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DOI
10.1109/TAC.2020.3003647

Publication date
2021

Document Version
Accepted author manuscript

Published in
IEEE Transactions on Automatic Control

Citation (APA)
Yuan, S., Lv, M., Baldi, S., & Zhang, L. (2021). Lyapunov-Equation-Based Stability Analysis for Switched Linear Systems and Its Application to Switched Adaptive Control. *IEEE Transactions on Automatic Control*, *66*(5), 2250-2256. https://doi.org/10.1109/TAC.2020.3003647

Lyapunov-equation-based stability analysis for switched linear systems and its application to switched adaptive control

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***Abstract*—This paper investigates the stability of continuous-time switched linear systems with dwell time constraints. A fresh insight into this established problem is provided via novel stability conditions that require the solution to a family of differential Lyapunov equations and algebraic Lyapunov equations. The proposed analysis, which leads to a peculiar Lyapunov function that is decreasing in between and at switching instants, enjoys the following properties: it achieves the same dwell time as the well-known result in [1]; it removes the increasing computational complexity of the linear interpolation method of [2]; it leads to a straightforward counterpart for discrete-time switched linear systems. We show the application of this methodology to the problem of adaptive control of switched linear systems with parametric uncertainties.**

***Index Terms*—Stability analysis; dwell time switching; switched linear systems; switched adaptive control.**

I. Introduction

Thanks to many potential applications, e.g., electrohydraulic systems [3] and robotics [4], systems switching among a family of dynamics, a.k.a. *switched systems*, have been attracting a significant amount of research in the last decades [5]. This note focuses on "time-driven" switched systems, which involve the design of switching signals by specifying the length of the time intervals between two consecutive switching instants.

Stability and stabilization of linear time-driven switched systems have been intensively studied [5], [6], [7]. Most stability results stem from two frameworks which, with some abuse of terminology, might be referred to as the slow switching framework [8], [9] and the multiple Lyapuonv functions framework [10]. For achieving stability under slow switching, two important properties of the Lyapunov function, the rate of exponential decrease in between switching instants and the possible increase at switching instants, have been exploited to derive dwell-time or average dwell-time switching signals. On the other hand, the framework of multiple Lyapunov functions may not explicitly exploit such properties. Typically, the following stability criterion is exploited: when switching to a certain stable subsystem, the value of its corresponding Lyapunov function at one exiting/entering instant of this subsystem is smaller than the value of the Lyapunov function at the previous exiting/entering instant of the same subsystem. This stability condition can be checked numerically only among two consecutive switching instants, resulting in the well-known dwell-time based condition in [1], which is, to date, the least conservative[1] stability result for time-driven switched linear systems via quadratic Lyapunov functions [1]. Later, the authors of [2] have proposed a stability condition based on a quadratic Lyapunov function with time-varying positive definite matrices in place of constant ones. These time-varying matrices are obtained by linear interpolation of constant positive define matrices defined over many subintervals. The resulting Lyapunov function (let us call it interpolated Lyapunov function) is endowed with a crucial property, i.e. it is *non-increasing at the switching instants*. Provided that a large enough number of subintervals over which to interpolate is selected [11], the stability conditions in [1] and [2] have been shown to be dwell time equivalent, i.e., in terms of minimum achievable dwell time. Despite the dwell time equivalence, the non-increasing behavior of the interpolated Lyapunov function at switching instants has been shown to be crucial in establishing a large number of new results. For example, a less conservative estimate for the output reachable set of switched systems was obtained in [12]; adaptive asymptotic tracking for uncertain switched linear systems was solved in [13]; a non-weighted $\mathscr{L}_2$ gain for switched linear systems was achieved in [14]. In other words, the approaches in [1] and [2] are dwell-time equivalent, but not equivalent in terms of achievable results. However, the dwell time equivalence of [1] and [2] can be achieved only after solving a sufficiently large number of matrix inequalities proportional to the numbers of interpolation subintervals, which dramatically increases the computational burden [11]. A similar comment applies also to the sum-of-squares approach of [7] which, in order to approximate [1] arbitrarily close (i.e. to be dwell-time equivalent), requires a sufficiently large degree of the polynomials. Therefore, in the following, allow us to take [1] and [2] as exemplifying frameworks from which our research stems, and allow us to introduce remarks involving [7] whenever appropriate. In view of the theoretical advantages and computational drawbacks of the approach in

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This work was partially supported by the "Fundamental Research Funds for The Central Universities" under Grant 4007019109 and the Special Guiding Funds for "Double First-Class" under Grant 4007019201.

[1]In this note, the term conservativeness is adopted to indicate the minimum achievable dwell time for which stability can be guaranteed.

[2], a question comes to light: is it possible to derive a new Lyapunov function based on time-varying positive definite matrices that, while guaranteeing the equivalent dwell time of [1], is non-increasing at switching instants and removes the computational drackbacks of the interpolation approach in [2]?

In this note, we give a positive answer to this question. The contribution of this work is twofold: *a*) a novel stability condition is formulated that, while being dwell-time equivalent with the result in [1], leads to a Lyapunov function with time-varying positive definite matrices which is *non-increasing at the switching instants*. Specifically, the time-varying positive definite matrices are derived from a family of differential Lyapunov equations and algebraic Lyapunov equations, without requiring subintervals and thus a large number of matrix inequalities as in [2]; *b*) while the approach in [1] cannot be adopted in adaptive closed loops due to their intrinsic nonlinear nature [13], the proposed stability analysis can find application in this setting, and it is exploited to improve the result about switched adaptive control for uncertain switched linear systems in [13]. In particular, a novel adaptive law is introduced, which reduces the computational burden of interpolation procedure exploited in [13] while guaranteeing asymptotic adaptive tracking error.

The rest of the paper is organized as follows: some preliminaries are introduced in Section II, followed by the stability results for continuous-time and discrete-time switched linear systems in Section III. In Section IV, the proposed stability analysis is used to study switched adaptive control. Numerical examples are used in Section V to illustrate the theoretical findings.

Notations: The notations used in this paper are standard: $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}^+$ represent the set of real numbers, positive real numbers, and positive natural numbers, respectively. For symmetric matrices, $P = P^T > 0$ indicates that P is positive definite with the superscript T representing the transpose. The operators $\mathrm{sgn}(\cdot)$ gives the sign of a number and $\mathrm{tr}(\cdot)$ returns the trace of a matrix. The function $\mathrm{diag}\{\cdots\}$ represents a block-diagonal matrix. The identity matrix of compatible dimensions is denoted by I.

II. Preliminaries

Continuous-time systems. Consider the following switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $A_\sigma \in \mathbb{R}^{n\times n}$. The switching signal $\sigma(\cdot)$ is defined as $\sigma(t) : \mathbb{R}^+ \to \mathcal{M} := \{1, 2, \dots, M\}$ where M is the number of subsystems. The switching instants for system (1) are denoted with $\mathcal{S}_{\mathrm{C}} := \{t_1, t_2, \dots\}$.

Discrete-time systems. Consider the following switched linear system

$$x(k + 1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \tag{2}$$

where $x \in \mathbb{R}^n$ is the state and $A_\sigma \in \mathbb{R}^{n\times n}$. The switching signal $\sigma(\cdot)$ is defined as $\sigma(k) : \mathbb{N}^+ \to \mathcal{M} := \{1, 2, \dots, M\}$ where M is the number of subsystems. The switching instants for system (2) are denoted with $\mathcal{S}_{\mathrm{D}} := \{k_1, k_2, \dots\}$.

In the following we provide the definition for the class of switching signals considered in this note.

*Definition 2.1 (**Dwell time**):* [9] Switching laws with switching instants $\mathcal{S}_{\mathrm{C}}$ ($\mathcal{S}_{\mathrm{D}}$) are said to belong to the dwell-time admissible set $\mathcal{D}(\tau_{\mathrm{d}})$ if there exists a number $\tau_{\mathrm{d}} > 0$ such that $t_{i+1} - t_i > \tau_{\mathrm{d}}$ ($k_{i+1} - k_i > \tau_{\mathrm{d}}$) holds for all $i \in \mathbb{N}^+$. Any positive number τ_{d}, for which these constraints hold for all $i \in \mathbb{N}^+$, is called *dwell time*.

The asymptotic stability condition introduced in [2] is recalled below for convenience.

Lemma 2.2: [2] Assume that there exist a collection of symmetric positive definite matrices $P_{p,k} \in \mathbb{R}^{n\times n}$, $p \in \mathcal{M}$, $k = 0, \dots, K$, and a sequence $\{\delta_k\}_{k=1}^K > 0$ with $\sum_{k=1}^K \delta_k = \tau_{\mathrm{d}}$ such that the following hold:

$$(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k}A_p + A_p^T P_{p,k} < 0 \tag{3a}$$

$$(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k+1}A_p + A_p^T P_{p,k+1} < 0 \tag{3b}$$

$$k = 0, \dots, K - 1$$

$$P_{p,K}A_p + A_p^T P_{p,K} < 0 \tag{3c}$$

$$P_{p,K} - P_{q,0} \geq 0 \tag{3d}$$

for any $q \neq p \in \mathcal{M}$. Then, the switched linear system (1) is globally asymptotically stable for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$.

The role of K is to partition the switching interval $[t_i,t_i + \tau_{\mathrm{d}})$ into a collection of subintervals. Then, a time-varying symmetric positive definite matrix $P_p(\cdot)$ can constructed by linear interpolation of the matrices $P_{p,k}$ over the switching interval $[t_i,t_{i+1})$. The resulting Lyapunov function $V(t) = x^T(t)P_{\sigma(t)}(t)x(t)$ can be used to prove stability of (1).

It has been shown in [11] that as K approaches a sufficiently large number K^*, the condition (3) is equivalent to the condition in [1, Theorem 1], i.e., the minimum achievable dwell time tends to the same value. However, as K increases, solving (3) requires additional computation burden. Therefore, K is an integer that needs to be chosen according to the allowed computational complexity. For a clear numerical comparison between [1, Theorem 1] and Lemma 2.2, the reader is referred to the example provided in [11].

III. Stability analysis

A novel stability analysis for system (1) is introduced hereafter.

Theorem 3.1: Assume that for some positive τ_{d}, there exists a family of symmetric positive definite matrices Ξ_p, $Q_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{M}$ such that

$$\Xi_p - P_p > 0 \tag{4a}$$

$$A_p^T \Xi_p + \Xi_p A_p < 0 \tag{4b}$$

$$e^{A_p^T \tau_{\mathrm{d}}}\Xi_p e^{A_p \tau_{\mathrm{d}}} - \Xi_q + P_p - e^{A_p^T \tau_{\mathrm{d}}}P_p e^{A_p \tau_{\mathrm{d}}} < 0 \tag{4c}$$

for any $q \neq p \in \mathcal{M}$, where P_p is a symmetric positive definite matrix, solution to the continuous-time Lyapuonv equation

$$A_p^T P_p + P_p A_p + Q_p = 0. \tag{5}$$

Then, the switched linear system (1) is globally asymptotically stable for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$.

Proof: Let the index of the subsystem which is active over the time interval $[t_{i-1},t_i)$ be denoted by q, and the index of the subsystem which is active over the time interval $[t_i,t_{i+1})$ be denoted by p, $q \neq p \in \mathcal{M}$. Then, let us define the following differential Lyapunov equation for $t \in [t_i,t_i+\tau_\mathrm{d}]$

$$-\dot{\mathscr{P}}_p(t) = A_p^T \mathscr{P}_p(t) + \mathscr{P}_p(t) A_p + Q_p \tag{6}$$

which is intended to be solved backwards starting from the condition at time $t_i+\tau_\mathrm{d}$. Subtracting (5) from (6) leads to

$$-\dot{Z}_p(t) = A_p^T Z_p(t) + Z_p(t) A_p \tag{7}$$

where $Z_p(\cdot) = \mathscr{P}_p(\cdot) - P_p$. In what follows, we will find a solution to (7) for $t \in [t_i,t_i+\tau_\mathrm{d}]$. Let $\Xi_p - P_p$ be the boundary condition for (7) when $t = t_i+\tau_\mathrm{d}$. Then, using (4a), the analytic solution to (7) can be shown to be positive definite and equal to

$$\begin{aligned}\mathscr{P}_p(t) = {} & e^{A_p^T(t_i+\tau_\mathrm{d}-t)}\Xi_p e^{A_p(t_i+\tau_\mathrm{d}-t)} + P_p \\ & - e^{A_p^T(t_i+\tau_\mathrm{d}-t)} P_p e^{A_p(t_i+\tau_\mathrm{d}-t)} \quad t \in [t_i,t_i+\tau_\mathrm{d}].\end{aligned} \tag{8}$$

We now consider the following Lyapunov function

$$V(t) = x^T(t)\mathcal{P}_{\sigma(t)}(t)x(t) \tag{9}$$

where the time-varying positive definite matrix $\mathscr{P}_p(\cdot)$ is defined as

$$\mathcal{P}_p(t) = \begin{cases} \mathscr{P}_p(t) & t \in [t_i,t_i+\tau_\mathrm{d}) \\ \Xi_p & t \in [t_i+\tau_\mathrm{d},t_{i+1}) \end{cases} \tag{10}$$

which can be shown to be continuous for $t = t_i+\tau_\mathrm{d}$. Since the Lyapunov function (9) is not differentiable when $t = t_i+\tau_\mathrm{d}$, its time derivative along (1) is studied over the two subintervals as (10). For $t \in [t_i,t_i+\tau_\mathrm{d})$, we have

$$\begin{aligned}\dot{V}(t) &= x^T(t)\left(A_p^T\mathscr{P}_p(t) + \mathscr{P}_p(t)A_p + \dot{\mathscr{P}}_p(t)\right)x(t) \\ &= -x^T(t)Q_p x(t) \\ &< 0\end{aligned} \tag{11}$$

for all $|x| \neq 0$, where the second equality is obtained via (6). For $t \in (t_i+\tau_\mathrm{d},t_{i+1})$, making use of (4b), we have

$$\begin{aligned}\dot{V}(t) &= x^T(t)\left(A_p^T\Xi_p + \Xi_p A_p\right)x(t) \\ &< 0\end{aligned} \tag{12}$$

for all $|x| \neq 0$. According to (11) and (12) and continuity of $\mathcal{P}_p(\cdot)$ for $t = t_i+\tau_\mathrm{d}$, (9) is decreasing for $t \in [t_i,t_{i+1})$ for all $|x| \neq 0$. At the switching instant t_i, the solution (8) gives us

$$\mathscr{P}_p(t_i) = e^{A_p^T\tau_\mathrm{d}}\Xi_p e^{A_p\tau_\mathrm{d}} + P_p - e^{A_p^T\tau_\mathrm{d}} P_p e^{A_p\tau_\mathrm{d}}$$

which, in conjunction with (4c), results in

$$\begin{aligned}V(t_i^-) - V(t_i) = {} & x^T(t_i)\Big(\Xi_q - e^{A_p^T\tau_\mathrm{d}}\Xi_p e^{A_p\tau_\mathrm{d}} \\ & -P_p + e^{A_p^T\tau_\mathrm{d}} P_p e^{A_p\tau_\mathrm{d}}\Big)x(t_i) \\ > {} & 0\end{aligned} \tag{13}$$

for all $|x| \neq 0$, where $V(t_i^-)$ represents the left-limit of $V(t)$ at $t = t_i$. In view of (11)–(13), we have that $V(\cdot)$ is decreasing in between the switching instants and decreasing at the switching instants, which implies that $x(t) \to 0$ as $t \to \infty$. This completes the proof. ■

It is worth mentioning that the proposed Lyapunov function (9) enjoys two properties: it is decreasing in between switching instants and also decreasing at switching instants. These two properties have been shown to be effective in obtaining results for switched systems that are consistent with non-switched systems, for example: non-weighted $\mathscr{L}_2$ gain for switched systems [14] and for asynchronously switched systems [15]; non-weighted $\mathscr{L}_1$ gain for switched positive systems [16]. It is important to notice that both the slow switching framework [17], and the multiple Lyapuonv functions framework leading to [1] show that: the crucial difference between a Lyapunov function for non-switched systems and one for switched systems is the Lyapunov function might increase at the switching instants, i.e. the switching can have a destabilizing effect to be compensated. However, the presence of non-increasing jumps at the switching instants departs from these frameworks, because the switching instants are not seen anymore as a destabilizing term. That is, one obtains a Lyapunov function for switched systems that essentially recovers all stability results for non-switched systems. Similarly to [2], the method proposed in Theorem 3.1 also enjoys this crucial property.

In fact, the mechanism of non-increasing jump has been also proven to be crucial in filling various gaps between switched systems and non-switched systems, for which the frameworks in [17] or [1] are unapplicable, for example: stability analysis of switched time-delay systems with no restriction on the bounds of the time delays [18]; adaptive control of switched systems with guaranteeing asymptotic stability [13].

Remark 1: Similar to the stability condition in [1, Theorem 1], also the inequalities (4) depend on both Ξ_p and τ_d. Therefore, the solution to (4) can be found by using the following line-search procedure borrowed from [1]: select $Q_p > 0$ and solve the Lyapunov equation (5) to obtain P_p, $p \in \mathcal{M}$; use a line search for τ_d so that the inequalities (4c) reduce to linear matrix inequalities which can be efficiently solved using convex optimization [19].

Note that, by letting $P_p = 0$ and $Q_p = 0$ for all $p \in \mathcal{M}$ in (5), the conditions in (4) degenerate to the following stability result.

Corollary 3.2: Assume that for some positive τ_d, there exists a family of symmetric positive definite matrices $\Xi_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{M}$ such that

$$A_p^T\Xi_p + \Xi_p A_p < 0 \quad \forall p \in \mathcal{M} \tag{14a}$$

$$e^{A_p^T\tau_\mathrm{d}}\Xi_p e^{A_p\tau_\mathrm{d}} - \Xi_q < 0 \quad \forall p \neq q \in \mathcal{M}. \tag{14b}$$

Then, the switched linear system (1) is globally asymptotically stable for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$.

The stability condition (14), which is the dual of [1, Theorem 1], where the system indices in (14b) are swapped, was shown in [11] to be equivalent to (3). Nevertheless, it is instructive to show its proof, in order to underline the drawback of not employing time-varying positive definite matrices as in (9).

Proof: In place of the Lyapunv function (9) with time-varying positive definite matrices, let us consider the Lyapunov function

$$V(t) = x^T(t)\Xi_{\sigma(t)}x(t). \tag{15}$$

Again, we assume that subsystem p is active over the time interval $[t_i, t_{i+1})$ and subsystem q is active over the time interval $[t_{i-1}, t_i)$. Then, according to (14), we have

$$\begin{aligned}\dot{V}(t) &= x^T(t)\left(A_p^T \Xi_p + \Xi_p A_p\right)x(t) \\ &< 0\end{aligned}$$

for all $|x| \neq 0$, and

$$\begin{aligned}V(t_i + \tau_\mathrm{d}) &= x^T(t_i + \tau_\mathrm{d})\Xi_p x(t_i + \tau_\mathrm{d}) \\ &\leq x^T(t_i) e^{A_p^T \tau_\mathrm{d}} \Xi_p e^{A_p \tau_\mathrm{d}} x(t_i) \\ &\leq x^T(t_i)\Xi_q x(t_i) \\ &= V(t_i^-).\end{aligned}$$

Since $V(\cdot)$ is decreasing over the time interval $[t_i, t_{i+1})$, it immediately follows

$$V(t_{i+1}^-) < V(t_i + \tau_\mathrm{d}) < V(t_i^-)$$

which implies, using the stability criterion in [10], that $x(t) \to 0$ for $t \to \infty$ for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$. This completes the proof. ■

Adopting quadratic Lyapunov functions, it has been shown in [2], [11] that, up to now, Theorem 1 in [1] (or equivalently Corollary 3.2) provides the least conservative stability result for switched systems under dwell-time switching. Hence, it is of interest to compare the stability results of Theorem 3.1 and of Corollary 3.2: despite using different Lyapunov functions, we have that the two results are equivalent in terms of achievable dwell time, as discussed in the following.

Theorem 3.3: [*Equivalence Theorem*] Let us consider the switched system (1) with $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$. The following statements are equivalent:

(a) There exist a family of matrices $\Xi_p > 0$ and a family of matrices $Q_p > 0$, $p \in \mathcal{M}$ such that conditions (4) hold.

(b) There exists a family of matrices $\Xi_p > 0$, $p \in \mathcal{M}$ such that conditions (14) hold.

Proof: (b) $\Rightarrow$ (a). For positive definite matrices Ξ_p satisfying (14a)–(14b), we can find positive numbers μ and ρ guaranteeing

$$\Xi_p - \rho I > 0 \tag{16a}$$

$$A_p^T \Xi_p + \Xi_p A_p < 0 \tag{16b}$$

$$e^{A_p^T \tau_\mathrm{d}} \Xi_p e^{A_p \tau_\mathrm{d}} - \Xi_q + \mu I < 0 \tag{16c}$$

for all $p, q \in \mathcal{M}$ with $p \neq q$. On the other hand, for any selection of $\mu > 0$ and $\rho > 0$, there exist matrices $Q_p > 0$ and $P_p > 0$ satisfying the Lyapunov equation (5) such that

$$P_p - e^{A_p^T \tau_\mathrm{d}} P_p e^{A_p \tau_\mathrm{d}} \leq \mu I \tag{17a}$$

$$P_p - \rho I < 0. \tag{17b}$$

Applying (16a) to (16c) and (17b) to (16a) reproduces the inequality (4c) and (4a), respectively.

(a) $\Rightarrow$ (b). By making use of (5), it follows

$$\begin{aligned}-\int_0^{\tau_\mathrm{d}} e^{A_p^T t} Q_p e^{A_p t} dt &= \int_0^{\tau_\mathrm{d}} e^{A_p^T t}\left(A_p^T P_p + P_p A_p\right) e^{A_p t} dt \\ &= e^{A_p^T \tau_\mathrm{d}} P_p e^{A_p \tau_\mathrm{d}} - P_p \\ &< 0\end{aligned}$$

upon which (4c) reduces to (14b). This completes the proof. ■

Some remarks are necessary to frame the proposed result in the state of the art.

Comparison with [1, Theorem 1]. According to *Theorem 3.3*, the condition (4) can achieve the equivalent minimum dwell time as [1, Theorem 1]. From a computational point of view, only requires to solve one extra Lyapunov equation for each subsystem, which does not increase the computational burden to a considerable extent. The main difference between the two methods is that the Lyapunov function (9), by incorporating time-varying positive definite matrices, enjoys the desirable properties of being decreasing at and between consecutive switching instants. In this regard, the advantage of (4) over the condition in [1, Theorem 1] is that the resulting Lyapunov function can be used to achieve all the results proposed in literature that require non-increasing property of the Lyapunov function at the switching instants.

Comparison with [2, Theorem 1]. For both methods, the resulting Lyapunov function enjoys the non-increasing property at the switching instants. However, for [2, Theorem 1], in order to approach the same minimum dwell time of (4), it is necessary to increase K to very large values, thus resulting in $3K - 1$ extra matrix inequalities for each subsystem. In view of this, the proposed stability condition (4) can serve as a significant improvement over [2, Theorem 1] in sense of computational complexity.

Comparison with [7, Theorem 8]. One merit of the proposed Theorem 3.1 is to provide an *analytic solution* via differential Lyapunov equations without any approximation involved. On the other hand, the LMI-based condition of [7, Theorem 8] lacks of such an analytic solution. Rather, the condition in [7, Theorem 8] is eventually approximated in a numerical way via sum-of-squares optimization, therefore requiring a sufficiently large degree of the polynomials in order to attain the equivalence with [1, Theorem 1]. That is, similarly to [2], equivalence is attained at the price of increasing the computational burden [20], while *Theorem* 3.3 provides an equivalence result obtained without increasing the computational burden.

In what follows, the counterpart of Theorem 3.1 for the discrete-time switched linear system (2) is presented, which leads to new and unexplored results for this class of systems.

Theorem 3.4: Assume that for some positive τ_d, there exists a family of symmetric positive definite matrices Ξ_p, $Q_p \in \mathbb{R}^{n \times n}$, $p \in \mathcal{M}$ such that

$$\Xi_p - P_p > 0 \tag{18a}$$

$$A_p^T \Xi_p A_p - \Xi_p < 0 \tag{18b}$$

$$\left(A_p^{\tau_\mathrm{d}}\right)^T \Xi_p A_p^{\tau_\mathrm{d}} - \Xi_q + P_p - \left(A_p^{\tau_\mathrm{d}}\right)^T P_p A_p^{\tau_\mathrm{d}} < 0 \tag{18c}$$

for all $q \neq p \in \mathcal{M}$, where P_p is a symmetric positive definite matrix, solution to the discrete-time Lyapunov equation

$$A_p^T P_p A_p - P_p + Q_p = 0. \tag{19}$$

Then, the switched linear system (2) is globally asymptotically stable for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$.

Proof: The proof is constructed using similar steps as the ones of Theorem 3.1. Consider the following difference Lyapunov equation over the time interval $[k_i, k_i+\tau_\mathrm{d}]$

$$A_p^T \mathscr{P}_p(k+1)A_p - \mathscr{P}_p(k) + Q_p = 0 \tag{20}$$

which is intended to be solved backwards starting from the condition at time $k_i+\tau_\mathrm{d}$. Subtracting (19) from (20) leads to

$$A_p^T Z_p(k+1)A_p - Z_p(k) = 0 \tag{21}$$

where $Z_p(\cdot) = \mathscr{P}_p(\cdot) - P_p$. In what follows, we will find a solution of (21) over the time interval $[k_i, k_i+\tau_\mathrm{d}]$. Let $\Xi_p - P_p$ be the boundary condition for (7) when $k = k_i+\tau_\mathrm{d}$. Then, using (18a), the analytic solution of (21) can be shown to be positive definite and equal to

$$\begin{aligned}\mathscr{P}_p(k) = &\left(A_p^{(k_i+\tau_\mathrm{d}-k)}\right)^T \Xi_p A_p^{(k_i+\tau_\mathrm{d}-k)} + P_p \\ &- \left(A_p^{(k_i+\tau_\mathrm{d}-k)}\right)^T P_p A_p^{(k_i+\tau_\mathrm{d}-k)} \quad k\in[k_i,k_i+\tau_\mathrm{d}].\end{aligned} \tag{22}$$

We consider the following Lyapunov function

$$V(k) = x^T(k)\mathcal{P}_{\sigma(k)}(k)x(k) \tag{23}$$

where the time-varying positive definite matrix $\mathscr{P}_p(\cdot)$ is defined as

$$\mathcal{P}_p(k) = \begin{cases}\mathscr{P}_p(k) & k\in[k_i,k_i+\tau_\mathrm{d}) \\ \Xi_p & k\in[k_i+\tau_\mathrm{d},k_{i+1}).\end{cases} \tag{24}$$

The two subintervals used in the definition of $\mathscr{P}_p(\cdot)$ in (24) are considered to study the time difference of (23). For $k\in[k_i,k_i+\tau_\mathrm{d})$, making use of (20), we have

$$\begin{aligned}V(k+1)-V(k) &= x^T(k)\left(A_p^T\mathscr{P}_p(k+1)A_p - \mathscr{P}_p(k)\right)x(k) \\ &= -x^T(k)Q_p x(k) \\ &< 0\end{aligned} \tag{25}$$

for all $|x|\neq 0$, while for $k\in[k_i+\tau_\mathrm{d},k_{i+1})$, making use of (18c)

$$\begin{aligned}V(k+1)-V(k) &= x^T(k)\left(A_p^T\Xi_p A_p - \Xi_p\right)x(k) \\ &< 0\end{aligned} \tag{26}$$

for all $|x|\neq 0$. At the switching instant k_i, it can be shown that using (22)

$$\mathscr{P}_p(k_i) = \left(A_p^{\tau_\mathrm{d}}\right)^T \Xi_p A_p^{\tau_\mathrm{d}} + P_p - \left(A_p^{\tau_\mathrm{d}}\right)^T P_p A_p^{\tau_\mathrm{d}}$$

which, in conjunction with (18b), results in

$$\begin{aligned}V(k_i^-)-V(k_i) = x^T(k_i)\Big(&\Xi_q - \left(A_p^{\tau_\mathrm{d}}\right)^T\Xi_p A_p^{\tau_\mathrm{d}} \\ &-P_p + \left(A_p^{\tau_\mathrm{d}}\right)^T P_p A_p^{\tau_\mathrm{d}}\Big)x(k_i) \\ > 0&\end{aligned} \tag{27}$$

for all $|x|\neq 0$. In view of (25)–(27), it follows that $V(\cdot)$ is decreasing in between the switching instants and decreasing at the switching instants, which implies that $x(k)\to 0$ as $k\to\infty$. This completes the proof. ■

Remark 2: Similar with [2], [21], the approach of Theorem 3.1 can be handily extended to discrete time as well. In this regard, the Lyapunov function (23) allows to close many theoretical gaps between discrete-time switched linear systems and discrete-time non-switched linear systems, e.g. the discrete-time counterparts of the stability results in [13], [18].

Similar with the continuous-time case, by imposing $P_p = 0$ and $Q_p = 0$ in (19), we arrive at the following result.

Corollary 3.5: Assume that for some τ_d, there exists a family of positive definite matrices $\Xi_p\in\mathbb{R}^{n\times n}$, $p\in\mathcal{M}$ such that

$$A_p^T\Xi_p A_p - \Xi_p < 0 \quad \forall p\in\mathcal{M} \tag{28a}$$

$$\left(A_p^{\tau_\mathrm{d}}\right)^T\Xi_p A_p^{\tau_\mathrm{d}} - \Xi_q < 0 \quad \forall p\neq q\in\mathcal{M}. \tag{28b}$$

Then, the switched system (2) is globally asymptotically stable for any switching signal $\sigma(\cdot)\in\mathcal{D}(\tau_\mathrm{d})$.

Proof: The proof follows the one of Corollary 3.2 and is thus omitted. ■

Remark 3: Similar to what shown in Corollary 3.2, (28) turns out to be equivalent to the condition in [22, Theorem 1]. In addition, the discrete-time counterpart of Theorem 3.3 (dwell-time equivalence of Theorem 3.4 and Corollary 3.5) can be derived along similar lines.

IV. Switched Adaptive Control of Uncertain Switched Linear Systems

In this section, the proposed stability analysis is exploited to improve the result of adaptive control of uncertain switched linear systems in [13], [23]. For the sake of convenience, let us revisit the problem formulation of adaptive control of switched systems. Consider the uncertain switched linear system described by:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{29}$$

where $u\in\mathbb{R}^m$ represents some piecewise continuous input. The matrices $A_p\in\mathbb{R}^{n\times n}$ and $B_p\in\mathbb{R}^{n\times m}$ are assumed to be unknown for all $p\in\mathcal{M}$. To develop the adaptive control scheme, a reference switched system representing the desired behavior of (29) is given as follows:

$$\dot{x}_\mathrm{m}(t) = A_{\mathrm{m}\sigma(t)}x_\mathrm{m}(t) + B_{\mathrm{m}\sigma(t)}r(t) \tag{30}$$

where the vector $x_\mathrm{m}\in\mathbb{R}^n$ is the desired state, and $r\in\mathbb{R}^m$ is a bounded reference input. The matrices $A_{\mathrm{m}p}\in\mathbb{R}^{n\times n}$ and $B_{\mathrm{m}p}\in\mathbb{R}^{n\times m}$ are known, and $A_{\mathrm{m}p}$ are Hurwitz matrices for all $p\in\mathcal{M}$. Define the tracking error $e(t) = x(t) - x_\mathrm{m}(t)$. To drive the tracking error to zero, a nominal state feedback controller that makes the switched system behave like the reference model is given as $u^*(t) = K_{\sigma(t)}^{*T}(t)x(t) + L_{\sigma(t)}^*(t)r(t)$, where the nominal parameters $K_p^*\in\mathbb{R}^{n\times m}$ and $L_p^*\in\mathbb{R}^{m\times m}$ exist under the assumption that the following matching conditions hold [24]:

$$A_p + B_pK_p^{*T} = A_{\mathrm{m}p}, \quad B_pL_p^* = B_{\mathrm{m}p}.$$

However, since A_p and B_p are unknown, we cannot obtain K_p^* and L_p^* via the matching conditions. Therefore, a controller with parameter estimates is developed as:

$$u(t) = K_{\sigma(t)}^T(t)x(t) + L_{\sigma(t)}(t)r(t). \tag{31}$$

The adaptive control problem for the switched linear system with parametric uncertainties (29) is formulated as:

[**Switched adaptive control**] Develop a family of adaptive laws for K_p and L_p in (31) and a dwell-time switching law $\sigma(\cdot)$ such that the switched system (29) with the state feedback controller (31) can asymptotically track the reference switched system (30), i.e., the tracking error $e(t) \to 0$ as $t \to \infty$.

Next, the stability condition in Theorem 3.1 is used to develop the adaptive laws. Before introducing the solution to the switched adaptive control problem, a conventional assumption is needed: there exists a family of matrices $S_p \in \mathbb{R}^{m\times m}$ such that $M_p := L_p^* S_p = \left(L_p^* S_p\right)^T = S_p^T L_p^{*T} > 0, \forall p \in \mathcal{M}$ [24]. Then, the following result holds.

Theorem 4.1: Assume that for some positive $\tau_{\rm d}$, there exists a family of symmetric positive definite matrices Ξ_p and Q_p, $p \in \mathcal{M}$ satisfying the conditions (4) and (5) for the switched system $\dot{x} = A_{{\rm m}\sigma}x$. The controller (31) with adaptive laws

$$\begin{aligned} \dot{K}^T_{\sigma(t)}(t) &= -\, S^T_{\sigma(t)} B^T_{{\rm m}\sigma(t)} \mathcal{P}_{\sigma(t)}(t) e(t) x^T(t) \\ \dot{L}_{\sigma(t)}(t) &= -\, S_{\sigma(t)} B^T_{{\rm m}\sigma(t)} \mathcal{P}_{\sigma(t)}(t) e(t) r^T(t) \end{aligned} \tag{32}$$

with

$$\mathcal{P}_p(t) = \begin{cases} e^{A^T_{{\rm m}p}(t_i+\tau_{\rm d}-t)}\Xi_p e^{A_{{\rm m}p}(t_i+\tau_{\rm d}-t)} + P_p \\ -e^{A^T_{{\rm m}p}(t_i+\tau_{\rm d}-t)} P_p e^{A_{{\rm m}p}(t_i+\tau_{\rm d}-t)}, & t \in [t_i, t_i+\tau_{\rm d}) \\ \Xi_p, & t \in [t_i+\tau_{\rm d}, t_{i+1}) \end{cases}$$

guarantees that the tracking error $e(t)$ tends to zero for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_{\rm d})$.

Proof: Let us define the parameter estimation errors $\tilde{K}_p = K_p - K_p^*$ and $\tilde{L}_p = L_p - L_p^*$. Consider the following Lyapunov function:

$$\begin{aligned} \upsilon(t) = {} & e^T(t)\mathcal{P}_{\sigma(t)}(t)e(t) + \sum_{p=1}^{M} {\rm tr}\left[\tilde{K}_p(t) M_p^{-1} \tilde{K}_p^T(t)\right] \\ & + \sum_{p=1}^{M} {\rm tr}\left[\tilde{L}_p^T(t) M_p^{-1} \tilde{L}_p(t)\right] \end{aligned} \tag{33}$$

and using the adaptive laws (32), its time derivative is given for $t \in [t_i, t_i+\tau_{\rm d})$ and for $t \in (t_i+\tau_{\rm d}, t_{i+1})$

$$\begin{aligned} \dot{\upsilon}(t) &= e^T(t)\left(A^T_{\rm mp}\mathcal{P}_p(t) + \dot{\mathcal{P}}_p(t) + \mathcal{P}_p(t)A_{\rm mp}\right)e(t) \\ &< 0 \end{aligned}$$

for all $|e| \neq 0$. By resorting to a similar analysis as in (11)–(12) and using continuity of (33) at $t = t_i + \tau_{\rm d}$, it can be shown that $\upsilon(t)$ is decreasing for all $|e(t)| \neq 0$ over the time interval $[t_i, t_{i+1})$. At the switching instant t_i, since the signals $e(t)$ and $\tilde{K}_p(t)$, $\tilde{L}_p(t)$ are continuous for all $t \geq 0$, using (4c), we have

$$\begin{aligned} \upsilon(t_i^-) - \upsilon(t_i) &= e^T(t_i)\left(\mathcal{P}_p(t_i^-) - \mathcal{P}_q(t_i)\right)e(t_i) \\ &> 0 \end{aligned} \tag{34}$$

for all $|e(t_i)| \neq 0$. Considering that $\upsilon(t)$ is decreasing for all $|e(t)| \neq 0$ and using Barbalat's Lemma as in [13], it can be concluded that $e(t) \to 0$ as $t \to \infty$. This completes the proof. ∎

Remark 4: The adaptive laws (32) can be regarded as an improvement over those in [13]. Since the adaptive laws in [13] exploit the stability conditions (3) to develop the time-varying definite matrices $\mathscr{P}_p(\cdot)$, more computation burden is needed to achieve the same minimal dwell time of the adaptive laws (32). Note that a discrete-time switched adaptive control (not shown for lack of space) can be formulated using the analysis of Theorem 3.4.

V. EXAMPLES

In this section, two examples are used to illustrate the stability results and switched adaptive control scheme.

A. Stability analysis

The switched system with two subsystems from [2] is adopted:

$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 6 \\ -1 & -1 & -5 \\ 0 & 1 & -1 \end{bmatrix}.$$

Let $Q_1 = 0.1I$, $Q_2 = 0.15I$. Solving the Lyapunov equation (5) and matrix inequalities (4), we obtain the minimum dwell time $\tau_{\rm d} = 0.4$, which recovers the result in [2], [11] using the stability condition (3) with $K = 95$, and is consistent with the result using the method of [1, Theorem 1] as well. It is important to note that we obtain the same result by using 8 matrix inequalities instead of 568 matrix inequalities. Given the initial condition $x_0 = [2; 1; -2]$, the resulting Lyapunov function in the form of (9) is plotted in Fig.1, which shows that the Lyapunov function is decreasing in between and at the switching instants.

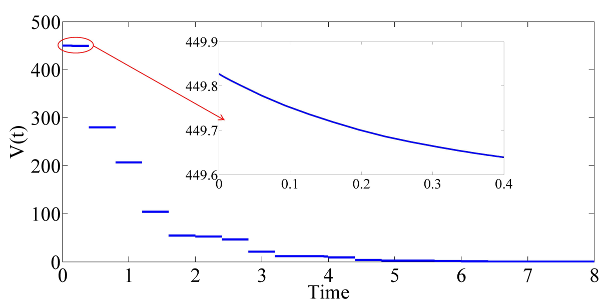

Fig. 1. The Lyapunov function (10).

B. Switched adaptive control

The electro-hydraulic system from [13] is adopted to illustrate the switched adaptive control method. The adaptive control approach is utilized to design a closed-loop controller and a switching signal for the unstable dynamics. Two different supply pressures, 11.0 MPa and 1.4 MPa, are selected, and the resulting subsystems are given as:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -4.58 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 62.4 \end{bmatrix} u(t), \quad 11.0 \text{ MPa}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -9.19 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 47.2 \end{bmatrix} u(t), \quad 1.4 \text{ MPa}$$

where the state $x = [x_1 \ x_2]^T$ with x_1, x_2 represents the displacement of arm and the velocity of the arm, respectively. The desired state is generated by:

$$\dot{x}_{\rm m}(t) = \begin{bmatrix} 0 & 1 \\ -15 & -8 \end{bmatrix} x_{\rm m}(t) + \begin{bmatrix} 0 \\ 31.2 \end{bmatrix} r(t), \quad 11.0 \text{ MPa}$$

$$\dot{x}_{\rm m}(t) = \begin{bmatrix} 0 & 1 \\ -27 & -12 \end{bmatrix} x_{\rm m}(t) + \begin{bmatrix} 0 \\ 23.6 \end{bmatrix} r(t), \quad 1.4 \text{ MPa}.$$

With $Q_1 = \text{diag}\{0.5, 1\}$, $Q_2 = \text{diag}\{1, 0.5\}$, and $\tau_{\rm d} = 1s$, we solve (4) and (5), resulting in

$$\Xi_1 = \begin{bmatrix} 4.6759 & 0.1060 \\ 0.1060 & 0.4138 \end{bmatrix}, \quad \Xi_2 = \begin{bmatrix} 4.9802 & 0.0660 \\ 0.0660 & 0.2569 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1.1021 & 0.0167 \\ 0.0167 & 0.0646 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.8264 & 0.0185 \\ 0.0185 & 0.0224 \end{bmatrix}.$$

Select the following initial conditions: $x_0 = [0 \ 0]^T$, $x_{\rm m0} = [2 \ -2]^T$, $K_1(0) = 0.5K_1^*$, $K_2(0) = 0.5K_2^*$, $L_1(0) = L_2(0) = 0.5$. Let the reference input $r(t) = 3\sin(\pi t) + 2\cos(2t)$ and the adaptive gains $S_1 = 10I$, $S_2 = 5I$. Then, using the controller (31) with the adaptive laws (32) and the same switching signal with $\tau_{\rm d} = 1s$, the resulting tracking error is shown in Fig. 2, which tends to zero.

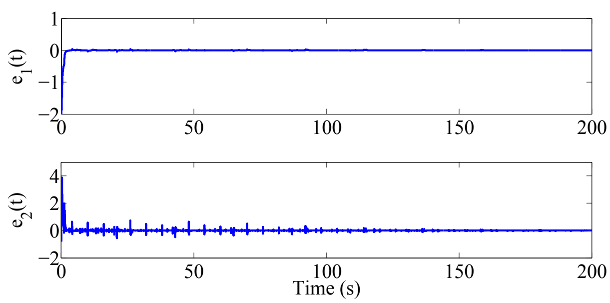

Fig. 2. The tracking error.

VI. Conclusion

In this paper, we have introduced a novel stability analysis for switched linear systems. By exploiting solutions to a family of differential Lyapunov equations and algebraic Lyapunov equations, a new Lyapunov function with a time-varying positive definite matrix has been developed, which is decreasing at the switching instants. The resulting stability condition for continuous-time switched linear systems has been shown to improve the one using the linear interpolation method of [2] by decreasing the computational burden. At the same time, the proposed analysis leads to the same minimum dwell time of the well-known result of [1]. Moreover, the stability analysis is amenable to a discrete-time counterpart, which is consistent with the interpolation approach. The proposed stability analysis has been applied to switched adaptive control problem for uncertain switched linear systems.

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