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Feedback Regulation of Elastically Decoupled Underactuated Soft Robots

Pietro Pustina, Cosimo Della Santina, Alessandro De Luca

***Abstract*—The intrinsically underactuated and nonlinear nature of continuum soft robots makes the derivation of provably stable feedback control laws a challenging task. Most of the works so far circumvented the issue either by looking at coarse fully-actuated approximations of the dynamics or by imposing quasi-static assumptions. In this letter, we move a step in the direction of controlling generic soft robots taking explicitly into account their underactuation. A class of soft robots that have no direct elastic couplings between the dynamics of actuated and unactuated coordinates is identified. Considering the actuated variables as output, we prove that the system is minimum phase. We then propose regulators that implement different levels of model compensation. The stability of the associated closed-loop systems is formally proven by Lyapunov/LaSalle techniques, taking into account the nonlinear and underactuated dynamics. Simulation results are reported for two models of 2D and 3D soft robots.**

***Index Terms*—Modeling, Control, and Learning for Soft Robots; Motion Control; Flexible Robotics**

I. Introduction

CONTINUUM soft robots are mechanical systems whose main body is entirely made of deformable soft materials [1]. This design choice allows safe human-robot interaction and provides to soft robots the ability to exhibit unprecedented adaptation, sensitivity, and agility [2]. However, to deliver on these high expectations, soft robots must be capable at the very least to control their shape in space in a reliable way. This is still an open challenge due to the peculiar dynamic characteristics of such systems.

Deriving exact dynamic equations for soft robots requires continuum mechanics methods (e.g., by Cosserat rod theory [3]), with the constitutive equations given by nonlinear partial differential equations. However, using infinite-dimensional formulations imposes substantial limitations to model-based control methods [4], [5]. To address this issue, researchers have proposed finite-dimensional descriptions of soft robots dynamics based on a discretization of rod models [6]–[9] or on direct volumetric FEM [10]–[12]. These formulations can be made physically consistent and accurate enough. In addition, their structure lends itself more directly to the design of model-based controllers [13].

Researchers have devoted much attention to developing model-based controllers within a purely kinematic approach [14]–[16]. These control laws work in practice when the actuator dynamics is dominant and under quasi-static regimes, e.g., for very lightweight soft robots moving slow. More recently, these hypotheses have been removed and controllers designed using full-fledged dynamic models have been proposed [11], [17], assuming that full actuation is available. However, soft robots are intrinsically underactuated mechanical systems. Fully actuated models are obtained only when considering coarse approximations of the continuum dynamics, leading in turn to possibly erroneous assessments of controller stability and performance. Thus, underactuation must be explicitly taken into account in a more formal control design.

In [18], local stabilization of a robot equilibrium is obtained within a linear approximation of the dynamics. A regulator compensating for higher-order deformation modes has been presented in [8] without a stability analysis. Posture regulation using an energy shaping method is considered in [19], but developed only for a single planar soft segment. An heuristic extension of computed torque to underactuated soft robots is tested by simulation in [20]. Finally, the soft inverted pendulum is proposed as a template model for nonlinear control of soft robots in [21], showing how the unstable equilibrium is stabilized by means of collocated or non-collocated feedback linearization. Underactuation is taken explicitly into account for control purposes only in [8] and [21], where stability analyses are performed for a single soft pendulum with affine or polynomial curvature. As a matter of fact, there is still no feedback control method that allows to formally guarantee closed-loop stability of desired equilibria for general underactuated soft robots.

In this letter, we consider a class of underactuated soft robots that we call *elastically decoupled*, for which there is no direct elastic coupling between actuated and unactuated variables (Sec. II). This class is reasonably large and contains, among others, fine piecewise constant curvature (or strain) discretizations of homogeneous segments [13]. Interestingly, the structure of the dynamic equations is similar to that of robots with flexible links [22], [23], other well-studied underactuated mechanical systems. For elastically decoupled soft robots, we prove first that the zero dynamics of the collocated control problem is stable, i.e., the system is minimum phase (Sec. III). Starting from this basic result, we present in Sec. IV the main contribution of the paper, i.e., a PD regulator with gravity cancellation in the actuated subsystem for (possibly,

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global) asymptotic stabilization. Few variants with different compensation/cancellation of gravity and of other dynamic terms are briefly presented in Sec. V. The stability properties of the various control laws are formally assessed via Lyapunov/LaSalle techniques. In Sec. VI, validation is performed through simulations on two models of 2D and 3D soft robots.

II. Dynamic Model

Consider a generic soft robot. A finite dimensional model for such systems can be derived by means of different discretization approaches. Such models can be obtained through the Euler-Lagrange formalism. It can be shown [13] that they are described by Ordinary Differential Equations (ODEs) of the form

$$B(\theta)\ddot{\theta} + S(\theta,\dot{\theta})\dot{\theta} + E(\theta) + H\theta + F(\theta)\dot{\theta} = A\tau, \tag{1}$$

where $\theta, \dot{\theta}, \ddot{\theta} \in \mathbb{R}^n$ are the vectors of configuration variables and their time derivatives, $B(\theta) > 0$ is the symmetric robot inertia matrix, $S(\theta,\dot{\theta})\dot{\theta}$ collects Coriolis and centrifugal terms, $E(\theta) = (\partial U(\theta)/\partial\theta)^T$ is the gravity vector being $U(\theta)$ the gravitational potential energy of the robot. The terms $H\theta$ and $F(\theta)\dot{\theta}$ model, respectively, elastic and (possibly nonlinear) dissipative effects, with $H > 0$ and $F(\theta) > 0$ both symmetric. Furthermore, the (constant) matrix $A \in \mathbb{R}^{n\times m}$, with $m < n$, projects the actuation torques $\tau \in \mathbb{R}^m$ into the configuration space. Without loss of generality, matrix A is always full column rank.

A. Underactuated model for control design

Model (1) can be conveniently rewritten by separating the dynamic equations of the actuated and unactuated variables. To this end, we introduce the linear change of coordinates $q = T\theta$ with

$$T^T = \begin{pmatrix} A & T_2 \end{pmatrix}, \tag{2}$$

being $T_2 \in \mathbb{R}^{n\times(n-m)}$ any orthogonal complement to A. We will refer with the term *elastically decoupled* to the class of soft robots that have block diagonal stiffness in these new coordinates, i.e., that are represented by the dynamic equations

$$\overbrace{\begin{pmatrix} M_{aa} & M_{au} \\ M_{ua} & M_{uu} \end{pmatrix}}^{M(q)} \begin{pmatrix} \ddot{q}_a \\ \ddot{q}_u \end{pmatrix} + \overbrace{\begin{pmatrix} C_{aa} & C_{au} \\ C_{ua} & C_{uu} \end{pmatrix}}^{C(q,\dot{q})} \begin{pmatrix} \dot{q}_a \\ \dot{q}_u \end{pmatrix} + \overbrace{\begin{pmatrix} G_a \\ G_u \end{pmatrix}}^{G(q)}$$
$$+ \underbrace{\begin{pmatrix} K_{aa} & 0 \\ 0 & K_{uu} \end{pmatrix}}_{K} \begin{pmatrix} q_a \\ q_u \end{pmatrix} + \underbrace{\begin{pmatrix} D_{aa} & D_{au} \\ D_{ua} & D_{uu} \end{pmatrix}}_{D(q)} \begin{pmatrix} \dot{q}_a \\ \dot{q}_u \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}, \tag{3}$$

where $q_a \in \mathbb{R}^m$ and $q_u \in \mathbb{R}^{n-m}$ denote, respectively, the actuated and unactuated variables, and the dynamic terms have been partitioned accordingly, omitting the dependence for the ease of reading. We assume the coordinate transformation (2) is such that the symmetric stiffness matrix $K = T^{-T}HT^{-1} > 0$ takes on the elastically decoupled form in (3) with zero off-diagonal blocks. Moreover, being $D(q) = T^{-T}F(T^{-1}q)T^{-1} > 0$, it is $D_{uu} > 0$. Also, symmetry of F implies $D_{ua} = D_{au}^T$.

Remark 1. *This is a reasonably general class of systems, which includes fine discretizations of sequences of continuum segments with homogeneous stiffness, and sequences of actuated and passive segments, moving either in 2D or 3D (including for example the affine and polynomial models in [8], [21]). Examples of 2D and 3D soft robots are provided in Sec. VI.*

B. Known structural properties

Model (3) verifies a set of classical properties of rigid robots with revolute joints, as inherited from eq. (1) [13].

Property 1. *The inertia matrix $M(q)$ is symmetric, positive definite and bounded for any $q \in \mathbb{R}^n$.*

Property 2. *If the matrix $C(q,\dot{q})$ is defined through Christoffel symbols, then $\dot{M}(q) - 2C(q,\dot{q})$ is a skew symmetric matrix. This is equivalent to $\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$.*

Property 3. *The matrix $C(q,\dot{q})$ is bounded in q and linear in the velocity $\dot{q}$. Thus, there exists a constant $\gamma_C > 0$ such that $\|C(q,\dot{q})\| \le \gamma_C\|\dot{q}\|$, for any $q, \dot{q} \in \mathbb{R}^n$.*

Property 4. *There exist constants $\alpha_U, \alpha_G, \alpha_{\partial G} > 0$ such that*

$$\|U(q)\| \le \alpha_U, \quad \|G(q)\| \le \alpha_G, \quad \left\|\frac{\partial G(q)}{\partial q}\right\| \le \alpha_{\partial G},$$

for any $q \in \mathbb{R}^n$. The latter implies also

$$\|G(q_1) - G(q_2)\| \le \alpha_{\partial G}\|q_1 - q_2\|,$$

for any $q_1, q_2 \in \mathbb{R}^n$.

III. Zero Dynamics Analysis

The role of the zero dynamics is fundamental in assessing the stability properties of a nonlinear feedback control system, and can be used as a guideline for the design of effective control laws [24]. The zero dynamics of a system is the residual dynamics left in the state space of x when the controlled output y is forced to be zero at all times (by a suitable control input u). A nonlinear control system is said to be minimum phase if the trajectories of its zero dynamics are bounded. To apply advanced control techniques, such as high-gain output feedback or input-output feedback linearization for trajectory tracking, it is necessary that the system is minimum phase w.r.t. the controlled output. If the zero dynamics is unstable, the system state will eventually diverge. Indeed, the stability properties of the zero dynamics may depend on the choice of the controlled output y, thus establishing what can be expected (or not) from a proposed feedback control design.

In particular, feedback control of linear or nonlinear mechanical systems turns out to be more problematic if the controlled output y is associated to an unstable zero dynamics. In fact, one should resort to a feedback from the full state x (or to a dynamic feedback law from the output y, e.g., using a state observer) in order to be able to stabilize the closed-loop system. As a result, energy-motivated control laws like a PD action on the error $e = y_d - y$ of a positional output y would fail in this case. Therefore, when considering a general underactuated model of a soft robot, it is relevant to investigate the nature of the zero dynamics for different possible controlled outputs of interest.

In this letter, we consider as controlled output of system (3) the actuated variables q_a, i.e., $y = q_a - q_{a,d}$, for a constant $q_{a,d}$. This is also known as the collocated case. Due to the presence of the unactuated dynamics, the system possesses a zero dynamics of dimension $2(n-m)$. This is easily found

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by looking at the residual dynamics in (3) when y, $\dot{y}$, and $\ddot{y}$ are forced to zero:

$$\begin{aligned} &M_{uu}(q_{a,d}, q_u)\ddot{q}_u + C_{uu}(q_{a,d}, q_u, 0, \dot{q}_u)\dot{q}_u \\ &+ G_u(q_{a,d}, q_u) + K_{uu}q_u + D_{uu}(q_{a,d}, q_u)\dot{q}_u = 0. \end{aligned} \tag{4}$$

The following result shows that the dynamic system (3) with the chosen output y is minimum phase. Intuitively, this happens because eq. (4) contains a damping term that dissipates the energy initially stored in the unactuated subsystem.

Lemma 1. *For any initial state, the trajectories of* (4) *are bounded and converge to* $(q_u, \dot{q}_u) = (q_{u,eq}, 0)$ *where* $q_{u,eq}$ *is a solution of*

$$K_{uu}q_u + G_u(q_{a,d}, q_u) = 0. \tag{5}$$

Proof. Consider the Lyapunov-like function

$$V(q_u, \dot{q}_u) = \frac{1}{2}\dot{q}_u^T M_{uu}(q_{a,d}, q_u)\dot{q}_u + \frac{1}{2}q_u^T K_{uu}q_u + U(q_{a,d}, q_u).$$

The gravitational potential $U(q_{a,d}, q_u)$ is lower bounded thanks to Property 4. This implies that also $V(q_u, \dot{q}_u)$ is such. Evaluating $\dot{V}$ along the trajectories of (4) yields (omitting dependence of the dynamic terms)

$$\begin{aligned} \dot{V}(q_u, \dot{q}_u) &= \frac{1}{2}\dot{q}_u^T \dot{M}_{uu}\dot{q}_u + \dot{q}_u^T M_{uu}\ddot{q}_u + \dot{q}_u^T K_{uu}q_u + \dot{q}_u^T G_u \\ &= \frac{1}{2}\dot{q}_u^T \dot{M}_{uu}\dot{q}_u + \dot{q}_u^T K_{uu}q_u + \dot{q}_u^T G_u \\ &\quad + \dot{q}_u^T\left(-C_{uu}\dot{q}_u - K_{uu}q_u - D_{uu}\dot{q}_u - G_u\right) \\ &= -\dot{q}_u^T D_{uu}\dot{q}_u \le 0, \end{aligned}$$

where the skew symmetry of $\dot{M}_{uu} - 2C_{uu}$ has been used. Being V both radially unbounded and lower bounded, it is possible to invoke the Corollary to LaSalle invariance principle in the Appendix from which the thesis follows. □

Remark 2. *The equilibrium reached by the unactuated variables is not unique in general. In Sec. IV-B, a sufficient condition for the uniqueness of* $q_{u,eq}$ *will be presented.*

IV. PD+ Control Under Gravity

We present here our main result for elastically decoupled underactuated soft robots under gravity. The law is an extension of the PD regulator of [13] which uses a constant gravity compensation term evaluated at the target equilibrium. We prove that regulation can be achieved also by fully cancelling gravity on the actuated variables at the current configuration. Additional conditions are provided for obtaining a global result and for tuning the (lowest) proportional gain in the control law sufficient for asymptotic stability.

A. PD control with gravity cancellation

Consider the collocated control law for the regulation of the actuated variables q_a,

$$\tau = K_P\left(q_{a,d} - q_a\right) - K_D\dot{q}_a + G_a(q) + K_{aa}q_{a,d}, \tag{6}$$

where $q_{a,d} \in \mathbb{R}^m$ is the desired set point, and $K_P > 0$ and $K_D \ge 0$ are gain matrices assumed to be symmetric.

Theorem 1. *There exists a finite constant* $\alpha_P > 0$ *such that, for all* $K_P > -K_{aa} + \alpha_\mathrm{P} I_m$*, the trajectories of the closed-loop system* (3), (6) *are bounded and converge asymptotically to the equilibrium state* $(q_a, q_u, \dot{q}_a, \dot{q}_u) = (q_{a,d}, q_{u,eq}, 0, 0)$*, where* $q_{u,eq}$ *is a solution of*

$$K_{uu}q_u + G_u(q_{a,d}, q_u) = 0. \tag{7}$$

Proof. The proof is based again on the Corollary to LaSalle reported in Appendix. Consider the Lyapunov-like function[1]

$$\begin{aligned} V(\bar{q}, \dot{q}) = \gamma_1\left(\frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\bar{q}^T\hat{K}\bar{q} - \frac{\tilde{q}_a^T G_a(q)}{1 + 2\tilde{q}_a^T\tilde{q}_a} + U(q)\right) \\ + \frac{2\tilde{q}_a^T\left(M_{aa}(q)\dot{q}_a + M_{au}(q)\dot{q}_u\right)}{1 + 2\tilde{q}_a^T\tilde{q}_a}, \end{aligned} \tag{8}$$

where $\gamma_1 > 0$ is a scalar and we defined

$$\bar{q} = \begin{pmatrix}\tilde{q}_a \\ \bar{q}_u\end{pmatrix} = \begin{pmatrix} q_a - q_{a,d} \\ q_u\end{pmatrix},\ \hat{K} = \begin{pmatrix} K_P + K_{aa} & 0 \\ 0 & K_{uu}\end{pmatrix},$$

$$\hat{D}(q) = \begin{pmatrix}\hat{D}_a(q) \\ \hat{D}_u(q)\end{pmatrix} = \begin{pmatrix} K_D + D_{aa}(q) & D_{au}(q) \\ D_{ua}(q) & D_{uu}(q)\end{pmatrix}.$$

We show first that hypothesis *i)* of the Corollary in Appendix holds. We have that $\frac{2\tilde{q}_a^T(M_{aa}\dot{q}_a + M_{au}\dot{q}_u)}{1+2\tilde{q}_a^T\tilde{q}_a} \ge -2\lambda_{\max}(M)\|\dot{q}\|$, $-\gamma_1\frac{\tilde{q}_a^T G_a}{1+2\tilde{q}_a^T\tilde{q}_a} \ge -\gamma_1\alpha_G$, and $\gamma_1 U \ge -\gamma_1\alpha_U$. Hence,

$$\begin{aligned} V(\bar{q}, \dot{q}) &\ge \frac{\gamma_1}{2}\lambda_{\min}(M)\|\dot{q}\|^2 - 2\lambda_{\max}(M)\|\dot{q}\| \\ &\quad -\gamma_1(\alpha_G + \alpha_U) + \frac{\gamma_1}{2}\lambda_{\min}(\hat{K})\|\bar{q}\| \\ &\ge \frac{\gamma_1}{2}\lambda_{\min}(M)\|\dot{q}\|^2 - 2\lambda_{\max}(M)\|\dot{q}\| + \gamma_1(\alpha_G + \alpha_U). \end{aligned} \tag{9}$$

The function on the right hand-side of the last inequality is convex and quadratic in $\|\dot{q}\|$. Its minimum is located at $\|\dot{q}\| = 2\lambda_{\max}(M)/(\gamma_1\lambda_{\min}(M))$, with value

$$\gamma_2 = -\frac{8\lambda_{\max}^2(M)}{\gamma_1\lambda_{\min}(M)} - \gamma_1(\alpha_G + \alpha_U). \tag{10}$$

Combining (10) with (9) yields

$$V(\bar{q}, \dot{q}) \ge \gamma_2 > -\infty. \tag{11}$$

Being $V(\bar{q}, \dot{q})$ lower bounded, hypothesis *i)* holds true. From (9), $V(\bar{q}, \dot{q})$ is radially unbounded and thus hypothesis *ii)* is also fulfilled. Consider the time derivative of (8):

$$\begin{aligned} \dot{V}(\bar{q}, \dot{q}) = \gamma_1\Bigg(&\frac{1}{2}\dot{q}^T\dot{M}\dot{q} + \dot{q}^T M\ddot{q} + \dot{q}^T\hat{K}\bar{q} - \frac{\dot{q}_a^T G_a}{1 + 2\tilde{q}_a^T\tilde{q}_a} \\ &- \frac{\tilde{q}_a^T\frac{\partial G_a}{\partial q}\dot{q}}{1 + 2\tilde{q}_a^T\tilde{q}_a} + \frac{4\tilde{q}_a^T G_a\dot{q}_a^T\tilde{q}_a}{(1 + 2\tilde{q}_a^T\tilde{q}_a)^2} + \dot{q}^T G\Bigg) \\ &+ \frac{2\dot{q}_a^T(M_{aa}\dot{q}_a + M_{au}\dot{q}_u)}{1 + 2\tilde{q}_a^T\tilde{q}_a} + \frac{2\tilde{q}_a^T(M_{aa}\ddot{q}_a + M_{au}\ddot{q}_u)}{1 + 2\tilde{q}_a^T\tilde{q}_a} \\ &+ \frac{2\tilde{q}_a^T(\dot{M}_{aa}\dot{q}_a + \dot{M}_{au}\dot{q}_u)}{1 + 2\tilde{q}_a^T\tilde{q}_a} - \frac{8\tilde{q}_a^T(M_{aa}\dot{q}_a + M_{au}\dot{q}_u)\tilde{q}_a^T\dot{q}_a}{(1 + 2\tilde{q}_a^T\tilde{q}_a)^2}. \end{aligned}$$

Simple algebraic manipulations lead to

$$\begin{aligned} \dot{V}(\bar{q}, \dot{q}) = \gamma_1\Bigg(&-\dot{q}^T\hat{D}\dot{q} + \frac{2\tilde{q}_a^T\tilde{q}_a\dot{q}_a^T G_a}{1 + 2\tilde{q}_a^T\tilde{q}_a} - \frac{\tilde{q}_a^T\frac{\partial G_a}{\partial q}\dot{q}}{1 + 2\tilde{q}_a^T\tilde{q}_a} \\ &+ \frac{4\tilde{q}_a^T G_a\dot{q}_a^T\tilde{q}_a}{(1 + 2\tilde{q}_a^T\tilde{q}_a)^2}\Bigg) + \frac{2\dot{q}_a^T(M_{aa}\dot{q}_a + M_{au}\dot{q}_u)}{1 + 2\tilde{q}_a^T\tilde{q}_a} \\ &- \frac{2\tilde{q}_a^T\hat{K}_a\tilde{q}_a}{1 + 2\tilde{q}_a^T\tilde{q}_a} - \frac{2\tilde{q}_a^T\hat{D}_a\dot{q}}{1 + 2\tilde{q}_a^T\tilde{q}_a} + \frac{2(\dot{q}_a^T C_{aa} + \dot{q}_u^T C_{ua})\tilde{q}_a}{1 + 2\tilde{q}_a^T\tilde{q}_a} \\ &- \frac{8\tilde{q}_a^T(M_{aa}\dot{q}_a + M_{au}\dot{q}_u)\tilde{q}_a^T\dot{q}_a}{(1 + 2\tilde{q}_a^T\tilde{q}_a)^2}. \end{aligned}$$

[1] The function (8) is inspired by a similar one used in the control of rigid manipulators, see [25].

All terms in the right-hand side, except for the first and the sixth one (the only that are certainly negative definite), can be easily upper bounded by positive functions using Prop. 1–4 of Sec. II. As a result, we can bound $\dot{V}(\bar{q},\dot{q})$ as

$$\dot{V}(\bar{q},\dot{q}) \leq -\begin{pmatrix} \|\dot{q}\| \\ \dfrac{\|\tilde{q}_a\|}{\sqrt{1+2\|\tilde{q}_a\|^2}} \end{pmatrix}^T Q \begin{pmatrix} \|\dot{q}\| \\ \dfrac{\|\tilde{q}_a\|}{\sqrt{1+2\|\tilde{q}_a\|^2}} \end{pmatrix},$$

with matrix Q given by

$$\begin{pmatrix} \gamma_1\lambda_{\min}(\hat{D}) - \frac{\gamma_C}{\sqrt{2}} - 4\lambda_{\max}(M) & -\gamma_1(2\alpha_G + \alpha_{\partial G} + \sigma_{\max}(\hat{D}_a)) \\ \text{symm} & 2\lambda_{\min}(K_P + K_{aa}) \end{pmatrix}.$$

Thus, $\dot{V} \leq 0$ for a matrix $Q > 0$. According to Sylvester criterion, this will be the case i.f.f.

$$\gamma_1\lambda_{\min}(\hat{D}) - \frac{\gamma_C}{\sqrt{2}} - 4\lambda_{\max}(M) > 0,$$

and

$$\begin{aligned} \det Q = 2\lambda_{\min}(K_P + K_{aa}) \left(\gamma_1\lambda_{\min}(\hat{D}) - \frac{\gamma_C}{\sqrt{2}} - 4\lambda_{\max}(M)\right) \\ - \left(\gamma_1(2\alpha_G + \alpha_{\partial G}) + \sigma_{\max}(\hat{D}_a)\right)^2 > 0. \end{aligned} \tag{12}$$

Both conditions are fulfilled by taking γ_1 and K_P such that

$$\gamma_1 > \frac{\gamma_C + 4\sqrt{2}\lambda_{\max}(M)}{\sqrt{2}\lambda_{\min}(\hat{D})}, \tag{13}$$

and

$$\lambda_{\min}(K_P + K_{aa}) > \frac{\left(\gamma_1(2\alpha_G + \alpha_{\partial G}) + \sigma_{\max}(\hat{D}_a)\right)^2}{2\left(\gamma_1\lambda_{\min}(\hat{D}) - \frac{\gamma_C}{\sqrt{2}} - 4\lambda_{\max}(M)\right)}. \tag{14}$$

The latter is verified by hypothesis taking α_P equal to the right-hand side of (14). Combining (12), (13) and (14), it follows that $\dot{V} \leq 0$. All three hypotheses of the Corollary to LaSalle are therefore verified. Furthermore, $Q > 0$ implies that $\dot{V}$ vanishes if and only if $\tilde{q}_a = 0$ and $\dot{q} = 0$. As a result, the trajectories of the closed-loop system will asymptotically converge to $\tilde{q}_a = 0$ and $\dot{q} = 0$, hence the thesis. □

Remark 3. *Assuming elastic decoupling between actuated and unactuated variables guarantees the absence of the term $\dot{q}_a^T K_{au} q_u$ in $\dot{V}(\bar{q},\dot{q})$, which is not definite in sign and possibly unbounded in q_u. However, the results from this section can be generalized to the case of* weakly *elastically coupled systems, i.e., having bounded elastic coupling.*

B. Uniqueness of the equilibrium

The same control law (6) is also sufficient to ensure global convergence to a single equilibrium, as soon as the stiffness of the field acting on underactuated variables is large enough.

Corollary 1. *Under the hypotheses of Theorem 1, if*

$$K_{uu} > -\frac{\partial^2 U(q_{a,d}, q_u)}{\partial q_u^2},$$

for all $q_u \in \mathbb{R}^{n-m}$, then the closed-loop system (3), (6) *has a unique globally asymptotically stable equilibrium.*

Proof. Consider the auxiliary function

$$P(q_u) = U(q_{a,d}, q_u) + \frac{1}{2} q_u^T K_{uu} q_u.$$

According to Theorem 1, the unactuated variables tend to a $q_{u,eq}$ such that $G_u(q_{a,d}, q_{u,eq}) + K_{uu} q_{u,eq} = 0$. The latter is the gradient of P evaluated at this closed-loop equilibrium, and thus $q_{u,eq}$ is an extremum of P. This point is unique since the Hessian of $P(q_u)$ is $\partial^2 U(q_{a,d}, q_u)/\partial q_u^2 + K_{uu}$, which is positive definite by hypothesis. □

C. Lower bounds on control gains

Asymptotic stability of the (single or multiple) closed-loop equilibria has been proven under the hypothesis that the proportional gain K_P in (6) is large enough. Still, it is useful to find a lower bound on this gain, that can be used to reduce the control effort and to avoid stiffening unnecessarily the soft robot [26].

Corollary 2. *All K_P such that*

$$\lambda_{\min}(K_P + K_{aa}) > \alpha_{th}, \tag{15}$$

verify Theorem 1, with

$$\begin{aligned} \alpha_{th} = {} & 2(2\alpha_G + \alpha_{\partial G})^2 \left(\tfrac{\gamma_C}{\sqrt{2}} + 4\lambda_{\max}(M)\right) / \lambda_{\min}^2(\hat{D}) \\ & + 2\,(2\alpha_G + \alpha_{\partial G})\,\sigma_{\max}(\hat{D}_a)/\lambda_{\min}(\hat{D}), \end{aligned}$$

where

$$\hat{D}(q) = \begin{pmatrix} \hat{D}_a(q) \\ \hat{D}_u(q) \end{pmatrix} = \begin{pmatrix} K_D + D_{aa}(q) & D_{au}(q) \\ D_{ua}(q) & D_{uu}(q) \end{pmatrix}.$$

Proof. Consider the right-hand side of inequality (14) as a function of γ_1 subject to the constraint (13), i.e., a function $f(\gamma_1) : D = ((k_4/k_3), \infty) \rightarrow \mathbb{R}^+$ defined as

$$f(\gamma_1) = \frac{(k_1\gamma_1 + k_2)^2}{2(k_3\gamma_1 - k_4)}, \tag{16}$$

with $k_1 = 2\alpha_G + \alpha_{\partial G}$, $k_2 = \sigma_{\max}(\hat{D}_a)$, $k_3 = \lambda_{\min}(\hat{D})$, and $k_4 = \left(\gamma_C/\sqrt{2}\right) + 4\lambda_{\max}(M)$. Function (16) is convex in its domain of definition D since the second-order derivative

$$\frac{\partial^2 f(\gamma_1)}{\partial \gamma_1^2} = \frac{(k_1k_4 + k_2k_3)^2}{(k_3\gamma_1 - k_4)^3}$$

is positive for $\gamma_1 > (k_4/k_3)$. Thus, its unique global minimum can be found analytically as the solution of

$$\frac{\partial f(\gamma_1)}{\partial \gamma_1} = \frac{\sqrt{2}k_1}{2\sqrt{k_3\gamma_1 - k_4}} - \frac{\sqrt{2}k_3(k_1\gamma_1 + k_2)}{4(k_3\gamma_1 - k_4)^{\frac{3}{2}}} = 0. \tag{17}$$

This is obtained at $\gamma_{1,\min} = (2k_1k_4 + k_2k_3)/(k_1k_3) \in D$, with minimum value

$$f(\gamma_{1,\min}) = \frac{2k_1(k_1k_4 + k_2k_3)}{k_3^2}. \tag{18}$$

The thesis follows by substituting back in (18) the values of k_1, k_2, k_3 and k_4. □

V. Variants for Handling Gravity

We can further simplify the regulator by updating online only the unactuated variables in the term that cancels gravity in (6), as in the following 'mixed' PD+ control law

$$\tau = K_P\,(q_{a,d} - q_a) - K_D\dot{q}_a + G_a(q_{a,d}, q_u) + K_{aa}q_{a,d}. \tag{19}$$

The closed-loop asymptotic stability properties are similar to the those obtained with the previous solution, with the exception of a higher proportional gain.

Corollary 3. *Under the hypothesis of Theorem 1, with* K_P *such that* $\lambda_{\min}(K_P + K_{aa}) > \alpha_{\partial G} + \alpha_{th}$*, the trajectories of the closed-loop system* (3), (19) *are bounded and converge asymptotically to the equilibrium state* $(q_a, q_u, \dot{q}_a, \dot{q}_u) = (q_{a,d}, q_{u,eq}, 0, 0)$*, where* $q_{u,eq}$ *is a solution of* (7).

Proof. We only sketch the proof due to lack of space. The first part proceeds along similar steps as in the proof of Theorem 1 using the Lyapunov-like function

$$V(\bar{q}, \dot{q}) = \gamma_1 \left(\frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} \bar{q}^T \hat{K} \bar{q} - \frac{\tilde{q}_a^T G_a(q_{a,d}, q_u)}{1 + 2\tilde{q}_a^T \tilde{q}_a} + U(q) \right) + \frac{2\tilde{q}_a^T (M_{aa}(q)\dot{q}_a + M_{au}(q)\dot{q}_u)}{1 + 2\tilde{q}_a^T \tilde{q}_a}.$$

Instead, the part that deals with defining a lower bound for K_P follows by exactly the same steps as in the proof of Corollary 2. □

Both regulators (6) and (19) guarantee that the trajectories of the closed-loop system converge asymptotically to an equilibrium. However, there is no clue about if and how a sufficiently fast convergence rate can be obtained in the large. Instead, this goal is automatically achieved by a controller designed using (collocated) Partial Feedback Linearization (PFL) theory [23]. When considering our class of underactuated soft robots, it is possible to compute the explicit expression of $\ddot{q}_{uu}$ from the last $n - m$ equations in (3) and to replace it in the first m equations, leading to

$$\begin{aligned}&\left(M_{aa} - M_{au} M_{uu}^{-1} M_{ua}\right) \ddot{q}_{aa} - M_{au} M_{uu}^{-1} \\ &\cdot \left(C_{ua}\dot{q}_a + C_{uu}\dot{q}_u + G_u + K_{uu} q_u + D_{ua}\dot{q}_a + D_{uu}\dot{q}_u\right) \\ &+ C_{aa}\dot{q}_a + C_{au}\dot{q}_u + G_a + K_{aa} q_a + D_{aa}\dot{q}_a + D_{au}\dot{q}_u = \tau.\end{aligned}$$

The above set of equations is exactly linearized by the law

$$\begin{aligned}\tau = &\left(M_{aa} - M_{au} M_{uu}^{-1} M_{ua}\right) u - M_{au} M_{uu}^{-1} \\ &\cdot \left(C_{ua}\dot{q}_a + C_{uu}\dot{q}_u + G_u + K_{uu} q_u + D_{ua}\dot{q}_a + D_{uu}\dot{q}_u\right) \\ &+ C_{aa}\dot{q}_a + C_{au}\dot{q}_u + G_a + K_{aa} q_a + D_{aa}\dot{q}_a + D_{au}\dot{q}_u,\end{aligned} \quad (20)$$

where u is the input acting on the linearized system. Choosing $u = K_P(q_{a,d} - q_a) - K_D \dot{q}_a$, yields the closed-loop dynamics for the actuated variables

$$\ddot{q}_a + K_D \dot{q}_a + K_P (q_a - q_{a,d}) = 0.$$

Thus, for any $K_P > 0$ and $K_D > 0$, q_a will converge exponentially fast to $q_{a,d}$, with a rate explicitly assigned by the choice of gains, as calculated using standard linear theory. Moreover, thanks to Lemma 1, the use of (20) will also induce a convergent behavior to the entire state of the soft robot. However, these nice properties come at the cost of a substantially more complex and potentially less robust controller when compared to (6), since the implementation of (20) requires full knowledge of the robot dynamics.

Remark 4. *To implement the proposed control laws the measure of both* θ *and* $\dot{\theta}$ *(or, equivalently, of* q *and* $\dot{q}$*) is needed. Although with some limitations, the first can be acquired through a motion capture system, while the latter estimated through backward differentiation [17]. Note also that, unlike the PFL law in* (20)*, the PD+ control laws* (6) *and* (19) *do not necessarily require velocity measures, as* K_D *can be set to zero without affecting the asymptotic stability of the closed-loop system.*

VI. Simulation Results

Two simulations are proposed to show that the considered class of systems encompasses different types of underactuation. In the first simulation (Sec. VI-A), we consider an inextensible 3D soft arm described through the state parametrization proposed in [27], while in the second one (Sec. VI-B) a planar soft robot modeled under the Piecewise Constant Curvature (PCC) formulation presented in [17] is used. In both cases, the base is rotated so that in the rest position $q = 0$ ([m] or [rad]) the arm is aligned with the gravitational field, with its tip pointing downwards. The PD law with gravity cancellation (6) and the partially feedback linearizing control law (20) are compared. We do not report results for the control law (19) since its performance were found comparable to those of (6).

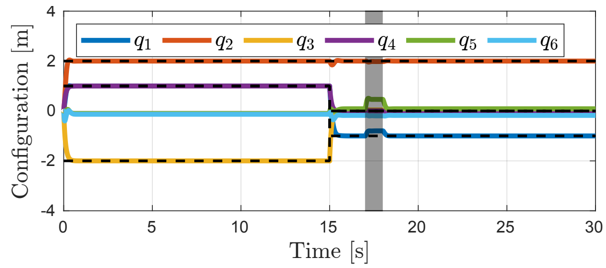
A. Simulation 1

Consider an inextensible 3D soft arm with 3 segments, where only the first and third ones are actuated. In this case, the change of coordinates (2) boils down to a reordering of the variables. Thus, the goal is to regulate the two configuration variables of each of the actuated segments, i.e., $q_a = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} & \theta_{3,1} & \theta_{3,2} \end{pmatrix}^T \in \mathbb{R}^4$. Each uniform segment has length $0.11\,[\mathrm{m}]$ and mass $0.1\,[\mathrm{kg}]$. The stiffness and damping are assumed uniform and equal to $h_i = 0.6\,[\mathrm{N/m}]$ and $f_i = 0.03\,[\mathrm{N\,s/m}]$, $i = 1, 2, 3$. The robot starts at rest and the simulation runs for $30\,[\mathrm{s}]$. The reference for the controllers has two successive targets (in $[\mathrm{m}]$):

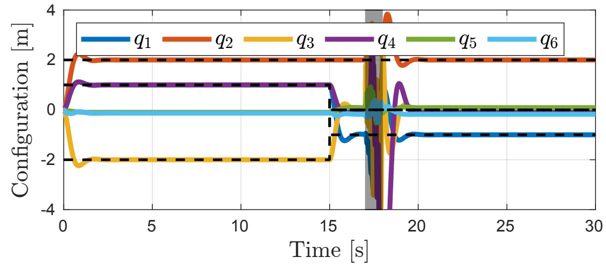
$$q_{(1,2,3,4),d}(t) = \begin{cases} \begin{pmatrix} 1 & 2 & -1 & 1 \end{pmatrix}^T, & 0 \le t < 15\,[\mathrm{s}] \\ \begin{pmatrix} -1 & 2 & 0 & 0 \end{pmatrix}^T, & t \ge 15\,[\mathrm{s}]. \end{cases} \quad (21)$$

At $t = 17\,[\mathrm{s}]$, an external force $f_{\mathrm{ext}} = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix}^T\,[\mathrm{N}]$ is applied for $1\,[\mathrm{s}]$ to the robot tip. The control gains are chosen as $K_P = 1 \cdot I_4\,[\mathrm{N/m}]$ and $K_D = 0.1 \cdot I_4\,[\mathrm{N\,s/m}]$ for the PD+ controller (6) and, respectively, as $K_P = 20 \cdot I_4\,[\mathrm{s}^{-2}]$ and $K_D = 5 \cdot I_4\,[\mathrm{s}^{-1}]$ for the nonlinear PFL regulator (20).

Figure 1 shows the time evolution of the configuration variables. Despite of the disturbance f_{ext}, both regulators yield zero error at steady state. The PFL law is characterized by a faster transient, but does not exhibit the same nice disturbance rejection capability of the PD+ regulator. This is expected since (20) makes the input-output behavior virtually equivalent to a unitary mass-spring-damper system subject to (the projection of) f_{ext}. On the other hand, the PD+ controller preserves the inertial properties of the soft robot, achieving regulation by cancelling only the necessary dynamic terms. Figure 2 shows the control effort requested to execute the motion. During the action of f_{ext} an oscillatory behaviour is observed in the output of the PFL control torque, which attains

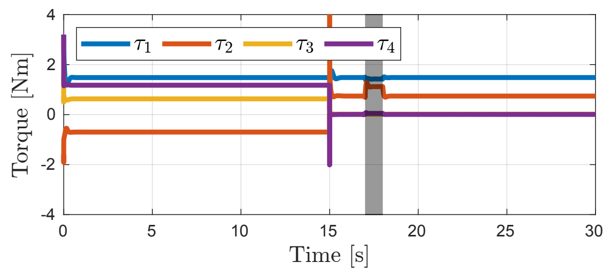


(a) PD with gravity cancellation

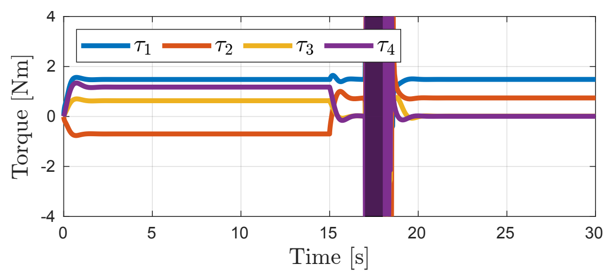


(b) Partial feedback linearization

Figure 1. Simulation 1 (3D soft arm). Time evolution of the configuration for the reference (21) (black dashed lines). A force $f_{\text{ext}} = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix}^T$ [N] is applied to the robot tip as a disturbance during motion in the time window spanned by the shaded gray area.



(a) PD with gravity cancellation



(b) Partial feedback linearization

Figure 2. Simulation 1 (3D soft arm). Time evolution of the control torques.

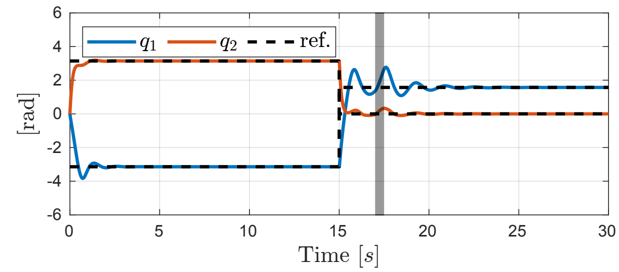
peaks that are one order of magnitude larger than those of the PD+ torque.

B. Simulation 2

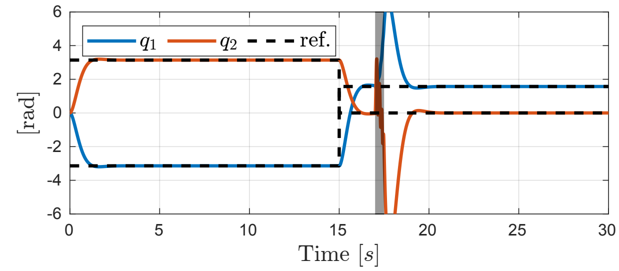
Consider a planar soft robot with 2 actuated segments. It is assumed that the shape of each segment is well described by 3 CC segments. However, including a higher number of segments yields similar results.

As discussed in [13], the actuation matrix takes the form

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}^T.$$

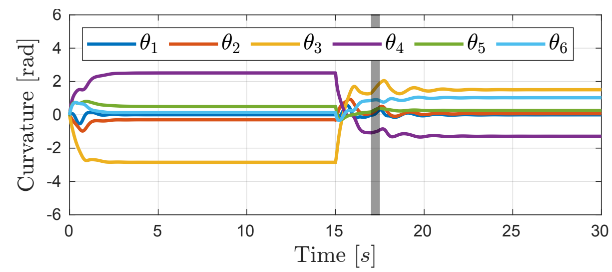


(a) PD with gravity cancellation

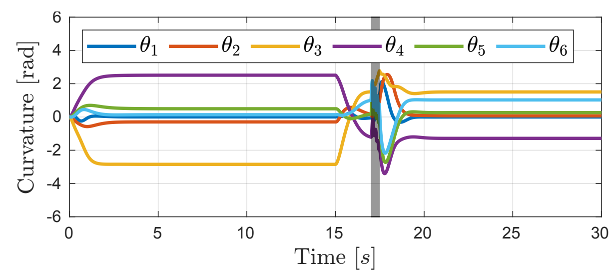


(b) Partial feedback linearization

Figure 3. Simulation 2 (2D soft robot). Time evolution of the actuated variables for the reference (22) (black dashed lines). A lateral force of $f_{\text{ext}} = \begin{pmatrix} 0 & 3 \end{pmatrix}^T$ [N] is applied to the robot tip as a disturbance during motion in the time window spanned by the gray shaded area.



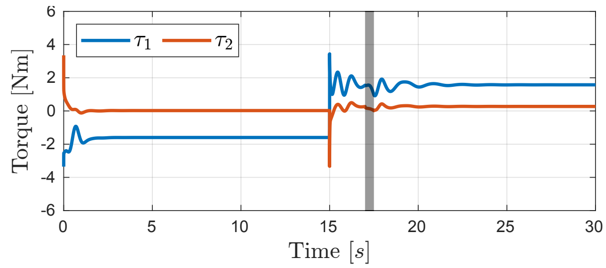
(a) PD with gravity cancellation



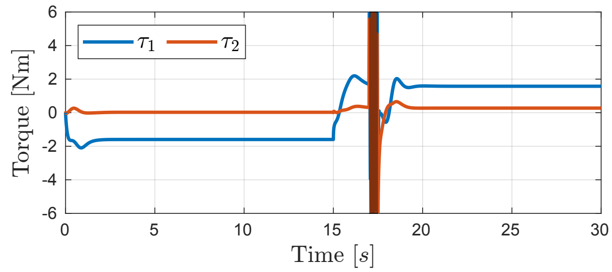
(b) Partial feedback linearization

Figure 4. Simulation 2 (2D soft robot). Time evolution of the curvatures of the six CC segments used for the discretization of the soft robot.

Hence, according to (2), the first and second variables in the new coordinates $q \in \mathbb{R}^6$ are the sum of the three curvatures of the first and, respectively, the second actuated segment, i.e., $q_a = \begin{pmatrix} \theta_1 + \theta_2 + \theta_3 & \theta_4 + \theta_5 + \theta_6 \end{pmatrix}^T$. As a result, through the commands $\tau \in \mathbb{R}^2$ it is possible to regulate the orientation of the tip of the two actuated segments. Each CC segment has length $0.1\,[\text{m}]$ and mass $0.3\,[\text{kg}]$. The stiffness and damping matrices are both taken diagonal with elements $0.2\,[\text{Nm/rad}]$ and $0.2\,[\text{Nm s/rad}]$, respectively. The robot starts at rest and the simulation runs for $30\,[\text{s}]$. The reference commanded to



(a) PD with gravity cancellation



(b) Partial feedback linearization

Figure 5. Simulation 2 (2D soft robot). Time evolution of the control torques.

the controllers has again two successive targets (in $[\mathrm{rad}]$):

$$q_{(1,2),d}(t) = \begin{cases} \begin{pmatrix} -\pi & \pi \end{pmatrix}^T, & 0 \leq t < 15\,[\mathrm{s}] \\ \begin{pmatrix} \pi/2 & 0 \end{pmatrix}^T, & t \geq 15\,[\mathrm{s}]. \end{cases} \tag{22}$$

To empirically evaluate the disturbance rejection capabilities, at $t = 17\,[\mathrm{s}]$ an external force $f_{\mathrm{ext}} = \begin{pmatrix} 0 & 3 \end{pmatrix}^T\,[\mathrm{N}]$ is applied to the robot tip for $0.5\,[\mathrm{s}]$. The chosen control gains are $K_P = 1 \cdot I_2\,[\mathrm{Nm/rad}]$ and $K_D = 0.1 \cdot I_2\,[\mathrm{Nms/rad}]$ for the PD+ law (6), and $K_P = 10 \cdot I_2\,[\mathrm{s}^{-2}]$, $K_D = 5 \cdot I_2\,[\mathrm{s}^{-1}]$ for the PFL controller (20).

Figure 3 shows the time evolution of the actuated variables, together with the corresponding references. The final curvature of each actuated segment is correctly regulated. The closed-loop system under the PD+ control exhibits a more oscillatory behavior compared to what obtained with the PFL law. However, also in this scenario, the latter control law is less robust to the presence of an external disturbance f_{ext}. Figure 4 shows the curvature of the six CC segments used to discretize the structure. As expected, these converge to finite values. Finally, Figure 5 shows the control torques required to perform the motion, for which similar conclusions as the ones drawn in Simulation 1 hold. In particular, the control action for the PFL controller attains a peak during the action of f_{ext} that is one order of magnitude larger than the peak of the PD+ law.

For better illustration of the dynamic behavior in the workspace, Figures 6 and 7 show stroboscopic views of the motion of the two soft robots in Simulation 1 and Simulation 2, for both the PD+ and the PFL controller.

VII. CONCLUSIONS

We have identified a new class of underactuated soft robots that we call *elastically decoupled*. For such systems we proved the asymptotic stability of the zero dynamics when the collocated variables are taken as controlled output. This has at least two important consequences. First, it allows the application of well-established nonlinear control techniques, such as input-output (partial) feedback linearization. Second, it serves as a guideline to look for simpler control laws allowing the regulation of the actuated coordinates. Along this line of thought, we proposed two PD+ controllers and provided sufficient conditions that guarantee the global asymptotic stability of the desired closed-loop equilibrium under gravity. The theoretical results have been validated through simulations. Future work will be devoted to the experimental validation of these controllers.

APPENDIX

We report here a trivial variation on the LaSalle invariance principle that we use to prove the main results of this paper.

Corollary (to LaSalle [28])**.** *Consider the system $\dot{x} = f(x)$, with $x \in \mathbb{R}^l$. Suppose that $\forall x(0) \in \mathbb{R}^l, t \geq 0$ there exists a unique solution $x(t, x(0))$ to $\dot{x} = f(x)$. Let $V(x) : \mathbb{R}^l \to \mathbb{R}$ be a continuously differentiable function, such that $\forall x \in \mathbb{R}^l$*

i) $V(x) \geq \gamma_V$, for some $\gamma_V > -\infty$,
ii) $V(x)$ is radially unbounded,
iii) $\dot{V}(x) \leq 0$.

Let E be the set of points in R^l where $\dot{V}(x) = 0$. Then, for all initial conditions $x(0) \in \mathbb{R}^l$, the evolution $x(t, x(0))$ approaches the largest invariant set in E as $t \to \infty$.

Proof. Consider $\Omega_c = \{x \in \mathbb{R}^l \mid \gamma_V \leq V(x) \leq c\}$, with $\gamma_V < c$. This set is bounded $\forall c < \infty$. Indeed, if that was not true, then there would be a $\eta \in \mathbb{R}^l$ such that $\alpha\eta \in \Omega_c$ for all $\alpha > 0$, and thus $\lim_{\alpha\to\infty} V(\alpha\eta) \leq c < \infty$. This is in contradiction with ii). Furthermore, Ω_c is also closed since the set $[\gamma_V; c]$ is closed and the inverse image of closed sets on continuous functions is closed. Hence, Ω_c is compact. This set is also positively invariant. Indeed, from iii) for all $x(0) \in \Omega_c$ and $t \geq 0$, $V(x(t, x(0))) \leq V(x(0)) \leq c$. In addition from i), $\gamma_V \leq V(x(t))$. Thus, if $x(0) \in \Omega_c$, then $x(t, x(0))$ stays in Ω_c at all the future instants. Finally, for any initial state $x(0) \in \mathbb{R}^l$ it is possible to choose c large enough so that $x(0) \in \Omega_c$. In particular, it is sufficient that $c \geq V(x(0))$. The thesis follows choosing Ω_c as the set Ω in Theorem 4.4 in [28]. □

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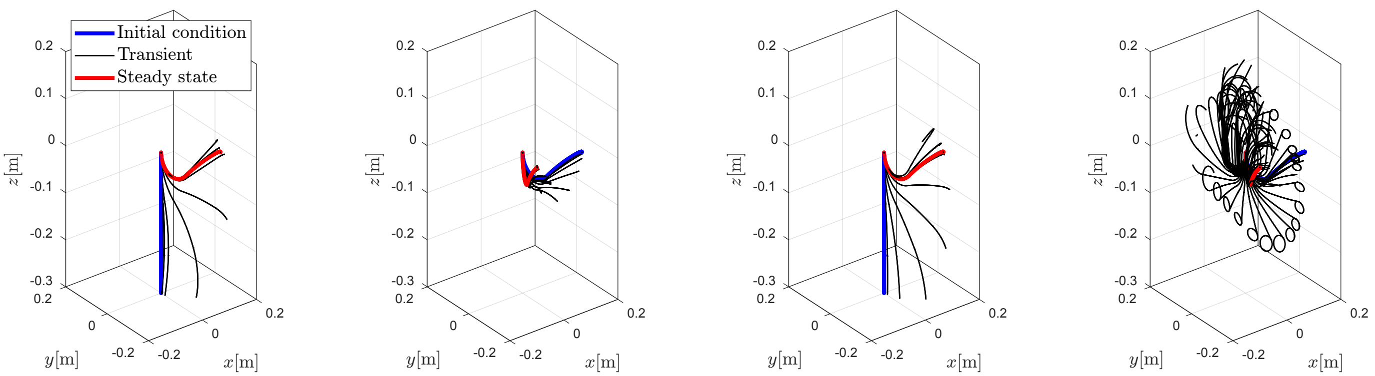


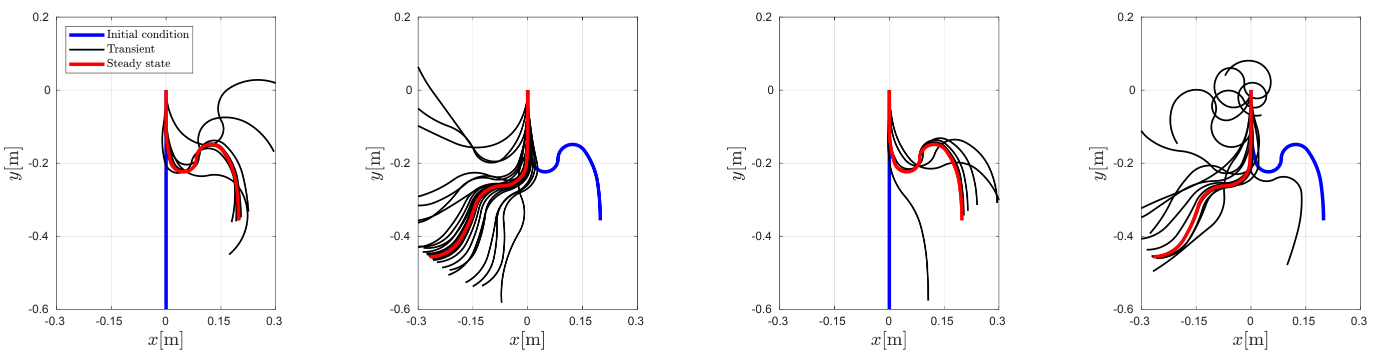
(a) PD with gravity cancellation for $t \in [0, 15]$ [s]

(b) PD with gravity cancellation for $t \in [15, 30]$ [s]

(c) Partial feedback linearization for $t \in [0, 15]$ [s]

(d) Partial feedback linearization for $t \in [15, 30]$ [s]

Figure 6. Simulation 1 (3D soft arm). Stroboscopic views of the robot motion in the workspace for the reference (21). At $t = 17$ [s], a disturbance force $f_{\text{ext}} = \begin{pmatrix} 1 & 3 & 0 \end{pmatrix}^T$ [N] is applied to the robot tip for 0.5 [s]. (a) and (b) show the robot motion under the PD+ regulator for $t \in [0, 15]$ [s] and $t \in [15, 30]$ [s], respectively. Similarly, (c) and (d) show the motion under the PFL regulator in the same time windows.



(a) PD with gravity cancellation for $t \in [0, 15]$ [s]

(b) PD with gravity cancellation for $t \in [15, 30]$ [s]

(c) Partial feedback linearization for $t \in [0, 15]$ [s]

(d) Partial feedback linearization for $t \in [15, 30]$ [s]

Figure 7. Simulation 2 (2D soft robot). Stroboscopic views of the robot motion in the workspace for reference (22), with the same organization as in Fig. 6.

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