \leq z'$ (in the partial order from Lemma 5.1.2), then $z = z'$.*

Proof. Let $z, z' \in \partial(\mathcal{T}, o)$ be elements with $z \leq z'$. By Proposition 5.1.9 there exist infinite geodesic paths α, β which represent z, z'. Assume that $\alpha_0 = \beta_0 = o$. We have $\alpha_1 \leq z$ and hence $\alpha_1 \leq z'$. Because geodesic paths between two points of a tree are unique, β passes α_1 and hence $\alpha_1 = \beta_1$. By the same argument we get $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, ..., therefore $z = z'$. □

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5.2. BOUNDARIES AND COMPACTIFICATIONS OF COXETER GROUPS

The most important graphs that we consider in this chapter are Cayley graphs of Coxeter systems. Even though some of the results hold in greater generality we restrict to finite rank Coxeter groups to avoid technical subtleties and to keep the statements consistent with each other.

Definition 5.2.1. Let (W, S) be a finite rank Coxeter system. As before, let $\mathrm{Cay}(W, S)$ be the Cayley graph of W with respect to the generating set S and view it as a connected rooted graph with root $e \in W$. We call $\partial(W, S) := \partial(\mathrm{Cay}(W, S), e)$ the *boundary of* (W, S) and $\overline{(W, S)} := \overline{(\mathrm{Cay}(W, S), e)}$ the *compactification of* (W, S). For convenience, we will often write ∂W and $\overline{W}$ if the generating set S is clear.

By what we have seen in Subsection 5.1.1, the spaces $\partial(W, S)$, $\overline{(W, S)}$ are metrizable compact spaces and $W \subseteq \overline{(W, S)}$ is both dense and discrete with $\partial(W, S) = \overline{(W, S)} \setminus W$. Further, by Corollary 5.1.12 every element in $\partial(W, S)$ is represented by an infinite geodesic path. In the following we will make use of these facts without any further mention.

5.2.1. LEFT ACTIONS OF COXETER GROUPS ON THEIR COMPACTIFICATION

Theorem 5.2.2. *Let (W, S) be a finite rank Coxeter system. Then the canonical action of W on itself via left multiplication extends to a continuous action $W \curvearrowright \overline{(W, S)}$ with $W.(\partial(W, S)) = \partial(W, S)$.*

Proof. It suffices to show that for every $s \in S$ the map $\mathbf{w} \mapsto s\mathbf{w}$ continuously extends to the boundary. First let α, β be equivalent infinite geodesic paths. It is clear that $(s.\alpha_n)_{n\in\mathbb{N}}$, $(s.\beta_n)_{n\in\mathbb{N}}$ are infinite geodesic paths as well, hence the elements $[s.\alpha] := [(s.\alpha_n)_{n\in\mathbb{N}}] \in \partial W$ and $[s.\beta] := [(s.\beta_n)_{n\in\mathbb{N}}] \in \partial W$ are well-defined. Without loss of generality we can assume that $\alpha_0 = \beta_0 = e$. Then, for every $n \in \mathbb{N}$ there exist minimal $k_n, l_n \in \mathbb{N}$ with $\alpha_n \leq \beta_{k_n}$ and $\beta_n \leq \alpha_{l_n}$.

- *Case 1:* Assume that $s \leq [\alpha] = [\beta]$ and let $\mathbf{v} \leq [s.\alpha]$. Then there exists $N \in \mathbb{N}$ such that $s \leq \alpha_n$ and $\mathbf{v} \leq s\alpha_n$ for all $n \geq N$. Since then $s \leq \alpha_n \leq \beta_{k_n}$, Lemma 2.7.5 implies that $\mathbf{v} \leq s\alpha_n \leq s\beta_{k_n} \leq [s.\beta]$ for all $n \geq N$. The same argument can be used to show that if $\mathbf{v} \leq [s.\beta]$, then $\mathbf{v} \leq [s.\alpha]$ which implies that $[s.\alpha] = [s.\beta]$.
- *Case 2:* Assume that $s \nleq [\alpha] = [\beta]$ and let $\mathbf{v} \leq [s.\alpha]$. Then, there exists $N \in \mathbb{N}$ such that $s \nleq \alpha_n$, $s \nleq \beta_{k_n}$ and $\mathbf{v} \leq s\alpha_n$ for all $n \geq N$. Again, an application of Lemma 2.7.5 to $s\alpha_n$ and $s\beta_{k_n}$ implies that $\mathbf{v} \leq s\alpha_n \leq s\beta_{k_n} \leq [s.\beta]$ for all $n \geq N$. The same argument implies that if $\mathbf{v} \leq [s.\beta]$, then $\mathbf{v} \leq [s.\alpha]$. We get $[s.\alpha] = [s.\beta]$.

We have shown that the map $\mathbf{w} \mapsto s\mathbf{w}$ extends to the boundary via $s.[\alpha] := [(s.\alpha_n)_{n\in\mathbb{N}}] \in \partial W$ for $[\alpha] \in \partial W$. It remains to show that the extension is continuous. Because $\overline{W}$ is metrizable, it suffices to consider sequences. Let $(z^i)_{i\in\mathbb{N}} \subseteq \overline{W}$ be a sequence converging to a boundary point $z \in \partial W$ and let α^i (resp. α) be (possibly finite) geodesic paths representing z^i (resp. z). Again, we can assume that $\alpha_0^i = \alpha_0 = e$.

- *Case 1:* Assume that $s \leq z = [\alpha] \in \partial W$ and let $\mathbf{v} \in W$ with $\mathbf{v} \leq s.z$. There exists $N \in \mathbb{N}$ with $s \leq \alpha_n$ and $\mathbf{v} \leq s\alpha_n$ for all $n \geq N$. Further, for $n \geq N$ there exists $i_0(n) \in \mathbb{N}$ with $s \leq \alpha_n \leq z^i$ for $i \geq i_0(n)$. Lemma 2.7.5 implies that $\mathbf{v} \leq s\alpha_n \leq s.z^i$ for all $n \geq N$, $i \geq i_0(n)$, so in particular $\mathbf{v} \leq s.z^i$ for all $i \geq i_0(N)$. Now, let $\mathbf{v} \in W$ with $\mathbf{v} \nleq s.z$. We have to show that $\mathbf{v} \nleq s.z^i$ for i large enough. Assume without loss of generality that $\mathbf{v} \leq s.z^i$ for all $i \in \mathbb{N}$. There exists $i_0 \in \mathbb{N}$ with $s \leq z^i$ for all $i \geq i_0$ and hence $s \nleq \mathbf{v}$. We get with Lemma 2.7.5 that $s\mathbf{v} \leq z^i$ for all $i \geq i_0$. But $z^i \to z$, so $s\mathbf{v} \leq z$ as well. Again, using Lemma 2.7.5 we get $\mathbf{v} \leq s.z$ in contradiction to our choice of $\mathbf{v}$. This implies that $s.z^i \to s.z$.

We have hence shown that if $z \in \partial W$ with $s \leq z$, then $s.z^i \to s.z$ for every sequence $(z^i)_{i\in\mathbb{N}} \subseteq \overline{W}$ with $z^i \to z$.

- *Case 2:* Assume that $s \nleq z = [\alpha] \in \partial W$ and let $\mathbf{v} \in W$ with $\mathbf{v} \leq s.z$. There exists $N \in \mathbb{N}$ with $s \nleq \alpha_n$ and $\mathbf{v} \leq s\alpha_n$ for all $n \geq N$. Further, for $n \in \mathbb{N}$ there exists $i_0(n) \in \mathbb{N}$ with $\alpha_n \leq z^i$ and $s \nleq z^i$ for all $i \geq i_0(n)$. Lemma 2.7.5 implies that $\mathbf{v} \leq s\alpha_n \leq s.z^i$ for all $n \geq N$, $i \geq i_0(n)$, so in particular $\mathbf{v} \leq s.z^i$ for all $i \geq i_0(N)$. Now, let $\mathbf{v} \in W$ with $\mathbf{v} \nleq s.z$. Again, we have to show that $\mathbf{v} \nleq s.z^i$ for i large enough. Assume without loss of generality that $\mathbf{v} \leq s.z^i$ for all $i \in \mathbb{N}$. Because $\overline{W}$ is (sequentially) compact, we find a subsequence $(s.z^{i_k})_{k\in\mathbb{N}}$ of $(s.z^i)_{i\in\mathbb{N}}$ converging to a boundary point $z' \in \partial W$. Then, $s \leq z'$ and $\mathbf{v} \leq z'$. By what we have shown in *Case 1*, we get that $z^{i_k} \to s.z'$ which implies $s.z' = z$. But then $\mathbf{v} \leq z' = s.z$ in contradiction to our choice of $\mathbf{v}$. This implies that $s.z^i \to s.z$.

The claim follows. □

An immediate implication of Theorem 5.1.15 is the following.

Corollary 5.2.3. *Let (W,S) be a word hyperbolic Coxeter system. Then the map $\tilde{\phi}$: $\overline{(W,S)} \to W \cup \partial_h W$ given by $\tilde{\phi}(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in W$ and $\tilde{\phi}([\alpha]) = [\alpha]_h$ for an infinite geodesic path α is well-defined, continuous, W-equivariant and surjective with $\tilde{\phi}(\partial(W,S)) = \partial_h W$.*

In particular, in the setting of Corollary 5.2.3, by Proposition 2.5.5 the action of W on the compactification $\overline{W}$ is amenable (in the sense of Definition 2.3.7). But we can do better, as we will see in Subsection 5.2.2 and Subsection 5.2.3.

5.2.2. COMBINATORIAL COMPACTIFICATIONS AND HOROFUNCTION COMPACTIFICATIONS

As it turns out, in the case of Cayley graphs of Coxeter groups our construction coincides with Caprace-Lécureux's minimal combinatorial compactification associated with the Coxeter complex of the system (see [37], [130]) and hence relates to Lam-Thomas' results in [129]. Let us elaborate on this, for details of the construction see [37]. Let X be a locally finite building of type (W,S) with chamber set $\mathrm{Ch}(X)$ and denote the corresponding set of spherical residues by $\mathrm{Res}_{\mathrm{sph}}(X)$. Given a spherical residue $\sigma \in \mathrm{Res}_{\mathrm{sph}}(X)$ the associated *combinatorial projection* $\mathrm{proj}_\sigma : \mathrm{Ch}(X) \to \mathrm{St}(\sigma)$ associates to a chamber C the chamber of the star $\mathrm{St}(\sigma)$ of σ (i.e. the set of all residues containing σ in their boundaries) closest to C. It may be extended to a map on the set of all (spherical) residues of X and hence induces a map

$$\pi_{\mathrm{Res}} : \mathrm{Res}_{\mathrm{sph}}(X) \to \prod_{\sigma \in \mathrm{Res}_{\mathrm{sph}}(X)} \mathrm{St}(\sigma), R \mapsto (\mathrm{proj}_\sigma(R))_{\sigma \in \mathrm{Res}_{\mathrm{sph}}(X)}.$$

Equip $\prod_{\sigma \in \mathrm{Res}_{\mathrm{sph}}(X)} \mathrm{St}(\sigma)$ with the product topology where each star is discrete. Then the *minimal combinatorial compactification* $\mathscr{C}_1(X)$ of X can be defined as the closure $\mathscr{C}_1(X) := \overline{\pi_{\mathrm{Res}}(\mathrm{Ch}(X))}$ (see [37, Proposition 2.12]) and the *maximal combinatorial compactification* of X is $\mathscr{C}_{\mathrm{sph}}(X) := \overline{\pi_{\mathrm{Res}}(\mathrm{Res}_{\mathrm{sph}}(X))}$. In particular, $\mathscr{C}_1(X)$ is a closed subset of $\mathscr{C}_{\mathrm{sph}}(X)$. One can show that the $\mathrm{Aut}(X)$-action on X extends in a canonical way to continuous actions on $\mathscr{C}_1(X)$ and $\mathscr{C}_{\mathrm{sph}}(X)$.

The following theorem builds a connection between Caprace-Lécureux's combinatorial compactifications and our construction. It holds in greater generality, but we restrict to compactifications of finite rank Coxeter systems. The theorem states that for a finite rank Coxeter system (W,S) and the *Coxeter complex* Σ Caprace-Lécureux's minimal combinatorial compactification $\mathscr{C}_1(W,S) := \mathscr{C}_1(\Sigma)$ coincides with the space $\overline{(W,S)}$. Its proof is based on the characterization [37, Theorem 3.1] of $\mathscr{C}_1(W,S)$ as the horofunction compactification of the chamber graph (i.e. the set of chambers with the gallery distance) of the locally finite building Σ. Recall that the chamber graph of Σ is just the Cayley graph $\mathrm{Cay}(W,S)$ with the usual metric.

The horofunction compactificiation is constructed as follows. Following [31, Chapter II.8], let (Y,d) be a metric space and consider the space $C(Y)$ of continuous functions on Y equipped with the topology of uniform convergence on bounded sets. Given a base point $y_0 \in Y$ define the subspace $C(Y, y_0) := \{f \in C(Y) \mid f(y_0) = 0\}$. It is homeomorphic to the quotient $C_*(Y)$ of $C(Y)$ by the 1-dimensional subspace of constant functions, so in particular $C(Y, y_0)$ does not depend on the choice of $y_0 \in Y$. The space Y (continuously and injectively) embeds into $C(Y, y_0)$ via $y \mapsto f_y := d(y, \cdot) - d(y, y_0)$. We can hence view Y as a subspace of $C(Y, y_0)$. The closure of Y in $C(Y, y_0)$ is then denoted by $\widehat{Y}$. If Y is proper, $\widehat{Y}$ is a compact Hausdorff space (see [31, Exercise 8.15]) which is called the *horofunction compactification* of Y.

Note that the chamber graph $\mathrm{Cay}(W,S)$ of Σ is a proper metric space and that in this case the topology of uniform convergence on bounded sets on $C(W, e) := C(\mathrm{Cay}(W,S), e)$ coincides with the topology of pointwise convergence. By [37, Theorem 3.1] the minimal combinatorial compactification $\mathscr{C}_1(W,S)$ is $\mathrm{Aut}(\Sigma)$-equivariantly homeomorphic to the horofunction compactification of $\mathrm{Cay}(W,S)$ via $\pi_{\mathrm{Res}}(\mathrm{Ch}(\Sigma)) \ni \pi_{\mathrm{Res}}(\mathbf{w}) \mapsto f_{\mathbf{w}} = \left|\mathbf{w}^{-1}(\cdot)\right| - |\mathbf{w}|$ where $\mathbf{w} \in \mathrm{Ch}(\Sigma) = W$.

Theorem 5.2.4. *Let (W,S) be a finite rank Coxeter system. Then the map*

$$W \to C(W,e), \mathbf{w} \mapsto f_{\mathbf{w}} := \left|\mathbf{w}^{-1}(\cdot)\right| - |\mathbf{w}|$$

induces a W-equivariant homeomorphism between $\overline{(W,S)}$ and $\widehat{\mathrm{Cay}(W,S)}$. In particular, $\overline{(W,S)}$ is W-equivariantly homeomorphic to the minimal combinatorial compactification $\mathscr{C}_1(W,S)$.

Proof. By the compactness of $\overline{W}$ it suffices to show that the map $W \to C(W,e)$, $\mathbf{w} \mapsto f_{\mathbf{w}} := \left|\mathbf{w}^{-1}(\cdot)\right| - |\mathbf{w}|$ extends to a well-defined, bijective and continuous map $\phi : \overline{W} \to \widehat{\mathrm{Cay}(W,S)}$ via $(\phi(z))(\mathbf{v}) := \lim_i \left(\left|\alpha_i^{-1}\mathbf{v}\right| - |\alpha_i|\right)$, where $\mathbf{v} \in W$ and α is a (possibly finite) geodesic path representing $z \in \overline{W}$.

Well-defined: Let α and β be equivalent infinite geodesic paths and consider $\mathbf{v} \in W$ with reduced expression $\mathbf{v} = s_1 \dots s_n$. We have that

$$\left|\alpha_i^{-1}\mathbf{v}\right| - |\alpha_i| = \sum_{j=0}^{n-1} \left(\left|\left(s_j \dots s_1 \alpha_i\right)^{-1} s_{j+1}\right| - \left|\left(s_j \dots s_1 \alpha_i\right)^{-1}\right|\right). \tag{5.2.1}$$

Theorem 5.2.2 implies that for every $j = 0, \dots, n-1$ the sequences $(s_j \dots s_1 \alpha_i)_{i \in \mathbb{N}}$ and $(s_j \dots s_1 \beta_i)_{i \in \mathbb{N}}$ are equivalent infinite geodesic paths. By (5.2.1) it hence suffices to show that $\lim_i \left(\left|\alpha_i^{-1}s\right| - |\alpha_i|\right) = \lim_i \left(\left|\beta_i^{-1}s\right| - |\beta_i|\right)$ for all $s \in S$. Because α and β are equivalent we either have $s \le \alpha_i, \beta_i$ for i large enough or $s \not\le \alpha_i, \beta_i$ for i large enough. In the first case,

$$\lim_{i \to \infty} \left(\left|\alpha_i^{-1}s\right| - |\alpha_i|\right) = (-1) = \lim_{i \to \infty} \left(\left|\beta_i^{-1}s\right| - |\beta_i|\right)$$

and in the second one

$$\lim_{i \to \infty} \left(\left|\alpha_i^{-1}s\right| - |\alpha_i|\right) = 1 = \lim_{i \to \infty} \left(\left|\beta_i^{-1}s\right| - |\beta_i|\right).$$

We get that ϕ is indeed well-defined.

Continuity: Let $\mathbf{v} \in W$ with reduced expression $\mathbf{v} = s_1 \dots s_n$. The equality (5.2.1) implies that for every $z \in \overline{W}$, $(\phi(z))(\mathbf{v}) = \sum_{j=0}^{n-1} \phi(s_j \dots s_1.z)(s_{j+1})$. It hence suffices to show that for every $s \in S$ and every sequence $(z^i)_{i\in\mathbb{N}} \subseteq \overline{W}$ converging to a point $z \in \overline{W}$ the equality $(\phi(z))(s) = \lim_i (\phi(z^i))(s)$ holds. A straightforward modification of the argument above implies the desired statement.

Surjectivity: The surjectivity is clear.

Injectivity: Let $z, z' \in \overline{W}$ with $z \neq z'$. Then there exists $\mathbf{v} \in W$ with $\mathbf{v} \leq z$ but $\mathbf{v} \not\leq z'$. Let α be a (possibly finite) geodesic path representing z and β a (possibly finite) geodesic path representing z' with $\alpha_0 = \beta_0 = e$. Then,

$$(\phi(z))(\mathbf{v}) = \lim_{i\to\infty} \left(\left|\alpha_i^{-1}\mathbf{v}\right| - |\alpha_i|\right) = -|\mathbf{v}| \neq \lim_{i\to\infty} \left(\left|\beta_i^{-1}\mathbf{v}\right| - \left|\beta_i\right|\right) = (\phi(z'))(\mathbf{v})$$

and hence $\phi(z) \neq \phi(z')$. The claim follows. □

Remark 5.2.5. In general it is not true that for a connected rooted graph (K, o) the map $K \to C(K, o)$, $x \mapsto f_x := d_K(x, \cdot) - d_K(x, o)$ extends to a homeomorphism between $\overline{(K, o)}$ and $\widehat{K}$; not even in the locally finite case or in the setting of Proposition 5.1.9. Indeed, as mentioned above the horofunction compactification does not depend on the choice of the base point whereas $\overline{(K, o)}$ can depend on the choice of the root o (see for instance Remark 5.1.10).

5.2.3. AMENABILITY OF THE CANONICAL ACTIONS

The main result in [130, Section 5] states that for a finite rank Coxeter system (W, S) the action of W on the maximal combinatorial compactification $\mathscr{C}_{\mathrm{sph}}(\Sigma)$ is amenable. Because $\mathscr{C}_1(W, S)$ is a closed subset of $\mathscr{C}_{\mathrm{sph}}(\Sigma)$ we deduce with Theorem 5.2.4 that the actions $W \curvearrowright \overline{(W, S)}$ and $W \curvearrowright \partial(W, S)$ are amenable. For the convenience of the reader, we will give a direct proof of this fact. The approach is similar to that in [130].

The following construction is due to Dranishnikov and Januszkiewicz, see [71]. Similar constructions appear in [80] and [141].

For every Coxeter system (W, S) there exists a cell complex $\Sigma(W, S)$, the *Davis complex of* (W, S), which is the geometric realization of a partially ordered set. The construction goes as follows. Consider the set

$$\mathscr{P} := \{\mathbf{w}W_T \mid \mathbf{w} \in W,\ T \subseteq S \text{ with } W_T \text{ finite}\}$$

of special cosets, partially ordered by inclusion. It gives rise to a simplicial complex whose vertex set is $\mathscr{P}$ and whose simplices are all finite chains (i.e. totally ordered subsets) of $\mathscr{P}$. Then, the cell complex $\Sigma(W, S)$ is defined to be the geometric realization of this simplicial complex. There is a canonical action of the group W on $\Sigma(W, S)$ coming from the left action of W on itself. Further, every reflection $t \in \{\mathbf{w}^{-1}s\mathbf{w} \mid s \in S, \mathbf{w} \in W\}$ has its mirror of fixed points and for every mirror the corresponding complement consists of exactly two connected components.

Assume that W is infinite, of finite rank and let $W_0 \lhd W$ be a finite-index normal torsion-free subgroup. By Selberg's Lemma [165] such a subgroup always exists. Let $\mathscr{H}$ be the finite set of orbits for the W_0-action on the set of all mirrors and fix $[h] \in \mathscr{H}$ where h is a mirror. Define the tree $\mathscr{T}_{[h]}$ whose vertices are the connected components of $\Sigma(W,S) \setminus \left(\bigcup_{\gamma \in W_0} \gamma h\right)$ and where two vertices are adjacent if and only if the corresponding connected components intersect after taking their closure in $\Sigma(W,S)$. This indeed defines a tree, as argued in [71]. Further, there exists a W_0-equivariant simplicial map $\Sigma(W,S) \to \mathscr{T}_{[h]}$ sending a vertex to the connected component of $\Sigma(W,S) \setminus \left(\bigcup_{\gamma \in W_0} \gamma h\right)$ it belongs to. The corresponding diagonal map

$$\mu \colon \Sigma(W,S) \to X := \prod_{\Lambda \in \mathscr{H}} \mathscr{T}_\Lambda$$

is a W-equivariant embedding and the ℓ^1-metric on $\prod_{\Lambda \in \mathscr{H}} \mathscr{T}_\Lambda$ restricted to the image of W under μ agrees with with the word metric on W, for details see [71]. Write d_Λ for the graph metric on $\mathscr{T}_\Lambda$, $\Lambda \in \mathscr{H}$ and $d_X := \sum_{\Lambda \in \mathscr{H}} d_\Lambda \circ p_\Lambda$ for the ℓ^1-metric on X where $p_\Lambda \colon X \twoheadrightarrow \mathscr{T}_\Lambda$ denotes the canonical projection. Further set $o := \mu(e)$ and $o_\Lambda := p_\Lambda(o)$ for $\Lambda \in \mathscr{H}$.

Lemma 5.2.6. *For every vertex $x \in X$ the W_0-stabilizer $W_0^x := \{\mathbf{w} \in W_0 \mid \mathbf{w}.x = x\}$ is trivial.*

Proof. For every $\mathbf{w} \in W_0 \setminus \{e\}$ we have

$$\sum_{\Lambda \in \mathscr{H}} d_\Lambda(\mathbf{w}^i.o_\Lambda, o_\Lambda) = d_X(\mu(\mathbf{w}^i), \mu(e)) = |\mathbf{w}^i| \to \infty,$$

because $\mathbf{w}$ is torsion-free. This implies that there exists $\Lambda \in \mathscr{H}$ with $d_\Lambda(\mathbf{w}^i.o_\Lambda, o_\Lambda) \to \infty$ and hence, since

$$\begin{aligned} d_\Lambda(\mathbf{w}^i.o_\Lambda, o_\Lambda) &\leq d_\Lambda(\mathbf{w}^i.o_\Lambda, \mathbf{w}^i.x) + d_\Lambda(\mathbf{w}^i.x, x) + d_\Lambda(x, o_\Lambda) \\ &= d_\Lambda(\mathbf{w}^i.x, x) + 2d_\Lambda(x, o_\Lambda) \end{aligned}$$

for $x \in \mathscr{T}_\Lambda$, $\mathbf{w}$ does not fix any vertex in X. □

The proof of the amenability of the actions $W \curvearrowright \overline{(W,S)}$ and $W \curvearrowright \partial(W,S)$ of a Coxeter system (W,S) requires the following statement from [33].

Proposition 5.2.7 ([33, Proposition 5.2.1])**.** *Let G be a countable group, X a compact G-space and K a countable G-space. Assume that for every $x \in K$ the restricted action of the stabilizer subgroup G^x on X is amenable. Further, assume that there exists a net of Borel maps $\zeta_i \colon X \to \mathrm{Prob}(K)$ (meaning that for every $y \in K$ the function $X \ni x \mapsto \zeta_i^x(y) \in \mathbb{R}$ is Borel) such that*

$$\lim_i \int_X \|g.\zeta_n^x - \zeta_n^{g.x}\|_1 \, dm(x) = 0$$

for every $g \in G$ and every regular Borel probability measure m on X. Then the action $G \curvearrowright X$ is amenable.

We will further need the following well-known statement whose proof we include for convenience.

Lemma 5.2.8. *Let G be a discrete group continuously acting on a compact Hausdorff space X and $N \lhd G$ a finite-index normal subgroup for which the restricted action $N \curvearrowright X$ is amenable. Then G acts amenably as well.*

Proof. By the finiteness of G/N its trivial action on the one-point space $\{\bullet\}$ is amenable. Because N acts amenably on X, we get with [33, Proposition 5.1.11] that the diagonal action of G on $X \times \{\bullet\} \cong X$ is amenable. □

Theorem 5.2.9. *Let (W,S) be a finite rank Coxeter system. Then the actions $W \curvearrowright \overline{(W,S)}$ and $W \curvearrowright \partial(W,S)$ are amenable.*

Proof. If W is finite, the statement is clear. So let us assume that W is infinite, let $W_0 \lhd W$ be a finite-index normal torsion-free subgroup and adopt the notation from before. As mentioned before, the restriction $\mu|_W$ of μ to W is a W-equivariant embedding where X is equipped with the metric d_X. In particular, for every $\Lambda \in \mathscr{H}$, $p_\Lambda \circ (\mu|_W)$ is monotone with respect to the graph order on $(\mathscr{T}_\Lambda, o_\Lambda)$ and the standard order on W. One checks that $p_\Lambda \circ (\mu|_W)$ extends to a well-defined map $\overline{p_\Lambda \circ (\mu|_W)}\colon \overline{W} \to \overline{(\mathscr{T}_\Lambda, o_\Lambda)}$ via $\overline{p_\Lambda \circ (\mu|_W)}(z) := \lim (p_\Lambda \circ (\mu|_W))(\alpha_i)$ where α is an infinite geodesic path representing z. For $z \in \overline{W}$ let $\alpha_z := (\alpha_z^i)_{i\in\mathbb{N}}$ be the unique geodesic path in $\mathscr{T}_\Lambda$ starting in o_Λ and ending in (resp. representing) $\overline{p_\Lambda \circ (\mu|_W)}(z)$, where the path is assumed to eventually become constant if $\overline{p_\Lambda \circ (\mu|_W)}(z) \in \mathscr{T}_\Lambda$. For $n \in \mathbb{N}$ define maps $\lambda_{\Lambda,n}\colon \overline{W} \to \mathrm{Prob}(\mathscr{T}_\Lambda)$ by $\lambda_{\Lambda,n}^z := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{\alpha_z^i} \in \mathrm{Prob}(\mathscr{T}_\Lambda)$. As in [33, Lemma 5.2.6] one checks that $\sup_z \|\mathbf{w}.\lambda_{\Lambda,n}^z - \lambda_{\Lambda,n}^{\mathbf{w}.z}\|_1 \leq 2d_\Lambda(\mathbf{w}.o_\Lambda, o_\Lambda)/n$ for every $\mathbf{w} \in W_0$. We further claim that $\lambda_{\Lambda,n}$ is Borel. Indeed, fix $x \in \mathscr{T}_\Lambda$ and consider the map $f\colon \overline{W} \to \mathbb{R}$ given by $z \mapsto \lambda_{\Lambda,n}^z(x)$. For $z \in \overline{W}$ we have $f(z) = 1/n$ if $d_\Lambda(x, o_\Lambda) < n$, $x \leq \overline{p_\Lambda \circ (\mu|_W)}(z)$ and $f(z) = 0$ in every other case. For $x \in \mathscr{T}_\Lambda$ with $d_\Lambda(x, o_\Lambda) < n$ one gets that for every open set $U \subseteq \mathbb{R}$, $f^{-1}(U) = \overline{W}$ if $\{0, \frac{1}{n}\} \subseteq U$, $f^{-1}(U) = \emptyset$ if $0, \frac{1}{n} \notin U$,

$$f^{-1}(U) = \left\{ z \in \overline{W} \mid x \not\leq \overline{p_\Lambda \circ (\mu|_W)}(z) \right\} = \bigcap_{\mathbf{w}\in W\colon\, p_\Lambda\circ\mu(\mathbf{w})\geq x} \mathscr{U}_{\mathbf{w}}^c$$

if $0 \in U$, $\frac{1}{n} \notin U$ and

$$f^{-1}(U) = \left\{ z \in \overline{W} \mid x \leq \overline{p_\Lambda \circ (\mu|_W)}(z) \right\} = \bigcup_{\mathbf{w}\in W\colon\, p_\Lambda\circ\mu(\mathbf{w})\geq x} \mathscr{U}_{\mathbf{w}}$$

if $0 \notin U$, $\frac{1}{n} \in U$. For $x \in \mathscr{T}_\Lambda$ with $d_\Lambda(x, o_\Lambda) \geq n$ one further has $f^{-1}(U) \in \{\emptyset, \overline{W}\}$. This implies that $\lambda_{\Lambda,n}$ is indeed a Borel map. Now, define Borel maps $\lambda_n\colon \overline{W} \to \mathrm{Prob}(X)$ by $\lambda_n^z(x) := \prod_{\Lambda\in\mathscr{H}} \lambda_{\Lambda,n}^z \circ p_\Lambda(x)$ for $x \in X$. We have

$$\begin{aligned}
\sup_{z\in\overline{W}} \|\mathbf{w}.\lambda_n^z - \lambda_n^{\mathbf{w}.z}\|_1 &= \sup_{z\in\overline{W}} \sum_{x\in X} \left|\lambda_n^z(\mathbf{w}^{-1}.x) - \lambda_n^{\mathbf{w}.z}(x)\right| \\
&= \sup_{z\in\overline{W}} \sum_{x\in X} \left| \prod_{\Lambda\in\mathscr{H}} \lambda_{\Lambda,n}^z \circ p_\Lambda(\mathbf{w}^{-1}.x) - \prod_{\Lambda\in\mathscr{H}} \lambda_{\Lambda,n}^{\mathbf{w}.z} \circ p_\Lambda(x) \right| \\
&\leq \sup_{z\in\overline{W}} \sum_{\Lambda\in\mathscr{H}} \sum_{x\in\mathscr{T}_\Lambda} \left|\lambda_{\Lambda,n}^z(\mathbf{w}^{-1}.x) - \lambda_{\Lambda,n}^{\mathbf{w}.z}(x)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{z\in\overline{W}} \sum_{\Lambda\in\mathcal{H}} \left\| \mathbf{w}.\lambda^{z}_{\Lambda,n} - \lambda^{\mathbf{w}.z}_{\Lambda,n} \right\|_1 \\
&\leq \sum_{\Lambda\in\mathcal{H}} \frac{2d_\Lambda(\mathbf{w}.o_\Lambda, o_\Lambda)}{n} \\
&\leq \frac{2d_X(\mathbf{w}.o,o)}{n} \\
&\to 0
\end{aligned}
$$

for every $\mathbf{w} \in W_0$. As by Lemma 5.2.6 all the stabilizer subgroups W_0^x are trivial, the above implies in combination with Proposition 5.2.7 the amenability of the action $W_0 \curvearrowright \overline{W}$. The amenability of the action $W \curvearrowright \overline{W}$ (resp. $W \curvearrowright \partial W$) then follows from Lemma 5.2.8. □

5.2.4. Smallness at infinity

Recall that an equivariant compactification of a discrete group G is a compact Hausdorff space $\overline{G}$ containing G as a dense open subset for which the left translation action of G on itself extends to a continuous action on $\overline{G}$ (see Subsection 2.3.2). It is further said to be small at infinity if for every net $(g_i)_{i\in I} \subseteq G$ with $g_i \to z \in \overline{G}\setminus G$ and $g' \in G$, one has that $g_i g' \to z$. From what we have seen before it is clear that for every finite rank Coxeter system (W,S) the corresponding space $\overline{(W,S)}$ is an equivariant compactification in the sense of Definition 2.3.9. In this subsection we will be concerned with the question for when $\overline{(W,S)}$ is also small at infinity, as this property will allow us to deduce rigidity properties of Hecke-von Neumann algebras in Chapter 9.

Definition 5.2.10. We call a finite rank Coxeter system (W,S) *small at infinity* if $\overline{(W,S)}$ is small at infinity. If the generating set S is clear, we will also say that W is small at infinity.

Theorem 5.2.11. *Let (W,S) be a finite rank Coxeter system. Then the following statements are equivalent:*

(1) W is small at infinity;

(2) $\#C_W(s) < \infty$ for every $s \in S$.

Here $C_W(s) := \{\mathbf{w} \in W \mid s\mathbf{w} = \mathbf{w}s\}$ denotes the centralizer of s in W.

Proof. "(1) ⇒ (2)": Let $s \in S$ be a generator with $\#C_W(s) = \infty$. By the compactness of $\overline{W}$ one can find a sequence $(\mathbf{w}_i)_{i\in\mathbb{N}} \subseteq C_W(s)$ converging to a boundary point $z \in \partial W$. It can be chosen in such a way that $s \nleq \mathbf{w}_i$ for every $i \in \mathbb{N}$. But then $\mathbf{w}_i s \not\to z$ since $s \nleq z$, i.e. W is not small at infinity.

"(2) ⇒ (1)": Let W not be small at infinity. Choose a convergent sequence $(\mathbf{w}_i)_{i\in\mathbb{N}} \subseteq W$ with limit point $z \in \partial W$ and an element $\mathbf{v} \in W$ such that $\mathbf{w}_i\mathbf{v} \not\to z$. One can assume that $\mathbf{v} = s$ for some $s \in S$ and that there exist $\mathbf{w} \in W$, $i_0 \in \mathbb{N}$ with $\mathbf{w} \leq \mathbf{w}_i$ and $\mathbf{w} \nleq \mathbf{w}_i s$ for all $i \geq i_0$. Further, we can assume that s always cancels the first letter of $\mathbf{w}_i$. Indeed, for $i \geq i_0$, $\mathbf{w}_i$ is of the form $\mathbf{w}_i = \mathbf{w}\mathbf{u}_i$ with $|\mathbf{w}\mathbf{u}_i| = |\mathbf{w}| + |\mathbf{u}_i|$ and the multiplication of $\mathbf{w}\mathbf{u}_i$ with s cancels some letter in the reduced expression $t_1 ... t_n$ for $\mathbf{w}$. As $\mathbf{w}$ consists of finitely many letters, by possibly going over to some subsequence, we can assume that multiplication by s always cancels the same letter, say t_j, in the expression. Then, by possibly replacing $\mathbf{w}_i$ by $(t_1 ... t_{j-1})^{-1}\mathbf{w}_i$, we can further assume that s cancels the first letter of $\mathbf{w}_i$. Call this letter t. We get that for $i \geq i_0$, $\mathbf{w}_i$ is of the form $\mathbf{w}_i = t\mathbf{v}_i$ where $|t\mathbf{v}_i| = |\mathbf{v}_i| + 1$ and $\mathbf{w}_i s = \mathbf{v}_i$. This implies

$$s = \mathbf{w}_i^{-1} t \mathbf{w}_i = (\mathbf{w}_{i_0}^{-1}\mathbf{w}_i)^{-1} s (\mathbf{w}_{i_0}^{-1}\mathbf{w}_i),$$

i.e. $\mathbf{w}_{i_0}^{-1}\mathbf{w}_i \in C_W(s)$ for every $i \geq i_0$. We get that $\#C_W(s) = \infty$. □

Reflection centralizers of Coxeter groups have been studied in [2] and [32]. The main theorem in [32] describes the centralizer $C_W(s)$ of a generator s in a Coxeter group W as a semidirect product of its reflection subgroup by the fundamental group of the connected component of the odd Coxeter diagram of W containing s. In combination with Theorem 5.2.11 this has the following immediate consequence.

Corollary 5.2.12. *Let (W,S) be a finite rank Coxeter system for which the corresponding odd Coxeter diagram contains a cycle. Then $\overline{(W,S)}$ is not small at infinity.*

Remark 5.2.13. In [24] the gradient $\mathscr{S}_p$-property, introduced in [47], [48], was studied by Borst, Caspers and Wasilewski in the context of group von Neumann algebras of Coxeter groups. As proved in [24, Theorem 5.15], for $p \in [1,\infty]$ the quantum Markov semi-group associated with the word length function of the system (W,S) is gradient-$\mathscr{S}_p$ if and only if (W,S) is small at infinity. In [24, Subsection 5.4] necessary and sufficient conditions are given for this to happen. Following [24, Definition 5.5] and [24, Definition 5.6], define $\mathrm{Graph}_S(W)$ to be the complete simplicial graph with vertex set $V = S$ and labels m_{st} associated with the edges (s,t). Let $k \geq 1$ and $s_1, t_1, ..., s_k, t_k \in S$ be given and let $P = (s_1, t_1, s_2, t_2, ..., s_k, t_k)$ be a path in $\mathrm{Graph}_S(W)$ that has even length. Then P is called a *parity path* if its edges all have finite labels, if $s_1 \neq t_1$, ..., $s_k \neq t_k$, if

$$s_{l+1} = \begin{cases} s_l & \text{, if } m_{s_l t_l} \text{ even} \\ t_l & \text{, if } m_{s_l t_l} \text{ odd} \end{cases}$$

for $l = 1, ..., k-1$, and if $t_{l+1} \notin \{s_l, t_l\}$. A parity path is called *cyclic* if

$$\overline{P} := (s_1, t_1, ..., s_k, t_k, s_1, t_1)$$

is a parity path. By [24, Theorem 5.8], if (W,S) admits a cyclic parity path $P = (s_1, t_1, ..., s_k, t_k)$ in $\mathrm{Graph}_S(W)$ for which all labels $m_{s_l t_l}, m_{s_l t_{l+1}}, m_{t_l t_{l+1}}$ are not equal to 2, then (W,S) is not small at infinity. Moreover, by [24, Theorem 5.9], if there does not exist a cyclic parity path in $\mathrm{Graph}_S(W)$ then (W,S) is small at infinity.

Let us collect some other consequences of Theorem 5.2.11.

The proof of the following proposition makes use of the fact that irreducible affine type Coxeter groups arise as subgroups generated by (affine) reflections associated with crystallographic root systems. We discussed the details of this representation in Subsection 2.7.3.

Proposition 5.2.14. *An irreducible Coxeter system of affine type is small at infinity if and only if it is the infinite dihedral group.*

Proof. Let (W, S) be an irreducible affine Coxeter system and denote the associated crystallographic root system by $\Phi \subseteq V$ where V is a finite-dimensional real Euclidean vector space with canonical inner product $\langle \cdot, \cdot \rangle$. The Coxeter diagram is of one of the following forms: $(\tilde{A}_n)_{n\geq 2}$, $(\tilde{B}_n)_{n\geq 3}$, $(\tilde{C}_n)_{n\geq 2}$, $(\tilde{D}_n)_{n\geq 4}$, $(\tilde{E}_n)_{6\leq n\leq 8}$, $\tilde{F}_4$, $\tilde{G}_2$, $\tilde{I}_1$.

- *Case 1:* If the Coxeter system is of the form $(\tilde{A}_n)_{n\geq 2}$, $(\tilde{B}_n)_{n\geq 3}$, $(\tilde{C}_n)_{n\geq 2}$, $(\tilde{D}_n)_{n\geq 4}$, $(\tilde{E}_n)_{6\leq n\leq 8}$, $\tilde{F}_4$ or $\tilde{G}_2$, then $\#S \geq 3$. Therefore, the reflection hyperplane $H_{\alpha,i}$ with $\alpha \in \Phi$, $i \in \mathbb{Z}$ corresponding to a generator $s \in S$ is at least 1-dimensional and one finds an element $\beta \in \Phi$ that is linearly independent from α. We have that $\gamma := \beta - \langle \alpha, \beta \rangle \alpha^\vee \in \Phi$ and the translation $t_{\beta^\vee + \gamma^\vee} = t_{\beta^\vee} t_{\gamma^\vee}$ corresponds to an infinite order element in the Coxeter group W. By

$$\beta^\vee + \gamma^\vee = \frac{2}{\langle \beta, \beta \rangle}\left(2\beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha\right) \in H_{\alpha,0},$$

  the translation $t_{\beta^\vee + \gamma^\vee}$ stabilizes the hyperplane $H_{\alpha,i}$, hence the element commutes with s. It follows from Theorem 5.2.11 that W is not small at infinity.

- *Case 2:* If (W, S) is infinite dihedral, i.e. $W = \langle s, t \mid s^2 = t^2 = e \rangle$, then obviously $\#C_W(s) = \#C_W(t) = 2$.

This finishes the proof. □

Recall that by Corollary 5.2.3 for every word hyperbolic Coxeter system (W, S) the map

$$\tilde{\phi} \colon \overline{(W, S)} \to W \cup \partial_h W$$

given by $\tilde{\phi}(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in W$ and $\tilde{\phi}([\alpha]) = [\alpha]_h$ for an infinite geodesic path α is well-defined, continuous, W-equivariant and surjective with $\tilde{\phi}(\partial(W, S)) = \partial_h W$. The injectivity of $\tilde{\phi}$ gives information on whether or not the system is small at infinity, as the next theorem illustrates.

Theorem 5.2.15. *Let (W, S) be a finite rank Coxeter system. Then (W, S) is small at infinity if and only if W is word hyperbolic and the map $\tilde{\phi}$ (resp. its restriction $\tilde{\phi}|_{\partial(W,S)}$) from Corollary 5.2.3 is a homeomorphism.*

Proof. "⇒": Let (W, S) be small at infinity and assume that the system is not word hyperbolic. By Moussong's characterization of word hyperbolic Coxeter groups Theorem 2.7.16 S contains a subset $T \subseteq S$ such that (W_T, T) is either of affine type with $\#T \geq 3$ or the Coxeter system decomposes as $(W_T, T) = (W_{T'} \times W_{T''}, T' \cup T'')$ with both $W_{T'}$ and $W_{T''}$ infinite. In the first case we deduce with Proposition 5.2.14

that W is not small at infinity which contradicts our assumption. In the second case the same contradiction follows from Theorem 5.2.11 and $C_W(s) \supseteq W_s \times W_{T''}$ for every $s \in T'$. Hence, (W,S) must be word hyperbolic. It remains to show that the map $\tilde{\phi}$ is injective. For this, let α and β be infinite geodesic paths with $[\alpha]_h = [\beta]_h$. By $\sup_i |\alpha_i^{-1}\beta_i| < \infty$ the set $\{\alpha_i^{-1}\beta_i \mid i \in \mathbb{N}\} \subseteq W$ is bounded with respect to the word metric on W. We hence find a strictly increasing sequence $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and an element $\mathbf{w} \in W$ with $\alpha_{i_k}^{-1}\beta_{i_k} = \mathbf{w}$ for all $k \in \mathbb{N}$. But (W,S) is small at infinity, so $[\beta] = \lim_k \beta_{i_k} = \lim_k \alpha_{i_k}\mathbf{w} = [\alpha]$. This implies that $\tilde{\phi}$ is indeed injective.

"$\Leftarrow$": By Proposition 2.5.6 the hyperbolic compactification of a word hyperbolic group is small at infinity. Hence, if (W,S) is word hyperbolic and the map $\tilde{\phi}$ is a homeomorphism, then (W,S) is small at infinity. □

Proposition 5.2.16. *Let (W,S) be a finite rank Coxeter system that is a free product of finite Coxeter groups, meaning that S is the disjoint union of non-empty subsets $S_1,\dots,S_n \subseteq S$ whose corresponding special subgroups $W_{S_1},\dots,W_{S_n}$ are all finite with $W = W_{S_1} \star \dots \star W_{S_n}$. Then (W,S) is small at infinity.*

Proof. Let (W,S) be an irreducible Coxeter system that is a free product of finite Coxeter groups. The corresponding Cayley graph $\mathrm{Cay}(W,S)$ is locally finite and hyperbolic. By Theorem 5.2.15 it suffices to prove the injectivity of the map $\tilde{\phi}$. Let α and β be two infinite geodesic paths with $\alpha \sim_h \beta$. For every $i \in \mathbb{N}$ let $s_1 \dots s_i$ be a reduced expression for α_i and let $t_1 \dots t_i$ be a reduced expression for β_i. It is clear that s_1 and t_1 must lie in the same component of the free product. The same is true for s_2, t_2, ... As the free product components W_{S_1}, ..., W_{S_n} are finite, there exists $i \in \mathbb{N}$ such that $s_1,\dots,s_i$ (and hence $t_1,\dots,t_i$) all lie in the same component and such that s_{i+1} (resp. t_{i+1}) lies in a different component. By $\sup_j |\alpha_j^{-1}\beta_j| < \infty$ we then get $s_i \dots s_1 t_1 \dots t_i = e$. Proceeding like this, one concludes that there exists an increasing sequence $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $\alpha_{i_k} = \beta_{i_k}$ for every $k \in \mathbb{N}$. This implies $[\alpha] = [\beta]$, i.e. $\tilde{\phi}$ is injective. □

Corollary 5.2.17. *An irreducible finite rank right-angled Coxeter system is small at infinity if and only if it is a free product of finite Coxeter groups.*

Proof. "$\Leftarrow$": This follows from Lemma 5.2.16. "$\Rightarrow$": Let (W,S) be an irreducible right-angled Coxeter system that is not a free product of finite Coxeter groups. One easily checks that S contains elements r,s,t with coefficients $m_{rs} = m_{rt} = 2$ and $m_{st} = \infty$. In particular, $C_W(r) \supseteq \langle s,t\rangle \cong \mathbf{D}_\infty$ where $\mathbf{D}_\infty$ denotes the infinite dihedral group. But then $\#C_W(r) = \infty$, so W is not small at infinity by Theorem 5.2.11. □

Remark 5.2.18. Not every Coxeter system that is small at infinity is a free product of finite Coxeter groups. Consider for instance the group W represented by

$$\langle r,s,t \mid m_{rr} = m_{ss} = m_{tt} = 2, m_{rs} = 3, m_{rt} = 2, m_{st} = \infty \rangle .$$

It is irreducible and non-affine. Obviously, all of its reflection centralizers are finite, so W is small at infinity. However, W can not be decomposed into a non-trivial free product because, by $m_{rs} = 3$ and $m_{rt} = 2$, then the generators r, s and t would all have to sit in the same component of that decomposition.

5.2.5. Boundary actions of Coxeter groups

In this subsection we study two classes of Coxeter systems (W,S) whose associated boundary $\partial(W,S)$ is a W-boundary in the sense of Furstenberg (see Subsection 2.3.3). We will further consider the question for topological freeness of the action $W \curvearrowright \partial(W,S)$. Recall that the notion of boundary actions (as well as topological freeness) plays a crucial rule in Kalantar-Kennedy's approach to the C*-simplicity problem for (discrete) groups in [120]. Our approach to the simplicity of Hecke C*-algebras of right-angled Coxeter groups in Section 6.2 has a similar flavor.

To simplify the statements and proofs of this and the later chapters, we introduce the following notion.

Definition 5.2.19. Let (W,S) be a right-angled finite rank Coxeter system. A *path* $s_1...s_n \in W$ *in the Coxeter diagram of* (W,S) is a product of generators $s_1,...,s_n \in S$ with $m_{s_i s_{i+1}} = \infty$ for $i = 1,...n-1$. We say that the path is *closed* if $m_{s_1 s_n} = \infty$ and that the path *covers the whole graph* if $\{s_1,...,s_n\} = S$.

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Remark 5.2.20. Let (W,S) be a right-angled finite rank Coxeter system. For a closed path $\mathbf{g} := s_1...s_n \in W$ in the Coxeter diagram of (W,S) that covers the whole graph we have that $|s\mathbf{g}| > |\mathbf{g}|$ for every $s \in S \setminus \{s_1\}$ and $C_W(\mathbf{g}) = \{\mathbf{g}^i \mid i \in \mathbb{Z}\}$. In particular, $|\mathbf{g}^n| = |n|\,|\mathbf{g}|$ for every $n \in \mathbb{Z}$.

In the case of an irreducible right-angled Coxeter system, we can completely characterize when the corresponding action on the boundary is a boundary action. Note that the only Coxeter group generated by one element is the finite group $\mathbb{Z}_2$ whose boundary is empty.

Theorem 5.2.21. *Let* (W,S) *be a finite rank right-angled irreducible Coxeter system. Then the following statements hold:*

- *If* $\#S = 2$*, then the action* $W \curvearrowright \partial(W,S)$ *is minimal but not strongly proximal;*
- *If* $\#S \geq 3$*, then the action* $W \curvearrowright \partial(W,S)$ *is a boundary action.*

Proof. In the case $\#S = 2$ the Coxeter group W is the infinite dihedral group

$$\mathbf{D}_\infty = \langle s,t \mid s^2 = t^2 = e \rangle$$

whose boundary $\partial\mathbf{D}_\infty$ consists of the two points $z_1 := stst...$ and $z_2 := tsts....$ It is clear that the action $\mathbf{D}_\infty \curvearrowright \partial\mathbf{D}_\infty$ is minimal. It is not strongly proximal because for the probability measure $\mu := \frac{1}{2}(\delta_{z_1} + \delta_{z_2}) \in \mathrm{Prob}(\partial\mathbf{D}_\infty)$ the equalities $s.\mu = t.\mu = \mu$ hold, i.e. $\overline{W.\mu} = \{\mu\}$.

Let us now assume that (W,S) is a right-angled irreducible Coxeter system with $\#S \geq 3$. Recall that if we have cancellation of the form $s_1...s_n = s_1...\widehat{s_i}...\widehat{s_j}...s_n$ for $s_1,...,s_n \in S$, then $s_i = s_j$ and s_i commutes with every letter in the reduced expression for $s_{i+1}...s_{j-1}$ (see Subsection 2.7.4). In the following we will often implicitly make use of this property.

Minimality: Let α and β be arbitrary infinite geodesic paths with $\alpha_0 = \beta_0 = e$. We have to show that $[\beta] \in \overline{W.[\alpha]}$. Since S is finite, we find $t \in S$ and a strictly increasing sequence $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $t \leq_L \beta_{i_k}$ for every $k \in \mathbb{N}$. Further, let $t' := \alpha_1 \in S$ and choose a path $s_0...s_{n+1}$ in the Coxeter diagram of (W,S) which covers the whole graph with $s_0 = t'$, $s_{n+1} = t$. We claim that $(\beta_{i_k} s_n...s_1).[\alpha] \to [\beta]$. Indeed, by the choice of $s_1,...,s_n$ one gets $\beta_{i_k} \leq (\beta_{i_k} s_n...s_1)\alpha_j \leq (\beta_{i_k} s_n...s_1).[\alpha]$ for all $j,k \in \mathbb{N}$, so for every $\mathbf{w} \in W$ with $\mathbf{w} \leq [\beta]$ one eventually has $(\beta_{i_k} s_n...s_1).[\alpha] \in \mathcal{U}_{\mathbf{w}} = \{z \in \overline{W} \mid \mathbf{w} \leq z\}$.

Now let $\mathbf{w} \in W$ with $\mathbf{w} \nleq [\beta]$ and let $\mathbf{w} = t_1...t_n$ be a reduced expression for $\mathbf{w}$. We have to show that $\mathbf{w} \nleq (\beta_{i_k} s_n...s_1).[\alpha]$ eventually. Assume that this is not the case. By possibly going over to a subsequence we can then assume that $\mathbf{w} \leq (\beta_{i_k} s_n...s_1).[\alpha]$ for all $k \in \mathbb{N}$. Let us proceed inductively:

- By the choice of $s_1,...,s_n$ one either has $t_1 \leq \beta_{i_k}$ or $t_1 = s_n$ and t_1 commutes with every letter of β_{i_k}. Only the first case is possible because $m_{s_n,t} = \infty$, so $t_1 \leq \beta_{i_k}$.
- Further, one either has $t_1 t_2 \leq \beta_{i_k}$ or $t_2 = s_n$ and t_2 commutes with every letter of $t_1\beta_{i_k}$. In the second case we would get that $t_1 = t$ and that t commutes with every letter of β_{i_k}. But for $k \geq 1$ the letter t appears more than once in the reduced expression for β_{i_k} which leads to a contradiction. Hence, $t_1 t_2 \leq \beta_{i_k}$ for $k \geq 1$.

Proceeding like this, we get that $\mathbf{w} \leq \beta_{i_k}$ for large enough k, in contradiction to $\mathbf{w} \nleq [\beta]$. Therefore, for every $\mathbf{w} \in W$ with $\mathbf{w} \nleq [\beta]$, $(\beta_{i_k} s_n...s_1).[\alpha] \in \mathcal{U}_{\mathbf{w}}^c$ eventually. This implies that indeed $(\beta_{i_k} s_n...s_1).[\alpha] \to [\beta]$, i.e. $[\beta] \in \overline{W.[\alpha]}$.

Strong proximality: We have to show that for every probability measure $\mu \in \mathrm{Prob}(\partial W)$ there exists $z \in \partial W$ with $\delta_z \in \overline{W.\mu}$ where the closure is taken in the weak-$*$ topology. The argument is similar to the one above. Choose a closed path $s_1...s_n$ in the Coxeter diagram of (W,S) that covers the whole graph. Obviously, the sequences $(\mathbf{g}^k)_{k\in\mathbb{N}}$ and $(\mathbf{g}^{-k})_{k\in\mathbb{N}}$ converge to boundary points $\mathbf{g}^\infty$ and $\mathbf{g}^{-\infty}$. For $z \in \partial W$ we either have $s_1 \leq \mathbf{g}^k.z$ for some $k \in \mathbb{N}$ or $z = \mathbf{g}^{-\infty}$. In the first case, $\mathbf{g}^k.z \to \mathbf{g}^\infty$ and in the second case $\mathbf{g}^k.z \to \mathbf{g}^{-\infty}$. This implies that for $\mu \in \mathrm{Prob}(\partial W)$ there exists $\lambda \in [0,1]$ with

$$\lambda\delta_{\mathbf{g}^\infty} + (1-\lambda)\delta_{\mathbf{g}^{-\infty}} = \lim_{k\to\infty} \mathbf{g}^k.\mu \in \overline{W.\mu}.$$

Now, choose a second closed path $t_1...t_m$ in the Coxeter diagram of (W,S) that covers the whole graph with $t_1 \notin \{s_1,s_n\}$ and set $\mathbf{h} := t_1...t_m$. Again, the sequences $(\mathbf{h}^k)_{k\in\mathbb{N}}$ and $(\mathbf{h}^{-k})_{k\in\mathbb{N}}$ converge to boundary points $\mathbf{h}^\infty$ and $\mathbf{h}^{-\infty}$. Further, $\mathbf{h}^k.\mathbf{g}^\infty \to \mathbf{h}^\infty$ and $\mathbf{h}^k.\mathbf{g}^{-\infty} \to \mathbf{h}^\infty$ from which we conclude that

$$\delta_{\mathbf{h}^\infty} = \lim_{k\to\infty} \left(\lambda\delta_{\mathbf{g}^\infty} + (1-\lambda)\delta_{\mathbf{g}^{-\infty}}\right) \in \overline{W.\mu}.$$

The claim follows. □

For Coxeter systems which are small at infinity a characterization of the form as in Theorem 5.2.21 is possible as well. Note that by Theorem 5.2.11 and Proposition 5.2.14 the only amenable finite rank irreducible Coxeter groups that are small at infinity are either the finite ones or the infinite dihedral group which is already covered by Theorem 5.2.21.

Theorem 5.2.22. *Let (W,S) be a non-amenable finite rank Coxeter system that is small at infinity. Then the action $W \curvearrowright \partial(W,S)$ is a boundary action.*

Proof. By Theorem 5.2.15 the group W is word hyperbolic and the boundary $\partial(W,S)$ coincides with the hyperbolic boundary $\partial_h W$. It is well-known that the action of a non-amenable word hyperbolic group is a boundary action (see for instance [120, Remark 5.6]). This proves the statement. □

Remark 5.2.23. Let (W,S) be a right-angled irreducible Coxeter system with $3 \leq \#S < \infty$. Note that by the same argument as in the proof of Theorem 5.2.21 the action $W \curvearrowright \overline{(W,S)}$ is strongly proximal. Indeed, the elements $\mathbf{g}$ and $\mathbf{h}$ appearing in the proof of Theorem 5.2.21 have the property that the limits $\mathbf{g}^{\pm\infty} := \lim \mathbf{g}^{\pm l}$ and $\mathbf{h}^{\pm\infty} := \lim \mathbf{h}^{\pm l}$ exist and that $\mathbf{g}^k.z \to \mathbf{g}^{\infty}$ for every $z \in \overline{(W,S)} \setminus \{\mathbf{g}^{-\infty}\}$ and $\mathbf{h}^k.z \to \mathbf{h}^{\infty}$ for every $z \in \overline{(W,S)} \setminus \{\mathbf{h}^{-\infty}\}$. Further, $\mathbf{h}^{-\infty} \neq \mathbf{g}^{\pm\infty}$. We deduce that the action $W \curvearrowright \overline{(W,S)}$ is strongly proximal. If the Coxeter system (W,S) is non-amenable and small at infinity, the strong proximality of the action $W \curvearrowright \overline{(W,S)}$ also holds. That follows from Theorem 5.2.15 and [88, Corollaire 20].

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We now turn our attention to the question for topological freeness of the natural action of a Coxeter group on its compactification and its boundary.

Lemma 5.2.24. *Let (W,S) be a finite rank Coxeter system. Then the natural action of W on its compactification $\overline{(W,S)}$ is topologically free.*

Proof. The statement immediately follows from the fact that W is a dense subset of $\overline{W}$. □

Again, in the right-angled case we can characterize when the corresponding action of the Coxeter group on its boundary is topologically free. The argument requires a technical lemma.

Lemma 5.2.25. *Let (W,S) be a finite rank right-angled irreducible Coxeter system. For $\mathbf{w} \in W \setminus \{e\}$, $z \in \partial(W,S)$ with $\mathbf{w}.z = z$ there exist elements $\mathbf{u},\mathbf{v} \in W$ with $\mathbf{w} = \mathbf{u}\mathbf{v}^{-1}$, $|\mathbf{w}| = |\mathbf{u}| + |\mathbf{v}|$ and $\mathbf{u},\mathbf{v} \leq z$.*

Proof. Let $\mathbf{w} \in W$, $z \in \partial(W,S)$ be elements with $\mathbf{w}.z = z$ and let $\mathbf{w} = s_1 \dots s_n$ be a reduced expression for $\mathbf{w}$. We claim that for every $1 \leq k \leq n$ we find integers $i_1 < \dots < i_l$ and $j_1 < \dots < j_m$ such that

$$\{n-k+1, \dots, n\} = \left\{i_1, \dots, i_l, j_1, \dots, j_m\right\}, \mathbf{w} = (s_1 \dots s_{n-k})(s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m})$$

is a reduced expression for $\mathbf{w}$, $s_{j_m} \dots s_{j_1} \leq z$ and $s_{i_1} \dots s_{i_l} \leq (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z$. We prove this by induction over k.

- For $k = 1$ we have that either $s_n \leq z$ or $s_n \nleq z$. In the first case set $l = 0$, $m = 1$ and $j_1 = n$. Then, $\mathbf{w} = (s_1 \dots s_{n-1})\, s_{j_1}$ is a reduced expression for $\mathbf{w}$ with $s_{j_1} \leq z$. In the second case set $l = 1$, $m = 0$ and $i_1 = n$. Then again, $\mathbf{w} = (s_1 \dots s_{n-1}) s_{i_1}$ is a reduced expression for $\mathbf{w}$ with $s_{i_1} \leq s_{i_1}.z$. We get that for $k = 1$ the claimed statement holds.

- Now assume that the claim holds for $k \in \mathbb{N}$, i.e. we have $i_1 < \dots < i_l$ and $j_1 < \dots < j_m$ with $\{n-k+1, \dots, n\} = \{i_1, \dots, i_l, j_1, \dots, j_m\}$ such that
$$\mathbf{w} = (s_1 \dots s_{n-k})(s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m})$$
is a reduced expression for $\mathbf{w}$, $s_{j_m} \dots s_{j_1} \leq z$ and $s_{i_1} \dots s_{i_l} \leq (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z$. Now, either $s_{n-k} \leq (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z$ or $s_{n-k} \not\leq (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z$. In the first case, since $s_{n-k}(s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m})$ is reduced and $s_{i_1} \dots s_{i_l} \leq (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z$, we get that s_{n-k} commutes with $s_{i_1} \dots s_{i_l}$ and $s_{n-k} \leq (s_{j_1} \dots s_{j_m}).z$. Hence,
$$\{n-k, \dots, n\} = \{i_1, \dots, i_l, n-k, j_1, \dots, j_m\},$$
$\mathbf{w} = (s_1 \dots s_{n-k-1})(s_{i_1} \dots s_{i_l})(s_{n-k} s_{j_1} \dots s_{j_m})$ is a reduced expression for $\mathbf{w}$, $s_{j_m} \dots s_{j_1} s_{n-k} \leq z$ and $s_{i_1} \dots s_{i_l} \leq (s_{i_1} \dots s_{i_l})(s_{n-k} s_{j_1} \dots s_{j_m}).z$. In the second case,
$$\{n-k, \dots, n\} = \{n-k, i_1, \dots, i_l, j_1, \dots, j_m\},$$
$\mathbf{w} = (s_1 \dots s_{n-k-1})(s_{n-k} s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m})$ is a reduced expression for $\mathbf{w}$, $s_{j_1} \dots s_{j_m} \leq z$ and $s_{n-k} s_{i_1} \dots s_{i_l} \leq (s_{n-k} s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z$. In both cases we get that the claim also holds for $k+1$.

This completes the induction argument.

For $k = n$ we get that there exist $i_1 < \dots < i_l$ and $j_1 < \dots < j_m$ with $\{1, \dots, n\} = \{i_1, \dots, i_l, j_1, \dots, j_m\}$ such that $\mathbf{w} = (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m})$ is a reduced expression for $\mathbf{w}$, $s_{j_m} \dots s_{j_1} \leq z$ and $s_{i_1} \dots s_{i_l} \leq (s_{i_1} \dots s_{i_l})(s_{j_1} \dots s_{j_m}).z = \mathbf{w}.z = z$. The lemma then follows via $\mathbf{u} := s_{i_1} \dots s_{i_l}$ and $\mathbf{v} := s_{j_m} \dots s_{j_1}$. □

Proposition 5.2.26. *Let (W, S) be a right-angled irreducible Coxeter system with $2 \leq \#S < \infty$. Then the action $W \curvearrowright \partial(W, S)$ is topologically free if and only if $\#S \geq 3$.*

Proof. "⇒": Again, for $\#S = 2$ the Coxeter group W is the infinite dihedral group
$$\mathbf{D}_\infty = \langle s, t \mid s^2 = t^2 = e \rangle$$
with boundary $\partial \mathbf{D}_\infty = \{z_1, z_2\}$ where $z_1 := stst\dots$ and $z_2 := tsts\dots$. Obviously, $\partial \mathbf{D}_\infty$ carries the discrete topology and $(\partial \mathbf{D}_\infty)^{st} = \{z_1, z_2\}$. Hence, the action is not topologically free.

"⇐": Let $\#S \geq 3$ and assume that the action is not topologically free. We find $\mathbf{w} \in W \setminus \{e\}$ such that $(\partial W)^{\mathbf{w}}$ contains an inner point. Without loss of generality we can assume that $\mathbf{w}$ with that property has minimal length. Fix some inner point $z \in (\partial W)^{\mathbf{w}}$. By Lemma 5.2.25 there exist $\mathbf{u}, \mathbf{v} \in W$ with $\mathbf{w} = \mathbf{u}\mathbf{v}^{-1}$, $|\mathbf{w}| = |\mathbf{u}| + |\mathbf{v}|$ and $\mathbf{u}, \mathbf{v} \leq z$. Let $\mathbf{u} = s_1 \dots s_n$, $\mathbf{v} = t_1 \dots t_m$ be reduced expressions for $\mathbf{u}$, $\mathbf{v}$. Without loss of generality one can assume that $n \leq m$. We claim that every letter of $\mathbf{u}$ commutes with every letter of $\mathbf{v}$ and that the letters are pairwise different.

- If $s_1 = t_1$, then $(\partial W)^{(s_2 \dots s_n)(t_m \dots t_2)} = s_1.(\partial W)^{\mathbf{w}}$. But we assumed $\mathbf{w}$ to have minimal length, so $s_1 \neq t_1$. By $s_1, t_1 \leq z$ for $z \in (\partial W)^{\mathbf{w}}$ we further get $m_{s_1 t_1} = 2$.
- If $s_2 = t_1$, then $(\partial W)^{(s_1 s_3 \dots s_n)(t_m \dots t_2)} = s_2.(\partial W)^{\mathbf{w}}$. Again, by the minimality of $\mathbf{w}$ we get $s_2 \neq t_1$ with $m_{s_2 t_1} = 2$. In the same way, $t_2 \neq s_1$, $m_{s_1 t_2} = 2$ and $s_2 \neq t_2$, $m_{s_2 t_2} = 2$.

- ....

Proceeding like this we find that every letter of $\mathbf{u}$ commutes with every letter of $\mathbf{v}$ and that the letters are pairwise different.

Claim. We have $|\mathbf{u}^n| = n|\mathbf{u}|$, $|\mathbf{v}^n| = n|\mathbf{v}|$ and $\mathbf{u}^n, \mathbf{v}^n \leq z$ for every $n \in \mathbb{N}$.

Proof of Claim. Let α be an infinite geodesic path representing z with $\alpha_0 = e$. By $\mathbf{u}, \mathbf{v} \leq z$ and the above we can assume that $\alpha_l = t_1 ... t_m s_1 ... s_n \mathbf{w}_l$ for $l \geq m+n+1$ with $|\alpha_l| = |\mathbf{u}| + |\mathbf{v}| + |\mathbf{w}_l|$. The identities $\mathbf{u} \leq z$ and $z = \mathbf{u}\mathbf{v}^{-1} z$ imply that for every $i \in \{1, ..., n-1\}$ one has $s_i s_{i+1} ... s_n (t_m ... t_1) z = (s_{i-1} ... s_1) z \geq s_i$, so for l large enough $s_i \leq s_i s_{i+1} ... s_n s_1 ... s_n \mathbf{w}_l = s_i s_{i+1} ... s_n \mathbf{u}\mathbf{w}_l$. We get $|s_i (s_{i+1} ... s_n) \mathbf{u}| = |(s_{i+1} ... s_n) \mathbf{u}| + 1$ and hence (via induction over i, starting with $i = n$) that $\left|\mathbf{u}^2\right| = 2|\mathbf{u}|$. This implies $|\mathbf{u}^n| = n|\mathbf{u}|$ for every $n \in \mathbb{N}$ and in a similar way $|\mathbf{v}^n| = n|\mathbf{v}|$ for every $n \in \mathbb{N}$. Now, because each letter of $\mathbf{u}$ commutes with each letter of $\mathbf{v}$, we have $\mathbf{u}^{-n} z = \mathbf{v}^{-n} z \geq \mathbf{u}$ for every $n \in \mathbb{N}$. Inductively we get that $\mathbf{u}^n \leq z$ for every $n \in \mathbb{N}$. In a similar way, $\mathbf{v}^n \leq z$ for every $n \in \mathbb{N}$. The claim follows.

The claim in particular implies that $\mathbf{v}^{-1}.z = z$ and hence $z \in (\partial W)^{\mathbf{u}}$. But then $\mathbf{w} = \mathbf{u}$ and $\mathbf{v} = e$ by the minimality of $\mathbf{w}$ and $n \leq m$. Heuristically, z starts with arbitrarily large powers of $\mathbf{w}$, but there can also appear other expressions in front of z. To make this precise, for every $i \in \mathbb{N}$ one can find $\mathbf{w}_i \in W$ with $\mathbf{w}_i \mathbf{w} = \mathbf{w}\mathbf{w}_i$ and $|\mathbf{w}_i \mathbf{w}| = |\mathbf{w}_i| + |\mathbf{w}|$ such that $\mathbf{w}_i \mathbf{w}^i \to z$. Let $s, t \in S$ with $s \leq_L \mathbf{w}$, $m_{st} = \infty$ and write $(ts)^\infty := \lim_k (ts)^k \in \partial W$. Assume that $\mathbf{w}$ is not of the form $\mathbf{w} = (st)^l$ for some $l \in \mathbb{N}$. Then $\mathbf{w}_i \mathbf{w}^i (ts)^\infty \notin (\partial W)^{\mathbf{w}}$ for every $i \in \mathbb{N}$. But $\mathbf{w}_i \mathbf{w}^i (ts)^\infty \in \partial W \setminus (\partial W)^{\mathbf{w}}$ is a sequence converging to z which contradicts our assumption that z is an inner point. Hence, $\mathbf{w} = (st)^l$ for some $l \in \mathbb{N}$. By the minimality of $\mathbf{w}$, $l = 1$, so in particular $\mathbf{w}_i (st)^i \to z$. Because $\#S \geq 3$ one can find $r \in S$ such that either $m_{sr} = \infty$ or $m_{tr} = \infty$. If $m_{sr} = \infty$, then $\mathbf{w}_i (st)^i s (rs)^\infty \in \partial W \setminus (\partial W)^{\mathbf{w}}$ is a sequence converging to z and if $m_{tr} = \infty$, then $\mathbf{w}_i (st)^i (rt)^\infty \in \partial W \setminus (\partial W)^{\mathbf{w}}$ is a sequence converging to z where $(rs)^\infty := \lim_k (rs)^k$ and $(rt)^\infty := \lim_k (rt)^k$. In both cases z turns out not to be an inner point, in contradiction to our assumption. Hence, the action $W \curvearrowright \partial W$ must be topologically free. □

Remark 5.2.27. The proof of Proposition 5.2.26 is direct and only uses combinatorial arguments. We chose to present it that way because of its self-containedness. However, the same statement can also be shown by an operator algebraic approach. Indeed, if (W, S) is an irreducible right-angled Coxeter system with $3 \leq \#S < \infty$, then W is C^*-simple (see for instance [80], [101], [61] or [49]). The C^*-simplicity and the minimality of the action $W \curvearrowright \partial(W, S)$ then imply with [30, Theorem 7.1] that the reduced crossed product $C(\partial(W, S)) \rtimes_r W$ is simple. By Theorem 5.2.9 and [33, Theorem 4.3.4] the reduced crossed product coincides with the universal one. The topological freeness of the action $W \curvearrowright \partial(W, S)$ hence follows with [9, Theorem 2].

Lemma 5.2.28. *Let (W, S) be a finite rank non-amenable Coxeter system that is small at infinity. Then the action $W \curvearrowright \partial(W, S)$ is topologically free.*

Proof. As in the proof of Theorem 5.2.22, the group W is word hyperbolic and the boundary $\partial(W,S)$ coincides with the hyperbolic boundary $\partial_h W$. The topological freeness then follows from [88, Corollaire 20]. □

An extension of the results above to broader classes (or even a complete characterization) of Coxeter systems (W,S) whose respective boundary defines a boundary in the sense of Furstenberg and whose respective action $W \curvearrowright \partial(W,S)$ is topologically free would be very interesting.

As discussed in Subsection 2.3.3 one of the main results in [120] states that a discrete group is C*-simple if and only if it admits a topologically free boundary action. Further, by [30, Corollary 4.3] and [95, Theorem 3.3], the reduced group C*-algebra of a discrete group carries a unique tracial state if and only if the group's *amenable radical* (i.e. its largest normal amenable subgroup) is trivial. Because C*-simplicity implies the triviality of the amenable radical, in the context of right-angled Coxeter groups Theorem 5.2.21 and Proposition 5.2.26 lead to a new proof of a well-known C*-simplicity and trace-uniqueness result (see [80], [101], [61]).

Corollary 5.2.29. *Let (W,S) be a right-angled irreducible Coxeter system with $3 \le \#S < \infty$. Then the reduced group C*-algebra $C_r^*(W)$ is simple and has unique tracial state.*

5

Note that an action of a group G on a compact Hausdorff space X is minimal if and only if $C(X)$ does not contain any non-trivial G-invariant ideal. We close this subsection with a result that relates to the ideal structure of the C*-algebra $C(\partial(W,S))$. Recall that by Proposition 5.1.6 (and Remark 5.1.7), $\pi(\mathcal{D}(W,S)) \cong C(\partial(W,S))$ via $\pi(P_{\mathbf{w}}) \mapsto \chi_{\mathcal{U}_{\mathbf{w}} \cap \partial(W,S)}$ where $\mathcal{D}(W,S) := \mathcal{D}(\mathrm{Cay}(W,S),e)$.

Proposition 5.2.30. *Let (W,S) be a finite rank Coxeter system and let I be a non-zero ideal in $\pi(\mathcal{D}(W,S))$. Then I intersects non-trivially with the $*$-algebra Span$\{\pi(P_{\mathbf{w}}) \mid \mathbf{w} \in W\} \subseteq \pi(\mathcal{D}(W,S))$.*

Proof. Let I be a non-zero ideal in $C(\partial W) \cong \pi(\mathcal{D}(W,S))$ and assume that I intersects the $*$-algebra Span$\{\chi_{\mathcal{U}_{\mathbf{w}} \cap \partial W} \mid \mathbf{w} \in W\}$ trivially. Denote the quotient map $C(\partial W) \to C(\partial W)/I$ by ρ. Let further $x := \sum_{\mathbf{w} \in W} \lambda_{\mathbf{w}} \chi_{\mathcal{U}_{\mathbf{w}} \cap \partial W} \in C(\partial W)$ with $\lambda_{\mathbf{w}} \in \mathbb{C}$ be a non-zero element where we assume that the sum is finite. The space ∂W is compact, therefore there exists $z \in \partial W$ with

$$\|x\| = \left| \sum_{\mathbf{w} \in W:\ \mathbf{w} \le z} \lambda_{\mathbf{w}} \right|.$$

Define the finite set $\mathfrak{S} := \{\mathbf{v} \in W \mid \lambda_{\mathbf{v}} \neq 0 \text{ and } \mathbf{v} \not\le z\}$ and let $(\alpha_i)_{i \in \mathbb{N}} \subseteq W$ be an infinite geodesic path representing the element z. Then, for every $i \in \mathbb{N}$ the continuous function $\mathbf{P}_i := \chi_{\mathcal{U}_{\alpha_i} \cap \partial W} \prod_{\mathbf{v} \in \mathfrak{S}} \chi_{\mathcal{U}_{\mathbf{v}}^c \cap \partial W} \in C(\partial W)$ is a projection with $\rho(\mathbf{P}_i) \neq 0$. Indeed, $\mathbf{P}_i(z) = 1$ implies that $\mathbf{P}_i \neq 0$ and hence $\rho(\mathbf{P}_i) \neq 0$ because $\mathbf{P}_i \in \mathrm{Span}\{\chi_{\mathcal{U}_{\mathbf{w}} \cap \partial W} \mid \mathbf{w} \in W\}$. We get that

$$\begin{aligned} \|\rho(x)\| &\geq \lim_{i \to \infty} \left\| \sum_{\mathbf{w} \in W} \lambda_{\mathbf{w}} \rho(\chi_{\mathcal{U}_{\mathbf{w}} \cap \partial W} \mathbf{P}_i) \right\| \\ &= \lim_{i \to \infty} \left\| \sum_{\mathbf{w} \in W:\ \mathbf{w} \notin \mathfrak{S}} \lambda_{\mathbf{w}} \rho(\chi_{\mathcal{U}_{\mathbf{w} \vee \alpha_i} \cap \partial W} \prod_{\mathbf{v} \in \mathfrak{S}} \chi_{\mathcal{U}_{\mathbf{v}}^c \cap \partial W}) \right\| \end{aligned}$$

$$= \lim_{i\to\infty} \left\| \left(\sum_{\mathbf{w}\in W:\, \mathbf{w}\le z} \lambda_{\mathbf{w}} \right) \rho(\mathbf{P}_i) \right\|$$
$$= \|x\|.$$

But then ρ must be isometric, i.e. $I = 0$ in contradiction to our assumption. We deduce the claim. □

5.2.6. Operator algebraic description of the canonical actions in the right-angled case

Proposition 5.1.6 and Remark 5.1.7 imply that for every Coxeter system (W,S) the C*-algebras $C(\overline{(W,S)})$ and $C(\partial(W,S))$ can be realized very concretely as spectra of commutative C*-algebras generated by projections. Indeed, if we denote by

$$\pi : \mathscr{B}(\ell^2(W)) \to \mathscr{B}(\ell^2(W))/\mathscr{K}(\ell^2(W))$$

the quotient map onto the Calkin algebra and if we define $P_{\mathbf{w}} \in \ell^\infty(K) \subseteq \mathscr{B}(\ell^2(K))$, $\mathbf{w}\in W$ to be the orthogonal projection onto the subspace

$$\overline{\mathrm{Span}\{\delta_{\mathbf{v}} \mid \mathbf{v}\in W \text{ with } \mathbf{w} \le_R \mathbf{v}\}} \subseteq \ell^2(W),$$

then $\mathscr{D}(W,S) \cong C(\overline{(W,S)})$ via $P_{\mathbf{w}} \mapsto \chi_{\mathscr{U}_{\mathbf{w}}}$ where $\mathscr{D}(W,S) := C^*(\{P_{\mathbf{w}} \mid \mathbf{w}\in W\}) \subseteq \mathscr{B}(\ell^2(W))$ and $\pi(\mathscr{D}(W,S)) \cong C(\partial(W,S))$ via $\pi(P_{\mathbf{w}}) \mapsto \chi_{\mathscr{U}_{\mathbf{w}}\cap\partial(W,S)}$. For brevity from now on we will write $\widetilde{P}_{\mathbf{w}} := \pi(P_{\mathbf{w}})$, $\mathbf{w}\in W$.

In some cases (such as in the setting of Subsection 5.2.3, Subsection 5.2.4 and Subsection 5.2.5) it is useful to work with the concrete description of the spaces $\overline{(W,S)}$ and $\partial(W,S)$ whereas in other cases it makes sense to work with the C*-algebras $\mathscr{D}(W,S)$ and $\pi(\mathscr{D}(W,S))$ instead. For instance, in the right-angled case the actions $W \curvearrowright \overline{(W,S)}$ and $W \curvearrowright \partial(W,S)$ have an accessible and useful description on the level of operator algebras. Note that for every Coxeter system (W,S) the induced action $W \curvearrowright \mathscr{D}(W,S)$ is inner and given by $\mathbf{v}.P_{\mathbf{w}} = T^{(1)}_{\mathbf{v}} P_{\mathbf{w}} T^{(1)}_{\mathbf{v}^{-1}}$ for $\mathbf{v},\mathbf{w}\in W$, i.e. the action is implemented by conjugation by the left regular representation operators.

Proposition 5.2.31. *Let (W,S) be a right-angled Coxeter system and $\mathbf{w}\in W$, $s\in S$. Then the following equalities hold:*

(1) $s.P_{\mathbf{w}} = P_{s\mathbf{w}}$ if $\mathbf{w}\notin C_W(s)$;

(2) $s.P_{\mathbf{w}} = P_{s\mathbf{w}} - P_{\mathbf{w}}$ if $\mathbf{w}\in C_W(s)$ and $s \le \mathbf{w}$;

(3) $s.P_{\mathbf{w}} = P_{\mathbf{w}}$ if $\mathbf{w}\in C_W(s)$ and $s \not\le \mathbf{w}$.

Here $C_W(s) := \{\mathbf{v}\in W \mid s\mathbf{v} = \mathbf{v}s\}$ denotes the centralizer of s in W.

Proof. First observe that by Proposition 2.7.5 for all $s \in S$ and $\mathbf{v}, \mathbf{w} \in W$ with $s \leq \mathbf{w}$, $s \not\leq \mathbf{v}$ or $s \not\leq \mathbf{w}$, $s \leq \mathbf{v}$,

$$(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = T_s^{(1)} P_{\mathbf{w}} \delta_{s\mathbf{v}} = \begin{cases} \delta_{\mathbf{v}} & \text{, if } \mathbf{w} \leq s\mathbf{v} \\ 0 & \text{, if } \mathbf{w} \not\leq s\mathbf{v} \end{cases} = \begin{cases} \delta_{\mathbf{v}} & \text{, if } s\mathbf{w} \leq \mathbf{v} \\ 0 & \text{, if } s\mathbf{w} \not\leq \mathbf{v} \end{cases} = P_{s\mathbf{w}}\delta_{\mathbf{v}}. \tag{5.2.2}$$

We will cover the remaining cases in the following.

(1): Assume that $\mathbf{w} \notin C_W(s)$. If $s \leq \mathbf{w}$ and $s \leq \mathbf{v}$, then $\mathbf{w} \not\leq s\mathbf{v}$ and $s\mathbf{w} \not\leq \mathbf{v}$. Indeed, if we assume that $\mathbf{w} \leq s\mathbf{v}$, then $s \leq s\mathbf{v}$ in contradiction to $s \not\leq s\mathbf{v}$. Further, if we assume that $s\mathbf{w} \leq \mathbf{v}$, then there exists $\mathbf{u} \in W$ with $\mathbf{v} = (s\mathbf{w})\mathbf{u}$ and $|\mathbf{v}| = |s\mathbf{w}| + |\mathbf{u}|$. Because (W,S) is right-angled $s \leq \mathbf{v}$ implies that $s \leq \mathbf{u}$ and $s\mathbf{w} \in C_W(s)$. But then $\mathbf{w} \in C_W(s)$ in contradiction to the assumption $\mathbf{w} \notin C_W(s)$. We get that $(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = 0 = P_{s\mathbf{w}}\delta_{\mathbf{v}}$.

If $s \not\leq \mathbf{w}$ and $s \not\leq \mathbf{v}$, then one obtains in the same way $\mathbf{w} \not\leq s\mathbf{v}$ and $s\mathbf{w} \not\leq \mathbf{v}$ which implies $(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = 0 = P_{s\mathbf{w}}\delta_{\mathbf{v}}$. With (5.2.2) this covers all possible cases. Hence, $s.P_{\mathbf{w}} = P_{s\mathbf{w}}$.

(2): Assume that $\mathbf{w} \in C_W(s)$ and $s \leq \mathbf{w}$. If $s \not\leq \mathbf{v}$, then (5.2.2) implies $(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = P_{s\mathbf{w}}\delta_{\mathbf{v}}$ and hence $(s.P_{\mathbf{w}})(1-P_s) = P_{s\mathbf{w}}(1-P_s)$. If $s \leq \mathbf{v}$, then $\mathbf{w} \not\leq s\mathbf{v}$ implies $(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = 0$ and hence $(s.P_{\mathbf{w}})P_s = 0$. Combined this leads to

$$s.P_{\mathbf{w}} = (s.P_{\mathbf{w}})(1-P_s) + (s.P_{\mathbf{w}})P_s = P_{s\mathbf{w}}(1-P_s) = P_{s\mathbf{w}} - P_{s\mathbf{w}\vee s} = P_{s\mathbf{w}} - P_{\mathbf{w}}.$$

(3): Assume that $\mathbf{w} \in C_W(s)$ and $s \not\leq \mathbf{w}$. If $s \leq \mathbf{v}$, then $(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = P_{s\mathbf{w}}\delta_{\mathbf{v}} = P_{\mathbf{w}\vee s}\delta_{\mathbf{v}} = P_{\mathbf{w}}P_s\delta_{\mathbf{v}} = P_{\mathbf{w}}\delta_{\mathbf{v}}$ by (5.2.2). So consider the case where $s \not\leq \mathbf{v}$. If $\mathbf{w} \leq \mathbf{v}$, then $\mathbf{v} = \mathbf{w}\mathbf{u}$ for some $\mathbf{u} \in W$ with $|\mathbf{v}| = |\mathbf{w}| + |\mathbf{u}|$ and $s \not\leq \mathbf{u}$. Hence, $s\mathbf{v} = \mathbf{w}(s\mathbf{u}) \geq \mathbf{w}$. Conversely, if $\mathbf{w} \leq s\mathbf{v}$, then $s\mathbf{v} = \mathbf{w}\mathbf{u}$ for some $\mathbf{u} \in W$ with $|s\mathbf{v}| = |\mathbf{w}| + |\mathbf{u}|$ and $s \leq \mathbf{u}$. We get that $\mathbf{v} = s(s\mathbf{v}) = \mathbf{w}(s\mathbf{u}) \geq \mathbf{w}$. Together this gives $(s.P_{\mathbf{w}})\delta_{\mathbf{v}} = P_{\mathbf{w}}\delta_{\mathbf{v}}$, that is $s.P_{\mathbf{w}} = P_{\mathbf{w}}$ as claimed. □

Remark 5.2.32. Let (W,S) be a Coxeter system. Recall that W equipped with the weak right Bruhat order defines a complete meet-semilattice. If existent, denote the corresponding join of two elements $\mathbf{v}, \mathbf{w} \in W$ by $\mathbf{v} \vee \mathbf{w}$. We then have $P_{\mathbf{v}}P_{\mathbf{w}} = P_{\mathbf{v}\vee\mathbf{w}}$ for all $\mathbf{v}, \mathbf{w} \in W$ where we assume that $P_{\mathbf{v}\vee\mathbf{w}} = 0$ if the join $\mathbf{v} \vee \mathbf{w}$ does not exist (compare with Theorem 5.1.13). In particular, the equalities $P_sP_t = 0$ (i.e. P_s and P_t are orthogonal to each other) if $m_{st} = \infty$ and $P_sP_t = P_{st}$ if $m_{st} = 2$ hold (this follows for instance from [67, Lemma 4.3.3]). Now assume that (W,S) is right-angled, that $q \in \mathbb{R}_{>0}^{(W,S)}$ and that $s \in S$, $\mathbf{w} \in W$. In combination with Proposition 5.2.31 the equality $T_s^{(q)} = T_s^{(1)} + p_s(q)P_s$ then leads to a description of the conjugation of the generating projections in $\mathcal{D}(W,S)$ with the Hecke operators $T_s^{(q)}$, $s \in S$. In particular, for $s \in S$, $\mathbf{w} \in W$ with $\mathbf{w} \notin C_W(s)$ and $s \not\leq \mathbf{w}$ the identities

$$T_s^{(q)}(1-P_s)T_s^{(q)} = T_s^{(1)}(1-P_s)T_s^{(1)} = P_s$$

and

$$T_s^{(q)}P_{\mathbf{w}}T_s^{(q)} = T_s^{(1)}P_{\mathbf{w}}T_s^{(1)} = P_{s\mathbf{w}}$$

hold.

5.2.7. Probability measures on the boundary and the compactification

Our characterization of the simplicity of the Hecke C*-algebras of a right-angled Coxeter system (W,S) is inspired by Haagerup's approach to the unique trace property of group C*-algebras in [95]. The translation of the techniques into the deformed setting requires the study of probability measures on the compactification $\overline{(W,S)}$ and the boundary $\partial(W,S)$. The aim of this subsection is to prove that such measures in a certain sense decrease very rapidly. Recall that probability measures on a given compact Hausdorff space correspond to states on the C*-algebra of continuous functions on that space. We can hence make use of the convenient description $\mathscr{D}(W,S) \cong C(\overline{(W,S)})$ and $\pi(\mathscr{D}(W,S)) \cong C(\partial(W,S))$ where the notation is the same as before.

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Lemma 5.2.33. *Let (W,S) be a right-angled, finite rank Coxeter system. For every $\mathbf{u} \in W$ and $0 < q < 1$ the operator $\mathbf{Q}_q^{\mathbf{u}}$ on $\ell^2(W)$ defined by*

$$\mathbf{Q}_q^{\mathbf{u}} := \sum_{l=|\mathbf{u}|}^{\infty} \sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l,\mathbf{u}\leq\mathbf{w}^{-1}} q^l P_{\mathbf{w}}$$

exists (where the limit is taken with respect to the operator norm) and is contained in the C-algebra $\mathscr{D}(W,S) \subseteq \mathscr{B}(\ell^2(W))$.*

Proof. It suffices to show that the sequence $(\mathbf{Q}_{q,i}^{\mathbf{u}})_{i\geq|\mathbf{u}|}$ with

$$\mathbf{Q}_{q,i}^{\mathbf{u}} := \sum_{l=|\mathbf{u}|}^{i} \sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l,\mathbf{u}\leq\mathbf{w}^{-1}} q^l P_{\mathbf{w}} \in \mathscr{D}(W,S)$$

is a Cauchy sequence. For $\mathbf{v} \in W$ and $l \in \mathbb{N}$ set $\kappa_{\mathbf{v}}(l) := \#\{\mathbf{w} \in W \mid \mathbf{w} \leq \mathbf{v} \text{ and } |\mathbf{w}| = l\}$. It has been shown in [46, Lemma 4.4] that $\kappa_{\mathbf{v}}(l) \leq C l^{\#S-2}$ for some constant $C > 0$. Using this in the third line of the following inequalities, we get that for $i < j$ with $i \geq |\mathbf{u}|$ and $\xi \in \ell^2(W)$,

$$\begin{aligned} \|(\mathbf{Q}_{q,j}^{\mathbf{u}} - \mathbf{Q}_{q,i}^{\mathbf{u}})\xi\|_2 &= \left\| \sum_{\mathbf{v}\in W} \left(\sum_{l=i+1}^{j} \sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l,\mathbf{w}\leq\mathbf{v},\mathbf{u}\leq\mathbf{w}^{-1}} q^l \right) \xi(\mathbf{v})\delta_{\mathbf{v}} \right\|_2 \\ &\leq \sqrt{\sum_{\mathbf{v}\in W} \left(\sum_{l=i+1}^{j} q^l \kappa_{\mathbf{v}}(l) \right)^2 |\xi(\mathbf{v})|^2} \\ &\leq C \left(\sum_{l=i+1}^{j} q^l l^{\#S-2} \right) \|\xi\|_2 . \end{aligned}$$

For $0 < q < 1$ the series $\sum_{l=1}^{\infty} q^l l^{\#S-2}$ converges. This implies the claim. □

Lemma 5.2.34. *Let (W,S) be an irreducible right-angled, finite rank Coxeter system, let $\mathbf{g} := s_1 \ldots s_n \in W$ be a path in the Coxeter diagram of (W,S) that covers the whole graph and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$. Then the series $\sum_{\mathbf{w}\in W} q_{\mathbf{w}}$ converges if and only if the series $\sum_{\mathbf{w}\in W:\ \mathbf{g}\leq\mathbf{w}^{-1}} q_{\mathbf{w}}$ converges.*

Proof. Because all summands of the series are positive it is clear that the convergence of $\sum_{\mathbf{w}\in W} q_{\mathbf{w}}$ implies the convergence of $\sum_{\mathbf{w}\in W:\ \mathbf{g}\leq\mathbf{w}^{-1}} q_{\mathbf{w}}$. So assume that the series $\sum_{\mathbf{w}\in W:\ \mathbf{g}\leq\mathbf{w}^{-1}} q_{\mathbf{w}}$ converges. For every $i, j \in \mathbb{N}$ with $i < j$ we have that

$$\begin{aligned}
&\left|\sum_{\mathbf{w}\in W:|\mathbf{w}|\leq i} q_{\mathbf{w}} - \sum_{\mathbf{w}\in W:\ |\mathbf{w}|\leq j} q_{\mathbf{w}}\right| \\
=\ & \sum_{\mathbf{w}\in W: i<|\mathbf{w}|\leq j} q_{\mathbf{w}} \\
=\ & \sum_{\mathbf{w}\in W: i<|\mathbf{w}|\leq j,\ s_n\leq \mathbf{w}^{-1}} q_{\mathbf{w}} + \sum_{\mathbf{w}\in W: i<|\mathbf{w}|\leq j,\ s_n\not\leq \mathbf{w}^{-1}} q_{\mathbf{w}} \\
=\ & \sum_{\mathbf{w}\in W: i-1<|\mathbf{w}|\leq j-1,\ s_n\not\leq \mathbf{w}^{-1}} q_{s_n} q_{\mathbf{w}} + \sum_{\mathbf{w}\in W: i<|\mathbf{w}|\leq j,\ s_n\not\leq \mathbf{w}^{-1}} q_{\mathbf{w}} \\
\leq\ & (1+q_{s_n}) \sum_{\mathbf{w}\in W: i-1<|\mathbf{w}|\leq j,\ s_n\not\leq \mathbf{w}^{-1}} q_{\mathbf{w}} \\
=\ & \frac{1+q_{s_n}}{q_{\mathbf{g}^{-1}}} \sum_{\mathbf{w}\in W: i-1<|\mathbf{w}|\leq j,\ s_n\not\leq \mathbf{w}^{-1}} q_{\mathbf{g}^{-1}} q_{\mathbf{w}} \\
=\ & \frac{1+q_{s_n}}{q_{\mathbf{g}}} \sum_{\mathbf{w}\in W: i-1<|\mathbf{g}^{-1}\mathbf{w}^{-1}|\leq j,\ \mathbf{g}\leq \mathbf{w}^{-1}} q_{\mathbf{w}} \\
=\ & \frac{1+q_{s_n}}{q_{\mathbf{g}}} \sum_{\mathbf{w}\in W: i-1+n<|\mathbf{w}|\leq j+n,\ \mathbf{g}\leq \mathbf{w}^{-1}} q_{\mathbf{w}},
\end{aligned}$$

where the fifth equality follows from the fact that

$$\{\mathbf{w}\mathbf{g}^{-1} \mid \mathbf{w}\in W \text{ with } s_n \not\leq \mathbf{w}^{-1}\} = \{\mathbf{w}\in W \mid \mathbf{g}\leq \mathbf{w}^{-1}\}$$

since $\mathbf{g} = s_1 \dots s_n$ is a path in the Coxeter diagram of (W,S). That implies that the sequence of partial sums of $\sum_{\mathbf{w}\in W} q_{\mathbf{w}}$ is a Cauchy sequence and hence that the series converges. □

The following proposition will play a crucial role in Subsection 6.1.2 and Section 6.2. Recall that $\overline{\mathscr{R}'(W,S)}$ is the closure of

$$\mathscr{R}'(W,S) := \left\{(q_s^{\epsilon_s})_{s\in S} \mid q\in \mathscr{R}(W,S)\cap \mathbb{R}_{>0}^{(W,S)},\ \epsilon\in\{-1,1\}^{(W,S)}\right\}$$

in $\mathbb{R}_{>0}^{(W,S)}$, where $\mathscr{R}(W,S)$ denotes the region of convergence of the growth series of (W,S) (compare with Subsection 2.7.7).

Proposition 5.2.35. *Let (W,S) be a right-angled, irreducible, finite rank Coxeter system, $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \overline{\mathscr{R}'(W,S)}$ and let $\mathbf{g} := s_1 \dots s_n \in W$ be a path in the Coxeter diagram of (W,S) that covers the whole graph. Then, for every state ϕ on $\mathscr{D}(W,S)$ there exists a sequence $(\mathbf{w}_i)_{i\in\mathbb{N}} \subseteq W$ of group elements with increasing word length such that $\mathbf{g}\leq \mathbf{w}_i^{-1}$ for all $i\in\mathbb{N}$ and $q_{\mathbf{w}_i}^{-1}\phi(P_{\mathbf{w}_i}) \to 0$.*

The same statement holds, if one replaces $\mathscr{D}(W,S)$ by $\pi(\mathscr{D}(W,S))$ and $P_{\mathbf{w}_i}$ by $\widetilde{P}_{\mathbf{w}_i}$.

Proof. The set $\mathbb{R}_{>0}^{(W,S)} \setminus \overline{\mathcal{R}'(W,S)}$ is open in $\mathbb{R}_{>0}^{(W,S)}$, so there exist positive real numbers $q', \lambda \in (0,1)$ such that $q'q := (q'q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \overline{\mathcal{R}'(W,S)}$ and $\lambda q'q := (\lambda q' q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \overline{\mathcal{R}'(W,S)}$. In particular, Lemma 5.2.34 implies that the series $\sum_{\mathbf{g}\leq \mathbf{w}^{-1}} \lambda^{|\mathbf{w}|}(q'q)_{\mathbf{w}}$ diverges. By the root test criterium for convergence,

$$\limsup_l \left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l, \mathbf{g}\leq \mathbf{w}^{-1}} \lambda^l (q'q)_{\mathbf{w}} \right)^{1/l} \geq 1$$

and hence

$$\limsup_l \left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l, \mathbf{g}\leq \mathbf{w}^{-1}} (q'q)_{\mathbf{w}} \right)^{1/l} > 1.$$

One can thus find a strictly increasing sequence $(l_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$ and a constant $C > 1$ such that for all $i \in \mathbb{N}$,

$$\left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l_i, \mathbf{g}\leq \mathbf{w}^{-1}} (q'q)_{\mathbf{w}} \right)^{1/l_i} \geq C. \tag{5.2.3}$$

For $\mathbf{w} \in W$ define the set

$$C_{\mathbf{w}} := \left\{ \mathbf{v} \in W \mid \mathbf{g} \leq \mathbf{v}^{-1} \text{ and } z_{\mathbf{v}} = z_{\mathbf{w}} \text{ for all } z = (z_s)_{s\in S} \in \mathbb{C}^{(W,S)} \right\}$$

and note that the elements in $C_{\mathbf{w}}$ all have the same length. Choose for every $i \in \mathbb{N}$ an element $\mathbf{w}_i \in W$ with $\#C_{\mathbf{w}_i}(q'q)_{\mathbf{w}_i} = \max_{|\mathbf{w}|=l_i, \mathbf{g}\leq\mathbf{w}^{-1}} \#C_{\mathbf{w}}(q'q)_{\mathbf{w}}$ that has length l_i and satisfies $\mathbf{g} \leq \mathbf{w}_i^{-1}$. Since by the definition of $C_{\mathbf{w}_i}$ the equality $\#C_{\mathbf{v}}(q'q)_{\mathbf{v}} = \#C_{\mathbf{w}_i}(q'q)_{\mathbf{w}_i}$ holds for all $\mathbf{v} \in C_{\mathbf{w}_i}$, this element can be chosen in such a way that $\phi(P_{\mathbf{w}_i}) \leq \phi(P_{\mathbf{v}})$ for all $\mathbf{v} \in C_{\mathbf{w}_i}$. Now, by picking a suitable subset $\mathcal{M} \subseteq W$ of elements $\mathbf{w} \in W$ with length l_i and $\mathbf{g} \leq \mathbf{w}^{-1}$, the sum $\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l_i, \mathbf{g}\leq\mathbf{w}^{-1}} (q'q)_{\mathbf{w}}$ can be written as $\sum_{\mathbf{w}\in\mathcal{M}} \#C_{\mathbf{w}}(q'q)_{\mathbf{w}}$. By the choice of $\mathbf{w}_i$ we hence have

$$\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l_i, \mathbf{g}\leq \mathbf{w}^{-1}} (q'q)_{\mathbf{w}} \leq (l_i+1)^{\#S} \#C_{\mathbf{w}_i}(q'q)_{\mathbf{w}_i}$$

which implies in combination with (5.2.3) that for all $i \in \mathbb{N}$,

$$\#C_{\mathbf{w}_i}(q')^{l_i} q_{\mathbf{w}_i} \geq \frac{C^{l_i}}{(l_i+1)^{\#S}}. \tag{5.2.4}$$

It follows from Lemma 5.2.33 that the series $\sum_{\mathbf{w}\in W:\ \mathbf{g}\leq\mathbf{w}^{-1}} (q')^{|\mathbf{w}|}\phi(P_{\mathbf{w}})$ converges. By the same argument as above we hence have that

$$\limsup_l \left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l, \mathbf{g}\leq \mathbf{w}^{-1}} (q')^l \phi(P_{\mathbf{w}}) \right)^{1/l} < 1.$$

One can therefore assume (by possibly going over to a further subsequence) that

$$\left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l_i,\ \mathbf{g}\leq \mathbf{w}^{-1}} (q')^{l_i} \phi(P_{\mathbf{w}}) \right)^{1/l_i} \leq L$$

for all $i \in \mathbb{N}$ where $0 < L < 1$. But then, by the choice of $\mathbf{w}_i$,

$$\#C_{\mathbf{w}_i}(q')^{l_i}\phi(P_{\mathbf{w}_i}) \leq (q')^{l_i}\sum_{\mathbf{w}\in C_{\mathbf{w}_i}}\phi(P_{\mathbf{w}}) \leq \sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l_i,\mathbf{g}\leq\mathbf{w}^{-1}}(q')^{l_i}\phi(P_{\mathbf{w}}) \leq L^{l_i}$$

and thus with (5.2.4)

$$0 \leq q_{\mathbf{w}_i}^{-1}\phi(P_{\mathbf{w}_i}) < \frac{L^{l_i}}{\#C_{\mathbf{w}_i}(q')^{l_i}q_{\mathbf{w}_i}} \leq (l_i+1)^{\#S}\left(\frac{L}{C}\right)^{l_i} \to 0.$$

This implies the first part of the statement. The second part is an immediate consequence since $\pi(\mathfrak{A}(W))$ is a quotient of $\mathfrak{A}(W)$. That finishes the proof. □

Remark 5.2.36. The proof of Proposition 5.2.35 significantly simplifies in the case of single-parameters q. Indeed, if we follow the notation of Proposition 5.2.35 and assume that $q_s = q_t$ for all $s,t \in S$, Lemma 5.2.33 implies that for $i \in \mathbb{N}$ and $0 < q' < 1$,

$$\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=i,\mathbf{g}\leq\mathbf{w}^{-1}}(q')^{i}\phi(P_{\mathbf{w}}) \leq \phi(\mathbf{Q}_{q'}^{\mathbf{g}}).$$

One can thus find an element $\mathbf{w}_i$ of length i with $\mathbf{g} \leq \mathbf{w}_i^{-1}$ such that

$$\phi(P_{\mathbf{w}_i}) \leq \left(\#L_i^{\mathbf{g}}(q')^i\right)^{-1}\phi(\mathbf{Q}_{q'}^{\mathbf{g}})$$

where $L_i^{\mathbf{g}} := \{\mathbf{w}\in W \mid |\mathbf{w}| = i,\ \mathbf{g}\leq\mathbf{w}_i^{-1}\}$. We get that

$$q_{\mathbf{w}_i}^{-1}\phi(P_{\mathbf{w}_i}) \leq \frac{\phi(\mathbf{Q}_{q'}^{\mathbf{g}})}{\#L_i^{\mathbf{g}}(q'q)_{\mathbf{w}_i}}.$$

The Cauchy-Hadamard formula (for radii of convergence of power series) implies that for increasing i, if q' is close enough to 1, the expression on the right approaches 0.

5.3. THE CONNECTION TO HECKE C*-ALGEBRAS

The main reason for our study of compactifications and boundaries of connected rooted graphs (or more precisely Cayley graphs of Coxeter systems) is the construction's intimate relationship with Hecke operator algebras of Coxeter systems which will be crucial for several results of the later chapters. In the following, we will adopt the notation of Subsection 5.2.6.

Let (W,S) be a finite rank Coxeter system. For every $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$, $s \in S$ the operator $T_s^{(q)}$ can be written as $T_s^{(q)} = T_s^{(1)} + p_s(q)P_s$ and the map $q_s \mapsto p_s(q) = q_s^{-\frac{1}{2}}(q_s - 1)$ is injective on $\mathbb{R}_{>0}$. This implies that for all $q = (q_s)_{s\in S}, q' = (q'_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $q_s \neq q'_s$ for all $s \in S$ the C*-subalgebra $\mathfrak{A}(W,S)$ of $\mathscr{B}(\ell^2(W))$ generated by $C_{r,q}^*(W)$ and $C_{r,q'}^*(W)$ does not depend on the choice of the different parameters q

and q'. It is the smallest C*-subalgebra of $\mathscr{B}(\ell^2(W))$ that contains all Hecke C*-algebras of the system (W,S) and there exists a natural isomorphism

$$\iota : \mathfrak{A}(W,S) \cong C(\overline{(W,S)}) \rtimes_r W \text{ via } P_{\mathbf{w}} \mapsto \chi_{\mathscr{U}_{\mathbf{w}}} \text{ and } T_{\mathbf{w}}^{(1)} \mapsto \lambda_{\mathbf{w}}$$

where (as before) $\mathscr{U}_{\mathbf{w}} := \{z \in \overline{W} \mid \mathbf{w} \leq z\}$. Here $C(\overline{(W,S)}) \rtimes_r W \subseteq \mathscr{B}(\ell^2(W) \otimes \ell^2(W))$ denotes the reduced crossed product C*-algebra associated with the canonical action $W \curvearrowright \overline{(W,S)}$. The isomorphism is being implemented by conjugation with the unitary $U \in \mathscr{B}(\ell^2(W) \otimes \ell^2(W))$ defined by $U(\delta_{\mathbf{v}} \otimes \delta_{\mathbf{w}}) := \delta_{\mathbf{v}} \otimes \delta_{\mathbf{wv}}$ for $\mathbf{v}, \mathbf{w} \in W$ in combination with the identification in Remark 5.1.7. Indeed, for $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{w}' \in W$,

$$\begin{aligned} U(P_{\mathbf{u}}\lambda_{\mathbf{v}})(\delta_{\mathbf{w}} \otimes \delta_{\mathbf{w}'}) &= U((\mathbf{vw}')^{-1}.P_{\mathbf{u}})\delta_{\mathbf{w}}) \otimes \delta_{\mathbf{vw}'}) \\ &= U(T^{(1)}_{(\mathbf{vw}')^{-1}} P_{\mathbf{u}} \delta_{\mathbf{vw}'\mathbf{w}}) \otimes \delta_{\mathbf{vw}'}) \\ &= \begin{cases} \delta_{\mathbf{w}} \otimes \delta_{\mathbf{vw}'\mathbf{w}} & \text{, if } \mathbf{u} \leq \mathbf{vw}'\mathbf{w} \\ 0 & \text{, if } \mathbf{u} \nleq \mathbf{vw}'\mathbf{w} \end{cases} \\ &= (1 \otimes P_{\mathbf{u}} T^{(1)}_{\mathbf{v}}) U(\delta_{\mathbf{w}} \otimes \delta_{\mathbf{w}'}) \end{aligned}$$

so $\mathfrak{A}(W,S) \cong \mathbb{C}1 \otimes \mathfrak{A}(W,S) \cong \mathscr{D}(W,S) \rtimes_r W$ where we identified in our calculation the element $P_{\mathbf{u}}$ with its image in $\mathscr{B}(\ell^2(W) \otimes \ell^2(W))$ (for the corresponding notation revisit Section 2.3). By Remark 5.1.7 the latter C*-algebra is isomorphic to $C(\overline{(W,S)}) \rtimes_r W$, so $\mathfrak{A}(W,S) \cong C(\overline{(W,S)}) \rtimes_r W$ as claimed. In the same way there exists an isomorphism

$$\kappa : \pi(\mathfrak{A}(W,S)) \cong C(\partial(W,S)) \rtimes_r W \text{ via } \widetilde{P}_{\mathbf{w}} \mapsto \chi_{\mathscr{U}_{\mathbf{w}} \cap C(\partial W)} \text{ and } T_{\mathbf{w}}^{(1)} \mapsto \lambda_{\mathbf{w}}.$$

Let us pick up some of the implications of the previous sections.

Corollary 5.2.9 and [33, Theorem 4.3.4] lead to the following statement. Recall that a C*-algebra is called *nuclear* if for every other C*-algebra the corresponding algebraic tensor product admits a unique C*-norm.

Corollary 5.3.1. *Let (W,S) be a finite rank Coxeter system. Then the C*-algebras $\mathfrak{A}(W,S)$ and $\pi(\mathfrak{A}(W,S))$ are nuclear.*

The results of Subsection 5.2.5 have the following consequences.

Corollary 5.3.2. *Let (W,S) be a finite rank Coxeter system. Assume that W is either small at infinity and non-amenable or that the system is irreducible and right-angled with $\#S \geq 3$. Then $\mathfrak{A}(W,S)$ contains $\mathfrak{A}(W,S) \cap \mathscr{K}(\ell^2(W))$ as its unique non-trivial ideal. Further, $C(\overline{(W,S)})$ and $C(\partial(W,S))$ carry no W-invariant tracial states and both $\mathfrak{A}(W,S)$ and $\pi(\mathfrak{A}(W,S))$ are traceless.*

Proof. The simplicity of $\pi(\mathfrak{A}(W,S))$ follows from the identification

$$\pi(\mathfrak{A}(W,S)) \cong C(\partial(W,S)) \rtimes_r W$$

in combination with Theorem 5.2.9, Theorem 5.2.22, Theorem 5.2.21 and [9]. Let $I \lhd \mathfrak{A}(W,S)$ be a non-trivial ideal. By Lemma 5.2.24, Corollary 5.2.9 and [9, Theorem

2], I intersects non-trivially with the unital C*-algebra $\mathscr{D}(W,S)$ generated by all $P_{\mathbf{w}}$, $\mathbf{w}\in W$. Because $\mathscr{D}(W,S)\cap\mathscr{K}(\ell^2(W))$ is a W-equivariant ideal in $\mathscr{D}(W,S)$ it is easy to see that $\mathscr{D}(W,S)\cap\mathscr{K}(\ell^2(W))\subseteq I\cap\mathscr{D}(W,S)$, hence $\mathfrak{A}(W,S)\cap\mathscr{K}(\ell^2(W))\subseteq I$. But by the simplicity of $\pi(\mathfrak{A}(W,S))$, $\mathfrak{A}(W,S)\cap\mathscr{K}(\ell^2(W))$ is a maximal non-trivial ideal, so $I=\mathfrak{A}(W,S)\cap\mathscr{K}(\ell^2(W))$. We get that the C*-algebra $\mathfrak{A}(W,S)$ contains $\mathfrak{A}(W,S)\cap\mathscr{K}(\ell^2(W))$ as its unique non-trivial ideal.

To show the remaining statements, it suffices to show that $C(\overline{W})$ carries no W-invariant tracial state. Indeed, if $C(\overline{W})$ carries no W-invariant tracial state then $C(\partial W)$ obviously also carries no W-invariant state. That $\mathfrak{A}(W,S)\cong C(\overline{W})\rtimes_r W$ and $\pi(\mathfrak{A}(W,S))\cong C(\partial W)\rtimes_r W$ are both traceless then follows with [125, Corollary 1.4]. So let us show that $C(\overline{W})$ carries no W-invariant state. For this, assume that τ is such a state. The strong proximality of the action $W\curvearrowright\overline{W}$ implies that $\delta_z\in\overline{W.\{\tau\}}=\{\tau\}$ for some $z\in\overline{W}$, i.e. $\tau=\delta_z$. But δ_z is obviously not W-invariant. This leads to a contradiction. □

We have seen that for every $q\in\mathbb{R}_{>0}^{(W,S)}$ the map ι restricts to a natural embedding of $C_{r,q}^*(W)$ into $C(\overline{(W,S)})\rtimes_r W$. Motivated by this, let us investigate for which Hecke C*-algebras a similar statement holds for $\kappa\circ\pi$.

The one-dimensional central projections appearing in the following proposition already occur in [76], [66], [67], [86] and [160]. As will be exploited in Subsection 6.1.2 they induce characters on the Hecke C*-algebras of the system and are in the case of right-angled Coxeter groups already contained in the corresponding Hecke C*-algebras (see Subsection 6.1.1). Following the notation in [160], for a Coxeter system (W,S) write $q_{s,\epsilon}:=\epsilon_s q_s^{\epsilon_s}$ and $q_{\mathbf{w},\epsilon}:=q_{s_1,\epsilon}...q_{s_n,\epsilon}$ for $q\in\mathbb{R}_{>0}^{(W,S)}$, $\epsilon\in\{-1,1\}^{(W,S)}$ and $\mathbf{w}\in W$. Further set $\sqrt{q}:=(\sqrt{q_s})_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$.

Proposition 5.3.3 ([67, Lemma 19.2.5]). *Let (W,S) be a finite rank Coxeter system, let $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$ with $0<q_s\leq 1$ for all $s\in S$ be a multi-parameter and let $W(z)=\sum_{\mathbf{w}\in W}z_{\mathbf{w}}$ be the growth series of (W,S) (see Subsection 2.7.7). Further let $\epsilon=(\epsilon_s)_{s\in S}\in\{-1,1\}^{(W,S)}$, $q_\epsilon:=(\epsilon_s q_s^{\epsilon_s})_{s\in S}$ and assume that $|q_\epsilon|:=(q_s^{\epsilon_s})_{s\in S}\in\mathscr{R}(W,S)$. Then the operator*

$$E_{q,\epsilon}:\ell^2(W)\to\ell^2(W),\delta_{\mathbf{w}}\mapsto\left(\sqrt{q}\right)_{\mathbf{w},\epsilon}\eta_{q,\epsilon}$$

with $\eta_{q,\epsilon}:=(W(|q_\epsilon|))^{-1}\sum_{\mathbf{w}\in W}\left(\sqrt{q}\right)_{\mathbf{w},\epsilon}\delta_{\mathbf{w}}$ is bounded, it is a central projection in $\mathcal{N}_q(W)$ and it satisfies

$$T_{\mathbf{w}}^{(q)}E_{q,\epsilon}=E_{q,\epsilon}T_{\mathbf{w}}^{(q)}=\left(\sqrt{q}\right)_{\mathbf{w},\epsilon}E_{q,\epsilon}$$

for all $\mathbf{w}\in W$. For distinct $\epsilon,\epsilon'\in\{-1,1\}^{(W,S)}$ with $|q_\epsilon|,|q_{\epsilon'}|\in\mathscr{R}(W,S)$ the projections $E_{q,\epsilon}$ and $E_{q,\epsilon'}$ are orthogonal to each other.

Proof. First note that indeed $\eta_{q,\epsilon}\in\ell^2(W)$ with $\|\eta_{q,\epsilon}\|_2=1$. By the Cauchy-Schwarz inequality, for every $\xi:=\sum_{\mathbf{w}\in W}\xi(\mathbf{w})\delta_{\mathbf{w}}\in\ell^2(W)$,

$$\|E_{q,\epsilon}\xi\|_2=\left\|\sum_{\mathbf{w}\in W}\epsilon_{\mathbf{w}}\left|q_\epsilon\right|_{\mathbf{w}}^{\frac{1}{2}}\xi(\mathbf{w})\eta_{q,\epsilon}\right\|_2\leq\left|\sum_{\mathbf{w}\in W}\epsilon_{\mathbf{w}}\left|q_\epsilon\right|_{\mathbf{w}}^{\frac{1}{2}}\xi(\mathbf{w})\right|\leq W(|q_\epsilon|)\,\|\xi\|_2$$

so $E_{q,\epsilon}$ is a bounded operator on $\ell^2(W)$. Now, for $\mathbf{w} \in W$,

$$\begin{aligned} T_s^{(q),r} E_{q,\epsilon}\delta_{\mathbf{w}} &= (\sqrt{q})_{\mathbf{w},\epsilon} T_s^{(q),r} \eta_{q,\epsilon} \\ &= \frac{(\sqrt{q})_{\mathbf{w},\epsilon}}{W(|q_\epsilon|)} \sum_{\mathbf{v}\in W} (\sqrt{q})_{\mathbf{v},\epsilon} T_s^{(q),r} \delta_{\mathbf{v}} \\ &= \frac{(\sqrt{q})_{\mathbf{w},\epsilon}}{W(|q_\epsilon|)} \sum_{\mathbf{v}\in W: s \not\leq_L \mathbf{v}} \left[(\sqrt{q})_{\mathbf{v},\epsilon} \delta_{\mathbf{v}s} + (\sqrt{q})_{\mathbf{v}s,\epsilon} \delta_{\mathbf{v}} + (\sqrt{q})_{\mathbf{v}s,\epsilon} p_s(q) \delta_{\mathbf{v}s} \right] \end{aligned}$$

In the case where $\epsilon_s = 1$ we thus conclude

$$\begin{aligned} T_s^{(q),r} E_{q,\epsilon}\delta_{\mathbf{w}} &= \frac{(\sqrt{q})_{\mathbf{w},\epsilon}}{W(|q_\epsilon|)} \sum_{\mathbf{v}\in W: s \not\leq_L \mathbf{v}} \sqrt{q_s} \left[(\sqrt{q})_{\mathbf{v}s,\epsilon} \delta_{\mathbf{v}s} + (\sqrt{q})_{\mathbf{v},\epsilon} \delta_{\mathbf{v}} \right] \\ &= \sqrt{q_s} (\sqrt{q})_{\mathbf{w},\epsilon} \eta_{q,\epsilon} \end{aligned}$$

whereas in the case where $\epsilon_s = (-1)$,

$$\begin{aligned} T_s^{(q),r} E_{q,\epsilon}\delta_{\mathbf{w}} &= \frac{(\sqrt{q})_{\mathbf{w},\epsilon}}{W(|q_\epsilon|)} \sum_{\mathbf{v}\in W: s \not\leq_L \mathbf{v}} -\frac{1}{\sqrt{q_s}} \left[(\sqrt{q})_{\mathbf{v}s,\epsilon} \delta_{\mathbf{v}s} + (\sqrt{q})_{\mathbf{v},\epsilon} \delta_{\mathbf{v}} \right] \\ &= -\frac{1}{\sqrt{q_s}} (\sqrt{q})_{\mathbf{w},\epsilon} \eta_{q,\epsilon}. \end{aligned}$$

A similar calculation implies that $E_{q,\epsilon} T_s^{(q),r} \delta_{\mathbf{w}} = \sqrt{q_s} (\sqrt{q})_{\mathbf{w},\epsilon} \eta_{q,\epsilon}$ if $\epsilon_s = 1$ and $T_s^{(q),r} E_{q,\epsilon}$ $\delta_{\mathbf{w}} = -(\sqrt{q_s})^{-1} (\sqrt{q})_{\mathbf{w},\epsilon} \eta_{q,\epsilon}$ if $\epsilon_s = (-1)$, so $T_s^{(q),r} E_{q,\epsilon} = E_{q,\epsilon} T_s^{(q),r}$. In an analogous way, $T_s^{(q)} E_{q,\epsilon} = E_{q,\epsilon} T_s^{(q)} = (\sqrt{q})_{s,\epsilon} E_{q,\epsilon}$ for all $s \in S$ from which we deduce that $E_{q,\epsilon} \in (\mathcal{N}_q^r(W))' = \mathcal{N}_q(W)$ is a central element of $\mathcal{N}_q(W)$. The calculations also imply that $E_{q,\epsilon}$ is an orthogonal projection onto the 1-dimensional subspace $\mathbb{C}\eta_{q,\epsilon}$ of $\ell^2(W)$. It is further easy to check that for distinct elements $\epsilon, \epsilon' \in \{-1,1\}^{(W,S)}$ with $|q_\epsilon|, |q_{\epsilon'}| \in \mathcal{R}(W,S)$ the projections $E_{q,\epsilon}$ and $E_{q,\epsilon'}$ are orthogonal to each other. □

Remark 5.3.4. In [160] Raum and Skalski introduced the notion of *Hecke eigenvectors* (for the parameter q). These are non-zero elements $\eta \in \ell^2(W)$ satisfying $T_s^{(q)}\eta \in \mathbb{C}\eta$ for all $s \in S$. In the case of right-angled Coxeter groups the central projections considered in Proposition 5.3.3 are exactly the orthogonal projections onto the Hecke eigenspaces. Note that they are always of finite rank.

Theorem 5.3.5. *Let (W,S) be a finite rank Coxeter system for which W is infinite and let $q \in \mathbb{R}_{>0}^{(W,S)}$. Then $\mathcal{N}_q(W) \cap \mathcal{K}(\ell^2(W)) \neq 0$ if and only if $q \in \mathcal{R}'(W,S)$ with $\mathcal{R}'(W,S)$ as in Subsection 2.7.7.*

Proof. "⇒": First assume that $q \in \mathcal{R}(W,S) \cap \mathbb{R}_{>0}^{(W,S)}$. The existence of the one-dimensional central projection $E_{q,1}$ appearing in Proposition 5.3.3 then implies that the intersection $\mathcal{N}_q(W) \cap \mathcal{K}(\ell^2(W))$ is non-trivial. For general $q \in \mathcal{R}'(W,S)$ note that the isomorphism in Proposition 3.5.2 is unitarily implemented. The non-triviality hence follows from the above.

"⇐": First let $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}(W,S)$ with $0 < q_s \leq 1$ for every $s \in S$ and assume that $\mathcal{N}_q(W) \cap \mathcal{K}(\ell^2(W)) \neq 0$. Then $\mathcal{N}_q(W) \cap \mathcal{K}(\ell^2(W))$ contains a non-zero positive operator and also its finite-rank spectral projections. Since $\mathcal{N}_q^r(W) = J\mathcal{N}_q(W)J$, where J is the (unitary) modular conjugation operator (see Section 3.4), this implies that $\mathcal{N}_q^r(W)$ also contains a finite-rank projection which we denote by P. Because P commutes with the elements in $\mathcal{N}_q(W)$, the Hilbert subspace $\mathcal{H} := P\ell^2(W)$ is invariant under $\mathcal{N}_q(W)$. Let $(\xi_i)_{i=1,\dots,n}$ be an orthonormal basis of $P\ell^2(W)$. Then, by Lemma 3.4.7,

$$\|P\delta_e\|_2^2 \leq q_{\mathbf{w}}^{-1} \|T_{\mathbf{w}}^{(q)} P\delta_e\|_2^2 = \sum_{i=1}^{n} q_{\mathbf{w}}^{-1} |\langle \xi_i, T_{\mathbf{w}}^{(q)} P\delta_e\rangle|^2 = \sum_{i=1}^{n} q_{\mathbf{w}}^{-1} |\langle \xi_i, \delta_{\mathbf{w}}\rangle|^2$$

for every $\mathbf{w} \in W$. Let us distinguish two cases:

- *Case 1:* Assume that there exists a constant $C > 0$ such that for every $\mathbf{w} \in W$ there exists some $1 \leq i \leq n$ with $q_{\mathbf{w}}^{-1} |\langle \xi_i, \delta_{\mathbf{w}}\rangle|^2 > C$. We get that

  $$\sum_{i=1}^{n} \|\xi_i\|_2^2 = \sum_{i=1}^{n} \sum_{\mathbf{w}\in W} |\langle \xi_i, \delta_{\mathbf{w}}\rangle|^2 > C \sum_{\mathbf{w}\in W} q_{\mathbf{w}}.$$

  But $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}(W,S)$, so the sum on the right-hand side diverges in contradiction to $\sum_{i=1}^{n} \|\xi_i\|_2^2 < \infty$. This implies that there exists no such constant C.

- *Case 2:* Assume that there exists a sequence $(\mathbf{w}_j)_{j\in\mathbb{N}} \subseteq W$ with $q_{\mathbf{w}_j}^{-1} |\langle \xi_i, \delta_{\mathbf{w}_j}\rangle|^2 \to 0$ for $1 \leq i \leq n$. Then,

  $$\|P\delta_e\|_2^2 \leq \sum_{i=1}^{n} q_{\mathbf{w}_j}^{-1} |\langle \xi_i, \delta_{\mathbf{w}_j}\rangle|^2 \to 0,$$

  i.e. $P\delta_e = 0$. But then $P = 0$ since $P\delta_{\mathbf{w}} = PT_{\mathbf{w}}^{(q)}\delta_e = T_{\mathbf{w}}^{(q)} P\delta_e = 0$ for every $\mathbf{w} \in W$. This is a contradiction to our assumption.

Since both cases lead to a contradiction, the intersection $\mathcal{N}_q(W) \cap \mathcal{K}(\ell^2(W))$ must be trivial. Again, for general $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$ the statement follows with Proposition 3.5.2. □

Remark 5.3.6. For right-angled Coxeter systems the statement in Theorem 5.3.5 is a consequence of the results in [160] (see also [86]).

Corollary 5.3.7. *Let (W,S) be a finite rank Coxeter system. For $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$ the map $\kappa \circ (\pi|_{\mathfrak{A}(W)}) : \mathfrak{A}(W) \to C(\partial(W,S)) \rtimes_r W$ restricts to an embedding of $C_{r,q}^*(W)$ into $C(\partial(W,S)) \rtimes_r W$.*

One of the main ideas in [120] is the observation of the fact that for a discrete group G the G-injective envelope $I_G(\mathbb{C})$ of $\mathbb{C}$ (i.e. the unique G-injective and G-essential extension of $\mathbb{C}$) carries a natural C*-algebra structure for which $I_G(\mathbb{C}) \cong$

$C(\partial_F G)$. Here, $\partial_F G$ denotes the Furstenberg boundary of the group G. The construction of $\partial_F G$ by means of Hamana's theory of G-injective envelopes implies some powerful rigidity results.

In [149] Ozawa conjectured that for every exact C*-algebra $\mathscr{A}$ there is a nuclear C*-algebra $N(\mathscr{A})$ such that $\mathscr{A} \subseteq N(\mathscr{A}) \subseteq I(\mathscr{A})$. Here $I(\mathscr{A})$ denotes the injective envelope of $\mathscr{A}$. (For more information on operator systems and (G-)injective envelopes we refer to [97], [98],[99], [100] and Paulsen's book [151].) Embeddings of this form have the striking advantage that properties of the larger C*-algebra (for instance simplicity and primeness) are reflected by the properties of $\mathscr{A}$. Ozawa proved his conjecture in the case of reduced group C*-algebras of word hyperbolic groups G by choosing $N(C_r^*(G))$ to be the crossed product $C(\partial_h G) \rtimes_r G$. Kalantar and Kennedy extended his result in [120, Section 4] to general exact group C*-algebras by replacing the crossed product by $C(\partial_h G)$ by the crossed product $C(\partial_F G) \rtimes_r G$. However, in full generality Ozawa's conjecture remains a major open problem.

The following corollary provides an embedding of certain Hecke C*-algebras which is similar to the one above.

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Proposition 5.3.8. *Let (W,S) be a finite rank Coxeter system. Assume that W is either small at infinity or that the system is irreducible and right-angled with $\#S \geq 3$. Then $I_W(C(\partial(W,S))) = C(\partial_F W)$ where $I_W(C(\partial(W,S)))$ is the W-injective envelope of $C(\partial(W,S))$. Further, for every $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$ there are natural embeddings*

$$C_{r,q}^*(W) \hookrightarrow C(\partial(W,S)) \rtimes_r W \hookrightarrow C(\partial_F W) \rtimes_r W \hookrightarrow I(C_r^*(W)).$$

In particular, $I(C_{r,q}^*(W)) \hookrightarrow I(C_r^*(W))$ for every $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$.

Proof. By Theorem 5.2.21 and Theorem 5.2.22 ∂W is a W-boundary. Hence ∂W is a continuous W-equivariant quotient of the Furstenberg boundary $\partial_F W$ (see [85, Proposition 4.6]). This induces a W-equivariant embedding $C(\partial W) \hookrightarrow C(\partial_F W)$. The equality $I_W(C(\partial(W,S))) = C(\partial_F W)$ then follows in the same way as in the proof of [120, Corollary 5.5]. We further deduce the existence of the chain $C_{r,q}^*(W) \hookrightarrow C(\partial W) \rtimes_r W \hookrightarrow C(\partial_F W) \rtimes_r W \hookrightarrow I(C_r^*(W))$ of inclusions from Corollary 5.3.7, from [100, Theorem 3.4] and by extending the W-equivariant embedding $C(\partial W) \hookrightarrow C(\partial_F W)$ to an embedding $C(\partial W) \rtimes_r W \hookrightarrow C(\partial_F W) \rtimes_r W$ of the corresponding crossed products.

It remains to show that $I(C_{r,q}^*(W)) \hookrightarrow I(C_r^*(W))$ for every $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$. But this is clear since by the injectivity of $I(C_r^*(W))$ and $C_{r,q}^*(W) \hookrightarrow I(C_r^*(W))$ the injective envelope $I(C_{r,q}^*(W))$ is contained in $I(C_r^*(W))$ as an operator system. Hence, every completely positive projection $\theta : \mathscr{B}(\ell^2(W)) \to I(C_r^*(W))$ restricts to the identity on $I(C_{r,q}^*(W))$. But the C*-algebra structure of $I(C_r^*(W))$ is given by the Choi-Effros product associated with θ, so this induces an embedding $I(C_{r,q}^*(W)) \hookrightarrow I(C_r^*(W))$. □

Remark 5.3.9. Proposition 5.3.8 holds for all Coxeter systems whose canonical action $W \curvearrowright \partial(W,S)$ is a boundary action. Considering Ozawa's conjecture it would be interesting to know if the embedding $I(C_{r,q}^*(W)) \hookrightarrow I(C_r^*(W))$, $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$

is always surjective, i.e. if $I(C^*_{r,q}(W))$ does not depend on the choice of the parameter q. In that case, $C^*_{r,q}(W)$ would turn out to be a prime C*-algebra for all $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$ (see [100, Corollary 3.5]).

6

Ideal structure and trace-uniqueness of Hecke C^*-algebras

The aim of this chapter, which contains some of the main results of this thesis and combines almost all the tools developed in the previous chapters, is the study of the (non-)simplicity and trace-uniqueness of Hecke C^*-algebras.

We begin by studying the central projections from Proposition 5.3.3 in combination with the Haagerup type inequality in Theorem 4.3.6. If present these projections induce characters on the corresponding Hecke C^*-algebra which leads to non-simplicity results, see Subsection 6.1.2. Much in the spirit of [86, Section 6], we further apply results by Dykema (see [75]) to free products of finite-dimensional right-angled Hecke C^*-algebras to fully characterize their ideal structure and trace-uniqueness. In Section 6.2 we will then use our findings from Subsection 5.2.5 to completely characterize the simplicity of right-angled Hecke C^*-algebras and (by again applying the Haagerup inequality in Theorem 4.3.6) we deduce trace-uniqueness results for right-angled Hecke C^*-algebras.

The content of this chapter is based on parts of the articles

- M. Caspers, M. Klisse, N.S. Larsen, *Graph product Khintchine inequalities and Hecke C^*-algebras: Haagerup inequalities, (non)simplicity, nuclearity and exactness*, J. Funct. Anal. 280 (2021), no. 1, Paper No. 108795, 41 pp.
- M. Klisse, *Simplicity of right-angled Hecke C^*-algebras*, to appear in Int. Math. Res. Not. IMRN.

6.1. NON-SIMPLICITY OF HECKE C*-ALGEBRAS

Motivated by the connection of Hecke operator algebras with the ℓ^2-cohomology of buildings (see [76], [66] and also [67]), in [67, Chapter 19] Davis formulated the question for a classification of factorial Hecke-von Neumann algebras. In the right-angled single-parameter setting such a classification was obtained by Garncarek (see [86] and also Remark 3.5.7) whose results were later extended to the multi-parameter case by Raum and Skalski in [160]. Using a combinatorial approach they proved that the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ of a right-angled irreducible Coxeter system (W,S) with $\#S \geq 3$ is a factor if and only if $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathcal{R}'(W,S)$ where $\mathcal{R}'(W,S)$ is defined as in Subsection 2.7.7. For general (not necessarily right-angled) Hecke-von Neumann algebras the factoriality problem is still wide open. The C*-algebraic analogs to factoriality are the notions of simplicity and the uniqueness of the tracial state. Recall that a C*-algebra is called *simple* if it does not contain any non-trivial closed two-sided ideal.

6.1.1. CENTRAL PROJECTIONS IN HECKE C*-ALGEBRAS

Let (W,S) be a finite rank Coxeter system, let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $0 < q_s \leq 1$ for all $s \in S$ be a multi-parameter and let $W(z) = \sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ be the growth series of (W,S). Recall that by Proposition 5.3.3, if there exists $\epsilon = (\epsilon_s)_{s\in S} \in \{-1,1\}^{(W,S)}$ with $|q_\epsilon| = (q_s^{\epsilon_s})_{s\in S} \in \mathcal{R}(W,S)$, then the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ contains a central projection $E_{q,\epsilon}$ with $T_{\mathbf{w}}^{(q)} E_{q,\epsilon} = (\sqrt{q})_{\mathbf{w},\epsilon} E_{q,\epsilon}$ for all $\mathbf{w} \in W$.

In [160] Raum and Skalski, generalizing the single-parameter results by Garncarek [86], proved that for a right-angled, irreducible, finite rank Coxeter system (W,S) with at least three generators and $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $0 < q_s \leq 1$ for all $s \in S$ the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ decomposes as

$$\mathcal{N}_q(W) \cong \mathcal{M} \oplus \bigoplus_{\epsilon\in\{-1,1\}^{(W,S)}:\, |q_\epsilon|\in\mathcal{R}'(W,S)} \mathbb{C}$$

where $\mathcal{M}$ is a factor and where the direct summands $\mathbb{C}$ correspond to the central projections $E_{q,\epsilon}$; that is the center of $\mathcal{N}_q(W)$ is the linear span of the unit and the projections $E_{q,\epsilon}$. Note that by Proposition 3.5.2 the assumption that $0 < q_s \leq 1$ for all $s \in S$ is not really restrictive. The statement further implies that $\mathcal{N}_q(W)$ is a factor if and only if $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathcal{R}'(W,S)$. It is a natural question whether for $q \in \mathcal{R}'(W,S)$ the central projections in $\mathcal{N}_q(W)$ are already contained in the corresponding Hecke C*-algebra $C_{r,q}^*(W)$. We will prove this by using the Haagerup-type inequality from Theorem 4.3.6. We will further need the following easy lemma.

Lemma 6.1.1. *Let (W,S) be a finite rank Coxeter system. Then the intersection $\mathcal{R}(W,S) \cap \mathbb{R}_{>0}^{(W,S)}$ of the region of convergence $\mathcal{R}(W,S)$ of the growth series $W(z) = \sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ with $\mathbb{R}_{>0}^{(W,S)}$ is open in $\mathbb{R}_{>0}^{(W,S)}$.*

Proof. Assume that the set $\mathcal{R}(W,S) \cap \mathbb{R}_{>0}^{(W,S)}$ is not open in $\mathbb{R}_{>0}^{(W,S)}$ and let $q \in \mathcal{R}(W,S) \cap \mathbb{R}_{>0}^{(W,S)}$ be a point on its boundary. Since $\sum_{\mathbf{w}\in W} q_{\mathbf{w}}$ converges, the power series

$f(z) := \sum_{\mathbf{w}\in W} q_{\mathbf{w}} z^{|\mathbf{w}|}$ absolutely converges for all $z \in \mathbb{C}$ with $|z| \leq 1$. But the radius of convergence of f coincides with the distance of the origin to the closest pole of f, hence there exists $\lambda > 1$ such that $\sum_{\mathbf{w}\in W} q_{\mathbf{w}} z^{|\mathbf{w}|}$ absolutely converges for all $z \in \mathbb{C}$ with $|z| < \lambda$. This implies that $(2^{-1}(1+\lambda)q_s)_{s\in S} \in \mathscr{R} \cap \mathbb{R}_{>0}^{(W,S)}$ which contradicts the choice of q. □

Proposition 6.1.2. *Let (W,S) be a right-angled, finite rank Coxeter system and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $0 < q_s \leq 1$ for all $s \in S$. Further, let $\epsilon \in \{-1,1\}^{(W,S)}$, $q_\epsilon := \left(\epsilon_s q_s^{\epsilon_s}\right)_{s\in S}$ and assume that $|q_\epsilon| := \left(q_s^{\epsilon_s}\right)_{s\in S} \in \mathscr{R}(W,S)$. Then the corresponding central projection $E_{q,\epsilon} \in \mathcal{N}_q(W) \subseteq \mathscr{B}(\ell^2(W))$ from Proposition 5.3.3 is already contained in the Hecke C^*-algebra $C^*_{r,q}(W)$ and is given by the norm limit*

$$\frac{1}{W(|q_\epsilon|)} \sum_{i=0}^{\infty} \sum_{\mathbf{w}:\ |\mathbf{w}|=i} (\sqrt{q})_{\mathbf{w},\epsilon}\, T_{\mathbf{w}}^{(q)}.$$

Proof. By the assumption $|q_\epsilon| \in \mathscr{R}(W,S)$, so Lemma 6.1.1 implies that there exists $\lambda > 1$ such that still $|\lambda q_\epsilon| := \left(\lambda q_s^{\epsilon_s}\right)_{s\in S} \in \mathscr{R}(W,S)$. Using the root test criterium for convergence,

$$\limsup_{l} \left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l} \lambda^l \left|q_{\mathbf{w},\epsilon}\right| \right)^{1/l} \leq 1$$

and hence

$$\limsup_{l} \left(\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l} \left|q_{\mathbf{w},\epsilon}\right| \right)^{1/l} < 1.$$

One can therefore find $l_0 \in \mathbb{N}$ and $0 < L < 1$ such that for all $l \geq l_0$,

$$\sum_{\mathbf{w}\in W:\ |\mathbf{w}|=l} \left|q_{\mathbf{w},\epsilon}\right| < L^l. \tag{6.1.1}$$

Now set $E_{q,\epsilon}^{(i)} := \left(W(|q_\epsilon|)\right)^{-1} \sum_{l=0}^{i} \sum_{\mathbf{w}:\ |\mathbf{w}|=l} (\sqrt{q})_{\mathbf{w},\epsilon} T_{\mathbf{w}}^{(q)}$. For $i,j \in \mathbb{N}$ with $i < j$ and $i \geq l_0$ we have by Theorem 4.3.6 and the inequality (6.1.1) that

$$\begin{aligned}
\left\| E_{q,\epsilon}^{(j)} - E_{q,\epsilon}^{(i)} \right\| &\leq \frac{1}{W(|q_\epsilon|)} \sum_{l=i+1}^{j} \left\| \sum_{\mathbf{w}:\ |\mathbf{w}|=l} (\sqrt{q})_{\mathbf{w},\epsilon}\, T_{\mathbf{w}}^{(q)} \right\| \\
&\leq \frac{1}{W(|q_\epsilon|)} \sum_{l=i+1}^{j} Cl \sqrt{\sum_{\mathbf{w}:\ |\mathbf{w}|=l} \left|q_{\mathbf{w},\epsilon}\right|} \\
&< \frac{1}{W(|q_\epsilon|)} \sum_{l=i+1}^{j} ClL^{\frac{l}{2}}
\end{aligned}$$

for some $C > 0$. The series $\sum_{l=0}^{\infty} lL^{\frac{l}{2}}$ converges, so $(E_{q,\epsilon}^{(i)})_{i\in\mathbb{N}} \subseteq C^*_{r,q}(W)$ converges to

$$\frac{1}{W(|q_\epsilon|)} \sum_{i=0}^{\infty} \sum_{\mathbf{w}:\ |\mathbf{w}|=i} (\sqrt{q})_{\mathbf{w},\epsilon}\, T_{\mathbf{w}}^{(q)} \in C^*_{r,q}(W).$$

The remaining statements follow from short calculations which are similar to the ones in the proof of Proposition 5.3.3. □

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The following corollary follows from the discussion above and Proposition 3.5.2.

Corollary 6.1.3. *Let (W,S) be a right-angled, finite rank Coxeter system with $\#S \geq 3$ and let $q \in \mathbb{R}_{>0}^{(W,S)}$. Then the center of the Hecke C*-algebra $C_{r,q}^*(W)$ coincides with the center of the Hecke-von Neumann algebra $\mathcal{N}_q(W)$.*

One other immediate consequence is that right-angled Hecke C*-algebras admit a decomposition that is analogous to the one of their von Neumann-algebraic counterparts.

Corollary 6.1.4. *Let (W,S) be a right-angled, finite rank Coxeter system with $\#S \geq 3$ and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$. Then the corresponding Hecke C*-algebra $C_{r,q}^*(W)$ decomposes as*

$$C_{r,q}^*(W) \cong \pi(C_{r,q}^*(W)) \oplus \bigoplus_{\epsilon\in\{-1,1\}^{(W,S)}:\ |q_\epsilon|\in\mathcal{R}'(W,S)} \mathbb{C},$$

where π denotes the quotient map of $\mathcal{B}(\ell^2(W))$ onto $\mathcal{B}(\ell^2(W))/\mathcal{K}(\ell^2(W))$.

Proof. By Proposition 6.1.2 $C_{r,q}^*(W)$ decomposes as $A \oplus \bigoplus_{\epsilon\in\{-1,1\}^{(W,S)}:\ |q_\epsilon|\in\mathcal{R}'(W,S)} \mathbb{C}$ where

$$A = C_{r,q}^*(W) \prod_{\epsilon\in\{-1,1\}^{(W,S)}:\ |q_\epsilon|\in\mathcal{R}'(W,S)} (1 - E_{q,\epsilon}) \subseteq \mathcal{B}(\ell^2(W)).$$

By [160, Theorem A] the von Neumann algebra $A'' \subseteq \mathcal{B}(\ell^2(W))$ is a factor, necessarily of type II_1, so A contains no compact operators. This implies that $A \cong \pi(C_{r,q}^*(W))$ from which the claim follows. □

6.1.2. CHARACTERS ON HECKE C*-ALGEBRAS

Recall that, since the (spatial) tensor product of two C*-algebras is simple if and only if both C*-algebras are simple, we may by Proposition 3.5.2 and Proposition 3.5.5 restrict in the treatment of the simplicity of Hecke C*-algebras to irreducible Coxeter systems (W,S) and parameters $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $0 < q_s \leq 1$. The same is true for the uniqueness of the tracial state, see e.g. [17, Proposition 14].

The operators appearing in Proposition 5.3.3 and Proposition 6.1.2 are projections onto one-dimensional subspaces of $\ell^2(W)$ and thus induce characters on the right-angled Hecke C*-algebras as the following lemma illustrates.

Lemma 6.1.5. *Let (W,S) be a finite rank Coxeter system and $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}$ a multi-parameter. Assume moreover that $\epsilon = (\epsilon_s)_{s\in S} \in \{-1,1\}^{(W,S)}$ with $|q_\epsilon| := (q_s^{\epsilon_s})_{s\in S} \in \overline{\mathcal{R}'(W,S)}$. Then $T_s^{(q)} \mapsto \epsilon_s q_s^{\frac{\epsilon_s}{2}}$ defines a* character *(i.e. a unital, linear, multiplicative functional) on the Hecke C*-algebra $C_{r,q}^*(W)$ that we denote by χ_{q_ϵ}. In particular, $C_{r,q}^*(W)$ is not simple and does not have unique tracial state.*

Proof. By Proposition 3.5.2 we can assume that $0 < q_s \leq 1$ for all $s \in S$. First consider the case where $|q_\epsilon| \in \mathcal{R}'(W,S)$. By Proposition 5.3.3 there exists a central projection

$E_{q,\epsilon} \in \mathcal{N}_q(W)$ with $E_{q,\epsilon} T_{\mathbf{w}}^{(q)} = (\sqrt{q})_{\mathbf{w},\epsilon} E_{q,\epsilon}$ for every $\mathbf{w} \in W$. Hence the map $\chi_{q_\epsilon}(\cdot) := \tau_q(\cdot E_{q,\epsilon}) / \|E_{q,\epsilon}\delta_e\|_2$ defines a character on $C^*_{r,q}(W)$ with $\chi_{q_\epsilon}(T_s^{(q)}) = \epsilon_s q_s^{\frac{\epsilon_s}{2}}$ for $s \in S$. Its kernel is a non-trivial maximal ideal in $C^*_{r,q}(W)$, i.e. $C^*_{r,q}(W)$ is not simple and does not have a unique tracial state.

For $q \in \overline{\mathcal{R}'(W,S)} \setminus \mathcal{R}'(W,S)$ choose a sequence $(q_n)_{n\in\mathbb{N}} \subseteq \mathcal{R}'(W,S)$ with $q_n \to q$ and $0 < (q_n)_s \leq q_s$ for all $s \in S$. One easily checks that the map $\chi\colon T_{\mathbf{w}}^{(q)} \mapsto q_{\mathbf{w}}^{\frac{1}{2}}$ defines a character on $\mathbb{C}_q[W]$. For finite sums $x := \sum_{\mathbf{w}\in W} x(\mathbf{w}) T_{\mathbf{w}}^{(q)} \in \mathbb{C}_q[W]$ and $x_n := \sum_{\mathbf{w}\in W} x(\mathbf{w}) T_{\mathbf{w}}^{(q_n)} \in \mathbb{C}_{q_n}[W]$ with complex coefficients we have $x_n \to x$ in $\mathcal{B}(\ell^2(W))$ and $\chi_{(q_n)_\epsilon}(x_n) \to \chi(x)$ with $\chi_{(q_n)_\epsilon}$ defined as above. This implies that

$$|\chi(x)| = \lim_{n\to\infty} |\chi_{q_n}(x_n)| \leq \lim_n \|x_n\| = \|x\|,$$

so χ extends to a character on $C^*_{r,q}(W)$. Again, this implies that $C^*_{r,q}(W)$ is not simple and does not have unique tracial state. □

It follows from Lemma 6.1.5 that Hecke C*-algebras coming from irreducible Coxeter systems of spherical or affine type are never simple and never have a unique tracial state.

Corollary 6.1.6. *Let (W,S) be an irreducible Coxeter system of spherical or affine type. Then for any choice of parameter $q \in \mathbb{R}_{>0}^{(W,S)}$ there exists a character on the corresponding Hecke C*-algebra $C^*_{r,q}(W)$. In particular, $C^*_{r,q}(W)$ is not simple and does not have unique tracial state.*

Proof. First assume that (W,S) has finite rank and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$ with $0 < q_s \leq 1$ for every $s \in S$. Being of spherical or affine type the Coxeter group W is amenable, see 2.7.20. Hence by Proposition 2.7.20 the radius of convergence of the power series $\sum_{\mathbf{w}\in W} z^{|\mathbf{w}|}$ is one. But then $q \in \overline{\mathcal{R}(W,S)} \cap \mathbb{R}_{>0}^{(W,S)}$, so $C^*_{r,q}(W)$ admits a character by Lemma 6.1.5. For general $q \in \mathbb{R}_{>0}^{(W,S)}$, the statement follows with Proposition 3.5.2.

Next assume that $|S| = \infty$ and so (W,S) is of spherical type. Again, the map $\chi\colon T_{\mathbf{w}}^{(q)} \mapsto q_{\mathbf{w}}^{\frac{1}{2}}$, $\mathbf{w} \in W$ defines a character on $\mathbb{C}_q[W]$. For every finite sum

$$x := \sum_{\mathbf{w}\in W} x(\mathbf{w})\, T_{\mathbf{w}}^{(q)} \in \mathbb{C}_q[W]$$

with complex-valued coefficients there exists a finite subset $T \subseteq S$ such that the support $\{\mathbf{w} \in W \mid x(\mathbf{w}) \neq 0\}$ of x is contained in the special subgroup W_T of W generated by T. Now W_T is also a Coxeter group with the same exponents as W (see e.g. [67, Theorem 4.1.6 (i), Theorem 3.4.2 (i)]) and by Proposition 3.4.5 the C*-algebra $C^*_{r,q_T}(W_T)$ with $q_T := (q_t)_{t\in T}$ canonically embeds into $C^*_{r,q}(W)$. Under this identification we have $x \in C^*_{r,q}(W_T)$. But the map $T_{\mathbf{w}}^{(q)} \mapsto q_{\mathbf{w}}^{\frac{1}{2}}$ is a character on the finite-dimensional C*-algebra $C^*_{r,q}(W_T)$, hence $\|\chi(x)\| \leq \|x\|$. As this holds for every $x \in \mathbb{C}_q[W]$, χ extends to a character on $C^*_{r,q}(W)$. □

Using the results from Section 5.2 in combination with Lemma 6.1.5, in the right-angled case we can completely characterize the character space of the corresponding Hecke C*-algebras.

Proposition 6.1.7. *Let (W,S) be a right-angled, irreducible, finite rank Coxeter system and $q=(q_s)_{s\in S}\in\mathbb{R}^{(W,S)}_{>0}$. Then the set of characters of the corresponding Hecke C*-algebra $C^*_{r,q}(W)$ is given by*

$$\{\chi_{q_\epsilon}\mid \epsilon\in\{-1,1\}^{(W,S)} \text{ with } |q_\epsilon|\in\overline{\mathscr{R}'(W,S)}\},$$

where $|q_\epsilon|:=(q_s^{\epsilon_s})_{s\in S}\in\mathbb{R}^{(W,S)}_{>0}$ and where χ_{q_ϵ} satisfies $\chi_{q_\epsilon}(T_s^{(q)}):=\epsilon_s q_s^{\frac{\epsilon_s}{2}}$ for all $s\in S$.

Proof. By Proposition 3.5.2 we can assume that $0<q_s\leq 1$ for all $s\in S$. It follows from Lemma 6.1.5 that for every $\epsilon\in\{-1,1\}^{(W,S)}$ with $|q_\epsilon|\in\overline{\mathscr{R}'(W,S)}$ the character χ_{q_ϵ} exists. Conversely, let χ be a state on $\mathfrak{A}(W,S)$ which restricts to a character on $C^*_{r,q}(W)$ where $\mathfrak{A}(W,S)$ is defined as in Section 5.3. For $s\in S$ the Hecke relation $(T_s^{(q)})^2=1+p_s(q)T_s^{(q)}$ implies $(\chi(T_s^{(q)}))^2-p_s(q)\chi(T_s^{(q)})=1$ and hence $\chi(T_s^{(q)})\in\{q_s^{\frac{1}{2}},-q_s^{-\frac{1}{2}}\}$. One can thus find $\epsilon\in\{-1,1\}^{(W,S)}$ with $\chi=\chi_{q_\epsilon}$. Now assume that $|q_\epsilon|\notin\overline{\mathscr{R}'(W,S)}$, fix $s\in S$ and choose a path $\mathbf{g}:=s_1...s_n\in W$ in the Coxeter diagram of (W,S) that covers the whole graph and for which $m_{ss_1}=\infty$. By Proposition 5.2.35 there exists a sequence $(\mathbf{w}_i)_{i\in\mathbb{N}}\subseteq W$ of increasing word length with $\mathbf{g}\leq\mathbf{w}_i^{-1}$ for all $i\in\mathbb{N}$ and $|q^{-1}_{\mathbf{w}_i,\epsilon}\chi(P_{\mathbf{w}_i})|\to 0$. We further have that $P_{\mathbf{w}_i s}\leq P_{\mathbf{w}_i}$, so $\chi(P_{\mathbf{w}_i s})\leq\chi(P_{\mathbf{w}_i})$. Using that $T^{(q)}_{\mathbf{w}_i}$ and $T^{(q)}_{\mathbf{w}_i^{-1}}$ lie in the multiplicative domain of χ (see for instance [33, Proposition 1.5.7]) in combination with Proposition 5.2.31 (as well as Remark 5.2.32) one concludes

$$\begin{aligned}
|\chi(T_s^{(q)})| &= |q^{-1}_{\mathbf{w}_i,\epsilon}|\left|\chi(T^{(q)}_{\mathbf{w}_i}(1-P_s)T_s^{(q)}T^{(q)}_{\mathbf{w}_i^{-1}})+\chi(T^{(q)}_{\mathbf{w}_i}P_sT_s^{(q)}T^{(q)}_{\mathbf{w}_i^{-1}})\right|\\
&\leq \left|q^{-1}_{\mathbf{w}_i,\epsilon}\chi(T^{(q)}_{\mathbf{w}_i}T_s^{(1)}T^{(1)}_{\mathbf{w}_i^{-1}}P_{\mathbf{w}_is})\right|+\left|q^{-1}_{\mathbf{w}_i,\epsilon}\chi(P_{\mathbf{w}_is}T^{(1)}_{\mathbf{w}_i}T_s^{(q)}T^{(q)}_{\mathbf{w}_i^{-1}})\right|\\
&= \left|q^{-1/2}_{\mathbf{w}_i,\epsilon}\chi(T^{(1)}_{s\mathbf{w}_i^{-1}}P_{\mathbf{w}_is})\right|+\left|q^{-1/2}_{s,\epsilon}q^{-1/2}_{\mathbf{w}_i,\epsilon}\chi(P_{\mathbf{w}_is}T^{(1)}_{\mathbf{w}_i})\right|.
\end{aligned}$$

The Cauchy-Schwarz inequality then implies

$$|\chi(T_s^{(q)})| \leq (1+q_s^{-1/2})\sqrt{|q^{-1}_{\mathbf{w}_i,\epsilon}\chi(P_{\mathbf{w}_is})|}\to 0.$$

This contradicts $\chi(T_s^{(q)})\in\{q_s^{\frac{1}{2}},-q_s^{-\frac{1}{2}}\}$. □

6.1.3. DECOMPOSITION OF HECKE C*-ALGEBRAS OF FREE PRODUCTS OF RIGHT-ANGLED ABELIAN COXETER GROUPS

In [75] Dykema studied the simplicity and the unique trace property of reduced free products of C*-algebras and carefully investigated the ideal structure of free products of finite-dimensional C*-algebras. In the case of free products of abelian Coxeter groups this allows us to fully characterize the simplicity and the uniqueness of the canonical tracial state of the corresponding Hecke C*-algebras. The statement that we will employ is the following.

Proposition 6.1.8 ([75, Corollary 4.10]). *Let $l \in \mathbb{N}$ with $l \geq 2$ and consider for every $1 \leq m \leq l$ a compact Hausdorff space X_m endowed with a regular Borel probability measure μ_m whose support is all of X_m. Assume that each X_m consists of at least two points and exclude the case where $N = 2$ and where the spaces X_1, X_2 do consist of exactly two points. Further assume that μ_m has at most finitely many atoms each of which is an isolated point and denote by $\phi_m : f \mapsto \int f d\mu_m$ the induced state on $C(X_m)$. Define*

$$L_+ := \left\{ (x_m)_{m=1}^l \in \prod_{m=1}^l X_m \mid l-1 < \sum_{m=1}^l \mu_m(\{x_m\}) \right\}$$

and

$$L_0 := \left\{ (x_m)_{m=1}^l \in \prod_{m=1}^l X_m \mid l-1 = \sum_{m=1}^l \mu_m(\{x_m\}) \right\}.$$

Then the reduced free product C-algebra $(A, \phi) := \star_{1 \leq m \leq l}(C(X_m), \phi_m)$ has stable rank 1 and decomposes as*

$$A \cong A_0 \oplus \bigoplus_{x \in L_+} \mathbb{C}.$$

If $L_0 = \emptyset$, then A_0 is simple and carries a unique tracial state. If $L_0 \neq \emptyset$, then there exist distinct characters $(\chi_x : A_0 \to \mathbb{C})_{x \in L_0}$ such that the intersection $\bigcap_{x \in L_0} \ker(\chi_x)$ is simple, non-unital and carries a unique tracial state.

Let us now assume that (W, S) is a Coxeter system where W is of the form $W = \mathbb{Z}_2^{k_1} * \cdots * \mathbb{Z}_2^{k_l}$ with $l, k_1 \geq 2$ and $k_2, \ldots, k_l \in \mathbb{N}$ (compare with Example 2.7.1). For each $1 \leq m \leq l$ denote by $s_1^{(m)}, \ldots, s_{k_m}^{(m)}$ the mutually commuting generators corresponding to the component $\mathbb{Z}_2^{k_m}$ of W and set $S_m := \{s_1^{(m)}, \ldots, s_{k_m}^{(m)}\}$, so in particular $S = \bigcup_{m=1}^l S_m$. Let $q = (q_s)_{s \in S} \in \mathbb{R}_{>0}^{(W,S)}$. By Proposition 3.5.5 and Proposition 4.3.1 the Hecke C*-algebra $C^*_{r,q}(W)$ decomposes as a reduced free product

$$(C^*_{r,q}(W), \tau_q) \cong \star_{1 \leq m \leq l} \left(\bigotimes_{i=1}^{k_m} (C^*_{r,q_{s_i^{(m)}}}(W_{s_i^{(m)}}), \tau_{q_{s_i^{(m)}}}) \right)$$

over the canonical traces. By Proposition 6.1.2, for every $s \in S$ the 2-dimensional commutative C*-algebra $C^*_{r,q_s}(W_s)$ contains orthogonal projections

$$\frac{\sqrt{q_s}}{q_s+1} T_s^{(q_s)} + \frac{1}{q_s+1} \quad \text{and} \quad -\frac{\sqrt{q_s}}{q_s+1} T_s^{(q_s)} + \frac{q_s}{q_s+1}$$

which sum to 1, hence $C^*_{r,q_{s_i^{(m)}}}(W_{s_i^{(m)}}) \cong C(\mathbb{Z}_2)$ where the canonical tracial state τ_{q_s} corresponds to the measure μ_s which maps the identity $e \in \mathbb{Z}_2$ to $(1+q_s)^{-1}$ and which maps the generator s to $q_s(1+q_s)^{-1}$. In the notation of Proposition 6.1.8 we get that

$$(C^*_{r,q}(W),\tau_q) \cong \star_{1\leq m\leq l}(C(\mathbb{Z}_2^{k_m}),\phi_m), \tag{6.1.2}$$

where $\mu_m(\mathbf{w}) = q_{\mathbf{w}}\prod_{i=1}^{k_m}(1+q_{s_i^{(m)}})^{-1}$ for $\mathbf{w}\in\mathbb{Z}_2^{k_m}$.

Proposition 6.1.9. *Let (W,S) be a Coxeter system of the form $W = \mathbb{Z}_2^{k_1}\star\cdots\star\mathbb{Z}_2^{k_l}$ where $l,k_1,\dots,k_l\in\mathbb{N}$ and $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$. Then the growth series $W(z)$ of W is equal to the Taylor expansion of the function*

$$z\mapsto\left(\sum_{m=1}^{l}\prod_{i=1}^{k_m}(1+z_{s_i^{(m)}})^{-1}-(l-1)\right)^{-1}. \tag{6.1.3}$$

The series converges for every element in

$$\Omega := \left\{(z_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}\cap[0,1]^S \mid \sum_{m=1}^{l}\prod_{i=1}^{k_m}(1+z_{s_i^{(m)}})^{-1}>l-1\right\}$$

i.e. $\Omega\subseteq\mathscr{R}(W,S)$ where $\mathscr{R}(W,S)$ is the region of convergence of $W(z)$, as defined in Subsection 2.7.7.

Proof. It has been shown in Example 2.7.19 that for all $z\in\mathbb{C}^{(W,S)}$ for which $\sum_{\mathbf{w}\in W} z_{\mathbf{w}}$ absolutely converges,

$$W(z)=\left(\sum_{m=1}^{l}\prod_{i=1}^{k_m}(1+z_{s_i^{(m)}})^{-1}-(l-1)\right)^{-1}. \tag{6.1.4}$$

This implies that the growth series $W(z)$ of W is equal to the Taylor expansion of (6.1.3) in 0. From (6.1.4) we also find that

$$W(z)=(1-F(z))^{-1}\prod_{s\in S}(1+z_s) \tag{6.1.5}$$

with the polynomial

$$F(z) := 1+(l-1)\prod_{s\in S}(1+z_s)-\sum_{m=1}^{l}\prod_{n\neq m}\prod_{j=1}^{k_n}(1+z_{s_j^{(n)}}).$$

One has $F(0)=0$. With $D^\alpha F$ denoting the higher order partial derivative of F with respect to $0\neq\alpha\in\{0,1\}^S$ we have

$$\begin{aligned} D^\alpha F\big|_{z=0} &= \left[(l-1)\prod_{s\in S:\ \alpha_s=0}(1+z_s)-\sum_{m\in J}\prod_{n\neq m}\prod_{j\in K_n}(1+z_{s_j^{(n)}})\right]_{z=0} \\ &= (l-1)-\sum_{m\in J}1 \end{aligned}$$

$$\geq \quad 0,$$

where $J := \{m \mid 1 \leq m \leq l \text{ with } \alpha_s = 0 \text{ for all } s \in S_m\}$ and

$$K_n := \{j \mid 1 \leq j \leq k_m \text{ with } \alpha_{s_j^{(n)}} = 0\}.$$

This implies that F has only positive coefficients. Hence for $z \in \Omega$ we have $F(z) \geq 0$. Moreover, for $z \in \Omega$ we have $W(z) > 0$ and so by (6.1.5), $0 \leq F(z) < 1$. In particular, the terms in the series $\sum_{m=0}^{\infty}(F(z))^m$ can be expanded and rearranged to get the (converging) Taylor series of $(1 - F(z))^{-1}$. The same is true for the product $(1 - F(z))^{-1}\prod_{s\in S}(1 + z_s)$. We get that the Taylor series of the function in (6.1.3) (which is the growth series $W(z)$) converges on Ω. □

In combination with Proposition 6.1.8, Proposition 6.1.9 implies the following.

Theorem 6.1.10. *Let (W,S) be a Coxeter system of the form $W = \mathbb{Z}_2^{k_1} \star \cdots \star \mathbb{Z}_2^{k_l}$ where $l, k_1 \geq 2$ and let $q = (q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)}$. Then the following statements are equivalent.*

(1) $C^*_{r,q}(W)$ is simple;

(2) $C^*_{r,q}(W)$ carries a unique tracial state;

(3) $q \notin \overline{\mathscr{R}'(W,S)}$, where $\mathscr{R}'(W,S)$ is defined as in Subsection 2.7.7.

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Proof. "(1) ⇔ (2)": The equivalence of the first two statements follows from Proposition 6.1.8.

"(2) ⇒ (3)": Assume that $q \in \overline{\mathscr{R}'(W,S)}$. Then $C^*_{r,q}(W)$ is not simple and does not have unique tracial state by Lemma 6.1.5.

"(3) ⇒ (1)": Assume that $C^*_{r,q}(W)$ is not simple. By Proposition 3.5.2 we may further assume that $0 < q_s \leq 1$ for every $s \in S$. Using Proposition 6.1.8 in combination with (6.1.2) we get that the set

$$\left\{(\mathbf{w}_1, \ldots, \mathbf{w}_l) \in \prod_{m=1}^{l} \mathbb{Z}_2^{k_m} \mid l - 1 \leq \sum_{m=1}^{l} q_{\mathbf{w}_m} \prod_{i=1}^{k_m}(1 + q_{s_i^{(m)}})^{-1}\right\}$$

is not empty, so in particular

$$l - 1 \leq \max_{(\mathbf{w}_1,\ldots,\mathbf{w}_l)\in\prod_{m=1}^{l}\mathbb{Z}_2^{k_m}} \left(\sum_{m=1}^{l} q_{\mathbf{w}_m} \prod_{i=1}^{k_m}(1 + q_{s_i^{(m)}})^{-1}\right) \leq \sum_{m=1}^{l}\prod_{i=1}^{k_m}(1 + q_{s_i^{(m)}})^{-1}.$$

Comparing this with Proposition 6.1.9, we get that $q \in \overline{\mathscr{R}'(W,S)}$. □

Remark 6.1.11. In the proof of Theorem 6.1.10 we only used the simplicity and trace-uniqueness part of Proposition 6.1.8. For (W,S) as above the full statement of the proposition also provides a detailed description of the ideal structure of $C^*_{r,q}(W)$ for $q \in \overline{\mathscr{R}'(W,S)}$ (which coincides with our findings in Corollary 6.1.4). Further we conclude that for $l \geq 3$ the Hecke C*-algebra $C^*_{r,q}(W)$ has stable rank 1 for every $q \in \mathbb{R}_{>0}^{(W,S)}$.

Remark 6.1.12. In [120] a new approach to C*-simplicity results was obtained via Furstenberg/Hamana boundaries (see [97], [98], [99], [100], [84], [85]), see also [30], [95], [125], [16] and [121]. In [16] the Furstenberg boundary of a general unitary representation of a discrete group was defined and investigated in relation to trace-uniqueness properties. Proposition 3.5.6 shows that Hecke C*-algebras of a right-angled Coxeter group are C*-algebras generated by such a unitary representation. It would be interesting to exploit this connection. However, in light of the results from [16], it is not clear how manageable the Furstenberg-Hamana boundary is.

6.2. SIMPLICITY OF RIGHT-ANGLED HECKE C*-ALGEBRAS

Using the results and ideas from Section 5.2 we will now extend the simplicity results from Subsection 6.1.3 to arbitrary right-angled Coxeter systems. Our approach is inspired by [95]. Recall that for a finite rank Coxeter system (W,S) and $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$ the Hecke C*-algebra $C^*_{r,q}(W)$ can be viewed as a C*-subalgebra of $\pi(\mathfrak{A}(W))$ where π and $\mathfrak{A}(W,S)$ are defined as in Section 5.3. We will use this observation frequently.

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Proposition 6.2.1. *Let (W,S) be a right-angled, irreducible, finite rank Coxeter system, $q=(q_s)_{s\in S} \in \mathbb{R}_{>0}^{(W,S)} \setminus \overline{\mathscr{R}'(W,S)}$ with $0<q_s\leq 1$ for all $s\in S$ and let $I \neq C^*_{r,q}(W)$ be an ideal of $C^*_{r,q}(W)$ where we view $C^*_{r,q}(W)$ as a C*-subalgebra of $\pi(\mathfrak{A}(W,S))$. Then, for every two elements $s,t\in S$ with $m_{st}=\infty$ there exists a state ϕ on $\pi(\mathfrak{A}(W,S))$ that vanishes on I and which satisfies $\phi(P_s)=1$, $\phi(P_t)=0$.*

Proof. Choose a state on $C^*_{r,q}(W)$ that vanishes on I. We can extend it to a state ψ on $\pi(\mathfrak{A}(W,S))$. Further let $\mathbf{g} := s_1...s_n \in W$ with $s_1 := s$, $s_2 := t$ be a path in the Coxeter diagram of (W,S) that covers the whole graph and let $(\mathbf{w}_i)_{i\in\mathbb{N}} \subseteq W$ be a sequence as in Proposition 5.2.35, i.e. the $\mathbf{w}_i$ have increasing word length, $\mathbf{g} \leq \mathbf{w}_i^{-1}$ for all $i\in\mathbb{N}$ and $q_{\mathbf{w}_i}^{-1}\psi(\widetilde{P}_{\mathbf{w}_i}) \to 0$. Note that $\psi(T^{(q)}_{\mathbf{w}_i} T^{(q)}_{\mathbf{w}_i^{-1}}) = 0$ is not possible since then Lemma 3.4.7 and the Cauchy-Schwarz inequality would imply

$$0=\psi(T^{(q)}_{\mathbf{w}_i} T^{(q)}_{\mathbf{w}_i^{-1}}) \geq q_{\mathbf{w}_i s_1}\psi((T^{(q)}_{s_1})^2) \geq q_{\mathbf{w}_i s_1}|\psi(T^{(q)}_{s_1})|^2$$

and thus $\psi((T^{(q)}_{s_1})^2)=\psi(T^{(q)}_{s_1})=0$. This contradicts the identity $(T^{(q)}_{s_1})^2 = 1+p_s(q)\,T^{(q)}_{s_1}$. With Proposition 5.2.31 (as well as Remark 5.2.32) and Lemma 3.4.7 we get that for $i\in\mathbb{N}$,

$$\left|\frac{\psi(T^{(q)}_{\mathbf{w}_i}\widetilde{P}_s T^{(q)}_{\mathbf{w}_i^{-1}})}{\psi(T^{(q)}_{\mathbf{w}_i} T^{(q)}_{\mathbf{w}_i^{-1}})}-1\right| = \left|\frac{\psi(T^{(q)}_{\mathbf{w}_i}(\widetilde{P}_s-1) T^{(q)}_{\mathbf{w}_i^{-1}})}{\psi(T^{(q)}_{\mathbf{w}_i} T^{(q)}_{\mathbf{w}_i^{-1}})}\right| = \left|\frac{\psi(\widetilde{P}_{\mathbf{w}_i})}{\psi(T^{(q)}_{\mathbf{w}_i} T^{(q)}_{\mathbf{w}_i^{-1}})}\right| \leq q_{\mathbf{w}_i}^{-1}\psi(\widetilde{P}_{\mathbf{w}_i}) \to 0.$$

The weak-$*$ compactness of the state space $\mathscr{S}(\pi(\mathfrak{A}(W,S)))$ implies that we can find a subsequence of

$$\left((\psi(T^{(q)}_{\mathbf{w}_i} T^{(q)}_{\mathbf{w}_i^{-1}}))^{-1}\psi(T^{(q)}_{\mathbf{w}_i}(\cdot) T^{(q)}_{\mathbf{w}_i^{-1}})\right)_{i\in\mathbb{N}} \subseteq \mathscr{S}(\pi(\mathfrak{A}(W,S)))$$

that weak-$*$ converges to a state ϕ. By construction, this state vanishes on the ideal I, we have $\phi(\widetilde{P}_s)=1$ and hence also $\phi(\widetilde{P}_t)=0$ since $0\le\widetilde{P}_t\le 1-\widetilde{P}_s$. □

The following corollary immediately follows from Proposition 3.4.6.

Corollary 6.2.2. *Let (W,S) be a right-angled Coxeter system, $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$, $l\in\mathbb{N}$ and let $\mathbf{g}:=s_1...s_n\in W$ be a closed path in the Coxeter diagram of (W,S). Then there exists an operator $x\in\mathfrak{A}(W,S)$ such that $T_{\mathbf{g}^l}^{(q)}$ decomposes as $T_{\mathbf{g}^l}^{(q)}=T_{\mathbf{g}^l}^{(1)}+P_{s_1}x$.*

Proof. Write $t_1...t_m$ for the reduced expression $\mathbf{g}^l=(s_1...s_n)(s_1...s_n)...(s_1...s_n)$ of $\mathbf{g}^l$ where $m=nl$. By Proposition 3.4.6 the operator $T_{\mathbf{g}^l}^{(q)}$ decomposes as

$$T_{\mathbf{g}^l}^{(q)}=T_{\mathbf{g}^l}^{(1)}+\sum_{i=1}^{m}p_{t_i}(q)\,T_{t_1...t_{i-1}}^{(1)}P_{t_i}T_{t_{i+1}...t_m}^{(1)},$$

where the first summand corresponds to the triple $(e,\emptyset,\mathbf{g}^l)\in A_{\mathbf{g}^l}$ and the other summands correspond to triples $(t_1...t_{i-1},t_i,t_{i+1}...t_m)\in A_{\mathbf{g}^l}$ with $i=1,...,m$ (since $\mathbf{g}$ is a closed path in the Coxeter diagram of (W,S) all triples in $A_{\mathbf{g}^l}$ are of these forms); here the set $A_{\mathbf{g}^l}$ is defined as in Proposition 3.4.6. Using the description in Proposition 5.2.31 we get that

$$T_{\mathbf{g}^l}^{(q)}=T_{\mathbf{g}^l}^{(1)}+\sum_{i=1}^{m}p_{t_i}(q)\,T_{t_1...t_{i-2}}^{(1)}P_{t_{i-1}t_i}T_{t_{i-1}t_{i+1}...t_m}^{(1)}=...=T_{\mathbf{g}^l}^{(1)}+\sum_{i=1}^{m}p_{t_i}(q)P_{t_1...t_i}T_{t_1...\widehat{t_i}...t_m}^{(1)},$$

so the claim follows by setting $x:=\sum_{i=1}^{m}p_{t_i}(q)P_{t_1...t_i}T_{t_1...\widehat{t_i}...t_n}^{(1)}$. □

Recall that the inner action of the group W on $\pi(\mathfrak{A}(W,S))$ defined by $\mathbf{w}.x:=T_{\mathbf{w}}^{(1)}xT_{\mathbf{w}^{-1}}^{(1)}$ for $\mathbf{w}\in W$, $x\in\pi(\mathfrak{A}(W,S))$ induces an action of W on the state space of $\pi(\mathfrak{A}(W,S))$ via $(\mathbf{w}.\phi)(x):=\phi(T_{\mathbf{w}^{-1}}^{(1)}xT_{\mathbf{w}}^{(1)})$ for $\phi\in\mathscr{S}(\pi(\mathfrak{A}(W,S)))$, $\mathbf{w}\in W$ and $x\in\pi(\mathfrak{A}(W,S))$.

We are now ready to characterize the simplicity of right-angled Hecke C*-algebras.

Theorem 6.2.3. *Let (W,S) be an irreducible, right-angled, finite rank Coxeter system and let $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter. Then the Hecke C*-algebra $C_{r,q}^*(W)$ is simple if and only if $q\in\mathbb{R}_{>0}^{(W,S)}\setminus\overline{\mathscr{R}'(W,S)}$.*

Proof. By Proposition 6.1.7 the Hecke C*-algebra $C_{r,q}^*(W)$ is not simple for $q\in\overline{\mathscr{R}'(W,S)}$. For the treatment of the case where $q\in\mathbb{R}_{>0}^{(W,S)}\setminus\overline{\mathscr{R}'(W,S)}$ by Proposition 3.5.2 it suffices to consider multi-parameters with $0<q_s\le 1$ for all $s\in S$. View $C_{r,q}^*(W)$ as a C*-subalgebra of $\pi(\mathfrak{A}(W,S))$ and assume that $I\neq C_{r,q}^*(W)$ is an ideal in $C_{r,q}^*(W)$. Further choose a closed path $\mathbf{g}:=s_1...s_n$ in the Coxeter diagram of (W,S) that covers the whole graph. Proposition 6.2.1 implies that we can find a state ϕ on $\pi(\mathfrak{A}(W,S))$ that vanishes on I and for which $\phi(\widetilde{P}_{s_1})=1$, $\phi(\widetilde{P}_{s_n})=0$ holds. In particular the projections $\widetilde{P}_{s_1}$, $\widetilde{P}_{s_n}$ are contained in the multiplicative domain of ϕ (see for instance [33, Proposition 1.5.7]).

By the identification $\pi(\mathscr{D}(W,S)) \cong C(\partial(W,S))$ and the equality $\phi(\widetilde{P}_{s_1}) = 1$ the restriction of ϕ to $\pi(\mathscr{D}(W,S))$ corresponds to a probability measure μ on the boundary $\partial(W,S)$ whose support is contained in the set of all $z \in \partial(W,S)$ with $s_1 \leq z$. The sequence $(\mathbf{g}^i.\mu)_{i\in\mathbb{N}}$ hence weak-$*$ converges to the point mass $\delta_{\mathbf{g}^\infty} \in \operatorname{Prob}(\partial(W,S))$ where $g^\infty := \lim_l \mathbf{g}^l \in \partial(W,S)$ (compare also with the proof of Theorem 5.2.21). This implies that there exists an increasing sequence $(i_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ for which $\left(\mathbf{g}^{i_k}.\phi\right)_{k\in\mathbb{N}}$ weak-$*$ converges to a state ψ whose restriction to $\pi(\mathscr{D}(W,S))$ is multiplicative. The product $s_n ... s_1$ also defines a path in the Coxeter diagram of (W,S). Using Corollary 6.2.2 and $\phi(\widetilde{P}_{s_n}) = 0$ one deduces that for $a \in I$

$$\psi(a) = \lim_k \phi(T^{(1)}_{\mathbf{g}^{-i_k}} a T^{(1)}_{\mathbf{g}^{i_k}}) = \lim_k \phi(T^{(q)}_{\mathbf{g}^{-i_k}} a T^{(q)}_{\mathbf{g}^{i_k}}) = 0,$$

so ψ vanishes on the ideal I.

Now, let J be the ideal in $\pi(\mathfrak{A}(W))$ generated by I. Since $\pi(\mathfrak{A}(W,S))$ identifies with the crossed product C*-algebra $C(\partial(W,S)) \rtimes_r W$, every element in $\pi(\mathfrak{A}(W,S))$ can be approximated by finite sums of the form $\sum_{\mathbf{w}\in W} f_{\mathbf{w}} T^{(1)}_{\mathbf{w}}$ where $f_{\mathbf{w}} \in \pi(\mathscr{D}(W,S))$. Using $T^{(1)}_s = T^{(q)}_s - p_s(q)P_s$ for $s \in S$, one concludes via induction that every such operator can be written as a finite sum the form $\sum_{\mathbf{w}\in W} g_{\mathbf{w}} T^{(q)}_{\mathbf{w}}$ for suitable $g_{\mathbf{w}} \in \pi(\mathscr{D}(W,S))$. But for all $a \in I$, $g, h \in \pi(\mathscr{D}(W,S))$ and $\mathbf{v}, \mathbf{w} \in W$ we have that

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$$\psi((gT^{(q)}_{\mathbf{w}})a(T^{(q)}_{\mathbf{v}}h)) = \psi(g)\psi(T^{(q)}_{\mathbf{w}} a T^{(q)}_{\mathbf{v}})\psi(h) = 0$$

since $T^{(q)}_{\mathbf{w}} a T^{(q)}_{\mathbf{v}} \in I$, so the state ψ vanishes on J. In particular, since $\psi \neq 0$, J can not coincide with the whole C*-algebra $\pi(\mathfrak{A}(W,S))$. But $\pi(\mathfrak{A}(W,S))$ is simple by Corollary 5.3.2, so $J = 0$. We get that $C^*_{r,q}(W)$ must be simple as well. This completes the proof. ☐

Corollary 6.2.4. *Let (W,S) be an irreducible, right-angled Coxeter system with $\#S = \infty$ and let $q = (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$. Then the Hecke C*-algebra $C^*_{r,q}(W)$ is simple if and only if there exists a finite subset $T \subseteq S$ such that the Hecke C*-algebra $C^*_{r,q_T}(W_T)$ with $q_T := (q_t)_{t\in T}$ is simple.*

Proof. Again, by Proposition 3.5.2 it suffices to consider multi-parameters with $0 < q_s \leq 1$ for $s \in S$. First assume that for all finite subsets $T \subseteq S$ the Hecke C*-algebra $C^*_{r,q_T}(W_T)$ is not simple. The map $\chi : T^{(q)}_{\mathbf{w}} \mapsto q^{\frac{1}{2}}_{\mathbf{w}}$, $\mathbf{w} \in W$ defines a character on $\mathbb{C}_q[W]$. Further, for every element $x := \sum_{\mathbf{w}\in W} x(\mathbf{w}) T^{(q)}_{\mathbf{w}} \in \mathbb{C}_q[W]$ with $x(\mathbf{w}) \in \mathbb{C}$ for all $\mathbf{w} \in W$ there exists a finite subset $T \subseteq S$ such that the support $\{\mathbf{w} \in W \mid x(\mathbf{w}) \neq 0\}$ of x is contained in the special subgroup W_T. By Proposition 3.4.5, $C^*_{r,q_T}(W_T)$ canonically embeds into $C^*_{r,q}(W)$. Since by the assumption $C^*_{r,q_T}(W_T)$ is not simple, Theorem 6.2.3 implies in combination with Proposition 6.1.7 that the restriction of χ to $\mathbb{C}_{q_T}[W_T]$ continuously extends to a character χ_T on $C^*_{r,q_T}(W_T)$. But then, $|\chi(x)| = |\chi_T(x)| \leq \|x\|$, so (as x was arbitrary) χ continuously extends to a character on $C^*_{r,q}(W)$. Hence $C^*_{r,q}(W)$ is not simple.

Conversely assume that there exists a finite subset $T \subseteq S$ for which $C^*_{r,q_T}(W_T)$ is simple. Then from Theorem 6.2.3 it follows that the C*-algebra $C^*_{r,q_{T'}}(W_{T'})$ is simple for all finite subsets $T' \subseteq S$ with $T \subseteq T'$. It is a standard fact that inductive limits of simple C*-algebras are simple (see e.g. [20, II.8.2.5]), so the simplicity of $C^*_{r,q}(W)$ follows from Proposition 3.5.3. □

The following example demonstrates that there exist infinitely generated right-angled, irreducible Coxeter systems and corresponding multi-parameters whose respective Hecke C*-algebras are non-simple.

Example 6.2.5. Let $S = \{s_1, s_2, ...\}$ be a countable set and consider the Coxeter group W generated by S subject to the relations defined by $m_{ss} = 2$ for all $s \in S$ and $m_{st} = \infty$ for all $s, t \in S$, $s \neq t$. Define $q := (q_s)_{s\in S} \in \mathbb{R}^{(W,S)}_{>0}$ by $q_{s_i} := 2^{-i}$ for $i \in \mathbb{N}_{\geq 1}$. Then for every finite subset $T \subseteq S$ one checks that

$$\sum_{s\in T} \frac{1}{1+q_s} \geq \sum_{i=1}^{\#T} \frac{1}{1+2^{-i}} \geq \#T - 1$$

and hence, by the analysis in Subsection 6.1.3, the C*-algebra $C^*_{r,q_T}(W_T)$ is not simple. Corollary 6.2.4 then implies that $C^*_{r,q}(W)$ is not simple.

6.3. THE UNIQUE TRACE PROPERTY

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Unfortunately, there seems to be no obvious way to treat the uniqueness of the canonical tracial state of right-angled Hecke C*-algebras with the methods employed in Section 6.2. In this section we will therefore walk an alternative path by using an approach inspired by Powers' averaging argument in [157]. This requires the introduction of suitable averaging operators on the corresponding Hecke algebra which average over a finite subset of the Coxeter group. The following statements apply in greater generality. Therefore we will formulate our approach for arbitrary discrete groups.

Let G be a discrete group. Denote the left regular representation of G by λ, let τ be the canonical tracial state on $C^*_r(G)$ and fix some finite subset $F \subseteq G$. Every such F defines an averaging operator Φ_F on $C^*_r(G)$ given by

$$\Phi_F(x) := \frac{1}{|F|} \sum_{g\in F} \lambda_{g^{-1}} x \lambda_g.$$

The map Φ_F is trace-preserving, unital and completely positive. In particular it induces a bounded operator $\widetilde{\Phi}_F$ on $\ell^2(G) \ominus \mathbb{C}\delta_e$ given by $\widetilde{\Phi}_F(x\delta_e) := \Phi_F(x)\delta_e$ for all $x \in C_c(G)$ with $\tau(x) = 0$. Indeed, the Kadison-Schwarz inequality implies that

$$\|\Phi_F(x)\delta_e\|_2^2 = \tau(\Phi_F(x)^*\Phi_F(x)) \leq \tau(\Phi_F(x^*x)) = \tau(x^*x) = \|x\delta_e\|_2^2.$$

Proposition 6.3.1. *Let G be a discrete group with the property that there exist three elements $g_1, g_2, g_3 \in G$ and a subset $D \subseteq G \setminus \{e\}$ such that $D \cup g_1 D g_1^{-1} = G \setminus \{e\}$ and the sets D, $g_2 D g_2^{-1}$ and $g_3 D g_3^{-1}$ are pairwise disjoint. Take $F := \{e, g_1, g_2, g_3\}$. Then the operator $\widetilde{\Phi}_F$ on $\ell^2(G) \ominus \mathbb{C}\delta_e$ has norm strictly smaller than one.*

The proof of Proposition 6.3.1 is based on Ching's following variation of Pukánszky's 14ε-argument in [158, Lemma 10].

Lemma 6.3.2 ([53, Lemma 4]). *Let G be a group as in Proposition 6.3.1. Then*

$$\|(x-\tau(x))\delta_e\|_2 \leq 14 \max_{i=1,2,3} \|(x-\lambda_{g_i^{-1}} x \lambda_{g_i})\delta_e\|_2$$

for every $x \in C_r^*(G)$.

Further, we shall need that for arbitrary vectors $\xi_0, \ldots, \xi_n$ in a Hilbert space $\mathscr{H}$ we have the following equality, which for $n = 1$ is known as the parallelogram law:

$$\|\sum_{i=0}^{n} \xi_i\|^2 + \sum_{0\leq i<j\leq n} \|\xi_i - \xi_j\|^2 = (n+1)\sum_{i=0}^{n} \|\xi_i\|^2. \tag{6.3.1}$$

Note that (6.3.1) can be verified directly by writing out all norms as inner products.

Proof of Proposition 6.3.1. The norm of $\widetilde{\Phi}_F : \ell^2(G) \ominus \mathbb{C}\delta_e \to \ell^2(G) \ominus \mathbb{C}\delta_e$ is clearly majorized by 1. Now suppose that $\|\widetilde{\Phi}_F\| = 1$ and take a sequence $x_k \in C_c(G)$ with $\tau(x_k) = 0$ and $\|x_k\delta_e\|_2 = 1$ such that $\|\widetilde{\Phi}_F(x_k\delta_e)\|_2 \nearrow 1$. Set $\xi_i^k = \lambda_{g_i^{-1}} x_k \lambda_{g_i}\delta_e$ with g_1, g_2, g_3 as in the proposition and $g_0 := e$. By (6.3.1) we have

$$\sum_{0\leq i<j\leq 3} \|\xi_i^k - \xi_j^k\|_2^2 = 4\sum_{i=0}^{3} \|\xi_i^k\|_2^2 - \|\sum_{i=0}^{3} \xi_i^k\|_2^2 = 4^2 - \|4\widetilde{\Phi}_F(x_k\delta_e)\|_2^2 \to 4^2 - 4^2 = 0.$$

Therefore each of the individual summands on the left-hand side converges to 0 as $k \to \infty$. Since $\xi_0^k = x_k\delta_e$ we see from Lemma 6.3.2 that $\|x_k\delta_e\|_2 = \|x_k\delta_e - \tau(x_k)\delta_e\|_2^2 \to 0$. This contradicts that $\|x_k\delta_e\|_2 = 1$. We conclude that $\|\widetilde{\Phi}_F\| < 1$. □

Let us now bring together Theorem 4.3.6 and Lemma 6.3.2.

Theorem 6.3.3. *Let (W,S) be a irreducible, right-angled, finite rank Coxeter system with $\#S \geq 3$. Then there exists an open neighborhood $\mathscr{U} \subseteq \mathbb{R}_{>0}^{(W,S)}$ of 1 such that for all $q = (q_s)_{s\in S} \in \mathscr{U}$ the Hecke C^*-algebra $C_{r,q}^*(W)$ has unique tracial state.*

Proof. As (W,S) is irreducible with $\#S \geq 3$ we find a path $st_0...t_n$ in the Coxeter diagram of (W,S) that covers the whole graph and for which $t_1 \neq s$. Then $\mathbf{w}_1 := t_0 \cdots t_n \cdots t_0$, $\mathbf{w}_2 := s$, $\mathbf{w}_3 := t_1$ and $D := \{\mathbf{w} \in W \mid t_0 \leq \mathbf{w}\}$ satisfy the conditions from Proposition 6.3.1. For every reduced expression $\mathbf{w} = s_1 \cdots s_m$ in W consider the operator

$$\prod_{i=1}^{m}\left(\frac{1-q_{s_i}}{1+q_{s_i}} + \frac{2\sqrt{q_{s_i}}}{1+q_{s_i}} T_{s_i}^{(q)}\right) \in C_{r,q}^*(W).$$

By the same arguments as in the proof of Proposition 3.5.6 this operator is unitary and does not depend on the reduced expression for $\mathbf{w}$. By abuse of notation we will denote it by $\pi_{q,1}(T_{\mathbf{w}}^{(1)})$. Choose a positive integer d with $|\mathbf{w}| \leq d$ for all $\mathbf{w}$ in $F := \{e, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and define a "deformed" averaging operator Φ_q on $C_{r,q}^*(W)$ by

$$\Phi_q(x) = \frac{1}{|F|}\sum_{\mathbf{w}\in F} \pi_{q,1}(T_{\mathbf{w}^{-1}}^{(1)})\, x\, \pi_{q,1}(T_{\mathbf{w}}^{(1)}).$$

Again, these maps are trace-preserving, unital, completely positive and they induce contractive linear operators $\widetilde{\Phi}_q$ on $\ell^2(W) \ominus \mathbb{C}\delta_e$ via $\widetilde{\Phi}_q(x\delta_e) := \Phi_q(x)\delta_e$ for $x \in \mathbb{C}_q[W]$. One easily checks that $\|\widetilde{\Phi}_q\| \to \|\widetilde{\Phi}_F\|$ for $q \to 1$. In particular, Proposition 6.3.1 implies that there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}_{>0}^{(W,S)}$ of 1 such that $\widetilde{\Phi}_q$ has norm strictly smaller than one for all $q \in \mathcal{U}$.

For $r \in \mathbb{N}$ denote by χ_r the word length projection on $C^*_{r,q}(W)$ from equation (4.1.1) and put $\chi_{\leq d} := \sum_{r=0}^{d} \chi_r$. Let $l \geq 1$ and $x \in \chi_{\leq d}(\mathbb{C}_q[W])$. Note that for every $\mathbf{w} \in F$ we have $\chi_r(\pi_{q,1}(T^{(1)}_{\mathbf{w}})) = 0$ for $r \geq d$. As Φ^l_q averages over $\{\pi_{q,1}(T^{(1)}_{\mathbf{w}}) \mid \mathbf{w} \in F\}$ and $\chi_r(x) = 0$ for $r \geq d$, we get that $\chi_r(\Phi^l_q(x)) = 0$ for $r \geq (2l+1)d$. In particular

$$\|\Phi^l_q(x) - \tau_q(x)\| \leq \sum_{r=1}^{(2l+1)d} \|\chi_r(\Phi^l_q(x))\|$$

by the triangle inequality and with $C := (\#\mathrm{Cliq}(K))^3 (\prod_{s\in S} p_s(q))$ (where K is defined as in Theorem 4.3.6) we have

$$\|\Phi^l_q(x) - \tau_q(x)\| \leq \sum_{r=1}^{(2l+1)d} Cr\|\chi_r(\Phi^l_q(x))\delta_e\|_2$$

by Theorem 4.3.6. The Cauchy-Schwarz inequality then implies

$$\begin{aligned}
\|\Phi^l_q(x) - \tau_q(x)\| &\leq ((2l+1)d)^{\frac{1}{2}} \left(\sum_{r=1}^{(2l+1)d} C^2 r^2 \|\chi_r(\Phi^l_q(x))\delta_e\|_2^2 \right)^{\frac{1}{2}} \\
&\leq C((2l+1)d)^{\frac{3}{2}} \left(\sum_{r=1}^{(2l+1)d} \|\chi_r(\Phi^l_q(x))\delta_e\|_2^2 \right)^{\frac{1}{2}} \\
&= C((2l+1)d)^{\frac{3}{2}} \|\Phi^l_q(x)\delta_e - \tau_q \circ \Phi^l_q(x)\delta_e\|_2 \\
&\leq C((2l+1)d)^{\frac{3}{2}} \|\widetilde{\Phi}_q\|^l \|x\delta_e - \tau_q(x)\delta_e\|_2.
\end{aligned}$$

For $q \in \mathcal{U}$ this converges to 0 as $l \to \infty$. The unique trace property of $C^*_{r,q}(W)$, $q \in \mathcal{U}$ now follows by a standard argument (see for instance [157]): Let $x \in C^*_{r,q}(W)$ be positive. For every $\varepsilon > 0$ we find $x_\varepsilon \in \mathbb{C}_q[W]$ with $\|x - x_\varepsilon\| < \frac{\varepsilon}{3}$. For l large enough this implies

$$\|\Phi^l_q(x) - \tau_q(x)\| \leq \|\Phi^l_q(x - x_\varepsilon)\| + \|\Phi^l_q(x_\varepsilon) - \tau_q(x_\varepsilon)\| + \|\tau_q(x_\varepsilon) - \tau_q(x)\| < \varepsilon,$$

so $\Phi^l_q(x) \to \tau_q(x)$. For every tracial state τ' on $C^*_{r,q}(W)$ we get that

$$\tau'(x) = \tau'(\Phi^l_q(x)) = \tau_q(x),$$

that is $C^*_{r,q}(W)$ carries τ_q as its unique tracial state. □

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The relative Haagerup property

The present chapter will be somewhat isolated from the rest of this dissertation. Its aim is the introduction and examination of a generalized version of the relative Haagerup property (see [23] and [155]) for a unital, expected inclusion of arbitrary σ-finite von Neumann algebras. We will show that if the smaller algebra is finite then the notion only depends on the inclusion itself, and not on the choice of the conditional expectation. Further, several variations of the definition are shown to be equivalent in this case, and in particular the approximating maps can be chosen to be unital and preserving the reference state. The concept is then applied to amalgamated free products of von Neumann algebras and used to deduce that the standard Haagerup property for a von Neumann algebra is stable under taking reduced free products with amalgamation over finite-dimensional subalgebras. These results are illustrated by examples coming from von Neumann algebras of quantum orthogonal groups and will play a role in Chapter 8.

The content of these sections is entirely based on the article

- M. Caspers, M. Klisse, A. Skalski, G. Vos, M. Wasilewski, *Relative Haagerup property for arbitrary von Neumann algebras*, arXiv preprint arXiv:2110.15078 (2021).

7.1. Preliminaries

Let us begin by recalling some facts regarding von Neumann algebras, their modular theory and completely positive approximations. Throughout the whole chapter we will assume that the von Neumann algebras we study are σ-finite, i.e. that they admit faithful normal states.

7.1.1. GENERAL VON NEUMANN ALGEBRA THEORY

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ a von Neumann algebra. A self-adjoint (possibly unbounded) operator h on $\mathscr{H}$ is said to be *affiliated with* $\mathscr{M}$ if for all $k \in \mathbb{N}$ the corresponding spectral projection $E_{[-k,k]}(h)$ is an element of $\mathscr{M}$. Equivalently, h is affiliated with $\mathscr{M}$ if and only if h commutes with all unitaries in the commutant $\mathscr{M}' \subseteq \mathscr{B}(\mathscr{H})$.

We will always assume inclusions of von Neumann algebras $\mathscr{N} \subseteq \mathscr{M}$ to be unital in the sense that $1_{\mathscr{M}} \in \mathscr{N}$, and conditional expectations to be faithful and normal. We will usually repeat these conditions throughout the chapter. For a functional $\varphi \in \mathscr{M}_*$ in the predual of $\mathscr{M}$ and elements $a, b \in \mathscr{M}$ we denote by $a\varphi b \in \mathscr{M}_*$ the normal functional given by $(a\varphi b)(x) := \varphi(bax)$ for $x \in \mathscr{M}$, and further write $a\varphi$ for $a\varphi 1$ and φb for $1\varphi b$. If $\varphi \in \mathscr{M}_*$ is faithful, normal and positive, as before we write $L^2(\mathscr{M}, \varphi)$ for the GNS-Hilbert space associated with φ and Ω_φ for the corresponding cyclic vector. We will usually identify $\mathscr{M}$ with its image under the GNS-representation, so that $\mathscr{M} \subseteq \mathscr{B}(L^2(\mathscr{M}, \varphi))$. We further write $\|x\|_{2,\varphi} := \varphi(x^*x)^{1/2}$ for $x \in \mathscr{M}$ or, if φ is clear from the context, $\|x\|_2 := \|x\|_{2,\varphi}$.

The following lemma is standard.

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Lemma 7.1.1. *Let φ be a faithful normal state on a von Neumann algebra $\mathscr{M}$. Then, on bounded subsets of $\mathscr{M}$ the strong topology coincides with the topology induced by the norm $\|x\|_{2,\varphi} = \varphi(x^*x)^{1/2}$, $x \in \mathscr{M}$.*

Proof. Assume $\mathscr{M} \subseteq \mathscr{B}(L^2(\mathscr{M}, \varphi))$ and recall that on bounded sets the strong topology of $\mathscr{M}$ does not change under the choice of a faithful representation. Now if $(x_i)_{i \in I} \subseteq \mathscr{M}$ is a net that strongly converges to x, then $\|x_i - x\|_{2,\varphi} = \|(x_i - x)\Omega_\varphi\|_2 \to 0$. Conversely, suppose that $(x_i)_{i \in I} \subseteq \mathscr{M}$ is a bounded net in $\mathscr{M}$ such that $\|x_i - x\|_{2,\varphi} = \|(x_i - x)\Omega_\varphi\|_2 \to 0$. Then for $a \in \mathscr{M}$ analytic for the modular group σ^φ (see Subsection 7.1.2) we have by [172, Lemma 3.18 (i)] that

$$\|(x_i - x)a\Omega_\varphi\|_2 \leq \|\sigma^\varphi_{i/2}(a)\| \|(x_i - x)\Omega_\varphi\|_2 \to 0.$$

Since by [172, Lemma VIII.2.3] such elements $a\Omega_\varphi$, $a \in \mathscr{M}$ are dense in $L^2(\mathscr{M}, \varphi)$ and $(x_i)_{i \in I}$ is bounded, we conclude by a 2ϵ-estimate that $x_i \to x$ strongly. □

Remark 7.1.2. Note that one implication in the proof of Lemma 7.1.1 does not require the uniform boundedness assumption, provided that we assume that $\mathscr{M}$ is represented on its standard Hilbert space $L^2(\mathscr{M}, \varphi)$.

7.1.2. TOMITA-TAKESAKI MODULAR THEORY

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal positive functional $\varphi \in \mathcal{M}_*$. We let S_φ be the closure of the operator

$$L^2(\mathcal{M},\varphi) \to L^2(\mathcal{M},\varphi), x\Omega_\varphi \mapsto x^*\Omega_\varphi, \qquad x \in \mathcal{M}.$$

Let $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ be the (anti-linear) polar decomposition where J_φ is the *modular conjugation* and Δ_φ the *modular operator*. We have the *modular automorphism group* $\sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}$, $t \in \mathbb{R}$. Then $(\mathcal{M}, L^2(\mathcal{M},\varphi), J_\varphi, L^2(\mathcal{M},\varphi)^+)$ is the *standard form* of $\mathcal{M}$ (see [91]), where the *positive cone* is given by

$$L^2(\mathcal{M},\varphi)^+ := \overline{\{x(JxJ)\Omega_\varphi \mid x \in \mathcal{M}\}} \subseteq L^2(\mathcal{M},\varphi).$$

The standard form is uniquely determined up to a unique (unitarily implemented) isomorphism. For $x \in \mathcal{M}$ and $\xi \in L^2(\mathcal{M},\varphi)$ we write $\xi x := J_\varphi x^* J_\varphi \xi$. An element $x \in \mathcal{M}$ is called *analytic* for σ^φ if the function $\mathbb{R} \ni t \mapsto \sigma_t^\varphi(x) \in \mathcal{M}$ extends to a (necessarily unique) analytic function on the complex plane $\mathbb{C}$. In this case we write $\sigma_z^\varphi(x)$ for the extension at $z \in \mathbb{C}$.

The *centralizer* of a von Neumann algebra with respect to a faithful normal state φ is the set $\mathcal{M}^\varphi := \{x \in \mathcal{M} \mid \varphi(xy) = \varphi(yx) \text{ for all } y \in \mathcal{M}\}$, by [172, Theorem VIII.2.6] equivalently described as $\{x \in \mathcal{M} \mid \sigma_t^\varphi(x) = x \text{ for all } t \in \mathbb{R}\}$. In the following we will often consider the situation where $\mathcal{N} \subseteq \mathcal{M}$ is a unital embedding, equipped with a faithful normal conditional expectation $\mathbb{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$, where $\tau \in \mathcal{N}_*$ is a faithful tracial state and where $\varphi = \tau \circ \mathbb{E}_\mathcal{N}$. Then an easy computation shows that $\mathcal{N} \subseteq \mathcal{M}^\varphi$.

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7.1.3. COMPLETELY POSITIVE MAPS

Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras with faithful normal positive functionals φ and ψ respectively. For a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ we say that its L^2*-implementation* $\Phi^{(2)}$ (with respect to φ and ψ) exists if the map $x\Omega_\varphi \mapsto \Phi(x)\Omega_\psi$ extends to a bounded operator $\Phi^{(2)} : L^2(\mathcal{A},\varphi) \to L^2(\mathcal{B},\psi)$. This is the case if and only if there exists a constant $C > 0$ such that for all $x \in \mathcal{A}$,

$$\psi(\Phi(x)^*\Phi(x)) \le C\varphi(x^*x).$$

In particular, if Φ is 2-positive with $\psi \circ \Phi \le \varphi$, the Kadison-Schwarz inequality implies that $\Phi^{(2)}$ exists with $\|\Phi^{(2)}\| \le \|\Phi(1)\|^{1/2}$. Indeed, for $x \in \mathcal{A}$

$$\begin{aligned} \|\Phi(x)\Omega_\psi\|_2^2 &= \psi(\Phi(x)^*\Phi(x)) \le \|\Phi(1)\|\psi(\Phi(x^*x)) \\ &\le \|\Phi(1)\|\varphi(x^*x) = \|\Phi(1)\|\|x\Omega_\varphi\|_2^2. \end{aligned}$$

This implies that if Φ is contractive then so is $\Phi^{(2)}$.

The general principle of the following lemma was used as part of a proof in [41] and [116] a number of times. Here we present it separately. We will also need several straightforward variations of this lemma. Because they can be proved in a very similar way, we shall not state them here. The essence of the result is that, given two nets of maps with suitable properties that strongly converge to the identity, the composition of these maps gives rise to a net that also converges to the identity in the strong operator topology.

Lemma 7.1.3. *Let $(\mathscr{A},\varphi)$ and $(\mathscr{B},\varphi_j)$, $j\in\mathbb{N}$ be pairs of von Neumann algebras equipped with faithful normal states. Consider a normal completely positive map $\pi:\mathscr{A}\to\mathscr{B}$, a bounded sequence of normal completely positive maps $(\Psi_j:\mathscr{B}\to\mathscr{A})_{j\in\mathbb{N}}$ and for every $j\in\mathbb{N}$ a bounded net of completely positive maps $(\Phi_{j,k}:\mathscr{B}\to\mathscr{B})_{k\in K_j}$. Assume that for all $j\in\mathbb{N}$, $k\in K_j$ the inequalities $\varphi_j\circ\pi\leq\varphi$, $\varphi\circ\Psi_j\leq\varphi_j$ and $\varphi_j\circ\Phi_{j,k}\leq\varphi_j$ hold, that $\Psi_j\circ\pi(x)\to x$ strongly in j for every $x\in\mathscr{A}$ and that for every $j\in\mathbb{N}$, $x\in\mathscr{B}$ we have $\Phi_{j,k}(x)\to x$ strongly in k. Then there exists a directed set $\mathscr{F}$ and a function $(\widetilde{j},\widetilde{k})$: $\mathscr{F}\to\{(j,k)\mid j\in\mathbb{N},k\in K_j\}$, $F\mapsto(\widetilde{j}(F),\widetilde{k}(F))$ such that $\Psi_{\widetilde{j}(F)}\circ\Phi_{\widetilde{j}(F),\widetilde{k}(F)}\circ\pi(x)\to x$ strongly in F for every $x\in A$.*

Proof. For $j\in\mathbb{N}$ and $k\in K_j$ write

$$\begin{aligned}
\pi_j^{(2)} &: L^2(\mathscr{A},\varphi)\to L^2(\mathscr{B},\varphi_j), x\Omega_\varphi\mapsto\pi(x)\Omega_{\varphi_j},\\
\Psi_j^{(2)} &: L^2(\mathscr{B},\varphi_j)\to L^2(\mathscr{A},\varphi), x\Omega_{\varphi_j}\mapsto\Phi_{j,k}(x)\Omega_\varphi,\\
\Phi_{j,k}^{(2)} &: L^2(\mathscr{B},\varphi_j)\to L^2(\mathscr{B},\varphi_j), x\Omega_{\varphi_j}\mapsto\Phi_{j,k}(x)\Omega_{\varphi_j}
\end{aligned}$$

for the corresponding L^2-implementations with respect to φ and φ_j. Let $C\geq 1$ be a bound for the norms of $(\Psi_j)_{j\in\mathbb{N}}$ and hence for the norms of $(\Psi_j^{(2)})_{j\in\mathbb{N}}$. We shall make use of the fact that on bounded sets the strong topology coincides with the L^2-topology determined by a state, see Lemma 7.1.1. Therefore we have strong limits $\Psi_j^{(2)}\pi_j^{(2)}\to 1$ in $\mathscr{B}(L^2(\mathscr{A},\varphi))$ and $\Phi_{j,k}^{(2)}\to 1$ in $\mathscr{B}(L^2(\mathscr{B},\varphi_j))$. Now let $F\subseteq L^2(\mathscr{A},\varphi)$ be a finite subset. We may find $j=\widetilde{j}(F)\in\mathbb{N}$ such that for all $\xi\in F$,

$$\|\Psi_j^{(2)}\pi_j^{(2)}\xi-\xi\|_2<|F|^{-1}.$$

In turn, we may find $k=\widetilde{k}(j,F)=\widetilde{k}(F)$ such that for all $\xi\in F$,

$$\|\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\pi_j^{(2)}\xi\|_2<|F|^{-1}.$$

From the triangle inequality and by using that the operator norm of $\Psi_j^{(2)}$ is bounded by C,

$$\begin{aligned}
\|\Psi_j^{(2)}\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\xi\|_2 &\leq \|\Psi_j^{(2)}\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\Psi_j^{(2)}\pi_j^{(2)}\xi\|_2+\|\Psi_j^{(2)}\pi_j^{(2)}\xi-\xi\|_2\\
&\leq \|\Psi_j^{(2)}\|\|\Phi_{j,k}^{(2)}\pi_j^{(2)}\xi-\pi_j^{(2)}\xi\|_2+\|\Psi_j^{(2)}\pi_j^{(2)}\xi-\xi\|_2\\
&< (1+C)|F|^{-1}.
\end{aligned}$$

This implies that $\Psi_{\widetilde{j}(F)}^{(2)}\Phi_{\widetilde{j}(F),\widetilde{k}(F)}^{(2)}\pi_{\widetilde{j}(F)}^{(2)}\to 1$ strongly in $\mathscr{B}(L^2(\mathscr{A},\varphi))$ where the net is indexed by all finite subsets of $L^2(\mathscr{A},\varphi)$ partially ordered by inclusion. Using once more Lemma 7.1.1, one sees that for $x\in\mathscr{A}$ we have that $\Psi_{\widetilde{j}(F)}\circ\Phi_{\widetilde{j}(F),\widetilde{k}(F)}\circ\pi(x)\to x$ strongly. The claim follows. □

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7.2. RELATIVE HAAGERUP PROPERTY

The Haagerup property can be traced back to Haagerup's celebrated article [93], in which he noted that the free group admits a sequence of positive-definite functions vanishing at infinity which pointwise converges to a constant function equal to 1; in other words, the free group von Neumann algebra admits a sequence of unital completely positive Herz-Schur multipliers which are in a certain sense "small" and yet converge to the identity operator. Soon after that Choda gave in [55] a definition of the Haagerup property for a von Neumann algebra $\mathscr{M}$ equipped with a faithful normal tracial state in terms of the existence of abstract approximating maps on $\mathscr{M}$, which behave well with respect to the trace in question.

Definition 7.2.1 ([55])**.** Let $\mathscr{M}$ be a von Neumann algebra equipped with a faithful normal tracial state τ. We say that $\mathscr{M}$ has the *Haagerup property* if there exists a net $(\Phi_i)_{i\in I}$ of τ-preserving unital completely positive normal maps $\Phi_i : \mathscr{M} \to \mathscr{M}$ such that for every $i \in I$ the L^2-implementation $\Phi_i^{(2)}$ is compact with $\Phi_i \to \mathrm{id}_{\mathscr{M}}$ in the point-ultraweak topology.

For several years the study focused on finite von Neumann algebras, mainly as the motivating examples came from discrete groups. This changed with the articles [29], [58], which established the Haagerup property for the von Neumann algebras of certain discrete quantum groups, and the paper [68], which introduced and studied the analogous property for quantum groups themselves. Soon after that Okayasu and Tomatsu on one hand, and Caspers and Skalski on another gave a definition of the Haagerup property for an arbitrary von Neumann algebra equipped with a faithful normal semifinite weight and proved that the notion does not depend on the choice of the weight in question (see [41], [42], [143] as well as [40] and references therein). In all the cases above the Haagerup property should be thought of as a natural weakening of amenability/injectivity, which permits applying several approximation ideas and techniques beyond the class of amenable groups or algebras.

In several group theoretic and operator algebraic contexts it is important to consider also "relative" properties; for example, relative Property (T) is key to showing that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ does not have the Haagerup property, which in turn has several von Neumann algebraic consequences (see e.g. [114]). In the context of finite von Neumann algebras the relative Haagerup property first appeared in [23] in the study of Jones' towers associated with irreducible finite index subfactors, and was later applied in [155] as a key tool to obtain deep structural results about algebras admitting a certain type of Cartan inclusion (i.e. maximal abelian subalgebras with a "sufficiently rich" normalizer). Such Cartan inclusions are deeply related to von Neumann algebras of equivalence relations by the celebrated results of Feldman and Moore in [78]. The case of Cartan subalgebras was also the first in which a definition of a relative Haagerup property was proposed beyond finite von Neumann algebras (see [176] and [6]). Notably the latter developments took place even before the usual Haagerup property for arbitrary von Neumann algebras was well understood.

In this section we introduce the relative Haagerup property for inclusions of general σ-finite von Neumann algebras and consider natural variations of the definition. For this, fix a triple $(\mathcal{M},\mathcal{N},\varphi)$ where $\mathcal{N}\subseteq\mathcal{M}$ is a unital inclusion of von Neumann algebras and where φ is a faithful normal positive functional on $\mathcal{M}$ whose corresponding modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ satisfies $\sigma_t^\varphi(\mathcal{N})\subseteq\mathcal{N}$ for all $t\in\mathbb{R}$. To keep the notation short, we will often just write $(\mathcal{M},\mathcal{N},\varphi)$ and will implicitly assume that the triple satisfies the mentioned conditions. By [172, Theorem IX.4.2] the assumption $\sigma_t^\varphi(\mathcal{N})\subseteq\mathcal{N}$, $t\in\mathbb{R}$ is equivalent to the existence of a (uniquely determined) φ-preserving (necessarily faithful) normal conditional expectation $\mathbb{E}_{\mathcal{N}}^\varphi:\mathcal{M}\to\mathcal{N}$. If the corresponding functional φ is clear, we will often just write $\mathbb{E}_{\mathcal{N}}$ instead of $\mathbb{E}_{\mathcal{N}}^\varphi$ (compare also with Subsection 7.2.2).

7.2.1. FIRST DEFINITION OF THE RELATIVE HAAGERUP PROPERTY

For a triple $(\mathcal{M},\mathcal{N},\varphi)$ as before define the *Jones projection*

$$e_{\mathcal{N}}^\varphi := \mathbb{E}_{\mathcal{N}}^{(2)} : L^2(\mathcal{M},\varphi)\to L^2(\mathcal{M},\varphi)$$

which is the orthogonal projection onto $L^2(\mathcal{N},\varphi)\subseteq L^2(\mathcal{M},\varphi)$ and let

$$\langle\mathcal{M},\mathcal{N}\rangle\subseteq\mathcal{B}(L^2(\mathcal{M},\varphi))$$

be the von Neumann subalgebra generated by $e_{\mathcal{N}}^\varphi$ and $\mathcal{M}$. This is the *Jones construction*. We will usually write $e_{\mathcal{N}}$ instead of $e_{\mathcal{N}}^\varphi$ if there is no ambiguity. Further set

$$\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi) := \mathrm{Span}\{xe_{\mathcal{N}}y\mid x,y\in\mathcal{M}\}\subseteq\mathcal{B}(L^2(\mathcal{M},\varphi))$$

and

$$\mathcal{K}(\mathcal{M},\mathcal{N},\varphi) := \overline{\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)}.$$

Then $\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$ is a (not necessarily closed) two-sided ideal in $\langle\mathcal{M},\mathcal{N}\rangle$ whose elements are called the *finite rank operators* relative to $\mathcal{N}$. Similarly, $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$ is a closed two-sided ideal in $\langle\mathcal{M},\mathcal{N}\rangle$ whose elements are called the *compact operators* relative to $\mathcal{N}$. Note that if $\mathcal{N}=\mathbb{C}1_{\mathcal{M}}$, then $e_{\mathcal{N}}$ is a rank one projection and the operators in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$ are precisely the compact operators on $L^2(\mathcal{M},\varphi)$.

In the following it is often convenient to identify a finite rank operator $ae_{\mathcal{N}}b\in\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$, $a,b\in\mathcal{M}$ with the map $a\mathbb{E}_{\mathcal{N}}(b\cdot):\mathcal{M}\to\mathcal{M}$. The latter does not depend on φ (but only on the conditional expectation $\mathbb{E}_{\mathcal{N}}$), and the notation is naturally compatible with the inclusion $\mathcal{M}\subseteq L^2(\mathcal{M},\varphi)$. We will often write $a\mathbb{E}_{\mathcal{N}}b := a\mathbb{E}_{\mathcal{N}}(b\,\cdot)$.

Definition 7.2.2. Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras and let φ be a faithful normal positive functional on $\mathcal{M}$ with $\sigma_t^\varphi(\mathcal{N})\subseteq\mathcal{N}$ for all $t\in\mathbb{R}$. We say that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the *relative Haagerup property* (or just *property (rHAP)*) if there exists a net $(\Phi_i)_{i\in I}$ of normal maps $\Phi_i:\mathcal{M}\to\mathcal{M}$ such that

(1) Φ_i is completely positive and $\sup_i\|\Phi_i\|<\infty$ for all $i\in I$;

(2) Φ_i is an $\mathcal{N}$-$\mathcal{N}$-bimodule map for all $i \in I$;

(3) $\Phi_i(x) \to x$ strongly for every $x \in \mathcal{M}$;

(4) $\varphi \circ \Phi_i \leq \varphi$ for all $i \in I$;

(5) For every $i \in I$ the L^2-implementation

$$\Phi_i^{(2)} : L^2(\mathcal{M}, \varphi) \to L^2(\mathcal{M}, \varphi), x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi,$$

is contained in $\mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.

Remark 7.2.3. *(a)* In many applications φ will be a faithful normal state, but for notational convenience we shall rather work in the more general setting. Note that we may always normalize φ to be a state and that the definition of the relative Haagerup property does not change under this normalization.
(b) Note that in [155] (see also [117]) a different notion of relative compactness is used to define the relative Haagerup property. It coincides with ours in the case where $\mathcal{N}' \cap \mathcal{M} \subseteq \mathcal{N}$. However, the alternative notion is not very suitable beyond the tracial situation since it requires the existence of finite projections. We will return to this issue in Subsection 7.5.1.
(c) In the case where $\mathcal{N} = \mathbb{C}1_{\mathcal{M}}$, Definition 7.2.2 recovers the usual definition of the (non-relative) Haagerup property, see [41, Definition 3.1].

There is a number of immediate variations of Definition 7.2.2. For instance, one may replace the condition (1) by one of the following stronger conditions:

(1′) For every $i \in I$ the map Φ_i is contractive completely positive.

(1″) For every $i \in I$ the map Φ_i is unital completely positive.

We may also replace the condition (4) by the following condition:

(4′) $\varphi \circ \Phi_i = \varphi$.

One of the results that we shall prove is that if the subalgebra $\mathcal{N}$ is finite, then condition (4) is redundant. We will further prove that in this setting the approximating maps Φ_i, $i \in I$ can be chosen to be unital and state-preserving implying that all the variations of the relative Haagerup property from above coincide. To simplify the statements of the following subsections, let us introduce the following auxiliary notion, which is a priori weaker.

Definition 7.2.4. Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras and let φ be a faithful normal positive functional on $\mathcal{M}$ with $\sigma_t^\varphi(\mathcal{N}) \subseteq \mathcal{N}$ for all $t \in \mathbb{R}$. We say that the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has *property (rHAP)*$^-$ if there exists a net $(\Phi_i)_{i \in I}$ of normal maps $\Phi_i : \mathcal{M} \to \mathcal{M}$ such that

(1) Φ_i is completely positive for all $i \in I$;

(2) Φ_i is an $\mathcal{N}$-$\mathcal{N}$-bimodule map for all $i \in I$;

(3) $\|\Phi_i(x) - x\|_{2,\varphi} \to 0$ for every $x \in \mathcal{M}$;

(4) For every $i \in I$ the L^2-implementation

$$\Phi_i^{(2)} \colon L^2(\mathcal{M},\varphi) \to L^2(\mathcal{M},\varphi), x\Omega_\varphi \mapsto \Phi_i(x)\Omega_\varphi,$$

exists and is contained in $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$.

7.2.2. DEPENDENCE ON THE POSITIVE FUNCTIONAL: REDUCTION TO THE DEPENDENCE ON THE CONDITIONAL EXPECTATION

Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Recall that for every faithful normal positive functional φ on $\mathcal{M}$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ the corresponding modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ satisfies $\sigma_t^\varphi(\mathcal{N}) \subseteq \mathcal{N}$ for all $t \in \mathbb{R}$. Note that such a functional always exists, as it suffices to pick a faithful normal state $\omega \in \mathcal{N}_*$ (which exists by our standing σ-finiteness assumption) and set $\varphi = \omega \circ \mathbb{E}_{\mathcal{N}}$. In this subsection we will examine the dependence of the relative Haagerup property of $(\mathcal{M},\mathcal{N},\varphi)$ on the functional φ. We shall prove that the property rather depends on the conditional expectation $\mathbb{E}_{\mathcal{N}}$ than on φ.

Lemma 7.2.5. *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be $\mathcal{N}$-$\mathcal{N}$-bimodular. Then the following statements are equivalent:*

(1) $\mathbb{E}_{\mathcal{N}} \circ \Phi \leq \mathbb{E}_{\mathcal{N}}$ (resp. $\mathbb{E}_{\mathcal{N}} \circ \Phi = \mathbb{E}_{\mathcal{N}}$).

(2) For all $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ we have $\varphi \circ \Phi \leq \varphi$ (resp. $\varphi \circ \Phi = \varphi$).

(3) There exists a faithful functional $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ such that $\varphi \circ \Phi \leq \varphi$ (resp. $\varphi \circ \Phi = \varphi$).

Further, the following statements are equivalent:

(4) There exists $C > 0$ such that $\mathbb{E}_{\mathcal{N}}(\Phi(x)^*\Phi(x)) \leq C\mathbb{E}_{\mathcal{N}}(x^*x)$ for all $x \in \mathcal{M}$.

(5) There exists $C > 0$ such that for all $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $x \in \mathcal{M}$ we have $\varphi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x)$.

(6) There exists $C > 0$ and a faithful functional $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ such that for all $x \in \mathcal{M}$ we have $\varphi(\Phi(x)^*\Phi(x)) \leq C\varphi(x^*x)$.

In particular, if the L^2-implementation of Φ with respect to φ exists, then it exists with respect to any other ψ with $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$.

Proof. We prove the statements for the inequalities; the respective cases with equalities follow similarly. The implications "(1) ⇔ (2) ⇒ (3)" of the first three statements are trivial. For the implication "(3) ⇒ (1)" take φ as in (3). For $x \in \mathcal{N}$ consider the positive functional $x^*\varphi x \in \mathcal{M}_*^+$ which again satisfies $(x^*\varphi x) \circ \mathbb{E}_{\mathcal{N}} = x^*\varphi x$. Then, for $y \in \mathcal{M}^+$

$$(x^*\varphi x) \circ \mathbb{E}_{\mathcal{N}} \circ \Phi(y) = \varphi \circ \mathbb{E}_{\mathcal{N}} \circ \Phi(xyx^*) \leq \varphi \circ \mathbb{E}_{\mathcal{N}}(xyx^*) = (x^*\varphi x) \circ \mathbb{E}_{\mathcal{N}}(y).$$

Since the restrictions of functionals $x^*\varphi x$, $x \in \mathcal{N}$ to $\mathcal{N}$ are dense in $\mathcal{N}_*^+$ we conclude that $\mathbb{E}_{\mathcal{N}} \circ \Phi \leq \mathbb{E}_{\mathcal{N}}$.

The equivalence of the statements (4), (5) and (6) follows similarly. □

The following lemma shows that in good circumstances compactness of the L^2-implementations does not depend on the choice of the state.

Lemma 7.2.6. *Let $\varphi, \psi \in \mathcal{M}_*^+$ be faithful with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$. Let further $\Phi : \mathcal{M} \to \mathcal{M}$ be a completely positive $\mathcal{N}$-$\mathcal{N}$-bimodule map whose L^2-implementation $\Phi_\varphi^{(2)}$ with respect to φ exists (hence, by Lemma 7.2.5, the L^2-implementation $\Phi_\psi^{(2)}$ of Φ with respect to ψ exists as well). Then, $\Phi_\varphi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$ if and only if $\Phi_\psi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \psi)$.*

Proof. Let U be the unique unitary mapping the standard form $(\mathcal{M}, L^2(\mathcal{M}, \varphi), J_\varphi, P_\varphi)$ to the standard form $(\mathcal{M}, L^2(\mathcal{M}, \psi), J_\psi, P_\psi)$, see [91, Theorem 2.3]. It restricts to the unique unitary map between the standard forms $(\mathcal{N}, L^2(\mathcal{N}, \varphi), J_{\varphi|_{\mathcal{N}}}, P_{\varphi|_{\mathcal{N}}})$ and $(\mathcal{N}, L^2(\mathcal{N}, \psi), J_{\psi|_{\mathcal{N}}}, P_{\psi|_{\mathcal{N}}})$. Indeed, for all $x \in \mathcal{M}$, $\varphi(x) = \langle xU\Omega_\varphi, U\Omega_\varphi\rangle$ and by [91, Lemma 2.10], $U\Omega_\varphi$ is the unique element in $L^2(\mathcal{M}, \psi)$ satisfying this equation. On the other hand, applying [91, Lemma 2.10] to $\varphi|_{\mathcal{N}}$ implies the existence of a unique vector $\xi \in L^2(\mathcal{N}, \psi)$ such that $\varphi(x) = \langle x\xi, \xi\rangle$ for all $x \in \mathcal{N}$. By approximating ξ by elements in $\mathcal{N}\Omega_\psi$ and by using the assumptions $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$ one deduces that for all $x \in \mathcal{M}$,

$$\varphi(x) = \varphi \circ \mathbb{E}_{\mathcal{N}}(x) = \langle \mathbb{E}_{\mathcal{N}}(x)\xi, \xi\rangle = \langle x\xi, \xi\rangle$$

and hence $U\Omega_\varphi = \xi \in L^2(\mathcal{N}, \psi)$. This implies $U(\mathcal{N}\Omega_\varphi) \subseteq L^2(\mathcal{N}, \psi)$ and therefore (by density and symmetry) $U(L^2(\mathcal{N}, \varphi)) = L^2(\mathcal{N}, \psi)$. Finally, using that $\sigma_t^\varphi(\mathcal{N}) = \mathcal{N}$ and hence $J_{\varphi|_{\mathcal{N}}} = (J_\varphi)|_{L^2(\mathcal{N},\varphi)}$ (and similarly $J_{\psi|_{\mathcal{N}}} = (J_\psi)|_{L^2(\mathcal{N},\psi)}$) it is straightforward to check that the restriction of U satisfies the other properties of the unique unitary mapping between the standard forms

$$(\mathcal{N}, L^2(\mathcal{N}, \varphi), J_{\varphi|_{\mathcal{N}}}, P_{\varphi|_{\mathcal{N}}}) \qquad \text{and} \qquad (\mathcal{N}, L^2(\mathcal{N}, \psi), J_{\psi|_{\mathcal{N}}}, P_{\psi|_{\mathcal{N}}}).$$

Since $e_{\mathcal{N}}^\varphi = (\mathbb{E}_{\mathcal{N}}^\varphi)^{(2)}$ is the orthogonal projection of $L^2(\mathcal{M}, \varphi)$ onto $L^2(\mathcal{N}, \varphi)$ and $e_{\mathcal{N}}^\psi = (\mathbb{E}_{\mathcal{N}}^\psi)^{(2)}$ is the orthogonal projection of $L^2(\mathcal{M}, \psi)$ onto $L^2(\mathcal{N}, \psi)$, we see that $U^* e_{\mathcal{N}}^\psi U = e_{\mathcal{N}}^\varphi$. Hence, for every map Λ of the form $\Lambda = a\mathbb{E}_{\mathcal{N}} b$ with $a, b \in \mathcal{M}$ the L^2-implementation $\Lambda_\varphi^{(2)}$ with respect to φ and the L^2-implementation $\Lambda_\psi^{(2)}$ with respect to ψ exist with $\Lambda_\varphi^{(2)} = ae_{\mathcal{N}}^\varphi b = U^* ae_{\mathcal{N}}^\psi bU = U^*\Lambda_\psi^{(2)} U$.

Now assume that $\Phi_\varphi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$. Then there exists a sequence $(\Phi_k : \mathcal{M} \to \mathcal{M})_{k\in\mathbb{N}}$ of maps of the form $\Phi_k = \sum_{i=1}^{N_k} a_{i,k}\mathbb{E}_{\mathcal{N}} b_{i,k}$ with $N_k \in \mathbb{N}$ and $a_{1,k}$,$b_{1,k}$, ..., $a_{N_k,k}$, $b_{N_k,k} \in \mathcal{M}$ whose L^2-implementations $\Phi_{k,\varphi}^{(2)} \in \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$ (with respect to φ) norm-converge to $\Phi_\varphi^{(2)}$. By the above, $U\Phi_{k,\varphi}^{(2)} U^* \in \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \psi)$ is given by $x\Omega_\psi \mapsto \Phi_k(x)\Omega_\psi$ for $x \in \mathcal{M}$. We claim that the sequence $(U\Phi_{k,\varphi}^{(2)} U^*)_{k\in\mathbb{N}} \subseteq \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \psi)$ norm-converges to $\Phi_\psi^{(2)}$. Indeed, by the density of the set of all elements of the form $x(\varphi|_{\mathcal{N}})x^*$, $x \in \mathcal{N}$ in $\mathcal{N}_*^+$ we find a net $(x_i)_{i\in I} \subseteq \mathcal{N}$ such that $x_i(\varphi|_{\mathcal{N}})x_i^* \to \psi|_{\mathcal{N}}$.

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In combination with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$ this also implies $x_i \varphi x_i^* \to \psi$. For $y \in \mathcal{M}$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\|(\Phi_{\psi}^{(2)} - U\Phi_{k,\varphi}^{(2)}U^*)y\Omega_{\psi}\|_{2,\psi}^2 &= \|(\Phi(y) - \Phi_k(y))\Omega_{\psi}\|_{2,\psi}^2 \\
&= \lim_i \|(\Phi(y) - \Phi_k(y))x_i\Omega_{\varphi}\|_{2,\varphi}^2 \\
&= \lim_i \|(\Phi_{\varphi}^{(2)} - \Phi_{k,\varphi}^{(2)})yx_i\Omega_{\varphi}\|_{2,\varphi}^2 \\
&\leq \lim_i \|\Phi_{\varphi}^{(2)} - \Phi_{k,\varphi}^{(2)}\|^2 \varphi(x_i^* y^* y x_i) \\
&= \|\Phi_{\varphi}^{(2)} - \Phi_{k,\varphi}^{(2)}\|^2 \psi(y^* y),
\end{aligned}$$

where in the third step we used the $\mathcal{N}$-$\mathcal{N}$-bimodularity of Φ and the right $\mathcal{N}$-modularity of Φ_k. Now, $\Phi_{k,\varphi}^{(2)} \to \Phi_{\varphi}^{(2)}$ and $(U\Phi_{k,\varphi}^{(2)}U^*)_{k\in\mathbb{N}}$ is a Cauchy sequence, hence the above inequality leads to $U\Phi_{k,\varphi}^{(2)}U^* \to \Phi_{\psi}^{(2)}$ as claimed. In particular, $\Phi_{\psi}^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\psi)$ which finishes the proof. □

Theorem 7.2.7. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Further, let $\varphi, \psi \in \mathcal{M}_*^+$ be faithful normal positive functionals with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$ and $\psi \circ \mathbb{E}_{\mathcal{N}} = \psi$. Then the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP) (resp. property (rHAP)$^-$) if and only if the triple $(\mathcal{M},\mathcal{N},\psi)$ has property (rHAP) (resp. property (rHAP)$^-$). In particular, property (rHAP) (resp. property (rHAP)$^-$) only depends on the triple $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$.*

Proof. It follows from Lemma 7.2.5 and Lemma 7.2.6 that if $(\Phi_j)_{j\in J}$ is a net of approximating maps witnessing the relative Haagerup property of $(\mathcal{M},\mathcal{N},\varphi)$ (resp. property (rHAP)$^-$ of $(\mathcal{M},\mathcal{N},\varphi)$), then it also witnesses the Haagerup property of $(\mathcal{M},\mathcal{N},\psi)$ (resp. property (rHAP)$^-$ of $(\mathcal{M},\mathcal{N},\psi)$) and vice versa. □

We will later see that in the case where the von Neumann subalgebra $\mathcal{N}$ is finite the statement in Theorem 7.2.7 can be strengthened: in this case property (rHAP) (and equivalently property (rHAP)$^-$) does not even depend on the choice of the conditional expectation $\mathbb{E}_{\mathcal{N}}$.

Motivated by Theorem 7.2.7 we introduce the following natural definition.

Definition 7.2.8. Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We say that the triple $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$ has the *relative Haagerup property* (or just *property (rHAP)*) if the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the relative Haagerup property for some (equivalently any) faithful normal positive functional $\varphi \in \mathcal{M}_*^+$ with $\varphi \circ \mathbb{E}_{\mathcal{N}} = \varphi$. The same terminology shall be adopted for property (rHAP)$^-$.

7.2.3. STATE-PRESERVATION, CONTRACTIVITY AND UNITALITY OF THE APPROXIMATING MAPS IN A SPECIAL CASE

In this subsection we will prove that the relative Haagerup property of certain triples $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$ may be witnessed by approximating maps that satisfy extra conditions, such as state-preservation, contractivity and unitality. This will play a crucial role in Section 7.4. The approach is inspired by [13, Section 2], where ideas from [116] were used.

Lemma 7.2.9. *Let $\mathcal{M}$ be a von Neumann algebra, $\varphi\in\mathcal{M}_*$ a faithful normal state and $y\in\mathcal{M}$. If $y\varphi=\varphi y$ (i.e. $y\in\mathcal{M}^{\varphi}$), then $y\Omega_\varphi=\Omega_\varphi y$.*

Proof. As mentioned in Subsection 7.1.2, by [172, Theorem VIII.2.6] we have that $\sigma_t^{\varphi}(y)=y$ for all $t\in\mathbb{R}$. But then y is analytic and moreover $\sigma^{\varphi}_{-i/2}(y)=y$. Hence

$$\Omega_\varphi y=J_\varphi y^*J_\varphi\Omega_\varphi=J_\varphi\sigma^{\varphi}_{-i/2}(y^*)J_\varphi\Omega_\varphi=J_\varphi\Delta_\varphi^{1/2}y^*\Delta_\varphi^{-1/2}J_\varphi\Omega_\varphi=S_\varphi y^*S_\varphi\Omega_\varphi=y\Omega_\varphi.$$

The claim follows. □

Proposition 7.2.10. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras that admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau\in\mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi:=\tau\circ\mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Let further $\Phi:\mathcal{M}\to\mathcal{M}$ be a normal, completely positive, $\mathcal{N}$-$\mathcal{N}$-bimodular map for which there exists $\delta>0$ such that $c:=\Phi(1)\le 1-\delta$ and $\varphi\circ\Phi\le(1-\delta)\varphi$. Then one can find $a,b\in\mathcal{N}'\cap\mathcal{M}$ such that $a\ge 0$, $\mathbb{E}_{\mathcal{N}}(a)=1$, $a\mathbb{E}_{\mathcal{N}}(b^*b)=\mathbb{E}_{\mathcal{N}}(b^*b)a=1-c$ and $b\varphi b^*=\varphi-\varphi\circ\Phi$.*

Proof. The complete positivity of Φ implies that $0\le\|\Phi\|=\|\Phi(1)\|=\|c\|\le 1-\delta$, hence the map Φ must be contractive. It is clear that $c=\Phi(1)\ge 0$. Further, since $\mathbb{E}_{\mathcal{N}}(1-c)>\mathbb{E}_{\mathcal{N}}(\delta)=\delta$, the element $\mathbb{E}_{\mathcal{N}}(1-c)\in\mathcal{N}$ is boundedly invertible. Additionally, the $\mathcal{N}$-$\mathcal{N}$-bimodularity of Φ implies that for every $n\in\mathcal{N}$,

$$nc=n\Phi(1)=\Phi(n)=\Phi(1)n=cn,$$

so $c\in\mathcal{N}'\cap\mathcal{M}$. The latter two observations imply that for

$$a:=(1-c)(\mathbb{E}_{\mathcal{N}}(1-c))^{-1}$$

we have $a\in\mathcal{N}'\cap\mathcal{M}$, $a\ge 0$ and $\mathbb{E}_{\mathcal{N}}(a)=1$.

Consider the positive normal functional $\varphi-\varphi\circ\Phi\in\mathcal{M}_*$. By [91, Lemma 2.10] there exists a unique vector $\xi\in L^2(\mathcal{M},\varphi)^+$ such that $(\varphi-\varphi\circ\Phi)(x)=\langle x\xi,\xi\rangle$ for all $x\in\mathcal{M}$. Note that $\{J_\varphi x\Omega_\varphi\mid x\in\mathcal{M}\}$ is dense in $L^2(\mathcal{M},\varphi)$ and define the linear map

$$b:L^2(\mathcal{M},\varphi)\to L^2(\mathcal{M},\varphi),\ J_\varphi x\Omega_\varphi\mapsto J_\varphi x\xi.$$

It is contractive since

$$\|b(J_\varphi x\Omega_\varphi)\|_2^2=\|J_\varphi x\xi\|_2^2=\|x\xi\|_2^2=(\varphi-\varphi\circ\Phi)(x^*x)\le\varphi(x^*x)=\|x\Omega_\varphi\|_2^2=\|J_\varphi x\Omega_\varphi\|_2^2$$

for all $x\in\mathcal{M}$. Further, for $x,y\in\mathcal{M}$,

$$bJ_\varphi xJ_\varphi(J_\varphi y\Omega_\varphi)=bJ_\varphi xy\Omega_\varphi=J_\varphi xy\xi=J_\varphi xJ_\varphi J_\varphi y\xi=J_\varphi xJ_\varphi b(J_\varphi y\Omega_\varphi).$$

It hence follows that b and $J_\varphi x J_\varphi$ commute and therefore that $b \in (J_\varphi \mathcal{M} J_\varphi)' = \mathcal{M}'' = \mathcal{M}$.

We claim that a and b from above satisfy the required conditions. It remains to show that $b \in \mathcal{N}' \cap \mathcal{M}$, $b\varphi b^* = \varphi - \varphi \circ \Phi$ and $a\mathbb{E}_{\mathcal{N}}(b^*b) = \mathbb{E}_{\mathcal{N}}(b^*b)a = 1 - c$.

- $b \in \mathcal{N}' \cap \mathcal{M}$: By the assumption we have $\varphi - \varphi \circ \Phi \geq \delta\varphi$ and therefore $\varphi - \varphi \circ \Phi$ is a faithful normal functional. For $x \in \mathcal{M}$, $n \in \mathcal{N}$ the $\mathcal{N}$-$\mathcal{N}$-bimodularity of Φ and the traciality of φ on $\mathcal{N}$ (implying that n is contained in the centralizer of φ) imply that $\varphi \circ \Phi(xn) = \varphi(\Phi(x)n) = \varphi(n\Phi(x)) = \varphi \circ \Phi(nx)$, hence $n(\varphi - \varphi \circ \Phi) = (\varphi - \varphi \circ \phi)n$. The unique isomorphism between the standard forms induced by φ and $\varphi - \varphi \circ \Phi$ maps ξ to the canonical cyclic vector in $L^2(\mathcal{M}, \varphi - \varphi \circ \Phi)$. Hence, from Lemma 7.2.9 applied to $\varphi - \varphi \circ \Phi$ we get $n\xi = \xi n$ for all $n \in \mathcal{N}$, which, together with the fact that $J_\varphi n \Omega_\varphi = n^* \Omega_\varphi$, implies that for $x \in \mathcal{M}$

$$\begin{aligned} bn(J_\varphi x \Omega_\varphi) &= bJ_\varphi x J_\varphi n \Omega_\varphi = bJ_\varphi x n^* \Omega_\varphi = J_\varphi x n^* \xi \\ &= J_\varphi x \xi n^* = J_\varphi x J_\varphi n J_\varphi \xi = nJ_\varphi x \xi = nb(J_\varphi x \Omega_\varphi), \end{aligned}$$

  so $b \in \mathcal{N}' \cap \mathcal{M}$ by the density of $\{J_\varphi x \Omega_\varphi \mid x \in \mathcal{M}\}$ in $L^2(\mathcal{M}, \varphi)$.

- $b\varphi b^* = \varphi - \varphi \circ \Phi$: For every $x \in \mathcal{M}$ the equality

$$(b\varphi b^*)(x) = \langle xb\Omega_\varphi, b\Omega_\varphi \rangle = \langle x\xi, \xi \rangle = (\varphi - \varphi \circ \Phi)(x)$$

  holds, i.e. $b\varphi b^* = \varphi - \varphi \circ \Phi$.

- $a\mathbb{E}_{\mathcal{N}}(b^*b) = \mathbb{E}_{\mathcal{N}}(b^*b)a = 1 - c$: For $x \in \mathcal{M}$ we find by $b \in \mathcal{N}' \cap \mathcal{M}$ and $b\varphi b^* = \varphi - \varphi \circ \Phi$ that

$$\begin{aligned} & \varphi(x\mathbb{E}_{\mathcal{N}}(b^*b)) = \varphi(\mathbb{E}_{\mathcal{N}}(x)b^*b) = \varphi(b^*\mathbb{E}_{\mathcal{N}}(x)b) = (\varphi - \varphi \circ \Phi)(\mathbb{E}_{\mathcal{N}}(x)) \\ = \; & \varphi(\mathbb{E}_{\mathcal{N}}(x)) - \varphi(\mathbb{E}_{\mathcal{N}}(x)\Phi(1)) = \varphi(x) - \varphi(x\mathbb{E}_{\mathcal{N}}(\Phi(1))) = \varphi(x\mathbb{E}_{\mathcal{N}}(1 - c)) \end{aligned}$$

  and hence $\mathbb{E}_{\mathcal{N}}(1-c) = \mathbb{E}_{\mathcal{N}}(b^*b)$. It follows by the definition of a that $a\mathbb{E}_{\mathcal{N}}(b^*b) = a\mathbb{E}_{\mathcal{N}}(1 - c) = 1 - c$ and similarly, as $a \in \mathcal{N}' \cap \mathcal{M}$, we have $\mathbb{E}_{\mathcal{N}}(b^*b)a = 1 - c$.

□

Lemma 7.2.11. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Let further $x \in \mathcal{N}' \cap \mathcal{M}$ be an element which is analytic for σ^φ. Then $\mathbb{E}_{\mathcal{N}}(yx) = \mathbb{E}_{\mathcal{N}}(\sigma_i^\varphi(x)y)$ for all $y \in \mathcal{M}$.*

Proof. For $n \in \mathcal{N}$ we have by the traciality of φ on $\mathcal{N}$ (implying that n is contained in the centralizer of φ) that

$$n\sigma_z^\varphi(x) = \sigma_z^\varphi(n)\sigma_z^\varphi(x) = \sigma_z^\varphi(nx) = \sigma_z^\varphi(xn) = \sigma_z^\varphi(x)\sigma_z^\varphi(n) = \sigma_z^\varphi(x)n.$$

for all $z \in \mathbb{C}$. Therefore, $\sigma_z^\varphi(x) \in \mathcal{N}' \cap \mathcal{M}$ and in particular $\sigma_i^\varphi(x) \in \mathcal{N}' \cap \mathcal{M}$. One further calculates that for $y \in \mathcal{M}$,

$$\begin{aligned}(\varphi n)(\mathbb{E}_{\mathcal{N}}(yx)) &= \varphi(\mathbb{E}_{\mathcal{N}}(nyx)) = \varphi(nyx) = \varphi(\sigma_i^\varphi(x)ny) \\ &= \varphi(n\sigma_i^\varphi(x)y) = (\varphi n)(\mathbb{E}_{\mathcal{N}}(\sigma_i^\varphi(x)y)).\end{aligned}$$

Since the set of all functionals of the form φn, $n \in \mathcal{N}$ is dense in $\mathcal{N}_*$ we find that $\mathbb{E}_{\mathcal{N}}(yx) = \mathbb{E}_{\mathcal{N}}(\sigma_i^\varphi(x)y)$, as claimed. □

Lemma 7.2.12. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras that admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Let $h_1, h_2 \in \mathcal{M}$ and let $h_3, h_4 \in \mathcal{N}' \cap \mathcal{M}$ be analytic for σ^φ. Suppose that $\Phi : \mathcal{M} \to \mathcal{M}$ is a normal map such that $\Phi_\varphi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$ and define the map $\widetilde{\Phi} := h_1\Phi(h_2 \cdot h_3)h_4$. Then we also have that $\widetilde{\Phi}_\varphi^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.*

Proof. Note first that by [172, VIII.3.18(i)] (and its proof) $\widetilde{\Phi}$ has a bounded L^2-implementation, with $\|\widetilde{\Phi}_\varphi^{(2)}\| \leq C\|\Phi_\varphi^{(2)}\|$, with the constant $C > 0$ depending on h_1, h_2, h_3, h_4. It thus suffices to show that the passage $\Phi \to \widetilde{\Phi}$ preserves the property of having a finite-rank implementation. Let then $a, b \in \mathcal{M}$ so that $ae_{\mathcal{N}}b$ is in $\mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$. So for $x \in \mathcal{M}$ we have by Lemma 7.2.11,

$$h_1(ae_{\mathcal{N}}b)(h_2xh_3)h_4\Omega_\varphi = h_1ah_4e_{\mathcal{N}}(\sigma_i^\varphi(h_3)bh_2x)\Omega_\varphi,$$

and so this map is in $\mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$. □

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We are now ready to formulate the main result of this subsection. In combination with Lemma 7.2.14 it will later allow us to deduce that the relative Haagerup property of a triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ with finite $\mathcal{N}$ may be witnessed by unital and state-preserving maps. Its proof is inspired by [13, Section 2].

Theorem 7.2.13. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite, let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$ and suppose that the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) witnessed by contractive approximating maps. Then, if all the elements of $\mathcal{M}$ are analytic with respect to the modular automorphism group of φ – for example if there exists a boundedly invertible element $h \in \mathcal{M}^+$ with $\sigma_t^\varphi(x) = h^{it}xh^{-it}$ for all $t \in \mathbb{R}$, $x \in \mathcal{M}$ - property (rHAP) of $(\mathcal{M}, \mathcal{N}, \varphi)$ may be witnessed by unital and state-preserving approximating maps, i.e. we may assume that (1″) and (4′) hold.*

Proof. Let $(\Phi_j)_{j \in J_1}$ be a net of contractive approximating maps witnessing property (rHAP) of the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ and choose a net $(\delta_j)_{j \in J_2}$ with $\delta_j \to 0$. We now set $J = J_1 \times J_2$ with the product partial order and for $j = (j_1, j_2) \in J$ we set $\Phi_j = \Phi_{j_1}$ and $\delta_j = \delta_{j_2}$. Then for all $j \in J$,

$$c_j := (1 - \delta_j)\Phi_j(1) \leq 1 - \delta_j \qquad \text{and} \qquad (1 - \delta_j)(\varphi \circ \Phi_j) \leq (1 - \delta_j)\varphi.$$

In particular, we may apply Proposition 7.2.10 to $(1-\delta_j)\Phi_j$ to find elements a_j, $b_j \in \mathcal{N}' \cap \mathcal{M}$ with $a_j \geq 0$, $\mathbb{E}_{\mathcal{N}}(a_j) = 1$, $a_j \mathbb{E}_{\mathcal{N}}(b_j^* b_j) = \mathbb{E}_{\mathcal{N}}(b_j^* b_j) a_j = 1 - c_j$ and $b_j \varphi b_j^* = \varphi - (1-\delta_j)(\varphi \circ \Phi_j)$. For $j \in J$ define

$$\Psi_j : \mathcal{M} \to \mathcal{M}, \Psi_j(x) := (1-\delta_j)\Phi_j(x) + a_j \mathbb{E}_{\mathcal{N}}(b_j^* x b_j).$$

It is clear that Ψ_j is normal completely positive and $\mathcal{N}$-$\mathcal{N}$-bimodular. Further,

$$\Psi_j(1) = (1-\delta_j)\Phi_j(1) + a_j \mathbb{E}_{\mathcal{N}}(b_j^* b_j) = c_j + (1-c_j) = 1$$

and for any $x \in \mathcal{M}$

$$\begin{aligned} \varphi \circ \Psi_j(x) &= (1-\delta_j)\varphi(\Phi_j(x)) + \varphi(a_j \mathbb{E}_{\mathcal{N}}(b_j^* x b_j)) \\ &= (1-\delta_j)\varphi(\Phi_j(x)) + \varphi(\mathbb{E}_{\mathcal{N}}(a_j) b_j^* x b_j) \\ &= (1-\delta_j)\varphi(\Phi_j(x)) + (b_j \varphi b_j^*)(x) \\ &= (1-\delta_j)\varphi(\Phi_j(x)) + \varphi(x) - (1-\delta_j)\varphi(\Phi_j(x)) \\ &= \varphi(x), \end{aligned}$$

so the Ψ_j are unital and φ-preserving.

For the relative compactness note that by the assumption that every element in M is analytic for σ^{φ}, Lemma 7.2.11 implies that for all $x \in \mathcal{M}$

$$\Psi_j(x) = (1-\delta_j)\Phi_j(x) + a_j \mathbb{E}_{\mathcal{N}}(\sigma_i^{\varphi}(b_j) b_j^* x),$$

hence,

$$\Psi_j^{(2)} = (1-\delta_j)\Phi_j^{(2)} + a_j e_{\mathcal{N}} \sigma_i^{\varphi}(b_j) b_j^* \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi).$$

It remains to show that for every $x \in \mathcal{M}$, $\Psi_j(x) \to x$ strongly. For this, estimate for $x \geq 0$,

$$\begin{aligned} (\Psi_j - (1-\delta_j)\Phi_j)(x) &= a_j^{1/2} \mathbb{E}_{\mathcal{N}}(b_j^* x b_j) a_j^{1/2} \\ &\leq \|x\| a_j^{1/2} \mathbb{E}_{\mathcal{N}}(b_j^* b_j) a_j^{1/2} \\ &= \|x\|(1-c_j). \end{aligned}$$

Since $c_j = (1-\delta_j)\Phi_j(1) \to 1$ and $(1-\delta_j)\Phi_j(x) \to x$ strongly it then follows that

$$\Psi_j(x) = (\Psi_j(x) - (1-\delta_j)\Phi_j(x)) + (1-\delta_j)\Phi_j(x) \to x$$

strongly for every $x \in \mathcal{M}$. This completes the proof. □

Another important statement that was proved in [42] in case of the usual (non-relative) Haagerup property is the following lemma. It will later ensure the contractivity of certain approximating maps and allow us to apply Theorem 7.2.13 in a suitable setting.

Lemma 7.2.14. *Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau \in \mathcal{M}_*$ and let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion. Assume that $h \in \mathcal{N}' \cap \mathcal{M}$ is a boundedly invertible self-adjoint element and define $\varphi \in \mathcal{M}_*$ by $\varphi(x) := \tau(hxh)$ for $x \in \mathcal{M}$. Then, if $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP), the approximating maps $(\Phi_i)_{i\in I}$ witnessing property (rHAP) may be chosen contractively, i.e. we may assume that $(1')$ holds.*

Proof. The proof is given in [42, Lemma 4.3]. One only needs to check that the condition $h \in \mathcal{N}' \cap \mathcal{M}^+$ ensures that the maps Φ'_k, Φ^l_k and Ψ_j defined there are $\mathcal{N}$-$\mathcal{N}$-bimodule maps that are compact relative to $\mathcal{N}$. Let us comment on this.

In Step 1 of the proof of [42, Lemma 4.3] it is shown that the approximating maps Φ_k witnessing the Haagerup property may be chosen such that $\sup_k \|\Phi_k\| < \infty$. In the current setup of (rHAP) this is automatic (see Definition 7.2.2) and so we may skip this step.

We now turn to Step 2 in the proof of [42, Lemma 4.3]. Let Φ_k be the approximating maps witnessing the (rHAP) for $(\mathcal{M}, \mathcal{N}, \varphi)$. In particular Φ_k is $\mathcal{N}$-$\mathcal{N}$-bimodular and $\Phi^{(2)}_k \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$. By [172, Theorem VIII.2.211] we have $\sigma^\varphi_t(x) = h^{it}xh^{-it}$, $t \in \mathbb{R}, x \in \mathcal{M}$. Now recall the map defined in [42, Lemma 4.3] given by

$$\begin{aligned}\Phi^l_k(x) &= \sqrt{\frac{1}{l\pi}} \int_{-\infty}^{\infty} e^{-t^2/l} \sigma^\varphi_t(\Phi_k(\sigma^\varphi_{-t}(x)))\,dt \\ &= \sqrt{\frac{1}{l\pi}} \int_{-\infty}^{\infty} e^{-t^2/l} h^{it}\Phi_k(h^{-it}xh^{it})h^{-it}\,dt.\end{aligned} \tag{7.2.1}$$

Since $h \in \mathcal{N}' \cap \mathcal{M}$ this map is $\mathcal{N}$-$\mathcal{N}$-bimodular. Since $\sigma^\varphi_t(h^{is}) = h^{is}$, $s, t \in \mathbb{R}$ it follows from Lemma 7.2.12 that the L^2-implementation of

$$x \mapsto \sigma^\varphi_t(\Phi_k(\sigma^\varphi_{-t}(x))) = h^{it}\Phi_k(h^{-it}xh^{it})h^{-it}, \qquad t \in \mathbb{R}, \tag{7.2.2}$$

exists and is compact, i.e. contained in $\mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$. By assumption h is boundedly invertible and so $t \mapsto h^{it}$ depends continuously (in norm) on t. Hence the map (7.2.2) depends continuously on t and it follows that (7.2.1) is compact.

Next, in the proof of [42, Lemma 4.3] the following operators were defined:

$$g^l_k = \Phi^l_k(1), \qquad f^{n,l}_k = F_n(g^l_k),$$

where $F_n(z) = e^{-n(z-1)^2}$, $z \in \mathbb{C}, n \in \mathbb{N}$. Since Φ^l_k is $\mathcal{N}$-$\mathcal{N}$-bimodular it follows that $g^l_k \in \mathcal{N}' \cap \mathcal{M}$. Therefore also $f^{n,l}_k \in \mathcal{N}' \cap \mathcal{M}$. Then the proof of [42, Lemma 4.3] defines for suitable $n(j), k(j), l(j) \in \mathbb{N}, \epsilon_j > 0$ depending on some j in a directed set the map $\Psi_j : \mathcal{M} \to \mathcal{M}$ via the formula:

$$\Psi_j(\cdot) = \frac{1}{(1+\epsilon_j)^2} f^{n(j),l(j)}_{k(j)} \Phi^{l(j)}_{k(j)}(\cdot) f^{n(j),l(j)}_{k(j)}.$$

Since $f^{n(j),l(j)}_{k(j)} \in \mathcal{N}' \cap \mathcal{M}$ it follows that Ψ_j is both $\mathcal{N}$-$\mathcal{N}$-bimodular and compact, i.e. $\Psi^{(2)}_j \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$. The last part of the proof of [42, Lemma 4.3] shows that $\Psi^{(2)}_j \to 1$ strongly and this holds true here as well with the same proof. By Lemma 7.1.1 this shows that for every $x \in \mathcal{M}$ we have $\Psi_j(x) \to x$ strongly. □

7.3. FOR FINITE $\mathscr{N}$: TRANSLATION INTO THE FINITE SETTING

As in [41], the key idea of the proof of the main results in Section 7.4 uses crossed products by modular actions and the passage to the semifinite setting that (Takai-) Takesaki duality permits. However, the relative context makes the technical details much more demanding and makes adapting the earlier methods – including those developed in [13] – significantly more complicated.

Let again $\mathscr{N} \subseteq \mathscr{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}}$. Assume moreover that $\mathscr{N}$ is a general σ-finite von Neumann algebra, though in many of the statements below we shall add the assumption that $\mathscr{N}$ is finite. This section aims to characterize the relative Haagerup property (resp. property (rHAP)$^-$) of the triple $(\mathscr{M},\mathscr{N},\mathbb{E}_{\mathscr{N}})$ in terms of the structure of certain corners of crossed product von Neumann algebras associated with the modular automorphism group of some faithful $\varphi \in M_*^+$ with $\varphi \circ \mathbb{E}_N = \varphi$. These statements will play a crucial role in Section 7.4.

7.3.1. CROSSED PRODUCTS

Let us first recall some of the theory of crossed product von Neumann algebras and their duality for which we refer to [172, Section X.2]. For this, fix an action $\mathbb{R} \curvearrowright^{\alpha} \mathscr{M}$ on $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$, define the corresponding *fixed point algebra*

$$\mathscr{M}^{\alpha} := \{x \in \mathscr{M} \mid \alpha_t(x) = x \text{ for all } t \in \mathbb{R}\}$$

and let $\mathscr{M} \overline{\rtimes}_{\alpha} \mathbb{R} \subseteq \mathscr{B}(L^2(\mathbb{R},\mathscr{H})) \cong \mathscr{B}(\mathscr{H} \otimes L^2(\mathbb{R}))$ be the corresponding crossed product von Neumann algebra (see also Section 2.3). Recall that it is generated by the operators $\pi(x)$, $x \in \mathscr{M}$ and λ_t, $t \in \mathbb{R}$ where

$$(\pi(x)\xi)(t) = \alpha_{-t}(x)(\xi(t)) \qquad \text{and} \qquad (\lambda_t \xi)(s) = \xi(s-t)$$

for $s,t \in \mathbb{R}$, $x \in \mathscr{M}$, $\xi \in L^2(\mathbb{R},\mathscr{H})$. We will also occasionally use λ to denote the left regular representation of $\mathbb{R}$, which should not cause any confusion. Recall that this construction does not depend on the choice of the embedding $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ and that $\mathscr{M} \cong \pi(M)$. For notational convenience we will therefore omit the faithful normal representation π in our notation and identify $\mathscr{M}$ with $\pi(\mathscr{M})$ and $\mathscr{N}$ with $\pi(\mathscr{N})$. Note that $\pi(x) = x \otimes 1$ for all $x \in \mathscr{M}^{\alpha}$. Set further $\lambda(f) := \int_{\mathbb{R}} f(t)\lambda_t dt$ for $f \in L^1(\mathbb{R})$ and

$$\mathscr{L}(\mathbb{R}) := \{\lambda(f) \mid f \in L^1(\mathbb{R})\}'' = \{\lambda_s \mid s \in \mathbb{R}\}'' \subseteq \mathscr{B}(L^2(\mathbb{R},\mathscr{H})).$$

Remark 7.3.1. For $f \in L^1(\mathbb{R})$ we denote by

$$\widehat{f}(s) := \int_{\mathbb{R}} f(t)e^{ist}dt \in L^{\infty}(\mathbb{R}),$$

its Fourier transform. Let $\mathscr{F}_2 : L^2(\mathbb{R}) \to L^2(\mathbb{R}), f \mapsto (2\pi)^{-\frac{1}{2}}\widehat{f}$ be the unitary Fourier transform operator on $L^2(\mathbb{R})$. Then $\mathscr{F}_2\lambda(f)\mathscr{F}_2^*$ is the operator of multiplication by

$\widehat{f}$. We shall occasionally extend our notation in the following way. Let $f \in L^2(\mathbb{R})$ be such that its Fourier transform $\widehat{f}$ is in $L^\infty(\mathbb{R})$. We shall write $\lambda(f)$ for $\mathcal{F}_2^* \widehat{f} \mathcal{F}_2$ where we view $\widehat{f}$ as a multiplication operator. This is naturally compatible with the earlier notation for $f \in L^1(\mathbb{R})$

Let $\mathbb{R} \overset{\widehat{\alpha}}{\curvearrowright} \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}$ be the *dual action* determined by

$$\widehat{\alpha}_t(x) = x, \qquad \text{and} \qquad \widehat{\alpha}_t(\lambda_s) = \exp(-ist)\lambda_s, \tag{7.3.1}$$

for $x \in \mathcal{M}$, $s, t \in \mathbb{R}$ and recall that its fixed point algebra is given by

$$\mathcal{M} = (\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R})^{\widehat{\alpha}}. \tag{7.3.2}$$

The expression

$$T_{\widehat{\alpha}}(x) := \int_{\mathbb{R}} \widehat{\alpha}_s(x) ds, \qquad x \in (\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R})^+,$$

defines a faithful normal semi-finite operator-valued weight on $\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}$ which takes values in the extended positive part of $\mathcal{M}$. Choose $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|f\|_2 = 1$ such that the support of the Fourier transform $\widehat{f}$ equals $\mathbb{R}$. We keep f fixed throughout the whole subsection. One has $T_{\widehat{\alpha}}(\lambda(f)^* \lambda(f)) = \|f\|_2^2 = 1$, hence we may define the unital normal completely positive map

$$T_f := T_{f,\widehat{\alpha}} : \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R} \to M, \, x \mapsto T_{\widehat{\alpha}}(\lambda(f)^* x \lambda(f)).$$

By Lemma 7.1.1 T_f is strongly continuous on the unit ball. For a given map $\Phi : \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R} \to \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}$ and a positive normal functional $\varphi \in \mathcal{M}_*$ we further define

$$\widetilde{\Phi}_f : \mathcal{M} \to \mathcal{M}, \, \widetilde{\Phi}_f(x) := T_f(\Phi(x))$$

and

$$\widehat{\varphi}_f : \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R} \to \mathbb{C}, \, \widehat{\varphi}_f(x) := \varphi(T_f(x)).$$

The functional $\widehat{\varphi}_f$ is normal and positive. It is moreover a state if φ is a state. Since we assumed the support of $\widehat{f}$ to be equal to $\mathbb{R}$, by Remark 7.3.1 the support projection of $\lambda(f)$ equals 1. It follows that $\widehat{\varphi}_f$ is faithful if and only if φ is faithful.

Lemma 7.3.2. *Assume that $\mathcal{N} \subseteq \mathcal{M}^\alpha$. Then T_f is $\mathcal{N}$-$\mathcal{N}$-bimodular, meaning that for $x, y \in \mathcal{N}$, $a \in \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}$ we have $T_f(xay) = xT_f(a)y$.*

Proof. As $\mathcal{N} \subseteq \mathcal{M}^\alpha$ we have that $\mathcal{N}$ and $\lambda(f)$ commute. From the definition of $T_{\widehat{\alpha}}$ and (7.3.2) we get that for $x, y \in \mathcal{N}$ and $a \in \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}$,

$$T_{\widehat{\alpha}}(\lambda(f)^* xay\lambda(f)) = T_{\widehat{\alpha}}(x\lambda(f)^* a\lambda(f)y) = xT_{\widehat{\alpha}}(\lambda(f)^* a\lambda(f))y.$$

This concludes the proof. □

We recall the following formula which was proved in [41, Lemma 5.2] (which extends [92, Theorem 3.1 (c)]) in case $k = g$; the general case then follows from the polarization identity. For $k, g \in L^2(\mathbb{R})$ such that $\widehat{k}, \widehat{g} \in L^\infty(\mathbb{R})$ and $x \in \mathscr{M}$ we have

$$T_{\widehat{\alpha}}(\lambda(k)^* x \lambda(g)) = \int_{\mathbb{R}} \overline{k(t)} g(t) \alpha_{-t}(x) dt. \tag{7.3.3}$$

We shall need the following consequence of it. For $g \in L^1(\mathbb{R})$ define $g^*(t) := \overline{g(-t)}$, which is the involution for the convolution algebra $L^1(\mathbb{R})$.

Lemma 7.3.3. *Let $h \in C_c(\mathbb{R})$ and let $x \in \mathscr{M}$. Then, for $k, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $a := \lambda(h)x$,*

$$T_{\widehat{\alpha}}(\lambda(k)^* a \lambda(g)) = \int_{\mathbb{R}} \int_{\mathbb{R}} k^*(s) g(t) h(-s-t) \alpha_{-t}(x) ds dt,$$

and

$$T_{\widehat{\alpha}}(\lambda(k)^* \lambda(g) a) = \int_{\mathbb{R}} \int_{\mathbb{R}} k^*(s) g(t) h(-s-t) x ds dt.$$

Proof. We have $\lambda(k)^* a = \lambda(h^* * k)^* x$. The equality (7.3.3) then implies

$$\begin{aligned} T_{\widehat{\alpha}}(\lambda(k)^* a \lambda(g)) &= T_{\widehat{\alpha}}(\lambda(h^* * k)^* x \lambda(g)) \\ &= \int_{\mathbb{R}} \overline{(h^* * k)(t)} g(t) \alpha_{-t}(x) dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} k^*(s) g(t) h(-s-t) \alpha_{-t}(x) ds dt. \end{aligned}$$

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This concludes the proof of the first formula. The second formula follows from the first after observing that $T_{\widehat{\alpha}}(\lambda(k)^* \lambda(g) a) = T_{\widehat{\alpha}}(\lambda(k)^* \lambda(g) \lambda(h)) x$. □

7.3.2. PASSAGE TO CROSSED PRODUCTS

Let us now study the stability of the relative Haagerup property with respect to certain crossed products. The setting is the same as in Subsection 7.3.1.

Proposition 7.3.4. *Let $\Phi : \mathscr{M} \overline{\rtimes}_\alpha \mathbb{R} \to \mathscr{M} \overline{\rtimes}_\alpha \mathbb{R}$ be a linear map and fix $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as before. Then the following statements hold:*

(1) *If Φ is completely positive then so is $\widetilde{\Phi}_f$.*

(2) *Assume that $\mathscr{N} \subseteq \mathscr{M}^\alpha$. If Φ is an $\mathscr{N}$-$\mathscr{N}$-bimodule map then $\widetilde{\Phi}_f$ is an $\mathscr{N}$-$\mathscr{N}$-bimodule map.*

In the remaining statements let $\varphi \in \mathscr{M}_*^+$ be a faithful normal positive functional with $\varphi \circ \mathbb{E}_{\mathscr{N}} = \varphi$ and $\varphi \circ \alpha_t = \varphi$ for all $t \in \mathbb{R}$. Then:

(3) *If $\widehat{\varphi}_f \circ \Phi \leq \widehat{\varphi}_f$ (resp. $\widehat{\varphi}_f \circ \Phi = \widehat{\varphi}_f$) then $\varphi \circ \widetilde{\Phi}_f \leq \varphi$ (resp. $\varphi \circ \widetilde{\Phi}_f = \varphi$).*

(4) *If the L^2-implementation of Φ with respect to $\widehat{\varphi}_f$ exists, then the L^2-implementation of $\widetilde{\Phi}_f$ with respect to φ exists as well.*

Now, if $\mathcal{N} \subseteq \mathcal{M}^\alpha$, $\mathbb{E}_\mathcal{N} \circ \alpha_t = \mathbb{E}_\mathcal{N}$ for all $t \in \mathbb{R}$ and f is continuous, then:

(5) If $\Phi \in \mathcal{K}_{00}(\mathcal{M}\overline{\rtimes}_\alpha\mathbb{R}, \mathcal{N}, \widehat{\varphi}_f)$, then $\widetilde{\Phi}_f \in \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$.

(6) If $\Phi \in \mathcal{K}(\mathcal{M}\overline{\rtimes}_\alpha\mathbb{R}, \mathcal{N}, \widehat{\varphi}_f)$, then $\widetilde{\Phi}_f \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.

Proof. (1) is straightforward from the constructions and (2) follows from Lemma 7.3.2.

(3): If $\widehat{\varphi}_f \circ \Phi \leq \widehat{\varphi}_f$ we have for $x \in \mathcal{M}^+$, using (7.3.3) and the α-invariance of φ,

$$\begin{aligned} \varphi \circ \widetilde{\Phi}_f(x) &= \varphi(T_f(\Phi(x))) = \widehat{\varphi}_f(\Phi(x)) \leq \widehat{\varphi}_f(x) \\ &= \varphi(T_{\widehat{\alpha}}(\lambda(f)^* x \lambda(f))) = \int_\mathbb{R} |f(t)|^2 \varphi(\alpha_{-t}(x))dt = \varphi(x). \end{aligned}$$

Moreover, if $\widehat{\varphi}_f \circ \Phi = \widehat{\varphi}_f$ then the inequality above is actually an equality.

(4): Assume that there exists a constant $C > 0$ such that $\widehat{\varphi}_f(\Phi(x)^*\Phi(x)) \leq C\widehat{\varphi}_f(x^*x)$ for all $x \in \mathcal{M}$. Then, by the Kadison-Schwarz inequality and (7.3.3),

$$\varphi(\widetilde{\Phi}_f(x)^*\widetilde{\Phi}_f(x)) = \varphi(T_f(\Phi(x))^* T_f(\Phi(x))) \leq \widehat{\varphi}_f(\Phi(x)^*\Phi(x)) \leq C\widehat{\varphi}_f(x^*x) = C\varphi(x^*x)$$

for all $x \in \mathcal{M}$, where we use the fact (proved above) that φ and $\hat{\varphi}_f$ coincide on $\mathcal{M}_+$. This implies that the L^2-implementation of $\widetilde{\Phi}_f$ with respect to φ exists.

(5): By Lemma 7.3.2 and the discussion before, $\mathbb{F}_\mathcal{N} = \mathbb{E}_\mathcal{N} \circ T_f$ is the unique faithful normal $\widehat{\varphi}_f$-preserving conditional expectation of $\mathcal{M}\overline{\rtimes}_\alpha\mathbb{R}$ onto $\mathcal{N}$. Let $a, b \in \mathcal{M}\overline{\rtimes}_\alpha\mathbb{R}$. By $\mathcal{N} \subseteq \mathcal{M}^\alpha$ we have for $x \in \mathcal{M}$,

$$\widetilde{(a\mathbb{F}_\mathcal{N} b)}_f(x) := T_f(a\mathbb{F}_\mathcal{N}(bx)) = T_f(a)\mathbb{F}_\mathcal{N}(bx) \tag{7.3.4}$$

We shall show that $\mathbb{F}_\mathcal{N}(bx) = \mathbb{E}_\mathcal{N}(\widetilde{b}x)$ for all $x \in \mathcal{M}$, where $\widetilde{b} := T_{\widehat{\alpha}}(\lambda(f)^*\lambda(f)b)$. For this it suffices to consider the case where $b = \lambda(h)y$ for some compactly supported function $h \in C_c(\mathbb{R})$ and $y \in \mathcal{M}$, since such elements span a σ-weakly dense subset of $\mathcal{M}\overline{\rtimes}_\alpha\mathbb{R}$ and the map $b \mapsto \widetilde{b}$ is σ-weakly continuous. Using Lemma 7.3.3 twice and the fact that $\mathbb{E}_\mathcal{N} \circ \alpha_t = \mathbb{E}_\mathcal{N}$ for all $t \in \mathbb{R}$ one has

$$\begin{aligned} \mathbb{F}_\mathcal{N}(bx) &= \mathbb{E}_\mathcal{N} \circ T_f(bx) \\ &= \mathbb{E}_\mathcal{N}\left(\int_\mathbb{R}\int_\mathbb{R} f^*(s)f(t)h(-s-t)\alpha_{-t}(yx)dsdt\right) \\ &= \int_\mathbb{R}\int_\mathbb{R} f^*(s)f(t)h(-s-t)\mathbb{E}_\mathcal{N}(yx)dsdt \\ &= \mathbb{E}_\mathcal{N}\left(\int_\mathbb{R}\int_\mathbb{R} f^*(s)f(t)h(-s-t)ydsdt\ x\right) \\ &= \mathbb{E}_\mathcal{N}(\widetilde{b}x), \end{aligned}$$

as claimed. Combining this equality and (7.3.4) we get that $\widetilde{(a\mathbb{E}_\mathcal{N} b)}_f = T_f(a)\mathbb{E}_\mathcal{N}\widetilde{b}$. By considering linear combinations of such expressions one gets that if Φ is contained in $\mathcal{K}_{00}(\mathcal{M}\overline{\rtimes}_\alpha\mathbb{R}, \mathcal{N}, \widehat{\varphi}_f)$ then also $\widetilde{\Phi}_f \in \mathcal{K}_{00}(\mathcal{M}, \mathcal{N}, \varphi)$. This proves (5).

(6): The statement follows directly from (5) by approximation and the fact that $\|\widetilde{\Phi}_f\| \leq \|\Phi\|$. □

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In the following we will direct our attention to certain choices of functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\|f\|_2 = 1$ whose support of the Fourier transform $\widehat{f}$ equals $\mathbb{R}$. For this, define for $j \in \mathbb{N}$ the L^2-normalized *Gaussian*

$$f_j : \mathbb{R} \to \mathbb{R},\ f_j(s) := \left(\frac{j}{\pi}\right)^{1/4} \exp(-js^2/2).$$

Further set for a given map $\Phi : \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R} \to \mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}$ and a positive normal functional $\varphi \in \mathcal{M}_*$

$$\widehat{\varphi}_j := \widehat{\varphi}_{f_j} \qquad \text{and} \qquad \widetilde{\Phi}_j := \widetilde{\Phi}_{f_j}.$$

Theorem 7.3.5. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$. Let further $\varphi \in \mathcal{M}_*^+$ be a faithful normal positive functional with $\varphi \circ \mathbb{E}_\mathcal{N} = \varphi$ and $\mathbb{R} \curvearrowright^\alpha \mathcal{M}$ be an action such that $\mathcal{N} \subseteq \mathcal{M}^\alpha$. Finally assume that $\mathbb{E}_\mathcal{N} \circ \alpha_t = \mathbb{E}_\mathcal{N}$ (or, equivalently under the earlier assumptions, that $\varphi = \varphi \circ \alpha_t$) for all $t \in \mathbb{R}$. Then the following statements hold:*

(1) If for all $j \in \mathbb{N}$ the triple $(\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$), then $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) (resp. property (rHAP)$^-$).

(2) If for all $j \in \mathbb{N}$ property (rHAP) of $(\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) approximating maps (see (1'') and (4') in Subsection 7.2.1), then also property (rHAP) of $(\mathcal{M}, \mathcal{N}, \varphi)$ may be witnessed by unital (resp. φ-preserving) approximating maps.

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Proof. *(1)*: For fixed $j \in \mathbb{N}$ let $(\Phi_{j,k})_{k \in K_j}$ be a bounded net of normal completely positive maps witnessing the relative Haagerup property of $(\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. In particular, $\Phi_{j,k} \to 1$ in the point-strong topology in k. Set $\widetilde{\Phi}_{j,k} := T_{f_j} \circ \Phi_{j,k}$. As $s \mapsto \alpha_s(x)$ is strongly continuous for $x \in \mathcal{M}$ and f_j is L^2-normalized with mass concentrated around 0, Lemma 7.3.3 shows that for $x \in \mathcal{M}$,

$$T_{f_j}(x) = \int_\mathbb{R} |f_j(s)|^2 \alpha_s(x)\, ds \to x$$

as $j \to \infty$ in the strong topology. Lemma 7.1.3 then shows that we may find a directed set $\mathcal{F}$ and a function $(\widetilde{j}, \widetilde{k}) : \mathcal{F} \to \{(j,k) \mid j \in \mathbb{N}, k \in K_j\}$, $F \mapsto (\widetilde{j}(F), \widetilde{k}(F))$ such that the net $(\widetilde{\Phi}_{\widetilde{j}(F), \widetilde{k}(F)})_{F \in \mathcal{F}}$ converges to the identity in the point-strong topology. By Proposition 7.3.4 these maps then witness the relative Haagerup property for $(\mathcal{M}, \mathcal{N}, \varphi)$. In the same way, using a variant of Lemma 7.1.3, we can deduce that if $(\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP)$^-$, then $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP)$^-$ as well.

(2): Note that if $\Phi_{j,k}$ is unital for all $k \in \mathbb{N}$, then $\widetilde{\Phi}_{j,k}$ is unital as well and if $\Phi_{j,k}$ is $\widehat{\varphi}_j$-preserving for all $k \in \mathbb{N}$, then $\widetilde{\Phi}_{j,k}$ is φ-preserving, c.f. Proposition 7.3.4. □

We will now apply this theorem to the modular automorphism group σ^φ of φ as well as its dual action.

Theorem 7.3.6. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state. Further define the faithful normal (possibly non-tracial) state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then the following statements hold:*

(1) If for all $j \in \mathbb{N}$ the triple $(\mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) (resp. property (rHAP)$^-$), then $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) (resp. property (rHAP)$^-$).

(2) If for all $j \in \mathbb{N}$ property (rHAP) of $(\mathcal{M} \overline{\rtimes}_\alpha \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital (resp. $\widehat{\varphi}_j$-preserving) approximating maps, then also property (rHAP) of $(\mathcal{M}, \mathcal{N}, \varphi)$ may be witnessed by unital (resp. φ-preserving) approximating maps.

Proof. This is Theorem 7.3.5 for $\alpha = \sigma^\varphi$; the assumptions are satisfied, as follows from the fact that $\mathcal{N} \subseteq \mathcal{M}^\varphi$. □

We will also prove the converse of Theorem 7.3.6 by using crossed product duality. We first recall the following well-known lemma. We will use the fact that every function $g \in L^\infty(\mathbb{R})$ may be viewed as a multiplication operator on $L^2(\mathbb{R})$.

Lemma 7.3.7. *For $g, h \in C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ we have that $g\lambda(h) \in \mathcal{B}(L^2(\mathbb{R}))$ is Hilbert-Schmidt with*

$$\mathrm{Tr}((g\lambda(h))^* g\lambda(h)) = \|g\|_2^2 \|h\|_2^2.$$

Proof. Let $\mathcal{S}_2(\mathcal{H})$ denote the Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$. We have linear identifications $\mathcal{H} \otimes \overline{\mathcal{H}} \cong \mathcal{S}_2(\mathcal{H})$ where $\xi \otimes \overline{\eta}$ corresponds to the rank 1 operator $v \mapsto \xi\eta^*(v)$. We identify $L^2(\mathbb{R})$ with $\overline{L^2(\mathbb{R})}$ linearly and isometrically through the pairing $\langle \xi, \eta \rangle = \int_{\mathbb{R}} \xi(s)\eta(s)ds$. Therefore we have isometric linear identifications

$$\mathcal{S}_2(L^2(\mathbb{R})) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R}^2), \tag{7.3.5}$$

where the rank 1 operator $\xi\eta^*$ corresponds to the function $(s,t) \mapsto \xi(s)\eta(t)$.

Now, $g\lambda(h)$ is an integral operator on $L^2(\mathbb{R})$ with a square-integrable kernel $K(x,y) := g(x)h(x-y)$. Then $g\lambda(h)$ is Hilbert-Schmidt and corresponds to $K \in L^2(\mathbb{R}^2)$ in (7.3.5), so that $\|g\lambda(h)\|_{\mathcal{S}_2} = \|K\|_2 = \|g\|_2 \|h\|_2$. □

Further recall that for $j \in \mathbb{N}$ the Gaussian f_j was defined by

$$f_j(s) := j^{1/4} \pi^{-1/4} \exp(-js^2/2)$$

$s \in \mathbb{R}$ and $\widehat{f}_j$ denotes its Fourier transform. Both these functions are L^2-normalized by definition and the Plancherel identity. Define for $i, j \in \mathbb{N}$ a positive linear functional $\psi_{i,j}$ on $\mathcal{B}(L^2(\mathbb{R}))$ by

$$\psi_{i,j}(x) := \mathrm{Tr}((\widehat{f}_i \lambda(f_j))^* x \widehat{f}_i \lambda(f_j)).$$

It is a state by Lemma 7.3.7. We will need the following elementary lemma for which we give a short non-explicit proof following from the results in [41].

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Lemma 7.3.8. *For all $i,j \in \mathbb{N}$ the pair $(\mathcal{B}(L^2(\mathbb{R})), \psi_{i,j})$ has the Haagerup property in the sense that the triple $(\mathcal{B}(L^2(\mathbb{R})), \mathbb{C}, \psi_{i,j})$ has the relative Haagerup property, see [41, Definition 3.1]. Moreover, the approximating maps may be chosen to be unital and $\psi_{i,j}$-preserving.*

Proof. According to [41, Proposition 3.4], $(\mathcal{B}(L^2(\mathbb{R}), \mathrm{Tr})$ has the Haagerup property. By [41, Theorem 1.3] the Haagerup property does not depend on the choice of the faithful normal semi-finite weight and hence $(\mathcal{B}(L^2(\mathbb{R})), \psi_{i,j})$ has the Haagerup property for all $i,j \in \mathbb{N}$. In [42, Theorem 5.1] it was proved that the approximating maps may be taken unital and state-preserving. This finishes the proof. □

As before, let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$ and fix a faithful normal state φ on $\mathcal{M}$ with $\varphi = \varphi \circ \mathbb{E}_{\mathcal{N}}$. Let σ^φ be the corresponding modular automorphism group, $\mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}$ the crossed product von Neumann algebra and let

$$\theta := \widehat{\sigma^\varphi} : \mathbb{R} \curvearrowright \mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}$$

be the dual action as defined in (7.3.1). Define for $j \in \mathbb{N}$ the state $\widehat{\varphi}_j := \varphi \circ T_{f_j,\theta}$ on $\mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}$ as before and recall that $\mathcal{M}$ (hence also $\mathcal{N}$) is invariant under θ. We may in turn consider the double crossed product which admits an isomorphism of von Neumann algebras (i.e. a bijective $*$-homomorphism, which is automatically normal by [164, Theorem 1.13.2]),

$$(\mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}) \overline{\rtimes}_\theta \mathbb{R} \cong \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R})). \tag{7.3.6}$$

Let us describe what this isomorphism looks like. For $g \in L^\infty(\mathbb{R})$ write $\mu(g) := 1_{\mathcal{M}} \otimes g \in \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}))$ for the multiplication operator acting in the second tensor leg. The double crossed product above is generated by $\mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}$ and the left regular representation of the second copy of $\mathbb{R}$, denoted here by λ_t^θ, $t \in \mathbb{R}$. Under the isomorphism, $\mathcal{M} \overline{\rtimes}_{\sigma^\varphi} \mathbb{R}$ is identified as a subalgebra of $\mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}))$ via inclusion. Further, λ_t^θ is identified for every $t \in \mathbb{R}$ with $\mu(e_t) = 1_{\mathcal{M}} \otimes e_t$ where $e_t(s) := \exp(-ist)$ for $s \in \mathbb{R}$. Under this correspondence, $\lambda^\theta(f_j) = \mu(\widehat{f_j})$. We find that for $x \in \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}))$,

$$\begin{aligned} (\varphi \circ T_{f_j,\theta} \circ T_{f_i,\widehat{\theta}})(x) &= \varphi\left(T_\theta\left(\lambda(f_j)^* T_{\widehat{\theta}}\left(\mu(\widehat{f_i})^* x \mu(\widehat{f_i})\right) \lambda(f_j)\right)\right) \\ &= \varphi\left(T_\theta\left(T_{\widehat{\theta}}\left(\lambda(f_j)^* \mu(\widehat{f_i})^* x \mu(\widehat{f_i}) \lambda(f_j)\right)\right)\right). \end{aligned}$$

By [172, Theorem X.2.3] and the fact that $\varphi \circ \sigma_t^\varphi = \varphi$ we have that (formally, being imprecise about domains) the normal semi-finite faithful weight $\varphi \circ T_\theta \circ T_{\widehat{\theta}}$ coincides with $\varphi \otimes \mathrm{Tr}$. Hence, for $i,j \in \mathbb{N}$ we have equality of states

$$\varphi \circ T_{f_j,\theta} \circ T_{f_i,\widehat{\theta}} = \varphi \otimes \psi_{i,j}.$$

The following theorem now provides a passage to study the relative Haagerup property on the continuous core of a von Neumann algebra, which is semi-finite.

Theorem 7.3.9. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$ and assume that $\mathcal{N}$ is finite with a faithful normal tracial state $\tau \in \mathcal{N}_*$. Set $\varphi = \tau \circ \mathbb{E}_{\mathcal{N}} \in \mathcal{M}_*$. Then the following two statements hold:*

(1) *The triple* $(\mathcal{M},\mathcal{N},\varphi)$ *has property (rHAP) (resp. property (rHAP)$^-$) if and only if* $(\mathcal{M}\overline{\rtimes}_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ *has property (rHAP) (resp. property (rHAP)$^-$) for all* $j\in\mathbb{N}$.

(2) *If property (rHAP) of* $(\mathcal{M},\mathcal{N},\varphi)$ *is witnessed by unital (resp.* φ*-preserving) maps, then for all* $j\in\mathbb{N}$ *property (rHAP) of* $(\mathcal{M}\overline{\rtimes}_\alpha\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ *is witnessed by unital (resp.* $\widehat{\varphi}_j$*-preserving) maps, and vice versa.*

Proof. The if statements were proven in Theorem 7.3.6. For the converse of (1) assume that $(\mathcal{M},\mathcal{N},\varphi)$ has the relative Haagerup property. $(\mathcal{B}(L^2(\mathbb{R})),\mathbb{C},\psi_{i,j})$ has the relative Haagerup property for all $i,j\in\mathbb{N}$, see Lemma 7.3.8. Therefore by a suitable modification of [41, Lemma 3.5], we see that $(\mathcal{M}\otimes\mathcal{B}(L^2(\mathbb{R})),\mathcal{N}\otimes\mathbb{C},\varphi\otimes\psi_{i,j})$ has the relative Haagerup property for all $i,j\in\mathbb{N}$. It follows from Theorem 7.3.5 and the discussion above that the triple $(\mathcal{M}\overline{\rtimes}_{\sigma^\varphi}\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$ has the relative Haagerup property.[1]

The statements in (2) and the statement about property (rHAP)$^-$ follow in the same way. □

7.3.3. PASSAGE TO CORNERS OF CROSSED PRODUCTS

In the last subsection we characterized the relative Haagerup property of the triple $(\mathcal{M},\mathcal{N},\varphi)$ for finite $\mathcal{N}$ with a faithful normal tracial state $\tau\in\mathcal{N}_*$ and $\varphi:=\tau\circ\mathbb{E}_\mathcal{N}\in\mathcal{M}_*$ in terms of the crossed product triples $(\mathcal{M}\rtimes_\alpha\mathbb{R},\mathcal{N},\widehat{\varphi}_j)$, $j\in\mathbb{N}$. In the following we will pass over to suitable corners of these crossed products which allows us to translate our investigations into the setting of finite von Neumann algebras. In this setting the following lemma will be useful.

Lemma 7.3.10. *Let* $\mathcal{N}\subseteq\mathcal{M}$ *be a unital inclusion of finite von Neumann algebras, let* $\tau\in\mathcal{M}_*$ *be a faithful normal tracial state and let* $\mathbb{E}_\mathcal{N}:\mathcal{M}\to\mathcal{N}$ *be the unique* τ*-preserving faithful normal conditional expectation onto* $\mathcal{N}$*. Further, let* $h\in\mathcal{N}'\cap\mathcal{M}$ *be self-adjoint and boundedly invertible. For a linear completely positive map* $\Phi:\mathcal{M}\to\mathcal{M}$ *set*

$$\Phi^h(x)=h^{-1}\Phi(hxh)h^{-1}.$$

Then, the L^2*-implementation* $\Phi^{(2)}$ *of* Φ *with respect to* τ *exists if and only if the* L^2*-implementation* $(\Phi^h)^{(2)}$ *of* Φ^h *with respect to* $h\tau h$ *exists. Further,* $\Phi^{(2)}\in\mathcal{K}(\mathcal{M},\mathcal{N},\tau)$ *if and only if* $(\Phi^h)^{(2)}\in\mathcal{K}(\mathcal{M},\mathcal{N},h\tau h)$.

Proof. Note first that the assumptions on h imply that $\mathbb{E}_\mathcal{N}(h^2)$ is a positive boundedly invertible element of the center $\mathcal{Z}(\mathcal{N})$ of $\mathcal{N}$. Indeed, we have for all $n\in\mathcal{N}$ the equality $n\mathbb{E}_\mathcal{N}(h^2)=\mathbb{E}_\mathcal{N}(nh^2)=\mathbb{E}_\mathcal{N}(h^2n)=\mathbb{E}_\mathcal{N}(h^2)n$, and if h is boundedly invertible, then $h^2\geq c1_M$ for some $c>0$, hence $\mathbb{E}_\mathcal{N}(h^2)\geq c1_M$.

The map $\mathbb{E}^h_\mathcal{N}:x\mapsto\mathbb{E}_\mathcal{N}(h^2)^{-1/2}\mathbb{E}_\mathcal{N}(hxh)\mathbb{E}_\mathcal{N}(h^2)^{-1/2}$ is the unique normal $h\tau h$-preserving conditional expectation onto N. Indeed, we can verify it is an idempotent, normal, unital, completely positive map with image equal to $\mathcal{N}$ and for

[1] Note that in the picture above $\pi(x)=x\otimes 1$ and hence $\pi(\mathcal{N})=\mathcal{N}\otimes\mathbb{C}$ since φ is tracial on $\mathcal{N}$. This is used implicitly in the identifications of $\mathcal{N}$ in the double crossed product isomorphism (7.3.6).

any $x \in \mathcal{M}$ we have

$$\begin{aligned}(h\tau h)(\mathbb{E}^h_{\mathcal{N}}(x)) &= \tau(h\mathbb{E}_{\mathcal{N}}(h^2)^{-1/2}\mathbb{E}_{\mathcal{N}}(hxh)\mathbb{E}_{\mathcal{N}}(h^2)^{-1/2}h) \\ &= \tau(\mathbb{E}_{\mathcal{N}}(h^2\mathbb{E}_{\mathcal{N}}(h^2)^{-1}\mathbb{E}_{\mathcal{N}}(hxh))) \\ &= \tau(\mathbb{E}_{\mathcal{N}}(h^2)\mathbb{E}_{\mathcal{N}}(h^2)^{-1}\mathbb{E}_{\mathcal{N}}(hxh)) \\ &= \tau(\mathbb{E}_{\mathcal{N}}(hxh)) \\ &= \tau(hxh) \\ &= (h\tau h)(x).\end{aligned}$$

Now assume that the L^2-implementation $\Phi^{(2)}$ of Φ with respect to τ exists, i.e. that there exists a constant $C > 0$ such that $\tau(\Phi(x)^*\Phi(x)) \leq C\tau(x^*x)$ for all $x \in \mathcal{M}$. Then

$$\begin{aligned}&(h\tau h)(\Phi^h(x)^*\Phi^h(x)) = \tau(\Phi(hx^*h)h^{-2}\Phi(hxh)) \leq \left\|h^{-2}\right\|\tau(\Phi(hx^*h)\Phi(hxh)) \\ \leq\ & C\left\|h^{-2}\right\|\tau(hx^*hhxh) \leq C\left\|h^{-2}\right\|\left\|h^2\right\|\tau(hx^*xh) = C\left\|h^{-2}\right\|\left\|h^2\right\|(h\tau h)(x^*x)\end{aligned}$$

for all $x \in \mathcal{M}$, so the L^2-implementation $(\Phi^h)^{(2)}$ exists as well.

The converse implication follows, as $\Phi = (\Phi^h)^{h^{-1}}$.

For elements $a, b, x \in \mathcal{M}$ the equality

$$\begin{aligned}(a\mathbb{E}_{\mathcal{N}}b)^h(x) &= h^{-1}a\mathbb{E}_{\mathcal{N}}(bhxh)h^{-1} \\ &= h^{-1}a\mathbb{E}_{\mathcal{N}}(h^2)^{1/2}\mathbb{E}^h_{\mathcal{N}}(h^{-1}bhx)\mathbb{E}_{\mathcal{N}}(h^2)^{1/2}h^{-1} \\ &= (h^{-1}a\mathbb{E}_{\mathcal{N}}(h^2)h^{-1})\mathbb{E}^h_{\mathcal{N}}(h^{-1}bhx) \\ &= (h^{-1}a\mathbb{E}_{\mathcal{N}}(h^2)h^{-1}\mathbb{E}^h_{\mathcal{N}}h^{-1}bh)(x)\end{aligned}$$

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implies by taking linear combinations and approximation that if $\Phi^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\tau)$, then $(\Phi^h)^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},h\tau h)$. The converse statement follows as before, which finishes the proof. □

Now, for a triple $(\mathcal{M},\mathcal{N},\varphi)$ let h be the unique (possibly unbounded) positive self-adjoint operator affiliated with $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ such that $h^{it} = \lambda_t$ for all $t \in \mathbb{R}$. If we further assume that $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ (which implies that $\mathcal{N}$ is finite with a tracial state $\varphi|_{\mathcal{N}}$) we have for $x \in \mathcal{N}$ that $\lambda_t x \lambda_t^* = \sigma_t^\varphi(x) = x$ and hence $\lambda_t \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$. This implies that h is affiliated with $\mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$ and so its finite spectral projections are elements in $\mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$. Set for $k \in \mathbb{N}$

$$p_k = \chi_{[k^{-1},k]}(h) \qquad \text{and} \qquad h_k = hp_k.$$

Here $\chi_{[k^{-1},k]}$ denotes the indicator function of $[k^{-1},k] \subseteq \mathbb{R}$ and p_k is the corresponding spectral projection. Then, for every $k \in \mathbb{N}$, h_k is boundedly invertible in the corner algebra $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ and we write h_k^{-1} for its inverse which we view as an operator in $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$.

Denote by $\widehat{\varphi} := \varphi \circ T_\theta$ the dual weight of φ and let $\tau_\rtimes$ be the unique faithful normal semi-finite weight on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ whose Connes cocycle derivative satisfies $(D\widehat{\varphi}/D\tau_\rtimes)_t = h^{it}$ for all $t \in \mathbb{R}$ (we refer to [94, Lemma 5.2]; the proofs below stay within the realm of bounded functionals). It is a trace on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ which is formally given by

$$\tau_\rtimes(x) = \varphi \circ T_\theta(h^{-\frac{1}{2}} x h^{-\frac{1}{2}}), \qquad x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})^+.$$

By construction we have

$$\widehat{\varphi}_j(p_k x p_k) = \tau_\rtimes(h_k^{\frac{1}{2}} \lambda(f_j)^* x \lambda(f_j) h_k^{\frac{1}{2}}), \qquad x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}. \tag{7.3.7}$$

for all $j \in \mathbb{N}$, where $\widehat{\varphi}_j$ and f_j are defined as in Subsection 7.3.2. Further note that the operators $\lambda(f_j)$ and h_k commute.

Remark 7.3.11. Following Remark 7.3.1, for $k \in \mathbb{N}$ the operators p_k and h_k can be described in terms of multiplication operators conjugated with the Fourier unitary $\mathcal{F}_2$. Indeed, $\mathcal{F}_2 \lambda_t \mathcal{F}_2^*$ is the multiplication operator on $L^2(\mathbb{R}, \mathcal{H})$ with the function $(s \mapsto e^{its})$ and therefore (under proper identification of the domains) $\mathcal{F}_2 h \mathcal{F}_2^*$ coincides with the multiplication operator with $(s \mapsto e^s)$. It follows that for all $k \in \mathbb{N}$, $\mathcal{F}_2 p_k \mathcal{F}_2^*$ is the multiplication with $(I_k : s \mapsto \chi_{[-\log(k), \log(k)]}(s))$ and $\mathcal{F}_2 h_k \mathcal{F}_2^*$ is the multiplication with $(J_k : s \mapsto \chi_{[-\log(k), \log(k)]}(s) e^s)$. Therefore, by Remark 7.3.1,

$$p_k = \lambda(\widehat{I}_k), \qquad h_k = \lambda(\widehat{J}_k), \qquad \text{and} \qquad h_k^{-1} = \lambda(\widehat{J_k^{-1}}),$$

where J_k^{-1} is the function $(s \mapsto \chi_{[-\log(k), \log(k)]} e^{-s})$. We also have that

$$\lambda(f_j) h_k = \lambda(f_j) \lambda(\widehat{J}_k) = \lambda(f_j * \widehat{J}_k) = \mathcal{F}_2^* \widehat{f}_j J_k \mathcal{F}_2, \tag{7.3.8}$$

where we view the product $\widehat{f}_j J_k$ as a multiplication operator. Since the Fourier transform of f_j is Gaussian we see that $\mathcal{F}_2^* \widehat{f}_j J_k \mathcal{F}_2$ is positive and boundedly invertible in the corner algebra $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}) p_k$. Further, by (7.3.3) and the Plancherel identity,

$$T_\theta(h_k^{-1}) = T_\theta(\lambda(\widehat{J_k^{-1/2}}) \lambda(\widehat{J_k^{-1/2}})) = \| \widehat{J_k^{-1/2}} \|_2^2 = \| J_k^{-1/2} \|_2^2 = k - k^{-1}.$$

It follows that

$$\tau_\rtimes(p_k) = \varphi(T_\theta(h^{-1/2} p_k h^{-1/2})) = \varphi(T_\theta(h_k^{-1})) = k - k^{-1}.$$

In particular, $\tau_\rtimes(p_k) < \infty$. Since $\tau_\rtimes$ is tracial we also have for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$,

$$\tau_\rtimes(p_k x p_k) = \varphi \circ T_\theta(h_k^{-1} p_k x p_k). \tag{7.3.9}$$

In the next statements it is notationally more convenient to work with property (rHAP) (resp. property (rHAP)$^-$) for general faithful normal positive functionals instead of just states, see Remark 7.2.3. Note that $p_k \widehat{\varphi}_j p_k$, $j \in \mathbb{N}$ is not a state, but a positive scalar multiple of a state.

We shall use the fact that the unique faithful normal $\widehat{\varphi}_j$-preserving conditional expectation $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}$ of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ onto $\mathcal{N}$ is given by $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j} = \mathbb{E}_{\mathcal{N}} \circ T_{f_j}$. This fact was used in the proof of Proposition 7.3.4 already.

Lemma 7.3.12. *For every $k \in \mathbb{N}$, $j \in \mathbb{N}$ there is a faithful normal $p_k\widehat{\varphi}_j p_k$-preserving conditional expectation of $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ onto $p_k\mathcal{N}p_k$ given by*

$$x \mapsto \mu_k^{-1} p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(x))p_k = \mu_k^{-1} p_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x)p_k \tag{7.3.10}$$

where $\mu_k := T_{f_j}(p_k) = \|\widehat{f}_j\chi_{[-\log(k),\log(k)]}\|_2^2$. In particular, $T_{f_j}(p_k)$ is a scalar multiple of the identity.

Proof. First note that by Remark 7.3.1 and Remark 7.3.11 the operator $p_k\lambda(f_j)$ coincides with $\lambda(g_{j,k})$ where $g_{j,k}$ is the inverse Fourier transform of the function $\widehat{f}_j\chi_{[-\log(k),\log(k)]}$. The equality (7.3.3) then implies that

$$T_{f_j}(p_k) = T_\theta(\lambda(f_j)^* p_k \lambda(f_j)) = T_\theta(\lambda(g_{j,k})^*\lambda(g_{j,k})) = \|g_{j,k}\|_2^2 = \mu_k \tag{7.3.11}$$

is a multiple of the identity.

For $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ expand

$$\begin{aligned}(p_k\widehat{\varphi}_j p_k)(p_k\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x)p_k) &= \widehat{\varphi}_j(p_k\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x)p_k)\\ &= (\varphi\circ T_{f_j})(p_k\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))p_k)\\ &= (\varphi\circ T_\theta)(\lambda(f_j)^* p_k\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))p_k\lambda(f_j)).\end{aligned}$$

Since $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ we see that

$$\begin{aligned}(p_k\widehat{\varphi}_j p_k)(p_k\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x)p_k) &= (\varphi\circ T_\theta)(\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))\lambda(f_j)^* p_k\lambda(f_j))\\ &= \varphi(\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))T_{f_j}(p_k)).\end{aligned}$$

With (7.3.11) we can continue as follows:

$$(p_k\widehat{\varphi}_j p_k)(p_k\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(x)p_k) = \mu_k\varphi(\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))) = \mu_k\varphi(T_{f_j}(x)) = \mu_k\widehat{\varphi}_j(x) = \mu_k\widehat{\varphi}_j(p_kxp_k).$$

This proves that (7.3.10) is $p_k\widehat{\varphi}_j p_k$-preserving, as claimed. For $x \in \mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ we have that x and p_k commute. Therefore, using the $\mathcal{N}$-module property of the maps involved,

$$p_k\mathbb{E}_{\mathcal{N}}(T_{f_j}(p_kxp_k))p_k = p_kxp_k\mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k)) = \mu_k p_kxp_k.$$

This shows that the map $x \mapsto \mu_k^{-1}p_k\mathbb{E}_{\mathcal{N}}(T_{f_j}(x))p_k$ is a unital (the unit being p_k) normal completely positive projection onto $p_k\mathcal{N}p_k$ (see [33, Theorem 1.5.10]). □

Lemma 7.3.13. *Let $\mathcal{N} \subseteq M$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau\circ\mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then we have $\mathbb{E}_{\mathcal{N}}(T_{f_j}(xa)) = \mathbb{E}_{\mathcal{N}}(T_{f_j}(ax))$ and $\mathbb{E}_{\mathcal{N}}(T_\theta(xa)) = \mathbb{E}_{\mathcal{N}}(T_\theta(ax))$ for every $j \in \mathbb{N}$, $a \in \mathcal{L}(\mathbb{R})$ and $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$.*

Proof. We first prove that $\mathbb{E}_{\mathcal{N}}(T_{f_j}(xa)) = \mathbb{E}_{\mathcal{N}}(T_{f_j}(ax))$. Suppose $a = \lambda(k)$ and $x = y\lambda(g)$ for $y \in M$, $k \in L^1(\mathbb{R})$ and $g \in C_c(\mathbb{R})$. Let us first compute $T_{f_j}(xa)$ and $T_{f_j}(ax)$. By the formula (7.3.3) we have

$$T_{f_j}(xa) = \int_{\mathbb{R}} f_j^*(-t)(g*k*f_j)(t)\sigma_{-t}^\varphi(y)dt.$$

By a similar computation we get

$$T_{f_j}(ax) = \int_{\mathbb{R}} (f_j^* * k)(-t)(g * f_j)(t)\sigma_{-t}^{\varphi}(y)dt.$$

We now apply $\mathbb{E}_{\mathscr{N}}$ to these expressions and use the fact that $\mathscr{N}$ is contained in the centralizer of φ, so $\mathbb{E}_{\mathscr{N}}(\sigma_{-t}^{\varphi}(y)) = \mathbb{E}_{\mathscr{N}}(y)$. It therefore suffices to prove the equality of the integrals $\int_{\mathbb{R}} f_j^*(-t)(g * k * f_j)(t)dt$ and $\int_{\mathbb{R}} (f_j^* * k)(-t)(g * f_j)(t)dt$. Using the commutativity of the convolution on $\mathbb{R}$, we can rewrite the first one as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_j^*(-t)(g * f_j)(t-s)k(s)dsdt$$

and the second one is equal to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_j^*(-t-s)k(s)(g * f_j(t))dsdt.$$

In the second integral we can introduce a new variable $t' := t+s$ and it transforms into

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_j^*(-t')(g * f_j)(t'-s)k(s)dsdt',$$

which is equal to the first one. For arbitrary $a \in \mathscr{L}(\mathbb{R})$ and $x \in \mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R}$ we can find bounded nets $(a_i)_{i\in I}$ and $(x_i)_{i\in I}$ formed by linear combinations of elements of the form discussed above that converge strongly to a and x, respectively, as a consequence of Kaplansky's density theorem. As multiplication is strongly continuous on bounded subsets, we have strong limits $\lim_{i\in I} a_i x_i = ax$ and $\lim_{i\in I} x_i a_i$. As both $\mathbb{E}_{\mathscr{N}}$ and T_{f_j} are strongly continuous on bounded subsets, we may conclude.

The equality $\mathbb{E}_{\mathscr{N}}(T_\theta(xa)) = \mathbb{E}_{\mathscr{N}}(T_\theta(ax))$ follows by a similar computation. □

The ideas appearing in the proof of the next statements are of a similar type.

Proposition 7.3.14. *Let $\mathscr{N} \subseteq \mathscr{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathscr{N}}$. Assume that $\mathscr{N}$ is finite and let $\tau \in \mathscr{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathscr{N}}$ on $\mathscr{M}$. Then, for every $j \in \mathbb{N}$, the following statements hold:*

(1) The triple $(\mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R}, \mathscr{N}, \widehat{\varphi}_j)$ satisfies property (rHAP) if and only if $(p_k(\mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP) for every $k \in \mathbb{N}$. Moreover, the (rHAP) may be witnessed by contractive maps, i.e. we may assume that (1′) holds.

(2) If for every $k \in \mathbb{N}$ the property (rHAP) of the triple $(p_k(\mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\widehat{\varphi}_j p_k)$ is witnessed by unital $p_k\widehat{\varphi}_j p_k$-preserving approximating maps, then the relative Haagerup property of $(\mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R}, \mathscr{N}, \widehat{\varphi}_j)$ is witnessed by unital $\widehat{\varphi}_j$-preserving maps.

(3) If the triple $(\mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R}, \mathscr{N}, \widehat{\varphi}_j)$ satisfies property (rHAP)$^-$ then the triple $(p_k(\mathscr{M} \rtimes_{\sigma^{\varphi}} \mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP)$^-$ for every $k \in \mathbb{N}$.

Proof. *First part of (1)*: For the "⇒" direction assume that $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ satisfies property (rHAP), that it is witnessed by a net of maps $(\Phi_i)_{i\in I}$ and fix $k \in \mathbb{N}$. We will show that $(p_k\Phi_i(\cdot)p_k)_{i\in I}$ is a net of approximating maps witnessing the relative Haagerup property of $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$.

It is clear that for every $i \in I$ the map $p_k\Phi_i(\cdot)p_k$ is completely positive, that the net $(p_k\Phi_i(\cdot)p_k)_{i\in I}$ admits a uniform bound on its norms and that $p_k\Phi_i(\cdot)p_k \to \mathrm{id}$ in the point-strong topology in i as maps on $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$.

By our assumptions, $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ and hence p_k and $\mathcal{N}$ commute. Hence for $a, b \in \mathcal{N}$, $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have

$$\begin{aligned} p_k\Phi_i(p_k a p_k x p_k b p_k)p_k &= p_k\Phi_i(a p_k x p_k b)p_k \\ &= p_k a\Phi_i(p_k x p_k)b p_k \\ &= p_k a p_k \Phi_i(p_k x p_k)p_k b p_k, \end{aligned}$$

which shows that $p_k\Phi_i(\cdot)p_k$ is a $p_k\mathcal{N}p_k$-$p_k\mathcal{N}p_k$-bimodule map for every $i \in I$.

We have by [172, Theorem VIII.3.19.(vi)], [172, Theorem X.1.17.(ii)] and the fact that p_k and $\lambda(f_j)$ commute that

$$\sigma_t^{\widehat{\varphi}_j}(p_k) = \lambda(f_j)^{it}\sigma_t^{\widehat{\varphi}}(p_k)\lambda(f_j)^{-it} = \lambda(f_j)^{it}p_k\lambda(f_j)^{-it} = p_k.$$

Therefore by [41, Lemma 2.3], for $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ positive,

$$(p_k\widehat{\varphi}_j p_k)(p_k\Phi_i(x)p_k) = \widehat{\varphi}_j(p_k\Phi_i(x)p_k) \leq \widehat{\varphi}_j(\Phi_i(x)) \leq \widehat{\varphi}_j(x) = (p_k\widehat{\varphi}_j p_k)(x),$$

i.e. $(p_k\widehat{\varphi}_j p_k) \circ (p_k\Phi_i(\cdot)p_k) \leq p_k\widehat{\varphi}_j p_k$.

Now, for every map Φ on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ of the form $\Phi = a\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(\cdot)b$ with $a, b \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ and $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ we have, using Lemma 7.3.13 (recalling that $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j} = \mathbb{E}_{\mathcal{N}} \circ T_{f_j}$) and the fact that p_k commutes with $\mathcal{N}$, that

$$p_k\Phi(x)p_k = p_k a\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b p_k x p_k)p_k = p_k a\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k b p_k x)p_k = (p_k a p_k)\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k b p_k x).$$

Lemma 7.3.12 then implies that $(p_k\Phi(\cdot)p_k)^{(2)} \in \mathcal{K}_{00}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$. By taking linear combinations and approximating we see that if Φ is a map on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ with $\Phi^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ then

$$(p_k\Phi(\cdot)p_k)^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k).$$

Therefore for the approximating maps Φ_i, $i \in I$ we conclude that

$$(p_k\Phi_i(\cdot)p_k)^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k).$$

This shows that $(p_k\Phi_i(\cdot)p_k)_{i\in I}$ indeed witnesses the relative Haagerup property of the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$.

(3): Note that if $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP)$^-$ witnessed by the net $(\Phi_i)_{i\in I}$, then property (rHAP)$^-$ of $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ follows in a

very similar way as above. The only condition that remains to be checked is that the L^2-implementation $(p_k\Phi_i(\cdot)p_k)^{(2)}$ exists. For this, assume that there exists $C>0$ with $\widehat{\varphi}_j(\Phi_i(x)^*\Phi_i(x)) \le C\widehat{\varphi}_j(x^*x)$ for all $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. Then, using again [41, Lemma2.3] for the second inequality,

$$\begin{aligned}(p_k\widehat{\varphi}_j p_k)((p_k\Phi_i(x)p_k)^*(p_k\Phi_i(x)p_k)) &= \widehat{\varphi}_j(p_k\Phi_i(x^*)p_k\Phi_i(x)p_k)\\ &\le \widehat{\varphi}_j(p_k\Phi_i(x)^*\Phi_i(x)p_k)\\ &\le \widehat{\varphi}_j(\Phi_i(x)^*\Phi_i(x))\\ &\le C\widehat{\varphi}_j(x^*x)\\ &= C(p_k\widehat{\varphi}_j p_k)(x^*x)\end{aligned}$$

for all $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. The claim follows.

Second part of (1): For the "$\Leftarrow$" direction assume that for every $k \in \mathbb{N}$ the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ satisfies property (rHAP) witnessed by approximating maps $(\Phi_{k,i})_{i\in I_k}$. We wish to apply Lemma 7.2.14 for which we check the conditions. By $\mathcal{N} \subseteq \mathcal{M}^{\sigma^\varphi}$ we have that $\mathcal{N}$ and λ_t commute for every $t \in \mathbb{R}$ and hence so do N and h_k. In particular, $h_k \in (p_k\mathcal{N}p_k)' \cap p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$. By (7.3.8) and the remarks after it, it follows that $\lambda(f_j)h_k$ is positive and boundedly invertible. Now, from (7.3.7) we see that the conditions of Lemma 7.2.14 are fulfilled and this lemma shows that the maps of the net $(\Phi_{k,i})_{i\in I_k}$ can be chosen contractively, i.e. we may assume that (1′) holds.

We shall prove that $(\Phi_{k,i}(p_k \cdot p_k))_{k\in\mathbb{N}, i\in I_k}$ induces a net witnessing property (rHAP) of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. This in particular shows that we may assume (1′).

By the contractivity of the $\Phi_{k,i}$ it is clear that the maps $\Phi_{k,i}(p_k \cdot p_k)$ are completely positive with a uniform bound on their norms.

Since $\mathcal{N}$ and p_k commute we see that for $a,b \in \mathcal{N}$ and $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$

$$\begin{aligned}\Phi_{k,i}(p_k axb p_k) &= \Phi_{k,i}(p_k a p_k x p_k b p_k)\\ &= p_k a p_k \Phi_{k,i}(p_k x p_k)p_k b p_k\\ &= a p_k \Phi_{k,i}(p_k x p_k)p_k b\\ &= a\Phi_{k,i}(p_k x p_k)b.\end{aligned}$$

Therefore $\Phi_{k,i}(p_k \cdot p_k)$ is an $\mathcal{N}$-$\mathcal{N}$-bimodule map for every $k \in \mathbb{N}$, $i \in I_k$.

We have, using again [41, Lemma 2.3], that for $x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})^+$

$$\begin{aligned}\widehat{\varphi}_j(\Phi_{k,i}(p_k x p_k)) &= \widehat{\varphi}_j(p_k\Phi_{k,i}(p_k x p_k)p_k)\\ &= (p_k\widehat{\varphi}_j p_k)(\Phi_{k,i}(p_k x p_k))\\ &\le (p_k\widehat{\varphi}_j p_k)(p_k x p_k)\\ &= \widehat{\varphi}_j(p_k x p_k) \le \widehat{\varphi}_j(x).\end{aligned}$$

i.e. $\widehat{\varphi}_j \circ \Phi_{k,i}(p_k \cdot p_k) \le \widehat{\varphi}_j$.

We claim that $(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ for all $k \in \mathbb{N}$, $i \in I_k$. Indeed, take an arbitrary map Φ of the form $\Phi(x) = p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k b p_k x))$ for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ where $a, b \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. The L^2-implementations of such operators span $\mathcal{K}_{00}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ by Lemma 7.3.12. Lemma 7.3.13 and the fact that p_k and $\mathcal{N}$ commute show that for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$,

$$\Phi(p_k x p_k) = p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k b p_k x p_k)) = p_k a p_k \mathbb{E}_{\mathcal{N}}(T_{f_j}(p_k b p_k x)).$$

Then, since $\mathbb{E}_{\mathcal{N}} \circ T_{f_j}$ is the faithful normal $\widehat{\varphi}_j$-preserving conditional expectation of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ onto $\mathcal{N}$, this implies that $(\Phi(p_k \cdot p_k))^{(2)} \in \mathcal{K}_{00}(\mathcal{M} \rtimes^{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. By taking linear combinations and approximation we see that if $\Phi^{(2)} \in \mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$, then $(\Phi(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes^{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. We conclude that

$$(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes^{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j).$$

Now, for $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ we see that

$$\lim_{k \to \infty} \lim_{i \in I_k} \Phi_{k,i}(p_k x p_k) = x,$$

in the strong topology. Then a variant of Lemma 7.1.3 shows that there is a directed set $\mathcal{F}$ and a function $(\widetilde{k}, \widetilde{i}) : \mathcal{F} \to \{(k,i) \mid k \in \mathbb{N}, i \in I_k\}$, $F \mapsto (\widetilde{k}(F), \widetilde{i}(F))$ such $(\Phi_{\widetilde{k}(F),\widetilde{i}(F)})_{F \in \mathcal{F}}$ witnesses the relative Haagerup property of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$.

(2): It only remains to show that if for every $k \in \mathbb{N}$ the property (rHAP) of the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ is witnessed by unital $p_k\widehat{\varphi}_j p_k$-preserving approximating maps, then the relative Haagerup property of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ is witnessed by unital $\widehat{\varphi}_j$-preserving maps. For this, assume that the maps $(\Phi_{k,i})_{i \in I}$ from before are unital and $p_k\widehat{\varphi}_j p_k$-preserving and choose a sequence $(\epsilon_k)_{k \in \mathbb{N}} \subseteq (0,1)$ with $\epsilon_k \to 0$. Recall that $p_k \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$ and note that $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\epsilon_k)p_k) \geq \epsilon_k$. We then have $\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\epsilon_k)p_k) \in \mathcal{N} \cap \mathcal{N}'$, the inverse $(\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\epsilon_k)p_k))^{-1} \in \mathcal{N} \cap \mathcal{N}'$ exists and $a_k := (1 - (1-\epsilon_k)p_k)(\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(1 - (1-\epsilon_k)p_k))^{-1} \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$ is positive. Set $b_k := 1 - (1-\epsilon_k)p_k \geq 0$. Define the maps

$$\widetilde{\Phi}_{k,i}(\cdot) := (1-\epsilon_k)\Phi_{k,i}(p_k \cdot p_k) + a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} \cdot b_k^{1/2}).$$

Obviously $\widetilde{\Phi}_{k,i}$ is normal, completely positive and $\mathcal{N}$-$\mathcal{N}$-bimodular. We may finish the proof as in Theorem 7.2.13 now; since the statement of that theorem is not directly applicable here we will give the complete proof for the convenience of the reader.

We have

$$\widetilde{\Phi}_{k,i}(1) = (1-\epsilon_k)\Phi_{k,i}(p_k) + a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k) = (1-\epsilon_k)p_k + (1 - (1-\epsilon_k)p_k) = 1.$$

Now, since $\Phi_{k,i}$ is $p_k\widehat{\varphi}_j p_k$-preserving we have that $\widehat{\varphi}_j \circ \Phi_{k,i}(p_k x p_k) = \widehat{\varphi_j}(p_k x p_k)$ for all $x \in \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$, and hence with Lemma 7.3.13 we deduce that

$$\widehat{\varphi}_j \circ \widetilde{\Phi}_{k,i}(x) \quad = \quad (1-\epsilon_k)\widehat{\varphi}_j(\Phi_{k,i}(p_k x p_k)) + \widehat{\varphi}_j(a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}))$$

$$\begin{aligned} &= (1-\epsilon_k)\widehat{\varphi}_j(p_k x p_k) + \widehat{\varphi}_j(\mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(a_k) b_k^{1/2} x b_k^{1/2}) \\ &= (1-\epsilon_k)\widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k x p_k) + \widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}) \\ &= (1-\epsilon_k)\widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(p_k x) + \widehat{\varphi}_j \circ \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k x) \\ &= (1-\epsilon_k)\widehat{\varphi}_j(p_k x) + \widehat{\varphi}_j(b_k x) \\ &= \widehat{\varphi}_j(x). \end{aligned}$$

By the fact that $(\Phi_{k,i}(p_k \cdot p_k))^{(2)} \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ and by Lemma 7.3.13, we have

$$\widetilde{\Phi}_{k,i}^{(2)} = (1-\epsilon_k)(\Phi_{k,i}(p_k \cdot p_k))^{(2)} + a_k e_{\mathcal{N}}^{\widehat{\varphi}_j} b_k \in \mathcal{K}(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j).$$

Further, for every $x \in (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})_+$,

$$\begin{aligned} \widetilde{\Phi}_{k,i}(x) - (1-\epsilon_k)\Phi_{k,i}(p_k x p_k) &= a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k^{1/2} x b_k^{1/2}) \\ &\leq \|x\|\, a_k \mathbb{E}_{\mathcal{N}}^{\widehat{\varphi}_j}(b_k) \\ &= \|x\|\,(1-(1-\epsilon_k)p_k), \end{aligned}$$

from which we deduce that $\lim_{F\in\mathcal{F}} \widetilde{\Phi}_{\widetilde{k}(F),\widetilde{i}(F)} = \mathrm{id}_{M\rtimes_{\sigma^\varphi}\mathbb{R}}$. This implies that the net $(\widetilde{\Phi}_{\widetilde{k}(F),\widetilde{i}(F)})_{F\in\mathcal{F}}$ of unital $\widehat{\varphi}_j$-preserving maps witnesses the relative Haagerup property of $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$. □

We are now ready to formulate the key statement of this subsection. Note that for every $k \in \mathbb{N}$ the von Neumann algebra $p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ is finite with a faithful normal tracial state $p_k \tau_\rtimes p_k$.

Proposition 7.3.15. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then the following are equivalent:*

1. *The triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP);*
2. *$(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) for every $j \in \mathbb{N}$;*
3. *$(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ has property (rHAP) for every $k \in \mathbb{N}$.*

Further, the following statement holds:

4. *If the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP)$^-$, then $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ has property (rHAP)$^-$ for every $k \in \mathbb{N}$.*

Proof. The equivalence "(1) ⇔ (2)" was proved in Theorem 7.3.9.

"(2) ⇒ (3)": Assume that for $j \in \mathbb{N}$ the triple $(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathcal{N}, \widehat{\varphi}_j)$ has property (rHAP) and fix $k \in \mathbb{N}$. Then by Proposition 7.3.14, the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ also has the (rHAP). Let $(\Phi_i)_{i\in I}$ be a net of suitable approximating maps and define the self-adjoint boundedly invertible operator $A_{j,k} := \lambda(f_j)h_k^{1/2} \in (p_k\mathcal{N}p_k)' \cap (p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k)$. By (7.3.7) for every $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ the equality

$$(p_k\widehat{\varphi}_j p_k)(x) = \tau_\rtimes(A_{j,k}^* x A_{j,k}) = (A_{j,k} p_k \tau_\rtimes p_k A_{j,k})(x)$$

holds and hence Lemma 7.3.10 implies that the L^2-implementation of the map $\Phi_i'(\cdot) := A_{j,k}\Phi_i(A_{j,k}^{-1} \cdot A_{j,k}^{-1})A_{j,k}$ exists and is contained in $\mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$. Similarly to the proof of Proposition 7.3.14 one checks that the net $(\Phi_i')_{i\in I}$ witnesses property (rHAP) of $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$. We omit the details.

"(2) $\Leftarrow$ (3)" Now assume that the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ has property (rHAP) for every $k \in \mathbb{N}$. It suffices to show that the triple $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ has property (rHAP) as it implies the desired statement by Proposition 7.3.14. So let $(\Phi_i)_{i\in I}$ be a net that witnesses property (rHAP) of $(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_\rtimes p_k)$ and set $\Phi_i' := A_{j,k}^{-1}\Phi_i(A_{j,k} \cdot A_{j,k})A_{j,k}^{-1}$. Lemma 7.3.10 and (7.3.7) imply that for every $i \in I$ the L^2-implementation $(\Phi_i')^{(2)}$ of Φ_i' with respect to the positive functional $p_k\widehat{\varphi}_j p_k$ is contained in $\mathcal{K}(p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$. Again, similarly to the proof of Proposition 7.3.14 one checks that the net $(\Phi_i')_{i\in I}$ witnesses property (rHAP).

It remains to show (4). The statement easily follows from Proposition 7.3.9, Proposition 7.3.14 and the arguments used in the proof of the implication "(2) $\Rightarrow$ (3)". □

7.4. MAIN RESULTS

7

After the main work has been done in Section 7.3 we can now put the pieces together. This allows us to show that in the case of a finite von Neumann subalgebra the notion of relative Haagerup property is independent of the choice of the corresponding faithful normal conditional expectation, that the approximating maps may be chosen to be unital and state-preserving and that property (rHAP) and property (rHAP)$^-$ are equivalent. The general notation will be the same as in Section 7.3.

7.4.1. INDEPENDENCE OF THE CONDITIONAL EXPECTATION

Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras for which $\mathcal{N}$ is finite with a faithful normal tracial state $\tau \in \mathcal{N}_*$. Let further $\mathbb{E}_\mathcal{N}, \mathbb{F}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ be two faithful normal conditional expectations and extend τ to states $\varphi := \tau \circ \mathbb{E}_\mathcal{N}$ and $\psi := \tau \circ \mathbb{F}_\mathcal{N}$ on $\mathcal{M}$. In this subsection we will prove that the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_\mathcal{N})$ has property (rHAP) if and only if the triple $(\mathcal{M}, \mathcal{N}, \mathbb{F}_\mathcal{N})$ does, i.e. the relative Haagerup property is an intrinsic invariant of the inclusion $\mathcal{N} \subseteq \mathcal{M}$. This extends results by Jolissaint, see [116]. Let us first introduce some notation.

As in Section 7.3 consider the crossed product von Neumann algebra $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ which contains the projections $p_k \in \mathcal{N}' \cap (\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})$, $k \in \mathbb{N}$ and carries the canonical normal semi-finite tracial weight $\tau_\rtimes$ which we will from now on denote by $\tau_{\rtimes,1}$. For $t \in \mathbb{R}$ write λ_t^φ for the left regular representation operators in $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. Similarly, we write $\tau_{\rtimes,2}$ for the canonical normal semi-finite tracial weight on $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ and denote the corresponding left regular representation operators by λ_t^ψ, $t \in \mathbb{R}$.

For $t \in \mathbb{R}$ let $u_t := (D\varphi/D\psi)_t \in \mathcal{M}$ be the Connes cocycle Radon-Nikodym derivative, so in particular $u_t\sigma_t^\varphi(u_s) = u_{t+s}$ and $\sigma_t^\psi(x) = u_t^*\sigma_t^\varphi(x)u_t$ hold for all $s, t \in \mathbb{R}$. Then (see [172, Proof of Theorem X.1.7]) there exists an isomorphism $\rho : \mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R} \to \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ of von Neumann algebras which restricts to the identity on $\mathcal{M}$ and for which $\rho(\lambda_t^\psi) = u_t\lambda_t^\varphi$ for all $t \in \mathbb{R}$. This implies that the dual actions θ^φ and θ^ψ of σ^φ and σ^ψ respectively are related by the equality $\theta_t^\varphi \circ \rho = \rho \circ \theta_t^\psi$, $t \in \mathbb{R}$. Further, $\tau_{\rtimes,1} \circ \rho = \tau_{\rtimes,2}$ (see the footnote [2]). Denote by h_ψ the unique unbounded self-adjoint positive operator affiliated with $\mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R}$ such that $h_\psi^{it} = \lambda_t^\psi$ for all $t \in \mathbb{R}$ and set

$$p_{\psi,k} := \chi_{[k^{-1},k]}(h_\psi) \qquad \text{and} \qquad q_k := \rho(p_{\psi,k}).$$

for $k \in \mathbb{N}$. Further, define

$$h_{\psi,k} := \rho(\chi_{[k^{-1},k]}(h_\psi)h_\psi) = \rho(p_{\psi,k}h_\psi).$$

Recall that for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we write $h^{it} = \lambda_t^\varphi$, $p_k := \chi_{[k^{-1},k]}(h)$, and $h_k := p_k h$.

The following statement compares to Lemma 7.3.12.

Lemma 7.4.1. *For every $k \in \mathbb{N}$ there is a (unique) faithful normal $p_k\tau_{\rtimes,1}p_k$-preserving conditional expectation $\mathbb{E}_{1,k} : p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k \to p_k\mathcal{N}p_k$ given by*

$$x \mapsto \nu_k^{-1} p_k \mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k,$$

where $\nu_k := T_{\theta^\varphi}(h_k^{-1}) = k - k^{-1}$. In particular, $T_{\theta^\varphi}(h_k^{-1})$ is a scalar multiple of the identity.

Proof. The proof is essentially the same as that of Lemma 7.3.12. First note that by Remark 7.3.11 the operator h_k coincides with $\lambda(\widehat{J_k})$ where $J_k(s) = \chi_{[-\log(k),\log(k)]}e^s$ and that $\nu_k = T_{\theta^\varphi}(h_k^{-1}) = k - k^{-1}$ is a multiple of the identity. For $x \in p_k(\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R})p_k$ one checks using (7.3.9) for the second and last equality, that

$$\begin{aligned}
(p_k\tau_{\rtimes,1}p_k)(p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k) &= \tau_{\rtimes,1}(p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k) \\
&= \varphi \circ T_{\theta^\varphi}(p_k h_k^{-1}\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))p_k) \\
&= \varphi \circ T_{\theta^\varphi}(h_k^{-1}\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))) \\
&= \varphi\left(T_{\theta^\varphi}(h_k^{-1})\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))\right) \\
&= \nu_k\varphi\left(\mathbb{E}_{\mathcal{N}}(T_{\theta^\varphi}(h_k^{-1}x))\right) \\
&= \nu_k\varphi \circ T_{\theta^\varphi}(h_k^{-1}x)) \\
&= \nu_k\tau_{\rtimes,1}(p_k x p_k),
\end{aligned}$$

[2] This is well-known to specialists, but it seems that the statement does not appear explicitly in [172]. The argument goes as follows. Firstly, as ρ intertwines the dual actions on $M \rtimes_{\sigma^\psi} \mathbb{R}$ and $M \rtimes_{\sigma^\varphi} \mathbb{R}$ we find that $\widehat{\varphi} \circ \rho$ is the dual weight of φ in the crossed product $M \rtimes_{\sigma^\psi} \mathbb{R}$. Let $t \in \mathbb{R}$. By [172, Theorem X.1.17] we have Connes cocycle derivative $\left(\frac{D\widehat{\psi}}{D\widehat{\varphi}\circ\rho}\right)_t = u_t = \rho(u_t)$. Then by the chain rule [172, Theorem VIII.3.7],

$$\left(\frac{D\tau_{\rtimes,2}}{D\tau_{\rtimes,1}\circ\rho}\right)_t = \left(\frac{D\tau_{\rtimes,2}}{D\widehat{\psi}}\right)_t \left(\frac{D\widehat{\psi}}{D\widehat{\varphi}\circ\rho}\right)_t \left(\frac{D\widehat{\varphi}\circ\rho}{D\tau_{\rtimes,1}\circ\rho}\right)_t = \lambda_{-t}^\psi\rho^{-1}(u_t\lambda_t^\varphi) = 1.$$

Hence $\tau_{\rtimes,1} \circ \rho = \tau_{\rtimes,2}$.

hence $\mathbb{E}_{1,k}$ is indeed $p_k\tau_{\rtimes,1}p_k$-preserving. Here we used in the fourth line that $\mathcal{N}$ is invariant under the dual action θ^{φ} and in the fifth line that $T_{\theta^{\varphi}}(h_k^{-1})$ is a multiple of the identity.

From Lemma 7.3.13 we see that

$$\nu_k^{-1}p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^{\varphi}}(h_k^{-1}\cdot))p_k = \nu_k^{-1}p_k\mathbb{E}_{\mathcal{N}}(T_{\theta^{\varphi}}(h_k^{-1/2}\cdot h_k^{-1/2}))p_k,$$

and from the right-hand side of this expression it is clear that the map is completely positive. The remaining statements (i.e. that $\mathbb{E}_{1,k}$ is a unital faithful normal $p_k\mathcal{N}p_k$-$p_k\mathcal{N}p_k$-bimodule map) are then easy to check. □

The following lemma provides the analogous statement for the functional $q_k\tau_{\rtimes,1}q_k$ and the inclusion $q_k\mathcal{N}q_k \subseteq q_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})q_k$. We omit the proof.

Lemma 7.4.2. *For every $k\in\mathbb{N}$ there is a (unique) faithful normal $q_k\tau_{\rtimes,1}q_k$-preserving conditional expectation $\mathbb{E}_{2,k}: q_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})q_k \to q_k\mathcal{N}q_k$ given by*

$$x\mapsto \nu_k^{-1}q_k\mathbb{E}_{\mathcal{N}}(T_{\theta^{\varphi}}(h_{\psi,k}^{-1}x))q_k,$$

where $\nu_k := k-k^{-1}$ as before.

Proposition 7.4.3. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras for which $\mathcal{N}$ is finite with a faithful normal tracial state $\tau\in\mathcal{N}_*$. Let further $\mathbb{E}_{\mathcal{N}},\mathbb{F}_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ be two faithful normal conditional expectations and extend τ to states $\varphi:=\tau\circ\mathbb{E}_{\mathcal{N}}$ and $\psi:=\tau\circ\mathbb{F}_{\mathcal{N}}$ on $\mathcal{M}$. Then the following statements are equivalent:*

(1) For every $k\in\mathbb{N}$ the triple $(p_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has property (rHAP).

(2) For every $k\in\mathbb{N}$ the triple $(q_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})q_k, q_k\mathcal{N}q_k, q_k\tau_{\rtimes,1}q_k)$ has property (rHAP).

Proof. By symmetry it suffices to consider the direction "(2) ⇒ (1)". For this, fix $k,l\in\mathbb{N}$ and let $(\Phi_{l,i})_{i\in I_l}$ be a net of maps witnessing the relative Haagerup property of the triple $(q_l(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})q_l, q_l\mathcal{N}q_l, q_l\tau_{\rtimes,1}q_l)$, which we can assume to be contractive by Lemma 7.2.14. Define for $i\in I_l$ the normal completely positive contractive map

$$\Phi'_{k,l,i}: p_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})p_k \to p_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})p_k, x\mapsto p_k\Phi_{l,i}(q_lxq_l)p_k.$$

As $\mathcal{N}\subseteq\mathcal{M}^{\sigma^{\varphi}}$ and $\mathcal{N}\subseteq\mathcal{M}^{\sigma^{\psi}}$, $\mathcal{N}$ commutes with both q_l and p_k. Thus we have that for $x\in p_k(\mathcal{M}\rtimes_{\sigma^{\varphi}}\mathbb{R})p_k$ and $a,b\in\mathcal{N}$

$$\begin{aligned}
\Phi'_{k,l,i}(p_kap_kxp_kbp_k) &= p_k\Phi_{l,i}(q_lp_kap_kxp_kbp_kq_l)p_k \\
&= p_k\Phi_{l,i}(q_laxbq_l)p_k \\
&= p_k\Phi_{l,i}(q_laq_lxq_lbq_l)p_k \\
&= p_kq_laq_l\Phi_{l,i}(q_lxq_l)q_lbq_lp_k \\
&= p_ka\Phi_{l,i}(q_lxq_l)bp_k \\
&= p_kap_k\Phi_{l,i}(q_lxq_l)p_kbp_k \\
&= p_kap_k\Phi'_{k,l,i}(x)p_kbp_k,
\end{aligned}$$

i.e. $\Phi'_{k,l,i}$ is $p_k\mathscr{N}p_k$-$p_k\mathscr{N}p_k$-bimodular. Further, $(p_k\tau_{\rtimes,1}p_k)\circ\Phi'_{k,l,i} \le p_k\tau_{\rtimes,1}p_k$ since for all positive $x \in p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ we have

$$\begin{aligned}(p_k\tau_{\rtimes,1}p_k)\circ\Phi'_{k,l,i}(x) &= \tau_{\rtimes_1}(p_k\Phi_{l,i}(q_lp_kxp_kq_l)p_k) \le \tau_{\rtimes,1}(\Phi_{l,i}(q_lp_kxp_kq_l))\\ &= \tau_{\rtimes,1}(q_l\Phi_{l,i}(q_lp_kxp_kq_l)q_l) \le \tau_{\rtimes,1}(q_lp_kxp_kq_l)\\ &\le (p_k\tau_{\rtimes,1}p_k)(x).\end{aligned}$$

For every map Φ on $q_l(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_l$ of the form $\Phi = a\mathbb{E}_{2,l}b$ with $a,b \in q_l(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_l$ and $x \in p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ we have by Lemma 7.4.2 that

$$\begin{aligned}p_k\Phi(q_lxq_l)p_k &= p_ka\mathbb{E}_{2,l}(bq_lxq_l)p_k\\ &= \nu_l^{-1}p_kaq_l\mathbb{E}_{\mathscr{N}}(T_{\theta^\varphi}(h_{\psi,l}^{-1}bq_lxq_l))q_lp_k\\ &= \nu_l^{-1}p_kap_k\mathbb{E}_{\mathscr{N}}(T_{\theta^\varphi}(h_{\psi,l}^{-1}bq_lxq_l)).\end{aligned}$$

Now we may use the isomorphism ρ and apply Lemma 7.3.13 to $\mathscr{M}\rtimes_{\sigma^\psi}\mathbb{R}$ to get

$$\begin{aligned}p_k\Phi(q_lxq_l)p_k &= \nu_l^{-1}p_kap_k\mathbb{E}_{\mathscr{N}}(T_{\theta^\psi}(p_{\psi,l}h_\psi^{-1}\rho^{-1}(b)p_{\psi,l}\rho^{-1}(x)p_{\psi,l}))\\ &= \nu_l^{-1}p_kap_k\mathbb{E}_{\mathscr{N}}(T_{\theta^\psi}(p_{\psi,l}h_\psi^{-1}\rho^{-1}(b)p_{\psi,l}\rho^{-1}(x))\\ &= \nu_l^{-1}p_kap_k\mathbb{E}_{\mathscr{N}}(T_{\theta^\varphi}(q_lh_{\psi,l}^{-1}bq_lx)).\end{aligned}$$

Then by Lemma 7.3.13 applied to $\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R}$ for the second equality and Lemma 7.4.1 for the last equality, we find

$$\begin{aligned}p_k\Phi(q_lxq_l)p_k &= \nu_l^{-1}p_kap_k\mathbb{E}_{\mathscr{N}}(T_{\theta^\varphi}(q_lh_{\psi,l}^{-1}bq_lxp_k))\\ &= \nu_l^{-1}p_kap_k\mathbb{E}_{\mathscr{N}}(T_{\theta^\varphi}(h_k^{-1}(h_kq_lh_{\psi,l}^{-1}bq_lx)))p_k\\ &= \nu_k\nu_l^{-1}p_ka\mathbb{E}_{1,k}((h_kq_lh_{\psi,l}^{-1}bq_l)x).\end{aligned}$$

Thus $(p_k\Phi(q_l\cdot q_l)p_k)^{(2)} \in \mathscr{K}_{00}(p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$. By taking linear combinations and approximation we see that if $\Phi^{(2)} \in \mathscr{K}(q_l(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_l, q_l\mathscr{N}q_l, q_l\tau_{\rtimes,1}q_l)$, then also $p_k\Phi(q_l\cdot q_l)p_k)^{(2)} \in \mathscr{K}(p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$. In particular, $(\Phi'_{k,l,i})^{(2)} \in \mathscr{K}(p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$ for $k,l \in \mathbb{N}$ and $i \in I_l$.

For every $x \in p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ we have that

$$\lim_{l\to\infty}\lim_{i\in I_l}\Phi'_{k,l,i}(x) = x$$

in the strong topology. A variant of Lemma 7.1.3 then shows that there is a directed set $\mathscr{F}$ and an increasing function $(\tilde{l},\tilde{i}) : \mathscr{F} \to \{(l,i) \mid k\in\mathbb{N}, i \in I_l\}$, $F \mapsto (\tilde{l}(F),\tilde{i}(F))$ such that $(\Phi'_{k,\tilde{l}(F),\tilde{i}(F)})_{F\in\mathscr{F}}$ witnesses the relative Haagerup property of $(p_k(\mathscr{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathscr{N}p_k, p_k\tau_{\rtimes,1}p_k)$. □

Theorem 7.4.4. *Let $\mathscr{N}\subseteq\mathscr{M}$ be a unital inclusion of von Neumann algebras with $\mathscr{N}$ finite. Let $\mathbb{E}_{\mathscr{N}}, \mathbb{F}_{\mathscr{N}} : \mathscr{M}\to\mathscr{N}$ be two faithful normal conditional expectations. Then the triple $(\mathscr{M},\mathscr{N},\mathbb{E}_{\mathscr{N}})$ has (rHAP) if and only if the triple $(\mathscr{M},\mathscr{N},\mathbb{F}_{\mathscr{N}})$ has (rHAP).*

Proof. Assume that the triple $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$ has the relative Haagerup property. Let τ be a faithful normal tracial state on $\mathcal{N}$ that we extend to a state $\varphi := \tau\circ\mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Theorem 7.3.15 implies that for every $k\in\mathbb{N}$, $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has the (rHAP). With Proposition 7.4.3 we get that for every $k\in\mathbb{N}$ the triple $(q_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_k, q_k\mathcal{N}q_k, q_k\tau_{\rtimes,1}q_k)$ has the (rHAP). The isomorphism ρ restricts to an isomorphism $q_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})q_k \cong p_{\psi,k}(\mathcal{M}\rtimes_{\sigma^\psi}\mathbb{R})p_{\psi,k}$ which maps $q_k\mathcal{N}q_k$ onto $p_{\psi,k}\mathcal{N}p_{\psi,k}$ and for which $(q_k\tau_{\rtimes,1}q_k)\circ\rho = p_{\psi,k}\tau_{\rtimes,2}p_{\psi,k}$. Combining this with Theorem 7.3.15 implies that $(\mathcal{M},\mathcal{N},\mathbb{F}_{\mathcal{N}})$ has the (rHAP). □

7.4.2. UNITALITY AND STATE-PRESERVATION OF THE APPROXIMATING MAPS

The following theorem states that for triples $(\mathcal{M},\mathcal{N},\varphi)$ with $\mathcal{N}$ finite the approximating maps may be assumed to be unital and state-preserving. The proof combines the passage to suitable crossed products and corners of crossed products from Section 7.3 with the case considered in Subsection 7.2.3.

Theorem 7.4.5. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite. Let $\tau\in\mathcal{N}_*$ be a faithful normal (possibly non-tracial) state that we extend to a state $\varphi := \tau\circ\mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$ and assume that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP). Then property (rHAP) may be witnessed by a net of unital and φ-preserving approximating maps, i.e. we may assume (1″) and (4′).*

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Proof. First assume that τ is tracial. Since the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP) we get with Theorem 7.3.9 and Proposition 7.3.14 that for all $j\in\mathbb{N}$, $k\in\mathbb{N}$ the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ has property (rHAP) as well and that it may be witnessed by a net of contractive approximating maps. As we have seen before, for every $k\in\mathbb{N}$ the element $h_k^{1/2}\lambda(f_j)\in p_k(M\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ is positive and boundedly invertible in $p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$. Further, by (7.3.7) and [172, Theorem VIII.2.11] the equality

$$\sigma_t^{p_k\widehat{\varphi}_j p_k}(x) = (h_k^{1/2}\lambda(f_j))^{it}x(h_k^{1/2}\lambda(f_j))^{-it}$$

holds for all $x\in p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$, $t\in\mathbb{R}$. Theorem 7.2.13 then implies that property (rHAP) of $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\widehat{\varphi}_j p_k)$ may for every $j,k\in\mathbb{N}$ be witnessed by a net of unital $(p_k\widehat{\varphi}_j p_k)$-preserving maps. By applying the converse directions of Proposition 7.3.14 and Theorem 7.3.9 we deduce the claimed statement.

Now we show that we may replace τ by any non-tracial faithful state in $\mathcal{N}_*$. Let still $\tau\in\mathcal{N}_*$ be a faithful tracial state. Let $(\Phi_i)_{i\in I}$ be approximating maps witnessing the (rHAP) for $(\mathcal{M},\mathcal{N},\tau\circ\mathbb{E}_{\mathcal{N}})$ which by the previous paragraph may be taken unital and $\tau\circ\mathbb{E}_{\mathcal{N}}$-preserving. The proof of Theorem 7.2.7, exploiting Lemmas 7.2.5 and 7.2.6 shows that $(\Phi_i)_{i\in I}$ also witness the (rHAP) for $(\mathcal{M},\mathcal{N},\varphi\circ\mathbb{E}_{\mathcal{N}})$ for any faithful state $\varphi\in\mathcal{N}_*$. Further Lemma 7.2.5 shows that Φ_i is $\varphi\circ\mathbb{E}_{\mathcal{N}}$-preserving and we are done. □

7.4.3. EQUIVALENCE OF (RHAP) AND (RHAP)$^-$

In [13] among other things Bannon and Fang prove that for triples $(\mathcal{M},\mathcal{N},\tau)$ of finite von Neumann algebras with a tracial state $\tau\in\mathcal{M}_*$ the subtraciality condition in Popa's notion of the relative Haagerup property is redundant. It is easy to check that their proof translates into our setting, which leads to the following variation of [13, Theorem 2.2].

Theorem 7.4.6 (Bannon-Fang)**.** *Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau\in\mathcal{M}_*$ and let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras. If the triple $(\mathcal{M},\mathcal{N},\tau)$ has property (rHAP)$^-$, then it has property (rHAP). Further, property (rHAP) may be witnessed by unital and trace-preserving approximating maps.*

In combination with Theorem 7.4.5 the following theorem provides a generalization of Theorem 7.4.6.

Theorem 7.4.7. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras which admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite. Let $\tau\in\mathcal{N}_*$ be a faithful normal state that we extend to a state $\varphi:=\tau\circ\mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP) if and only if it has property (rHAP)$^-$.*

Proof. By Theorem 7.2.7 we may without loss of generality assume that τ is tracial on $\mathcal{N}$. It is clear that property (rHAP) implies property (rHAP)$^-$. Conversely, if the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP)$^-$, then we deduce from Theorem 7.3.15 that for every $k\in\mathbb{N}$ the triple $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k, p_k\tau_{\rtimes,1}p_k)$ has property (rHAP)$^-$ as well. Recall that $p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k$ is finite since $p_k\tau_\rtimes p_k$ is a faithful normal tracial state. We can hence apply Theorem 7.4.6 to deduce that $(p_k(\mathcal{M}\rtimes_{\sigma^\varphi}\mathbb{R})p_k, p_k\mathcal{N}p_k,$ $p_k\tau_{\rtimes,1}p_k)$ has (rHAP) for every $k\in\mathbb{N}$. In combination with Theorem 7.3.15 this implies that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP). □

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We finish this subsection with an easy lemma which will be needed later on. It could be formulated in greater generality, but this is the form we will use in Section 7.7.

Lemma 7.4.8. *Let $\mathcal{N}\subseteq\mathcal{M}_1\subseteq\mathcal{M}$ be a unital inclusion of von Neumann algebras with $\mathcal{N}$ finite. Assume that we have faithful normal conditional expectations $\mathbb{E}_1:\mathcal{M}_1\to\mathcal{N}$ and $\mathbb{F}_1:\mathcal{M}\to\mathcal{M}_1$ and a faithful tracial state $\tau\in\mathcal{N}_*$. Set $\varphi=\tau\circ\mathbb{E}_1\circ\mathbb{F}_1$. Then if the triple $(\mathcal{M},\mathcal{N},\varphi)$ has property (rHAP) then the triple $(\mathcal{M}_1,\mathcal{N},\varphi|_{\mathcal{M}_1})$ also has property (rHAP).*

Proof. Suppose that $(\Phi_i)_{i\in I}$ is a net of approximations (unital, φ-preserving maps on $\mathcal{M}$) satisfying the conditions in the property (rHAP) for the triple $(\mathcal{M},\mathcal{N},\varphi)$. For each $i\in I$ define $\Psi_i:=\mathbb{F}_1\circ\Phi_i|_{\mathcal{M}_1}$. Our conditions guarantee that $\mathbb{F}_1$ is φ-preserving, so Ψ_i is a normal, unital, completely positive, $\mathcal{N}$-bimodular, $\varphi|_{\mathcal{M}_1}$ preserving map on $\mathcal{M}_1$. Due to the last theorem, we need only to check that $(\Psi_i)_{i\in I}$ satisfy the conditions in the property (rHAP)$^-$ (for the triple $(\mathcal{M}_1,\mathcal{N},\varphi|_{\mathcal{M}_1})$). Condition (iii) holds as for $x\in\mathcal{M}_1$ we have $\Psi_i(x)-x=\mathbb{F}_1(\Phi_i(x)-x)$ and $\mathbb{F}_1^{(2)}$ is the orthogonal projection from $L^2(\mathcal{M},\varphi)$ onto $L^2(\mathcal{M}_1,\varphi|_{\mathcal{M}_1})$.

To verify the last condition we assume first that Φ_i is of the form $a(\mathbb{E}_1 \circ \mathbb{F}_1)(b\cdot)$ for some $a, b \in \mathscr{M}$. But then for $x \in \mathscr{M}_1$ we have

$$\Psi_i(x) = \mathbb{F}_1(a(\mathbb{E}_1 \circ \mathbb{F}_1)(bx))) = \mathbb{F}_1(a)(\mathbb{E}_1 \circ \mathbb{F}_1)(bx))) = \mathbb{F}_1(a)\mathbb{E}_1(\mathbb{F}_1(b)x),$$

so we get that $\Psi_i^{(2)} \in \mathscr{K}_{00}(\mathscr{M}_1, \mathscr{N}, \varphi|_{\mathscr{M}_1})$. Taking linear combinations and approximation ends the proof. □

7.5. FIRST EXAMPLES

In this section we first put our definitions and main results in concrete context, discussing examples of the Haagerup (and non-Haagerup) inclusions arising in the framework of Cartan subalgebras, as studied in [116], [176] and [6] and then present the case of the bigger algebra being just $\mathscr{B}(\mathscr{H})$. The examples related to the latter situation show that the relative Haagerup property is not implied by coamenability as defined in [153].

7.5.1. EXAMPLES FROM EQUIVALENCE RELATIONS AND GROUPOIDS

In this subsection we will discuss examples of inclusions of von Neumann algebras which satisfy the relative Haagerup property and have already appeared in the literature. As mentioned in the introduction, the notion of the Haagerup property regarding the von Neumann inclusions beyond the finite context first appeared in the study of von Neumann algebras associated with groupoids/equivalence relations.

The first result here is due to [116], still in the finite context. Note that Jolissaint uses the definition of the Haagerup inclusion $\mathscr{N} \subseteq \mathscr{M}$ due to Popa in [155], namely the one using the larger ideal of "generalized compacts" than the one employed in this paper, but also note that due to [155, Proposition 2.2] both notions coincide if $\mathscr{N}' \cap \mathscr{M} \subseteq \mathscr{N}'$, so for example if $\mathscr{N}$ is a maximal abelian subalgebra in $\mathscr{M}$, which is the case of interest for the result below.

Theorem 7.5.1 ([116, Theorem 2.1]). *Let $\mathscr{R}$ be a measure-preserving standard equivalence relation on a set X (with the measure ν on $\mathscr{R}$ induced by the invariant probability measure μ on X). Then the following are equivalent:*

(1) *$\mathscr{R}$ has the* Haagerup property, *i.e. it admits a sequence of positive definite functions $(\varphi_n : \mathscr{R} \to \mathbb{C})_{n\in\mathbb{N}}$ which are bounded by 1 on the diagonal, converge to 1 ν-almost everywhere and satisfy the* vanishing property, *meaning that for every $n \in \mathbb{N}$ and $\epsilon > 0$ one has $\nu(\{(x,y) \in \mathscr{R} \mid |\varphi_n(x,y)| > \epsilon\}) < \infty$;*

(2) *The von Neumann inclusion $L^\infty(X,\mu) \subseteq \mathscr{L}(\mathscr{R})$ of finite von Neumann algebras has the relative Haagerup property.*

The definition beyond the finite case has first been considered in [176]; a more detailed study has been conducted by Anantharaman-Delaroche in [6]. Note that both these papers use the notion of the relative Haagerup property for arbitrary (expected) von Neumann inclusions identical to the one studied here. We will now describe the setup.

Let $\mathcal{G}$ be a measured groupoid with countable fibers, equipped with a quasi-invariant probability measure μ on the unit space $\mathcal{G}^{(0)}$ (note that a measure-preserving standard equivalence relation as considered above is one source of such examples). Again μ induces a measure ν on $\mathcal{G}$; we further obtain a (not necessarily finite) von Neumann algebra $\mathcal{L}(\mathcal{G}) \subset \mathcal{B}(L^2(\mathcal{G},\nu))$. The following result holds.

Theorem 7.5.2 ([6, Theorem 1]). *Let $\mathcal{G}$ be a measured groupoid with countable fibers, as above. Then the following conditions are equivalent:*

(1) $\mathcal{G}$ has the Haagerup property, *i.e. it admits a sequence of positive-definite functions $(F_n : \mathcal{G} \to \mathbb{C})_{n\in\mathbb{N}}$ which are equal to 1 on $\mathcal{G}^{(0)}$, converge to 1 ν-almost everywhere and satisfy the* vanishing property, *meaning that for every $n \in \mathbb{N}$ and $\epsilon > 0$ one has $\nu(\{g \in \mathcal{G} : |\varphi_n(g)| > \epsilon\}) < \infty$;*

(2) The von Neumann inclusion $L^\infty(\mathcal{G}^{(0)},\mu) \subseteq \mathcal{L}(\mathcal{G})$ has the relative Haagerup property.

Ueda shows in [176, Lemma 5] (and then Anantharaman-Delaroche reproves it in [6, Theorem 3]) that a property of a groupoid as above called *treeability* implies the Haagerup property. [6, Theorem 5] also shows that for ergodic measured groupoid with countable fibers the Haagerup property is incompatible with Property (T); we are however not aware of explicit examples of such Property (T) groupoids leading to von Neumann algebras which are not finite, and a general intuition regarding Property (T) objects says that these should naturally lead to finite von Neumann algebras (for example, discrete property (T) quantum groups are necessarily unimodular, see [81]).

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7.5.2. EXAMPLES AND COUNTEREXAMPLES WITH $\mathcal{M} = \mathcal{B}(\mathcal{H})$

We end this subsection with the example where $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and study which triples $(\mathcal{B}(\mathcal{H}), \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ have (rHAP). Since the conditional expectation $\mathbb{E}_{\mathcal{N}}$ is assumed to be normal it follows by a result of Tomiyama from [174] that $\mathcal{N}$ must be a direct sum of type I factors, so $\mathcal{N} \cong \bigoplus_{i\in I} \mathcal{B}(\mathcal{K}_i)$ for some index set I. Note that each $\mathcal{B}(\mathcal{K}_i)$ may occur in $\mathcal{B}(\mathcal{H})$ with a certain multiplicity $m_i \in \mathbb{N} \cup \{\infty\}$. In general, we have that $\mathcal{N}$ is spatially isomorphic to $\bigoplus_{i\in I} \mathcal{B}(\mathcal{K}_i) \otimes \mathbb{C}1_{m_i}$ where 1_{m_i} is the identity acting on a Hilbert space of dimension m_i. For simplicity in the examples below we assume that all multiplicities m_i equal 1 and ignore the spatial isomorphism. In that case the normal conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\bigoplus_{i\in I} \mathcal{B}(\mathcal{K}_i)$ is unique and determined by $\mathbb{E}_{\mathcal{N}}(x) = \sum_{i\in I} p_i x p_i$ where p_i is the projection onto $\mathcal{K}_i$. Therefore, in this case we can speak not only of the Haagerup property of the inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, but also about maps being compact and of finite index relative to this inclusion.

Theorem 7.5.3. *Assume that $\mathcal{H}$ is a separable Hilbert space, that $\mathcal{H} = \bigoplus_{i\in I}\mathcal{K}_i$, where I is an index set and that the dimension of $\mathcal{K}_i$ does not depend on $i \in I$. Put $\mathcal{N} = \bigoplus_{i\in I}\mathcal{B}(\mathcal{K}_i) \subseteq \mathcal{B}(\mathcal{H})$. Then the triple $(\mathcal{B}(\mathcal{H}), \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ has the property (rHAP).*

Proof. We may assume that $\mathcal{K}_i = \mathcal{K}$ for a single (separable) Hilbert space $\mathcal{K}$. The inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ is then isomorphic to the inclusion $\ell^\infty(I) \otimes \mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}(\ell^2(I)) \otimes \mathcal{B}(\mathcal{K})$. In the case where I is finite $\ell^\infty(I) \subseteq \mathcal{B}(\ell^2(I))$ is a finite-dimensional inclusion which clearly has (rHAP). In the case where I is infinite we may assume that $I = \mathbb{Z}$ and the inclusion $\ell^\infty(\mathbb{Z}) \subseteq \mathcal{B}(\ell^2(\mathbb{Z}))$ has the (rHAP) with approximating maps given by the (Fejér-)Herz-Schur multipliers T_n with

$$T_n((x_{i,j})_{i,j\in\mathbb{Z}}) = (W(i-j)x_{i,j})_{i,j\in\mathbb{Z}}, \qquad W(k) := \max(1 - \frac{|k|}{n}, 0).$$

Since $W = \frac{1}{n}(\chi_{[0,n]})^* * \chi_{[0,n]}$ is positive definite and converges to the identity pointwise it follows that T_n is completely positive and $T_n^{(2)}$ converges to the identity strongly. Further $T_n^{(2)}$ is finite rank relative to $\ell^\infty(\mathbb{Z})$, so certainly compact. In both cases (I being finite or infinite), we tensor the approximating maps with $\mathrm{Id}_{\mathcal{B}(\mathcal{K})}$ and find that $\ell^\infty(I) \otimes \mathcal{B}(\mathcal{K}) \subseteq \mathcal{B}(\ell^2(I)) \otimes \mathcal{B}(\mathcal{K})$ has (rHAP). □

With a bit more work Theorem 7.5.3 could be proved in larger generality by relaxing the assumption that the multiplicities are trivial and that the dimension is constant (as opposed to say for example uniformly bounded). However, we cannot admit just any subalgebra $\mathcal{N}$ as the following counterexample shows.

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Theorem 7.5.4. *Let $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where $\mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces such that* $\dim(\mathcal{K}_1) < \infty$ *and* $\dim(\mathcal{K}_2) = \infty$. *Set $\mathcal{N} = \mathcal{B}(\mathcal{K}_1) \oplus \mathcal{B}(\mathcal{K}_2)$. Then the triple $(\mathcal{B}(\mathcal{H}), \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ does not have the property (rHAP).*

Proof. Let p be the projection of $\mathcal{H}$ onto $\mathcal{K}_1$. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a normal linear map. The proof is based on two claims.

Claim 1: If Φ is an $\mathcal{N}$-$\mathcal{N}$-bimodule map then $\mathcal{B}(\mathcal{H})p$ is an invariant subspace. Moreover, the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ lies in the linear span of the two maps $xp \mapsto pxp$ and $xp \mapsto (1-p)xp$.
Proof of Claim 1. Note that p is contained in $\mathcal{N}$ from which the first statement follows. For the second part let $E^i_{k,l}$ be matrix units with respect to some basis of $\mathcal{K}_i$. Then for $x \in \mathcal{B}(\mathcal{H})$ we have $\Phi(E^i_{k,k}xE^i_{l,l}) = E^i_{k,k}\Phi(x)E^i_{k,k}$ so that $E^i_{k,k}\mathcal{B}(\mathcal{H})E^i_{k,k}$ is an eigenspace of Φ (i.e. Φ is a Schur multiplier). Moreover $\Phi(E^i_{k',k'}xE^i_{l',l'}) = E^i_{k',k}\Phi(E^i_{k,k'}xE^i_{l',l})E^i_{l,l'}$ so that the eigenvalues of these spaces only depend on i. This in particular implies the claim.

Claim 2: If Φ is compact relative to the inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ then $\mathcal{B}(\mathcal{H})p$ is an invariant subspace. Moreover, the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ is compact (in the non-relative sense).
Proof of Claim 2. By approximation it suffices to prove Claim 2 with 'compact' replaced by 'finite rank'. So assume that $\Phi = a\mathbb{E}_{\mathcal{N}}b$ with $a, b \in \mathcal{B}(\mathcal{H})$. Note that

$p \in \mathcal{N} \cap \mathcal{N}'$ and therefore $a\mathbb{E}_{\mathcal{N}}(bxp) = ap\mathbb{E}_{\mathcal{N}}(bx)p = a\mathbb{E}_{\mathcal{N}}(pbxp)$. The first of these equalities shows that $\mathcal{B}(\mathcal{H})p$ is invariant. Further $x \mapsto (pxp)$ is finite rank as p projects onto a finite-dimensional space. This proves the claim.

Remainder of the proof. Suppose that Φ is both $\mathcal{N}$-$\mathcal{N}$-bimodular and compact relative to $\mathcal{N}$. By Claim 1 we know that there are scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\Phi(xp) = \lambda_1 pxp + \lambda_2(1-p)xp$. If $\lambda_2 \neq 0$ then the associated L^2-map is not compact (in the non-relative sense) since $(1-p)$ projects onto an infinite-dimensional Hilbert space. This contradicts Claim 2 because the restriction of Φ to $\mathcal{B}(\mathcal{H})p$ is compact. We conclude that $\lambda_2 = 0$ for any normal map $\Phi : \mathcal{M} \to \mathcal{M}$ that is $\mathcal{N}$-$\mathcal{N}$-bimodular and compact relative to $\mathcal{N}$. But then we can never find a net of such maps that approximates the identity map on $\mathcal{B}(\mathcal{H})$ in the point-strong topology. Hence the inclusion $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ fails to have (rHAP). □

Remark 7.5.5. Recall that a unital inclusion of von Neumann algebras $\mathcal{N} \subseteq \mathcal{M}$ is said to be *co-amenable* if there exists a (not necessarily normal) conditional expectation from $\mathcal{N}'$ onto $\mathcal{M}'$, where the commutants are taken with respect to any Hilbert space realization of $\mathcal{M}$. Theorem 7.5.4 shows – surprisingly – that a co-amenable inclusion in general need not have (rHAP). Note that this also means that a naive extension of the definition of relative Haagerup property in terms of correspondences, modeled on the notion of *strictly mixing bimodules* (see [144, Theorem 9]) valid for the non-relative Haagerup property, cannot be equivalent to the definition studied in our paper. Indeed, the last fact, together with the examples above, would contradict [14, Theorem 2.4].

7.6. PROPERTY (RHAP) FOR FINITE-DIMENSIONAL SUBALGEBRAS

In this section we consider the case of finite-dimensional subalgebras and show equivalence of the relative Haagerup property and the non-relative Haagerup property. For this, we fix a unital inclusion $\mathcal{N} \subseteq \mathcal{M}$ of von Neumann algebras and assume that it admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}$. Assume that $\mathcal{N}$ is finite-dimensional and let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state on $\mathcal{N}$ that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. We will prove that the triple $(\mathcal{M}, \mathcal{N}, \varphi)$ has property (rHAP) if and only if $(\mathcal{M}, \mathbb{C}, \varphi)$ does. Recall that by Theorem 7.2.7 the Haagerup property of $(\mathcal{M}, \mathcal{N}, \varphi)$ does not depend on the choice of the state τ.

Denote by $z_1, ..., z_n \in \mathcal{Z}(\mathcal{N})$ the minimal central projections of $\mathcal{N}$. There exist natural numbers $n_1, ..., n_k \in \mathbb{N}$ such that $z_k\mathcal{N} \cong M_{n_k}(\mathbb{C})$ for $k = 1, ..., n$. Let $(f_i^k)_{1\le i\le n_k}$ be an orthonormal basis of $\mathbb{C}^{n_k}$, write $E_{i,j}^k$, $1 \le i, j \le n_k$ for the matrix units with respect to this basis and set $E_i^k := E_{i,i}^k$ for the diagonal projections. We have that $E_{i,j}^k f_l^k = \delta_{j,l} f_i^k$ for all $k \in \mathbb{N}$, $1 \le i, j, l \le n_k$ and $\sum_{k=1}^n \sum_{i=1}^{n_k} E_i^k = 1$. Set $d := \sum_{k=1}^n n_k$, choose an orthonormal basis $(f_{k,i})_{1\le k\le n,\, 1\le i\le n_k}$ of $\mathbb{C}^d$ with corresponding matrix units $e_{(k,i),(l,j)} \in M_d(\mathbb{C})$ where $1 \le k,l \le n$, $1 \le i \le n_k$, $1 \le j \le n_l$ and define

$$p := \sum_{k=1}^{n} E_1^k. \tag{7.6.1}$$

For a general linear map $\Phi\colon p\mathcal{M}p \to p\mathcal{M}p$ we may define a linear map $\widetilde{\Phi}\colon \mathcal{M} \to \mathcal{M}$ by

$$\widetilde{\Phi}(E_i^k x E_j^l) := E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l \tag{7.6.2}$$

for all $1 \le k,l \le n$, $1 \le i \le n_k$, $1 \le j \le n_l$ and $x \in \mathcal{M}$.

Let us study the properties of $\widetilde{\Phi}$.

Lemma 7.6.1. *Let $\Phi : p\mathcal{M}p \to p\mathcal{M}p$ be a linear map. Define*

$$U := \sum_{k=1}^{n}\sum_{i=1}^{n_k} e_{(1,1),(k,i)} \otimes E_{i,1}^k \in M_d(\mathbb{C}) \otimes \mathcal{M}, \qquad V := \sum_{k=1}^{n}\sum_{i=1}^{n_k} f_{k,i} \otimes E_{1,i}^k \in \mathbb{C}^d \otimes \mathcal{M}.$$

Then,

$$\widetilde{\Phi}(x) = V^*(id_{\mathcal{B}(L^2(\mathcal{N},\tau))} \otimes \Phi)\left(U^*(1 \otimes x)U\right)V.$$

Proof. We have for $x \in z_k\mathcal{M}z_l$ with $1 \le k,l \le n$ that

$$U^*(1 \otimes x)U = \sum_{i=1}^{n_k}\sum_{j=1}^{n_l} e_{(k,i),(l,j)} \otimes E_{1,i}^k x E_{j,1}^l$$

so that

$$V^*(\mathrm{id}_{\mathcal{B}(L^2(\mathcal{N},\tau))} \otimes \Phi)\left(U^*(1 \otimes x)U\right)V = \sum_{i=1}^{n_k}\sum_{j=1}^{n_l} E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l) E_{1,j}^l.$$

By definition this expression coincides with $\widetilde{\Phi}(x)$. The claim follows. □

Lemma 7.6.2. *If $\Phi\colon pMp \to pMp$ is a unital normal completely positive map, then $\widetilde{\Phi}$ is contractive, normal and completely positive.*

Proof. The normality and the complete positivity follow from Lemma 7.6.1. We further have

$$\begin{aligned} \|\widetilde{\Phi}\| &= \widetilde{\Phi}(1) = \widetilde{\Phi}\left(\sum_{k=1}^{n}\sum_{i=1}^{n_k} E_i^k\right) = \sum_{k=1}^{n}\sum_{i=1}^{n_k} E_{i,1}^k \Phi(E_1^k) E_{1,i}^k \\ &\le \sum_{k=1}^{n}\sum_{i=1}^{n_k} E_{i,1}^k E_{1,i}^k = \sum_{k=1}^{n}\sum_{i=1}^{n_k} E_i^k = 1, \end{aligned}$$

i.e. $\widetilde{\Phi}$ is contractive. □

Lemma 7.6.3. *Let $\Phi\colon p\mathcal{M}p \to p\mathcal{M}p$ be a linear map. Then $\widetilde{\Phi}$ is an $\mathcal{N}$-$\mathcal{N}$-bimodule map.*

Proof. Let $x \in \mathcal{M}$. For $1 \leq l, k, m \leq n$ and $1 \leq r, s \leq n_l$, $1 \leq i \leq n_k$, $1 \leq j \leq n_m$ we have

$$\begin{aligned} E^l_{r,s}\widetilde{\Phi}(E^k_i x E^m_j) &= E^l_{r,s}E^k_{i,1}\Phi(E^k_{1,i}xE^m_{j,1})E^m_{1,j} \\ &= \delta_{s,i}\delta_{l,k}E^k_{r,1}\Phi(E^k_{1,i}xE^m_{j,1})E^m_{1,j} \\ &= E^k_{r,1}\Phi(E^k_{1,r}E^l_{r,s}E^k_i x E^m_j E^m_{j,1})E^m_{1,j}. \end{aligned}$$

We hence find that for $y \in E^k_i x E^m_j$, $E^l_{r,s}\widetilde{\Phi}(y) = \widetilde{\Phi}(E^l_{r,s}y)$. The linearity of $\widetilde{\Phi}$ then implies that it is a left $\mathcal{N}$-module map. A similar argument applies to the right-handed case. □

Proposition 7.6.4. *Define the map*

$$\mathit{Diag}\colon p\mathcal{M}p \to p\mathcal{M}p,\ x \mapsto \sum_{k=1}^{n} \frac{\varphi(E^k_1 x E^k_1)}{\varphi(E^k_1)} E^k_1.$$

Then $\widetilde{\mathit{Diag}} = \mathbb{E}_{\mathcal{N}}$.

Proof. It is clear that the map Diag is linear unital normal and completely positive. Hence, by Lemma 7.6.2 and Lemma 7.6.3, $\widetilde{\mathrm{Diag}}$ is contractive normal completely positive and $\mathcal{N}$-$\mathcal{N}$-bimodular. It is easy to check that $\widetilde{\mathrm{Diag}}$ is even unital. In particular, $\widetilde{\mathrm{Diag}}$ restricts to the identity on $\mathcal{N}$. It is further clear that $\widetilde{\mathrm{Diag}}$ is faithful and that it maps $\mathcal{M}$ onto $\mathcal{N}$, so $\widetilde{\mathrm{Diag}}$ is a faithful normal conditional expectation. For $x \in \mathcal{M}$ and $1 \leq k, l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ we have

$$\begin{aligned} \varphi \circ \widetilde{\mathrm{Diag}}(E^k_i x E^l_j) &= \varphi\left(E^k_{i,1}\mathrm{Diag}(E^k_{1,i}xE^l_{j,1})E^l_{1,j}\right) \\ &= \sum_{m=1}^{n} \frac{\varphi(E^m_1 E^k_{1,i} x E^l_{j,1} E^m_1)}{\varphi(E^m_1)} \varphi\left(E^k_{i,1}E^m_1 E^l_{1,j}\right) \\ &= \frac{\varphi(E^l_1 E^k_{1,i} x E^l_{j,1} E^l_1)}{\varphi(E^l_1)} \varphi(E^k_{i,1}E^l_1 E^l_{1,j}) \\ &= \delta_{k,l}\frac{\varphi(E^l_{1,i} x E^l_{j,1})}{\varphi(E^l_1)} \varphi(E^l_{i,1}E^l_{1,j}) \\ &= \delta_{k,l}\frac{\varphi(E^l_{1,i} x E^l_{j,1})}{\tau(E^l_1)} \tau(E^l_{i,j}). \end{aligned}$$

But then, since τ is tracial,

$$\begin{aligned} \varphi \circ \widetilde{\mathrm{Diag}}(E^k_i x E^l_j) &= \delta_{i,j}\delta_{k,l}\varphi(E^l_{1,i}xE^l_{i,1}) = \delta_{i,j}\delta_{k,l}\tau(\mathbb{E}_{\mathcal{N}}(E^l_{1,i}xE^l_{i,1})) \\ &= \delta_{i,j}\delta_{k,l}\tau(E^l_{1,i}\mathbb{E}_{\mathcal{N}}(x)E^l_{i,1}) = \tau(E^k_i\mathbb{E}_{\mathcal{N}}(x)E^l_j) \\ &= \varphi(E^k_i x E^l_j), \end{aligned}$$

i.e. $\widetilde{\mathrm{Diag}}$ is φ-preserving. Since $\mathbb{E}_{\mathcal{N}}$ is the unique faithful normal φ-preserving conditional expectation onto $\mathcal{N}$, we get that $\widetilde{\mathrm{Diag}} = \mathbb{E}_{\mathcal{N}}$. □

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Lemma 7.6.5. *Let $\Phi\colon p\mathcal{M}p \to p\mathcal{M}p$ be a normal completely positive map with $\varphi\circ\Phi \leq \varphi$ and assume that the L^2-implementation $\Phi^{(2)}$ of Φ with respect to $\varphi|_{p\mathcal{M}p}$ is a compact operator. Then $\widetilde{\Phi}$ satisfies $\varphi\circ\widetilde{\Phi}\leq\varphi$ and $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$.*

Proof. For $1 \leq k,l \leq n$, $1 \leq i \leq n_k$, $1 \leq j \leq n_l$ and $x \in \mathcal{M}$ positive we have by the traciality of τ,

$$\begin{aligned}\varphi\circ\widetilde{\Phi}(E_i^k x E_j^l) &= \varphi\left(E_{i,1}^k \Phi(E_{1,i}^k x E_{j,1}^l)E_{1,j}^l\right)\\ &= \tau\left(E_{i,1}^k \mathbb{E}_{\mathcal{N}}\left(\Phi(E_{1,i}^k x E_{j,1}^l)\right)E_{1,j}^l\right)\\ &= \delta_{i,j}\delta_{k,l}\tau\left(E_1^k\mathbb{E}_{\mathcal{N}}(\Phi(E_{1,i}^k x E_{i,1}^k))\right),\end{aligned}$$

so in particular $\varphi\circ\widetilde{\Phi}(E_i^k x E_j^l)\geq 0$. We get (as $\mathcal{N}$ is contained in the centralizer $\mathcal{M}^{\varphi}$)

$$\begin{aligned}\varphi\circ\widetilde{\Phi}(E_i^k x E_j^l) &= \delta_{i,j}\delta_{k,l}\tau\left(E_1^k\mathbb{E}_{\mathcal{N}}(\Phi(E_{1,i}^k x E_{i,1}^k))\right)\\ &\leq \delta_{i,j}\delta_{k,l}\tau\left(\mathbb{E}_{\mathcal{N}}(\Phi(E_{1,i}^k x E_{i,1}^k))\right)\\ &\leq \delta_{i,j}\delta_{k,l}\varphi(E_{1,i}^k x E_{i,1}^k)\\ &= \varphi(E_i^k x E_j^l).\end{aligned}$$

This implies that $\widetilde{\Phi}$ indeed satisfies $\varphi\circ\widetilde{\Phi}\leq\varphi$. In particular, the L^2-implementation of $\widetilde{\Phi}$ with respect to φ exists.

It remains to show that $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. For this, let $\Psi\colon p\mathcal{M}p \to p\mathcal{M}p$ be a map with $\Psi^{(2)} = ae_{\mathbb{C}}b$ where $a,b\in p\mathcal{M}p$ and $e_{\mathbb{C}}$ denotes the rank one projection $(\varphi|_{p\mathcal{M}p}(\cdot)p)^{(2)} \in \mathcal{B}(L^2(p\mathcal{M}p,\varphi|_{p\mathcal{M}p}))$. For $1\leq k,l\leq n$, $1\leq i\leq n_k$, $1\leq j\leq n_l$ and $x\in\mathcal{M}$ we then have

$$\widetilde{\Psi}(E_i^k x E_j^l) = E_{i,1}^k a\varphi(bE_{1,i}^k x E_{j,1}^l)E_{1,j}^l = E_{i,1}^k a\varphi(bE_{1,i}^k x E_{j,1}^l)E_1^l E_{1,j}^l.$$

Note that by Proposition 7.6.4,

$$\begin{aligned}\mathbb{E}_{\mathcal{N}}(bE_{1,i}^k x E_{j,1}^l) &= \sum_{r=1}^n \mathbb{E}_{\mathcal{N}}(E_1^r bE_{1,i}^k x E_{j,1}^l)\\ &= \sum_{r=1}^n \widetilde{\mathrm{Diag}}(E_1^r bE_{1,i}^k x E_{j,1}^l)\\ &= \sum_{r=1}^n E_1^r\mathrm{Diag}(E_1^r bE_{1,i}^k x E_{j,1}^l)E_1^l\\ &= \sum_{r=1}^n\sum_{m=1}^n \frac{\varphi(E_1^m E_1^r bE_{1,i}^k x E_{j,1}^l E_1^m)}{\varphi(E_1^m)}E_1^r E_1^m E_1^l\\ &= \frac{\varphi(E_1^l bE_{1,i}^k x E_{j,1}^l)}{\varphi(E_1^l)}E_1^l\end{aligned}$$

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$$= \frac{\varphi(bE^k_{1,i}xE^l_{j,1})}{\varphi(E^l_1)}E^l_1,$$

where in the last equality we again used that τ is tracial. Hence

$$\widetilde{\Psi}(E^k_i xE^l_j) = \varphi(E^l_1)E^k_{i,1}a\mathbb{E}_{\mathcal{N}}(bE^k_{1,i}xE^l_{j,1})E^l_{1,j} = \varphi(E^l_1)E^k_{i,1}a\mathbb{E}_{\mathcal{N}}(bE^k_{1,i}E^k_i xE^l_j). \tag{7.6.3}$$

Fix now suitable t_0, j_0, k_0, l_0 and $x \in \mathcal{M}$ and compute the following expression:

$$\begin{aligned}
&\left(\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{n}\sum_{t=1}^{n_k} \varphi(E^r_1)E^k_{t,1}aE^l_1 e_{\mathcal{N}}E^r_1 bE^k_{1,t}\right)(E^{k_0}_{t_0}xE^{l_0}_{j_0}\Omega_\varphi)\\
&\qquad = \sum_{l=1}^{n}\sum_{r=1}^{n}\varphi(E^r_1)E^{k_0}_{t_0,1}aE^l_1\mathbb{E}_{\mathcal{N}}(E^r_1 bE^{k_0}_{1,t_0}xE^{l_0}_{j_0})\Omega_\varphi\\
&\qquad = \sum_{l=1}^{n}\varphi(E^l_1)E^{k_0}_{t_0,1}aE^l_1\mathbb{E}_{\mathcal{N}}(bE^{k_0}_{1,t_0}xE^{l_0}_{j_0,1})E^{l_0}_{1,j_0}\Omega_\varphi.
\end{aligned}$$

Now the equality (7.6.3) implies that the value of the conditional expectation appearing in the last formula is a scalar multiple of $E^{l_0}_1$, so the whole expression equals

$$\varphi(E^{l_0}_1)E^{k_0}_{t_0,1}a\mathbb{E}_{\mathcal{N}}(bE^{k_0}_{1,t_0}xE^{l_0}_{j_0})\Omega_\varphi = \widetilde{\Psi}(E^{k_0}_{t_0}xE^{l_0}_{j_0})\Omega_\varphi.$$

Hence we arrive at

$$(\widetilde{\Psi})^{(2)} = \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{n}\sum_{t=1}^{n_k}\varphi(E^r_1)E^k_{t,1}aE^l_1 e_{\mathcal{N}}E^r_1 bE^k_{1,t} \in \mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi).$$

By taking linear combinations this implies that for every map Ψ with $\Psi^{(2)} \in \mathcal{K}_{00}(p\mathcal{M}p,\mathbb{C},\varphi)$ the L^2-implementation of $\widetilde{\Psi}$ is contained in $\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$. Via approximation we then see that $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$. □

We are now ready to prove the main theorem of this section.

Theorem 7.6.6. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras and assume that it admits a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}}\colon \mathcal{M} \to \mathcal{N}$. Assume that $\mathcal{N}$ is finite-dimensional and let $\tau \in \mathcal{N}_*$ be a faithful state on $\mathcal{N}$ that we extend to a state $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$ on $\mathcal{M}$. Then $\mathcal{M}$ has the Haagerup property (in the sense that the triple $(\mathcal{M},\mathbb{C},\varphi)$ has the relative Haagerup property) if and only if the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the relative Haagerup property.*

Proof. By Theorem 7.2.7 we may assume without loss of generality that τ is tracial.

"$\Leftarrow$": Assume that the triple $(\mathcal{M},\mathcal{N},\varphi)$ has the relative Haagerup property and let $(\Phi_i)_{i\in I}$ be a net of normal completely positive maps witnessing it. Since $\mathcal{N}$ is finite-dimensional, $e_{\mathcal{N}}$ is a finite rank projection. In particular, $\mathcal{K}_{00}(\mathcal{M},\mathcal{N},\varphi)$ consists of finite rank operators and hence $\mathcal{K}(\mathcal{M},\mathcal{N},\varphi) \subseteq \mathcal{K}(\mathcal{M},\mathbb{C},\varphi)$. In particular, $\Phi_i^{(2)} \in \mathcal{K}(\mathcal{M},\mathbb{C},\varphi)$ for every $i \in I$. Further, $\Phi_i(x) \to x$ strongly for every $x \in \mathcal{M}$. This implies that the net $(\Phi_i)_{i\in I}$ also witnesses the relative Haagerup property of the triple $(\mathcal{M}, \mathbb{C}, \varphi)$.

"⇒": Assume that $\mathcal{M}$ has the Haagerup property. Recall that the projection p was defined in (7.6.1). By [41, Lemma 4.1] the triple $(p\mathcal{M}p, \mathbb{C}, \varphi|_{p\mathcal{M}p})$ also has the relative Haagerup property and by Theorem 7.4.5 we find a net $(\Phi_i)_{i\in I}$ of unital normal completely positive φ-preserving maps witnessing it. By Lemma 7.6.2, Lemma 7.6.3 and Lemma 7.6.5 we find that $\widetilde{\Phi}_i$ is a contractive normal completely positive $\mathcal{N}$-$\mathcal{N}$-bimodule map with $\varphi\circ\widetilde{\Phi} \le \varphi$ and $(\widetilde{\Phi})^{(2)} \in \mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$ for every $i \in I$. It follows directly from the prescription (7.6.2) that $\widetilde{\Phi}_i(x) \to x$ strongly for every $x \in \mathcal{M}$. It follows that the net $(\widetilde{\Phi}_i)_{i\in I}$ witnesses the relative Haagerup property of the triple $(\mathcal{M}, \mathcal{N}, \varphi)$. □

7.7. THE RELATIVE HAAGERUP PROPERTY FOR FREE PRODUCTS WITH AMALGAMATION

The class of discrete groups enjoying the Haagerup property has good permanence properties, one of which is that it is closed under taking free products amalgamated over finite subgroups (see [7, Section 6]). The following theorem demonstrates that in the setting of Section 7.4 the relative Haagerup property is preserved under taking amalgamated free products (see Subsection 3.6.1). For finite inclusions of von Neumann algebras this has been proved in [23, Proposition 3.9].

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Theorem 7.7.1. *Let $\mathcal{N} \subseteq \mathcal{M}_1$ and $\mathcal{N} \subseteq \mathcal{M}_2$ be unital embeddings of von Neumann algebras which admit faithful normal conditional expectations $\mathbb{E}_1 : \mathcal{M}_1 \to \mathcal{N}$, $\mathbb{E}_2 : \mathcal{M}_2 \to \mathcal{N}$ and for which $\mathcal{N}$ is finite. Denote by $\mathcal{M} := (\mathcal{M}_1, \mathbb{E}_1) \star_{\mathcal{N}} (\mathcal{M}_2, \mathbb{E}_2)$ the amalgamated free product von Neumann algebra of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to the expectations $\mathbb{E}_1$, $\mathbb{E}_2$ and let $\mathbb{E}_{\mathcal{N}}$ be the corresponding conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Then $(\mathcal{M}_1, \mathcal{N}, \mathbb{E}_1)$ and $(\mathcal{M}_2, \mathcal{N}, \mathbb{E}_2)$ have the relative Haagerup property if and only if the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_N)$ has the relative Haagerup property.*

Proof. "⇒": Assume that both $(\mathcal{M}_1, \mathcal{N}, \mathbb{E}_1)$ and $(\mathcal{M}_2, \mathcal{N}, \mathbb{E}_2)$ have the relative Haagerup property, let $\tau \in \mathcal{N}_*$ be a faithful normal tracial state and set $\varphi_1 := \tau \circ \mathbb{E}_1$, $\varphi_2 := \tau\circ\mathbb{E}_2$. Then the triples $(\mathcal{M}_1, \mathcal{N}, \varphi_1)$ and $(\mathcal{M}_2, \mathcal{N}, \varphi_2)$ have the relative Haagerup property. Without loss of generality we can assume that the corresponding nets $(\Phi_{i,1})_{i\in I}$ and $(\Phi_{i,2})_{i\in I}$ witnessing the relative Haagerup property are indexed by the same set I. By Theorem 7.4.5 we can also assume that the maps are unital with $\varphi_1\circ\Phi_{i,1} = \varphi_1$, $\varphi_2\circ\Phi_{i,2} = \varphi_2$ for all $i \in I$, which then implies that $\Phi_{i,1}|_{\mathcal{N}} = \Phi_{i,2}|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ and that $\mathbb{E}_1 \circ \Phi_{i,1} = \mathbb{E}_1$, $\mathbb{E}_2 \circ \Phi_{i,2} = \mathbb{E}_2$. Choose a net $(\varepsilon_i)_{i\in I}$ (we can use the same indexing set, modifying it if necessary) with $\varepsilon_i \to 0$ and define unital normal completely positive $\mathcal{N}$-$\mathcal{N}$-bimodular maps $\Phi'_{i,1} := \frac{1}{1+\varepsilon_i}(\Phi_{i,1} + \varepsilon_i\mathbb{E}_1)$, $\Phi'_{i,2} := \frac{1}{1+\varepsilon_i}(\Phi_{i,2} + \varepsilon_i\mathbb{E}_2)$.

In the following we will need to work with certain sets of multi-indices: for each $n \in \mathbb{N}$ set $\mathcal{J}_n = \{\mathbf{j} = (j_1, \dots, j_n) : j_k \in \{1,2\}$ and $j_k \neq j_{k+1}$ for $k = 1, \dots, n-1\}$; put also $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{J}_n$.

Set $\varphi := \tau \circ \mathbb{E}_{\mathcal{N}}$, let $\Psi_i := \Phi_{i,1} \star \Phi_{i,2}$: $\mathcal{M} \to \mathcal{M}$ be the unital normal completely positive map with $\Psi_i|_{\mathcal{N}} = \mathrm{id}_{\mathcal{N}}$ and $\Psi_i(x_1 ... x_n) = \Phi'_{i,j_1}(x_1)...\Phi'_{i,j_n}(x_n)$ for $\mathbf{j} \in \mathcal{J}_n$ and

$x_k \in \mathcal{M}_{j_k} \cap \ker(\mathbb{E}_{j_k})$ for $k = 1, \ldots, n$ (see [21, Theorem 3.8]) and define $\Psi'_i := \Phi'_{i,1} \star \Phi'_{i,2}$ analogously. We claim that the net $(\Psi'_i)_{i \in I}$ witnesses the relative Haagerup property of the triple $(\mathcal{M}, \mathcal{N}, \varphi)$. Indeed, it is clear that the maps satisfy the conditions (1), (2) and (4) of Definition 7.2.2. It remains to show that $\Psi'_i(x) \to x$ strongly for every $x \in \mathcal{M}$ and that the L^2-implementations $(\Psi'_i)^{(2)}$ are contained in $\mathcal{K}(\mathcal{M}, \mathcal{N}, \varphi)$.

Define for $n \in \mathbb{N}$ and $\mathbf{j} \in \mathcal{J}_n$ the Hilbert subspace

$$\mathcal{H}_{\mathbf{j}} := \overline{\text{Span}\{x_1 \ldots x_n \Omega_\varphi \mid x_1 \in \ker(\mathbb{E}_{j_1}), \ldots, x_n \in \ker(\mathbb{E}_{j_n})\}} \subseteq L^2(\mathcal{M}, \varphi)$$

and let $P_{\mathbf{j}} \in \mathcal{B}(L^2(M, \varphi))$ be the orthogonal projection onto $\mathcal{H}_{\mathbf{j}}$. Note that these Hilbert subspaces are pairwise orthogonal for different multi-indices $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{J}$, orthogonal to $\mathcal{N}\Omega_\varphi \subseteq L^2(\mathcal{M}, \varphi)$, one has inclusions $\Psi_i^{(2)} \mathcal{H}_{\mathbf{j}} \subseteq \mathcal{H}_{\mathbf{j}}$, $(\Psi'_i)^{(2)} \mathcal{H}_{\mathbf{j}} \subseteq \mathcal{H}_{\mathbf{j}}$ and the span of the union of all $\mathcal{H}_{\mathbf{j}}$, $\mathbf{j} \in \mathcal{J}$ with $N\Omega_\varphi$ is dense in $L^2(M, \varphi)$.

For the strong convergence it suffices to show that $\|\Psi_i^{(2)} \xi - \xi\|_2 \to 0$ for all $\xi \in \mathcal{H}_{\mathbf{j}}$, $\mathbf{j} \in \mathcal{J}$. So let $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$, $x_1 \in \ker(\mathbb{E}_{j_1})$, ..., $x_n \in \ker(\mathbb{E}_{j_n})$. Then,

$$\begin{aligned}
& \|(\Psi'_i)^{(2)}(x_1 \ldots x_n \Omega_\varphi) - x_1 \ldots x_n \Omega_\varphi\|_2 = \|\Phi'_{i,j_1}(x_1) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi - x_1 \ldots x_n \Omega_\varphi\|_2 \\
\leq\; & \|(\Phi'_{i,j_1}(x_1) - x_1)\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 + \|x_1\| \|\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi - x_2 \ldots x_n \Omega_\varphi\|_2 \\
\leq\; & \ldots \\
\leq\; & \|(\Phi'_{i,j_1}(x_1) - x_1)\Phi'_{i,j_2}(x_2) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\
& \quad + \|x_1\| \|(\Phi'_{i,j_2}(x_2) - x_2)\Phi'_{i,j_3}(x_3) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi\|_2 \\
& \quad\quad + \ldots + \|x_1\| \ldots \|x_{n-1}\| \|\Phi'_{i,j_n}(x_n)\Omega_\varphi - x_n \Omega_\varphi\|_2 \to 0.
\end{aligned}$$

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This implies that indeed $\Psi_i(x) \to x$ strongly for every $x \in \mathcal{M}$.

To treat the relative compactness, express the operators $(\Phi'_{i,1})^{(2)} \in \mathcal{K}(\mathcal{M}_1, \mathcal{N}, \varphi_1)$, $(\Phi'_{i,2})^{(2)} \in \mathcal{K}(\mathcal{M}_2, \mathcal{N}, \varphi_2)$ as norm-limits

$$(\Phi'_{i,1})^{(2)} = \lim_{l \to \infty} \sum_{k=1}^{N_l^{(i,1)}} a_{k,l}^{(i,1)} e_{\mathcal{N}}^{\varphi_1} b_{k,l}^{(i,1)} \quad \text{and} \quad (\Phi'_{i,2})^{(2)} = \lim_{l \to \infty} \sum_{k=1}^{N_l^{(i,2)}} a_{k,l}^{(i,2)} e_{\mathcal{N}}^{\varphi_2} b_{k,l}^{(i,2)}$$

for suitable $N_l^{(i,1)}, N_l^{(i,2)} \in \mathbb{N}$, $a_{k,l}^{(i,1)}, b_{k,l}^{(i,1)} \in \mathcal{M}_1$ and $a_{k,l}^{(i,2)}, b_{k,l}^{(i,2)} \in \mathcal{M}_2$.

Claim. For $n \in \mathbb{N}$, $\mathbf{j} \in \mathcal{J}_n$, we have

$$\left\|(\Psi'_i)^{(2)} P_{\mathbf{j}}\right\| \leq \left(\frac{1}{1 + \varepsilon_i}\right)^n \tag{7.7.1}$$

and

$$(\Psi'_i)^{(2)} P_{\mathbf{j}} = \lim_{l_1, \ldots, l_n \to \infty} \sum_{k_1, \ldots, k_n} a_{k_1,l_1}^{(i,j_1)} \ldots a_{k_n,l_n}^{(i,j_n)} e_N b_{k_n,l_n}^{(i,j_n)} \ldots b_{k_1,l_1}^{(i,j_1)}, \tag{7.7.2}$$

where the convergence is in norm.
Proof of the claim. For $x_1 \in \ker(\mathbb{E}_{j_1}), \ldots, x_n \in \ker(\mathbb{E}_{j_n})$ one calculates

$$\begin{aligned}(\Psi_i')^{(2)} P_{\mathbf{j}}(x_1 \ldots x_n \Omega_\varphi) &= \Phi'_{i,j_1}(x_1) \ldots \Phi'_{i,j_n}(x_n)\Omega_\varphi \\ &= \left(\frac{1}{1+\varepsilon_i}\right)^n \Phi_{i,j_1}(x_1)\ldots\Phi_{i,j_n}(x_n)\Omega_\varphi \\ &= \left(\frac{1}{1+\varepsilon_i}\right)^n \Psi_i^{(2)}(x_1\ldots x_n\Omega_\varphi)\end{aligned}$$

and hence $(\Psi_i')^{(2)}P_{\mathbf{j}} = (1+\varepsilon_i)^{-n}\Psi_i^{(2)}P_{\mathbf{j}}$. By the unitality of $\Phi_{i,1}$ and $\Phi_{i,2}$ the inequality (7.7.1) then follows from

$$\left\|(\Psi_i')^{(2)}P_{\mathbf{j}}\right\| = \left(\frac{1}{1+\varepsilon_i}\right)^n \left\|\Psi_i^{(2)}P_{\mathbf{j}}\right\| \leq \left(\frac{1}{1+\varepsilon_i}\right)^n \left\|\Psi_i^{(2)}\right\| \leq \left(\frac{1}{1+\varepsilon_i}\right)^n \|\Psi_i\| = \left(\frac{1}{1+\varepsilon_i}\right)^n.$$

We proceed by induction over n. For $n=1$ the equality (7.7.2) is clear. Assume that the equality (7.7.2) holds for $\mathbf{j} \in \mathscr{J}_{n-1}$ and let $j_n \in \{1,2\}$ with $j_n \neq j_{n-1}$, $\mathbf{j}' := (\mathbf{j}, j_n)$. One easily checks that the left- and right-hand side of (7.7.2) both vanish on the orthogonal complement of $\mathscr{H}_{\mathbf{j}'}$. Further, for $x_1 \in \ker(\mathbb{E}_{j_1}), \ldots, x_n \in \ker(\mathbb{E}_{j_n})$, we get by the assumption

$$\begin{aligned}&(\Psi_i')^{(2)}(x_1\ldots x_n\Omega_\varphi) = \Psi_i'(x_1\ldots x_{n-1})\Phi'_{i,j_n}(x_n)\Omega_\varphi = \Psi_i'(x_1\ldots x_{n-1})(\Phi'_{i,j_n})^{(2)}(x_n\Omega_\varphi) \\ &= \lim_{l_1,\ldots,l_n\to\infty}\sum_{k_1,\ldots,k_{n-1}} a_{k_1,l_1}^{(i,j_1)}\ldots a_{k_{n-1},l_{n-1}}^{(i,j_{n-1})}\mathbb{E}_N\left(b_{k_{n-1},l_{n-1}}^{(i,j_{n-1})}\ldots b_{k_1,l_1}^{(i,j_1)}x_1\ldots x_{n-1}\right)\left(\sum_{k_n} a_{k_n,l_n}^{(i,j_n)}e_N b_{k_n,l_n}^{(i,j_n)}\right)x_n\Omega_\varphi.\end{aligned}$$

Since the $\Phi'_{i,1}$ and $\Phi'_{i,2}$ are $\mathcal{N}$-$\mathcal{N}$-bimodular, we have $(\Phi'_{i,j_n})^{(2)} \in \mathcal{N}' \cap \langle\mathcal{N},\mathcal{M}\rangle$ and hence

$$(\Psi_i')^{(2)}(x_1\ldots x_n\Omega_\varphi) = \lim_{l_1,\ldots,l_n\to\infty}\sum_{k_1,\ldots,k_n} a_{k_1,l_1}^{(i,j_1)}\ldots a_{k_n,l_n}^{(i,j_n)}\mathbb{E}_{\mathcal{N}}\left(b_{k_n,l_n}^{(i,j_n)}\ldots b_{k_1,l_1}^{(i,j_1)}x_1\ldots x_n\right)\Omega_\varphi,$$

i.e.

$$\sum_{k_1,\ldots,k_n} a_{k_1,l_1}^{(i,j_1)}\ldots a_{k_n,l_n}^{(i,j_n)}e_{\mathcal{N}}b_{k_n,l_n}^{(i,j_n)}\ldots b_{k_1,l_1}^{(i,j_1)} \to (\Psi_i')^{(2)}$$

strongly in $l_1,\ldots,l_n$. The second part of the claim, i.e. (7.7.2), then follows from noticing that

$$\left(\sum_{k_1,\ldots,k_n} a_{k_1,l_1}^{(i,j_1)}\ldots a_{k_n,l_n}^{(i,j_n)}e_{\mathcal{N}}b_{k_n,l_n}^{(i,j_n)}\ldots b_{k_1,l_1}^{(i,j_1)}\right)_{l_1,\ldots,l_n}$$

is a Cauchy sequence (compare with [23, Section 3]).

The (in)equalities (7.7.1) and (7.7.2) in particular imply that $(\Psi_i')^{(2)}$ can be expressed as a norm limit

$$(\Psi_i')^{(2)} = e_{\mathcal{N}} + \lim_{n\to\infty}\sum_{\mathbf{j}\in\mathscr{J}_n}\Psi_i^{(2)}P_{\mathbf{j}}$$

and hence $(\Psi_i')^{(2)} \in \mathcal{K}(\mathcal{M},\mathcal{N},\varphi)$ for all $i \in I$. This finishes the direction "$\Rightarrow$".

"$\Leftarrow$": It suffices to prove the result for $\mathcal{M}_1$. Note first that [21, Lemma 3.5] shows that we have a normal conditional expectation $\mathbb{F}_1 : \mathcal{M} \to \mathcal{M}_1$ such that $\mathbb{E}_1 \circ \mathbb{F}_1 = \mathbb{E}_{\mathcal{N}}$. Hence Lemma 7.4.8 ends the proof. □

In combination with Theorem 7.6.6, Theorem 7.7.1 leads to the following corollary. This generalizes a result by Freslon [83, Theorem 2.3.19] who showed this corollary in the realm of von Neumann algebras of discrete quantum groups, and the analogous property for classical groups was first shown in [115] (see also [7, Section 6]). To the author's best knowledge even for inclusions of finite von Neumann algebras the statement is new.

Corollary 7.7.2. *Let $\mathcal{N} \subseteq \mathcal{M}_1$ and $\mathcal{N} \subseteq \mathcal{M}_2$ be unital embeddings of von Neumann algebras which admit faithful normal conditional expectations $\mathbb{E}_1 : \mathcal{M}_1 \to \mathcal{N}$, $\mathbb{E}_2 : \mathcal{M}_2 \to \mathcal{N}$ and assume that $\mathcal{N}$ is finite-dimensional. Assume moreover that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the Haagerup property. Then the amalgamated free product von Neumann algebra $(\mathcal{M}_1,\mathbb{E}_1) *_{\mathcal{N}} (\mathcal{M}_2,\mathbb{E}_2)$ has the Haagerup property as well.*

7.8. INCLUSION OF FINITE INDEX

In this section we will discuss finite index inclusions, defined in [11], for not necessarily tracial von Neumann algebras. We will pick one of the (possibly non-equivalent) definitions, which is most suitable in our context, and then we will illustrate this notion using certain compact quantum groups, namely free orthogonal quantum groups.

Definition 7.8.1. Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We say that a family of elements $(m_i)_{i\in I}$ is an *orthonormal basis* of the right $\mathcal{N}$-module $L^2(\mathcal{M})_{\mathcal{N}}$ if:

(1) For each $i, j \in I$ we have $\mathbb{E}_{\mathcal{N}}(m_i^* m_j) = \delta_{ij} p_j$, where p_j is a projection in $\mathcal{N}$;

(2) $\overline{\sum_{i\in I} m_i \mathcal{N}} = L^2(\mathcal{M})$.

We say that the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is *strongly of finite index* if it admits a finite orthonormal basis.

Lemma 7.8.2. *If an inclusion $\mathcal{N} \subseteq \mathcal{M}$ is strongly of finite index then it has the Haagerup property.*

Proof. Let $m_1,\ldots,m_n$ be a finite orthonormal basis for our inclusion. It suffices to show that $x = \sum_{i=1}^n m_i \mathbb{E}_{\mathcal{N}}(m_i^* x)$ for each $x \in \mathcal{M}$. Indeed, this would show that the identity map on $L^2(\mathcal{M})$ is relatively compact with respect to $\mathcal{N}$, so clearly the triple $(\mathcal{M},\mathcal{N},\mathbb{E}_{\mathcal{N}})$ satisfies the relative Haagerup property. The equality $x = \sum_{i=1}^n m_i \mathbb{E}_{\mathcal{N}}(m_i^* x)$ has been already observed by Popa (see [154, Section 1]) in a more general context. □

7.8.1. FREE ORTHOGONAL QUANTUM GROUPS

We will now present a certain inclusion arising in the theory of compact quantum groups that has the relative Haagerup property. For information about compact quantum groups we refer the reader to the excellent book [139].

Definition 7.8.3 ([178])**.** Let $n \geq 2$ be an integer and let $F \in M_n(\mathbb{C})$ be a matrix such that $F\overline{F} = c1$ for some $c \in \mathbb{R} \setminus \{0\}$. Let $\mathrm{Pol}(O_F^+)$ be the universal $*$-algebra generated by the entries of a unitary matrix $U \in M_n(\mathrm{Pol}(O_F^+))$, denoted u_{ij}, subject to the condition $U = F\overline{U}F^{-1}$, where $(\overline{U})_{ij} := (u_{ij})^*$ for all $i, j = 1, \ldots, n$. Then the unique $*$-homomorphic extension of the map $\Delta(u_{ij}) := \sum_{k=1}^n u_{ik} \otimes u_{kj}$ makes $\mathrm{Pol}(O_F^+)$ into a Hopf $*$-algebra, whose universal C*-algebra completion yields a compact quantum group.

Remark 7.8.4. As every compact quantum group admits a Haar state, we can use the GNS-construction to construct a von Neumann algebra $L^\infty(O_F^+)$.

In [12] Banica classified irreducible representations of the compact quantum group O_F^+. He showed that they are indexed by natural numbers, U^k, where U^0 is the trivial representation and $U^1 = U$ is the fundamental representation U. Moreover, the fusion rules satisfied by these representations are the following:

$$U^k \otimes U^l \cong U^{k+l} \oplus U^{k+l-2} \oplus \cdots \oplus U^{|k-l|}, \quad k, l \in \mathbb{N},$$

just like for the classical compact group $SU(2)$. From the fusion rules one can infer that the coefficients of representations indexed by even numbers form a subalgebra. Further, one can use the defining relation $U = F\overline{U}F^{-1}$ to show that they form a $*$-subalgebra.

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Definition 7.8.5. Let $\mathcal{M} := L^\infty(O_F^+)$. We define the *even subalgebra* $\mathcal{N}$ to be the von Neumann subalgebra of $\mathcal{M}$ generated by the elements $(u_{ij}u_{kl})_{1 \leq i,j,k,l \leq n}$. It is equal to the von Neumann algebra generated by the coefficients of the even representations; in fact it is related to the *projective version* of O_F^+, usually denoted PO_F^+.

Remark 7.8.6. It has been shown by Brannan in [28] that $\mathcal{N} \subseteq \mathcal{M}$ is a subfactor of index 2 in case that $F = 1$ (it is then an inclusion of finite von Neumann algebras).

We now roughly outline Brannan's argument and then mention why it cannot immediately be translated into our setting. There is an automorphism Φ of $\mathcal{M}$ such that $\Phi(u_{ij}) = -u_{ij}$; Φ can be first defined on $\mathrm{Pol}(O_F^+)$ by the universal property but it also preserves the Haar state, so can be extended to an automorphism of $L^\infty(O_F^+)$. The fixed point subalgebra of Φ is equal to the even subalgebra $\mathcal{N}$ and therefore $\mathbb{E}_{\mathcal{N}} := \frac{1}{2}(\mathrm{id} + \Phi)$ is a conditional expectation onto $\mathcal{N}$ that preserves the Haar state. As a consequence $\mathbb{E}_{\mathcal{N}} - \frac{1}{2}\mathrm{id}$ is a completely positive map, so one can use the Pimsner-Popa inequality, which works for II$_1$-factors, to conclude that the index of $\mathcal{N} \subseteq \mathcal{M}$ is at most 2. On the other hand, any proper inclusion has index of at least 2, so the result follows. Unfortunately in the non-tracial case it is not clear if the condition that $\mathbb{E}_{\mathcal{N}} - \frac{1}{2}\mathrm{id}$ is completely positive implies that the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is strongly of finite index; so far it is only known that it implies being of finite index in a weaker sense (see [11, Théorème 3.5]). Fortunately, in our case it is possible to explicitly define a finite orthonormal basis.

Proposition 7.8.7. *Let $n \geqslant 2$ be an integer and let $F \in M_n(\mathbb{C})$ be a matrix such that $F\overline{F} = c1$ for some $c \in \mathbb{R} \setminus \{0\}$. Let $\mathcal{M} := L^\infty(O_F^+)$ and let $\mathcal{N}$ be the even von Neumann subalgebra of $\mathcal{M}$. Then the inclusion $\mathcal{N} \subseteq \mathcal{M}$ is strongly of finite index. Moreover, one can find an orthonormal basis consisting of at most $n^2 + 1$ elements.*

Proof. One can verify by an explicit computation that $\mathcal{N}$ is left globally invariant by the modular automorphism group of the Haar state h of $L^\infty(O_F^+)$, so we do have a faithful normal h-preserving conditional expectation $\mathbb{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. We start with $n^2 + 1$ elements of $\mathcal{M}$, namely 1 and all the u_{ij}'s. Since our set contains all the coefficients of the fundamental representation, it follows from the fusion rules of O_F^+ that $\mathcal{N} \oplus \sum_{i,j=1}^n u_{ij}\mathcal{N}$ is a dense submodule of $L^2(\mathcal{M})_{\mathcal{N}}$.

Note that all the elements u_{ij} are odd, i.e. $\Phi(u_{ij}) = -u_{ij}$ for $i, j = 1, \ldots, n$. Suppose that we have a family $x_1, \ldots, x_k$ of odd elements. Then we can perform a Gram-Schmidt process to make this set orthonormal. To do it, first notice that $x_i^* x_i$ is an even element, hence so is $|x_i|$ – we conclude that the partial isometry in the polar decomposition $x_i = v_i|x_i|$ is odd as well. Our process works as follows: we first replace x_1 by the corresponding partial isometry v_1. Then we define $\tilde{x}_2 := x_2 - v_1 v_1^* x_2$. Because v_1 is a partial isometry, we get $v_1^* \tilde{x}_2 = v_1^* x_2 - v_1^* v_1 v_1^* x_2 = 0$. We then define v_2 to be the partial isometry appearing in the polar decomposition of $\tilde{x}_2$; it still holds that v_2 is odd and $v_1^* v_2 = 0$. We can continue this process just like the usual Gram-Schmidt process and obtain an orthonormal set of odd partial isometries v_i such that $\sum_{i=1}^k x_i\mathcal{N} \subseteq \sum_{i=1}^k v_i\mathcal{N}$; note that the projections $v_i^* v_i$ belong to $\mathcal{N}$. If we apply this procedure to the family $(u_{ij})_{1 \leqslant i,j \leqslant n}$, we obtain a finite orthonormal basis for the inclusion $\mathcal{N} \subseteq \mathcal{M}$. □

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Corollary 7.8.8. *The inclusion $\mathcal{N} \subseteq \mathcal{M} := L^\infty(O_F^+)$ has the relative Haagerup property.*

8

Approximation properties of Hecke operator algebras

Approximation properties of C*-algebras and von Neumann algebras play an important role in the theory of operator algebras. The idea of approximating complicated structures by simpler building blocks appears in lots of mathematical fields. The aim of this short chapter is to study such approximation properties of Hecke C*-algebras. More precisely, by using the results from Section 3.3, Subsection 5.2.3 and Section 7.7, we will prove that Hecke C*-algebras are exact, we characterize their nuclearity and we consider classes of Hecke-von Neumann algebras which satisfy the Haagerup property.

The content is based on the articles

- M. Caspers, M. Klisse, N.S. Larsen, *Graph product Khintchine inequalities and Hecke C*-algebras: Haagerup inequalities, (non)simplicity, nuclearity and exactness*, J. Funct. Anal. 280 (2021), no. 1, Paper No. 108795, 41 pp.

- M. Caspers, M. Klisse, A. Skalski, G. Vos, M. Wasilewski, *Relative Haagerup property for arbitrary von Neumann algebras*, arXiv preprint arXiv:2110.15078 (2021).

8.1. APPROXIMATION PROPERTIES OF HECKE OPERATOR ALGEBRAS

Recall that a C*-algebra is called *exact* if spatial tensoring with it preserves exact sequences. By a theorem of Kirchberg this is equivalent to the embeddability of the C*-algebra into a nuclear C*-algebra (which in the separable case can be chosen to be the Cuntz algebra $\mathcal{O}_2$). We make use of this characterization in the proof of the following theorem.

Theorem 8.1.1. *Let (W,S) be a Coxeter system and let $q=(q_s)_{s\in S}\in\mathbb{R}_{>0}^{(W,S)}$. Then $C^*_{r,q}(W)$ is an exact C*-algebra.*

Proof. First assume that (W,S) has finite rank. Recall that $\mathfrak{A}(W,S)$ is the smallest C*-subalgebra of $\mathcal{B}(\ell^2(W))$ that contains all Hecke C*-algebras of the system (W,S) (see Section 5.3), so in particular $C^*_{r,q}(W)\subseteq\mathfrak{A}(W,S)$. By Corollary 5.3.1 $\mathfrak{A}(W,S)$ is nuclear which implies the exactness of $C^*_{r,q}(W)$. If (W,S) has infinite rank, the exactness follows from the above by an inductive limit argument (see Proposition 3.5.3 and [20, IV.3.4.5]). □

The following theorem generalizes [46, Theorem 3.6], where the injectivity of right-angled Hecke-von Neumann algebras was characterized. By comparing it with Theorem 2.3.8 one confirms that no unexpected behaviour occurs. Recall that a von Neumann algebra $\mathcal{M}\subseteq\mathcal{B}(\mathcal{H})$ is called *injective* if there exists a conditional expectation $\mathbb{E}_{\mathcal{M}}:\mathcal{B}(\mathcal{H})\to\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$.

Theorem 8.1.2. *Let (W,S) be an irreducible Coxeter system. Then the following statements are equivalent:*

(1) (W,S) is of spherical or affine type;

(2) $C^*_{r,q}(W)$ is nuclear for all $q\in\mathbb{R}_{>0}^{(W,S)}$;

(3) $C^*_{r,q}(W)$ is nuclear for some $q\in\mathbb{R}_{>0}^{(W,S)}$;

(4) $\mathcal{N}_q(W)$ is injective for all $q\in\mathbb{R}_{>0}^{(W,S)}$;

(5) $\mathcal{N}_q(W)$ is injective for some $q\in\mathbb{R}_{>0}^{(W,S)}$.

Proof. "(1) ⇒ (2)": If (W,S) is of spherical type and if $q\in\mathbb{R}_{>0}^{(W,S)}$, then $C^*_{r,q}(W)$ is finite-dimensional or an inductive limit over finite-dimensional C*-algebras (see Proposition 3.5.3), and in either case nuclear (see e.g. [20, II.9.4.5]). So let us assume that (W,S) is of affine type, let W_0 be its corresponding Weyl group (see Subsection 2.7.3) and let $\pi:C^*_{r,q}(W)\to\mathcal{B}(\mathcal{H})$ be an irreducible representation of $C^*_{r,q}(W)$, that is $(\pi(C^*_{r,q}(W)))'=\mathbb{C}1$. Bernstein's decomposition (see Theorem 3.3.1) states that the Iwahori-Hecke algebra $\mathbb{C}_q[W]$ is finitely generated over its center. The representation π maps the center to $\mathbb{C}1$ and therefore $\pi(C^*_{r,q}(W))\subseteq\mathcal{B}(\mathcal{H})$ is finite-dimensional where the dimension is bounded by $\#W_0$. It follows that $\mathscr{H}$ must also be of finite dimension with $\dim(\mathscr{H}) \leq \#W_0$, thus $C^*_{r,q}(W)$ is subhomogeneous. From [33, Proposition 2.7.7] one then deduces that $C^*_{r,q}(W)$ is nuclear.

"(2) ⇒ (3)": Clear.

"(3) ⇒ (5)": If $C^*_{r,q}(W)$ is nuclear for fixed $q \in \mathbb{R}^{(W,S)}_{>0}$, then the bicommutant $\mathcal{N}_q(W) = (C^*_{r,q}(W))''$ must be injective by [33, Exercise 3.6.4, Corollary 3.8.6 and Theorem 9.3.3].

"(5) ⇒ (1)": Assume that (W,S) is of non-affine type. By Proposition 2.7.20, W contains the free group on two generators. Denote the corresponding generators by $\mathbf{a}_1$, $\mathbf{a}_2$ and let $\mathscr{M}$ be the von Neumann subalgebra of $\mathcal{N}_q(W)$ generated by $T^{(q)}_{\mathbf{a}_1}$ and $T^{(q)}_{\mathbf{a}_2}$. Let further $\mathscr{M}_1$ be the von Neumann algebra generated by $T^{(q)}_{\mathbf{a}_1}$ and $\mathscr{M}_2$ the one generated by $T^{(q)}_{\mathbf{a}_2}$. One checks that $\mathscr{M}$ is isomorphic to the free product $(\mathscr{M}_1, \tau_{q,1}) \star (\mathscr{M}_2, \tau_{q,2})$ over the canonical traces. The dimensions of $\mathscr{M}_1$, $\mathscr{M}_2$ are infinite, so $\mathscr{M}$ is non-injective by [177, Theorem 4.1 and Remark 4.2 (5)]. But there exists a trace-preserving normal conditional expectation $\mathbb{E}_{\mathscr{M}} \colon \mathcal{N}_q(W) \to \mathscr{M}$ (see for example [33, Lemma 1.5.11] or [46, Corollary 3.3]), so $\mathcal{N}_q(W)$ must be non-injective as well.

"(2) ⇔ (4)": Clear from the arguments above. □

By exploiting the graph product structure of right-angled Hecke-von Neumann algebras (see Subsection 4.3.1), in [46] Caspers proved the Haagerup property for single-parameter right-angled Hecke-von Neumann algebras. The proof translates verbatim to the multi-parameter setting.

Theorem 8.1.3 ([46, Theorem 3.9])**.** *Let (W,S) be a right-angled Coxeter system and let $q \in \mathbb{R}^{(W,S)}_{>0}$ be a multi-parameter. Then the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ has the Haagerup property.*

By applying the results of Section 7.7 we can complement Caspers' results with Hecke-von Neumann algebras induced by virtually free Coxeter groups.

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Corollary 8.1.4. *Let (W,S) be a Coxeter system, let $q \in \mathbb{R}^{(W,S)}_{>0}$ be a multi-parameter and assume that W is virtually free. Then the corresponding Hecke-von Neumann algebra $\mathcal{N}_q(W)$ has the Haagerup property.*

Proof. Recall that by the discussion in Subsection 2.7.5 the smallest class $\mathscr{G}$ of Coxeter groups which contains all finite rank spherical type Coxeter groups and which is closed under taking amalgamated free products over spherical special subgroups, coincides with the class of all virtually free Coxeter groups. Since injectivity implies the Haagerup property, from Theorem 8.1.2 in combination with Proposition 3.6.5 and Corollary 7.7.2 it therefore follows that $\mathcal{N}_q(W)$ has the Haagerup property. □

9

Fundamental properties of Hecke operator algebras

The notion of smallness at infinity (see Subsection 2.3.2) that we studied in Subsection 5.2.4 in the context of compactifications of Coxeter groups has a number of interesting operator algebraic implications. In this chapter we will apply a method used by Higson and Guentner (see [105]) in the context of word hyperbolic groups, to deduce rigidity properties for Hecke-von Neumann algebras of Coxeter systems which are small at infinity.

The content of this chapter is based on Section 4 of

- M. Klisse, *Topological boundaries of connected graphs and Coxeter groups*, to appear in the Journal of Operator Theory.

9.1. The Akemann-Ostrand property for Hecke-von Neumann algebras

In [105] Higson and Guentner used suitable compactifications (namely hyperbolic compactifications, see Subsection 2.5.2) of groups that are small at infinity in the sense of Definition 2.3.9 to prove that for a word hyperbolic discrete group G the map $C_r^*(G) \odot JC_r^*(G)J \to \mathcal{B}(\ell^2(G))/\mathcal{K}(\ell^2(G))$, $x \otimes y \mapsto xy + \mathcal{K}(\ell^2(G))$ where J denotes the modular conjugation operator is continuous with respect to the minimal tensor norm. The same statement has earlier been shown by Akemann and Ostrand in [1] for free groups by using a different method. The notion of the *property Akemann-Ostrand* (property $(\mathcal{AO})$) was introduced in [148] and was famously applied by

Ozawa to rigidity questions of von Neumann algebras. Variations of property $(\mathscr{AO})$ have later been introduced in [111] and [108]. Ozawa proved in [148] that finite von Neumann algebras that satisfy property $(\mathscr{AO})$ are *solid* in the sense that the relative commutant of any diffuse von Neumann subalgebra is injective. Using the stronger notion of strong solidity, Ozawa and Popa (see [150]) were able to find classes of von Neumann algebras that have no (von Neumann algebraic) Cartan subalgebras. Their approach has been advanced by Popa and Vaes in [156] (see also Chifan-Sinclair [52]). Isono [111] later proved that finite factors with the weak-$*$ completely bounded approximation property that satisfy condition $(\mathscr{AO})^+$ are strongly solid.

Using a method similar to that of Higson and Guentner (see also [46, Section 5]), we prove that Hecke-von Neumann algebras of Coxeter systems that are small at infinity (in the sense of Definition 5.2.10) satisfy Isono's strong condition $(\mathscr{AO})$ (see [108]). The same statement was claimed in [46] in the case of right-angled hyperbolic Coxeter groups. However, the proof presented there contains a gap since for general word hyperbolic Coxeter systems (W,S) the boundary $\partial(W,S)$ does not identify with the hyperbolic boundary $\partial_h W$. We correct it in the case of Coxeter groups which are small at infinity. Further results which go in a similar direction appear in [24].

Definition 9.1.1 ([108, Definition 103])**.** Let $\mathscr{M}$ be a von Neumann algebra and $(\mathscr{M},\mathscr{H},J,\mathfrak{P})$ a standard form for $\mathscr{M}$. We say that $\mathscr{M}$ satisfies the *strong Akemann-Ostrand condition* (*strong condition* $(\mathscr{A}O)$) if there exist unital C^*-subalgebras $A\subseteq\mathscr{M}$, $\mathscr{C}\subseteq\mathscr{B}(\mathscr{H})$ such that:

(1) A is exact and σ-weakly dense in $\mathscr{M}$;

(2) $\mathscr{C}$ is nuclear and contains A;

(3) The set of commutators $[\mathscr{C},JAJ]:=\{[c,JaJ]\mid c\in\mathscr{C},a\in A\}$ is contained in the compact operators $\mathscr{K}$.

Theorem 9.1.2. *Let (W,S) be a finite rank Coxeter system that is small at infinity and let $q\in\mathbb{R}_{>0}^{(W,S)}$ be a multi-parameter. Then the Hecke-von Neumann algebra $\mathscr{N}_q(W)\subseteq\mathscr{B}(\ell^2(W))$ satisfies the strong condition $(\mathscr{AO})$.*

Proof. Set $A:=C^*_{r,q}(W)$ and $\mathscr{C}:=\mathfrak{A}(W,S)$ where $\mathfrak{A}(W,S)$ is defined as in Section 5.3 as the smallest C^*-subalgebra of $\mathscr{B}(\ell^2(W))$ that contains all Hecke C^*-algebras of the system (W,S). The first property of Definition 9.1.1 follows from Theorem 8.1.2. Further, the nuclearity of $\mathscr{C}$ is clear by Corollary 5.3.1 (or follows from Corollary 5.2.3 and Theorem 5.2.15). It remains to show that $[C,JAJ]\subseteq\mathscr{K}(\ell^2(W))$ where $JAJ=C^{*,r}_{r,q}(W)$. Note that $\mathfrak{A}(W,S)$ is the unital C^*-subalgebra of $\mathscr{B}(\ell^2(W))$ generated by all operators $T_s^{(1)}$, $s\in S$ and $P_{\mathbf{w}}$, $\mathbf{w}\in W$. One can further write $T_s^{(q),r}=T_s^{(1),r}+p_s(q)P_s^r$ for all $s\in S$ where $P_s^r\in\mathscr{B}(\ell^2(W))$ is the orthogonal projection onto the subspace

$$\overline{\mathrm{Span}\{\delta_{\mathbf{w}}\mid \mathbf{w}\in W \text{ with } s\leq\mathbf{w}^{-1}\}}$$

of $\ell^2(W)$. We assumed W to be small at infinity, so by the discussion in Subsection 2.3.2,

$$[T_s^{(1)}, P_{\mathbf{w}}^r] \in \mathcal{K}(\ell^2(W)) \quad \text{and} \quad [P_{\mathbf{w}}, T_s^{(1),r}] \in \mathcal{K}(\ell^2(W))$$

for $s \in S$, $\mathbf{w} \in W$ and further

$$\left[T_s^{(1)}, T_t^{(1),r}\right] = [P_{\mathbf{v}}, P_{\mathbf{w}}^r] = 0$$

for all $s, t \in S$ and $\mathbf{v}, \mathbf{w} \in W$. Therefore, $[C, JAJ] \subseteq \mathcal{K}(\ell^2(W))$ which finishes the proof. □

Remark 9.1.3. Theorem 9.1.2 implies in combination with [108, Remark 2.7] that Hecke-von Neumann algebras of Coxeter systems that are small at infinity satisfy Ozawa's property $(\mathcal{AO})$ (see [148]) and Isono's property $(\mathcal{AO})^+$ (see [111]). Hence, we get from [148, Theorem 6] that these von Neumann algebras are solid, meaning that the relative commutant of any diffuse von Neumann subalgebra is injective. Further, if the Hecke-von Neumann algebra is a II_1-factor satisfying the weak-$*$ completely bounded approximation property, then [111, Theorem A] implies that it is strongly solid. The results in [111] rely on [156] and [150]. For right-angled Hecke-von Neumann algebras the weak-$*$ completely bounded approximation property has been studied in [46].

As discussed in Remark 3.5.7, Dykema's interpolated free group factors $\mathcal{L}(\mathbb{F}_t)$, $t \in \mathbb{R}_{>1}$ (cf. [73], [159]) can be realized as Hecke-von Neumann algebras of free products of finite right-angled Coxeter groups. Ozawa and Popa showed in [150] that the interpolated free group factors are strongly solid which strengthens earlier indecomposability results by Voiculescu [181] and Ozawa [148].

The following corollary is an immediate consequence of Proposition 5.2.16, Theorem 9.1.2 and the discussion above. The statement is known to experts.

Corollary 9.1.4. *For every $t \in \mathbb{R}_{>1}$ the interpolated free group factor $\mathcal{L}(\mathbb{F}_t)$ satisfies the strong condition $(\mathcal{AO})$.*

In the context of group algebras, property $(\mathcal{AO})$ has a number of interesting applications (see for instance [4]). In particular, it relates to Connes's notion of fullness, introduced in [59]. Recall that a factor $\mathcal{M}$ is said to be *full* if for every bounded net $(x_i)_{i\in I} \subseteq \mathcal{M}$ with $\lim_i \|\varphi(x_i\cdot) - \varphi(\cdot x_i)\| = 0$ for all $\varphi \in \mathcal{M}_*$ there exists a bounded net $(z_i)_{i\in I} \subseteq \mathbb{C}$ with $x_i - z_i \to 0$ in the strong operator topology. In the case of type II_1-factors this definition is equivalent to $\mathcal{M}$ not having Murray and von Neumann's *property Gamma* (see [138]). In [60] Connes proved that a II_1-factor $\mathcal{M}$ is full if and only if $C^*(\mathcal{M}, \mathcal{M}') \cap \mathcal{K}(L^2(\mathcal{M})) \neq 0$ where $L^2(\mathcal{M})$ denotes the GNS-space corresponding to the tracial state of $\mathcal{M}$.

Compare the following proposition with the results in [166] and [4]. The proof is close to [4, Proposition 6.19].

Proposition 9.1.5. *Let (W, S) be a finite rank non-amenable Coxeter system which is small at infinity and let $q \in \mathbb{R}_{>0}^{(W,S)}$. Then,*

$$C^*\left(C_{r,q}^*(W), C_{r,q}^{*,r}(W)\right) \cap \mathcal{K}(\ell^2(W)) \neq 0.$$

9

If the corresponding Hecke-von Neumann algebra is a II_1*-factor, then* $\mathcal{N}_q(W)$ *is full and*

$$C^*_{r,q}(W) \otimes C^{*,r}_{r,q}(W) \cong \pi\left(C^*\left(C^*_{r,q}(W), C^{*,r}_{r,q}(W)\right)\right).$$

Proof. Set $\mathscr{A} := C^*(C^*_{r,q}(W), C^{*,r}_{r,q}(W)) \subseteq \mathscr{B}(\ell^2(W))$. By (the proof of) Theorem 9.1.2 the map $C^*_{r,q}(W) \odot C^{*,r}_{r,q}(W) \to \mathscr{B}(\ell^2(W))/\mathscr{K}(\ell^2(W))$ given by $x \otimes y \mapsto xy + \mathscr{K}(\ell^2(W))$ is continuous with respect to the minimal tensor norm. Denote the corresponding extension by ρ. Let $\mu : C^*_{r,q}(W) \otimes_{\max} C^{*,r}_{r,q}(W) \to \mathscr{B}(\ell^2(W))$ and $Q : C^*_{r,q}(W) \otimes_{\max} C^{*,r}_{r,q}(W) \to C^*_{r,q}(W) \otimes C^{*,r}_{r,q}(W)$ be the canonical maps. Then, $\pi \circ \mu = \rho \circ Q$. Since by our assumption W is non-affine one can find an element $x \in C^*_{r,q}(W) \otimes_{\max} C^{*,r}_{r,q}(W)$ with $\mu(x) \neq 0$ and $Q(x) = 0$. Indeed, if no such element exists then $\ker(Q) \subseteq \ker(\mu)$ and therefore the map $C^*_{r,q}(W) \odot C^{*,r}_{r,q}(W) \to \mathscr{B}(\ell^2(W))$ given by $T^{(q)}_{\mathbf{v}} \otimes T^{(q),r}_{\mathbf{w}} \mapsto T^{(q)}_{\mathbf{v}} T^{(q),r}_{\mathbf{w}}$ is continuous with respect to the minimal tensor norm. With [33, Theorem 6.2.7] and Theorem 8.1.2 this leads to a contradiction. So let x be an element with $\mu(x) \neq 0$, $Q(x) = 0$. Then, $0 \neq \mu(x) \in \mathscr{A} \cap \mathscr{K}(\ell^2(W))$ because $\pi \circ \mu(x) = \rho \circ Q(x) = 0$.

In the case of a II_1-factor the fullness of $\mathcal{N}_q(W)$ follows from the discussion above. For the deduction of the existence of the isomorphism it suffices to show that ρ is isometric. It is well-known (see for instance [70, Cor. 4.1.10]) that a C*-algebra acting irreducibly on a Hilbert space $\mathscr{H}$ that intersects non-trivially with the compact operators on $\mathscr{H}$ contains all compact operators. Since by the factoriality of $\mathcal{N}_q(W)$ the commutant of $\mathscr{A}$ is trivial, one hence gets that $\mathscr{K}(\ell^2(W)) \subseteq \mathscr{A}$. We claim that

$$|||\cdot||| : \mathcal{N}_q(W) \odot \mathcal{N}_q(W) \to \mathbb{R}_+, \ \sum_i x_i \otimes y_i \mapsto \left\| \sum x_i (J y_i J) + \mathscr{K}(\ell^2(W)) \right\|$$

defines a C*-norm on $\mathcal{N}_q(W) \odot \mathcal{N}_q(W)$ where J is the modular conjugation operator. Indeed, the only property that is not obvious is the definiteness of $|||\cdot|||$. It follows from the fact that the norm closure of

$$\left\{x \in \mathcal{N}_q(W) \odot \mathcal{N}_q(W) \mid |||x||| = 0\right\} \subseteq \mathcal{N}_q(W) \odot \mathcal{N}_q(W)$$

is an ideal in $\mathcal{N}_q(W) \otimes \mathcal{N}_q(W)$, that $\mathcal{N}_q(W)$ is a II_1-factor (i.e. simple as a C*-algebra) and that the spatial tensor product of two simple C*-algebras is simple. Hence, $|||\cdot|||$ defines a C*-norm. In particular, it majorizes the minimal tensor norm on $\mathcal{N}_q(W) \odot \mathcal{N}_q(W)$ so ρ is indeed isometric. □

9

We have already seen that a complete classification of Coxeter systems (and ranges of multi-parameters q) that give rise to Hecke-von Neumann algebras which are II_1-factors is still an open problem where partial results have been obtained in [86] and [160]. Considering Proposition 9.1.5, a factoriality result would be particularly interesting in the case of systems which are small at infinity. We close this chapter with the following proposition which treats a similar question.

Proposition 9.1.6. *Let* (W, S) *be a finite rank Coxeter system that is small at infinity. Then* $C^*_{r,q}(W) \cap C^{*,r}_{r,q}(W) = \mathbb{C}1$ *for every* $q \in \mathbb{R}^{(W,S)}_{>0} \setminus \mathscr{R}'(W,S)$ *where* $\mathscr{R}'(W,S)$ *is defined as in Subsection 2.7.7.*

Proof. Let $q \in \mathbb{R}_{>0}^{(W,S)} \setminus \mathscr{R}'(W,S)$ and $x \in C_{r,q}^{*}(W) \cap C_{r,q}^{*,r}(W)$. By the same argument as in the proof of Theorem 9.1.2 we have that for every $y \in \mathfrak{A}(W,S)$ the commutator $[x,y]$ is a compact operator in $\mathscr{B}(\ell^2(W))$. This implies that $\pi(x)$ is in the center of $\pi(\mathfrak{A}(W))$. But by Corollary 5.3.2 the C*-algebra $\pi(\mathfrak{A}(W,S))$ is simple, so in particular its center must be trivial. We deduce that $\pi(x) \in \mathbb{C}1$. Since by Corollary 5.3.7 the quotient map $\pi : \mathscr{B}(\ell^2(W)) \to \mathscr{B}(\ell^2(W))/\mathscr{K}(\ell^2(W))$ restricts to an embedding of $C_{r,q}^{*}(W)$ into $\pi(\mathfrak{A}(W))$, we get that $x \in \mathbb{C}1$. □

10

Open problems and questions

In this final part of the dissertation we collect some open problems and questions which arise from our studies in the previous chapters. These are meant to guide potential future research on Hecke operator algebras.

Haagerup type inequalities

In Section 4.3 we proved that every right-angled Hecke C^*-algebra admits a Haagerup type inequality which estimates the norm of an operator of length n by its 2-norm up to a polynomial (even linear) bound depending on n. A similar inequality appears in [80, Theorem 1] in the context of group C^*-algebras of Coxeter groups. It is natural to ask whether such Haagerup type inequalities can be found for general (not necessarily right-angled) Hecke C^*-algebras.

Question. Does an analogue of the Haagerup inequality in Theorem 4.3.6 hold for arbitrary finite rank Coxeter systems?

Boundary actions

In Subsection 5.2.5 we studied Furstenberg's notion of boundary actions in the context of boundaries of Coxeter groups and characterized the minimality, the strong proximality and the topological freeness of the canonical action in the case of right-angled Coxeter groups and in the case of Coxeter groups which are small at infinity (in the sense of Definition 5.2.10). Similar arguments were used in Section 6.2 to characterize the simplicity of right-angled Hecke C^*-algebras. As discussed before, the reduced group C^*-algebra of a Coxeter group is simple (and has a unique tracial state) if and only if the corresponding Coxeter group only has non-affine factors. In combination with [120, Theorem 6.2] (and the results in Section 6.2) this suggests that in the irreducible case the canonical action of a Coxeter group on its boundary is a topologically free boundary action if and only if the Coxeter group is of non-affine type. Proposition 5.2.30 further suggests that the action of a Coxeter group on its boundary is always minimal.

Question. Let (W,S) be a Coxeter system. Can we characterize the minimality of the canonical action $W \curvearrowright \partial W$?

Question. Let (W,S) be a Coxeter system. Can we characterize the strong proximality of the canonical action $W \curvearrowright \partial W$?

Question. Let (W,S) be a Coxeter system. Can we characterize the topological freeness of the canonical action $W \curvearrowright \partial W$?

INJECTIVE ENVELOPES

In Section 5.3 we briefly discussed injective envelopes of Hecke C*-algebras and proved that for Coxeter systems (W,S) whose boundary $\partial(W,S)$ defines a W-boundary in the sense of Furstenberg the injective envelopes of the corresponding Hecke C*-algebras embed into the injective envelope of $C_r^*(W)$. As discussed in Remark 5.3.9, considering Ozawa's conjecture and its implications (see Section 5.3), it would be interesting to know if $I(C_{r,q}^*(W))$, $q \in \mathscr{R}'(W,S)$ does not depend on the choice of q.

Question. Let (W,S) be a Coxeter system. Does for $q \in \mathscr{R}'(W,S)$ the injective envelope $I(C_{r,q}^*(W))$ of the Hecke C*-algebra $C_{r,q}^*(W)$ depend on q?

CENTRAL PROJECTIONS AND IDEAL STRUCTURE

It is not difficult to show that central projections of group von Neumann algebras of discrete groups are already contained in the corresponding reduced group C*-algebra. By Proposition 6.1.2 a similar phenomenon occurs in the case of right-angled Hecke-von Neumann algebras. Especially with respect to the open question for a characterization of the factoriality of general Hecke-von Neumann algebras and concerning Proposition 9.1.6 answering the following question would be relevant.

Question. Let (W,S) be a Coxeter system, let $q \in \mathbb{R}_{>0}^{(W,S)}$ and let $P \in \mathcal{Z}(\mathcal{N}_q(W))$ be a central projection. Is P then already contained in the Hecke C*-algebra $C_{r,q}^*(W)$?

In the case of a right-angled, finite rank Coxeter system (W,S) with $\#S \geq 3$ the answer to the question above leads to a decomposition of the Hecke C*-algebra $C_{r,q}^*(W)$, $q \in \mathbb{R}_{>0}^{(W,S)}$ of the form $C_{r,q}^*(W) \cong \pi(C_{r,q}^*(W)) \oplus \bigoplus_{\epsilon \in \{-1,1\}^{(W,S)}:\, |q_\epsilon| \in \mathscr{R}'(W,S)} \mathbb{C}$ which is analogous to the one in the von Neumann algebraic setting, see Corollary 6.1.4.

It has a flavour similar to Dykema's decomposition of certain free product C*-algebras in Proposition 6.1.8. It would be interesting to characterize the simplicity of $\pi(C^*_{r,q}(W))$, as this would lead to a complete description of the ideal structure of $C^*_{r,q}(W)$ in the right-angled case.

Question. Let (W,S) be an irreducible, right-angled, finite rank Coxeter system with $\#S \geq 3$ and let $q \in \mathbb{R}^{(W,S)}_{>0}$. Denote by $\pi : \mathscr{B}(\ell^2(W)) \to \mathscr{B}(\ell^2(W))/\mathscr{K}(\ell^2(W))$ the quotient map. Is it then true that $\pi(C^*_{r,q}(W))$ is simple for all $q \in \mathbb{R}^{(W,S)}_{>0} \setminus \overline{\mathscr{R}'(W,S)}$?

Simplicity

With respect to the general setting it is of course of interest to characterize the simplicity and the unique trace property of arbitrary Coxeter systems.

Question. Let (W,S) be an irreducible, finite rank Coxeter system and $q \in \mathbb{R}^{(W,S)}_{>0}$. Is it then true that the corresponding Hecke C*-algebra $C^*_{r,q}(W)$ is simple and has unique tracial state if and only if $q \in \mathbb{R}^{(W,S)}_{>0} \setminus \overline{\mathscr{R}'(W,S)}$? Can the trace-uniqueness be obtained from similar methods as in Section 6.2?

Haagerup property

In Chapter 7 we proved that in the case where the smaller von Neumann algebra is finite, the (generalized) relative Haagerup property (see Definition 7.2.2) does not depend on the choice of the conditional expectation, i.e. the relative Haagerup property is an invariant of the corresponding inclusion. The finiteness assumption is crucial in our proof since it allows to apply modular theory to the problem and to pass over into a finite setting. A natural question is whether or not the finiteness assumption can be dropped.

Question. Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of von Neumann algebras and let $\mathbb{E}_{\mathcal{N}}, \mathbb{F}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ be two faithful normal conditional expectations. Is it true that the triple $(\mathcal{M}, \mathcal{N}, \mathbb{E}_{\mathcal{N}})$ has Haagerup property if and only if the triple $(\mathcal{M}, \mathcal{N}, \mathbb{F}_{\mathcal{N}})$ does?

From the results in Chapter 7 we deduced in Chapter 8 that Hecke-von Neumann algebras associated with Coxeter groups which are virtually free have the Haagerup property. This complements results by Caspers who proved in [46] that right-angled Hecke-von Neumann algebras also share this property. It is further known that group von Neumann algebras of arbitrary Coxeter groups have the Haagerup property.

Question. Let (W,S) be a Coxeter system let $q \in \mathbb{R}^{(W,S)}_{>0}$. Does $C^*_{r,q}(W)$ have the Haagerup property?

PROPERTY ($\mathcal{AO}$)

In Chapter 8 we saw that the notion of smallness at infinity of a Coxeter system can be used to deduce the Akemann-Ostrand property of the corresponding Hecke-von Neumann algebras. The argument breaks down as soon as the system is not small at infinity anymore. However, the Definition 9.1.1 is flexible in the sense that the unital C*-algebras A and $\mathscr{C}$ can be chosen very freely. It is therefore natural to ask if the Hecke-von Neumann algebras of the system might still satisfy the strong condition ($\mathcal{AO}$).

Question. Let (W,S) be a Coxeter system and let $q \in \mathbb{R}_{>0}^{(W,S)}$. Can we characterize when the Hecke-von Neumann algebra $\mathcal{N}_q(W)$ satisfies the (strong) condition ($\mathcal{AO}$)?

DISTINGUISHING HECKE OPERATOR ALGEBRAS

We finish this collection of problems by getting back to the free factor problem. Recall that for every Coxeter group $W = \mathbb{Z}_2^{*l}$, $l \geq 3$ and $q \in [(l-1)^{-1}, 1]$ the II$_1$-factor $\mathcal{N}_q(W)$ is isomorphic to $\mathscr{L}(\mathbb{F}_{2lq(1+q)^{-2}})$ where $\mathscr{L}(\mathbb{F}_t)$, $t \in \mathbb{R}_{>1}$ denotes Dykema's interpolated free group factors. Further recall that the interpolated free group factors are either all isomorphic or they are all non-isomorphic and that the problem which of the two in this dichotomy is true is the free factor problem. The distinction of different Hecke-von Neumann algebras is hence closely related to this major open problem in the field of operator algebras. But also on the C*-algebraic level a distinction is of huge interest, see Lemma 3.5.1. In this setting partial results have been obtained by Raum and Skalski in [161].

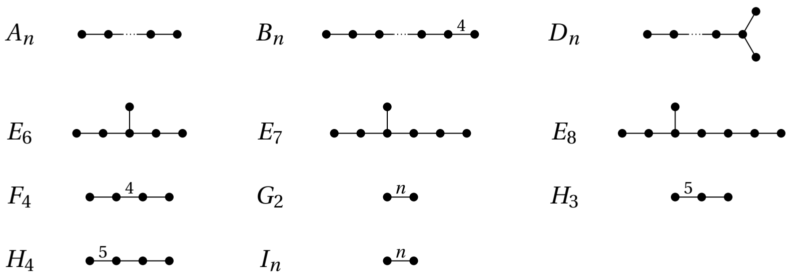
Question. Can one distinguish different Hecke operator algebras from each other?

A

Appendix

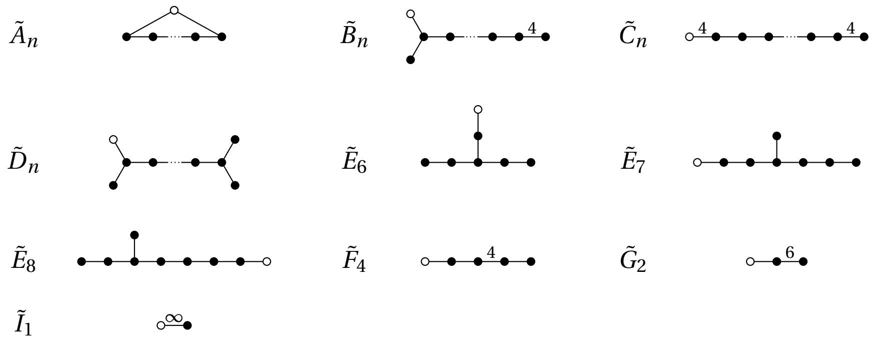
Classification of spherical Coxeter groups

Spherical type Coxeter systems are entirely classified by their Coxeter diagrams. In the following we list all Coxeter diagrams of irreducible spherical type Coxeter systems.



Classification of affine Coxeter groups

Similar to spherical type Coxeter systems, also affine type Coxeter systems are entirely classified by their Coxeter diagrams. In the following we list all Coxeter diagrams of irreducible affine type Coxeter systems.



$\tilde{A}_n$
$\tilde{B}_n$
4
$\tilde{C}_n$
4
4
$\tilde{D}_n$
$\tilde{E}_6$
$\tilde{E}_7$
$\tilde{E}_8$
$\tilde{F}_4$
4
$\tilde{G}_2$
6
$\tilde{I}_1$
∞

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Index

SUMMARY

This dissertation is concerned with the study of the structure of certain deformations of operator algebras associated with Coxeter groups. These operator algebras, called Hecke C^*-algebras and Hecke-von Neumann algebras, are operator algebraic completions of Iwahori-Hecke algebras. They occur as natural abstractions of certain endomorphism rings occurring in the representation theory of Lie groups and play a role in knot theory, combinatorics, the theory of buildings, quantum group theory, non-commutative geometry, and the local Langlands program. In this thesis we mainly focus on the ideal structure of Hecke C^*-algebras, on approximation properties, and the rigidity of Hecke-von Neumann algebras. On our way we encounter and study several other concepts such as (Khintchine inequalities of) graph products of operator algebras, topological dynamics associated with boundaries and compactifications of graphs and (Coxeter) groups, C^*-simplicity methods, the relative Haagerup property of σ-finite unital inclusions of von Neumann algebras, approximation properties of operator algebras, and the rigidity theory of von Neumann algebras.

The thesis consists of ten chapters. In the first one, we present the background material on C^*-algebras and von Neumann algebras, dynamical systems and crossed product operator algebras, partially ordered sets, graphs, trees, Gromov's notion of hyperbolicity, and Coxeter groups, which is necessary for the later chapters.

In Chapter 3 we begin with a detailed exposition to Iwahori-Hecke algebras and their operator algebraic analogs. Many of the results in this chapter are not new but we include them for the convenience of the reader. This in particular concerns Section 3.2 and Section 3.3 where we discuss isomorphism classes of Iwahori-Hecke algebras associated with finite Coxeter groups and the Bernstein decomposition of Iwahori-Hecke algebras of affine type. In Section 3.5 we prove isomorphism results for certain Iwahori-Hecke algebras, discuss their relevance for the free factor problem and deduce amalgamated free product decompositions in Section 3.6.

In Chapter 4 we study Caspers and Fima their operator algebraic analog to Green's graph products of groups by deducing a Khintchine type inequality for graph product C^*-algebras. This inequality estimates the operator norm of an operator of a given length with the norm of certain Haagerup tensor products of column and row Hilbert spaces and generalizes a result by Ricard and Xu in the context of free products of C^*-algebras. By decomposing Hecke C^*-algebras of right-angled Coxeter groups as graph products of two-dimensional Hecke C^*-algebras, we obtain a Haagerup type inequality which estimates the norm of an operator of length n by its 2-norm up to a polynomial bound depending on n.

In Chapter 5 we prepare a new dynamical approach to the study of Hecke C^*-algebras by defining and exploring natural boundaries and compactifications associated with connected rooted graphs. Our construction, which reflects combinatorial and order theoretic properties of the underlying graph, covers several interesting examples and (for hyperbolic graphs) nicely relates to Gromov's construction of hyperbolic compactifications and boundaries. Of particular interest is the case of Cayley graphs of Coxeter systems for which the natural action of the Coxeter group on its Cayley graph extends to a continuous action on the compactification and the boundary. We prove that these actions are amenable, we characterize when the action on the compactification is small at infinity, and we find classes of Coxeter groups for which the action on the boundary is a boundary action in the sense of Furstenberg. We further identify our construction with Caprace-Lécureux's combinatorial compactification as well as Gromov's horofunction compactification. In Section 5.3 we then build a bridge to the Hecke operator algebra setting by embedding (suitable) Hecke C^*-algebras into the C^*-algebraic crossed products associated with the Coxeter group its boundary and compactification. This will allow us to apply geometrical ideas to the study of these operator algebras.

By employing the dynamical approach from Chapter 5, in Chapter 6 we initiate the study of the ideal structure and the trace-uniqueness of Hecke C^*-algebras. We begin by studying central projections in Hecke-von Neumann algebras and show, by using the Haagerup type inequality obtained in Chapter 4, that in the right-angled case these are already contained in the corresponding Hecke C^*-algebras. If present these projections induce characters on the corresponding Hecke C^*-algebra from which we deduce non-simplicity results. By studying the crossed product embeddings from Chapter 5 we obtain a complete characterization of the simplicity of right-angled Hecke C^*-algebras. Further, again with the help of the Haagerup inequality from Chapter 4 as well as results by Dykema, we obtain partial answers regarding the trace-uniqueness of right-angled Hecke C^*-algebras.

In Chapter 7 we step back from the study of Hecke operator algebras to introduce a notion of the relative Haagerup property for general expected unital inclusions of σ-finite von Neumann algebras. We prove that our definition, which involves the choice of a state, rather depends on the conditional expectation of the inclusion than the state in question. In the case where the smaller von Neumann algebra is finite, we even prove that the relative Haagerup property does not depend on the choice of the conditional expectation. Further, several variations of the definition are shown to be equivalent in this case, and in particular, the approximating maps may be chosen to be unital and preserving the reference state. The concept is then applied to amalgamated free products of von Neumann algebras and used to deduce that the standard Haagerup property for a von Neumann algebra is stable under taking free products with amalgamation over finite-dimensional subalgebras. The general results are illustrated by examples coming from von Neumann algebras of free orthogonal quantum groups.

In Chapter 8 and Chapter 9 we pick up the results of the earlier chapters by deducing approximation properties for Hecke operator algebras (such as exactness, nuclearity and the Haagerup property) as well as rigidity properties for Hecke-von Neumann algebras. Our findings in Chapter 9 in particular concern Dykema's interpolated free group factors which are shown to satisfy the Akemann-Ostrand property.

The last chapter is devoted to the formulation of questions that naturally arise in the context of this thesis. It is meant to guide potential future research on Hecke operator algebras.

Samenvatting

Dit proefschrift gaat over deformaties van operatoralgebra's die op een natuurlijke manier geconstrueerd worden uit een Coxetergroep. Deze operatoralgebra's, genaamd Hecke-C*-algebra's en Hecke-von Neumann-algebra's, zijn operatoralgebraïsche afsluitingen van Iwahori-Hecke-algebra's. Ze komen voor als abstracte endomorfismeringen in de representatietheorie van Lie groepen. Deze algebra's spelen een belangrijke rol in knopentheorie, combinatoriek, de theorie van gebouwen (Engels: "buildings"), kwantumgroepen, niet-commutatieve meetkunde en het lokale Langlands programma. In dit proefschrift ligt de voornaamste focus op de structuur van de idealen van Hecke-C*-algebra's, approximatie-eigenschappen en rigiditeit van Hecke-von Neumann-algebra's. Tegelijkertijd onderzoeken we andere concepten zoals Khintchine-ongelijkheden van graafproducten, topologische dynamica van randen en compactificaties van grafen en Coxetergroepen, methoden om C*-simpliciteit te bewijzen, de relatieve Haagerupeigenschap van σ-eindige von Neumann-algebra's, approximatie-eigenschappen van operatoralgebra's en rigiditeitstheorie van von Neumann-algebra's.

Dit proefschrift bestaat buiten de introductie uit negen hoofdstukken. Hoofdstuk 2 bevat achtergrondmateriaal over C*-algebra's, von Neumann-algebra's, dynamische systemen, gekruisde producten, partieel geordende verzamelingen, grafen, bomen, Gromov's notie van hyperboliciteit en Coxetergroepen.

In Hoofdstuk 3 geven we een meer gedetailleerde beschrijving van Iwahori-Hecke-algebra's en de geassocieerde operatoralgebra's. De meeste resultaten in dit hoofdstuk komen uit de literatuur en zijn toegevoegd om de thesis toegankelijk voor de lezer te maken. Dit betreft voornamelijk Hoofdstukken 3.2 en 3.3 die gaan over isomorfismen van Iwahori-Hecke-algebra's van eindige Coxetergroepen en de Bernsteindecompositie van Iwahori-Hecke-algebra's van affien type. In Hoofdstuk 3.5 bewijzen we isomorfisme-resultaten voor bepaalde Iwahori-Hecke-algebra's. We beschouwen hun relatie en relevantie voor het vrije factoren probleem (in het Engels bekend als het "free factor problem"). We bewijzen ook een decompositiestelling in termen van geamalgameerde vrije producten in Hoofdstuk 3.6.

In Hoofdstuk 4 bestuderen we de constructie van Caspers en Fima van operatoralgebraïsche graafproducten; parallel aan de constructie van Green voor groepen. We bewijzen een Khintchine-ongelijkheid voor C*-algebra's van graafproducten. Deze ongelijkheid schat de operatornorm van een operator van zekere lengte af met de norm van zekere Haageruptensorproducten van kolom- en rij-Hilbertruimten. Dit generaliseert een resultaat van Ricard en Xu voor vrije producten van C*-algebra's. Door Hecke-C*-algebra's van rechthoekige (Engels: "right-angled") Coxetergroepen te ontbinden als graafproducten van tweedimensionale

Hecke-C*-algebra's vinden we een Haagerupongelijkheid die de norm van een operator van lengte n afschat met zijn 2-norm tot op een polynomiale grens die afhangt van n.

In Hoofdstuk 5 intoduceren we een nieuwe dynamische aanpak voor de studie van Hecke-C*-algebra's. We definiëren en bestuderen daarbij natuurlijke randen en compactificaties geassocieerd met samenhangende gewortelde grafen. Onze constructie reflecteert combinatorische en ordeningseigenschappen van de onderliggende graaf en is toepasbaar op verschillende belangrijke voorbeelden. Het relateert (voor hyperbolische) grafen aan Gromov's constructie van hyperbolische compactificaties en randen. Van bijzonder belang is het geval van Cayleygrafen van Coxetersystemen waarvoor de natuurlijke actie van de Coxetergroep op de Cayleygraaf uitbreidt naar een continue actie op de compactificatie en de rand. We bewijzen dat deze acties amenabel zijn, we karakteriseren wanneer de actie op de compactificatie klein is op oneindig en we vinden klassen van Coxetergroepen waarvoor de actie op de rand een echte randactie is in de zin van Furstenberg. We identificeren in speciale gevallen onze constructie met andere compactificaties gegeven door Caprace-Lécureux en Gromov. Hierdoor kunnen we operatoralgebra's met meetkundige technieken bestuderen. In Sectie 5.3 maken we een bruggetje naar Hecke-operatoralgebra's door (geschikte) Hecke-C*-algebraísche gekruisde product geassocieerd met de rand en compactificatie van de Coxetergroup.

Met behulp van de dynamische concepten van Hoofdstuk 5, bestuderen we in Hoodstuk 6 de ideaalstructuur en uniciteit van het spoor van Hecke-C*-algebra's. We beginnen met het bestuderen van centrale projecties in Hecke-von Neumann-algebra's en laten zien, door gebruik te maken van de Haagerupongelijkheden van Hoofdstuk 4, dat in het rechthoekige geval deze bevat zijn in de corresponderende Hecke-C*-algebra's. Als deze projecties bestaan dan leggen ze karakters op de corresponderende Hecke-C*-algebra's vast. Dit impliceert vervolgens dat de C*-algebra's niet simpel zijn. Door het bestuderen van de inbeddingen van gekruisde producten van Hoofdstuk 5 verkrijgen we een complete karakterisatie van simpele rechthoekige Hecke-C*-algebra's. Verder gebruiken we de Haagerupongelijkheden samen met resultaten van Dykema om deelresultaten te verkrijgen over de uniciteit van het spoor op rechthoekige Hecke-C*-algebra's.

In Hoofdstuk 7 bestuderen we de relatieve Haagerupeigenschap. We doen dit in de context van algemene σ-eindige von Neumann-algebra's met een unitale deelalgebra die een conditionele verwachting toelaten. De relatieve Haagerupeigenschap is gedefinieerd in termen van een toestand. We laten echter zien dat de definitie niet zozeer van de toestand afhangt, maar alleen van de conditionele verwachtingswaarde. Als de deel-von Neumann-algebra eindig is, dan bewijzen we zelfs dat de relatieve Haagerupeigenschap niet van de conditionele verwachtingswaarde afhangt. We beschouwen ook enkele alternatieve definities en laten zien dat deze equivalent zijn. In het bijzonder volgt dat in de definitie de afbeeldingen die de identiteitsafbeelding benaderen zo gekozen kunnen worden dat ze unitaal zijn en toestandsbewarend. We passen dit toe op geamalgameerde vrije producten van von Neumann-algebra's. Er volgt dat de standaard

Haagerupeigenschap stabiel is onder het nemen van geamalgameerde vrije producten over eindig-dimensionale deelalgebra's. Deze algemene resultaten worden toegepast op voorbeelden van von Neumann-algebra's van vrije orthogonale kwantumgroepen.

In Hoofdstuk 8 en Hoofdstuk 9 bewijzen we approximatie-eigenschappen voor Hecke-operatoralgebra's zoals exactheid, nucleariteit en de Haagerupeigenschap. We verkrijgen ook rigiditeitseigenschappen. Onze resultaten hebben toepassing op Dykema's geinterpoleerde vrije groepsfactoren waarvan we bewijzen dat ze de Akemann-Ostrandeigenschap hebben.

In het laatste hoofdstuk sommen we enkele vragen op die op een natuurlijke manier naar voren komen in de context van deze thesis. Deze zijn bedoeld als startpunt voor toekomstig onderzoek.

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CURRICULUM VITÆ

Mario KLISSE

Mario Klisse was born on December 23, 1992 in Essen, Germany. He completed his secondary education in 2012 at the St. Pius Gymnasium Coesfeld. In the same year he started his studies in Mathematics at the Westfälische Wilhelms-Universität Münster, obtaining his BSc degree and MSc degree in 2015 and 2018 respectively. He wrote his bachelor's thesis on "*Hardy spaces and Toeplitz operators*" and his master's thesis on "*Cartan subalgebras in* C^**-algebras*", both under the supervision of prof. dr. W. Winter. In 2018 he started his PhD research under the supervision of dr. M.P.T. Caspers and prof. dr. J.M.A.M. van Neerven at the Delft University of Technology. Part of his research was carried out during a three month research visit to prof. dr. A. Skalski and prof. dr. P. Nowak at the Institute of Mathematics of the Polish Academy of Sciencies in Warsaw.

List of Publications

This thesis is based on (parts of) the following five articles:

Published / accepted papers

(P1) M. Borst, M. Caspers, M. Klisse, M. Wasilewski, *On the isomorphism class of q-Gaussian C^*-algebras for infinite variables*, to appear in Proc. Amer. Math. Soc.

(P2) M. Klisse, *Simplicity of right-angled Hecke C^*-algebras*, to appear in Int. Math. Res. Not. IMRN.

(P3) M. Klisse, *Topological boundaries of connected graphs and Coxeter groups*, to appear in the Journal of Operator Theory.

(P4) M. Caspers, M. Klisse, N.S. Larsen, *Graph product Khintchine inequalities and Hecke C*-algebras: Haagerup inequalities, (non)simplicity, nuclearity and exactness*, J. Funct. Anal. (2021) 280, no. 1, 108795.

Submitted preprints

(P5) M. Caspers, M. Klisse, A. Skalski, G. Vos, M. Wasilewski, *Relative Haagerup property for arbitrary von Neumann algebras*, arXiv preprint arXiv:2110.15078 (2021).

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