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Resilience of epidemics for SIS model on networks

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Epidemic propagation on complex networks has been widely investigated, mostly with invariant parameters. However, the process of epidemic propagation is not always constant. Epidemics can be affected by various perturbations and may bounce back to its original state, which is considered resilient. Here, we study the resilience of epidemics on networks, by introducing a different infection rate λ_2 during SIS (susceptible-infected-susceptible) epidemic propagation to model perturbations (control state), whereas the infection rate is λ_1 in the rest of time. Noticing that when λ_1 is below λ_c, there is no resilience in the SIS model. Through simulations and theoretical analysis, we find that even for $\lambda_2 < \lambda_c$, epidemics eventually could bounce back if the control duration is below a threshold. This critical control time for epidemic resilience, i.e., cd_{max}, seems to be predicted by the diameter (d) of the underlying network, with the quantitative relation $cd_{max} \sim d^{\alpha}$. Our findings can help to design a better mitigation strategy for epidemics. © 2017 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).
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Recently, the dynamic modeling of complex networks has become an important means for the analysis of epidemic propagation. However, in the field of epidemiology, most studies of epidemic spreading mainly focus on the phase of epidemic outbreak on networks with nearly invariant parameters. Epidemics can be affected by various perturbations and may bounce back to its original state, presenting corresponding resilient behaviors, which have hardly been studied. In this paper, we perform studies on the resilience of epidemics on networks by lowering the infection rate during control state. After adding the "control" stage, the simulation results on different types of networks show that the epidemic can restore to the original steady state in the finite network size under certain conditions. We find that the resilience of epidemic propagation depends on the infection rate λ_2 with duration cd of control stage. In addition, the threshold, cd_{max}, is strongly related to the network structure, which appears to scale with network diameters. The discovery of cd_{max} can provide advanced indicator for the resilience of epidemics, which can help to design protection strategy keeping systems from a secondary epidemic outbreaks.

I. INTRODUCTION

Complex systems[1] in various fields, ranging from natural to engineering systems, such as ecosystems, financial markets, and electric grids, can be viewed as complex networks. Such complex networks[2–4] are frequently subject to environmental changes or internal fluctuations. The dynamics on networks[5] may possess the capacity to retain the original state essentially after perturbations. Such an adaptive capability is defined as resilience.[6–10] For example, it is

shown that the food chain[11] in a biological network can withstand the shocks from a dramatic fall of one species and reorganize into a connected ecological web.

The resilience of epidemics here means that the spreading of epidemics recovers after various perturbations. Most studies on epidemic spreading mainly focus on the phase of the epidemic outbreak on networks.[12–19] Correspondingly, the parameters characterizing the processes of epidemic transmission on networks are nearly invariant in most mathematical epidemic models.[20–22] In some cases,[23–25] the epidemic spreading may be influenced or reduced by external control of self-repairing mechanism.[26,27] However, epidemic may continue to spread by absorbing the perturbations and recover to a stable trajectory, thereby presenting the above-mentioned resilient behaviors.

Resilient behaviors in the context of epidemic propagation have rarely been studied. In this paper, we study the resilience of epidemics on networks based on the classical epidemic model. Three well-known epidemic mathematical models are usually used in the study of epidemic transmission: SI (susceptible-infected) model, SIS (susceptible-infected-susceptible) model, and SIR (susceptible-infected-recovered) model.[12,28,29] The classical SIS epidemic model is one of the most general way to model the epidemic dynamical behaviors on networks such as Erdős-Rényi (ER) networks[30] and scale-free (SF) networks.[31] It is well known that there exists a nonzero epidemic threshold λ_c[12,32] for ER networks in the dynamics of epidemic outbreak. For a given infection rate $\lambda \geq \lambda_c$, the epidemic will spread out, and the system will reach a stationary state with a finite stable density ρ of the infected population. However, if the infection rate λ is below the epidemic threshold λ_c, the epidemic will ultimately die out, with no infected individuals (i.e., $\rho = 0$). In contrast, it has been demonstrated that the epidemic threshold does not exist for SF networks with $2 < \gamma \leq 3$.[14]

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This has prompted the propagation of viruses in such networks. When the parameter $\gamma > 3$, the epidemic will spread on these SF networks with an epidemic threshold λ_c.[33,34]

In this paper, we perform studies on the resilience of epidemic transmission with λ varying over time in the classical SIS epidemic model, to model the perturbations (i.e., "control"). When the epidemic propagation is controlled since a certain time instant (i.e., ct) for a duration cd, the infection rate (λ_2) is smaller than λ_1 in the uncontrolled state. Based on different durations of control (i.e., cd), the transmission level of the epidemic would reflect distinct resilient behaviors after the control. The issue being tackled in this paper is whether the epidemic propagation can present resilient behaviors and the corresponding critical condition.

According to the above-mentioned model, we analyze epidemic spreading on three types of real networks (Facebook network, Internet, and social network) and two types of network models (ER networks and SF networks). The simulation results have demonstrated that under certain conditions, the epidemic can bounce back to the initial steady state in the finite network scale. It is shown on ER networks that even for $\lambda_2 < \lambda_c$, the epidemic may eventually bounce back when control time is below a threshold. This is verified by our theoretical analysis of the recovery probability for epidemics, P, calculated by the equation $\rho(ct + cd) \rightarrow \frac{1}{N}$. The critical control time of the resilience (i.e., the critical time for the extinction of epidemics, cd_{max}) seems to be predicted by the diameter (d) of networks, with the quantitative relation $cd_{max} \sim d^{\alpha}$.

The contents of this paper are arranged as follows. Section II is devoted to introducing the model used to study the resilience of epidemics on networks based on the classic SIS epidemic model. In Sec. III, we perform simulations on different types of networks. Theoretical analysis is also performed on the obtained results. In Sec. IV, we draw the conclusions and present the discussions.

II. MODEL

In the SIS epidemic model, nodes in the network are divided into two compartments: susceptible individuals (S) and infected individuals (I). Initially, a fraction of nodes, which are randomly selected in the network, are infected. At each time step, each susceptible node is infected by each of its infected neighbors in the network with probability β. Each infected node is cured and becomes susceptible again with probability δ simultaneously. The effective infection rate is defined as $\lambda = \beta/\delta$.

In this paper, we study the resilience of epidemics under perturbations, by changing the infection rate λ to model the perturbations (i.e., "control"). The model applied in networks describing the whole propagation process can be classified into three sub-phases by adding "control," where the starting time and the duration of the "control" are defined as ct and cd, respectively:

Phase 1: The epidemic spreads with an infection rate λ_1 = β/δ_1 lasting ct time steps since the beginning $t = 0$. The probability β is set to a constant value in the entire process of epidemic transmission.

Phase 2: When the "control" is introduced in the second phase since time ct, the recovery rate δ_2 in the controlled network is given larger than δ_1, i.e., λ_2 is smaller than λ_1. This process will last for cd time steps.

Phase 3: The control is removed at time instant $ct + cd$ and the epidemic propagates with the infection rate λ_3, equal to λ_1 in phase 1.

In addition, the infection densities that we mainly observe are defined as ρ_1, ρ_c, and ρ_r for each stage. The model can be explained by the following example. In the Internet, some computers may become infected by a certain virus. Accordingly, anti-virus efforts ("control"), i.e., operations including certain immunization process and restoration with anti-virus software,[35] will be implemented to stop or lower the epidemic spreading, thus leading to a dramatic decline of the infection rate. When the control is removed due to the limited budgets or adaptation of virus, for different control durations cd, the epidemic may lose its resilience or resume to spread, therein presenting different resilient behaviors. We study the critical condition by both simulation and theoretical results that the epidemic can continue propagating after the control phase.

We show the results of epidemic resilience on three types of real networks (Facebook network, Internet, and social network) and two types of network models (ER networks and scale-free networks). The Facebook network originates from the database on Stanford Large Network Dataset Collection site.[36] It contains 4039 nodes and 88 234 edges, with the average degree of $\langle k \rangle \approx 43.691$. The data of Internet come from the UCI Network Data Repository,[41] whose average degree is $\langle k \rangle \approx 4.219$, including 22 963 nodes and 48 436 edges. The dataset of social network (Google+) with $\langle k \rangle \approx 3.322$ was collected by the Koblenz Network Collection,[36,42] which contains 23 628 nodes and 39 242 edges. In addition, these real networks have degree distribution, following a power-law distribution with the exponent $\gamma \approx 1.3$, 2.5, and 2.6, respectively. For an ER network with N nodes, each node pair is independently connected with a probability p. Then, we create an ER graph with a Poisson degree distribution[37] described as follows:

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}. \tag{1}$$

We create scale-free networks via the configuration model,[3] following a scale-free distribution $P(k) \sim k^{-\gamma}$, where γ is the degree exponent.

In this paper, we first focus on the key quantity about the epidemics, i.e., the infection density ρ to study the resilient behaviors of epidemic propagation. Meanwhile, the critical condition for the emergence of the resilient behaviors will be studied. We also explore the probability of recovery for epidemics, P, as a function of infection rate λ_2 in the second phase and control duration cd. In order to find the critical recovery time of resilience for the epidemic to "bounce back," we calculate the infection density ρ_r in the third stage, as a function of control duration (cd). In addition, understanding the relation between the resilience of epidemics and network structure is essential to design the resilience
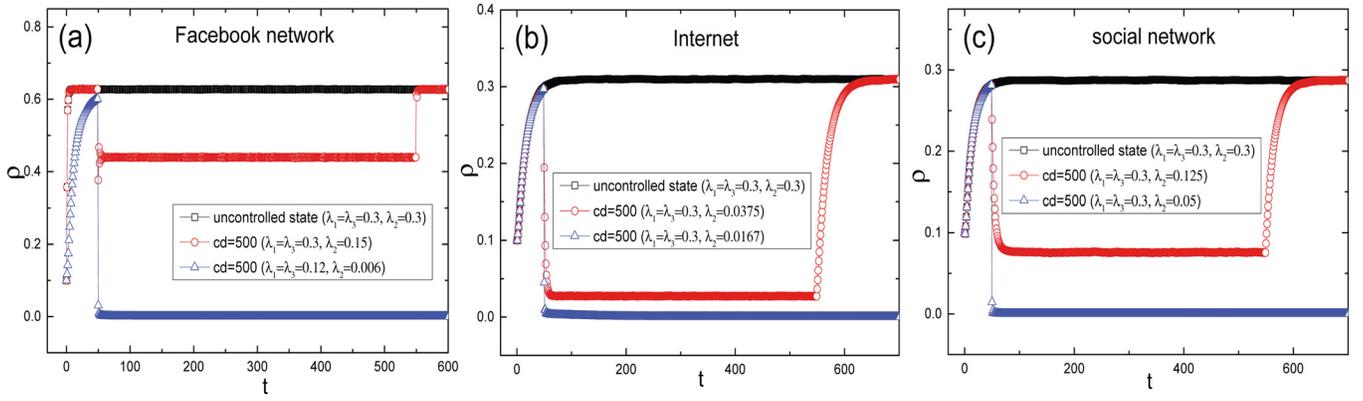
FIG. 1. The infection density ρ as a function of time on real networks. (a) Facebook network with $\langle k \rangle \approx 43.691$ and $N = 4039$. The infection rates are $\lambda_1 = \lambda_3 = 0.3$ and $\lambda_2 = 0.15$ (circle). The infection rates are $\lambda_1 = \lambda_3 = 0.12$ and $\lambda_2 = 0.006$ (triangle). (b) Internet with $\langle k \rangle \approx 4.219$ and $N = 22963$. The infection rates are $\lambda_1 = \lambda_3 = 0.3$ and $\lambda_2 = 0.0375$ (circle) and $\lambda_2 = 0.0167$ (triangle). (c) Social network with $\langle k \rangle \approx 3.322$ and $N = 23628$. The infection rates are $\lambda_1 = \lambda_3 = 0.3$ and $\lambda_2 = 0.125$ (circle) and $\lambda_2 = 0.05$ (triangle). Initially, 10% of nodes, which are randomly selected in network, are infected. The starting time of control is $ct = 50$. The control duration is set to $cd = 500$. The numerical results are averaged over 300 iterations.

strategy. The relation between the critical control time of resilience cd_{max} and diameters (d) seems to be identified on ER and SF networks.

III. RESULTS

To study the resilient behaviors of epidemic propagation, we first perform simulations to obtain the infection density ρ as a function of time on different types of networks. To observe the processes of epidemic transmission on real networks, Facebook network, Internet, and social network are examined with the SIS epidemic model, respectively. As shown in Fig. 1(a), the epidemic spreads rapidly in the uncontrolled stage, since there exits hub nodes in the topological structure of Facebook network. The infection density is decreased when the epidemic is controlled with a lower infection rate since time instant ct. When the control stage is finished and infection rate is recovered, the epidemic can soon restore to the same state before the control. For the Internet shown in Fig. 1(b), the epidemic can bounce back to the original steady state when the "control" is removed. As shown in Fig. 1(c), the simulations on social network show that the infection density can return to the initial stationary state after the "control," when λ_2 is below that of uncontrolled state. The infected individuals are decreased to zero with a quite small λ_2 and the epidemic propagation comes to an end without exhibiting resilience. The findings on the real networks illustrate that under certain conditions, the epidemic can bounce back to the original steady state in the finite network scale.

Then, we carry out simulations of the epidemic propagation on ER and SF networks. In the case of ER networks, we acquire the results obtained from simulations shown in Fig. 2(a). It shows that the infection density ρ is decreased quickly once the "control" is added. When the "control" is removed, the epidemic can also continue to spread and recover to a steady state. It is known that the epidemic threshold $\lambda_c = 1/\langle k \rangle$ through the theoretical calculation,[38] when there is no control in ER networks. Here, the infection rate for control phase is smaller than λ_c. Therefore, the transmission capability of epidemics decreases with "control" added. Afterwards, it recovers in the uncontrolled stage
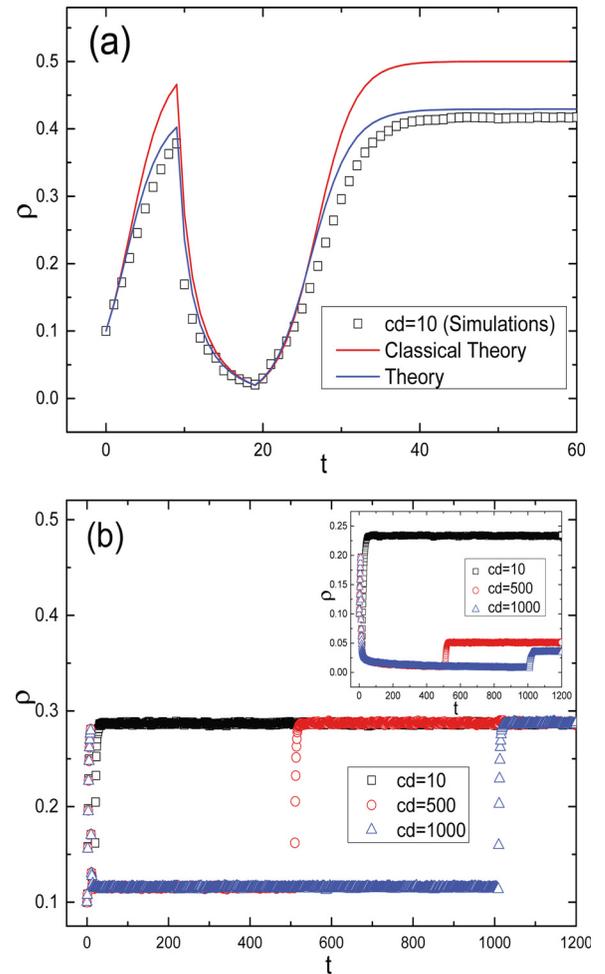

FIG. 2. The infection density ρ as a function of time on ER and SF networks. (a) Simulation results (splashes) on ER networks with $\langle k \rangle = 10$ for $cd = 10$, compared with the theoretical values (solid line) as obtained from the classical mean-field rate equation (red line) and Eqs. (3) and (4) (blue line). The infection rates are $\lambda_1 = \lambda_3 = 0.2$ for the uncontrolled period and $\lambda_2 = 0.08 < \lambda_c = 0.1$ for the controlled period, respectively. (b) Simulation results (splashes) for SF networks with $\langle k \rangle \approx 5.4$ ($m = 2$ and $\gamma = 2.5$) [SF networks with $\langle k \rangle \approx 4.5$ ($m = 3$, $\gamma = 3.5$) in the inset] for $cd = 10$, 500, and 1000. The infection rates are $\lambda_1 = \lambda_3 = 0.3$ for the uncontrolled period and $\lambda_2 = 0.15$ for the controlled period. Initially, 10% of nodes randomly selected in two networks are infected. The starting time of control is $ct = 10$. The size of networks is $N = 10^3$. The numerical results are averaged over 300 iterations.

showing resilient behaviors. For SF networks with $\gamma = 2.5$ illustrated in Fig. 2(b), epidemics can always bounce back to its original state in our simulations at different control durations. This may be due to the fact that the network has no epidemic threshold for $2 < \gamma \leq 3$. The findings explicitly demonstrate the existence of the resilient behaviors for the epidemic propagation on SF networks. In addition, in the inset of Fig. 2(b), the changes of infected density ρ with various given cd on SF networks with $\gamma = 3.5$ show that the epidemic could not exhibit resilience with a quite large cd. Meanwhile, the dynamical mean-field rate equation is applied to describe the processes analytically. For comparison, the classical mean-filed rate equation is shown below:

$$\frac{\partial \rho(t)}{\partial t} = \beta \langle k \rangle \rho(t)(1 - \rho(t)) - \delta \rho(t). \quad (2)$$

The first quantity we study here in ER networks is the density of infected nodes $\rho(t)$. The governing equation, which is mainly used for the analysis of this work, follows:

$$\frac{\partial \rho(t)}{\partial t} = \left(1 - (1 - \beta)^{\langle k \rangle \rho(t)}\right)(1 - \rho(t)) - \delta_1 \rho(t), \quad (3)$$

when $t \leq ct$ or $t > ct + cd$.

$$\frac{\partial \rho(t)}{\partial t} = \left(1 - (1 - \beta)^{\langle k \rangle \rho(t)}\right)(1 - \rho(t)) - \delta_2 \rho(t), \quad (4)$$

when $ct < t \leq ct + cd$.

It is found that the simulation results on ER networks can be well matched by the theoretical analysis obtained from Eqs. (3) and (4). However, we found that the classical mean-field rate equation can generate different results compared with our simulations. Through the analysis of each part of two equations, we found that the problem mainly comes from the infection process of classical mean-field theory, which is not consistent with that of governing equation in this work. Because, in the process of infection, we assume that a node being infected by each of its infectious neighbors is not independent. And the further explanation is that item $\beta \langle k \rangle \rho(t)$ in the classical mean-field theory of SIS model has

different probability consideration from item $1 - (1 - \beta)^{\langle k \rangle \rho(t)}$ in Eqs. (3) and (4).

In order to find the critical condition for the emergence of the above-mentioned resilient behaviors, we perform the theoretical and simulation analysis for the probability of epidemics returning to the steady state in ER networks, P, as a function of infection rate λ_2 in the controlled stage. For theoretical analysis, the probability of recovery for epidemics P can be calculated by the mean-field equation written as

$$P = \begin{cases} 0, & \rho(ct + cd) \leq \frac{1}{N} \\ 1, & \rho(ct + cd) > \frac{1}{N}. \end{cases} \quad (5)$$

In Fig. 3(a), the probability of epidemic restoration, P, grows to 1.0 eventually with increasing λ_2 for a given cd. There exists a critical infection rate λ_2^c in control stage for epidemic resilience. Meanwhile, even for $\lambda_2 < \lambda_c$, where λ_c is the epidemic threshold ($\lambda_c = 1/\langle k \rangle$) shown by dash line, the epidemic may still bounce back, which is due to the fact that the infected density did not drop below the necessary threshold during the time cd. When the equation satisfies $\rho(ct + cd) \leq \frac{1}{N}$ in theory, the epidemic is considered to lose the resilience. The theoretical results, based on the above-mentioned calculation method, can predict the critical λ_2^c, verified by the simulation results. It illustrates that the epidemics will bounce back with $\lambda_2 \geq \lambda_2^c$ ($\lambda_2 < \lambda_c$) for a given control duration cd. This threshold for epidemic resilience depends on the combined effect of epidemic spread and control processes.

To explore the effect of the control duration cd on the probability for epidemics to return to the steady state, we perform the theoretical and simulation analysis for P as a function of cd in ER networks. As shown in Fig. 3(b), the probability P is decreased with the increasing cd for a given infection rate λ_2 ($< \lambda_c = 0.1$). When the control duration (cd) is small, the probability maintains 1.0. As cd increases, the recovery probability is decreased and finally reaches to zero. It is shown that there exists a threshold (i.e., cd_{max}) that makes the epidemic lose the resilience completely. And cd_{max} can also be regarded as the
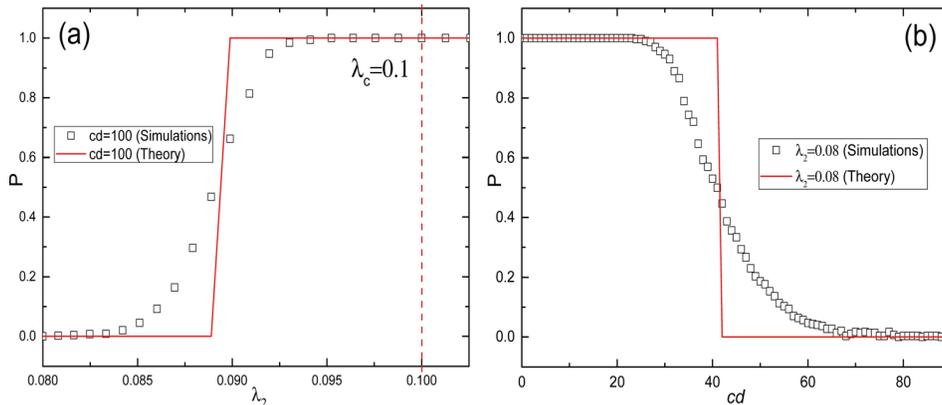

FIG. 3. The probability of epidemics returning to the steady state P as a function of the infection rate $\lambda_2 = \beta/\delta_2$ in control stage shown in (a) and as a function of control duration (cd) shown in (b). Simulation results (splashes) for ER networks with size of $N = 10^4$, compared with the theory (solid line). (a) ER networks with $\langle k \rangle = 10$. The control duration is set to $cd = 100$. The infection rate is $\lambda_1 = \lambda_3 = 0.2$ in the uncontrolled stage. The dashed line represents the value of epidemic threshold, i.e., $\lambda_c = 1/10$. (b) ER networks with $\langle k \rangle = 10$. The infection rates are $\lambda_1 = \lambda_3 = 0.2$ for the uncontrolled period and $\lambda_2 = 0.08 < \lambda_c = 0.1$ for the controlled period, respectively. Initially, 10% of nodes, which are randomly selected, are infected. The starting time of the control is $ct = 10$. The results above have been averaged over 300 realizations.
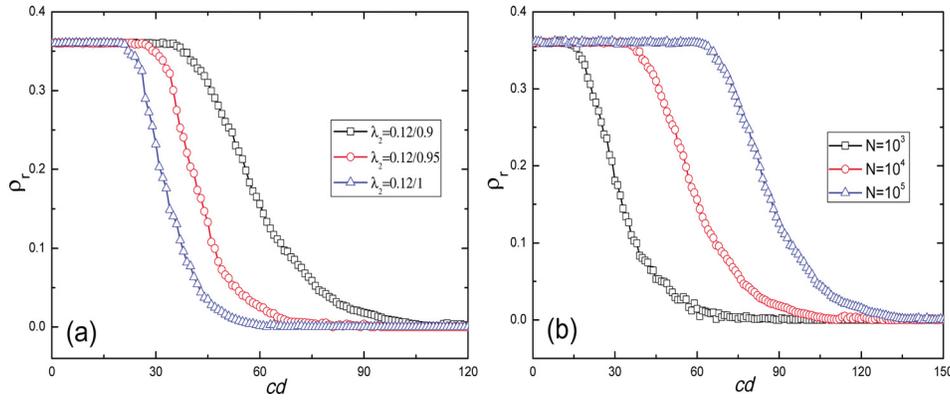
FIG. 4. The infection density after control stage, ρ_r, obtained in the steady state, as a function of control duration (cd). (a) ER networks with $\langle k \rangle = 6$ and $N = 10^4$. The infection rate in a controlled state is $\lambda_2 = 0.12/0.9$, $0.12/0.95$, and $0.12/1$. (b) ER networks with $\langle k \rangle = 6$ by setting $N = 10^3$, 10^4, and 10^5, given $\lambda_2 = 0.12/0.9$. Initially, 10% of nodes, which are randomly selected, are infected. The starting time of the control is $ct = 10$. The infection rates are $\lambda_1 = \lambda_3 = 0.3$ in the uncontrolled stage. The results above have been averaged over 300 realizations.

critical recovery time of resilience. The theoretical results (solid line) are obtained based on the Eq. (5). Our theoretical analysis can also predict the critical control duration, cd_{max}, above which the epidemic will lose its resilience. It further illustrates that even for $\lambda_2 < \lambda_c$, the epidemic is still possible to bounce back, when the control duration, cd, is not long enough.

As the resilience for the epidemic propagation is affected by the infection rate λ_2 in the second stage of transmission and the control duration (cd), we study the resilience output with the steady-state density in the third stage, ρ_r, as a function of cd. As shown in Fig. 4(a), for a fixed λ_2, the infection density ρ_r is decreased as cd is increased for an ER network. When cd is increased to a certain value, ρ_r is almost reduced to zero. This confirms our finding for cd_{max} in Fig. 3. It can also be seen that ρ_r falls to zero earlier with a smaller infection rate λ_2 than that with a large λ_2. To study the effect of network size on the resilience of epidemics, we perform simulations to obtain ρ_r as a function of cd in ER networks with different sizes. As shown in Fig. 4(b), in the case of ER networks, ρ_r is reduced to zero gradually with increasing cd time steps for various N. Infection density after control stage, ρ_r, is decreased more quickly with a relatively small N. It is considered that cd_{max} is larger for a large N.

It is meaningful to understand the relation between epidemic resilience and network structure, which can help to design the resilience strategy. Therefore, we perform simulations for cd_{max} as a function of diameters (d),[1–3,39,40] which is calculated by averaging the number of hops between each pair of nodes in ER networks and SF networks, respectively. In Fig. 5(a), for a given λ_2, the cd_{max} seems to be found to scale with network diameters in ER networks, $cd_{max} \sim d^\alpha$. In the case of SF networks with $\gamma = 3.5$, as shown in Fig. 5(b), there exits the critical recovery time of resilience (cd_{max}), which has a different quantitative relation with diameters (d) from that of ER networks. The findings on ER and SF networks illustrate that the time (distance) that epidemics take to spread over the entire network is critical to estimate the resilience of this process. In addition, α is increasing with increasing λ_2.

IV. CONCLUSIONS

By adding a "control" stage in the original SIS model, we model the resilience of epidemic propagation under perturbations.

When the network is in the controlled state, the epidemic transmits with a smaller infection rate λ_2 than that in the uncontrolled state. When the control is removed, the epidemic may restore to a steady state exhibiting resilient behaviors.
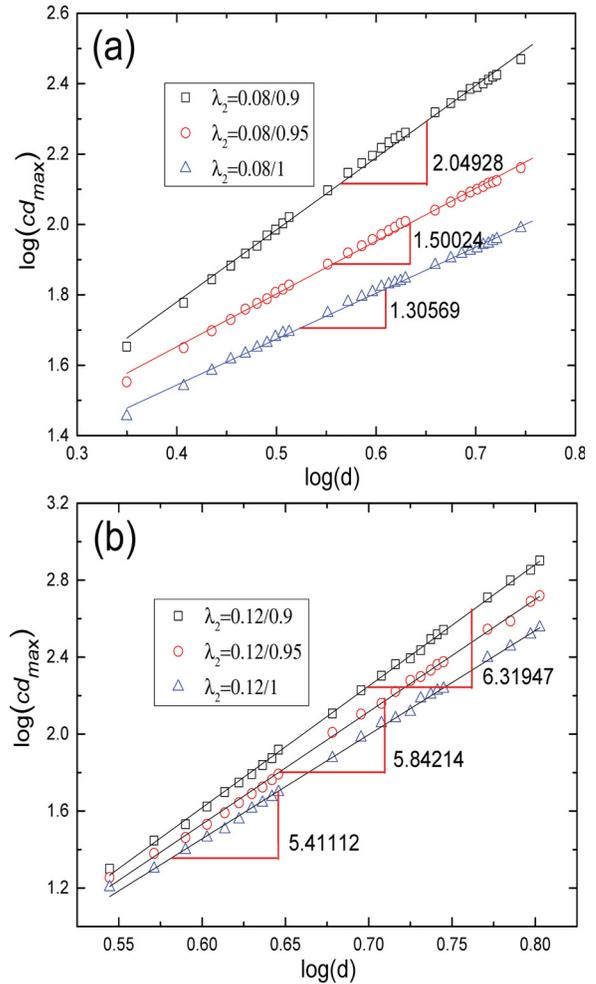

FIG. 5. The threshold cd_{max} as a function of diameters (d) of networks. (a) ER networks with $\langle k \rangle = 10$. The infection rate is given as $\lambda_2 = 0.08/0.9$, $0.08/0.95$, and $0.08/1$ in control stage. The infection rates are $\lambda_1 = \lambda_3 = 0.2$ in the uncontrolled stage. (b) SF networks with $\langle k \rangle \approx 4.5$ ($m = 3$ and $\gamma = 3.5$). The infection rate is given as $\lambda_2 = 0.12/0.9$, $0.12/0.95$, and $0.12/1$ in control stage. The infection rates are $\lambda_1 = \lambda_3 = 0.3$ in the uncontrolled stage. Initially, 10% of nodes, which are randomly selected, are infected. The starting time of the control is $ct = 10$. The results are averaged over 100 realizations.

Based on the above-mentioned model, we performed numerical simulations on three types of real networks (Facebook network, Internet, and social networks) and two types of network models (ER and SF networks). The simulation results indicate that under certain condition, the epidemic can restore to the original steady state in the finite network size. Through the simulations and theoretical analysis on ER networks, it is shown that even for $\lambda_2 < \lambda_c$, the epidemic may eventually bounce back when the control duration, cd, is smaller than a threshold cd_{max}. It can be verified by the theoretical results of the recovery probability for epidemics, P, computed by the equation, $\rho(ct + cd) \rightarrow \frac{1}{N}$. The critical value cd_{max} is strongly related to the network structure, where cd_{max} appears to be predicted by the diameter d of networks with the quantitative relation, written as $cd_{max} \sim d^\alpha$. Note that the maximum time of control for epidemics extinction is increasing with system size and will diverge for an infinite system. The resilience is an intrinsic property for epidemics to adapt to the external perturbations and the changes of internal conditions. It can also issue signals for systems to mitigate the epidemic rapidly and accurately when they have been infected. The discovery of cd_{max} may provide advanced indicator for the resilience of the epidemic, which can help to design protection strategy keeping systems from a secondary epidemic outbreaks.

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