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Abstract

Discrete networks is a kind of form-active structural system which actively change its shape under varying load conditions. And for this kind of structural system, form-finding is the initial and essential part in the their design process. Before the computer age, people complete the form-finding process using physical models, while with the advances in computational techniques, the research has focused on the numerical form-finding methods since the 1960s. A brief discussion on several numerical form-finding methods is presented in this paper. Firstly, two relatively mature numerical method, Dynamic Relaxation method and Force Density method, are introduced conceptually. And then, a newly developed numerical method, the Vector Form Intrinsic Finite Element method, is presented in more detail. At last, with a replacement of the calculation of the internal force of the element which obeys the Hooke's Law by the product of the force density and the length of the element, two derived methods based on the above three methods are proposed in this paper. Moreover, several numerical examples of hanging networks are shown to illustrate the validity and characteristic of the VFIFE method and the two newly proposed derived methods.

Keywords: form-finding; general networks; Dynamic Relaxation method; Force Density method; Vector Form Intrinsic Finite Element method; derived methods

1. Introduction

Discrete networks is a kind of form-active structural system which actively change its shape under varying load conditions.

From a view of different equilibrium states, discrete networks can be divided into two groups, one is the 'funicular' system in which form depends on the external system of actions applied (ex. hanging models under their self-weights), and the other one is the 'self-stressing' system whose stiffness results from a system of internal stresses in static equilibrium (ex. tensile structures).

From a view of different stress states of the components, discrete networks can be divided into three groups, one is the system which consists of components in compression (ex. unstrained gridshells), one is the system which consists of components in tension (ex. cable-nets), and the other one is the system which consists of components both in compression and tension (ex. tensegrity).

For this kind of form-active structural system, form-finding is the initial and essential part in the their design process. And adapted from Lewis W.J.[1], form-finding means that finding an optimal shape of a form-active structure that is in or approximates a static equilibrium.

Before the computer age, people complete the form-finding process using physical models, which include the hanging chain models of Antoni Gaudi, hanging membranes of Heinz Isler, and hanging, tensile or soap film models of Frei Otto, etc. Since the 1960s, with the advances in computational techniques, many numerical methods have been developed and are developing nowadays, which initially applied to design of cable-net structures. Adapted from Veenendaal D. and Block P.[2], the numerical methods developed in the last five decades can be categorized in three main families, stiffness matrix methods (ex. Finite Element method), geometric stiffness methods (ex. Force Density method) and dynamic equilibrium methods (Dynamic Relaxation method).

This paper presents a brief discussion on several form-finding methods for general networks from a conceptual standpoint. Firstly, two relatively mature numerical method, Dynamic Relaxation method and Force Density method, are introduced briefly. And then, a newly developed numerical method, the Vector Form Intrinsic Finite Element method, is presented in detail. At last, with a replacement of the calculation of the internal force of the element which obeys the Hooke's Law by the product of the force density and the length of the element, two derived methods based on the above three methods are proposed here. For this paper, all the methods are applied to hanging networks as numerical examples.

2. Dynamic Relaxation method (DR) and Force Density method (FD)

In this section, two relatively mature form-finding methods, the Dynamic Relaxation method and the Force Density method, are introduced briefly and conceptually. It introduces the basic principle and the basic procedure each form-finding method briefly, and a conceptual analysis of the result of each method is presented after that.

2.1. Dynamic Relaxation method

The Dynamic Relaxation method (DR, for short) was first proposed by British engineer A.S.Day [3], after the development by British scholar M.R.Barnes [6] and other scholars, it was widely used in the form-finding of tension structures.

The basis of the method is to trace step-by-step for small time increments, *dt*, the motion of each node of a structure until, due to artificial damping (kinetic damping is widely used nowadays), the structure comes to rest in static equilibrium.

Figure 1 shows the flowchart of the DR method, in which the undamped motion of the structure is traced and when a local peak in the total kinetic energy of the system is detected, all velocity components are set to zero. The process is then restarted from the current geometry and repeated through further (generally decreasing) peaks until the energy of all modes of vibration has been dissipated and static equilibrium is achieved.
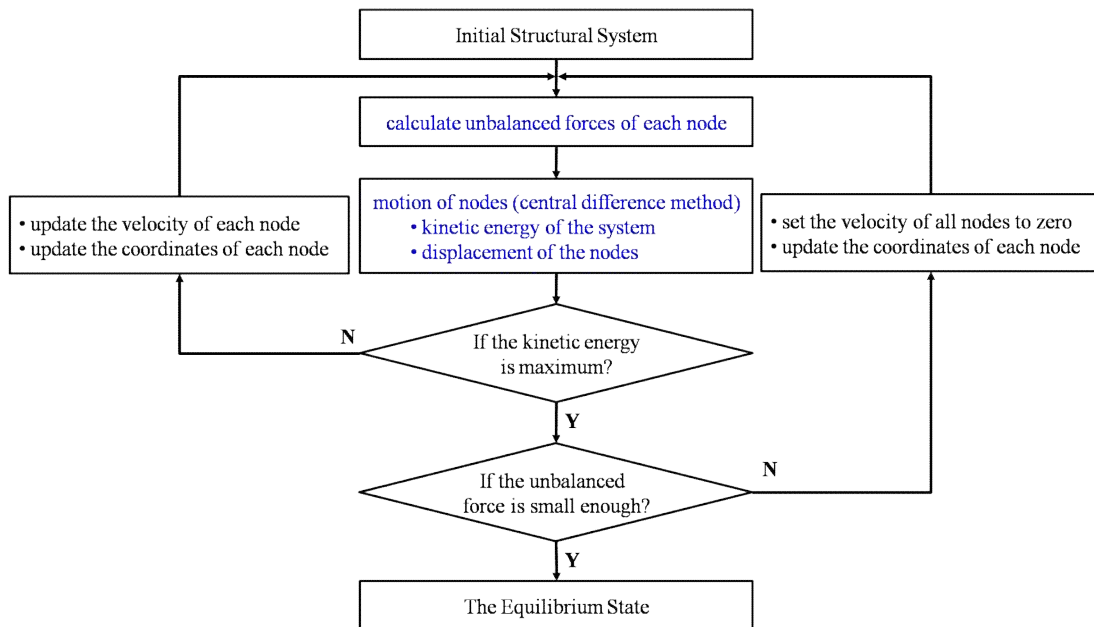

Figure 1: Flowchart of the DR method

And here, two key steps are pointed in more detail to illustrate the characteristics of the DR method.

1) One is the calculation of unbalanced force of each node, which is the residual force at each node. The residual force is the sum of all the forces acting on a node from the members connected to it and the applied loading. And it should be noticed that the internal force of the element obeys the Hooke's Law in the iteration process.

For instance, the internal force of one element F_n in the step *n* can be calculated by the formula (1), which obeys the Hooke's Law,

$$F_n = \frac{EA(l_n - l_{n-1})}{l_{n-1}} \tag{1}$$

where *EA* means the tensile stiffness of the element, l_n means the length of the element in step n, and l_{n-1} means the length of the element in step *n-1*.

2) The other one is the governing equation of the motion of nodes, which based on a central difference expression of the Newton's Second Law. For instance, in order to describe the motion of one node in one direction (the *x* direction here), the iteration formulas of the velocity and displacement of the node can be deduced by the following steps.

$$F_n = ma_n \tag{2}$$

$$a_n = \frac{v_{n+1/2} - v_{n-1/2}}{h} \tag{3}$$

$$v_{n+1/2} = v_{n-1/2} + a_n \cdot h = v_{n-1/2} + \frac{F_n \cdot h}{m} \tag{4}$$

$$x_{n+1} = x_n + v_{n+1/2} \cdot h \tag{5}$$

where *m* means the quality value of the node, *F* means the residual force at the node, *a* means the acceleration of the node, *v* means the velocity of the node, *x* means the displacement of the node, *n* means the step, and *h* means the step length.

2.2. Force Density method

The Force Density method (FD, for short), also known as the '(Stuttgart) direct approach', was first proposed by Schek H.J. and Linkwitz K. [5][6][7], it is commonly used in engineering to find the equilibrium shape of a structure consisting of a network of cables with different elasticity properties when stress is applied. The method assumes a straight cable and hinge between the cables, and relies on the assumption that the ratio of tension force to length of each cable can be constant, transforming a system of non-linear equations to a set of linear equations which can be solved directly.

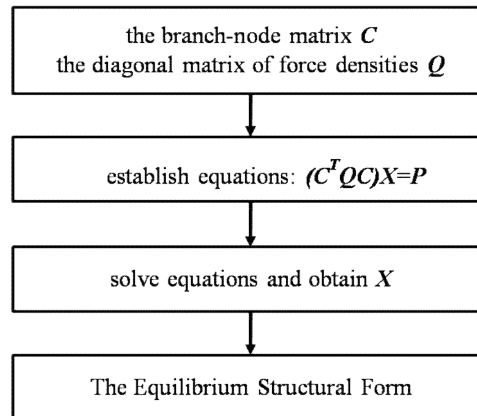Figure 2: Flowchart of the FD method

Figure 2 shows the flowchart of the FD method. To start the form-finding process, given the positions of nodes which connect the cables of the network, the topology of it is encoded in the branch-node matrix *C*. Given a load vector *P* and the diagonal matrix of force densities *Q*, the equilibrium location can be deduced by solving for *X* in formula (6),

$$(C^TQC)X=P \qquad\qquad (6)$$

Compared with the DR method, the FD method does not need an initial structural geometry, and the finial equilibrium structural form obtained by it just depends on the topology of the network and the given force densities.

3. Vector Form Intrinsic Finite Element method (VFIFE)

The Vector Form Intrinsic Finite Element method (VFIFE, for short) is a relatively new numerical method proposed by Ting E. C.[8][9][10]. Nowadays, many scholars are doing research by using this method on the field of complicated behavior analysis of structures, including dynamic responses, geometric nonlinearity, material nonlinearity, buckling or wrinkles failure, mechanism motion, contact and collision, etc. And base on this method, Luo etc. [11] proposed a method called Finite Particle Method (FPM) to do research on the behavior analysis spatial structures.

Different from the traditional methods generated from continuum mechanics and variational principles, the VFIFE is based on point value description and the vector mechanics theory. It uses particles to describe structure characteristic, and describes the deformation of the structural system by observing the motion of the particles, whose calculation procedure is a circular computation step-by-step and point-by-point.

The VFIFE method separates the pure deformation from the rigid body motions by introducing the concept of 'reverse rigid body motion', and then solves the internal forces of the element based on the Hooke's Law. After obtaining the unbalanced force of the particle, which is the sum of all the forces acting on the particle from the members connected to it and the applied loading. In this section, for the form-finding problems of general networks, basic concepts and basic procedure of the VFIFE method are introduced in brief which takes the cable/link element as an example.

3.1. Basic Concepts

In this part, three basic concepts are introduced here, including the point value description, the path element and pure deformation of the element.

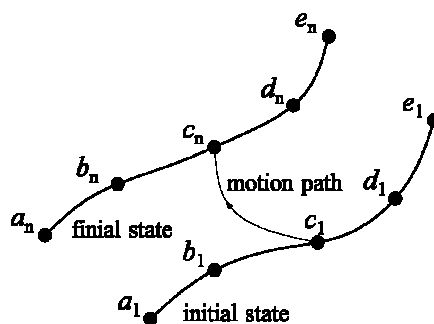1) Point value description

Figure 3: Discretion of the structural system

The point value description can be regarded as a body composed of spatial particles linked by a set of elements. The motion and configuration of a body are determined by the particles. The elements are

deformed following the moving particle, thus where internal forces arise from, to make the body remain a body. The body configuration depends on the choice of interpolation functions. Therefore, the point value description is an approximation of real structure. Shown in Figure 3, the motion and configuration (including the geometry and the spatial position) of one structural system can be described by discrete particles (a, b, c, d, e) and elements between the conected two particles.

2) Path element

To simplify the motion process of the particles, the VFIFE method disperses the continuous and complicated motion trajectory of particles into several simple motion processes, which is called path element. Shown in Figure 4, take particle c in Figure 3 as an example, it has a motion path moving from the initial state c_1 to the finial state c_2, disperse the whole time into finite time instants t_1、 $t_2 \cdots t_n$, and the structural behavior between two successive time instants is called a path element. It should be noted that the motion of the particle is continuous in one path element and obeys the govening equation, while the motion of the particle can be discontinuous between the two successive path elements in which the element type can even be changed.
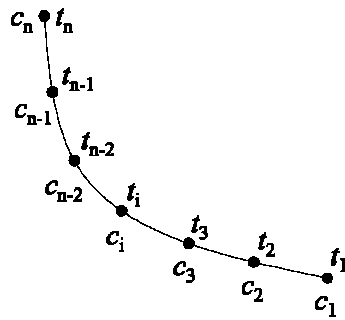

Figure 4: Schematic diagram of the path element

3) Pure deformation of the element

There is such a complicatedly coupling relationship between the rigid body motion and the pure deformation of the element, and how to get the pure deformation is always the core of the calculation of internal forces. The VFIFE method estimates the rigid body motion which satisfy the required precision, and throuh using the concept of reverse rigid body motion it gets the pure deformation of the element by deducting the rigid body motion from the whole displacement. Shown in Figure 5, take element ab in Figure 3 as an example, the pure deformation of it in time t_i can be obtained in the following steps: translate and rotate the element a_ib_i reversely to $a_{i-1}b_{i2}$ in time t_{i-1} firstly, and then get the value of the pure deformation of the element easily.
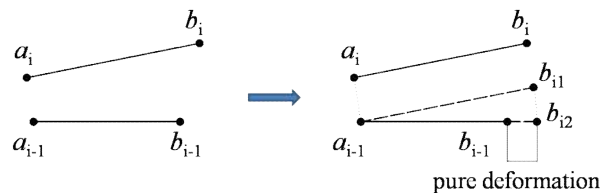

Figure 5: Schematic diagram of the path element

3.2. Basic Procedure

Shown in Figure 6, it is the flowchart of the VFIFE method, similarly to Section 2.1, two key steps are pointed here, including the calculation of unbalanced force of each node and the governing equation of the motion of nodes.
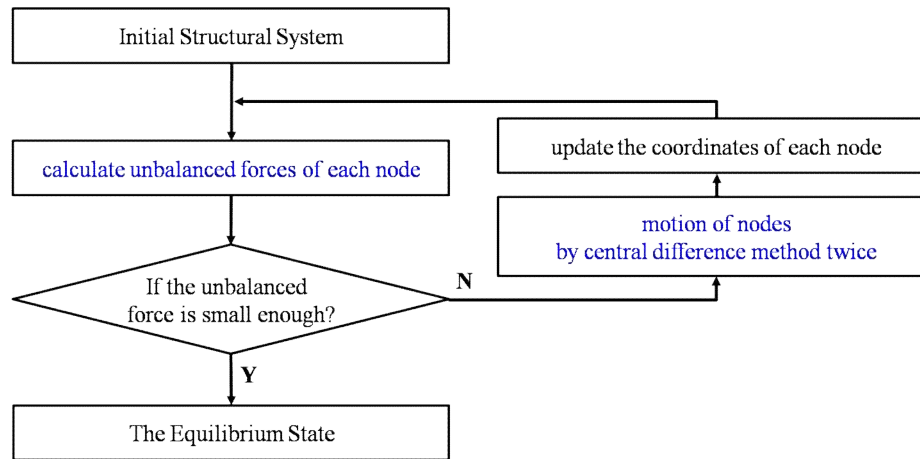

Figure 6: Flowchart of VFIFE method

1) The calculation of unbalanced force of each node

The residual force has the same meaning with the DR method, and the internal force of the element obeys the Hooke's Law in the iteration process but using the concept of pure deformation. For instance, the internal force of one element F_n in the step n can be calculated by formula (7),

$$F_n = \frac{EAl_{pure}}{l_{n-1}} \tag{7}$$

where EA means the tensile stiffness of the element, l_{pure} means the length of the element in step n, and l_{n-1} means the length of the element in step n-1.

2) The governing equation of the motion of nodes

The governing equation of the VFIFE method is also based on a central difference expression of the Newton's Second Law. For instance, in order to describe the motion of one node in one direction, the iteration formulas of the displacement of the node can be deduced by the following steps.

The Newton's Second Law,

$$F_n = ma_n \tag{8}$$

The acceleration with a central difference expression,

$$a_n = \frac{v_{n+1/2} - v_{n-1/2}}{h} \tag{9}$$

The velocities with a central difference expression,

$$v_{n+1/2} = \frac{x_{n+1} - x_n}{h} \qquad (10)$$

$$v_{n-1/2} = \frac{x_n - x_{n-1}}{h} \qquad (11)$$

Plug (10) and (11) into (9), and then into (8), get the iteration equation of the displacement of the node,

$$x_{n+1} = \frac{h^2}{m} F_n + 2x_n - x_{n-1} \qquad (12)$$

While $n=1$, x_0 does not exist, however,

$$v_1 = \frac{1}{2h}(x_2 - x_0) \qquad (13)$$

And then it gets the iteration equations of the VFIFE method,

$$\begin{cases} x_2 = \dfrac{h^2}{m} F_1 + x_1 + h v_1 & n = 1 \\[2mm] x_{n+1} = \dfrac{h^2}{m} F_n + 2x_n - x_{n-1} & n \geq 2 \end{cases} \qquad (14)$$

When there exists the damping, and the damping factor is ξ, and then,

$$F_n - \xi v_n = m a_n \qquad (15)$$

With some deduction, finally, it gets the iteration equations of the VFIFE method with damping,

$$\begin{cases} x_2 = \dfrac{h^2}{m} F_1 + x_1 + h v_1 & n = 1 \\[2mm] x_{n+1} = \dfrac{c_1 h^2}{m} F_n + 2c_1 x_n - c_2 x_{n-1} & n \geq 2 \end{cases} \qquad (16)$$

$$C_1 = \frac{1}{1 + 0.5\xi h}, \quad C_2 = \frac{C_1}{1 - 0.5\xi h} \qquad (17)$$

where m means the quality value of the node, F means the residual force at each node, a means the acceleration of the node, v means the velocity of the node, x means the displacement of the node, n means the step, and h means the step length.

The formula (16) is an explicit equation, which can get the unknowns from the known quantities, and in the VFIFE method, if the initial coordinates and the initial velocities of the particles are known, it can describe the motion path of the particles by stepwise derivation using formula (16).

Moreover, it should be noted that when considering the structural behavior under the dead load, the form-finding problems for instance, two strategies could be adapted, one is taking the dead load as a very slowly increased living load (ex. using a step-loading method), and the other one is adding virtual damping into the equations which aims to eliminate the dynamic effect.

3.2. Numerical example

In this section, one numerical example of a hanging network is shown here to illustrate the validity and advantages of the VFIFE method. A rectangular grid mesh with 30 squares in each direction is restrained at four nodes near each corner. All the nodes are in one plane and applied with a vertical force of 1N, and the distance of two adjacent points is 1.00m. All the cable elements have a same cross-sectional area of 1.0E-04m^2, a same elastic modulus of 1.0E10N/m^2, and no prestress. Each support is located five squares from the nearest edge.
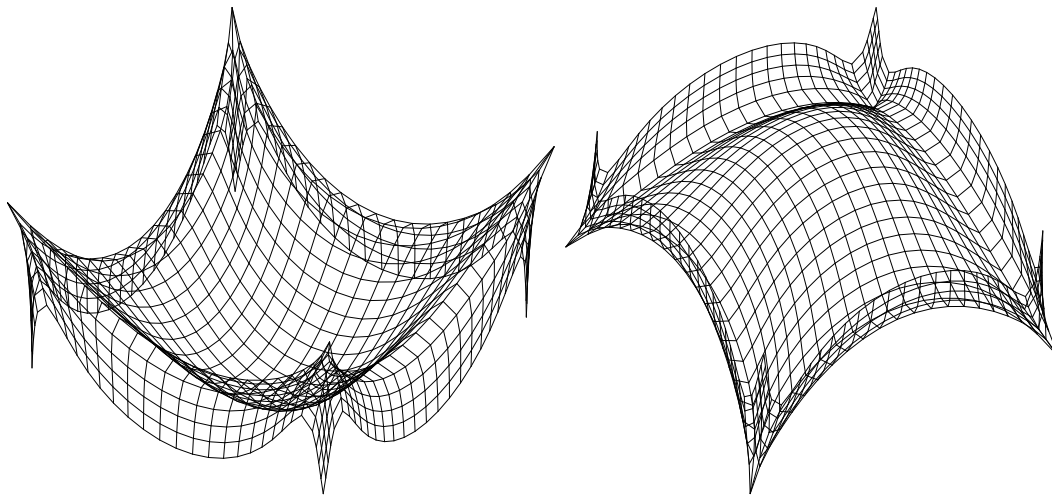

Figure 7: Hanging network supported near the corners

After form-finding by using the VFIFE method, shown in Figure 7, it illustrates the final equilibrium state of the hanging network and the inverted structural form which is in pure compression. From this numerical example, it can be seen that the VFIFE method is valid in form-finding problem of the hanging network even the initial structural system is a geometrical unstable system. Moreover, it also compares the equilibrium form in Figure 7 with the form-finding result of the DR method, and the two equilibrium forms have very little error with each other.

In conclusion, the VFIFE method can also generate the equilibrium structural form from any unbalanced state with arbitrary and inaccurate specification of geometry as the DR method. And it can also be seen clearly that in the case of form-finding of general networks, there seems little difference between the two methods. However, the VFIFE method still has some advantages than the DR method in two following aspects:

1) The governing equation of the VFIFE method uses the central difference method twice to describe the Newton's Second Law, which can simulate a real physical process if all the properties (ex. the elastic modulus, the damping factor , etc.) are given real values.

2) With a new concept of path element, in which the motion of the particle can be continuous in one path and while can be discontinuous between the two successive path elements, the VFIFE method can do more complicated behavior analysis, the elastic-plastic analysis for instance, which the elastic modulus can be changed in the iteration process when start a new path element.

4. Two derived form-finding methods (FD-DR, FD-VFIFE)

From the above two sections, three form-finding methods are introduced conceptually, and it can be seen clearly that the form-finding results of the DR method and the VFIFE method are almost the same because they calculate the internal force of the element obeying the Hooke's Law, while that of the FD method is different from the other two methods because it depends on the topology of the network and the given force densities.

In the iteration process of the DR method or the VFIFE method, it realises that there is a different way to calculate the internal force of the element when considering the concept of force density.For instance, the internal force of one element F_n in the step n can be calculated by the formula (18),

$$F_n = ql_n \qquad\qquad (18)$$

where q means the force density of the element which is given as a constant, and l_n means the length of the element in step n.

By using the formula (18) to replace the formula (1) of the DR method and formula (7) of the VFIFE method but other steps remain unchanged, two derived form-finding methods, the Force Density-Dynamic Relaxation method (FD-DR method, for short) and the Force Density-Vector Form Intrinsic Finite Element method (FD-VFIFE method, for short), are proposed here.

Taking the FD-DR method as an example, it illustrates the characteristics of the two derived methods in the following four aspects:

1) The form-finding process of the derived method needs an initial stuctural geometry, loads condition and restrained points just as the DR method, and it also needs the force densities of the elements.

2) There is no need to calculate the pure deformation of the element in the process, because it just need the length of the element in each step which is much easier to calculate.

3) The process of the form-finding of the derived method is solving the explicit equation by iteration, it does not need to integrate the matrixes needed in the FD method.

4) The result of the form-finding of the derived methods is same with the FD method, and the equilibrium structural form just depends on the force densities given in the initial structural system.

In order to show the validity of the derived method, three hanging networks form-found by the FD-DR method are shown here.
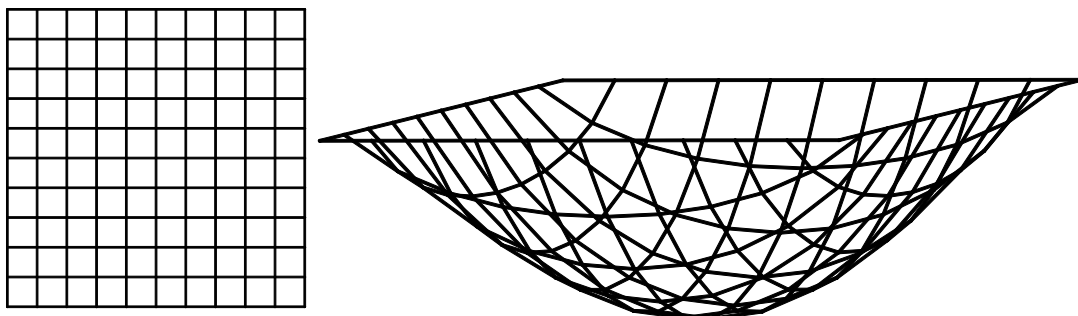

Figure 8: Hanging network with a force density of 2 of all elements supported in the boundary lines
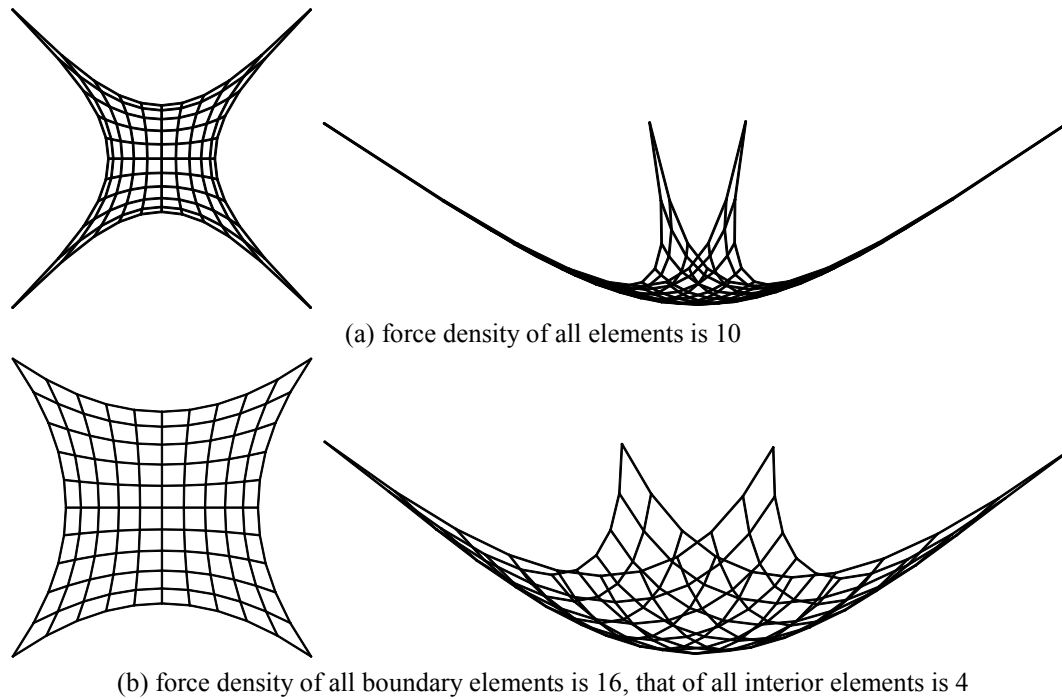
(a) force density of all elements is 10

(b) force density of all boundary elements is 16, that of all interior elements is 4

Figure 9: Hanging network supported in the corners

Shown in Figure 8 and 9, all the hanging networks have the same ininital geomertry with a 10×10 grids, all the nodes are applied with a vertical force of 1N, and the distance of two adjacent points is 1.00m. It can be seen clearly that the equilibrium structural forms are same with the form-found results by FD method, and they depend on the force densities given in the initial structural system and the boudaries conditions.

5. Conclusions

This paper presents a brief discussion on several numerical form-finding methods.

It introduces the DR method and the FD method which are relatively mature conceptually, and then the VFIFE method in more detail. After replacing the calculation method of the internal force of the element which obeys the Hooke's Law by the product of the force density and the length of the element, two derived methods, the FD-DR method and the FD-VFIFE method, are proposed.

Among the five above methods, the DR method and the VFIFE method can generate almost the same form-finding result, while the FD method, the FD-DR method and the FD-VFIFE method can obtain almost the same result.

Moreover, the VFIFE method, which is a relatively new and promising method, has many advantages than traditional methods in the complicated behavior analysis of structures, and there is significant future work to be done for this method.

While for the FD-DR method and FD-VFIFE method, apart from the characteristic introduced in this paper, the advantages of them are still being found in the research.

Acknowledgements

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