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Koppensteiner, Sarah; van Velthoven, Jordy Timo; Voigtlaender, Felix

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Anisotropic Triebel-Lizorkin spaces and wavelet coefficient decay over one-parameter dilation groups, II

Sarah Koppensteiner[1] · Jordy Timo van Velthoven[2] · Felix Voigtlaender[3]

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Abstract

Continuing previous work, this paper provides maximal characterizations of anisotropic Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{p,q}$ for the endpoint case of $p = \infty$ and the full scale of parameters $\alpha \in \mathbb{R}$ and $q \in (0, \infty]$. In particular, a Peetre-type characterization of the anisotropic Besov space $\dot{\mathbf{B}}^{\alpha}_{\infty,\infty} = \dot{\mathbf{F}}^{\alpha}_{\infty,\infty}$ is obtained. As a consequence, it is shown that there exist dual molecular frames and Riesz sequences in $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$.

Keywords Anisotropic Triebel-Lizorkin spaces · Maximal functions · Anisotropic wavelet systems · Coorbit molecules · Frames · Riesz sequences · One-parameter groups

Mathematics Subject Classification 42B25 · 42B35 · 42C15 · 42C40 · 46B15

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✉ Jordy Timo van Velthoven
j.t.vanvelthoven@tudelft.nl

Sarah Koppensteiner
sarah.koppensteiner@univie.ac.at

Felix Voigtlaender
felix.voigtlaender@ku.de

1 Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

2 Delft University of Technology, Mekelweg 4, Building 36, 2628 CD Delft, The Netherlands

3 Mathematical Institute for Machine Learning and Data Science (MIDS), Katholische Universität Eichstätt-Ingolstadt, Research group Reliable Machine Learning, Ostenstrasse 26, 85072 Eichstätt, Germany

1 Introduction

In a previous paper [20], we obtained characterizations of anisotropic Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{p,q}$, with $\alpha \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$, in terms of Peetre-type maximal functions and continuous wavelet transforms. In addition, as an application of these characterizations, it was shown that these spaces admit molecular dual frames and Riesz sequences. The purpose of the present paper is to provide analogous results for the endpoint case of $p = \infty$.

For defining the anisotropic Triebel-Lizorkin spaces, let $A \in \mathrm{GL}(d,\mathbb{R})$ be an expansive matrix, i.e., $|\lambda| > 1$ for all $\lambda \in \sigma(A)$, and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that it has compact Fourier support

$$\operatorname{supp}\widehat{\varphi} = \overline{\{\xi \in \mathbb{R}^d : \widehat{\varphi}(\xi) \neq 0\}} \subset \mathbb{R}^d \setminus \{0\} \tag{1.1}$$

and satisfies

$$\sup_{j\in\mathbb{Z}} |\widehat{\varphi}((A^*)^j \xi)| > 0, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \tag{1.2}$$

with A^* denoting the transpose of A. For $j \in \mathbb{Z}$, denote its dilation by $\varphi_j = |\det A|^j \varphi(A^j \cdot)$. The associated (homogeneous) *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}^{\alpha}_{\infty,q} = \dot{\mathbf{F}}^{\alpha}_{\infty,q}(A,\varphi)$, with $\alpha \in \mathbb{R}$ and $q \in (0,\infty]$, is defined as the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ (modulo polynomials) that satisfy

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} := \sup_{\ell\in\mathbb{Z},k\in\mathbb{Z}^d} \left(\frac{1}{|\det A|^{\ell}} \int_{A^{\ell}([0,1]^d+k)} \sum_{j=-\ell}^{\infty} \left(|\det A|^{\alpha j} |(f * \varphi_j)(x)|\right)^q dx \right)^{1/q} < \infty \tag{1.3}$$

if $q < \infty$, and

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} := \sup_{\ell\in\mathbb{Z},k\in\mathbb{Z}^d} \sup_{j\in\mathbb{Z},j\geq-\ell} \left(\frac{1}{|\det A|^{\ell}} \int_{A^{\ell}([0,1]^d+k)} |\det A|^{\alpha j} |(f * \varphi_j)(x)|\, dx \right) < \infty. \tag{1.4}$$

Note that the $\ell^{\infty}(\mathbb{Z}_{\geq-\ell})$-norms in Eq. (1.4) are positioned outside of the integral, whereas the $\ell^{q}(\mathbb{Z}_{\geq-\ell})$-norms in Eq. (1.3) are part of the integrand.

In contrast to the usual quasi-norms defining (anisotropic) Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{p,q}$ for $p < \infty$ (see, e.g., [3–5, 20]), the quantities (1.3) and (1.4) consider only averages over small scales. The quasi-norms (1.3) and (1.4) can therefore be considered as "localized versions"of the immediate analogue of the quasi-norms defining $\dot{\mathbf{F}}^{\alpha}_{p,q}$ for $p < \infty$, which would lead to an unsatisfactory definition of $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$, see [11, Sect. 5] and the references therein.

The above definition of $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$ follows Bownik [3, Sect. 3] (see also [11, Sect. 5]) for $q < \infty$, but differs from [3, 11] for $q = \infty$, where $\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}$ is instead defined via the quasi-norm

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} = \|f\|_{\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}} := \sup_{j\in\mathbb{Z}} |\det A|^{\alpha j} \|f * \varphi_j\|_{L^\infty} < \infty, \tag{1.5}$$

with $\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}$ denoting the (anisotropic) Besov space [2]. The quasi-norm (1.4) is an anisotropic version of the definition given by Bui and Taibleson [6], which (as a consequence of our main results) will be shown to be equivalent to (1.5), like for isotropic dilations [6, Theorem 3].

1.1 Maximal characterizations

As in [20], we assume additionally that the expansive matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is *exponential*, in the sense that $A = \exp(B)$ for some matrix $B \in \mathbb{R}^{d\times d}$. The power of A is then defined as $A^s := \exp(sB)$ for $s \in \mathbb{R}$.

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $\beta > 0$, the associated Peetre-type maximal function of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\varphi^{**}_{s,\beta} f : \mathbb{R}^d \to [0,\infty], \quad x \mapsto \sup_{z\in\mathbb{R}^d} \frac{|f * \varphi_s(x+z)|}{(1+\rho_A(A^s z))^\beta},$$

where $\varphi_s := |\det A|^s \varphi(A^s \cdot)$ and where $\rho_A : \mathbb{R}^d \to [0,\infty)$ denotes the step homogeneous quasi-norm associated with A (cf. Sect. 2.1).

The following Peetre-type maximal characterizations of $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$ will be proven in Sect. 3. Here, the notation $\fint_Q$ means the average integral over a measurable set $Q \subseteq \mathbb{R}^d$ of positive measure.

Theorem 1.1 *Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and exponential and let $\alpha \in \mathbb{R}$. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ has compact Fourier support satisfying the support conditions (1.1) and (1.2).*

For $q \in (0,\infty)$ and $\beta > 1/q$, the norm equivalences

$$\begin{aligned} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} &\asymp \sup_{\ell\in\mathbb{Z}, k\in\mathbb{Z}^d} \left(\fint_{A^\ell([0,1]^d+k)} \int_{-\ell}^{\infty} \big(|\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x)\big)^q \, ds \, dx \right)^{1/q} \\ &\asymp \sup_{\ell\in\mathbb{Z}, k\in\mathbb{Z}^d} \left(\fint_{A^\ell([0,1]^d+k)} \sum_{j=-\ell}^{\infty} \big(|\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x)\big)^q \, dx \right)^{1/q} \end{aligned} \tag{1.6}$$

hold for all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$. For $q = \infty$ and $\beta > 1$, the following equivalences hold

$$\begin{aligned} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} &\asymp \sup_{\ell\in\mathbb{Z}, k\in\mathbb{Z}^d} \sup_{s\in\mathbb{R}, s\geq -\ell} \left(\fint_{A^\ell([0,1]^d+k)} |\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x) \, dx \right) \\ &\asymp \sup_{\ell\in\mathbb{Z}, k\in\mathbb{Z}^d} \sup_{j\in\mathbb{Z}, j\geq -\ell} \left(\fint_{A^\ell([0,1]^d+k)} |\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x) \, dx \right) \end{aligned} \tag{1.7}$$

for all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$.

Theorem 1.1 provides an extension of [6, Theorem 1] to arbitrary expansive dilations, and appears to be new even for the commonly studied setting of diagonal dilation matrices $A = \mathrm{diag}(\alpha_1, ..., \alpha_d)$ with given anisotropy $(\alpha_1, ..., \alpha_d) \in (1, \infty)^d$.

The proof method of Theorem 1.1 is modeled on the proof of the maximal characterizations of Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{p,q}$ with $p < \infty$ given in [20]. In particular, it combines a sub-mean-value property of the Peetre-type maximal function (Proposition 3.2) with maximal inequalities. See also [22, 23, 27, 31] for similar approaches. Besides the similarities in the approach, the calculations in the proof of Theorem 1.1 differ non-trivially from these in [20, Theorem 3.5] as only averages over small scales appear in the definition of $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$.

As a consequence of Theorem 1.1, we show in Sect. 4 the coincidence $\dot{\mathbf{F}}^{\alpha}_{\infty,\infty} = \dot{\mathbf{B}}^{\alpha}_{\infty,\infty}$ mentioned above.

1.2 Molecular decompositions

For an expansive and exponential matrix $A \in \mathrm{GL}(d, \mathbb{R})$, denote by $G_A = \mathbb{R}^d \rtimes_A \mathbb{R}$ the associated semi-direct product group. Then G_A acts unitarily on $L^2(\mathbb{R}^d)$ via the quasi-regular representation π, defined by

$$\pi(x, s) f = |\det A|^{-s/2} f(A^{-s}(\cdot - x)), \quad (x, s) \in \mathbb{R}^d \times \mathbb{R}, \ f \in L^2(\mathbb{R}^d). \tag{1.8}$$

A vector $\psi \in L^2(\mathbb{R}^d)$ is called *admissible* if the associated wavelet transform

$$W_\psi : L^2(\mathbb{R}^d) \to L^\infty(G_A), \quad W_\psi f = \langle f, \pi(\cdot)\psi\rangle,$$

defines an isometry into $L^2(G_A)$. The existence of admissible vectors and associated Calderón-type reproducing formulae for this representation have been studied, among others, in [9, 12, 17, 21]. The assumption that A is expansive is essential for the existence of admissible vectors $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfying $\widehat{\psi} \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ (cf. Lemma 5.2), which play an important role in this paper.

A countable family $(\phi_\gamma)_{\gamma\in\Gamma}$ of functions $\phi_\gamma \in L^2(\mathbb{R}^d)$ parametrized by a discrete set $\Gamma \subset G_A$ is called a *molecular system* if there exists a function $\Phi \in \mathcal{W}(L^r_w) \subset L^1(G_A)$ such that

$$|W_\psi \phi_\gamma(g)| = |\langle \phi_\gamma, \pi(g)\psi\rangle| \le \Phi(\gamma^{-1}g), \quad \gamma \in \Gamma, \ g \in G. \tag{1.9}$$

The space $\mathcal{W}(L^r_w)$ denotes a weighted Wiener amalgam space for $r = \min\{1, q\}$ and the standard control weight $w = w^{\alpha',\beta}_{\infty,q} : G_A \to [1, \infty)$; see Sect. 6 for further details.

It should be mentioned that any family $(\pi(\gamma)\phi)_{\gamma\in\Gamma}$ for suitable $\phi \in L^2(\mathbb{R}^d)$ defines a molecular system in the sense of (6.1) with $\Phi = |W_\psi \phi|$, but that generally a molecular system $(\phi_\gamma)_{\gamma\in\Gamma}$ does not need to consist of translates and dilates of a fixed function ϕ. Nevertheless, general molecules $(\phi_\gamma)_{\gamma\in\Gamma}$ share many properties with atoms $(\pi(\gamma)\phi)_{\gamma\in\Gamma}$, see, e.g., [18, 32].

The following theorem provides decomposition theorems of $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$ in terms of molecules.

Theorem 1.2 *Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and exponential. For $\alpha \in \mathbb{R}$, $q \in (0, \infty]$, let $r := \min\{1, q\}$ and let $\alpha' = \alpha + 1/2 - 1/q$ if $q < \infty$ and $\alpha' = \alpha + 1/2$, otherwise. Let $\beta > 1/q$ if $q < \infty$ and $\beta > 1$ otherwise.*

Suppose $\psi \in L^2(\mathbb{R}^d)$ is admissible satisfying $W_\psi \psi \in \mathcal{W}(L^r_w)$ for the standard control weight $w = w^{-\alpha',\beta}_{\infty,q} : G_A \to [1, \infty)$ defined in Lemma 5.8. Additionally, suppose that $W_\varphi \psi \in \mathcal{W}(L^r_w)$ for some (equivalently, all) admissible $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$. Then there exists a compact unit neighborhood $U \subset G_A$ with the following property: For any discrete set $\Gamma \subset G_A$ satisfying

$$G_A = \bigcup_{\gamma \in \Gamma} \gamma U \quad \text{and} \quad \sup_{g \in G_A} \#(\Gamma \cap gU) < \infty, \tag{1.10}$$

there exists a molecular system $(\phi_\gamma)_{\gamma \in \Gamma}$ such that any $f \in \dot{\mathbf{F}}^\alpha_{\infty,q}$ admits the expansion

$$f = \sum_{\gamma \in \Gamma} \langle f, \pi(\gamma)\psi \rangle \phi_\gamma = \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \pi(\gamma)\psi,$$

where the series converges unconditionally in the weak-topology of $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$.*

Atomic decompositions of the anisotropic spaces $\dot{\mathbf{F}}^\alpha_{\infty,q}$ have been obtained earlier by Bownik [3]. However, Theorem 1.2 provides a frame decomposition of all elements $f \in \dot{\mathbf{F}}^\alpha_{\infty,q}$ in terms of the atoms $(\pi(\gamma)\psi)_{\gamma \in \Gamma}$ and molecules $(\phi_\gamma)_{\gamma \in \Gamma}$, whereas the atoms in [3, Theorem 5.7] depend on the element $f \in \dot{\mathbf{F}}^\alpha_{\infty,q}$ that is represented. For anisotropic Triebel-Lizorkin spaces $\dot{\mathbf{F}}^\alpha_{p,q}$ with $p < \infty$, decompositions as in Theorem 1.2 were obtained in [19, 20], but they appear to be new for the case of $p = \infty$. In fact, Theorem 1.2 seems even valuable for merely isotropic dilations, where the state-of-the-art [13] excludes the case $p = \infty$.

Theorem 1.2 will be obtained from the recent results on dual molecules [26, 32] through the identification of $\dot{\mathbf{F}}^\alpha_{\infty,q}$ with a coorbit space; see Proposition 5.10. This identification appears to be new for the full scale of $\dot{\mathbf{F}}^\alpha_{\infty,q}$ with $\alpha \in \mathbb{R}$ and $q \in (0, \infty]$, even for isotropic dilations.

In addition to the existence of dual molecular frames, we also obtain a corresponding result for Riesz sequences. Here, the space $\dot{\mathbf{p}}^{-\alpha',\beta}_{\infty,q}$ denotes a sequence space associated to $\dot{\mathbf{F}}^\alpha_{\infty,q}$; see Definition 6.3 for its precise definition.

Theorem 1.3 *With assumptions and notations as in Theorem 1.2, the following holds:*

There exists a compact unit neighborhood $U \subset G_A$ with the following property: For any discrete set $\Gamma \subset G_A$ satisfying

$$\gamma U \cap \gamma' U = \emptyset \quad \text{for all} \quad \gamma, \gamma' \in \Gamma \quad \text{with} \quad \gamma \neq \gamma', \tag{1.11}$$

there exists a molecular system $(\phi_\gamma)_{\gamma \in \Gamma}$ in $\overline{\operatorname{span}}\{\pi(\gamma)\psi : \gamma \in \Gamma\}$ such that the distribution $f := \sum_{\gamma \in \Gamma} c_\gamma \phi_\gamma \in \dot{\mathbf{F}}^\alpha_{\infty,q}$ forms a solution to the moment problem

$$\langle f, \pi(\gamma)\psi \rangle = c_\gamma, \quad \gamma \in \Gamma$$

for any given $(c_\gamma)_{\gamma\in\Gamma} \in \dot{\mathbf{p}}^{-\alpha',\beta}_{\infty,q} \leq \mathbb{C}^\Gamma$.

Riesz sequences in $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$ seem not to have appeared in the literature before, which makes Theorem 1.3 new even for isotropic dilations. Similarly to Theorem 1.2, we obtain Theorem 1.3 by applying results of [26, 32] to the coorbit realization of the Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$.

Notation

We denote by $s^+ := \max\{0, s\}$ and $s^- := -\min\{0, s\}$ the positive and negative part of $s \in \mathbb{R}$. If f_1, f_2 are positive functions on a common base set X, the notation $f_1 \lesssim f_2$ is used to denote the existence of a constant $C > 0$ such that $f_1(x) \leq Cf_2(x)$ for all $x \in X$. The notation $f_1 \asymp f_2$ is used whenever both $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$. We will sometimes use $\lesssim_\alpha$ to indicate that the implicit constant depends on a quantity α.

For a function $f : \mathbb{R}^d \to \mathbb{C}$ and a matrix $A \in \mathbb{R}^{d\times d}$, the dilation of $f : \mathbb{R}^d \to \mathbb{C}$ is denoted by $f_j := |\det A|^j f(A^j\cdot)$, where $j \in \mathbb{Z}$. Similarly, we write $f_s := |\det A|^s f(A^s\cdot)$ for $s \in \mathbb{R}$, provided that A^s is well-defined.

The class of Schwartz functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual space is simply denoted by $\mathcal{S}'(\mathbb{R}^d)$. Moreover, the notation $\mathcal{P}(\mathbb{R}^d)$ will be used for the collection of polynomials on $\mathbb{R}^d$, and we write $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ for the quotient space of tempered distributions modulo polynomials. The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is defined as $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\, dx$ with inverse $\check{f} := \mathcal{F}^{-1}f := \widehat{f}(-\,\cdot)$. Similar notations will be used for the extension of the Fourier transforms to $L^2(\mathbb{R}^d)$.

The Lebesgue measure on $\mathbb{R}^d$ is denoted by m. For a measurable set $Q \subset \mathbb{R}^d$ of finite, positive measure, it will be written $⨍_Q f(x)\, dx := \mathrm{m}(Q)^{-1}\int_Q f(x)\, dx$ for $f : \mathbb{R}^d \to \mathbb{C}$. The closure of a set $Q \subset \mathbb{R}^d$ is denoted by $\overline{Q}$.

If G is a group, then the left and right translation of a function $F : G \to \mathbb{C}$ by $h \in G$ will be denoted by $L_h F = F(h^{-1}\cdot)$ and $R_h F = F(\cdot\, h)$, respectively. In addition, we write $F^\vee(x) = F(x^{-1})$.

2 Anisotropic Triebel-Lizorkin spaces with $p = \infty$

This section provides preliminaries on expansive matrices and Triebel-Lizorkin spaces.

2.1 Expansive matrices

A matrix $A \in \mathbb{R}^{d\times d}$ is said to be *expansive* if $|\lambda| > 1$ for all eigenvalues $\lambda \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A. Equivalently, a matrix $A \in \mathbb{R}^{d\times d}$ is expansive if $\|A^{-j}\| \to 0$ as $j \to \infty$.

The following lemma collects several basic properties of expansive matrices that will be used in the sequel, see, e.g., [1, Definitions 2.3 and 2.5] and [1, Lemma 2.2]. For a more general background on spaces of homogeneous type on $\mathbb{R}^d$, we refer to [7, 8].

Lemma 2.1 *Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive.*

(i) There exist an ellipsoid $\Omega = \Omega_A$ (that is, $\Omega = P(B_1(0))$ for some $P \in \mathrm{GL}(d, \mathbb{R})$) and $r > 1$ such that

$$\Omega \subset r\Omega \subset A\Omega$$

and $\mathrm{m}(\Omega) = 1$. The function $\rho_A : \mathbb{R}^d \to [0, \infty)$ defined by

$$\rho_A(x) = \begin{cases} |\det A|^j, & \text{if } x \in A^{j+1}\Omega \setminus A^j\Omega, \\ 0, & \text{if } x = 0, \end{cases} \tag{2.1}$$

is Borel measurable and forms a quasi-norm, i.e., there exists $C \geq 1$ such that

$$\begin{aligned} \rho_A(-x) &= \rho_A(x), && x \in \mathbb{R}^d, \\ \rho_A(x) &> 0, && x \in \mathbb{R}^d \setminus \{0\}, \\ \rho_A(Ax) &= |\det A|\rho_A(x), && x \in \mathbb{R}^d, \\ \rho_A(x+y) &\leq C\big(\rho_A(x) + \rho_A(y)\big), && x, y \in \mathbb{R}^d. \end{aligned} \tag{2.2}$$

(ii) The function $d_A : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$, $(x, y) \mapsto \rho_A(x - y)$ forms a quasi-metric. The triple $(\mathbb{R}^d, d_A, \mathrm{m})$ forms a space of homogeneous type.

For an expansive matrix $A \in \mathrm{GL}(d, \mathbb{R})$, a function $\rho_A : \mathbb{R}^d \to [0, \infty)$ defined by Eq. (2.1) will be called a *step homogeneous quasi-norm* associated to A. Given $y \in \mathbb{R}^d$ and $r > 0$, its associated metric ball will be denoted by

$$B_{\rho_A}(y, r) := \{x \in \mathbb{R}^d : \rho_A(x - y) < r\}.$$

It is readily verified that $B_{\rho_A}(0, 1) = \Omega$. Hence, its metric balls are of the form

$$B_{\rho_A}(y, r) = A^\ell\Omega + y,$$

where $\ell \in \mathbb{Z}$ is such that $|\det A|^{\ell-1} < r \leq |\det A|^\ell$.

Throughout this paper, given an expansive $A \in \mathrm{GL}(d, \mathbb{R})$, we will fix an ellipsoid $\Omega = \Omega_A$ as appearing in Lemma 2.1 (i). This choice is not unique. Any other choice of ellipsoid will yield an equivalent quasi-norm, see, e.g., [1, Lemma 2.4].

2.2 Schwartz seminorms

Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive with associated step homogeneous quasi-norm ρ_A. For $\beta > 0$, the quasi-norm properties (2.2) imply that

$$(1 + \rho_A(x + y))^\beta \lesssim_{A,\beta} (1 + \rho_A(x))^\beta (1 + \rho_A(y))^\beta \tag{2.3}$$

for $x, y \in \mathbb{R}^d$. For ease of notation, we will often write $\nu_\beta(x) = (1 + \rho_A(x))^\beta$ for $x \in \mathbb{R}^d$.

Let λ_- and λ_+ be such that $1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda| \leq \max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+$. Put

$$\zeta_- := \frac{\ln \lambda_-}{\ln |\det A|} \in (0, d^{-1}) \quad \text{and} \quad \zeta_+ := \frac{\ln \lambda_+}{\ln |\det A|} \in (d^{-1}, \infty).$$

Then [1, Lemma 3.2] (see also [20, Lemma 2.2]) implies that

$$\begin{aligned} \rho_A(x) &\lesssim \|x\|^{1/\zeta_-} + \|x\|^{1/\zeta_+}, \qquad && x \in \mathbb{R}^d, \\ \|x\| &\lesssim \rho_A(x)^{\zeta_-} + \rho_A(x)^{\zeta_+}, \qquad && x \in \mathbb{R}^d. \end{aligned}$$

Therefore, the collection

$$p_{M,N}(\varphi) := \max_{\substack{|\alpha| \leq M, \\ 0 < \beta \leq N}} \sup_{y \in \mathbb{R}^d} (1 + \rho_A(y))^\beta \cdot |\partial^\alpha \varphi(y)|, \quad M, N \in \mathbb{N}, \tag{2.4}$$

defines an equivalent family of seminorms for the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

2.3 Analyzing vectors

Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive. Choose a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with compact Fourier support

$$\operatorname{supp} \widehat{\varphi} := \overline{\{\xi \in \mathbb{R}^d : \widehat{\varphi}(\xi) \neq 0\}} \subset \mathbb{R}^d \setminus \{0\} \tag{2.5}$$

satisfying, in addition,

$$\sup_{j \in \mathbb{Z}} \left| \widehat{\varphi}((A^*)^j \xi) \right| > 0, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{2.6}$$

Then the function $\psi \in \mathcal{S}(\mathbb{R}^d)$ defined by

$$\widehat{\psi}(\xi) = \begin{cases} \overline{\widehat{\varphi}(\xi)} / \sum_{k \in \mathbb{Z}} |\widehat{\varphi}((A^*)^k \xi)|^2, & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \text{if } \xi = 0, \end{cases}$$

is well-defined and satisfies

$$\sum_{j \in \mathbb{Z}} \widehat{\varphi}((A^*)^j \xi)\, \widehat{\psi}((A^*)^j \xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{2.7}$$

For more details and further properties, see, e.g., [5, Lemma 3.6]

2.4 Anisotropic Triebel-Lizorkin spaces

Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and fix an analyzing vector $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with compact Fourier support satisfying (2.5) and (2.6).

For $\alpha \in \mathbb{R}$ and $0 < q < \infty$, the associated (homogeneous) *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}^{\alpha}_{\infty,q} = \dot{\mathbf{F}}^{\alpha}_{\infty,q}(A, \varphi)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ for which

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} := \sup_{\ell \in \mathbb{Z}, k \in \mathbb{Z}^d} \left(\fint_{A^{\ell}([0,1]^d + k)} \sum_{j=-\ell}^{\infty} \big(|\det A|^{\alpha j} |(f * \varphi_j)(x)| \big)^q \, dx \right)^{1/q} < \infty.$$

The definition of $\dot{\mathbf{F}}^{\alpha}_{\infty,q} = \dot{\mathbf{F}}^{\alpha}_{\infty,q}(A, \varphi)$ is independent of the choice of analyzing vector φ, with equivalent quasi-norms for different choices, cf. [3, Corollary 3.13]. In addition, the space $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$, and is complete with respect to $\|\cdot\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}}$. See [3, Corollary 3.14] for both claims.

For the case $q = \infty$, the space $\dot{\mathbf{F}}^{\alpha}_{\infty,\infty} = \dot{\mathbf{F}}^{\alpha}_{\infty,\infty}(A, \varphi)$ will be defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ satisfying

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} := \sup_{\ell \in \mathbb{Z}, k \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}, j \geq -\ell} \left(\fint_{A^{\ell}([0,1]^d + k)} |\det A|^{\alpha j} |(f * \varphi_j)(x)| \, dx \right) < \infty.$$

For our purposes, it will be convenient to use the metric ball $\Omega = B_{\rho_A}(0, 1)$ instead of the cube $[0, 1]^d$ in defining $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$. The independence of this choice is guaranteed by the following lemma, whose simple proof follows from a standard covering argument and is hence omitted.

Lemma 2.2 *Let $F : \mathbb{R}^d \to [0, \infty)$ and $F_j : \mathbb{R}^d \to [0, \infty)$, $j \in \mathbb{Z}$, be measurable functions. Then*

$$\sup_{\ell \in \mathbb{Z}, k \in \mathbb{Z}^d} \fint_{A^{\ell}([0,1]^d + k)} F(x) \, dx \asymp \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \fint_{A^{\ell}\Omega + w} F(x) \, dx$$

and

$$\sup_{\ell \in \mathbb{Z}, k \in \mathbb{Z}^d} \sup_{j \in \mathbb{Z}, j \geq -\ell} \fint_{A^{\ell}([0,1]^d + k)} F_j(x) \, dx \asymp \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \sup_{j \in \mathbb{Z}, j \geq -\ell} \fint_{A^{\ell}\Omega + w} F_j(x) \, dx$$

with implicit constants only depending on d, A.

3 Maximal function characterizations

Throughout this section, $A \in \mathrm{GL}(d, \mathbb{R})$ will denote an expansive matrix.

3.1 Peetre-type maximal function

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. For $j \in \mathbb{Z}$ and $\beta > 0$, the associated *Peetre-type maximal function* of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as

$$\varphi_{j,\beta}^{**} f(x) = \sup_{z \in \mathbb{R}^d} \frac{|(f * \varphi_j)(x+z)|}{(1+\rho_A(A^j z))^\beta}, \quad x \in \mathbb{R}^d, \tag{3.1}$$

where $\rho_A : \mathbb{R}^d \to [0, \infty)$ denotes the step homogeneous quasi-norm.

The Peetre-type maximal function has the following basic properties.

Lemma 3.1 *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Let $j \in \mathbb{Z}$ and $\beta > 0$. For any $f \in \mathcal{S}'(\mathbb{R}^d)$, the following holds:*

(i) If $\varphi_{j,\beta}^{**} f(x)$ is finite for some $x \in \mathbb{R}^d$, then it is finite for all $x \in \mathbb{R}^d$.
(ii) There exists an $N = N(f, A) \in \mathbb{N}$ such that for all $\beta \geq N$

$$\varphi_{j,\beta}^{**} f(x) < \infty, \quad x \in \mathbb{R}^d.$$

Proof (i) Let $x_0 \in \mathbb{R}^d$ be such that $\varphi_{j,\beta}^{**} f(x_0) < \infty$. The symmetry of ρ_A and the inequality (2.3) for $\nu_\beta(y) = (1+\rho_A(y))^\beta$ yield that

$$\begin{aligned}\varphi_{j,\beta}^{**} f(x) &= \sup_{z \in \mathbb{R}^d} \frac{|(f * \varphi_j)(z)|}{\nu_\beta\big(A^j(x-z)\big)} \lesssim \nu_\beta\big(A^j(x-x_0)\big) \sup_{z \in \mathbb{R}^d} \frac{|(f * \varphi_j)(z)|}{\nu_\beta\big(A^j(x_0-z)\big)} \\ &= \nu_\beta\big(A^j(x-x_0)\big)\, \varphi_{j,\beta}^{**} f(x_0) < \infty.\end{aligned}$$

for arbitrary $x \in \mathbb{R}^d$.

(ii) Fix $x, z \in \mathbb{R}^d$. Let $\langle \cdot, \cdot \rangle_{\mathcal{S}',\mathcal{S}}$ denote the bilinear dual pairing. Since $f \in \mathcal{S}'(\mathbb{R}^d)$ and the seminorms (2.4) are ordered, there exist $M = M(f)$, $N = (f, A) \in \mathbb{N}$ and a constant $C = C(f) > 0$ such that

$$\begin{aligned}|(f * \varphi_j)(z)| &= |\langle f, T_z \varphi_j^\vee \rangle_{\mathcal{S}',\mathcal{S}}| \\ &\leq C \max_{\substack{|\alpha| \leq M,\\ 0<\beta \leq N}} \sup_{y \in \mathbb{R}^d} \nu_\beta(y)\, |\partial^\alpha\big(|\varphi_j^\vee|\big)(y-z)| \\ &\lesssim C \max_{\substack{|\alpha| \leq M,\\ 0<\beta \leq N}} \sup_{\tilde{y} \in \mathbb{R}^d} \nu_\beta(\tilde{y}+x)\nu_\beta(z-x)\, |\partial^\alpha\big(|\varphi_j^\vee|\big)(\tilde{y})| \\ &\lesssim C \max_{\substack{|\alpha| \leq M,\\ 0<\beta \leq N}} \sup_{\tilde{y} \in \mathbb{R}^d} \nu_\beta(\tilde{y}+x) \max\{1, |\det A|^{-j\beta}\}\, \nu_\beta\big(A^j(z-x)\big)\, |\partial^\alpha\big(|\varphi_j^\vee|\big)(\tilde{y})|.\end{aligned}$$

Consequently,

$$\frac{|(f * \varphi_j)(z)|}{\nu_N\big(A^j(z-x)\big)} \lesssim \max_{\substack{|\alpha| \leq M,\\ 0<\beta \leq N}} \sup_{\tilde{y} \in \mathbb{R}^d} \nu_\beta(\tilde{y}+x)\, |\partial^\alpha\big(|\varphi_j^\vee|\big)(\tilde{y})| \lesssim p_{M,N}(\varphi_j^\vee).$$

Since the constants are independent of $z \in \mathbb{R}^d$, the claim follows easily. □

3.2 Sub-mean-value property

The following type of result is often referred to as a "sub-mean-value property"and will play an essential role in deriving the main results. It forms an anisotropic analogue of the isotropic result [30, Theorem 5].

Proposition 3.2 *Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ have compact Fourier support satisfying (2.5) and (2.6). Then, for all $q \in (0, \infty)$ and $\beta > 0$, there exists a constant $C = C(A, \varphi, q, \beta) > 0$ such that*

$$(\varphi_{j,\beta}^{**} f(x))^q \leq C |\det A|^j \int_{\mathbb{R}^d} \frac{|(f * \varphi_j)(y)|^q}{\big(1 + \rho_A(A^j(x-y))\big)^{\beta q}} \, dy, \quad x \in \mathbb{R}^d, \tag{3.2}$$

for all $f \in \mathcal{S}'(\mathbb{R}^d)$ and all $j \in \mathbb{Z}$.

Proof The claim is trivial whenever $\varphi_{j,\beta}^{**} f(x) = 0$, so we assume throughout that $\varphi_{j,\beta}^{**} f(x) > 0$. The proof will be split into two steps dealing with $q \in [1, \infty)$ and $q \in (0, 1)$ separately.

Step 1. (*The case $q \in [1, \infty)$*). By the compact Fourier support condition (2.5) and the identity (2.7), it follows that there exists $N \in \mathbb{N}$ (depending on φ and A) such that the function $\Phi := \sum_{k=-N}^{N} \varphi_k * \psi_k$ satisfies

$$\varphi_j * \Phi_j = \varphi_j, \quad j \in \mathbb{Z}, \tag{3.3}$$

see, e.g., the proof of [20, Theorem 3.5] for a detailed verification.

Using that $\Phi_j = \sum_{k=-N}^{N} \varphi_{j+k} * \psi_{j+k}$ for $j \in \mathbb{Z}$, the equality (3.3) gives

$$\begin{aligned}
&\frac{|(f * \varphi_j)(x+z)|}{\nu_\beta(A^j z)} \\
&\quad \leq \sum_{k=-N}^{N} \frac{|(f * \varphi_j * \varphi_{j+k} * \psi_{j+k})(x+z)|}{\nu_\beta(A^j z)} \\
&\quad \leq \sum_{k=-N}^{N} \int_{\mathbb{R}^d} \frac{|(f * \varphi_j)(y)|}{\nu_\beta(A^j z)} |(\varphi_{j+k} * \psi_{j+k})(x+z-y)| \, dy \\
&\quad \lesssim \sum_{k=-N}^{N} \int_{\mathbb{R}^d} \frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \nu_\beta\big(A^j(x+z-y)\big) \\
&\qquad |(\varphi_{j+k} * \psi_{j+k})(x+z-y)| \, dy
\end{aligned}$$

for all $x, z \in \mathbb{R}^d$, $j \in \mathbb{Z}$ and $\beta > 0$. Combining Hölder's inequality for $\frac{1}{q} + \frac{1}{q'} = 1$ and the translation invariance of $L^{q'}(\mathbb{R}^d)$ yields

$$\begin{aligned}
&\frac{|(f*\varphi_j)(x+z)|}{\nu_\beta(A^j z)} \\
&\quad \lesssim \sum_{k=-N}^{N} \left\| \frac{(f*\varphi_j)}{\nu_\beta\big(A^j(x-\cdot)\big)} \right\|_{L^q} \\
&\qquad \left\| \nu_\beta\big(A^j(x+z-\cdot)\big)\,(\varphi_{j+k}*\psi_{j+k})(x+z-\cdot) \right\|_{L^{q'}} \\
&\quad = \sum_{k=-N}^{N} \left\| \frac{(f*\varphi_j)}{\nu_\beta\big(A^j(x-\cdot)\big)} \right\|_{L^q} \left\| \nu_\beta\big(A^j(\,\cdot\,)\big)\,(\varphi_{j+k}*\psi_{j+k})(\,\cdot\,) \right\|_{L^{q'}}.
\end{aligned}$$

If $q \in (1,\infty)$, applying the transformation $A^j y \mapsto \tilde{y}$ in the $L^{q'}$-norm above gives

$$\begin{aligned}
\left\| \nu_\beta\big(A^j(\,\cdot\,)\big)\,(\varphi_{j+k}*\psi_{j+k})(\,\cdot\,) \right\|_{L^{q'}}^{q'} &= \int_{\mathbb{R}^d} \big(\nu_\beta(A^j y)\big)^{q'}\,|\det A|^{jq'}\,|(\varphi_k*\psi_k)(A^j y)|^{q'}\,dy \\
&= |\det A|^{jq'-j} \int_{\mathbb{R}^d} \big(\nu_\beta(\tilde{y})\big)^{q'}\,|(\varphi_k*\psi_k)(\tilde{y})|^{q'}\,d\tilde{y}.
\end{aligned}$$

Using the seminorms defined in Eq. (2.4) and the fact that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, the last integral can be estimated by a constant $C_k = C_k(A,\varphi,q,\beta) > 0$. Combining the above gives

$$\begin{aligned}
\frac{|(f*\varphi_j)(x+z)|}{\nu_\beta(A^j z)} &\lesssim \left\| \frac{(f*\varphi_j)}{\nu_\beta\big(A^j(x-\cdot)\big)} \right\|_{L^q} \sum_{k=-N}^{N} |\det A|^{j(1-1/q')}\,C_k^{1/q'} \\
&\lesssim |\det A|^{j/q} \left\| \frac{(f*\varphi_j)}{\nu_\beta\big(A^j(x-\cdot)\big)} \right\|_{L^q},
\end{aligned}$$

with implicit constant depending on A, β, q, φ. Taking the supremum over $z \in \mathbb{R}^d$ and the q-th power yields the claim for $q \in (1,\infty)$. The case $q = 1$ follows by the same arguments with the usual modifications.

Step 2. (*The case* $q \in (0,1)$). For $f \in \mathcal{S}'(\mathbb{R}^d)$, the estimate obtained in Step 1 (for $q = 1$) gives

$$\begin{aligned}
\varphi_{j,\beta}^{**} f(x) &\lesssim |\det A|^j \int_{\mathbb{R}^d} \frac{|(f*\varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)}\,dy \\
&= |\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f*\varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{1-q} \left(\frac{|(f*\varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{q} dy \\
&\le (\varphi_{j,\beta}^{**} f(x))^{1-q}\,|\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f*\varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{q} dy. \qquad (3.4)
\end{aligned}$$

By Lemma 3.1, there exists $N' = N'(f) \in \mathbb{N}$ such that $\varphi_{j,\beta}^{**} f(x) < \infty$ for all $\beta \ge N'$. Hence, if $\beta \ge N'$, the claim follows immediately from the inequality (3.4).

For the case $\beta < N'$, we use the already proven result for N' to obtain

$$
\begin{aligned}
|(f * \varphi_j)(z)|^q &\lesssim |\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f * \varphi_j)(y)|}{\nu_{N'}\big(A^j(z-y)\big)} \right)^q dy \\
&\leq |\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f * \varphi_j)(y)|}{\nu_{\beta}\big(A^j(z-y)\big)} \right)^q dy \\
&\lesssim |\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f * \varphi_j)(y)|}{\nu_{\beta}\big(A^j(x-y)\big)} \right)^q \nu_\beta\big(A^j(x-z)\big)^q \, dy,
\end{aligned}
$$

where the second inequality used that $\nu_{N'}(x)^{-q} \leq \nu_\beta(x)^{-q}$ for $0 < \beta < N'$. Consequently,

$$
\left(\frac{|(f * \varphi_j)(z)|}{\nu_\beta\big(A^j(x-z)\big)} \right)^q \lesssim |\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^q dy.
$$

The right-hand side being independent of $z \in \mathbb{R}^d$, taking the supremum yields the claim for $\beta < N'$. Overall, this completes the proof. ◻

3.3 Maximal function characterizations

In this section, the matrix $A \in \mathrm{GL}(d, \mathbb{R})$ is often additionally assumed to be *exponential*, i.e., it is assumed that A admits the form $A = \exp(B)$ for some $B \in \mathbb{R}^{d \times d}$. Then the power $A^s = \exp(sB)$ is well-defined for $s \in \mathbb{R}$.

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $s, \beta \in \mathbb{R}$ with $\beta > 0$, the associated Peetre-type maximal function of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as in (3.1) by

$$
\varphi^{**}_{s,\beta} f(x) = \sup_{z \in \mathbb{R}^d} \frac{|(f * \varphi_s)(x+z)|}{(1 + \rho_A(A^s z))^\beta}, \quad x \in \mathbb{R}^d
$$

whenever A is exponential.

The following theorem forms a main result of this paper. It characterizes the anisotropic Triebel-Lizorkin spaces $\dot{\mathbf{F}}^\alpha_{\infty,q}$, with $0 < q \leq \infty$, in terms of Peetre-type maximal functions. The result forms an extension of [6, Theorem 1] to possibly anisotropic dilations.

Theorem 3.3 *Suppose $A \in \mathrm{GL}(d, \mathbb{R})$ is expansive and exponential. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ has compact Fourier support and satisfies (2.5) and (2.6).*

Then, for all $q \in (0, \infty)$, $\alpha \in \mathbb{R}$ and $\beta > 1/q$, the norm equivalences

$$
\begin{aligned}
\|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,q}} &\asymp \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\fint_{A^\ell \Omega + w} \int_{-\ell}^{\infty} \big(|\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x)\big)^q \, ds \, dx \right)^{1/q} \\
&\asymp \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\fint_{A^\ell \Omega + w} \sum_{j=-\ell}^{\infty} \big(|\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x)\big)^q \, dx \right)^{1/q}
\end{aligned}
\tag{3.5}
$$

hold for all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$. *For* $q = \infty$, $\alpha \in \mathbb{R}$ *and* $\beta > 1$, *the following equivalences hold*

$$\begin{aligned} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} &\asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \sup_{s\in\mathbb{R}, s\geq -\ell} \fint_{A^\ell\Omega+w} |\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x)\, dx \\ &\asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \sup_{j\in\mathbb{Z}, j\geq -\ell} \fint_{A^\ell\Omega+w} |\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x)\, dx. \end{aligned} \tag{3.6}$$

for all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$.

(The function $\varphi^{**}_{s,\beta} f : \mathbb{R}^d \to [0, \infty]$ *is well-defined for* $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$*, since* φ *has infinitely many vanishing moments and hence* $P * \varphi_s = 0$ *for every* $P \in \mathcal{P}(\mathbb{R}^d)$*.)*

Remark 3.4 The proof of Theorem 3.3 shows that the discrete characterizations

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} \asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{A^\ell\Omega+w} \sum_{j=-\ell}^{\infty} \left(|\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x)\right)^q dx \right)^{1/q},$$

and

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} \asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \sup_{j\in\mathbb{Z}, j\geq -\ell} \fint_{A^\ell\Omega+w} |\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x)\, dx,$$

also hold without the assumption that $A \in \mathrm{GL}(d, \mathbb{R})$ is exponential.

Proof of Theorem 3.3 Only the cases $q \in (0, \infty)$ will be treated; the case $q = \infty$ follows by the arguments for $q = 1$, with the usual modification to accommodate the supremum. The proof is split into three steps and for some parts we refer to calculations from the proof of [20, Theorem 3.5].

Throughout the proof, we will make use of the equivalent norms

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} \asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{A^\ell\Omega+w} \sum_{j=-\ell}^{\infty} \left(|\det A|^{\alpha j} |(f * \varphi_j)(x)|\right)^q dx \right)^{1/q}$$

and

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} \asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \sup_{j\in\mathbb{Z}, j\geq -\ell} \fint_{A^\ell\Omega+w} |\det A|^{\alpha j} |(f * \varphi_j)(x)|\, dx$$

provided by Lemma 2.2.

Step 1. In this step it will be shown that $\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}}$ can be bounded by the middle term of (3.5). This step is modeled on Step 1 of the proof of [20, Theorem 3.5]. By the calculations constituting [20, Eqs. (3.7)–(3.11)], it follows that, for $t \in [0, 1]$, there exists $N = N(A, \varphi) \in \mathbb{N}$ such that

$$\left\| \left(|\det A|^{\alpha j} |(f * \varphi_j)(x)| \right)_{j=-\ell}^{\infty} \right\|_{\ell^q} \lesssim \sum_{k=-N}^{N} \left\| \left(|\det A|^{\alpha(j+k+t)}\, \varphi^{**}_{j+k+t,\beta} f(x) \right)_{j=-\ell}^{\infty} \right\|_{\ell^q}$$

$$= \sum_{k=-N}^{N} \left\| \left(|\det A|^{\alpha(j+t)} \varphi^{**}_{j+t,\beta} f(x) \right)_{j=k-\ell}^{\infty} \right\|_{\ell^q}$$
$$\lesssim \left\| \left(|\det A|^{\alpha(j+t)} \varphi^{**}_{j+t,\beta} f(x) \right)_{j=-\ell-N}^{\infty} \right\|_{\ell^q}. \tag{3.7}$$

If $q < \infty$, then raising (3.7) to the q-th power and integrating over $t \in [0, 1]$ gives

$$\sum_{j=-\ell}^{\infty} \Big(|\det A|^{\alpha j} |(f * \varphi_j)(x)| \Big)^q \lesssim \int_0^1 \sum_{j=-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha(j+t)} \varphi^{**}_{j+t,\beta} f(x) \Big)^q \, dt$$
$$= \int_{-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x) \Big)^q \, ds.$$

Let $w \in \mathbb{R}^d$ be arbitrary and set $Q_\ell = A^\ell \Omega + w$. By Lemma 2.1, it follows that $Q_\ell \subset Q_{\ell+N}$. Therefore, averaging over Q_ℓ gives

$$\fint_{Q_\ell} \sum_{j=-\ell}^{\infty} \Big(|\det A|^{\alpha j} |(f * \varphi_j)(x)| \Big)^q \, dx \lesssim \fint_{Q_\ell} \int_{-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x) \Big)^q \, ds \, dx$$
$$\leq \frac{\mathrm{m}(Q_{\ell+N})}{\mathrm{m}(Q_\ell)} \fint_{Q_{\ell+N}} \int_{-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x) \Big)^q \, ds \, dx,$$

where $\frac{\mathrm{m}(Q_{\ell+N})}{\mathrm{m}(Q_\ell)} = |\det A|^N \lesssim 1$. Consequently, taking the q-th root and the supremum over $\ell \in \mathbb{Z}$ and $w \in \mathbb{R}^d$ yields the desired estimate.

Step 2. In this step we estimate the middle term by the right-most term of (3.5). This requires discretizing the inner-most integral, which works analogously to Step 2 in the proof of [20, Theorem 3.5]. By [20, Eq. (3.15)], for $t \in [0, 1]$, there exists $N = N(A, \varphi) \in \mathbb{N}$ such that

$$\Big(|\det A|^{\alpha(j+t)} \varphi^{**}_{j+t,\beta} f(x) \Big)^q \lesssim \sum_{k=-N}^{N} \Big(|\det A|^{\alpha(j+k)} \varphi^{**}_{j+k,\beta} f(x) \Big)^q. \tag{3.8}$$

Starting with the inner-most integral of the middle term in (3.5), we use a simple periodization argument and (3.8) to obtain

$$\int_{-\ell}^{\infty} \Big(|\det A|^{\alpha s} \varphi^{**}_{s,\beta} f(x) \Big)^q \, ds = \sum_{j=-\ell}^{\infty} \int_0^1 \Big(|\det A|^{\alpha(j+t)} \varphi^{**}_{j+t,\beta} f(x) \Big)^q \, dt$$
$$\lesssim \sum_{j=-\ell}^{\infty} \sum_{k=-N}^{N} \Big(|\det A|^{\alpha(j+k)} \varphi^{**}_{j+k,\beta} f(x) \Big)^q$$
$$\lesssim \sum_{j=-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x) \Big)^q.$$

Taking the averaged integral over $Q_\ell = A^\ell \Omega + w$ yields

$$\fint_{Q_\ell} \int_{-\ell}^{\infty} \Big(|\det A|^{\alpha s} \varphi_{s,\beta}^{**} f(x) \Big)^q \, ds \, dx \lesssim \fint_{Q_\ell} \sum_{j=-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha j} \varphi_{j,\beta}^{**} f(x) \Big)^q \, dx$$
$$\lesssim \fint_{Q_{\ell+N}} \sum_{j=-(\ell+N)}^{\infty} \Big(|\det A|^{\alpha j} \varphi_{j,\beta}^{**} f(x) \Big)^q \, dx,$$

where we used $Q_\ell \subset Q_{\ell+N}$ and $\frac{1}{\mathrm{m}(Q_\ell)} = |\det A|^N \frac{1}{\mathrm{m}(Q_{\ell+N})}$ in the last step. Taking the supremum over all $\ell \in \mathbb{Z}$ and $w \in \mathbb{R}^d$ and the q-th root yields the claim for $q \in (0, \infty)$.

Step 3. Lastly, it will be shown that the right-most term of (3.5) can be bounded by $\| f \|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}}$. We start with using Proposition 3.2 for the exponent $0 < q/r < \infty$, where $r := \sqrt{\beta q} > 1$ by assumption. This gives for all $x \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}$

$$\sum_{j=-\ell}^{\infty} |\det A|^{\alpha j q} \Big[\big(\varphi_{j,\beta}^{**} f(x) \big)^{q/r} \Big]^r$$
$$\lesssim \sum_{j=-\ell}^{\infty} |\det A|^{\alpha j q} \bigg[|\det A|^j \int_{\mathbb{R}^d} \bigg(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \bigg)^{q/r} dy \bigg]^r.$$

For fixed, but arbitrary $x \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}$, we partition $\mathbb{R}^d = Q_\ell(x) \cup \bigcup_{k=0}^{\infty} \mathring{Q}_{\ell+k+1}(x)$, where

$$Q_\ell(x) := A^\ell \Omega + x \quad \text{and} \quad \mathring{Q}_{\ell+k+1}(x) := Q_{\ell+k+1}(x) \setminus Q_{\ell+k}(x).$$

Combining this with the simple fact that $(a+b)^r \lesssim a^r + b^r$ for $a, b \geq 0$ yields

$$\sum_{j=-\ell}^{\infty} |\det A|^{\alpha j q} \Big[\big(\varphi_{j,\beta}^{**} f(x) \big)^{q/r} \Big]^r$$
$$\lesssim \sum_{j=-\ell}^{\infty} |\det A|^{\alpha j q} \bigg[|\det A|^j \int_{Q_\ell(x)} \bigg(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \bigg)^{q/r} dy \bigg]^r$$
$$+ \sum_{j=-\ell}^{\infty} |\det A|^{\alpha j q} \bigg[|\det A|^j \sum_{k=0}^{\infty} \int_{\mathring{Q}_{\ell+k+1}(x)} \bigg(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \bigg)^{q/r} dy \bigg]^r$$
$$=: S_1 + S_2.$$

In the remainder, the series defining S_1 and S_2 will be estimated.

Step 3.1. We look at the sum S_1 first. Note that since $|\det A| > 1$, there exists $M \in \mathbb{N}$ such that $|\det A|^M \geq 2C$, where $C > 0$ denotes the constant in the triangle-inequality for ρ_A (cf. Lemma 2.1). A straightforward computation shows that $Q_\ell(x) \subset Q_{\ell+M}(w)$

for all $w \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}$ whenever $x \in Q_\ell(w)$. Therefore, for all $w \in \mathbb{R}^d$, $\ell \in \mathbb{Z}$ and $x \in Q_\ell(w)$, it follows that

$$\begin{aligned}
&|\det A|^j \int_{Q_\ell(x)} \left(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{q/r} dy \\
&\leq |\det A|^j \int_{\mathbb{R}^d} \left(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}(y)\, dy \\
&= |\det A|^j \int_{Q_{-j}(x)} \left(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}(y)\, dy \\
&\quad + \sum_{m=0}^{\infty} |\det A|^j \int_{\mathring{Q}_{m-j+1}(x)} \left(\frac{|(f * \varphi_j)(y)|}{\nu_\beta\big(A^j(x-y)\big)} \right)^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}(y)\, dy \\
&=: I_1 + I_2.
\end{aligned}$$

To estimate the terms I_1 and I_2, we will use the (anisotropic) Hardy-Littlewood maximal operator for locally integrable $f : \mathbb{R}^d \to \mathbb{C}$ given by

$$M_{\rho_A} f(x) = \sup_{B \ni x} \fint_B |f(y)|\, dy, \quad x \in \mathbb{R}^d,$$

where $B = B_{\rho_A}(z, s)$ ranges over all metric balls containing x.

For estimating I_1, note that $\nu_\beta\big(A^j(x-y)\big)^{-q/r} = \big(1 + \rho_A(A^j(x-y))\big)^{-\beta q/r} \leq 1$ yields

$$\begin{aligned}
I_1 &\leq |\det A|^j \int_{Q_{-j}(x)} |(f * \varphi_j)(y)|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}(y)\, dy \\
&\leq M_{\rho_A}\big(|f * \varphi_j|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}\big)(x). \qquad (3.9)
\end{aligned}$$

For estimating I_2, note that $\rho_A(A^j(x-y)) = |\det A|^m$ for $y \in \mathring{Q}_{m-j+1}(x)$ by definition of ρ_A (see (2.1)). This and setting $\delta := \beta q/r - 1$ implies

$$\begin{aligned}
I_2 &\leq \sum_{m=0}^{\infty} |\det A|^{j - \beta q m/r} \int_{\mathring{Q}_{m-j+1}(x)} |(f * \varphi_j)(y)|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}(y)\, dy \\
&\leq \sum_{m=0}^{\infty} |\det A|^{-m\delta+1} |\det A|^{j-m-1} \int_{Q_{m-j+1}(x)} |(f * \varphi_j)(y)|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}(y)\, dy \\
&\leq M_{\rho_A}\big(|f * \varphi_j|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}\big)(x) \sum_{m=0}^{\infty} |\det A|^{-m\delta+1} \\
&\lesssim M_{\rho_A}\big(|f * \varphi_j|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}\big)(x), \qquad (3.10)
\end{aligned}$$

where the last inequality used that $\delta = \beta q/r - 1 > 0$. Here, the implicit constant only depends on A, q and β.

Combining (3.9) and (3.10) shows for $x \in Q_\ell(w)$ that

$$S_1 \lesssim \sum_{j=-\ell}^{\infty} |\det A|^{\alpha jq} \Big[M_{\rho_A}\big(|f * \varphi_j|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}\big)(x) \Big]^r.$$

Thus, averaging over $x \in Q_\ell(w)$ and applying the maximal inequalities for $L^r(\mathbb{R}^d)$ (see, e.g., [15, Theorem 1.2]), yield

$$\begin{aligned} ⨍_{Q_\ell(w)} S_1 \, dx &\lesssim \sum_{j=-\ell}^{\infty} |\det A|^{\alpha jq} ⨍_{Q_\ell(w)} \Big[M_{\rho_A}\big(|f * \varphi_j|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}\big)(x) \Big]^r dx \\ &\lesssim \sum_{j=-(\ell+M)}^{\infty} |\det A|^{\alpha jq} \frac{1}{\mathrm{m}(Q_{\ell+M}(w))} \int_{\mathbb{R}^d} \Big[M_{\rho_A}\big(|f * \varphi_j|^{q/r} \mathbb{1}_{Q_{\ell+M}(w)}\big)(x) \Big]^r dx \\ &\lesssim \sum_{j=-(\ell+M)}^{\infty} |\det A|^{\alpha jq} \frac{1}{\mathrm{m}(Q_{\ell+M}(w))} \int_{\mathbb{R}^d} |(f * \varphi_j)(x)|^q \mathbb{1}_{Q_{\ell+M}(w)}(x) \, dx. \end{aligned}$$

Lastly, taking the suprema over $w \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}$ yields

$$\sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} ⨍_{Q_\ell(w)} S_1 \, dx \lesssim \| f \|^q_{\dot{\mathbf{F}}^\alpha_{\infty,q}}, \tag{3.11}$$

with implicit constant depending on A, φ, q, and β.

Step 3.2. In this step, we deal with the sum S_2. Recall again that, for $y \in \mathring{Q}_{\ell+k+1}(x)$, $\rho_A((A^j(x-y))) = |\det A|^{j+k+\ell}$. Hence,

$$\begin{aligned} &|\det A|^j \sum_{k=0}^{\infty} \int_{\mathring{Q}_{\ell+k+1}(x)} \left(\frac{|(f * \varphi_j)(y)|}{v_\beta\big(A^j(x-y)\big)} \right)^{q/r} dy \\ &\quad \le |\det A|^j \sum_{k=0}^{\infty} |\det A|^{-(j+k+\ell)\beta q/r} \int_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^{q/r} \, dy \\ &\quad = |\det A|^{-\delta(j+\ell)+1} \sum_{k=0}^{\infty} |\det A|^{-\delta k} ⨍_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^{q/r} \, dy, \end{aligned}$$

where again $\delta := \beta q/r - 1 > 0$. Note that $|\det A|^{-\delta(j+\ell)} \le 1$ for $j \ge -\ell$, which implies that

$$\begin{aligned} S_2 &\lesssim \sum_{j=-\ell}^{\infty} |\det A|^{\alpha jq} \left[\sum_{k=0}^{\infty} |\det A|^{-\delta k} ⨍_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^{q/r} \, dy \right]^r \\ &\lesssim \left[\sum_{k=0}^{\infty} |\det A|^{-\delta k} \left[\sum_{j=-\ell}^{\infty} \left(|\det A|^{\alpha jq/r} ⨍_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^{q/r} \, dy \right)^r \right]^{1/r} \right]^r, \end{aligned}$$

where we used Minkowski's integral inequality (see, e.g., [29, Appendix 1]) to obtain the last line. An application of Jensen's inequality to the integral yields

$$\left(\fint_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^{q/r} \, dy \right)^r \leq \fint_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^q \, dy,$$

and consequently

$$\begin{aligned} S_2 &\lesssim \left[\sum_{k=0}^{\infty} |\det A|^{-\delta k} \left[\sum_{j=-\ell}^{\infty} |\det A|^{\alpha j q} \fint_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^q \, dy \right]^{1/r} \right]^r \\ &\leq \left[\sum_{k=0}^{\infty} |\det A|^{-\delta k} \left[\sum_{j=-(\ell+k+1)}^{\infty} |\det A|^{\alpha j q} \fint_{Q_{\ell+k+1}(x)} |(f * \varphi_j)(y)|^q \, dy \right]^{1/r} \right]^r \\ &\leq \sup_{\ell' \in \mathbb{Z}, x \in \mathbb{R}^d} \left(\sum_{j=-\ell'}^{\infty} |\det A|^{\alpha j q} \fint_{Q_{\ell'}(x)} |(f * \varphi_j)(y)|^q \, dy \right) \left[\sum_{k=0}^{\infty} |\det A|^{-\delta k} \right]^r \\ &\lesssim \sup_{\ell' \in \mathbb{Z}, x \in \mathbb{R}^d} \left(\sum_{j=-\ell'}^{\infty} |\det A|^{\alpha j q} \fint_{Q_{\ell'}(x)} |(f * \varphi_j)(y)|^q \, dy \right), \end{aligned} \tag{3.12}$$

where we used the index shift $\ell' = \ell + k + 1$ in the penultimate estimate. Since the implicit constants are independent of $w \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}$, it follows that

$$\sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \fint_{Q_\ell(w)} S_2 \, dx \lesssim \|f\|^q_{\dot{\mathbf{F}}^\alpha_{\infty,q}}. \tag{3.13}$$

Overall, combining the estimates (3.11) and (3.13) finishes the proof. □

4 The case $p = q = \infty$

Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and suppose that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ has compact Fourier support and satisfies (2.5) and (2.6). Following [2], for fixed $\alpha \in \mathbb{R}$, the associated *homogeneous anisotropic Besov space* $\dot{\mathbf{B}}^\alpha_{\infty,\infty} = \dot{\mathbf{B}}^\alpha_{\infty,\infty}(A, \varphi)$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ for which

$$\|f\|_{\dot{\mathbf{B}}^\alpha_{\infty,\infty}} := \sup_{j \in \mathbb{Z}} \sup_{x \in \mathbb{R}^d} |\det A|^{\alpha j} |(f * \varphi_j)(x)| < \infty.$$

The space $\dot{\mathbf{B}}^\alpha_{\infty,\infty}$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ and complete with respect to the quasi-norm $\|\cdot\|_{\dot{\mathbf{B}}^\alpha_{\infty,\infty}}$, cf. [2, Proposition 3.3]. In addition, it is independent of the choice of defining vector $\varphi \in \mathcal{S}(\mathbb{R}^d)$ by [2, Corollary 3.7].

The following theorem relates the anisotropic Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{\infty,q}$, with $0 < q \leq \infty$, to the anisotropic Besov space $\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}$. In particular, it shows that $\dot{\mathbf{B}}^{\alpha}_{\infty,\infty} = \dot{\mathbf{F}}^{\alpha}_{\infty,\infty}$, providing a characterization of $\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}$ in terms of Peetre-type maximal functions. It also shows that the definition of $\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}$ given in Sect. 2.4 coincides with the definition in [3, Sect. 3].

Theorem 4.1 *Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$ has compact Fourier support and satisfies (2.5) and (2.6).*

Then, for all $q \in (0, \infty]$ and $\alpha \in \mathbb{R}$, we have the continuous embedding

$$\dot{\mathbf{F}}^{\alpha}_{\infty,q}(A, \varphi) \subset \dot{\mathbf{F}}^{\alpha}_{\infty,\infty}(A, \varphi).$$

In addition, $\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}(A, \varphi) = \dot{\mathbf{B}}^{\alpha}_{\infty,\infty}(A, \varphi)$.

Proof The inequality $\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}} \leq \|f\|_{\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}}$ is immediate. To show that $\|f\|_{\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}} \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}}$, the norm equivalences of Theorem 3.3 will be used; see also Remark 3.4. For this, suppose $\beta > 1$. Then, for all $\ell \in \mathbb{Z}$, $w \in \mathbb{R}^d$, we see that

$$\begin{aligned} \fint_{Q_\ell(w)} |\det A|^{-\alpha\ell} \varphi^{**}_{-\ell,\beta} f(x)\, dx &\leq \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \sup_{j \in \mathbb{Z}, j \geq -\ell} \left(\fint_{Q_\ell(w)} |\det A|^{\alpha j} \varphi^{**}_{j,\beta} f(x)\, dx \right) \\ &\lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}}, \end{aligned} \tag{4.1}$$

In particular, the inequality (4.1) implies that, for every $\ell \in \mathbb{Z}$ and $w \in \mathbb{R}^d$ there exists $x_w \in Q_\ell(w)$ such that

$$|\det A|^{-\alpha\ell} \varphi^{**}_{-\ell,\beta} f(x_w) \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}}. \tag{4.2}$$

If $z \in Q_\ell(w)$, then $\rho_A\big(A^{-\ell}(z - x_w)\big) \lesssim \rho_A\big(A^{-\ell}(z - w)\big) + \rho_A\big(A^{-\ell}(w - x_w)\big) \lesssim 1$ with implicit constant depending only on A. Hence, for all $z \in Q_\ell(w)$, it holds that

$$\varphi^{**}_{-\ell,\beta} f(x_w) \geq \frac{|(f * \varphi_{-\ell})(z)|}{\big(1 + \rho_A\big(A^{-\ell}(z - x_w)\big)\big)^{\beta}} \gtrsim_{A,\beta} |(f * \varphi_{-\ell})(z)|. \tag{4.3}$$

Since $\bigcup_{w \in \mathbb{R}^d} Q_\ell(w) = \mathbb{R}^d$ for fixed, but arbitrary, $\ell \in \mathbb{Z}$, we see by combining (4.2) and (4.3) that

$$|\det A|^{-\alpha\ell} |(f * \varphi_{-\ell})(z)| \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}}$$

for all $z \in \mathbb{R}^d$ and $\ell \in \mathbb{Z}$, which shows $\dot{\mathbf{F}}^{\alpha}_{\infty,\infty}(A, \varphi) = \dot{\mathbf{B}}^{\alpha}_{\infty,\infty}(A, \varphi)$ with equivalent norms.

An analogous argument using (3.5) gives $\|f\|_{\dot{\mathbf{B}}^{\alpha}_{\infty,\infty}} \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}}$, which completes the proof. □

Theorem 4.1 is an extension of [6, Theorem 3] to the anisotropic setting.

5 Wavelet coefficient decay and Peetre-type spaces

Throughout this section, let $A \in \mathrm{GL}(d, \mathbb{R})$ be an exponential matrix. Define the associated semi-direct product $G_A = \mathbb{R}^d \rtimes_A \mathbb{R} = \{(x, s) : x \in \mathbb{R}^d, s \in \mathbb{R}\}$ with group operations

$$(x, s)(y, t) = (x + A^s y, s + t) \quad \text{and} \quad (x, s)^{-1} = (-A^{-s}x, -s). \tag{5.1}$$

Left Haar measure on G_A is given by $d\mu_{G_A}(x, s) = |\det A|^{-s} ds\, dx$ and the modular function on G_A is $\Delta_{G_A}(x, s) = |\det A|^{-s}$. To ease notation, we will often write $\mu := \mu_{G_A}$.

5.1 Wavelet transforms

The group $G_A = \mathbb{R}^d \rtimes_A \mathbb{R}$ acts unitarily on $L^2(\mathbb{R}^d)$ by means of the *quasi-regular representation* π, defined by

$$\pi(x, s) f = |\det A|^{-s/2} f(A^{-s}(\cdot - x)), \quad (x, s) \in G_A.$$

For a fixed vector $\psi \in L^2(\mathbb{R}^d)\setminus\{0\}$, the associated wavelet transform $W_\psi : L^2(\mathbb{R}^d) \to L^\infty(G_A)$ is defined through the coefficients

$$W_\psi f(x, s) = \langle f, \pi(x, s)\psi \rangle, \quad f \in L^2(\mathbb{R}^d).$$

The vector ψ is called *admissible* if $W_\psi : L^2(\mathbb{R}^d) \to L^2(G_A)$ is an isometry.

The following lemma is a special case of [12, Theorem 1] and [21, Theorem 1.1].

Lemma 5.1 ([12, 21]) *Let $\psi \in L^2(\mathbb{R}^d)$. Then ψ is admissible if and only if*

$$\int_{\mathbb{R}} |\widehat{\psi}((A^*)^s \xi)|^2 \, ds = 1$$

for a.e. $\xi \in \mathbb{R}^d$.

The significance of an admissible vector ψ is that $W_\psi^* W_\psi = I_{L^2(\mathbb{R}^d)}$, and hence that the weak-sense integral formula

$$f = \int_{G_A} W_\psi f(g) \pi(g)\psi \, d\mu_{G_A}(g)$$

holds for every $f \in L^2(\mathbb{R}^d)$. This, combined with fact that $W_\psi : L^2(\mathbb{R}^d) \to L^2(G_A)$ intertwines the action of π and left translation $L_h F = F(h^{-1}\cdot)$ on $L^2(G_A)$, also yields that

$$W_\varphi f = W_\psi f * W_\varphi \psi, \tag{5.2}$$

for all $f, \varphi \in L^2(\mathbb{R}^d)$. The identity (5.2) will be referred to as a *reproducing formula*.

Henceforth, it will always be assumed that $A \in \mathrm{GL}(d, \mathbb{R})$ is both exponential and expansive. This is essential for the existence of admissible vectors $\psi \in \mathcal{S}(\mathbb{R}^d)$ with compact Fourier support, as the following result shows.

Lemma 5.2 ([1, 9, 16, 28]) *Let $A \in \mathrm{GL}(d, \mathbb{R})$ be an exponential matrix. The following assertions are equivalent:*

(i) The matrix A or its inverse A^{-1} is expansive.
(ii) There exists an admissible $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\widehat{\psi} \in C_c^\infty(\mathbb{R}^d)$.

In addition, if A is an expansive matrix, then the admissible vector $\psi \in L^2(\mathbb{R}^d)$ can be chosen such that $\widehat{\psi} \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ satisfies condition (2.6).

We will also need to use wavelet transforms of distributions. For this, consider the subspace of $\mathcal{S}(\mathbb{R}^d)$ given by

$$\mathcal{S}_0(\mathbb{R}^d) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \varphi(x) x^\alpha \, dx = 0, \quad \forall \alpha \in \mathbb{N}_0^d \right\},$$

and equip it with the subspace topology of $\mathcal{S}(\mathbb{R}^d)$. Its topological dual space will be denoted by $\mathcal{S}_0'(\mathbb{R}^d)$ and will often be identified with $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$, see, e.g., [14, Proposition 1.1.3]. The dual bracket between $\mathcal{S}_0$ and $\mathcal{S}_0'(\mathbb{R}^d)$ is denoted by

$$\langle \cdot, \cdot \rangle : \mathcal{S}_0'(\mathbb{R}^d) \times \mathcal{S}_0(\mathbb{R}^d), \quad \langle f, h \rangle := f(\overline{h}).$$

Note that this pairing is conjugate-linear in the second variable.

For a fixed $\psi \in \mathcal{S}_0(\mathbb{R}^d) \setminus \{0\}$, the extended wavelet transform of $f \in \mathcal{S}_0'(\mathbb{R}^d)$ is defined as

$$W_\psi f(x, s) = \langle f, \pi(x, s)\psi \rangle, \quad (x, s) \in G_A.$$

By the continuity of the map $(x, s) \mapsto \pi(x, s)\varphi$ from $\mathbb{R}^d \times \mathbb{R}$ into $\mathcal{S}(\mathbb{R}^d)$, the transform $W_\psi : \mathcal{S}_0'(\mathbb{R}^d) \to C(G_A)$ is well-defined.

The reproducing formula (5.2) can be naturally extended to $\mathcal{S}_0'(\mathbb{R}^d)$. See [20, Lemma 4.7 and Lemma 4.8] for a proof of the following result.

Lemma 5.3 ([20]) *Let $\psi \in \mathcal{S}_0(\mathbb{R}^d)$ be an admissible vector. Then, for arbitrary $f \in \mathcal{S}_0'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$,*

$$\langle f, \varphi \rangle = \int_{G_A} W_\psi f(g) \overline{W_\psi \varphi(g)} \, d\mu_{G_A}(g).$$

In particular, the identity $W_\varphi f = W_\psi f * W_\varphi \psi$ holds.

5.2 Peetre-type spaces

As in [20], we define an auxiliary class of Peetre-type spaces $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}$ on the semi-direct product $G_A = \mathbb{R}^d \rtimes \mathbb{R}$. These spaces are an essential ingredient for identifying Triebel-Lizorkin spaces with associated coorbit spaces [10, 32].

In contrast to the spaces $\dot{\mathbf{P}}^{\alpha,\beta}_{p,q}$ defined in [20, Definition 5.1] for $p < \infty$, the spaces $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}$ will only be defined through averages over small scales.

Definition 5.4 Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and exponential and let $\Omega = \Omega_A$ be an associated ellipsoid as provided by Lemma 2.1. For $\ell \in \mathbb{Z}$ and $w \in \mathbb{R}^d$, set $Q_\ell(w) := A^\ell \Omega + w$.

For $\alpha \in \mathbb{R}$, $\beta > 0$, and $q \in (0, \infty)$, the Peetre-type space $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}(G_A)$ is defined as the space of all (equivalence classes of a.e. equal) measurable $F : G_A \to \mathbb{C}$ such that

$$\|F\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} := \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell} \left[|\det A|^{\alpha s} \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|F(x+z,s)|}{(1+\rho_A(A^{-s}z))^\beta} \right]^q \frac{ds\, dx}{|\det A|^s} \right)^{1/q} < \infty.$$

For $q = \infty$, the space $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,\infty}(G_A)$ consists of all measurable $F : G_A \to \mathbb{C}$ satisfying

$$\|F\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,\infty}} := \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \sup_{s\in\mathbb{R}, s\leq\ell} \left(\fint_{Q_\ell(w)} \left[|\det A|^{\alpha s} \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|F(x+z,s)|}{(1+\rho_A(A^{-s}z))^\beta} \right] dx \right) < \infty.$$

The following lemma collects some basic properties of $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}(G_A)$ and gives explicit estimates for the operator norm of left and right translation. The estimates involve the following weight function

$$v : G_A \to [1, \infty), \quad v(y,t) = \sup_{(z,u)\in G_A} \frac{1+\rho_A(A^{-u}z)}{1+\rho_A(A^{-u}A^t z - y)}.$$

The function v is measurable and submultiplicative by [20, Lemma 5.2].

Lemma 5.5 *Let $\alpha \in \mathbb{R}$, $\beta > 0$, and $q \in (0, \infty]$. Then the Peetre-type space $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}(G_A)$ is a solid quasi-Banach function space (Banach function space if $q \geq 1$). Moreover, the space $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}(G_A)$ is left- and right-translation invariant and there exists $N = N(A, \Omega) \in \mathbb{N}$ such that, for all $(y,t) \in G_A$, the operator norms can be bounded by*

$$|||L_{(y,t)}||| \leq |\det A|^{t(\alpha-1/q)+(N+1)/q}, \quad |||R_{(y,t)}||| \leq \begin{cases} |\det A|^{-t(\alpha-2/q)+1/q}(v(y,t))^\beta & \text{if } t > 0, \\ |\det A|^{-t(\alpha-1/q)}(v(y,t))^\beta & \text{if } t \leq 0, \end{cases}$$

if $q < \infty$, and

$$|||L_{(y,t)}||| \leq |\det A|^{t\alpha+N+1}, \quad |||R_{(y,t)}||| \leq \begin{cases} |\det A|^{-t(\alpha-1)+1}(v(y,t))^\beta & \text{if } t > 0, \\ |\det A|^{-t\alpha}(v(y,t))^\beta & \text{if } t \leq 0, \end{cases}$$

otherwise. Here, the operator norm is written $|||\cdot||| := \|\cdot\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q} \to \dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}}$.

Proof Most of the quasi-norm properties for $\|\cdot\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}}$ can be easily verified from the definition, while the positive definiteness follows from [20, Lemma B.1]. It is clear that $\|\cdot\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}}$ is solid and for $q \geq 1$ a norm. The completeness follows from $\|\cdot\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}}$ satisfying the *Fatou property* (see [34, Sect. 65, Theorem 1] and [33, Lemma 2.2.15]), which is easily verified by a straightforward computation using Fatou's lemma, see, e.g., the proof of [20, Lemma 5.3].

It remains to prove the translation invariance and associated norm estimates. We will only consider $q \in (0, \infty)$, the arguments for $q = \infty$ are analogous. To this end, let $F \in \dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}(G_A)$ and $(y,t) \in \mathbb{R}^d \times \mathbb{R}$ be arbitrary. Then the substitutions $\widetilde{x} = A^{-t}(x-y)$, $\widetilde{w} = A^{-t}(w-y)$, and $\widetilde{z} = A^{-t}z$, as well as $\widetilde{s} = s - t$, yield

$$\begin{aligned}
&\|L_{(y,t)}F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} \\
&= \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell} \left[|\det A|^{\alpha s} \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|F(A^{-t}(x+z-y), s-t)|}{(1+\rho_A(A^{-s}z))^\beta} \right]^q \frac{ds\,dx}{|\det A|^s} \right) \\
&= \sup_{\ell\in\mathbb{Z}, \widetilde{w}\in\mathbb{R}^d} \left(|\det A|^{-\ell} \int_{Q_{\ell-t}(\widetilde{w})} \int_{-\infty}^{\ell} \left[|\det A|^{\alpha s} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(\widetilde{x}+\widetilde{z}, s-t)|}{(1+\rho_A(A^{t-s}\widetilde{z}))^\beta} \right]^q \frac{ds\,d\widetilde{x}}{|\det A|^{s-t}} \right) \\
&= \sup_{\ell\in\mathbb{Z}, \widetilde{w}\in\mathbb{R}^d} \left(|\det A|^{-\ell} \int_{Q_{\ell-t}(\widetilde{w})} \int_{-\infty}^{\ell-t} \left[|\det A|^{\alpha(\widetilde{s}+t)} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(\widetilde{x}+\widetilde{z}, \widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,d\widetilde{x}}{|\det A|^{\widetilde{s}}} \right),
\end{aligned}$$

where the transformation of the domain is $A^{-t}(Q_\ell(w) - y) = A^{-t}(A^\ell\Omega + w - y) = A^{\ell-t}\Omega + \widetilde{w}$. To estimate this further, we decompose $t = k + t'$ with $k \in \mathbb{Z}$ and $t' \in [0,1)$. By [20, Lemma 2.4], there exists $N = N(A,\Omega) \in \mathbb{N}$ such that $A^{-t'}\Omega \subset A^N\Omega$ for all $t' \in [0,1)$, and hence $Q_{\ell-t}(\widetilde{w}) \subset Q_{\ell-k+N}(\widetilde{w})$. Increasing the upper limit of the inner integral from $\ell - t$ to $\ell - k + N$ and substituting $\widetilde{\ell} = \ell - k + N \in \mathbb{Z}$ gives

$$\begin{aligned}
&\|L_{(y,t)}F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} \\
&\leq \sup_{\widetilde{\ell}\in\mathbb{Z}, \widetilde{w}\in\mathbb{R}^d} \left(|\det A|^{-\widetilde{\ell}-k+N} \int_{Q_{\widetilde{\ell}}(\widetilde{w})} \int_{-\infty}^{\widetilde{\ell}} \left[|\det A|^{\alpha(\widetilde{s}+t)} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(\widetilde{x}+\widetilde{z}, \widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,d\widetilde{x}}{|\det A|^{\widetilde{s}}} \right) \\
&\leq \sup_{\widetilde{\ell}\in\mathbb{Z}, \widetilde{w}\in\mathbb{R}^d} \left(|\det A|^{-t+N+1} \fint_{Q_{\widetilde{\ell}}(\widetilde{w})} \int_{-\infty}^{\widetilde{\ell}} \left[|\det A|^{\alpha(\widetilde{s}+t)} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(\widetilde{x}+\widetilde{z}, \widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,d\widetilde{x}}{|\det A|^{\widetilde{s}}} \right) \\
&= |\det A|^{t(\alpha q-1)+N+1} \|F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}}.
\end{aligned}$$

For the right-translation, a direct calculation using the substitutions $\widetilde{z} = z + A^s y$ and $\widetilde{s} = s + t$ shows that

$$\begin{aligned}
&\|R_{(y,t)}F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} \\
&= \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell} \left[|\det A|^{\alpha s} \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|F(x+z+A^s y, s+t)|}{(1+\rho_A(A^{-s}z))^\beta} \right]^q \frac{ds\,dx}{|\det A|^s} \right) \\
&= \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell+t} \left[|\det A|^{\alpha(\widetilde{s}-t)} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(x+\widetilde{z}, \widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}A^t\widetilde{z}-y))^\beta} \right]^q \frac{d\widetilde{s}\,dx}{|\det A|^{\widetilde{s}-t}} \right) \\
&\leq |\det A|^{t-\alpha q t} v(y,t)^{\beta q}
\end{aligned}$$

$$\cdot \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell+t} \left[|\det A|^{\alpha\widetilde{s}} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(x+\widetilde{z},\widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,dx}{|\det A|^{\widetilde{s}}} \right),$$

where we used the fact that $\left(1+\rho_A(A^{-\widetilde{s}}A^t\widetilde{z}-y)\right)^{-1} \leq v(y,t)(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^{-1}$ for all $(\widetilde{z},\widetilde{s}),(y,t)\in\mathbb{R}^d\times\mathbb{R}$ by definition.

For $t \leq 0$, the claimed estimate follows immediately. For $t > 0$, we again write $t = k + t'$ with $k \in \mathbb{N}_0$ and $t' \in [0,1)$. Then $\ell + t \leq \ell + k + 1$ and clearly $Q_\ell(w) = A^\ell\Omega + w \subset A^{\ell+k+1}\Omega + w = Q_{\ell+k+1}(w)$ by Lemma 2.1. Hence,

$$\begin{aligned}
&\sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell+t} \left[|\det A|^{\alpha\widetilde{s}} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(x+\widetilde{z},\widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,dx}{|\det A|^{\widetilde{s}}} \right)\\
&\leq \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(|\det A|^{-\ell} \int_{Q_{\ell+k+1}(w)} \int_{-\infty}^{\ell+k+1} \left[|\det A|^{\alpha\widetilde{s}} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(x+\widetilde{z},\widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,dx}{|\det A|^{\widetilde{s}}} \right)\\
&= |\det A|^{k+1} \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_{\ell+k+1}(w)} \int_{-\infty}^{\ell+k+1} \left[|\det A|^{\alpha\widetilde{s}} \operatorname{ess\,sup}_{\widetilde{z}\in\mathbb{R}^d} \frac{|F(x+\widetilde{z},\widetilde{s})|}{(1+\rho_A(A^{-\widetilde{s}}\widetilde{z}))^\beta} \right]^q \frac{d\widetilde{s}\,dx}{|\det A|^{\widetilde{s}}} \right)\\
&\leq |\det A|^{t+1} \|F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}},
\end{aligned}$$

and consequently

$$\|R_{(y,t)}F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} \leq |\det A|^{2t-\alpha q t+1} v(y,t)^{\beta q} \|F\|^q_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}},$$

for $t > 0$, which completes the proof. □

Lastly, we mention the following r-norm property of Peetre-type spaces.

Lemma 5.6 *Let $\alpha \in \mathbb{R}$ and $\beta > 0$. For $q \in (0,\infty]$, set $r := \min\{1,q\}$. Then $\|\cdot\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}}$ is an r-norm, i.e.,*

$$\|F_1+F_2\|^r_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} \leq \|F_1\|^r_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} + \|F_2\|^r_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} \quad \textit{for} \quad F_1, F_2 \in \dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}.$$

Proof The claim follows immediately from the triangle inequality for $q \in [1,\infty]$, respectively a straightforward computation using the subadditivity of $x \mapsto |x|^q$ for $q \in (0,1)$. □

5.3 Standard envelope and control weight

We recall the definition of a standard envelope given in [20, Definition 5.5].

Definition 5.7 Let $\sigma = (\sigma_1,\sigma_2) \in (0,\infty)^2$ and let $L \in \mathbb{R}$. Then the *standard envelope* $\Xi_{\sigma,L} : G_A \to (0,\infty)$ is given by $\Xi_{\sigma,L}(x,s) := \theta_\sigma(s)\cdot\eta_L(x,s)$, where

$$\eta_L(x,s) := \left(1+\min\{\rho_A(x),\rho_A(A^{-s}x)\}\right)^{-L} \quad \text{and} \quad \theta_\sigma(s) := \begin{cases} \sigma_1^s, & \text{if } s \geq 0,\\ \sigma_2^s, & \text{if } s < 0.\end{cases}$$

A central notion in the theory of coorbit spaces [10, 32] is that of a so-called *control weight*. In the following lemma, we show the existence of such a weight for Peetre-type spaces and show that it can be estimated by standard envelopes as defined in Definition 5.7. The construction of the control weight follows [20, Lemma 5.7], but besides the slightly different parameters, the case distinction for the right translation needs to be accommodated with a few extra terms. The details are as follows.

Lemma 5.8 *Let $\alpha \in \mathbb{R}$ and $\beta > 0$. For $q \in (0, \infty]$, set $r := \min\{1, q\}$. Then there exists a continuous, submultiplicative weight $w = w^{\alpha,\beta}_{\infty,q} : G_A \to [1, \infty)$ such that*

$$w(g) = \Delta^{1/r}(g^{-1})\, w(g^{-1}), \qquad \left|\!\left|\!\left| L_{g^{-1}} \right|\!\right|\!\right| \lesssim w(g), \qquad \left|\!\left|\!\left| R_g \right|\!\right|\!\right| \lesssim w(g), \quad g \in G_A,$$

with implicit constant depending on A, β. The weight w will be referred to as the standard control weight.

Moreover, for $q \in (0, \infty)$, set $\sigma_1 := |\det A|^{1/r+|\alpha-1/q|}$ and $\sigma_2 := |\det A|^{-|\alpha-1/q|}$, as well as

$$\kappa_1 := \begin{cases} |\det A|^{1/r+\alpha+\beta-1/q}, & \text{if } \alpha \geq -\frac{1/r+\beta-3/q}{2}, \\ |\det A|^{-(\alpha-2/q)}, & \text{otherwise}, \end{cases}$$

and

$$\kappa_2 := \begin{cases} |\det A|^{-(\alpha+\beta-1/q)}, & \text{if } \alpha \geq -\frac{1/r+\beta-3/q}{2}, \\ |\det A|^{1/r+\alpha-2/q}, & \text{otherwise}. \end{cases}$$

For $q = \infty$, set $\sigma_1 := |\det A|^{1+|\alpha|}$ and $\sigma_2 := |\det A|^{-|\alpha|}$, as well as

$$\kappa_1 := \begin{cases} |\det A|^{1+\alpha+\beta}, & \text{if } \alpha \geq -\frac{\beta}{2}, \\ |\det A|^{-(\alpha-1)}, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \kappa_2 := \begin{cases} |\det A|^{-(\alpha+\beta)}, & \text{if } \alpha \geq -\frac{\beta}{2}, \\ |\det A|^{\alpha}, & \text{otherwise}. \end{cases}$$

Then the standard control weight w satisfies $w \asymp \Xi_{\sigma,0} + \Xi_{\kappa,-\beta}$.

Proof By [20, Lemma 5.2], the weight $v : G_A \to [0, \infty)$ appearing in Lemma 5.5 is submultiplicative, measurable, and locally bounded. It also satisfies $v \geq 1$, and is therefore a weight function in the sense of [25, Definition 3.7.1]. Hence, there exists a *continuous*, submultiplicative function $v_0 : G_A \to [1, \infty)$ with $v \asymp v_0$ by the proof of [25, Theorem 3.7.5].

Let $\tau \in \mathbb{R}$ and define $a_\tau(g) = a_\tau(x, s) := |\det A|^{s\tau}$ for $g = (x, s) \in G_A$. Then the weight function $w_{\gamma,\delta,\zeta} : G_A \to [1, \infty)$ with respect to $\gamma, \delta, \zeta \in \mathbb{R}$ is given by

$$\begin{aligned} w_{\gamma,\delta,\zeta} := \max \Big\{1,\ \ & a_{1/r},\ \ a_\gamma,\ \ a_{-\gamma},\ \ a_{\gamma+1/r},\ \ a_{1/r-\gamma}, \\ & a_{\delta+1/r} \cdot (v_0^\vee)^\beta,\ \ a_{-\delta} \cdot v_0^\beta,\ \ a_{\zeta+1/r} \cdot (v_0^\vee)^\beta,\ \ a_{-\zeta} \cdot v_0^\beta \Big\}. \end{aligned}$$

Clearly $w_{\gamma,\delta,\zeta}$ is again continuous and submultiplicative. As $\Delta = a_{-1}$ and $a_\tau^\vee = a_{-\tau}$, it is easily verified that

$$(\Delta^{1/r})^\vee \cdot w^\vee_{\gamma,\delta,\zeta} = w_{\gamma,\delta,\zeta}.$$

We choose the parameters γ, δ, ζ according to Lemma 5.5, i.e., $\gamma := \alpha - 1/q$, $\delta := \alpha - 2/q$, $\zeta := \alpha - 1/q$ if $q \in (0, \infty)$, and $\gamma := \alpha$, $\delta := \alpha - 1$, $\zeta := \alpha$ otherwise. Set $w = w^{\alpha,\beta}_{\infty,q} := w_{\gamma,\delta,\zeta}$. For $g = (x, s) \in G_A$, an application of Lemma 5.5 implies $|||L_{g^{-1}}||| \lesssim a_{-\gamma}(g) \leq w(g)$ and

$$|||R_g||| \lesssim \begin{Bmatrix} a_{-\delta}(g)v_0(g)^\beta, & \text{if } s > 0 \\ a_{-\zeta}(g)v_0(g)^\beta, & \text{if } s \leq 0 \end{Bmatrix} \leq w(g).$$

It remains to show that the control weight can be estimated by standard envelopes. To this end, note that $w \asymp w_1 + w_2$ for the weights given by $w_1 := \max\{a_0, a_{1/r}, a_\gamma, a_{-\gamma}, a_{1/r+\gamma}, a_{1/r-\gamma}\}$ and $w_2 := \max\big\{a_{\delta+1/r} \cdot (v_0^\vee)^\beta,\ a_{-\delta} \cdot v_0^\beta,\ a_{\zeta+1/r} \cdot (v_0^\vee)^\beta,\ a_{-\zeta} \cdot v_0^\beta\big\}$. A straightforward computation, also done in the proof of [20, Lemma 5.7], reveals that

$$w_1(g) = w_1(x, s) \leq \begin{Bmatrix} |\det A|^{s(1/r+|\gamma|)}, & \text{if } s \geq 0, \\ |\det A|^{-s|\gamma|}, & \text{if } s < 0, \end{Bmatrix} = \theta_\sigma(s) = \Xi_{\sigma,0}(x, s),$$

by the choice of σ. For estimating w_2, we use the fact that $v_0(x, s) \asymp |\det A|^{s^-}\ \eta_{-1}(x, s)$ and $v_0^\vee(x, s) \asymp |\det A|^{s^+}\ \eta_{-1}(x, s)$ as shown in [20, Lemma 5.7]. For $s \geq 0$, this gives

$$\begin{aligned} w_2(x, s) &\asymp \big(\eta_{-1}(x, s)\big)^\beta \max\big\{|\det A|^{(1/r+\delta+\beta)s},\ |\det A|^{-\delta s},\ |\det A|^{(1/r+\zeta+\beta)s},\ |\det A|^{-\zeta s}\big\} \\ &= \eta_{-\beta}(x, s)\kappa_1^s = \Xi_{\kappa,-\beta}(x, s), \end{aligned}$$

since, in the case of $q \in (0, \infty)$, we have

$$\max\{-\delta,\ 1/r + \zeta + \beta\} = \max\{-\alpha + 2/q,\ 1/r + \alpha + \beta - 1/q,\} = 1/r + \alpha + \beta - 1/q$$

if and only if $\alpha \geq -\frac{1/r+\beta-3/q}{2}$, with a similar case distinction for $q = \infty$. The estimate for $s < 0$ follows analogously. □

5.4 Coorbit spaces

The aim of this section is to show that Triebel-Lizorkin spaces $\dot{\mathbf{F}}^\alpha_{\infty,q}$ can be identified with so-called *coorbit spaces* [10, 32] by use of Theorem 3.3.

The definition of coorbit spaces requires the notion of a local maximal function. For a function $F \in L^\infty_{\text{loc}}(G_A)$, its (left-sided) *maximal function* is defined by

$$M^L_Q F(g) = \operatorname{ess\,sup}_{u \in Q} |F(gu)|, \quad g \in G_A, \tag{5.3}$$

where $Q \subset G_A$ is a relatively compact unit neighborhood.

Definition 5.9 Let $q \in (0, \infty]$, $\alpha \in \mathbb{R}$, and $\beta > 0$. Let $A \in \mathrm{GL}(d, \mathbb{R})$ be expansive and exponential, and let $Q \subset G_A$ be a relatively compact, symmetric unit neighborhood.

For admissible $\psi \in \mathcal{S}_0(\mathbb{R}^d)$, the *coorbit space* $\mathrm{Co}(\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}) = \mathrm{Co}_\psi(\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q})$ is the collection of all $f \in \mathcal{S}_0'(\mathbb{R}^d)$ satisfying

$$\|f\|_{\mathrm{Co}(\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q})} = \|f\|_{\mathrm{Co}_\psi(\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q})} = \left\| M_Q^L(W_\psi f)\right\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} < \infty.$$

The space will be equipped with the (quasi-)norm $\| \cdot \|_{\mathrm{Co}(\dot{\mathbf{P}}^{\alpha,\beta}_{p,q})}$.

The space $\mathrm{Co}(\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q})$ as defined in Definition 5.9 is complete with respect to the quasi-norm $\|\cdot\|_{\mathrm{Co}(\dot{\mathbf{F}}^{\alpha}_{\infty,q})}$. In addition, its definition is independent of the chosen defining vector ψ and unit neighborhood Q, with equivalent norms for different choices. These basic properties follow from the general theory [32]; see [20, Remark 5.10] for details and references.

The following is the key result of this section.

Proposition 5.10 *Let $\alpha \in \mathbb{R}$ and $q \in (0, \infty]$. Let $\beta > 1/q$ if $q \in (0, \infty)$, and $\beta > 1$ if $q = \infty$. Then*

$$\dot{\mathbf{F}}^{\alpha}_{\infty,q} = \begin{cases} \mathrm{Co}\big(\dot{\mathbf{P}}^{-(\alpha+1/2-1/q),\beta}_{\infty,q}\big), & q \in (0,\infty), \\ \mathrm{Co}\big(\dot{\mathbf{P}}^{-(\alpha+1/2),\beta}_{\infty,\infty}\big), & q = \infty. \end{cases}$$

Proof Throughout, let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be an admissible vector with $\widehat{\psi} \in C_c^\infty(\mathbb{R}^d \setminus \{0\}$ satisfying the support condition (2.6). The existence of such vectors follows by [9, Proposition 10]; see also Lemma 5.2. Furthermore, define $Q := [-1, 1)^d \times [-1, 1)$.

We split the proof into three steps and consider only $q \in (0, \infty)$. The case $q = \infty$ follows with the usual adjustments. Set $\alpha' := \alpha + 1/2 - 1/q$. The first two steps are essentially identical to the first part of the proof of [20, Proposition 5.11], but included for completeness.

Step 1. Note that $\psi^* \in \mathcal{S}_0(\mathbb{R}^d)$ also satisfies (2.6), where $\psi^*(t) = \overline{\psi(-t)}$. A direct calculation gives

$$W_\psi f(x, s) = \langle f, \pi(x, s)\psi\rangle = \langle f, |\det A|^{s/2} T_x \psi_{-s}\rangle = |\det A|^{s/2} f * \psi^*_{-s}(x). \tag{5.4}$$

By [20, Lemma A.1], it holds for arbitrary $\beta > 0$ and $s \in \mathbb{R}$ that

$$(\psi^*)^{**}_{s,\beta} f(x) := \sup_{z\in\mathbb{R}^d} \frac{|(f * \psi^*_s)(x+z)|}{(1+\rho_A(A^s z))^\beta} = \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|(f * \psi^*_s)(x+z)|}{(1+\rho_A(A^s z))^\beta}, \quad x \in \mathbb{R}^d.$$

Therefore, an application of Theorem 3.3 yields

$$\begin{aligned} \|f\|^q_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} &\asymp \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\ell}^{\infty} \left[|\det A|^{\alpha s} \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|(f * \psi^*_s)(x+z)|}{(1+\rho_A(A^s z))^\beta} \right]^q ds\, dx \right) \\ &= \sup_{\ell\in\mathbb{Z}, w\in\mathbb{R}^d} \left(\fint_{Q^\ell(w)} \int_{-\infty}^{\ell} \left[|\det A|^{-\alpha s} \operatorname{ess\,sup}_{z\in\mathbb{R}^d} \frac{|(f * \psi^*_{-s})(x+z)|}{(1+\rho_A(A^{-s} z))^\beta} \right]^q ds\, dx \right) \end{aligned}$$

$$= \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell} \left[|\det A|^{-(\alpha+1/2-1/q)s} \operatorname{ess\,sup}_{z \in \mathbb{R}^d} \frac{|W_\psi f(x+z,s)|}{(1+\rho_A(A^{-s}z))^\beta} \right]^q \frac{ds\,dx}{|\det A|^s} \right)$$
$$= \|W_\psi f\|^q_{\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}} \tag{5.5}$$

for any $f \in \mathcal{S}_0'(\mathbb{R}^d)$.

Step 2. The estimate $|W_\psi f| \leq M_Q^L W_\psi f$ a.e. implies that

$$\|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}} \asymp \|W_\psi f\|_{\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}} \leq \|M_Q^L(W_\psi f)\|_{\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}} = \|f\|_{\mathrm{Co}(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})},$$

for $f \in \mathcal{S}_0'(\mathbb{R}^d)$.

Step 3. We prove the remaining estimate $\|f\|_{\mathrm{Co}(\dot{\mathbf{P}}^{-\alpha',\beta}_{p,q})} \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,q}}$ for $f \in \mathcal{S}_0'(\mathbb{R}^d)$. In [20, Equation (5.12)], we already showed that

$$\left(|\det A|^{-(\alpha+1/2)s} \operatorname{ess\,sup}_{z \in \mathbb{R}^d} \frac{|M_Q^L(W_\psi f)(x+z,s)|}{(1+\rho_A(A^{-s}z))^\beta} \right)^q$$
$$\lesssim \sum_{k=-N}^{N} \left(|\det A|^{-\alpha(s+k)} (\psi^*)^{**}_{-(s+k),\beta} f(x) \right)^q,$$

with implicit constant depending on A, α, β, q and N, where $N \in \mathbb{N}$ depends on the support of ψ. Combining this with Theorem 3.3 yields

$$\|f\|^q_{\mathrm{Co}(\dot{\mathbf{P}}^{-\alpha',\beta}_{p,q})}$$
$$= \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell} \left[|\det A|^{-(\alpha+1/2-1/q)s} \operatorname{ess\,sup}_{z \in \mathbb{R}^d} \frac{|M_Q^L(W_\psi f)(x+z,s)|}{(1+\rho_A(A^{-s}z))^\beta} \right]^q \frac{ds}{|\det A|^s}\,dx \right)$$
$$\lesssim \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell} \sum_{k=-N}^{N} \left(|\det A|^{-\alpha(s+k)} (\psi^*)^{**}_{-(s+k),\beta} f(x) \right)^q ds\,dx \right)$$
$$\lesssim_N \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\fint_{Q_\ell(w)} \int_{-\infty}^{\ell+N} \left(|\det A|^{-\alpha s} (\psi^*)^{**}_{-s,\beta} f(x) \right)^q ds\,dx \right)$$
$$= \sup_{\ell \in \mathbb{Z}, w \in \mathbb{R}^d} \left(\frac{\mathrm{m}(A^{\ell+N}\Omega)}{\mathrm{m}(A^{\ell}\Omega)} \fint_{Q_{\ell+N}(w)} \int_{-(\ell+N)}^{\infty} \left(|\det A|^{\alpha s} (\psi^*)^{**}_{s,\beta} f(x) \right)^q ds\,dx \right)$$
$$\lesssim_N \|f\|^q_{\dot{\mathbf{F}}^{\alpha}_{\infty,q}},$$

where we used the fact that $\frac{\mathrm{m}(A^{\ell+N}\Omega)}{\mathrm{m}(A^{\ell}\Omega)} = |\det A|^N \lesssim_N 1$ in the last step. □

6 Molecular decompositions

In this section we use the identification of Triebel-Lizorkin spaces and associated coorbit spaces (cf. Proposition 5.10) to obtain proofs of Theorems 1.2 and 1.3.

6.1 Molecular systems

In addition to the left-sided local maximal function defined in Eq. (5.3), we will also need a two-sided version, defined by

$$M_Q F(g) = \operatorname{ess\,sup}_{u,v\in Q} |F(ugv)|, \quad g \in G_A,$$

for $F \in L^\infty_{\mathrm{loc}}(G_A)$, with $Q \subset G_A$ being a fixed relatively compact unit neighborhood.

For $r = \min\{1, q\}$ with $q \in (0, \infty]$ and the standard control weight $w = w^{\alpha,\beta}_{\infty,q} : G_A \to [1, \infty)$ provided by Lemma 5.8, define the associated *Wiener amalgam space* $\mathcal{W}(L^r_w)$ by

$$\mathcal{W}(L^r_w) = \Big\{ F \in L^\infty_{\mathrm{loc}}(G_A) : M_Q F \in L^r_w(G_A) \Big\}.$$

The space $\mathcal{W}(L^r_w)$ is independent of the choice of neighborhood Q and is complete with respect to the quasi-norm $\|F\|_{\mathcal{W}(L^r_w)} := \|M_Q F\|_{L^r_w}$; see, e.g., [24, Sect. 2] and [32, Sect. 2].

The space $\mathcal{W}(L^r_w)$ provides the class of envelopes that will be used for defining molecules.

Definition 6.1 Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero vector such that $W_\psi\psi \in \mathcal{W}(L^r_w)$ for the standard control weight $w : G_A \to [1, \infty)$ defined in Lemma 5.8.

A countable family $(\phi_\gamma)_{\gamma\in\Gamma}$ in $L^2(\mathbb{R}^d)$ is called an L^r_w*-molecular system* if there exists $\Phi \in \mathcal{W}(L^r_w)$ such that

$$|W_\psi \phi_\gamma(x)| \le \Phi(\gamma^{-1}x), \quad \gamma \in \Gamma, x \in G. \tag{6.1}$$

A function $\Phi \in \mathcal{W}(L^r_w)$ satisfying (6.1) is called an *envelope* for $(\phi_\gamma)_{\gamma\in\Gamma}$.

The following lemma provides an extended duality pairing for $\mathcal{S}'_0(\mathbb{R}^d)$. See [20, Lemma 6.8] for its proof.

Lemma 6.2 [20] *Let $\psi \in \mathcal{S}_0(\mathbb{R}^d)$ be an admissible vector and let $w : G_A \to [1, \infty)$ be the standard control weight as defined in Lemma 5.8.*

Suppose $f \in \mathcal{S}'_0(\mathbb{R}^d)$ satisfies $W_\psi f \in L^\infty_{1/w}(G_A)$ and $\phi \in L^2(\mathbb{R}^d)$ satisfies $W_\psi \phi \in L^1_w(G_A)$. Then the extended pairing *defined by*

$$\langle f, \phi\rangle_\psi := \int_{G_A} W_\psi f(g) \overline{W_\psi \phi(g)}\, d\mu_{G_A}(g) \in \mathbb{C}$$

is well-defined and independent of the choice of admissible vector $\psi \in \mathcal{S}_0(\mathbb{R}^d)$.

6.2 Sequence spaces

The following definition provides a class of sequence spaces that will be used in the molecular decomposition of anisotropic Triebel-Lizorkin spaces.

Definition 6.3 Let $U \subset G_A$ be a relatively compact unit neighborhood with non-void interior and let Γ be any family in G_A such that $\sup_{g \in G_A} \#(\Gamma \cap gU) < \infty$.

For $\alpha \in \mathbb{R}$, $\beta > 0$ and $q \in (0, \infty]$, the *Peetre-type sequence space* $\dot{\mathbf{p}}^{\alpha,\beta}_{\infty,q}(\Gamma, U)$ consists of all sequences $c \in \mathbb{C}^\Gamma$ satisfying

$$\|c\|_{\dot{\mathbf{p}}^{\alpha,\beta}_{\infty,q}} := \Big\| \sum_{\gamma \in \Gamma} |c_\gamma| \mathbb{1}_{\gamma U} \Big\|_{\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}} < \infty,$$

where $\dot{\mathbf{P}}^{\alpha,\beta}_{\infty,q}(G_A)$ denotes the Peetre-type space defined in Definition 5.4.

The Peetre-type sequence space $\dot{\mathbf{p}}^{\alpha,\beta}_{\infty,q}(\Gamma, U)$ forms a quasi-Banach space with respect to the quasi-norm $\| \cdot \|_{\dot{\mathbf{p}}^{\alpha,\beta}_{\infty,q}}$. In addition, it is independent of the choice of the defining neighborhood U. For proofs of both facts, see, e.g., [10, 24] or [33, Lemma 2.3.16].

6.3 Proofs of Theorems 1.2 and 1.3

Theorems 1.2 and 1.3 will be obtained from corresponding results for abstract coorbit spaces [32].

Proof of Theorem 1.2 For the fact that if $W_\varphi \psi \in \mathcal{W}(L^r_w)$ for some admissible $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$, then $W_\phi \psi \in \mathcal{W}(L^r_w)$ for all $\phi \in \mathcal{S}_0(\mathbb{R}^d)$, cf. the proof of [20, Lemma 6.9]. Throughout, we fix an admissible $\varphi \in \mathcal{S}_0(\mathbb{R}^d)$ with $\widehat{\varphi} \in C^\infty_c(\mathbb{R}^d \setminus \{0\})$ satisfying condition (2.6). Such an admissible vector exists by Lemma 5.2; see [9, Proposition 10].

Step 1. Under the assumptions, it follows by an application of Proposition 5.10 that $\dot{\mathbf{F}}^\alpha_{\infty,q} = \mathrm{Co}_\varphi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$. For applying the relevant results of [32], it will next be shown that $\mathrm{Co}_\varphi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$ can be identified with the abstract coorbit spaces used in [32]. To this end, note that $\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}$ is a solid, translation invariant quasi-Banach space by Lemma 5.5, which satisfies the r-norm property by Lemma 5.6. The standard control weight $w = w^{-\alpha',\beta}_{\infty,q} : G_A \to [1, \infty)$ of Lemma 5.8 is a *strong control weight* in the sense of [32, Definition 3.1], which can be dominated by a linear combination of standard envelopes.

Define the space $\mathcal{H}^1_w(\psi) := \big\{ f \in L^2(\mathbb{R}^d) : W_\psi f \in L^1_w(G_A) \big\}$ and equip it with the norm $\|f\|_{\mathcal{H}^1_w(\psi)} := \|W_\psi f\|_{L^1_w}$. Set $\mathcal{R}_w(\psi) := (\mathcal{H}^1_w(\psi))^*$ to be its anti-dual space. Since both $W_\psi \psi, W_\varphi \psi \in \mathcal{W}(L^r_w)$, it follows by similar arguments as proving [20, Lemma D.1] that

$$\mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}) := \big\{ f \in \mathcal{R}_w(\psi) : M^L_Q V_\psi f \in \dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q} \big\} = \mathrm{Co}_\varphi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}),$$

where $V_\psi f = \langle f, \pi(\cdot)\psi\rangle_{\mathcal{R}_w,\mathcal{H}^1_w}$ denotes the conjugate-linear pairing. In addition, for the unique extension $\widetilde{f} \in \mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$ of $f \in \mathrm{Co}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$, it holds that

$$\langle \widetilde{f}, \phi\rangle_{\mathcal{R}_w,\mathcal{H}^1_w} = \langle f, \phi\rangle_\varphi, \qquad \phi \in \mathcal{H}^1_w(\varphi), \tag{6.2}$$

where $\langle\cdot,\cdot\rangle_\varphi$ denotes the extended pairing of Lemma 6.2.

Step 2. Applying [32, Theorem 6.14] yields a compact unit neighborhood $U \subset G_A$ such that for any set $\Gamma \subset G_A$ satisfying (1.10), there exists a family $(\phi_\gamma)_{\gamma\in\Gamma}$ of L^r_w-localized molecules in $L^2(\mathbb{R}^d)$ such that any $\widetilde{f} \in \mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$ can be represented as

$$\widetilde{f} = \sum_{\gamma\in\Gamma} \langle \widetilde{f}, \pi(\gamma)\psi\rangle_{\mathcal{R}_w,\mathcal{H}^1_w}\, \phi_\gamma = \sum_{\gamma\in\Gamma} \langle \widetilde{f}, \phi_\gamma\rangle_{\mathcal{R}_w,\mathcal{H}^1_w}\, \pi(\gamma)\psi, \tag{6.3}$$

with unconditional convergence with respect to the weak*-topology on $\mathcal{R}_w(\psi)$.

Given any $f \in \dot{\mathbf{F}}^\alpha_{\infty,q} = \mathrm{Co}_\varphi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$, we apply (6.3) to its unique extension $\widetilde{f} \in \mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$. Since $\mathcal{W}(L^r_w) \hookrightarrow L^1_w$ by [32, Lemma 3.3(ii)], we have that $\psi \in \mathcal{H}^1_w$ and that $\phi_\gamma \in \mathcal{H}^1_w$. Consequently, the dual pairings in (6.3) agree with the extended pairing of Lemma 6.2; see Eq. (6.2). Since $\mathcal{S}_0 \hookrightarrow \mathcal{H}^1_w$, restricting both sides of (6.3) to $\mathcal{S}_0$ yields the desired expansion with unconditional convergence in the weak*-topology of $\mathcal{S}_0' \cong \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$. □

Proof of Theorem 1.3 As shown in Step 1 of the proof of Theorem 1.2, the map $f \mapsto f|_{\mathcal{S}_0}$ is a well-defined bijection from $\mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$ into $\dot{\mathbf{F}}^\alpha_{\infty,q}$, with $\mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$ denoting the abstract coorbit space as defined in [32]. Therefore, applying [32, Theorem 6.15] yields a compact unit neighborhood $U \subset G_A$ such that for any set $\Gamma \subset G_A$ satisfying (1.11), there exists a family $(\phi_\gamma)_{\gamma\in\Gamma}$ in $\overline{\mathrm{span}}\{\pi(\gamma)\psi : \gamma \in \Gamma\} \subset L^2(\mathbb{R}^d)$ of L^r_w-localized molecules such that the moment problem

$$\langle \widetilde{f}, \phi(\gamma)\psi\rangle_{\mathcal{R}_w,\mathcal{H}^1_w} = c_\gamma, \qquad \gamma \in \Gamma,$$

admits the solution $\widetilde{f} := \sum_{\gamma\in\Gamma} c_\gamma \phi_\gamma \in \mathrm{Co}^{\mathcal{H}}_\psi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q})$ for any sequence $(c_\gamma)_{\gamma\in\Gamma} \in \dot{\mathbf{p}}^{-\alpha',\beta}_{\infty,q}(\Gamma)$. Furthermore, by arguing as in the proof of Theorem 1.2, we can restrict to

$$f := \widetilde{f}|_{\mathcal{S}_0} \in \mathrm{Co}_\varphi(\dot{\mathbf{P}}^{-\alpha',\beta}_{\infty,q}) = \dot{\mathbf{F}}^\alpha_{\infty,q}$$

to obtain the desired claim. □

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