On an integration rule for products of barycentric coordinates over simplexes in R^n

Vermolen, F.J.; Segal, A.

DOI
10.1016/j.cam.2017.09.013

Publication date
2018

Document Version
Accepted author manuscript

Published in
Journal of Computational and Applied Mathematics

On an integration rule for products of barycentric coordinates over simplexes in $\mathbb{R}^n$ ☆

F.j. Vermolen, A. Segal[1]

Delft Institute of Applied Mathematics, Delft University of Technology, Delft, The Netherlands

Abstract

In finite-element computations, one often needs to calculate integrals of products of powers of monomials over simplexes. In this manuscript, we prove a generalisation of the exact integration formula that was reported and proved for two-dimensional simplexes by Holand & Bell in 1969. We extend the proof to n-dimensional simplexes and to simplexes on d-dimensional manifolds in n-dimensional space. The results are used to develop finite-element and boundary-element simulation tools. The proofs of the theorems are based on mathematical induction and coordinate mappings.

Keywords: barycentric coordinates, integration rule, factorisations, finite element methods

2010 MSC: 00-01, 99-00

1. Introduction

Finite-element methods are based on a weak form of a partial differential equation. The solution of the (initial) boundary value problem in consideration, which is a (partial) differential equation with (initial and) boundary conditions, is written in terms of a linear combination of a chosen set of basis functions. The resulting system of equations contains integrals over factorisations of these basis functions. Since the finite-element method further contains the division of the domain into elements, the integrals over basis functions need to be processed over the elements, after which an assembly step follows. In many cases, these elements have a triangular shape, or a tetrahedral shape in two and three dimensions, respectively. Of course different shapes of elements are also possible. The finite-element basis functions are smooth over the elements, however, they may have a discontinuity in their (higher-order) derivatives or even be discontinuous across the element boundaries, depending on the partial differentiation one is working with. How finite-element methods work can be read in textbooks written by Bræss [2], Brenner and Scott [3], Atkinson and Han [1], Strang & Fix [12], Zienkiewics [15], Quarteroni [13] or in Van Kan *et al.* [14], to mention a few. In most classical (nodal) finite-element methods, the basis functions are often expressed as combinations of piecewise linear functions over the domain of computation. In the elements, the linear functions are represented by the so-called barycentric coordinates, which are defined as follows:

Definition 1.1. *Given a simplex s_n in $\mathbb{R}^n$ with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$, then the barycentric coordinates $\lambda_i(\mathbf{x})$, $i \in \{1, \ldots, n+1\}$ are linear on s_n and satisfy $\lambda_i(\mathbf{x}_j) = \delta_{ij}$.*

Therewith it is often necessary to compute integrals with integrands that consist of factorisations of linear functions over triangles in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ or simplexes in $\mathbb{R}^n$. In spaces with dimensionality two, the so-called integration formula that was derived in Holand & Bell [6] is used. Holand & Bell proved the assertion in their work and in Brenner & Scott [3] the proof of the two-dimensional version is asked for in an exercise. However, for dimensionalities higher than three, the proof of the relation has not been reported in the literature as far as we know.

In the last decades, financial mathematics has gained a lot popularity in the numerical analysis community. Since some mathematical models from finance involve the solution of partial differential equations with high dimensionality, where the 'curse of dimensionality' is experienced, the need for higher dimensional finite-element methods has increased, see [10, 9] for instance. With the increase in computational power, these higher dimensional finite elements have become feasible from a computational point of view. To this extent, higher dimensional (exact) integration rules with a mathematical basis that facilitate the necessary integration of finite element functions over simplexes has become more important. Next to the increase of dimensionality of present day calculations, finite element techniques over surfaces have gained a lot of popularity in the communities of mathematical biology and electro-magnetics. In many of the models that are common in mathematical biology, see [4, 7] for instance, one wants to understand the development of certain patterns on moving surfaces. Although finite-element methods are more and more enriched with isogeometric analysis (IGA) [8], classical finite elements remain popular and hence integration over triangles on surfaces in $\mathbb{R}^3$ remains important. Therefore, integration over simplexes on (moving) manifolds in higher-dimensional spaces can and will be common. In this manuscript, we derive the main results in terms of exact computation for simplexes in $\mathbb{R}^n$ and for simplexes on d-dimensional manifolds in $\mathbb{R}^n$. The main results are often used to derive Newton-Cotes type integration formulas over internal and boundary elements. The proof is constructed via a sequence of several lemmas, where we first consider the one-dimensional case, and then we extend the proof to higher dimensionality through mathematical induction. We also base the proof of the final result on a coordinate transformation from a generic simplex to a unit simplex. Finally, we extend the proof so that simplexes on d-dimensional manifolds in an n-dimensional space can be dealt with.

2. Integration over simplexes

In this paper, we deal with (non-degenerate) simplexes in $\mathbb{R}^n$ that are denoted by s_n. The coordinates of the vertices of simplex s_n are represented by $\mathbf{x}_j$, for $j \in \{1, \ldots, n+1\}$. Next to the coordinates of the vertices, we denote the vectors in $\mathbb{R}^n$ pointing from the origin to point $\mathbf{x}_j$ by the column vector $\vec{x}_j$. The vertices of simplex s_n can have any coordinates in $\mathbb{R}^n$. We also use the unit simplex in $\mathbb{R}^n$, denoted by $\tilde{s}_n$ with vertices on the coordinate axes on a distance of one from the origin, as well as the last vertex being on the origin.

Next we deal with simplexes whose vertices are points that are located on a d-dimensional manifold in $\mathbb{R}^n$ where $d < n$, these simplexes are denoted by bs_d. This notation has been chosen since in an n-dimensional framework integration over manifolds is usually done in order to incorporate natural boundary conditions.

For the one-dimensional case, we drop the coordinate and vector notation, and there we simply use x_1 and x_2 to denote the vertices of the line-segment. First, the 1D-case is consideblack, which is necessary for the proof of the generalised results in n dimensions. Subsequently, the extension to $\mathbb{R}^n$ is carried out.

2.1. The 1D-case: Integration over a line segment

The one-dimensional case is given since it is needed in the mathematical induction-based proof of the general n-dimensional version of the theorem. We start with the unit interval.

Lemma 2.1. *Let $\tilde{s}_1 = (0, 1)$, be a line segment in $\mathbb{R}$, and $m_1,\ m_2 \in \mathbb{N}$, then*

$$\int_{\tilde{s}_1} (1-t)^{m_1} t^{m_2} dt = \frac{\prod_{j=1}^{2} m_j!}{(1 + \sum_{j=1}^{2} m_j)!} \tag{1}$$

Proof. *We proceed by integration by parts:*

$$\begin{aligned} \int_{\tilde{s}_1} (1-t)^{m_1} t^{m_2} dt = \quad & \frac{1}{m_2+1} [(1-t)^{m_1} t^{m_2+1}]_0^1 + \frac{m_1}{m_2+1} \int_{\tilde{s}_1} (1-t)^{m_1-1} t^{m_2+1} dt = \\ & \frac{m_1}{m_2+1} \int_{\tilde{s}_1} (1-t)^{m_1-1} t^{m_2+1} dt. \end{aligned} \tag{2}$$

In the above expression, the boundary term vanishes since $\tilde{s}_1 = (0, 1)$. Carrying out integration by parts recurrently, combined with the vanishing boundary terms, gives

$$\begin{aligned} \int_{\tilde{s}_1} (1-t)^{m_1} t^{m_2} dt = \quad & \frac{m_1!}{(m_2+1)\ldots(m_2+m_1)} \int_{\tilde{s}_1} t^{m_2+m_1} dt = \\ & \frac{m_1!}{(m_2+1)\ldots(m_2+m_1)(m_1+m_2+1)} = \frac{m_1! m_2!}{(m_1+m_2+1)!}. \end{aligned} \tag{3}$$

This proves Lemma 2.1. □

Subsequently using a coordinate transformation, we arrive at the following result for a line-segment in $\mathbb{R}^n$:

Theorem 2.1. *Let s_1 be a line-segment in $\mathbb{R}^n$with vertices $\mathbf{x}_1$ and $\mathbf{x}_2$ and suppose that $\lambda_1(\mathbf{x})$ and $\lambda_2(\mathbf{x})$ are the barycentric coordinates on s_1 with $m_1,\ m_2 \in \mathbb{N}$, then*

$$\int_{s_1} \lambda_1^{m_1} \lambda_2^{m_2} d\Gamma = \frac{||\mathbf{x}_2 - \mathbf{x}_1|| \prod_{j=1}^{2} m_j!}{(1 + \sum_{j=1}^{2} m_j)!} \tag{4}$$

Proof. *We use the coordinate transformation* $\mathbf{x}(t) = \mathbf{x}_1 + t(\mathbf{x}_2 - \mathbf{x}_1)$*, hence to this end, we get* $d\Gamma = \|\mathbf{x}'(t)\| dt = \|\mathbf{x}_2 - \mathbf{x}_1\| dt$ *and* $\lambda_1(\mathbf{x}(t)) = 1 - t$ *and* $\lambda_2(\mathbf{x}(t)) = t$*, and therewith we get*

$$\int_{s_1} \lambda_1^{m_1} \lambda_2^{m_2} d\Gamma = \|\mathbf{x}_2 - \mathbf{x}_1\| \int_0^1 (1-t)^{m_1} t^{m_2} dt = \frac{\|\mathbf{x}_2 - \mathbf{x}_1\| \cdot m_1! m_2!}{(1 + m_1 + m_2)!}. \tag{5}$$

In the last equality, we used Lemma 2.1. This proves Theorem 2.1. □

Remark 2.1. *Upon using any two points, say* x_1 *and* x_2 *(*$x_1 < x_2$*) in* $\mathbb{R}$*, with corresponding barycentric coordinates* $\lambda_1(x)$ *and* $\lambda_2(x)$*, Theorem 2.1 gives*

$$\int_{s_1} \lambda_1^{m_1} \lambda_2^{m_2} dx = (x_2 - x_1) \frac{m_1! m_2!}{(m_1 + m_2 + 1)!}, \qquad \text{for } m_1,\ m_2 \in \mathbb{N}. \tag{6}$$

2.2. The nD-case: Integration over simplexes in $\mathbb{R}^n$

First, we consider integration over a unit simplex in $\mathbb{R}^n$:

Lemma 2.2. *We define* $\tilde{s}_n$ *as the unit simplex in* $\mathbb{R}^n$ *with vertices that have coordinates* $\mathbf{x}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$*, where the one is positioned on the j-th position,* $j \in \{1, \ldots, n\}$*, and* $\mathbf{x}_{n+1}$ *is located on the origin. Let* $\lambda_j(\mathbf{t}) = t_j$ *for* $j \in \{1, \ldots, n\}$ *and* $\lambda_{n+1}(\mathbf{t}) = 1 - \sum_{j=1}^{n} t_j$ *(note that the barycentric coordinates are a partition of unity), and* $m_j \in \mathbb{N}$ *for* $j \in \{1, \ldots, n+1\}$*, then*

$$\int_{\tilde{s}_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n = \frac{\prod_{j=1}^{n+1} m_j!}{(n + \sum_{j=1}^{n+1} m_j)!}. \tag{7}$$

Proof. *We proceed by mathematical induction. For* $n = 1$*, we proved Lemma 2.1, which exactly gives equation (7). Suppose that equation (7) holds for* $n - 1$*, that is in* $\mathbb{R}^{n-1}$ *(starting at* $n = 2$*) then the equation should hold for n, that is in* $\mathbb{R}^n$*. Hence, equation (7) becomes*

$$\int_{\tilde{s}_{n-1}} \prod_{j=1}^{n} \lambda_j^{m_j} d\Omega_{n-1} = \frac{\prod_{j=1}^{n} m_j!}{(n - 1 + \sum_{j=1}^{n} m_j)!}. \tag{8}$$

The above equation is referblack to as the Induction Hypothesis. For the unit simplex $\tilde{s}_n$*, we have* $\lambda_j = t_j$ *for* $j \in \{1, \ldots, n\}$*, and* $\lambda_{n+1} = 1 - \sum_{j=1}^{n} t_j$*. The unit simplex* $\tilde{s}_n$ *is described by* $0 < t_1 < 1$, $0 < t_2 < 1 - t_1$, $\ldots$, $0 < t_{n-1} < 1 - \sum_{j=1}^{n-2} t_j$, $0 < t_n < 1 - \sum_{j=1}^{n-1} t_j$*. Further let* $d\Omega_n$ *be a hyper-volume element in* $\mathbb{R}^n$*, then*

$$\int_{\tilde{s}_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n = \int_{\tilde{s}_{n-1}} \int_0^{1-\sum_{j=1}^{n-1} t_j} (1 - \sum_{j=1}^{n} t_j)^{m_{n+1}} t_n^{m_n} \prod_{j=1}^{n-1} t_j^{m_j} dt_n d\Omega_{n-1}. \tag{9}$$

Carrying out integration by parts over the inner integral, yields

$$\int_{\tilde{s}_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n = \int_{\tilde{s}_{n-1}} [\frac{1}{m_n + 1}(1 - \sum_{j=1}^{n} t_j)^{m_{n+1}} t_n^{m_n+1}]_0^{1-\sum_{j=1}^{n-1} t_j} \prod_{j=1}^{n-1} t_j^{m_j} d\Omega_{n-1}$$
$$+ \int_{\tilde{s}_{n-1}} \frac{m_{n+1}}{m_n + 1} \int_0^{1-\sum_{j=1}^{n-1} t_j} t_n^{m_n+1} (1 - \sum_{j=1}^{n} t_j)^{m_{n+1}-1} dt_n \prod_{j=1}^{n-1} t_j^{m_j} d\Omega_{n-1}. \tag{10}$$

The boundary term in the above expression vanishes since it is zero on $t_n = 0$ *and on* $t_n = 1 - \sum_{j=1}^{n-1} t_j$*. Next, we proceed in the same way as in the proof of Lemma 2.1, where the inner integration over* t_n *is carried out recurrently, to arrive at*

$$\int_{\tilde{s}_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n = \frac{m_{n+1}! m_n!}{(m_n + m_{n+1} + 1)!} \int_{\tilde{s}_{n-1}} \prod_{j=1}^{n-1} t_j^{m_j} (1 - \sum_{j=1}^{n-1} t_j)^{m_n + m_{n+1} + 1} d\Omega_{n-1}. \tag{11}$$

Via the induction hypothesis, equation (8), and by observing that simplex $\tilde{s}_{n-1}$ *is described by* $0 < t_1 < 1$, $0 < t_2 < 1 - t_1$, $\ldots$, $0 < t_{n-1} < 1 - \sum_{j=1}^{n-2} t_j$*, with* $\lambda_j = t_j$ *for* $j \in \{1, \ldots, n-1\}$ *and* $\lambda_n = 1 - \sum_{j=1}^{n-1} t_j$*, we arrive at*

$$\begin{aligned} \int_{\tilde{s}_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n &= \frac{m_{n+1}! m_n!}{(m_n + m_{n+1} + 1)!} \cdot \frac{(m_n + m_{n+1} + 1)! \prod_{j=1}^{n-1} m_j!}{(n - 1 + m_1 + \ldots + m_{n-1} + m_n + m_{n+1} + 1)!} \\ &= \frac{\prod_{j=1}^{n+1} m_j!}{(n + \sum_{j=1}^{n+1} m_j)!}. \end{aligned} \tag{12}$$

This proves Lemma 2.2. □

Before we give and prove the generalisation of Lemma 2.2 by the use of a coordinate mapping, we state and prove the following equation that we need for the coordinate transformation:

Lemma 2.3. *Let $\vec{x}_1, \ldots, \vec{x}_n$ and $\vec{x}_{n+1}$ be column vectors pointing from the origin to the non-overlapping points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and $\mathbf{x}_{n+1}$ in $\mathbb{R}^n$,respectively, then*

$$det\begin{pmatrix} \vec{x}_1 - \vec{x}_{n+1} & \ldots & \vec{x}_n - \vec{x}_{n+1} \end{pmatrix} = det\begin{pmatrix} 1 & \vec{x}_1^{\,T} \\ \ldots & \ldots \\ 1 & \vec{x}_{n+1}^{\,T} \end{pmatrix},$$

where $\vec{x}_i^{\,T}$ denotes the row counterpart of the column vector $\vec{x}_i$ (hence the transpose).

Proof. *Since the value of a determinant does not change under addition of a multiple of a row to another row, it follows that if one subtracts the last row from each row in the right-hand side of the above equation in Lemma 2.3, then the value of the determinant does not change. From developing the determinant from the first column, and noting that the value of the determinant does not change under transposing of the matrix, the lemma follows immediately.* □

Note that the above determinant gives the volume of the hyperparallelepiped with vertices $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$. Next we state and prove the generalisation of Lemma 2.2:

Theorem 2.2. *We define s_n as a simplex in $\mathbb{R}^n$ with vertices that have coordinates $\mathbf{x}_j$, for $j \in \{1, \ldots, n+1\}$. Let $\lambda_j(\mathbf{x})$ be the barycentric coordinates on s_n, and $m_j \in \mathbb{N}$ for $j \in \{1, \ldots, n+1\}$, then*

$$\int_{s_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n = \frac{|\Delta_{s_n}| \prod_{j=1}^{n+1} m_j!}{(n + \sum_{j=1}^{n+1} m_j)!}, \quad where\ \Delta_{s_n} = det\begin{pmatrix} 1 & \vec{x}_1^{\,T} \\ 1 & \vec{x}_2^{\,T} \\ \ldots & \ldots \\ 1 & \vec{x}_{n+1}^{\,T} \end{pmatrix}. \qquad (13)$$

Proof. *We use the following parametrisation*

$$\mathbf{x}(t_1, t_2, \ldots, t_n) = \sum_{j=1}^{n} t_j \mathbf{x}_j + (1 - \sum_{j=1}^{n} t_j)\mathbf{x}_{n+1}.$$

Then the hypersimplex s_n *transforms into the unit simplex* $\tilde{s}_n$*, defined by* $0 < t_1 < 1$*,* $0 < t_2 < 1 - t_1, \ldots, 0 < t_n < 1 - \sum_{j=1}^{n-1} t_j$*, and the Jacobian matrix of the transformation becomes*

$$J = \begin{pmatrix} \vec{x}_1 - \vec{x}_{n+1} & \ldots & \vec{x}_n - \vec{x}_{n+1} \end{pmatrix}$$

Then the integral becomes

$$\int_{s_n} \prod_{j=1}^{n+1} \lambda_j^{m_j} d\Omega_n = |det\begin{pmatrix} \vec{x}_1 - \vec{x}_{n+1} & \ldots & \vec{x}_n - \vec{x}_{n+1} \end{pmatrix}| \cdot \int_{\tilde{s}_n} (1 - \sum_{j=1}^{n} t_j)^{m_{n+1}} \prod_{j=1}^{n} t_j^{m_j} d\Omega_n =$$

$$\left| det\begin{pmatrix} 1 & \vec{x}_1^{\,T} \\ 1 & \vec{x}_2^{\,T} \\ ... & ... \\ 1 & \vec{x}_{n+1}^{\,T} \end{pmatrix} \right| \cdot \int_{\tilde{s}_n} (1 - \sum_{j=1}^{n} t_j)^{m_{n+1}} \prod_{j=1}^{n} t_j^{m_j} d\Omega_n = |\Delta_{s_n}| \int_{\tilde{s}_n} (1 - \sum_{j=1}^{n} t_j)^{m_{n+1}} \prod_{j=1}^{n} t_j^{m_j} d\Omega_n$$

$$= |\Delta_{s_n}| \frac{\prod_{j=1}^{n+1} m_j!}{(n + \sum_{j=1}^{n+1} m_j)!}. \tag{14}$$

In the second equality above, Lemma 2.3 was used and in the second last equality we used Lemma 2.2. Therewith Theorem 2.2 has been proved. □

We note that the proof of the two-dimensional version can be found in [6], see page 84. Holand & Bell also base their proof on a coordinate transformation to a unit simplex (or triangle). Brenner and Scott [3] consider the proof of the two-dimensional version on page 92 in an exercise. In this exercise, it is also asked to give the three-dimensional version (without a proof). Bræss [2] and Van Kan *et al.* [14] give the two-dimensional version without the proof, where it should be noted that Van Kan *et al.* [14] also give the extension to *n*-dimensional spaces, though the proof is not given. Epstein [5] sketched some elements of the proof of Lemma 2.2 on a website, however, the proof did not appear in any refereed journal as far as we know and further the proof was not extended to Theorem 2.2. In the next subsection, we extend the theorem and proof to a simplex on a *d*-dimensional manifold in an *n*-dimensional space.

3. Integration over simplexes on manifold in $\mathbb{R}^n$

We quote our last result, which should be considered as a generalisation of Theorem 2.1, for the integration over a simplex on a d-dimensional manifold in an n-dimensional space $\mathbb{R}^n$ $(0 < d < n)$:

Theorem 3.1. *We define bs_d as a simplex on a d-dimensional manifold in $\mathbb{R}^n$ with vertices $\mathbf{x}_j$, $m_j \in \mathbb{N}$. Let $\lambda_j(\mathbf{x})$ be the barycentric coordinates on bs_d for $j \in \{1, \ldots, d+1\}$, then*

$$\int_{bs_d} \prod_{j=1}^{d+1} \lambda_j^{m_j} dS_d = \sqrt{|det(J^T J)|} \frac{\prod_{j=1}^{d+1} m_j!}{(d + \sum_{j=1}^{d+1} m_j)!}, \tag{15}$$

where

$$J = \begin{pmatrix} \vec{x}_1 - \vec{x}_{d+1} & \ldots & \vec{x}_d - \vec{x}_{d+1} \end{pmatrix} \in \mathcal{M}_{n \times d}(\mathbb{R}).$$

Proof. *Considering a simplex on a d-dimensional manifold in $\mathbb{R}^n$, we introduce the transformation*

$$\mathbf{x}(t_1, \ldots, t_d) = \sum_{j=1}^{d} t_j \mathbf{x}_j + (1 - \sum_{j=1}^{d} t_j)\mathbf{x}_{d+1}.$$

Then the Jacobian matrix, J, is given by

$$J = \begin{pmatrix} \vec{x}_1 - \vec{x}_{d+1} \ldots \vec{x}_1 - \vec{x}_{d+1}, \end{pmatrix}$$

which is an $n \times d$ matrix, and since it is not square, the metric is given by $\sqrt{|det(J^T J)|}$, see Spivak [11]. Further, $\lambda_j(\mathbf{x}(t_1, \ldots, t_d)) = t_j$ for $j \in \{1, \ldots, d\}$ and $\lambda_{d+1}(\mathbf{x}(t_1, \ldots, t_d)) = 1 - \sum_{j=1}^{d} t_j$ on the unit simplex $\tilde{bs}_d$. Then, it follows that

$$\int_{bs_d} \prod_{j=1}^{d+1} \lambda_j^{m_j} dS_d = \int_{\tilde{bs}_d} \sqrt{|det(J^T J)|} (1 - \sum_{j=1}^{d} t_j)^{m_{d+1}} \prod_{j=1}^{d} t_j^{m_j} dS_d =$$

$$\sqrt{|det(J^T J)|} \int_{\tilde{bs}_d} (1 - \sum_{j=1}^{d} t_j)^{m_{d+1}} \prod_{j=1}^{d} t_j^{m_j} dS_d = \sqrt{|det(J^T J)|} \frac{\prod_{j=1}^{d+1} m_j!}{(d + \sum_{j=1}^{d+1} m_j)!}. \tag{16}$$

Here we used Lemma 2.2 in the last step. This proves Theorem 3.1. □

4. Discussion and conclusions

We proved the generalised version of Holand & Bell's Integration rule, both for generic simplexes in $\mathbb{R}^n$ and for generic simplices on d-dimensional manifolds in $\mathbb{R}^n$. We note that Lemma 2.2 was also proved for the n-dimensional unit simplex by David Epstein on the *math overflow* website in 2015, see [5]. However in the literature, the proof for the generalisation to simplexes in $\mathbb{R}^n$, and for the extension to d-dimensional simplexes in $\mathbb{R}^n$, seems to be lacking. Possible reasons for the lack of such a proof in the literature, is its being straightforward or that many authors use Gauss-like numerical quadrature rules to approximate the integrals that are to be determined for the element matrices and vectors in the finite-element formalisms. Since this rule can be used to develop discretisation matrices without additional error in n-dimensional finite-element problems, and since it is found in many finite-element books as an unproved theorem, we decided to publish it. The results from this paper can be used to derive Newton-Cotes integration rules for the approximation of integrals over simplexes that are needed for the construction of higher dimensional finite-element methods. Theorem 3.1 allows the generalisation to simplexes on d-dimensional manifolds in n-dimensional space, which can be used to compute discretisation matrices over surface elements that either originate from natural boundary conditions or from surface (boundary) element methods. Currently, finite-element methods are being used more and more often in higher dimensional problems and also on manifolds. Herewith these integration rules for high dimensionality and on manifolds become more practical. Finally, note that formally Theorem 2.1 can be seen as a particular case of Theorem 3.1. We decided to state it explicitly in the manuscript because this type of integral often has to be dealt with when dealing with natural boundary conditions in 2-and 3-dimensional finite-element methods.

Acknowledgement: The authors thank one of the referees for his or her careful reading and for notifying us regarding some notational sloppiness.

References

[1] K. Atkinson, W. Han (2009) Theoretical numerical analysis. Springer, 3rd edition

[2] D. Bræss (2007) Finite Elements, theory, fast solvers, and applications in solid mechanics. Cambridge University Press, 3rd edition

[3] S.C. Brenner, L.R. Scott (2008) The mathematical theory of finite element methods. Springer, 3rd edition

[4] G. Dzuik, C.M. Elliott (2013) Finite element methods for surface PDEs. Acta Numerica, 22, 289–296

[5] D. Epstein (2015) Integrating barycentric monomial over a simplex. http://mathoverflow.net/questions/202820/integrating-a-barycentric-monomial-over-a-simplex/202821

[6] I. Holand, K. Bell (1969) Finite element methods in stress analysis. Tapir, Trondheim

[7] A. Madzvamuse, A. Wathen, P.K. Maini (2003) A moving grid finite element method applied to a biological pattern generator. Journal of Computational Physics, 190, 478–500

[8] M. Möller (2013) Algebraic flux correction for non-conforming finite element discretisations of scalar transport equations. Computing, 95 (5), 425–448

[9] H. bin Zubair, C.W. Oosterlee, R. Wienands (2007) Multigrid for high-dimensional elliptic partial differential equations on non-equidistant grids. SIAM Journal of Scientific Computing, 29 (4), 1613–1636

[10] A.J. Roberts (2009) Elementary calculus of financial mathematics. Society of Industrial and Applied Mathematics (SIAM), Philadelphia

[11] M. Spivak (1999) A comprehensive introduction to differential geometry. Publish or Perish Inc., Houston, Texas, 3rd edition

[12] G. Strang, G. Fix (1973) An analysis of the finite element method. Prentice-Hall, 2nd edition

[13] A. Quarteroni (2014) Numerical methods for differential problems. Springer, 2nd edition

[14] J. van Kan, A. Segal, F. Vermolen (2014) Numerical methods in scientific computing. Delft Academic Press, 2nd edition

[15] O.C. Zienkiewics, R.L. Taylor (2000) The finite element method. MPG Books Ltd, 5th edition