On the domination number and cop number of Erdős–Rényi random graphs

by

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Abstract

We consider the game cops and robbers, which is a pursuit-evasion game played on a graph G. The cops and the robber take turns moving across the vertices of G, where the goal for the cops is to eventually catch the robber. Specifically, we study the cop number of G, i.e. the minimum number of cops that is needed to catch the robber on G. We investigate the relation between the cop number of the Erdős-Rényi (ER) random graph $G(n, p_n)$ and compare it to another graph parameter, the domination number. Our goal is to collect results from previous research and create an overview. Throughout this thesis, we provide some examples of graphs for which the cop number is equal to the domination number. Furthermore, we provide proofs for some results on the domination number of the ER random graph. The main takeaway is that the cop number is asymptotic to the domination number for particular values of p_n.

Layman Abstract

We consider the game cops and robbers. This game is played on a graph, which is a collection of nodes and links connecting some of these nodes. The cops and the robber take turns moving across the nodes, where the cops try to capture the robber. We want to know what is the minimum number of cops that we need to make sure that at some point we catch the robber. This number is called the cop number of the graph. We are mainly interested in a particular type of graph, the Erdős-Rényi (ER) random graph. This type of graph has a set of n nodes. For the links we flip a coin for every two nodes: heads meaning we draw a link between the two nodes, tails meaning we do not. We investigate what the cop number of this ER random graph is and we compare it to the domination number. The domination number of a graph is the size (i.e. number of nodes) in the smallest possible dominating set. A dominating set is a set of nodes of the graph, such that all the nodes in the graph have a neighbor in the set. We collect results from previous research to create an overview. Throughout this thesis, we provide some examples of graphs for which the cop number is equal to the domination number. We also give proofs for some results on the domination number and cop number of the ER random graph. The main takeaway is that the domination number and cop number are strongly related.

Preface

This thesis is written as part of the Bachelor End Project, a final assignment to obtain the degree of Bachelor of Science in Applied Mathematics at the Delft University of Technology.

Given that I assumed the reader to have some knowledge of mathematics, this report aims to address readers with a mathematical background. However, readers without a mathematical background should also understand a great deal of what is done in this report, since the most important concepts are discussed in chapter 2. Readers of this thesis need not be familiar with the concept of an ER random graph prior to reading the report, as this will be discussed in chapter 3.

Firstly, I would like to thank my supervisors Anurag Bishnoi and Júlia Komjáthy for providing useful input during our weekly meetings and for supporting me during my project. Secondly, I would like to thank Carolina Urzúa Torres for being part of my graduation committee.

A.M. Primavera
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Introduction

When wanting to represent elements and their connections, we use a graph, which is essentially a network of vertices and edges connecting vertices. Graphs are used constantly in real life. If we for instance want to represent a set of people and who among them are friends, then we could use a graph. The people are the vertices and we draw an edge between two people if they are friends. Another example is a road network, where the vertices are cities and the edges are highways. It is very common to depict a graph using dots for the vertices and lines for an edge, see for example Figure 1.1. We denote a graph by $G = (V, E)$, where V is the set of vertices of the graph and E is the set of edges.

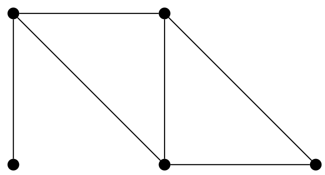


Figure 1.1: Example of a graph

The first paper on graph theory ever published is about the seven bridges in Königsberg and is written by Leonhard Euler [16]. The problem that was presented is the following. One wants to walk through Königsberg (which is now Kalinigrad), crossing each of the seven bridges exactly once. Euler showed using graph theory that this problem has no solution. Such a walk, where one crosses each edge exactly once, is now known as an Euler walk. Since then, graphs have been studied extensively [11].

1.1. Graph parameters

In order to be able to express characteristics of different graphs, we use graph parameters, such as the independence number, clique number or domination number [33]. The latter is a parameter that is used a lot throughout this thesis. The domination number, together with the dominating set is a concept that was first introduced in 1962 by O. Ore [25]. We say that $S \in V$ is a dominating set of G if all vertices in V are either in S or a neighbor of a vertex in S. The domination number $\gamma(G)$ is the size of the smallest possible dominating set.

Graph theory has applications in many different areas, such as organic chemistry, electrical engineering and computer science [3], [28], [38]. In this thesis, we consider the subject of graph games. Specifically, we focus on the game cops and robbers.

1.2. Graph games

In the field of graph games, there are many different types of games that follow different rules. An example of such a game is Tic-Tac-Toe, which takes place on a grid of 3 by 3 vertices [9]. It is a two player game that is won by the player that is able to first claim the elements of a winning set (of vertices). Another example of a graph game is the matroid secretary problem [17], which goes as follows. A set of n elements, each having a certain value, is presented to the player of the game. After the presentation of each element, the player has to decide whether they are satisfied with the value of the element and to stop the process. The goal here is to maximize the value of the element that the player chooses. The problem is of course, that we do not know the value of the future elements and thus do not know when to stop.

In this thesis, as mentioned before, we study another type of graph game. We will consider cops and robbers, which is played on an undirected graph G. The basics of the game are as follows. There exist two teams, namely one robber and a set of cops. The game starts with the cops taking their position on the vertices of the graph. It is possible to occupy a vertex with more than one cop. Subsequently, the robber selects their starting position. Cops and robbers is a perfect information game, meaning that it is assumed that the players know the current positions of each other constantly. Then the cops and the robber start moving across the vertices of the graph, alternating turns. They can only move to neighboring vertices. If eventually one of the cops and the robber end up at the same vertex, the robber is caught and we say that the cops have won the game. If, however, the robber can evade the cops indefinitely, we say that the robber wins.

One of the main questions this game raises is: how many cops do we at least need in order to catch the robber? The answer to this question is the cop number, denoted by $c(G)$, which was introduced by Aigner and Fromme in 1984 [1]. The cop number is equal to the minimum number of cops needed to catch the robber on the graph G.

The game cops and robbers itself was first introduced by Nowakowski and Winkler [30] and Quilliot [36] around 1980. Since then, the game has been studied substantially. For previous studies on cops and robbers, see for example [18], [29].

Depending on what rules you set and assumptions you make for the game, there can be many different variants of cops and robbers. For instance, one could assume that the robber is invisible to the cops [24]. Or we could consider a variant where the cops have limited visibility, i.e. they can only locate the robber when the distance between them is at most l [8]. Another rather well-known variant is lazy cops and robbers [34]. Here only one of the cops can make a move during each turn.

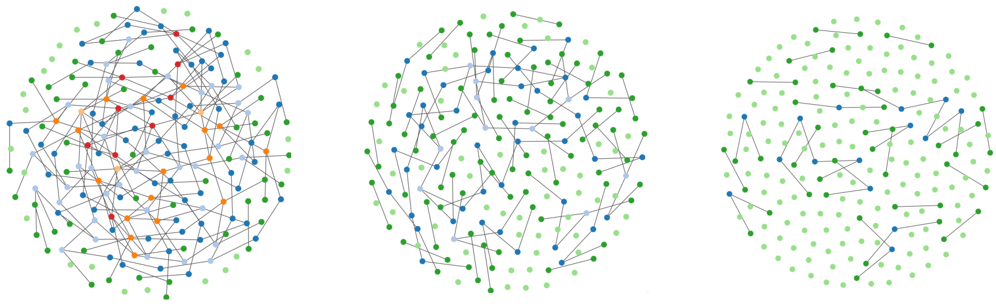
The best-known open problem for the game is Meyniel's conjecture, which states that, if G is a graph on n vertices, then $c(G) \le d\sqrt{n}$, where $d > 0$ is a constant [2]. Bollobás, Kun and Leader have then investigated Meyniel's conjecture for random graphs [5]. Furthermore, in [27], a theorem about the cop number of random graphs, graphs where vertices are adjacent with a certain probability, is stated. We mention and discuss this theorem in Chapter 3.

1.3. Random graphs

Another concept that is used throughout this thesis is the random graph, specifically the Erdős-Rényi model. This model was introduced independently in 1959 by Gilbert [20] and by Erdős and Rényi [13] as an extension of the probabilistic method, which is a method we can use to prove the existence of certain combinatorial objects [22]. Erdős and Rényi received credit for the model, considering that with their contribution the research of the random graph really took off [13],[14],[15].

There are two versions of the Erdős-Rényi model [19]. The first version considers the graph $G(n, M)$, where n is the number of vertices of the graph and we choose at random M edges from all possible edges. The other variant, which we will use in this thesis, considers the graph $G(n, p)$, with n again the number of vertices, but here p is the probability that there exists an edge between two vertices.

Since the introduction several decades ago, the random graph has become an important subject of modern mathematics. Random graphs are used in areas other than mathematics as well, for instance in computer science, life sciences or social sciences [23]. A typical aspect of random graphs that is used in different research areas is the phase transition model. This model describes a sudden change in the properties of a large structure when altering some critical parameter [23]. For example, take the phase transition of a molecular structure. At low temperatures, the structure is in solid state and the molecules (vertices) interact (edges) strongly with each other. At high temperatures, i.e. gas state, the molecules hardly interact with each other. At temperatures in-between, the molecules have some interaction. In this phase transition we thus have two critical temperatures at which the structure changes. See the figure below for visualizations of the structures.



(a) Solid (b) Liquid (c) Gas

Figure 1.2: Organization of molecules in different states, visualized using ER random graphs with $n = 200$ and (a) $p = 0,01$, (b) $p = 0,005$ (critical value), (c) $p = 0,002$.

1.4. Thesis outline

The goal of this thesis is the following. We combine the world of graph games with random graphs and investigate the relation between the domination number and the cop number of the Erdős-Rényi random graph. We aim to give an overview of already existing results in this area. Before we get to this, we will study graphs for which the cop number is equal to the domination number.

Firstly, in Chapter 2, we will define several important concepts, which will be used in the rest of this study. We will also study the domination number and cop number of a graph and mention graphs for which the cop number equals the domination number. Chapters 3 and 4 will cover the already available results on the domination number and cop number of random graphs. Lastly, in Chapter 5 some open problems will be mentioned and discussed.

2

Graphs and graph parameters

2.1. Basics of graph theory

A *graph* $G = (V, E)$ consists of a set of *vertices* (V) and a set of *edges* (E). The elements of E are two-element subsets of V. For $e \in E$ we write $e = uv$, when e is the two-element subset $\{u, v\} \subseteq V$. The edges of G are *undirected*, i.e. the cops and robber can move in either direction across an edge e.

It is very common to represent a graph with a *drawing*, as done in Figure 2.1. Every vertex is represented by a dot and the edges are represented by a line between the dots.
The number of vertices, i.e. the size of V, is called the *order* of the graph and is written as $|V|$. In like manner, the number of edges of the graph is denoted by $|E|$. The *degree* $deg(v)$ of a vertex $v \in G$ is the number of edges that share the vertex v, i.e. $|E(v)|$.

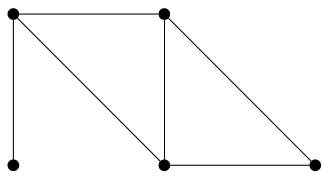


Figure 2.1: Example of a graph $G = (V, E)$ with $|V| = 5$ and $|E| = 6$.

We will now give several definitions, which we will use later.

Definition 2.1.1. *Two vertices $u, v \in V$ are adjacent if $e = uv \in E$. We then say that u and v are neighbors.*

Definition 2.1.2. *A path between two vertices is a graph $P = (V, E)$ with*

$$V = \{v_0, v_1, \ldots, v_k\} \quad \textit{and} \quad E = \{v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k\}.$$

The *length* of the path, denoted $len(P)$, is equal to $|E(P)|$. A graph G is *connected* if there exists a path in G from every vertex to another vertex of the graph.

Definition 2.1.3. *The distance $d_G(u, v)$ between two vertices $u, v \in G$ is the length of the shortest possible path between u and v. If there is no path between u and v, then we say that $d_G(u, v) := \infty$.*

Definition 2.1.4. *The diameter of a graph G, denoted by $diam(G)$, is the largest distance between any two vertices in G.*

We will now give some examples of well-known graphs.

Example 2.1.1 (Cycle)**.** The first well-known graph that we consider is the *cycle* (see Figure 2.2), which is a path where the vertices at the ends are the same vertex, i.e. $v_0 = v_k$. A cycle of length k (a k−cycle) is denoted by C_k.

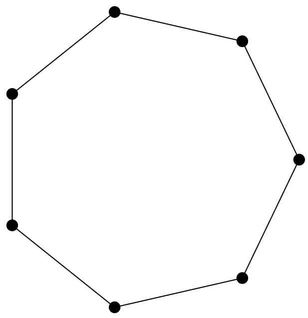


Figure 2.2: The cycle C_7.

Example 2.1.2 (Tree)**.** A graph is called a *tree* if the graph is connected and does not contain any cycles, see e.g. Figure 2.3. The vertices of a tree that have degree 1 are called the *leaves* of the tree.

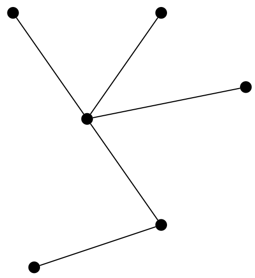


Figure 2.3: Example of a tree.

Example 2.1.3 (Complete graph)**.** We call a graph G *complete* if all vertices are pairwise adjacent and we denote a complete graph on n vertices by K_n. See below the complete graph K_5.

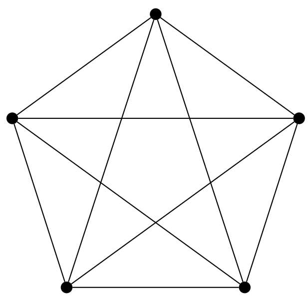


Figure 2.4: The complete graph on 5 vertices K_5.

Example 2.1.4 (Petersen graph)**.** The Petersen graph is shown below in Figure 2.5. In this graph, all vertices have degree 3.

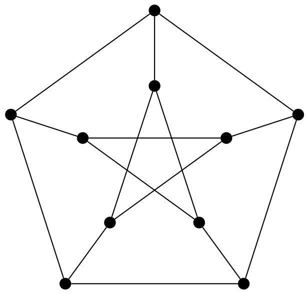


Figure 2.5: The Petersen graph

2.2. The domination number

Before we discuss the cop number and give some examples of it, we first define another important graph parameter, the domination number. Let $G = (V, E)$ be a graph and let $S \subseteq V$. Let us define $N(S)$ as the set of vertices that are adjacent to a vertex $s \in S$. If $N(S) \cup S = V$, then we say that S is a *dominating set* of G. The size of the smallest possible dominating set is called the *domination number* of G, and is denoted $\gamma(G)$.

Example 2.2.1 (Cycle)**.** In Figure 2.6 the smallest possible dominating set of an 8−cycle is shown. We find that the domination number $\gamma(C_8) = 3$. For the cycle on n vertices, C_n, we have $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.

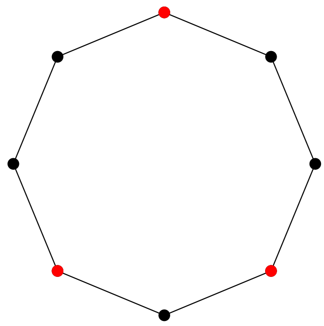


Figure 2.6: Smallest dominating set (red) of an 8−cycle.

Example 2.2.2 (Tree)**.** In a tree, the smallest dominating set can be very large, depending on the composition of the tree. If the diameter of the tree is large, the domination number will also be large. Several bounds on the domination number of trees have been established, for example by Desormeaux, Haynes and Henning [10].

Example 2.2.3 (Complete graph)**.** Since in the complete graph, every vertex is adjacent to every other vertex in the graph, the smallest dominating set consists of one single vertex. Hence, the domination number of a complete graph $\gamma(K_n) = 1$.

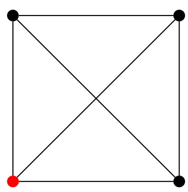


Figure 2.7: Smallest dominating set (red) of the complete graph K_4.

Example 2.2.4 (Petersen graph)**.** The domination number of the Petersen graph is equal to 3, see Figure 2.8 below. Later in this thesis we will see that, for the Petersen graph, the domination number is equal to the cop number (minimum number of cops needed to catch the robber).

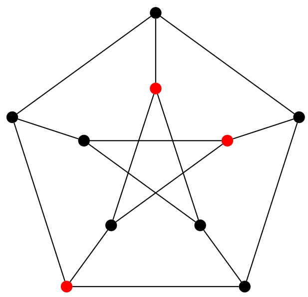


Figure 2.8: Smallest dominating set (red) of the Petersen graph.

2.3. The cop number

The cop number of a graph is the minimum number of cops that is needed to catch the robber on a graph G, and is denoted by $c(G)$. We will now give some examples of the cop number of well-known graphs.

Example 2.3.1 (Cycle)**.** When one single cop and a robber are proceeding through a cycle, the cop will continue to pursue the robber and the robber will continue to move away from the cop. Hence, it is not difficult to observe that two cops are needed and enough in this case. So $c(C_k) = 2$ for all $k \geq 4$.

Example 2.3.2 (Tree)**.** If cops and robbers is played on a tree, then one single cop can follow the robber until arriving at a leaf. The cop has then caught the robber. This will always be the case for a tree, hence we have $c(T) = 1$, for T a tree.

Example 2.3.3 (Complete graph)**.** Placing a robber and one cop on a complete graph causes both players to always be on adjacent vertices of the graph. Hence, the cop can catch the robber in one single turn. So for a complete graph K_n we have $c(K_n) = 1$.

One of the known bounds on the cop number of a graph G is the following.

Theorem 2.3.1. *Let G be a graph. Then for the cop number of G we have*

$$c(G) \leq \gamma(G)\ , \tag{2.1}$$

where $\gamma(G)$ *is the domination number of G.*

It is easy to see that this bound holds. If we place the cops on the vertices of a dominating set of G, then there is always a cop that is on a vertex adjacent to the robber's position. In many cases, this bound is not very tight. Take, for example, the cycle C_k, which has $\gamma(C_k) = \lceil k/3 \rceil$. But the cop number of the cycle, as mentioned in Example 2.3.1, is $c(C_k) = 2$ for all $k \geq 4$. However, there exist graphs, for which the bound in Theorem 2.3.1 is tight, or for which there even is exact equality. It remains an open question [31] whether the following can be obtained for all k.

Problem 2.3.1. *Given a positive integer k, does there exist a connected graph G, such that*

$$c(G) = \gamma(G) \geq k? \tag{2.2}$$

Definition 2.3.1. *The girth of a graph G, denoted by* $g(G)$, *is the minimum length of a cycle contained in G.*

A theorem on the lower bound of the cop number, that is important for showing that (2.2) holds for particular values of k, was found and proved by Aigner and Fromme [1]. The theorem is as follows.

Theorem 2.3.2. *Let G be a graph of girth* ≥ 5 *(no 3- and 4-cycles) with minimum degree* $\delta(G) := \min_{v \in G} deg(v)$*. Then*

$$c(G) \geq \delta(G). \tag{2.3}$$

Proof. Let $G = (V, E)$ be a graph of girth ≥ 5 with minimum degree $\delta(G)$. We start by showing that the robber can choose a starting position, where he will not be caught immediately. Assume that the cops take their starting positions on the vertices $v_1, v_2, \ldots, v_{\delta(G)-1}$. Then, since $|V| > \delta(G)$, there exists a $u \in V$ that has no cop on it. Using now that G has girth ≥ 5, there is a neighbor of u, say w, that is not a neighbor to any of the cops. The robber can thus choose w as a starting position and not get caught immediately.

Let us now assume that we have $\delta(G) - 1$ cops. Then at every point in the game there is at least one neighbor of the robber that is not a neighbor to any cop. This means that the robber can move to such a vertex and can escape every time (recall that G has girth ≥ 5). Hence we need at least $\delta(G)$ cops. □

We will now give some examples of graphs that satisfy the property in (2.2) for particular values of k. The examples that we mention for $k = 5$ and $k = 6$ were pointed out to the authors of [31] by Jérémie Turcotte and my supervisor Anurag Bishnoi pointed them to the example for $k = 7$.

Example 2.3.4 (Petersen graph)**.** For the Petersen graph P, which is given in Example 2.1.4, the property in (2.2) is satisfied with $k = 3$.

Lemma 2.3.1. *Let P be the Petersen graph, see Example 2.1.4. Then*

$$c(P) = \gamma(P) = 3. \tag{2.4}$$

Proof. We showed in Example 2.2.4 that the domination number of the Petersen graph satisfies $\gamma(P) = 3$. With the use of Theorem 2.3.1 we get that $c(P) \leq 3$. For the lower bound, we use Theorem 2.3.2. Since P does not contain any 3- or 4-cycles and has minimum degree $\delta(P) = 3$, we obtain that $c(P) \geq 3$. Taking the two bounds together results in $c(P) = \gamma(P) = 3$. □

Example 2.3.5 (Wong graph)**.** The Wong graph, see Figure 2.9 below, satisfies the property in (2.2) for $k = 5$.

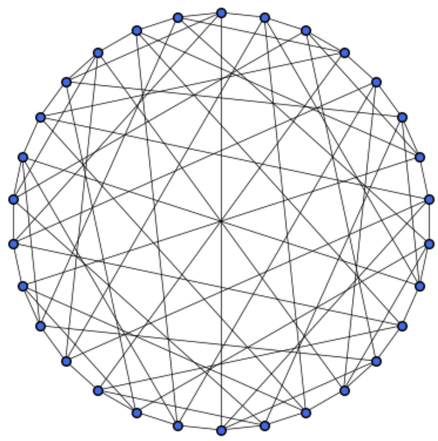


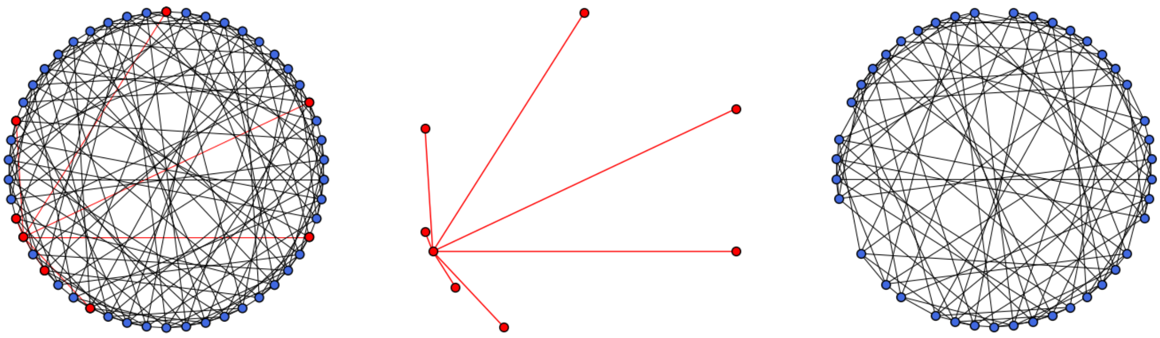
Figure 2.9: The Wong graph

Lemma 2.3.2. *Let W be the Wong graph, see Figure 2.9. Then*

$$c(W) = \gamma(W) = 5. \tag{2.5}$$

Proof. Let us first look at an upper bound for $c(W)$. By Theorem 2.3.1 we have $c(W) \leq \gamma(W) = 5$, see section A.1.1 for how we obtained values of the parameters. For the lower bound we use Theorem 2.3.2. The Wong graph has minimum degree $\delta(W) = 5$ and is of girth 5. Hence, using Theorem 2.3.2, we find $c(W) \geq 5$. Combining now the upper and lower bound on the cop number, we obtain $c(W) = 5$. □

Example 2.3.6 (Hoffman-Singleton graph minus star)**.** The graph that has the property in (2.2) for $k = 6$ is the Hoffman-Singleton graph minus star. This is the graph that we get if we take the Hoffman-Singleton graph, see Figure 2.10a, and we remove a star. A star is a vertex with all its neighbors and the edges between them. So, since in the Hoffman-Singleton graph all vertices have degree 7, we remove a component that has 8 vertices in total, see Figure 2.10b. The resulting graph is the Hoffman-Singleton graph minus star in Figure 2.10c.



(a) Hoffman-Singleton graph with star in red (b) A star taken out of the graph (c) Hoffman-Singleton graph minus star

Figure 2.10: Removing a star from the Hoffman-Singleton graph, visualized in steps.

Lemma 2.3.3. *Let $\hat{H}$ be the Hoffman-Singleton graph minus star. Then*

$$c(\hat{H}) = \gamma(\hat{H}) = 6. \tag{2.6}$$

Proof. We will prove this in the same way as we did before. For the upper bound on the cop number of the Hoffman-Singleton graph minus star we use Theorem 2.3.1. Since the domination number of the graph is equal to 6, we have that $c(\hat{H}) \leq \gamma(\hat{H}) = 6$. See section A.1.2 for how we derived the parameters. Since the Hoffman-Singleton graph minus star has minimum degree $\delta(\hat{H}) = 6$ and is of girth 5, by Theorem 2.3.2 we have that $c(\hat{H}) \geq 6$. Combining the upper and lower bound on the cop number, we find that $c(\hat{H}) = 6$. □

Example 2.3.7 (Hoffman-Singleton graph)**.** The Hoffman-Singleton graph H, which is given in Figure 2.11, also satisfies the property in (2.2), for $k = 7$.

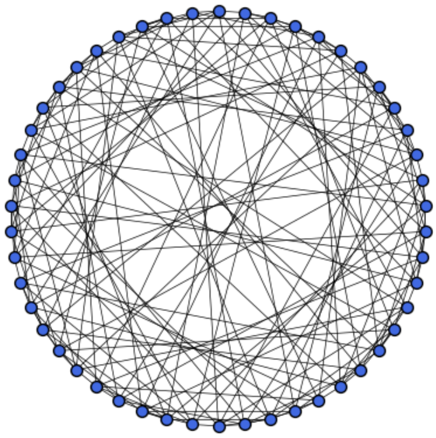


Figure 2.11: The Hoffman-Singleton graph

Lemma 2.3.4. *Let H be the Hoffman-Singleton graph. Then*

$$c(H) = \gamma(H) = 7. \tag{2.7}$$

Proof. We will prove the statement in the way as before. The upper bound on the cop number of the Hoffman-Singleton graph is obtained by Theorem 2.3.1, stating that $c(H) \leq \gamma(H) = 7$. Again, see section A.1.3 for how we obtained values of the parameters. Since the Hoffman-Singleton graph has minimum degree $\delta(H) = 7$ and has girth 5, by Theorem 2.3.2 we have that $c(H) \geq 7$. Again, combining the upper and lower bound on the cop number, we find that $c(H) = 7$. $\square$

All the proofs of the examples with equal cop number and domination number that we mentioned above were constructed in the same way. Since all graphs had girth 5, we were able to find a lower bound, using the minimum degree of the graph. The upper bound followed from the domination number. It could be interesting to see if we are able to do this for more values of k than those that are mentioned in this section. We mention this again in Chapter 5.

3

Domination number of Erdős-Rényi random graphs

In this chapter, we take a look at the domination number of Erdős-Rényi (ER) random graphs. This model for a random graph was introduced independently by Erdős and Rényi [13] and Gilbert [20], around 1960.

Definition 3.0.1. *An Erdős-Rényi (ER) random graph, denoted by $G(n, p_n)$, is a graph on n vertices, where two vertices are adjacent (i.e. form an edge) with probability p_n, independently from other edges.*

It is important to first establish some notation for these graphs. When p does not depend on n, we write $p_n = p$. Furthermore, we often express p as a power of n.

$$np = n^{\alpha} \quad \Rightarrow \quad p = n^{\alpha-1}. \tag{3.1}$$

Throughout this thesis, when we write $\log n$ (without subscript), we mean the natural logarithm of n. We will now present some results about the domination number of ER random graphs. The first result is due to Dryer [12] and describes the case where p is fixed.

3.1. The domination number of ER random graphs for fixed p

Theorem 3.1.1 is a result of the research of Dryer [12].

Definition 3.1.1. *We say that an event holds asymptotically almost surely (a.a.s.) if it holds with probability $1 - o(1)$, meaning that the probability of the event happening tends to 1 as n tends to infinity.*

Theorem 3.1.1. *Consider an ER random graph $G(n, p_n)$, as given in Definition 3.0.1, with fixed parameter $p_n = p > 0$ and the number of vertices n tending to infinity. Then a.a.s. the domination number of $G(n, p)$ satisfies*

$$\gamma(G(n,p)) = (1 + o(1)) \log_q n, \tag{3.2}$$

where $q = \frac{1}{1-p}$.

We will now take a close look at the following result, that is mentioned in [37] and provide a proof.

Proposition 3.1.1. *Let $G(n, p)$ be the ER random graph with fixed p. Let a_n be any sequence approaching infinity. Then*

$$\mathbb{P}\big(\gamma(G(n,p)) \leq \lceil \log_q n + a_n \rceil\big) \to 1 \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Proof. Let a_n be any sequence approaching infinity. Then

$$\begin{aligned} \mathbb{P}\big(\gamma(G(n,p)) \leq \lceil \log_q n + a_n \rceil\big) &= \mathbb{P}\big(\exists \text{ a dominating set of size } r := \lceil \log_q n + a_n \rceil\big) \\ &\geq \mathbb{P}\big(\text{vertices } \{1, 2, \ldots, r\} \text{ form a dominating set}\big). \end{aligned} \tag{3.4}$$

We now have a set consisting of r vertices, which we will call A_r. Let B_r be the set with the remaining $n - r$ vertices, as in Figure 3.1 below.

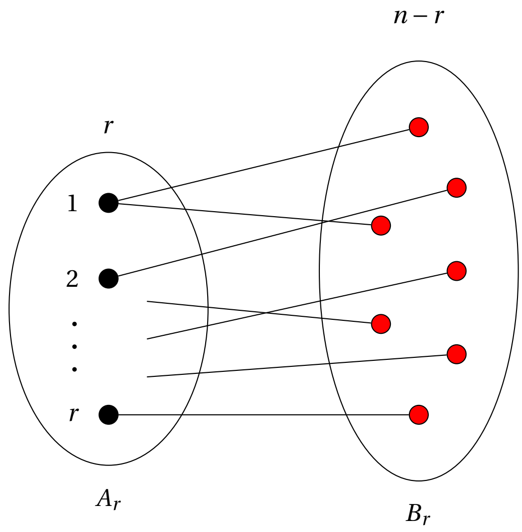


Figure 3.1: Visualization of the sets A_r and B_r, consisting of r and $n-r$ vertices, respectively.

It becomes clear that the vertices $\{1,2,\ldots,r\}$ forming a dominating set is equivalent to every vertex in B_r having at least one neighbor in A_r, as illustrated in Figure 3.1. This is because in that case $A_r \cup N(A_r) = V(G(n,p))$. So we obtain the following.

$$\begin{aligned}\mathbb{P}\big(\gamma(G(n,p)) \le \lceil \log_q n + a_n \rceil\big) &\ge \mathbb{P}\big(\forall \text{vertex in } B_r \text{ is adjacent to } \ge 1 \text{ vertex in } A_r\big) \\ &= \prod_{i=1}^{n-r} \mathbb{P}\big(\text{vertex } v \text{ in } B_r \text{ is adjacent to } u \text{ in } A_r\big) \\ &= \mathbb{P}\big(\text{vertex } v \text{ in } B_r \text{ is adjacent to } u \text{ in } A_r\big)^{n-r}.\end{aligned} \tag{3.5}$$

Where we use the fact that the probability of vertices forming an edge is independent. Next, we use the probability of a vertex v in B_r not having any neighbors in A_r for determining $\mathbb{P}(\text{vertex } v \text{ in } B_r \text{ is adjacent to } u \text{ in } A_r)$, as follows.

$$\begin{aligned}\mathbb{P}\big(\gamma(G(n,p)) \le \lceil \log_q n + a_n \rceil\big) &\ge \big(1 - \mathbb{P}(\text{vertex } v \text{ in } B_r \text{ has no neighbor in } A_r)\big)^{n-r} \\ &= \big(1-(1-p)^r\big)^{n-r}.\end{aligned} \tag{3.6}$$

Where the last step is due to the fact that an edge from a vertex v in B_r to a vertex u in A_r is non-existent with probability $1-p$, and we have r possibilities for vertices u in A_r. Next, we use that $(1-x)^a \ge 1-ax$, with $a = n-r$ and $x = (1-p)^r$, to obtain the following.

$$\begin{aligned}\mathbb{P}\big(\gamma(G(n,p)) \le \lceil \log_q n + a_n \rceil\big) &\ge 1-(n-r)(1-p)^r \\ &= 1 - n(1-p)^r + r(1-p)^r \\ &\ge 1 - n(1-p)^r.\end{aligned} \tag{3.7}$$

Substituting $r := \lceil \log_q n + a_n \rceil$ back in yields:

$$\begin{aligned}\mathbb{P}\big(\gamma(G(n,p)) \le \lceil \log_q n + a_n \rceil\big) &\ge 1 - n(1-p)^{\lceil \log_q n + a_n \rceil} \\ &\ge 1 - n(1-p)^{\frac{\log n}{-\log(1-p)} + a_n} \\ &= 1 - n\Big(e^{\frac{\log(1-p)\log n}{-\log(1-p)} + a_n \log(1-p)}\Big).\end{aligned} \tag{3.8}$$

Where we used that $1-p = e^{\log(1-p)}$ and $e^{a+b} = e^a e^b$. Rewriting the statement gives:

$$\begin{aligned}\mathbb{P}\big(\gamma(G(n,p)) \le \lceil \log_q n + a_n \rceil\big) &\ge 1 - n\big(e^{-\log n} e^{a_n \log(1-p)}\big) \\ &= 1 - n n^{-1}(1-p)^{a_n} \\ &= 1-(1-p)^{a_n}.\end{aligned} \tag{3.9}$$

Previously, we assumed that $a_n \to \infty$ as $n \to \infty$ and since $p > 0$, we have that $1-p < 1$. This results in $(1-p) \to 0$ as $n \to \infty$. Hence we get:

$$\mathbb{P}\big(\gamma(G(n,p)) \leq \lceil \log_q n + a_n \rceil\big) \geq 1-(1-p)^{a_n} \to 1-0 = 1. \tag{3.10}$$

So $\log_q n + a_n$ vertices is sufficient for any a_n tending to infinity. □

Let us now see why $\log_q n - a_n$ vertices is not enough. This is a result from [12], where the proof is left out, so we will give it here. It is a nice result that we can use to prove Theorem 3.1.1. In order to do this, we will prove the following proposition.

Proposition 3.1.2. *Let $G(n,p)$ be the ER random graph with fixed p. Then*

$$\mathbb{P}\big(\gamma(G(n,p)) \geq \lceil \log_q n - a_n \rceil\big) \to 1 \quad \text{as} \quad n \to \infty \tag{3.11}$$

for all $a_n = \frac{2\log(\log n) - \log(|\log(1-p)|)}{|\log(1-p)|} + g(n)$, *where* $g(n) = \omega(1)$.

Proof.

$$\begin{aligned} \mathbb{P}\big(\gamma(G(n,p)) \geq \lceil \log_q n - a_n \rceil\big) &= \mathbb{P}\big(\text{no set of size } r := \lceil \log_q n - a_n \rceil \text{ is dominating}\big) \\ &= 1 - \mathbb{P}\big(\exists \text{ a dominating set of size } r\big). \end{aligned} \tag{3.12}$$

In order to prove the statement in (3.11), we need to show that the probability that there exists a dominating set of size r tends to 0. Since there are $\binom{n}{r}$ possibilities of a set of size r and every set could be a dominating set, using the same reasoning as before, we have:

$$\begin{aligned} \mathbb{P}\big(\exists \text{ a dominating set of size } r\big) &= \binom{n}{r}\mathbb{P}\big(\text{vertices } \{1,2,\ldots,r\} \text{ form a dominating set}\big) \\ &= \binom{n}{r}\prod_{i=1}^{n-r}\mathbb{P}\big(\forall \text{ vertex in } B_r \text{ has a neighbor in } A_r\big) \\ &= \binom{n}{r}\Big(\mathbb{P}\big(\forall \text{ vertex in } B_r \text{ has a neighbor in } A_r\big)\Big)^{n-r}. \end{aligned} \tag{3.13}$$

Again, an edge from a vertex v in B_r to a vertex u in A_r is non-existent with probability $1-p$, and there are r possible neighbors in A_r for vertices in B_r. Furthermore, we use that $1-x \leq e^{-x}$ to obtain the following.

$$\begin{aligned} \mathbb{P}(\exists \text{ a dominating set of size } r) &= \binom{n}{r}\big(1-(1-p)^r\big)^{n-r} \\ &\leq \binom{n}{r}e^{-(n-r)(1-p)^r} \\ &\leq n^r\, e^{-(n-r)(1-p)^{\log_q n - a_n}}. \end{aligned} \tag{3.14}$$

Where in the last step we used that $\binom{n}{r} \leq n^r$ and we substituted $r := \lceil \log_q n - a_n \rceil$ back in. Rewriting so that we lose the logarithm in the e-power gives the following.

$$\begin{aligned} \mathbb{P}\big(\exists \text{ a dominating set of size } r\big) &\leq n^r\, e^{-(n-r)\frac{1}{n}(1-p)^{-a_n}} \\ &= e^{r\log n - (n-r)\frac{1}{n}(1-p)^{-a_n}}. \end{aligned} \tag{3.15}$$

If we look at (3.15), we see that $r\log n$ tends to infinity as n tends to infinity, $(n-r)\frac{1}{n}$ tends to 1 as n tends to infinity. Furthermore, $(1-p)^{-a_n}$ will tend to infinity as n tends to infinity. Taking all this together, we see that, in order for the whole e-power to tend to 0, we need that $(1-p)^{-a_n} \gg r\log n$.

It is elementary computation to check that $(1-p)^{-a_n} \gg r\log n$ is equivalent to requiring $a_n \gg \frac{2\log(\log n)}{|\log(1-p)|} + C(p)$, where $C(p)$ is a constant. In order for a_n to meet this requirement, we can take $a_n = \frac{2\log(\log n) - \log(|\log(1-p)|)}{|\log(1-p)|} + g(n)$, where we take $g(n) = \omega(1)$, which is any function tending to infinity.

So now we have shown that $\mathbb{P}\big(\exists$ a dominating set of size $r := \lceil \log_q n - a_n \rceil\big) \to 0$, for all $a_n = \frac{2\log(\log n) - \log(|\log(1-p)|)}{|\log(1-p)|} + g(n)$. This is, if we look at the beginning of our proof, equivalent to the following statement.

$$1 - \mathbb{P}\big(\exists \text{ a dominating set of size } r\big) \to 1\ , \text{ for all } a_n = \frac{2\log\big(\log n\big) - \log\big(|\log(1-p)|\big)}{|\log(1-p)|} + g(n)\ , \text{with } g(n) = \omega(1)$$

Hence, we see that for particular a_n, having $\log_q n - a_n$ vertices is enough, but not for all a_n, which is what we wanted to show. □

Proof of Theorem 3.1.1. From Proposition 3.1.1 and Proposition 3.1.2 it easily follows that a.a.s. we have

$$\gamma(G(n,p)) = (1+o(1))\log_q n, \tag{3.16}$$

where $q = \frac{1}{1-p}$. □

3.2. The domination number of ER random graphs for *p* depending on *n*

In the previous section we explained a result for the domination number of ER random graphs for fixed p. In this section we look at the case where p is not fixed, so p depends on n. Glebov, Liebenau and Szabó [21] established the following result on the domination number of $G(n,p_n)$.

Theorem 3.2.1. *Let* p_n *be such that* $\frac{\log^2 n}{\sqrt{n}} \ll p_n < 1$. *Then there exists an* $\hat{r}$ *, see* (3.17)*, such that* $\gamma(G(n,p_n)) = \lfloor \hat{r} \rfloor + 1$ *or* $\gamma(G(n,p_n)) = \lfloor \hat{r} \rfloor + 2$*, with*

$$\hat{r} = \log_q\left(\frac{n\log q}{\log^2 np_n}(1+o(1))\right) \text{ and } q = \frac{1}{1-p_n}. \tag{3.17}$$

4

Cop number of Erdős-Rényi random graphs

In this chapter we will discuss important results established by Łuczak and Prałat [27] on the cop number of the ER random graph. We divide this chapter into two parts, one where we discuss the case that p is fixed and one for p depending on n.

4.1. Fixed p

For fixed p, it has been shown [6] that for the cop number of ER random graphs we have the following.

Theorem 4.1.1. *Let $G(n,p)$ be the ER random graph with fixed p. Then*

$$c(G(n,p)) = (1+o(1))\log_q n, \tag{4.1}$$

where $q = \frac{1}{1-p}$.

Proof. The proof of the upper bound follows from the result of the previous chapter. We namely showed that $\gamma(G(n,p)) = (1+o(1))\log_q n$, which is Theorem 3.1.1. Combining this with Theorem 2.3.1, we obtain $c(G(n,p)) \leq (1+o(1))\log_q n$. We will now show that the lower bound also holds, as Bonato, Hahn and Wang did [6].

First, let us state the following. If k is a positive integer, we say that a graph is $(1,k)-e.c.$ if for each subset $S \subseteq V$, with $|S| = k$, and vertex u, there exists a vertex $z \notin S$, not joined to any vertex in S and joined to u.

If G is $(1,k)-e.c.$, then we need $\geq k$ cops. Write $k = \lfloor (1-o(1))\log_q n \rfloor$, with $q = \frac{1}{1-p}$. In order to show that $c(G(n,p)) \geq (1-o(1))\log_q n$, we will determine the probability that G is not $(1,k)-e.c.$. This is done in the following way.

Let $S \subseteq V$ such that $|S| = k$, let $u \notin S$. Then take $z \in V \setminus S$, $z \neq u$. Then

$$\begin{aligned} &\mathbb{P}\big(z \text{ is not adjacent to } u \text{ or } z \text{ is not adjacent to} \geq 1 \text{ element of } S.\big) \\ &= 1 - \mathbb{P}\big(z \text{ is adjacent to } u \text{ and } z \text{ is non-adjacent to each element of } S.\big) \\ &= 1 - p(1-p)^k. \end{aligned} \tag{4.2}$$

Where the last step is due to S having k elements. Since there are $n-k-1$ possibilities for z, the probability that G is not $(1,k)-e.c.$ for this particular choice of S and k, is equal to $\big(1-p(1-p)^k\big)^{n-k-1}$. The probability of having ≥ 1 such S and k is then

$$\begin{aligned} \mathbb{P}\big(G \text{ is not } (1,k)-e.c.\big) &\leq \binom{n}{k+1}\big(1-p(1-p)^k\big)^{n-k-1} \\ &\leq n^{k+1}\big(1-p(1-p)^k\big)^{n-k-1} = f(n). \end{aligned} \tag{4.3}$$

From Lemma 3.5 of [6] it then follows that

$$\lim_{n\to\infty} f(n) = 0.$$

So, with probability tending to 1, we find that $c(G(n,p)) \geq (1+o(1))\log_q n$, as desired. □

4.2. Zigzag theorem

We will now consider the case where p is not fixed, so $p = p_n$. The cop number in this case is more complicated than having a fixed p. The goal of this section is to summarize the result of Łuczak and Prałat [27]. The main result that they proved is in the following theorem.

Theorem 4.2.1. *Let* $0 < \alpha < 1$ *(see* (3.1) *for notation) and let* $d(n) = np_n = n^{\alpha+o(1)}$*, i.e.* $p_n = n^{\alpha-1+o(1)}$.

$$\begin{aligned} &\text{(i) If } \frac{1}{k+1} < \alpha < \frac{1}{k} \text{ for an even integer } k(\alpha)\text{, then} \\ &\qquad c(G(n,p_n)) = \Theta\left(d(n)^{j(\alpha)}\right), \\ &\quad \text{where } j(\alpha) = \frac{k(\alpha)}{2}. \\ &\text{(ii) If } \frac{1}{k+1} < \alpha < \frac{1}{k} \text{ for an odd integer } k(\alpha)\text{, then} \\ &\qquad \Omega\left(\frac{n}{d(n)^{j(\alpha)}}\right) = c(G(n,p_n)) = O\left(\frac{n\log n}{d(n)^{j(\alpha)}}\right), \\ &\quad \text{where } j(\alpha) = \frac{k(\alpha)+1}{2}. \end{aligned} \tag{4.4}$$

We will now explore how Theorem 4.2.1 leads to the zigzag function in Figure 4.1. Let us first study the case where $k(\alpha)$ is *even*. For $k(\alpha) = 2j(\alpha)$ we have

$$\begin{aligned} c(G(n,p_n)) &= \Theta\left(d(n)^{j(\alpha)}\right) \\ &= \Theta\left(n^{(\alpha+o(1))j(\alpha)}\right) \\ &= \Theta\left(n^{\alpha j(\alpha)+o(1)j(\alpha)}\right). \end{aligned} \tag{4.5}$$

For α we have the following.

$$\frac{1}{2j(\alpha)+1} < \alpha < \frac{1}{2j(\alpha)}. \tag{4.6}$$

Multiplying with $j(\alpha)$ gives

$$\frac{j(\alpha)}{2j(\alpha)+1} < \alpha j(\alpha) < \frac{1}{2}. \tag{4.7}$$

But when $\alpha = \frac{1}{2j(\alpha)}$, i.e. when $\alpha = \frac{1}{2\cdot 2m}$, where m is an integer, then $\alpha j(\alpha) = \frac{1}{2}$. So we obtain the following for the cop number

$$c(G(n,p_n)) = c\left(G(n,n^{\alpha-1+o(1)})\right) = n^{\frac{1}{2}+o(1)j(\alpha)}. \tag{4.8}$$

For the case where $k(\alpha)$ is *odd* we have that

$$\begin{aligned} c(G(n,p_n)) &= O\left(\frac{n\log n}{d(n)^{j(\alpha)}}\right) \\ &= O\left(\frac{n\log n}{n^{\alpha j(\alpha)+o(1)j(\alpha)}}\right) \\ &\leq O\left(n^{1-\alpha j(\alpha)-o(1)j(\alpha)}\right). \end{aligned} \tag{4.9}$$

For the other side of the equation we have

$$\begin{aligned}
c(G(n,p_n)) &= \Omega\left(\frac{n}{d(n)^{j(\alpha)}}\right) \\
&= \Omega\left(\frac{n}{n^{\alpha j(\alpha)+o(1)j(\alpha)}}\right) \\
&= \Omega\left(n^{1-\alpha j(\alpha)-o(1)j(\alpha)}\right).
\end{aligned} \tag{4.10}$$

Taking together (4.9) and (4.10) gives us the following for odd $k(\alpha)$.

$$c(G(n,p_n)) = n^{1-\alpha j(\alpha)-o(1)j(\alpha)}. \tag{4.11}$$

Putting together everything for both cases yields

$$c\left(G(n,n^{\alpha-1+o(1)})\right) = \begin{cases} n^{\alpha j(\alpha)-o(1)j(\alpha)} & \text{, if } k(\alpha) \text{ is even} \\ n^{1-\alpha j(\alpha)-o(1)j(\alpha)} & \text{, if } k(\alpha) \text{ is odd.} \end{cases} \tag{4.12}$$

Next, we take the logarithm of both sides of (4.12). For the even case this gives

$$\begin{aligned}
\log\left(c\left(G(n,n^{\alpha-1+o(1)})\right)\right) &= \log\left(n^{\alpha j(\alpha)-o(1)j(\alpha)}\right) \\
&= \left(\alpha j(\alpha)-o(1)j(\alpha)\right)\log n.
\end{aligned} \tag{4.13}$$

This gives that

$$\alpha j(\alpha)-o(1)j(\alpha) = \frac{\log\left(c\left(G(n,n^{\alpha-1+o(1)})\right)\right)}{\log n}. \tag{4.14}$$

For the odd case we can do the same, to obtain

$$1-\alpha j(\alpha)-o(1)j(\alpha) = \frac{\log\left(c\left(G(n,n^{\alpha-1+o(1)})\right)\right)}{\log n}. \tag{4.15}$$

So let the function $f:(0,1)\to\mathbb{R}$ be defined as follows.

$$f(x) = \frac{\log \bar{c}(G(n,n^{x-1}))}{\log n}, \tag{4.16}$$

where $\bar{c}(G(n,n^{x-1}))$ denotes the median of the cop number of $G(n,n^{x-1})$. From our calculations above, see (4.14) and (4.15), it then follows that

$$f(x) = \begin{cases} \alpha j(\alpha) & \text{, if } k(\alpha) \text{ is even} \\ 1-\alpha j(\alpha), & \text{if } k(\alpha) \text{ is odd,} \end{cases} \tag{4.17}$$

resulting in the zigzag function, shown in Figure 4.1 below.

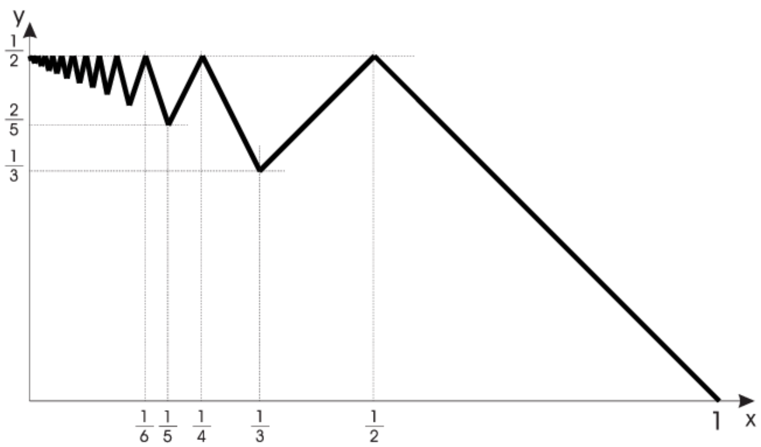


Figure 4.1: The zigzag function

This zigzag shape is caused by $k(\alpha)$ being an integer (odd or even) and the cops and robber taking turns, where the cops take the first turn. Having an odd $k(\alpha)$ is beneficial, in the sense that we need less cops.

4.3. Comparison of domination number and cop number of ER random graphs

In this section we will compare chapters 3 and 4, to see how the domination number and cop number of the Erdős-Rényi random graph are related. We will look at the case where p is fixed an where p depends on n separately.

Let us first look at the case where p is fixed. From section 3.1 we have that the domination number satisfies $\gamma(G(n,p)) = (1+o(1))\log_q n$, where $q = \frac{1}{1-p}$. Section 4.1 tells us that the cop number $c(G(n,p)) = (1+o(1))\log_q n$. So, when p is fixed, the cop number asymptotically equals the domination number, with probability tending to 1.

Next, we consider the other case, p depends on n. In section 3.2 we mentioned a theorem from Glebov, Liebenau and Szabó [21] on the domination number, see Theorem 3.2.1. For the cop number, we discussed in section 4.2 the zigzag theorem by Łuczak and Prałat, see Theorem 4.2.1. What we take from these sections is the following: The cop number of $G(n,p_n)$ is determined by the zigzag function, see Figure 4.1. From this it follows that the cop number is asymptotically equal to the domination number for $p_n = n^{\alpha-1}$, when $\alpha \in \left(\frac{1}{2},1\right)$. For other values of α, it is not.

5 Open problems

In this chapter we will mention some open problems that exist in the area of cops and robbers on graphs. The first open problem that is worth mentioning is Meyniel's conjecture, which is as follows.

Problem 5.0.1 (Meyniel's conjecture). *If G is a graph of order n, then*

$$c(G) = O(\sqrt{n}) \tag{5.1}$$

For n sufficiently large, this is equivalent to saying that there exists a constant $d > 0$ such that

$$c(G) \leq d\sqrt{n} \tag{5.2}$$

In this thesis we have explored the cop number of ER random graphs, which gives an example of a graph sequence that achieves an almost sharp approximation of the conjecture. Because, if we let $p_n = n^{\alpha-1+o(1)}$ and we take $\alpha = 0$ or $\alpha = \frac{1}{2}$, then $c(G(n, p_n)) = n^{\frac{1}{2}+o(1)}$ asymptotically almost surely.

In 2011 Lu and Peng [26] proved that Meyniel's Conjecture holds for diameter 2 graphs and for bibartite graphs of diameter 3. However, the conjecture has not yet been proved for general graphs.

The next problem that we want to address is about graphs that have equal domination number and cop number, as mentioned before in chapter 2, see Problem 2.3.1. In chapter 2 we already explored the problem and we mentioned some specific cases where (2.2) holds. However we would like to know if it is also true for all k, as Petr, Portier and Versteegen mentioned in their survey on cops and robbers [31]. My supervisor Anurag Bishnoi pointed out to me the following: If we combine Theorem 1.1 and 2.1 from [32], then we see that for every $k \in \mathbb{Z}_+$ there exists a graph such that $c(G) = \gamma(G) = k$. This would be an answer to Question 4.2 from [31]. It remains a question what other sequences of such graphs there are.

A natural weaker case of Problem 2.3.1, which could be interesting to look at, would be the following.

Problem 5.0.2. *Can we find a sequence of graphs G_n with $c(G_n) = \gamma(G_n)(1 - o(1))$?*

Looking at Chapter 4, we can conclude that this holds for the ER random graph with fixed parameter p.

In section 2.3 of Chapter 2 we looked at some examples of graphs with equal domination number and cop number. For all examples we had that the graphs had girth 5. For $k = 3, 5, 6, 7$ we had that if the graph is k-regular (all vertices have degree k), then $c(G) = \gamma(G) = k$. So it would be natural to wonder if we can do this for more values of k.

Problem 5.0.3. *Can we find an infinite sequence of k's, for which there exists a k-regular graph G of girth 5 with $\gamma(G) = k$?*

6 Conclusion

The goal of this thesis was twofold. Firstly, we wanted to investigate how the domination number and the cop number of Erdős-Rényi random graphs are related. Secondly, we aimed to provide an overview of the so far existing results.

In order to make a comparison of the domination and cop number of ER random graphs, we first presented some basic notions on graph theory. We then looked at both the domination number and cop number of graphs and provided some examples. We also mentioned graphs that have $\gamma(G) = c(G) = k$ and provided proofs for $k = 3, 5, 6$ and 7.

Subsequently, we studied the ER random graph and we gathered results from Dreyer [12] and from Glebov, Liebenau and Szabó [21] to get a clear view of what the domination number of an ER random graph looks like, when p is fixed and when p depends on n, respectively.

Afterwards, we studied the cop number of the ER random graph and used results from Łuczak and Prałat [27] to establish what the cop number roughly looks like. From chapters 3 and 4 it becomes clear that the domination number and cop number of the ER random graph are strongly related. To be precise, for fixed p the aforementioned graph parameters are a.a.s. equal.

However, for $p = p_n$ the result is more complicated. For $p_n = n^{\alpha-1}$, with $\alpha \in \left(\frac{1}{2}, 1\right)$, the cop number is asymptotically equal to the domination number. For other values, we use the zigzag function to obtain the cop number.

To be able to provide an overview of the so far existing results, we took the results and delivered proofs to some theorems, see sections 3.1 and 4.2 in particular.

Altogether, we can conclude that, when considering the ER random graph $G(n, p_n)$, the domination number and cop number are strongly related. For fixed p they are even asymptotically equal, which is interesting to see. There still are a lot of open questions and areas of research about this subject, partially because cops and robbers is quite a recent research field. For some open problems, see section 5. An interesting possible following study would be to look into the relation between the cop number and other graph parameters than the domination number. Another possible area of research lies in comparing cop number and other graph parameters for different types of random graphs [19].

A

Graphs with equal domination number and cop number

A.1. Sagemath code for determining graph parameters and plotting graphs

A.1.1. Wong graph

```
from sage.graphs.graph_plot import GraphPlot

###Creating the Wong graph with its adjacency matrix
A_w = [[0,1,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1],\
[1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0],\
[0,1,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0],\
[0,0,1,0,1,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0],\
[1,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0],\
[0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0],\
[0,0,0,0,0,1,0,1,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0],\
[0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0],\
[1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0],\
[0,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0],\
[0,1,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0],\
[0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1],\
[0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,0],\
[0,0,0,0,1,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0],\
[0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0],\
[1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,1,0,0,0,0,0,0],\
[0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,1,0,0,0,0],\
[0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1],\
[0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,1,0,0,0],\
[0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,0,0,1,0],\
[0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0],\
[0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,0,0,0,1],\
[0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0,1,0,0,0,0,0,0],\
[0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0,0,0,0],\
[0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,1,0],\
[0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0,1,0,0,0],\
[0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0,1,0,0],\
[0,1,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0],\
[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,1,0,1],\
[1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,1,0]]
```

```
rows = len(A_w)
columns = rows
W = graphs.EmptyGraph()
for i in range(rows):
    for j in range(columns):
        if A_w[i][j] == 1:
            W.add_edge(i,j)

###Plotting the Wong graph
options = {'layout':'circular','vertex_size': 100, 'vertex_labels': True,
    'vertex_color':'royalblue', 'edge_thickness': 0.5}
GP = GraphPlot(W, options)
GP.show()
```

A.1.2. Hoffman-Singleton graph minus star

```
from sage.graphs.graph_plot import GraphPlot

###Starting off with the Hoffman-Singleton graph
HS = graphs.HoffmanSingletonGraph()

###Taking a vertex from the HS graph and defining its star
s = 5
star_center = HS.vertices()[s]; star_center
star_endpoints = HS.neighbors(s); star_endpoints
star = HS.subgraph([star_center]+star_endpoints)

###Plotting the star
star.show(edge_thickness = 1, vertex_labels = False, vertex_color='red',
    edge_color = 'red', vertex_size= 50)

###Plotting the Hoffman-Singleton graph with a star highlighted in red
HS.show(edge_thickness = 0.7, vertex_labels = False, vertex_color={'royalblue':
    HS.vertices(),'red':star.vertices()}, edge_colors={'red':star.edges()},
    vertex_size= 50)
star.show(edge_thickness = 1, vertex_labels = False, vertex_color='red',
    edge_color='red', vertex_size= 50)

###Removing the star from the Hoffman-Singleton graph
HS_min_star = HS.copy()
HS_min_star.delete_vertex(star_center)
HS_min_star.delete_vertices(star_endpoints)

###Plotting the Hoffman-Singleton graph minus star
options = {'vertex_size': 50, 'vertex_labels': False,
    'vertex_color':'royalblue', 'edge_thickness': 0.7}
GP = GraphPlot(HS_min_star, options)
GP.show()

###Studying graph parameters in order to find the cop number
dom_HS_min_star = HS_min_star.dominating_set(value_only = True); dom_HS_min_star
min_degree = min(HS_min_star.degree_sequence()); min_degree
girth = len(min(HS_min_star.minimum_cycle_basis())); girth
```

A.1.3. Hoffman-Singleton graph

```
from sage.graphs.graph_plot import GraphPlot

###Importing the Hoffman-Singleton graph
HS = graphs.HoffmanSingletonGraph()

###Plotting the Hoffman-Singleton graph
options = {'vertex_size': 50, 'vertex_labels': False, 'vertex_color':'royalblue',
    'edge_thickness': 0.5}
GP = GraphPlot(HS, options)
GP.show()

###Studying graph parameters in order to find the cop number
dom_HS = HS.dominating_set(value_only = True); dom_HS
min_degree = min(HS.degree_sequence()); min_degree
girth = len(min(HS.minimum_cycle_basis())); girth
```

Bibliography

[1] M. Aigner, M. Fromme. (1984). A game of cops and robbers. *Discrete Applied Mathematics, 8*(1), 1–12.

[2] W. Baird, A. Bonato. (2012). Meyniel's conjecture on the cop number: A survey. *Journal of Combinatorics, 3*(2), 225–238.

[3] A. T. Balaban. (1985). Applications of graph theory in chemistry. *Journal of Chemical Information and Computer Sciences, 25*(3), 334–343.

[4] N. Biggs, E. K. Lloyd, R. J. Wilson. (1986). *Graph Theory, 1736–1936*. Amsterdam University Press.

[5] B. Bollobás,G. Kun, I. Leader. (2013). Cops and robbers in a random graph. *Journal of Combinatorial Theory, Series B, 103*(2), 226–236.

[6] A. Bonato, G. Hahn, C. Wang. (2007). The cop density of a graph. *Contributions to Discrete Mathematics, 2*(2).

[7] A. Bonato, R. J. Nowakowski. (2011). *The Game of Cops and Robbers on Graphs*. American Mathematical Society.

[8] N. Clarke, D. Cox, C. Duffy, D. Dyer, S. Fitzpatrick, M. Messinger. (2020). Limited visibility Cops and Robber. *Discrete Applied Mathematics, 282*, 53–64.

[9] D. Clemens. (2014). *Two-player games on graphs*.

[10] W. J. Desormeaux, T. W. Haynes, M. A. Henning. (2014). Improved bounds on the domination number of a tree. *Discrete Applied Mathematics, 177*, 88–94.

[11] R. Diestel. (2017). *Graph Theory* (5th ed. 2017 ed.). Springer Medizin Verlag.

[12] P. A. Dreyer, Applications and variations of domination in graphs, Ph.D.thesis, *Department of Mathematics, Rutgers University*, 2000.

[13] P. Erdős, A. Rényi (1959). On random graphs I. *Publ. Math. Debrecen 6*, 290–297.

[14] P. Erdős, A. Rényi. (1960). On the evolution of random graphs. *Publ. Math. Inst. Hungar. Acad. Sci. 5*, 17–61.

[15] P. Erdős, A. Rényi. (1964). On the strength of connectedness of a random graph. *Acta Mathematica Academiae Scientiarum Hungaricae, 12*(1–2), 261–267.

[16] L. Euler. (1736). Solutio problematis ad geometriam situs pertinentis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 128–140.

[17] M. Feldman, O. Svensson, R. Zenklusen. (2018). A Simple O(log log(rank))-Competitive Algorithm for the Matroid Secretary Problem. *Mathematics of Operations Research, 43*(2), 638–650.

[18] P. Frankl. (1987). Cops and robbers in graphs with large girth and Cayley graphs. *Discrete Applied Mathematics, 17*(3), 301–305.

[19] A. Frieze, M. Karoński. (2016). *Introduction to Random Graphs*. Cambridge University Press.

[20] E. N. Gilbert. (1959). Random Graphs. *The Annals of Mathematical Statistics, 30*(4), 1141–1144.

[21] R. Glebov, A. Liebenau, T. Szabó. (2015). On the Concentration of the Domination Number of the Random Graph. *SIAM Journal on Discrete Mathematics, 29*(3), 1186–1206.

[22] F. Havet. (2012). *Introduction to the probabilistic method*.

[23] M. Kang, Z. Petrášek. (2014). Random Graphs: Theory and Applications from Nature to Society to the Brain. *Internat. Math. Nachrichten, 227*, 1–24.

[24] A. Kehagias, D. Mitsche, P. Prałat. (2013). Cops and invisible robbers: The cost of drunkenness. *Theoretical Computer Science, 481*, 100–120.

[25] V. P. Kozyrev. (1974). Graph theory. *Journal of Soviet Mathematics, 2*(5), 489–519.

[26] L. Lu, X. Peng, (2011). On Meyniel's conjecture of the cop number. *Journal of Graph Theory, 71*(2), 192–205.

[27] T. Łuczak, P. Prałat. (2010). Chasing robbers on random graphs: Zigzag theorem. *Random Structures & Algorithms, 37*(4)

[28] A. Majeed, I. Rauf. (2020). Graph Theory: A Comprehensive Survey about Graph Theory Applications in Computer Science and Social Networks. *Inventions, 5*(1), 10.

[29] S. Neufeld, R. Nowakowski. (1998). A game of cops and robbers played on products of graphs. *Discrete Mathematics, 186*(1–3), 253–268.

[30] R. Nowakowski, P. Winkler. (1983). Vertex-to-vertex pursuit in a graph. *Discrete Mathematics, 43*(2–3), 235–239.

[31] J. Petr, J. Portier, L. Versteegen. (2022, February 25). *A note on Cops and Robbers, independence number, domination number*. arXiv.Org. Retrieved 1 June 2022, from https://arxiv.org/abs/2202.12740

[32] P. Prałat. (2010). When does a random graph have constant cop number? *Australasian Journal of Combinatorics, 46*, 285–296.

[33] M. Sorge. (2013). *The graph parameter hierarchy*.

[34] B. W. Sullivan, N. Townsend, M. L. Werzanski. (2017). An Introduction to Lazy Cops and Robbers on Graphs. *The College Mathematics Journal, 48*(5), 322–333.

[35] Turcotte, J. (2019). *Jérémie Turcotte*. JÉRÉMIE TURCOTTE. Retrieved 2 June 2022, from https://www.jeremieturcotte.com/index-en.html

[36] A. Quilliot. (1978). *Jeux et pointes fixes sur les graphes* (Doctoral dissertation, Ph. D. Dissertation, Université de Paris VI).

[37] B. Wieland, A. P. Godbole. (2001). On the Domination Number of a Random Graph. *The Electronic Journal of Combinatorics, 8*(1).

[38] Z. Zahedi, H. Mawengkang, M. Masri, H. Ramon, Y. M. (2019). Mathematical Fallacies and Applications of Graph Theory in Electrical Engineering. *Journal of Physics: Conference Series, 1255*(1), 012085.