

# *Market consistent valuation of deferred taxes*

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# Abstract

This thesis develops a continuous time framework to value deferred taxes using [Black and Scholes \(1973\)](#) type option pricing techniques. The valuation renders a market consistent pricing procedure, which avoids the necessity of subjective accounting principles. Our framework is flexible enough to value deferred taxes like carry forward, carry back or liabilities arising from temporary differences relying solely on quantities observed in the market. A simulation study over multiple time horizons shows that carry forward value is negatively influenced by leverage, whereas carry back and tax liability values increase. Two empirical applications serve to illustrate the practical use of our model: the loss absorbing capacity of deferred taxes for European insurers and an estimate of BP's loss of deferred taxes following the U.S. tax overhaul.

**Keywords:** Deferred tax valuation, martingale pricing, optimal capital structures, loss absorbing capacity of deferred taxes

**2010 Mathematics Subject Classification:** 91G20 (Derivative securities), 91G40 (Credit risk), 91G60 (Numerical methods), 91G70 (Statistical methods, econometrics)



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# Contents

Abstract	iii
Introduction	ix
Literature overview	xi
Chapter 1. Introduction to deferred taxes	1
§1.1. Fiscal and market consistent accounting	1
§1.2. Carry Forward	3
§1.3. Carry Back	4
§1.4. DTA from temporary differences	5
§1.5. DTL from temporary differences	6
Chapter 2. Extensions of the one-period model	9
§2.1. One-period model levered firms	9
2.1.1. Carry forward	9
2.1.2. Carry back	10
2.1.3. DTL	11
§2.2. Two-period model unlevered firms	11
2.2.1. Carry forward	11
2.2.2. Carry back	12
2.2.3. DTL	14
Chapter 3. Models	15
§3.1. Modeling unlevered firms	15
§3.2. Modeling levered firms	20
Chapter 4. Results	23

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§4.1. One-period model unlevered firms	23
4.1.1. Carry forward	23
4.1.2. Carry back	26
4.1.3. The trade-off between carry forward and carry back	28
4.1.4. DTL temporary differences	30
4.1.5. Comparison to current valuation methods	31
§4.2. Valuation of debt	36
§4.3. One-period model levered firms	38
4.3.1. Carry forward	39
4.3.2. Carry back	42
4.3.3. DTL	44
4.3.4. Valuation interest tax shield	46
§4.4. Multi-period model unlevered firms	49
4.4.1. Simulation set-up	49
4.4.2. Carry forward	50
4.4.3. Carry Back	57
4.4.4. DTL	61
§4.5. Multi-period model levered firms	63
4.5.1. Simulation set-up	63
4.5.2. Carry forward	64
4.5.3. Carry back	65
4.5.4. DTL	67
§4.6. Optimal capital structures	67
Chapter 5. Empirical applications and agency problems	71
§5.1. Loss absorbing capacity of deferred taxes for European insurers	71
5.1.1. Market consistent approach	72
5.1.2. Implication for policymakers	72
§5.2. BP tax loss estimates	72
5.2.1. Simulation with random tax regimes	76
§5.3. Tax smoothing effects and agency problems	78
Chapter 6. Conclusion	83
Shortcomings of approach	84
Suggestions for further research	85
Appendix A. Elements of financial mathematics	87
§A.1. Martingale pricing	87
Appendix B. Valuation of carry forward in 2-period model	91

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§B.1. No fiscal history	91
§B.2. One year duration carry forward	95
§B.3. Two year duration carry forward	97
Appendix C. Proof coupon payments	99
§C.1. No deferred taxes	99
§C.2. Carry forward	101
§C.3. Carry back	102
§C.4. DTL	104
Bibliography	107
List of Figures	111
List of Tables	115



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# Introduction

In this thesis we describe the market consistent valuation of deferred taxes. Deferred taxes are balance sheet items of undertakings with a certain history of fiscal profits and losses. They reflect the advantage or disadvantage of such undertakings to pay less or more taxes compared to a hypothetical, similar firm without any history of fiscal profits and losses.

Deferred taxes are commonly valued according to accounting standards, which recognize and value them using a single deterministic scenario. This, however, does not reflect the contingent characteristics of deferred taxes that their payoff, i.e. the tax advantage or disadvantage, is a non-linear function of the future fiscal profits and losses of the undertaking. In this thesis we construct a model to take account of this non-linear payoff by using basic option pricing techniques. Thereby, the model yields pricing formulas that depend on market observable parameters, resulting in market consistent valuations of deferred taxes.

The market consistent valuation of deferred taxes differ from the common valuations according to the accounting standards in the following ways:

- The market consistent valuation of *deferred tax assets* (DTA) and *deferred tax liabilities* (DTL) is typically lower than those obtained with accounting methods. This is because a market consistent valuation reflects scenarios in which DTA's/DTL's do not materialize, while a single deterministic scenario is more or less all or nothing.
- Profits in previous years and the possibility in a tax regime to carry back future losses to previous years also results in a DTA

under the market consistent valuation principles. However, this does not result in a DTA under conventional accounting standards. The reason for this comes from the deterministic scenario, which only considers a profit situation in which the carry back potential is not being realized. On the contrary, a market consistent valuation does attach value to the advantage in negative scenarios.

- Accounting standards (partially) recognize DTA's if an undertaking can prove future profitability using subjective assumptions. Market consistent valuation uses market data to encompass all future scenarios and thereby solely relies on objective parameters observed in the market.

The main novelty of this thesis is to attach a market consistent valuation to deferred taxes arising from carry forward, carry back and temporary differences in a continuous time framework. Hereby, we find that market consistent valuation techniques yield significantly different estimates in comparison with conventional accounting valuation methods. This is because the market consistent approach takes all future profit and loss scenarios into account. Moreover, we show that extending the firm's activities for an additional year, may, in some cases, lead to a value reduction of DTA's. The reason is that undertakings with an initial tax benefit expect to pay more taxes after the DTA is settled, compared to a similar undertaking that doesn't have the initial tax advantage. In addition, we show that the market consistent valuation of deferred taxes is influenced by leverage. Higher leverage reduces the value of carry forward, whereas it positively influences the value of carry back and DTL's.

The valuation of deferred taxes is in close liaison with tax shield pricing, as the tax advantage of coupon payments are contingent on profit streams. The option component that arises herein leads to pricing formulas similar to DTA's/DTL's, in lieu of the Modigliani-Miller theorem, which is contained as a special case. Additionally, we develop a credit model that dovetails tax benefits and bankruptcy cost. The tax benefits and loss in case of bankruptcy prompts the existence of optimal capital structures, which depend on the lifespan of the company. The credit model yields lower agency cost in the asset substitution problem compared to the [Merton \(1974\)](#) credit model, since debt and equity holders do not have a split claim on the assets.

In a final step, our methodology is applied to empirical investigations. The model in this thesis supports how insurance undertakings can value their deferred taxes according to the market consistent valuation principles of Solvency II. On top of that, the model provides insights in the loss absorbing

capacity of deferred taxes, an element that lowers the Solvency II capital requirements. The market consistent valuation model indicates the following regarding the loss absorbing capacity of deferred taxes:

- DTL's indeed have loss absorbing capacity; when an insurance, or any other undertaking suffers a loss, part of this loss is being compensated by a lower market valuation of the DTL after the loss.
- DTA's may have loss absorbing capacity if the undertaking has sufficient potential, resources and/or own funds to generate future profits; in that case the value of the total DTA increases after a loss. However, if insufficient own funds are available, such an undertaking would experience a decrease in its DTA. In these situations, a reduction of the Solvency II capital requirements for European insurance undertakings does not reflect the actual loss due to a decrease in their DTA.

We find that the loss absorbing capacity is, on average, less than extant estimates and under some circumstances can even be negative, since so much potential is lost after a severe (negative) shock. Not only is the market consistent valuation of deferred taxes important for insurers, but it is also relevant to mergers and acquisitions, when buyers have to value an undertaking. The common accounting valuation of deferred taxes does not necessarily reflect the market price of the tax advantage or disadvantage.

The remainder of this thesis is structured as follows. [Chapter 1](#) introduces the non-linear payoff structure of various deferred taxes over one time period. The basic ideas are extended to include leverage and multiple time horizons in [Chapter 2](#). In [Chapter 3](#), we outline the stochastic underpinnings of the pricing approach, which are used in [Chapter 4](#) to obtain the market consistent DTA/DTL values. [Chapter 5](#) covers two empirical applications: the loss absorbing capacity of European insurers and BP's loss of deferred taxes following the U.S. tax overhaul. In addition, agency problems are discussed. Finally, [Chapter 6](#) concludes and suggests further directions of research.

## Literature overview

Current valuation methods of deferred taxes are generally based on GAAP (Generally Accepted Accounting Principles) or IAS12 (International Accounting Standard, Income Taxes). [Sansing \(1998\)](#) remarks that these approaches tend to overestimate the true value of deferred taxes appearing on

financial statements, as they are future benefits but not discounted. Moreover, the appropriate discount factor is an open question, since the materialization of deferred taxes is not risk free. [Sansing \(1998\)](#) derives a discount factor for deferred tax liabilities, however assuming an average tax liability, thereby ignoring the dynamics over longer periods of time. [De Waegenare et al. \(2003\)](#) obtain closed form formulas for tax carry forward, including additional parameters like the duration period. This framework leads to the surprising conclusion that the market-to-book ratio of carry forward can exceed one, depending on the skewness of the underlying income distribution. This suggests that discounting deferred tax assets may not always be appropriate. But, [De Waegenare et al. \(2003\)](#) use the stringent assumption that income is generated in perpetuity, which is rather unrealistic. Empirical studies of deferred taxes are conducted by [Givoly and Hayn \(1992\)](#), [Amir et al. \(1997\)](#) and [Ayers \(1998\)](#). [Givoly and Hayn \(1992\)](#) use a linear regression approach, where abnormal returns are regressed on the reduction in deferred tax liabilities during a period of tax reforms. Hereby, [Givoly and Hayn \(1992\)](#) find that investors discount the liability based on likelihood and timing of the settlement. [Amir et al. \(1997\)](#) use a regression approach as well, but splitting the deferred taxes into seven categories, which renders a more precise estimate of the influence of deferred taxes on equity. All regression coefficients of the deferred tax assets are found to be greater than one, which contradicts the hypothesis that DTA's ought to be discounted, as this would imply a regression coefficient between zero and one. A similar approach and conclusion is reached by [Ayers \(1998\)](#). However, as [De Waegenare et al. \(2003\)](#) point out, this only holds if the disparity between book and market value is solely due to discounting.

The emphasis in this thesis centers around the valuation of deferred taxes and their influence on firm value. However, taxes also enact positive benefits to society that are not taken into consideration, cf. [Burda and Wyplosz \(2013\)](#) Chapter 18.4.

# Introduction to deferred taxes

## 1.1. Fiscal and market consistent accounting

The reason that deferred taxes arise stems from the difference in valuation principles between the accounting and fiscal balance sheet. The fiscal balance sheet is typically based on historical cost, whereas accounting principles are based on a market consistent approach, which values balance sheet items on the basis of current market prices. To get a better understanding of the matter at hand, we look at a fictitious example with market consistent accounting principles and historical cost as fiscal valuation principles. Consider a company without any fiscal history, so that the market consistent and fiscal account are exactly the same as in the T-account below.

Market consistent Account		Fiscal Account	
Stock 100	Equity 100	Stock 100	Equity 100

Next, assume that the company's assets rise in value to 120 due to an increase of stock investment, so that a profit of 20 is realized. However, this profit is not recognized in the fiscal account, because stocks are valued on the basis of historical costs. The company now faces a deferred tax liability (DTL), because it does not yet pay taxes, but is obliged to do so in the future. Suppose that the tax rate  $\tau = 25\%$ , so that the amount of tax paid on profit would be 5. This amount needs to be reserved on the market

account, because it needs to be paid in the future. The fiscal balance sheet remains unchanged. The new situation leads to the following T-account.

Market consistent Account		Fiscal Account	
Stock 120	Equity 115	Stock 100	Equity 100
	DTL 5		

Now, suppose a negative shock occurs, reducing total assets by 40%. As a result, total assets fall to 72, which means that the earlier profit of 20 and DTL of 5 disappear. When selling the assets, a loss of 28 would be realized. This loss can be carried forward to offset the next 28 (taxable) profit, creating a tax advantage of  $28\tau = 7$ . This amount is put on the market consistent account as a deferred tax asset (DTA). The new situation leads to the following T-accounts.

Market consistent Account		Fiscal Account	
Stock 72	Equity 79	Stock 100	Equity 100
DTA 7			

Because of deferred taxes, the loss is *not* 48 but 36. This is the *loss absorbing capacity of deferred taxes* (LAC DT), because part of the loss can be transferred to the tax authority, thereby mitigating the overall loss. However, the DTA can only be settled if future profits are sufficient to claim the full amount from the tax authority. Two of the most widely used accounting principles, GAAP and IAS12, require that a firm proves it is likely to make sufficient profit in the future to materialize the DTA. If not, the DTA will not be recognized on the balance sheet. This is the conventional way in which undertakings value DTA's/DTL's and use them to change equity. This valuation method is *not* correct, since one does not incorporate the possibility that future profits are unable to compensate the entire DTA (or DTL). Since these DTA's/DTL's are conditional claims on the tax authority, we can interpret them as a financial option. Methods from financial option pricing theory can be employed to find a market consistent value of deferred taxes.

## 1.2. Carry Forward

Carry forward is the allowance to carry forward losses to offset future taxable income. An undertaking recognizes that losses can be seen as an asset, since part of the loss will result in lower tax payments compared to a similar undertaking without any tax history. The extent to which an undertaking is able to settle the carry forward in the future determines the eventual value of this type of DTA. Throughout this thesis we make the following *important* assumption about corporate tax payments, which is needed to model the contingent characteristics of DTA's/DTL's.

**Assumption 1.1.** *Net profit is measured by the difference in asset value over two consecutive periods. Taxable income consists of net profit if this is a positive quantity and is zero otherwise.*

**Remark 1.2.** This assumption essentially implies market consistent accounting introduced in [Section 1.1](#), except that we exclude tax exempted earnings, like income outside tax jurisdiction. These items are excluded for modeling purposes. Moreover, the assumption leads to a convenient analytical interpretation in subsequent results.

Consider a company with no fiscal history, so that it does not have any deferred taxes on the balance sheet. Let  $A_t$  denote the asset value of a company at time  $t$  before taxes are levied. At  $t = 1$ , a firm pays taxes only if  $A_1 > A_0$  ([Assumption 1.1](#)), in which case the total asset value is reduced. Let  $\tau$  denote the tax rate, then the asset value at  $t = 1$  after tax can be expressed as

$$(1.1) \quad \tilde{A}_1 = A_1 - \tau(A_1 - A_0)^+.$$

Here  $\tilde{A}_t$  denotes the value of an asset after taxes in period  $t$  and  $(x)^+ \triangleq \max(x, 0)$ . The second term in (1.1) has the same structure as the payoff of a European call option with strike  $A_0$ .<sup>1</sup>

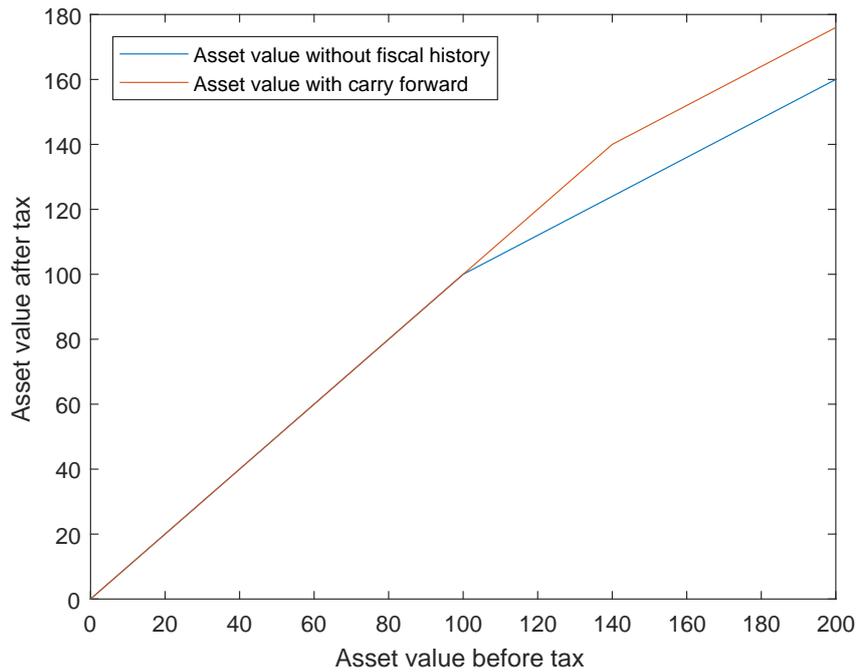
Consider a firm with an identical balance sheet, but with the additional benefit of carry forward as a deferred tax asset. Carry forward can be used to offset taxable income in case net profit is positive. Let  $CF_t$  be the loss carry forward available in year  $t$ . Then in period  $t = 1$  we have the following post-tax asset value

$$(1.2) \quad \tilde{A}_1 = A_1 - \tau(A_1 - A_0 - CF_1)^+.$$

An undertaking with carry forward pays taxes from the moment  $A_1 > A_0 + CF_1$ , which differs from a firm without having this tax asset, as they pay taxes as soon as  $A_1 > A_0$ . The asset value after tax (1.2) versus (1.1) is

<sup>1</sup>In particular, this is an at-the-money call option.

shown in [Figure 1.1](#). The difference between the two asset values becomes relevant as soon as  $A_1 > A_0$ , corresponding to the moment that a firm without fiscal history has to pay taxes. The difference between the two graphs is what fundamentally determines the DTA (or DTL) value.



**Figure 1.1.** Asset value after tax. Blue line denotes asset without fiscal history with  $A_0 = 100$ , while orange line depicts asset with 40 carry forward.

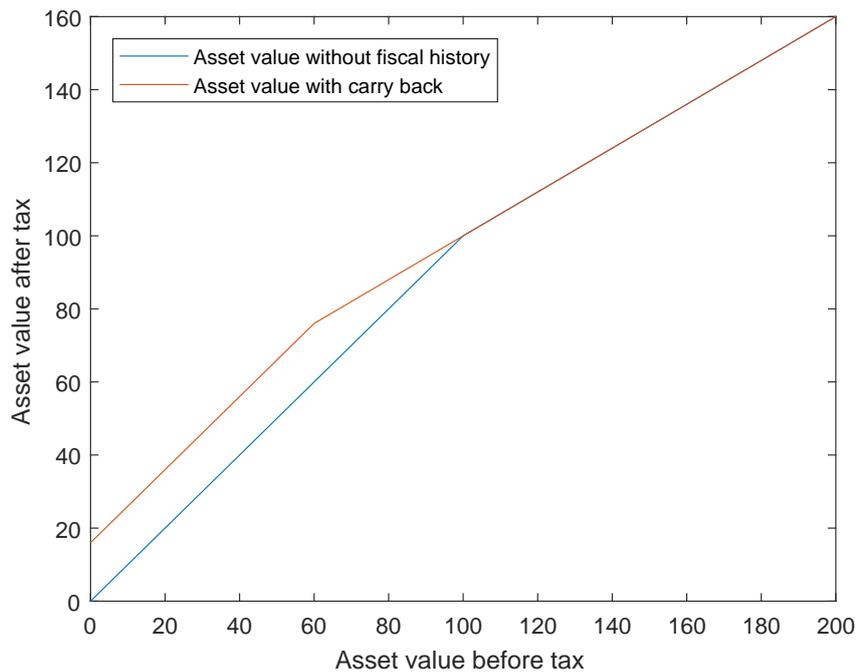
### 1.3. Carry Back

Tax carry back is the possibility to receive a refund of corporate tax paid in the past, due to current losses. The maximum amount that can be claimed as refund equals the current loss times the tax rate, but may be less when historical profits are insufficient to offset the current loss. Carry back renders the firm with an immediate cash flow that (partly) compensates current loss. Carry back expires worthless when an undertaking makes profit in period one. This gives rise to the following asset value of a firm in period one

$$(1.3) \quad \tilde{A}_1 = A_1 + \tau CB_1 - \tau(A_1 - A_0 + CB_1)^+,$$

where  $CB_t$  denotes the loss carry back in year  $t$ . The asset value after tax (1.3) and (1.1) are shown in [Figure 1.2](#). The value of the DTA manifests itself

when the company incurs a loss, in which one observes a positive difference between the two graphs. This difference vanishes as soon as  $A_1$  is bigger than or equal to  $A_0$ , which corresponds to a factual profit. In this situation, there is no loss that can be carried back anymore.



**Figure 1.2.** Asset value after tax. Blue line denotes asset without fiscal history with  $A_0 = 100$ , while orange line depicts asset with 40 carry back.

#### 1.4. DTA from temporary differences

Temporary differences arise as a consequence of the difference between the applicable and fiscal valuation principles. Fiscal accounting principles are typically based on historical costs. Initially, assets are valued at their market price on the fiscal balance sheet. If increases in the market price are not reflected on the fiscal balance sheet, the undertaking does not need to pay taxes on this profit now, but only when it sells. In other cases, like for fixed income, the tax advantage materializes over the lifetime of the fixed income asset. As a result, an undertaking can make an accounting profit/loss, which is not recognized yet under fiscal accounting principles. An example is given in [Section 1.1](#). A deferred tax asset arising from temporary differences might occur due to shocks in the asset value. In this case, the loss incurred now

can offset future taxable income. Hence, the after-tax value of a firm having a DTA from temporary differences is equal to

$$\tilde{A}_1 = A_1 - \tau(A_1 - A_0 - TD_1)^+,$$

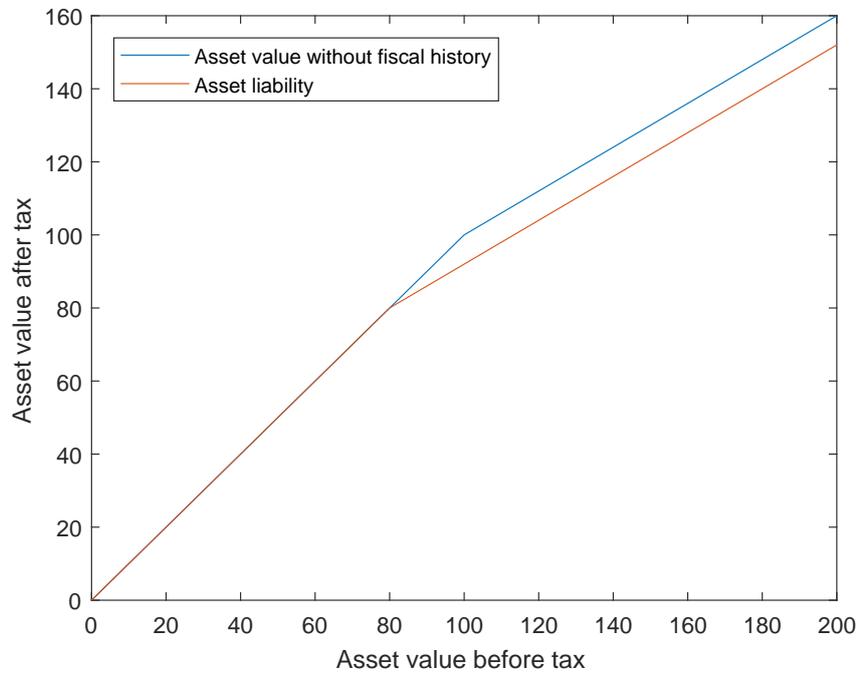
where  $TD_1$  is the nominal DTA value. We observe that the post-tax value of such a firm has the exact same structure as a firm having some carry forward available (see Equation (1.2)). Hence, in those circumstances, the analysis of DTA's arising from temporary differences is equivalent to finding a market consistent value of carry forward. When going beyond one year time periods, differences might emerge due to the absence of regulations on settlement terms of DTA's arising from temporary differences. Despite this, we *ignore* further valuation analysis of such DTA's, as they are still similar in spirit to DTA's coming from carry forward.

### 1.5. DTL from temporary differences

Finally, we consider scenarios in which a firm makes profit under applicable valuation principles, which is not recognized under fiscal valuation principles. The undertaking knows it is obliged to pay extra taxes in the future and reserves an appropriate amount on the balance sheet. In the one period case, the asset value after tax of a company having a DTL arising from temporary differences can be expressed as

$$(1.4) \quad \tilde{A}_1 = A_1 - \tau(A_1 - A_0 + gain_1)^+.$$

The variable  $gain_1$  is the amount of taxable profit recognized under market consistent accounting, but not under fiscal accounting standards in period one. In general, we write  $gain_t$  for the (total) unrecognized fiscal profit in time period  $t$ . The asset value (1.4) is always less than or equal to (1.1), because of deferred tax liabilities. The two graphs corresponding to (1.4) and (1.1) are shown in Figure 1.3. The difference between the two graphs unfolds whenever some of the untaxed profit still persists in period one.



**Figure 1.3.** Asset value after tax. Blue line denotes asset without fiscal history with  $A_0 = 100$ , while orange line depicts asset with 20 tax liabilities.



# Extensions of the one-period model

## 2.1. One-period model levered firms

So far, we ignored the capital structure of an undertaking. However, the way in which an undertaking is financed has repercussions for tax payments. Our approach to give a market consistent valuation of deferred taxes depends on comparing tax payments of a reference undertaking without fiscal history and a firm having the same characteristics with deferred taxes. Introducing debt financing alters tax payments, since coupon payments can be deducted from taxable income, creating the so-called tax shield. In the following subsections we examine the effect of coupon payments on DTA's/DTL's.

**2.1.1. Carry forward.** We assume that coupon payments are deducted from taxable income before deferred taxes are used. A reference undertaking (without deferred taxes), making yearly coupon payments due to leverage, has the following post-tax asset value

$$(2.1) \quad \tilde{A}_1 = A_1 - \mathcal{C} - \tau(A_1 - A_0 - \mathcal{C})^+,$$

where  $\mathcal{C}$  is the coupon payment on debt. The rationale behind (2.1) is the following; part of taxable income is reduced by coupon payments, this is the *tax shield* and appears in the  $(\cdot)^+$  term. Equation (2.1) contains (1.1) as a special case when debt ( $D$ ) is zero, since  $\mathcal{C} = 0$  in that case.

**Remark 2.1.** The amount of coupon payment a firm can deduct from taxable income is quite country specific. For example, countries like Italy have a limit on the amount of coupon an undertaking can deduct, in order to

eschew perverse incentives arising from debt financing. Because we aim for some generality in our analysis, we model the amount of interest payments the firm can deduct by an exogenous parameter  $\gamma \in [0, 1]$ . By doing so, (2.1) is replaced by

$$(2.2) \quad \tilde{A}_1 = A_1 - \mathcal{C} - \tau(A_1 - A_0 - \gamma\mathcal{C})^+.$$

In analogy with Section 1.2 and by Remark 2.1 we get the following asset value after tax for levered firms having some carry forward

$$(2.3) \quad \tilde{A}_1 = A_1 - \mathcal{C} - \tau(A_1 - A_0 - CF_1 - \gamma\mathcal{C})^+.$$

Deducting interest payments from net profits has repercussions for the carry forward value, as the following example shows.

**Example 2.2.** Suppose an unlevered firm has 20 carry forward available ( $CF_1 = 20$ ) and makes 10 profit in period one, i.e.  $A_1 - A_0 = 10$ . The firm can use 10 of the carry forward to offset all taxable income. Now consider an identical firm, which is levered and pays 10 interest each year, i.e.  $\mathcal{C} = 10$  (and  $\gamma = 1$ ). This means that the factual profit in period one is zero, because  $A_1 - A_0 - \mathcal{C} = 0$ . As a result, none of the carry forward can be used and expires worthless. Hence, the DTA arising from carry forward is less valuable for levered firms.

Thus, with leverage, generating fiscal loss becomes more likely, which decreases the probability of settling carry forward.

**2.1.2. Carry back.** The post-tax asset value of a levered firm with some carry back is equal to

$$(2.4) \quad \tilde{A}_1 = A_1 - \mathcal{C} + \tau CB_1 - \tau(A_1 - A_0 + CB_1 - \gamma\mathcal{C})^+.$$

In contrast to carry forward, tax carry back increases in value as a result of leverage. This is because fiscal profit/loss, as calculated by the difference in asset value less coupon payments, is always less for a levered undertaking. Thus, it becomes more likely that carry back is materialized, as the following example shows.

**Example 2.3.** Suppose that an unlevered firm has  $CB_1 = 20$  and incurs a loss of 15 in period one, i.e.  $A_1 - A_0 = -15$ . It can use 15 of the carry back to neutralize the loss in period one. Now consider an identical levered firm carrying interest cost of  $\mathcal{C} = 10$  each year. For this firm, the net “profit” is  $-15 - 10 = -25$ . With 20 carry back available, it can materialize the complete DTA. Hence, in this case carry back is more valuable when a firm is levered.

**2.1.3. DTL.** Finally, for firms carrying some tax liabilities (DTL), the post-tax asset value is given by

$$(2.5) \quad \tilde{A}_1 = A_1 - \mathcal{C} - \tau(A_1 - A_0 + \text{gain}_1 - \gamma\mathcal{C})^+.$$

The following example shows that levered undertakings are less likely to repay the entire DTL compared to unlevered undertakings.

**Example 2.4.** Take a firm having a deferred tax liability of 20 (i.e.  $\text{gain}_1 = 20$ ). If a firm makes a profit of 50 ( $A_1 - A_0 = 50$ ) in the next period, then it has to pay taxes over 70, instead of paying taxes over 50 if it did not have a tax obligation. Consider again an identical firm, which is levered and makes interest payments of  $\mathcal{C} = 10$ . Consequentially, taxable income is equal to  $A_1 - A_0 - \mathcal{C} + \text{gain}_1 = 60$ . Hence, the amount of taxes paid is less for levered firms, so that the DTL value is greater for such undertakings.

## 2.2. Two-period model unlevered firms

In a multi-period framework, the payoff structure of post-tax asset values becomes significantly more complicated, due to the presence of additional parameters. For example, the post-tax asset value depends on the settlement term of carry forward, whether carry back is allowed or not, settlement term of carry back etc.<sup>1</sup> In general, one would expect that deferred tax assets become more valuable over longer time periods, since the probability that the entire DTA is settled increases. An important change is that taxes are settled every year, which creates path dependency. In this section we present some formulas for the post-tax asset value of undertakings in a two-period model, which serve to illustrate the dynamics of the asset process over longer periods of time.

**2.2.1. Carry forward.** At the end of year two, the following post-tax asset value holds for an undertaking without deferred taxes and excluding carry back possibilities

$$(2.6) \quad \tilde{A}_2 = A_2 - \tau(A_2 - \tilde{A}_1 - \underbrace{1_{A_1 < A_0}(A_0 - A_1)}_{=CF_2})^+.$$

In this formula,  $\tilde{A}_1$  is given by (1.1) and the indicator function is included to account for carry forward possibilities.<sup>2</sup> In case a firm has carry forward ( $= CF_1$ ) which has a settlement term of one year, the formula is similar, except that  $\tilde{A}_1$  is now given by (1.2) Allowing carry back results in a linear

<sup>1</sup>The settlement term denotes the number of years that losses can be carried forward/back.

<sup>2</sup>Alternatively, we could rewrite the last term as  $1_{A_1 < A_0}(A_0 - A_1) = (A_0 - A_1)^+$  to highlight the *nested* option like nature of the payoff. This would be more consistent with previous notation, but we refrain from doing so for notational convenience.

combination of options multiplied by indicator functions to keep track of carry forward and carry back situations.

$$\begin{aligned}\tilde{A}_2 = & A_2 - 1_{A_1 < A_0} \tau(A_2 - A_0)^+ - 1_{A_0 < A_1 < A_0 + CF_1} \tau(A_2 - A_1)^+ \\ & + 1_{A_1 > A_0 + CF_1} (\tau CB_1 - \tau(A_2 - \tilde{A}_1 + CB_1)^+),\end{aligned}$$

where  $\tilde{A}_1$  is given by (1.2). This formula follows from considering three separate cases. If  $A_1 < A_0$ , a loss is incurred, carry forward expires worthless and no taxes are paid. However, the loss incurred in period one ( $A_0 - A_1$ ) can be carried forward, so that the strike of the call option in year two equals  $A_1 + (A_0 - A_1) = A_0 > A_1$ . The third term considers a situation in which profit is made which is less than the total carry forward, so that no taxes are paid and no carry forward nor carry back is taken to period two. Finally, for the last term, if profit is greater than carry forward, the entire carry forward is used and corporate tax is paid over the amount  $A_1 - A_0 - CF_1$ . This amount can subsequently be taken to period two, where it can be used as carry back. Hence, in the two-period model, the payoff structure of the assets after tax already becomes quite involved. Generalizing the formulas above for time periods  $t \geq 3$  is certainly possible, but will not be pursued here.

Excluding carry back and assuming a two year settlement term of carry forward still results in (2.6) for undertakings without deferred taxes, but a firm with carry forward in period  $t = 0$  gives rise to

$$\tilde{A}_2 = A_2 - \tau(A_2 - \tilde{A}_1 - 1_{A_1 < A_0 + CF_1} (A_0 + CF_1 - A_1))^+.$$

This formula holds, since, if at time  $t = 1$  the asset value is less than  $A_0 + CF_1$ , some part of carry forward has not been used and can be carried over to period two. Also,  $\tilde{A}_1$  is given by (1.2). The payoff after tax still has the structure of a call option, but there is path dependency that complicates the tractability of analytical solutions.

**2.2.2. Carry back.** When determining the value of carry back, we always assume that loss carry forward can be created in subsequent years, as there is no country in the world that allows carry back but no carry forward. The opposite situation is ubiquitous in many tax regimes (see also ??). Along similar lines, we can express the asset value after tax of an undertaking with

carry back in the two year model as

$$(2.7) \quad \begin{aligned} \tilde{A}_2 &= A_2 - 1_{A_1 < A_0 - CB_1} \left( \tau(A_2 - \tilde{A}_1^{(1)} - \underbrace{(A_0 - CB_1 - A_1)}_{=CF_2})^+ \right) \\ &\quad - 1_{A_0 - CB_1 \leq A_1 \leq A_0} \left( \tau(A_2 - \tilde{A}_0)^+ \right) \\ &\quad + 1_{A_1 > A_0} \left( \tau \underbrace{(A_1 - A_0)}_{=CB_2} - \tau(A_2 - \tilde{A}_1^{(2)}) + \underbrace{(A_1 - A_0)}_{=CB_2} \right)^+. \end{aligned}$$

where  $\tilde{A}_1^{(1)}, \tilde{A}_2^{(2)}$  are given by (1.3) and (1.1) respectively. The first line follows since  $A_1 < A_0 - CB_1$ , which means that the complete carry back can be settled and additional carry forward in the amount of  $A_0 - CB_1 - A_1$  is taken to period two. The second line considers  $A_0 - CB_1 \leq A_1 \leq A_0$ , which means that only part of the carry back is settled. Since carry back is only one year valid, the remaining carry back expires worthless as it cannot be taken to period two. The last line treats the condition  $A_1 > A_0$ , which means the undertaking made profit and the entire carry back expires worthless. However, additional carry back in the amount of  $A_1 - A_0$  can be taken to period two and can be used when incurring a loss in that period.

At last, we consider the situation in which carry back is two years valid. This yields the after-tax asset value in period two

$$(2.8) \quad \begin{aligned} \tilde{A}_2 &= A_2 - 1_{A_1 < A_0 - CB_1} \left( \tau(A_2 - \tilde{A}_1^{(1)} - \underbrace{(A_0 - CB_1 - A_1)}_{=CF_2})^+ \right) \\ &\quad + 1_{A_1 - CB_1 \leq A_1 \leq A_0} \left( \tau \underbrace{(CB_1 - (A_0 - A_1))}_{=CB_2} - \tau(A_2 - A_0 + \underbrace{(CB_1 - (A_0 - A_1))}_{=CB_2})^+ \right) \\ &\quad + 1_{A_1 > A_0} \left( \tau \underbrace{(A_1 - A_0 + CB_1)}_{=CB_2} - \tau(A_2 - \tilde{A}_1^{(2)}) + \underbrace{(A_1 - A_0 + CB_1)}_{=CB_2} \right)^+. \end{aligned}$$

Again  $\tilde{A}_1^{(1)}, \tilde{A}_2^{(2)}$  are given by (1.3) and (1.1). The difference between (2.7) and (2.8) comes from the last two lines. The first line in (2.8) considers a situation in which the entire carry forward is used in year one and additional carry forward can be taken to year two. This situation is identical to the one where carry back is one year valid. The second line results from the situation in which part of carry back is used, but unlike (2.7), this time the remaining carry back can be taken to year two. The total amount of carry back left for period two equals  $CB_1 - (A_0 - A_1)$ . Finally, if an undertaking makes a profit in period one, then  $CB_1$  cannot be used, but because it can be settled

in two years, the amount can be taken to period two. Tax is levied over the amount  $A_1 - A_0$ , and these tax payments can also be taken to period two and used as carry back. This means that  $CB_2 = CB_1 + A_1 - A_0$  and explains the last line of (2.8).

**2.2.3. DTL.** As opposed to DTA's arising from carry forward/back, there are no regulations on settlement terms of DTL's. A DTL is put on the balance sheet to reflect future tax expenses, but it depends on the specific characteristic of the profit stream when those untaxed profits are materialized. As there are no regulations to guide us here, we consider two different scenarios. In a basic setup, we assume that no intermediate tax payments occur and the DTL is settled at maturity. This gives the post-tax asset value

$$\tilde{A}_2 = A_2 - (A_2 - A_0 + gain_1)^+$$

In a more realistic setup, we assume that the DTL is reduced in period one whenever the firm incurs a loss in that period. The reduction in (nominal) DTL value is equal to the corresponding loss. If the loss in period one exceeds the entire DTL value, the DTL disappears in its entirety and carry forward is created over the remaining loss. In case the undertaking makes a profit in period one =  $(A_1 - A_0)$ , taxes are paid over that profit and the DTL remains the same. The DTL value left is taken to period two, in which settlement is due. Considering each of these three scenarios yields the post-tax asset

$$\begin{aligned} \tilde{A}_2 &= A_2 - 1_{A_1 < A_0 - gain_1} (A_2 - A_1 - \underbrace{(A_0 - gain_1 - A_1)}_{=CF_2})^+ \\ &\quad - 1_{A_0 - gain_1 \leq A_1 \leq A_0} (A_2 - A_1 + \underbrace{(gain_1 - A_0 + A_1)}_{=gain_2})^+ \\ &\quad - 1_{A_1 > A_0} (A_2 - \tilde{A}_1 + gain_1)^+, \end{aligned}$$

where  $\tilde{A}_1$  is the post-tax asset value in period one, as given by (1.1).

# Models

## 3.1. Modeling unlevered firms

The valuation of deferred taxes is based on comparing the value of two hypothetical firms having the same assets, with the only difference that one firm has deferred taxes on the balance sheet. We frequently refer to the undertaking without deferred taxes as the *reference undertaking*. This approach, by comparing two firms who differ in just one characteristic, essentially started with Modigliani and Miller (1963) and has been applied in various other settings (e.g. Arzac and Glosten (2005) on tax shield valuation). The value of a firm is gauged by using the *firm value* measure, which takes the market value of all assets as a proxy to total value of a firm. This is one of the conventional ways in which total firm value is measured, besides *market value of equity* and *enterprise value*.<sup>1</sup> Therefore, the following definition is used to determine the value of deferred taxes.

**Definition 3.1.** The market consistent value of a DTA/DTL is defined as the difference in firm value between an undertaking with deferred taxes and a reference undertaking with the same assets, without having deferred taxes. The precise value of the DTA/DTL then follows from comparing the discounted payoff of the post-tax assets at final time  $t = T$  under martingale measure. We will henceforth refer to this quantity by  $\xi_a^b$ , where superscript and subscript refer to the option model and type of DTA/DTL respectively.

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<sup>1</sup>See Professor Damodaran's blog at <http://aswathdamodaran.blogspot.nl/2013/06/a-tangled-web-of-values-enterprise.html>

In the model, we always work with *aggregated quantities*, which means that a firm cannot have a DTA and DTL at the same time.<sup>2</sup> Following the Merton (1974) methodology, we assume that the asset value follows a geometric Brownian motion

$$(3.1) \quad dA_t = \mu A_t dt + \sigma A_t dW_t^P, \quad \mu \in \mathbb{R}, \sigma > 0,$$

but only *in between* periods of tax payment. In this formula,  $W_t$  is a standard Brownian motion under  $P$ -measure. By Girsanov's theorem (Theorem A.9) we can switch to  $Q$ -measure, which renders the following dynamics for the asset process

$$(3.2) \quad dA_t = r A_t dt + \sigma A_t dW_t^Q.$$

This reveals that the discounted asset process under  $Q$ -measure is a martingale. Under the Black-Scholes assumptions it also follows that the market is *complete* (Definition A.6) and the risk-neutral measure  $Q$  is unique. That means we take

$$(3.3) \quad V = e^{-rT} \mathbb{E}^Q(\tilde{A}_T | \mathcal{F}_0), \quad \mathcal{F}_t = \sigma(B_s : s \leq t)$$

as the fair price of an unlevered undertaking with or without fiscal history. Saying that  $A_t$  follows a geometric Brownian motion in between periods of tax payments creates path dependency. In particular,  $\tilde{A}_T$  is calculated inductively. Given  $A_0$ ,  $A_1$  (before tax) is calculated by  $A_1 = A_0 \exp(r - \sigma^2/2 + \sigma W_1)$ , which is the SDE solution to (3.2) for  $t = 1$ . Depending on the availability of deferred taxes,  $\tilde{A}_1$  is calculated according to one of the four formulas (1.1), (1.2), (1.3) or (1.4). The asset process restarts at  $\tilde{A}_1$ , so that  $A_2 = \tilde{A}_1 \exp(r - \sigma^2/2 + \sigma(W_2 - W_1))$ . In general, given the after-tax value  $\tilde{A}_t$ , the pre-tax asset value is calculated by  $A_{t+1} = \tilde{A}_t \exp(r - \sigma^2/2 + \sigma(W_{t+1} - W_t))$ , where  $W_{t+1} - W_t \sim N(0, 1)$  is the increment of the Brownian motion. Depending on the deferred tax available at time  $t$ , the post-tax asset value is calculated by<sup>3</sup>

$$(3.4) \quad \begin{aligned} \tilde{A}_{t+1} &= A_{t+1} - \tau(A_{t+1} - A_t)^+ \quad (\text{No deferred tax}) \\ \tilde{A}_{t+1} &= A_{t+1} - \tau(A_{t+1} - A_t - CF_{t+1})^+ \quad (\text{Carry forward}) \\ \tilde{A}_{t+1} &= A_{t+1} + \tau CB_{t+1} - \tau(A_{t+1} - A_t + CB_{t+1})^+ \quad (\text{Carry back}). \end{aligned}$$

After taxes have been levied,  $CF_t$  (resp.  $CB_t$ ) is increased if the firm incurs a loss (resp. makes profit) and is reduced in case of profit (resp. loss). The total increase or reduction is equal to the loss/profit in the respective period. If profit exceeds the outstanding carry forward in a specific year, the entire DTA arising from carry forward is settled and carry back is created over

<sup>2</sup>Basically, we look at the net DTA; if a firm has both a DTA and DTL, both quantities are aggregated and the resultant is either a DTA or DTL which will be used in our analysis.

<sup>3</sup>Dividend policies are not taken into account in this model.

the remaining taxable income. The same holds for carry back if the loss in a certain year  $t$  exceeds the outstanding  $CB_t$ . This process is repeated until time  $t = T$ , after which the undertaking liquidates all the assets and ceases to exist. Equation (3.3) then yields the firm value at time  $t = 0$ . A typical realization of the asset process with and without taxes is shown in Figure 3.1. We refer to the lifespan  $T$  of the undertaking as the  $T$ -period model. During the calculation of the asset paths, we always assume that an undertaking settles the deferred tax asset in *chronological order*, meaning that the deferred tax created first is settled first (first in first out). The following example shows why this assumption matters.

**Example 3.2.** Suppose an undertaking has  $CF_1 = 10$  and creates additional carry forward in year one (say  $A_1 - A_0 = -10$ ) so that  $CF_2 = 20$ . If the undertaking makes a profit in period two equal to 15, then it first uses all carry forward from period zero (= 10) and then uses 5 carry forward created in year one to offset all taxable income. The remaining carry forward equals  $CF_3 = 20 - 15 = 5$ . This is important when the settlement term of carry forward is limited. If one could carry forward losses for only two years, and the firm would settle the carry forward created in period one first, the remaining carry forward in period three equals zero, since the 5 carry forward from period zero would expire worthless due to the two year settlement constraint.

The time losses can be carried forward (or profits carried back) is referred to as the *settlement term* or *duration time* of the DTA. There are no guidelines on the settlement of deferred tax liabilities over longer periods of time, as the DTL is rather an informative accounting measure not abide to tax regulations. Therefore, we examine two possibilities leading to the asset value of a firm having a DTL in the  $T$ -period model:

- (i) **Base case:** The undertaking pays no taxes until it liquidates the assets in period  $T$ . The asset value for time  $t < T$  is simply

$$A_0 \exp((r - \sigma^2/2)t + \sigma W_t),$$

and the post-tax asset value at time  $T$  equals

$$\tilde{A}_T = A_T - \tau(A_T - A_0 + \text{gain}_1)^+.$$

- (ii) **More realistic case:** In most cases, the undertaking can decide themselves when to settle the DTL because it depends on untaxed profit. Tax must be paid as soon as the profit is realized, but the timing of realization is at the mercy of the firm. One could think of realizing fiscal

profits underlying the DTL by selling stock.<sup>4</sup> In the  $T$ -period model, we assume that the firm realizes the profit at the end of year  $T$ . In intermediate years  $t < T$ , the untaxed profit  $gain_t$  is reduced when the firm incurs a loss (deduction equal to the loss). In case of profit, taxes are levied, the DTL is unaltered and tax expenses can be carried back prior to maturity. In all European member states where carry back is allowed, it is restrained to a one year duration (see ??). Therefore, when a firm incurs a loss and has tax carry back, we assume that carry back is used first (instead of reducing the DTL). The DTL is reduced accordingly in case losses exceed the tax carry back. When, due to losses, the untaxed profit  $gain_t$  has disappeared completely, carry forward can be created as well. Whenever the loss in a certain year exceeds the outstanding  $gain_t$ , the tax liability disappears and carry forward is created over the remaining loss. In summary, if  $gain_t > 0$  for  $t < T$ , the post-tax asset value is calculated by

$$(3.5) \quad \tilde{A}_t = \begin{cases} A_t - \tau(A_t - A_{t-1})^+, & \text{if } CB_t = 0 \\ A_t + \tau CB_t - \tau(A_t - A_{t-1} + CB_t)^+, & \text{otherwise.} \end{cases}$$

$$(3.6) \quad gain_{t+1} = \begin{cases} \max(gain_t - \max(A_{t-1} - A_t, 0), 0), & \text{if } CB_t = 0 \\ \max(gain_t - \max(A_{t-1} - A_t - CB_t, 0), 0), & \text{otherwise.} \end{cases}$$

Equation (3.6) is used to ensure that  $gain_t$  never turns negative and is only reduced in case of a loss, with the precise amount depending on the availability of carry back. Otherwise, if  $gain_t = 0$ , the post-tax asset values are calculated according to (3.4). If  $gain_T$  is still positive at the end of final period  $T$ , the remaining untaxed profit is realized and the post-tax asset value follows from

$$\tilde{A}_T = A_T - \tau(A_T - A_{T-1} + gain_T)^+.$$

For future reference, we recall the explicit expression of the Black-Scholes formula

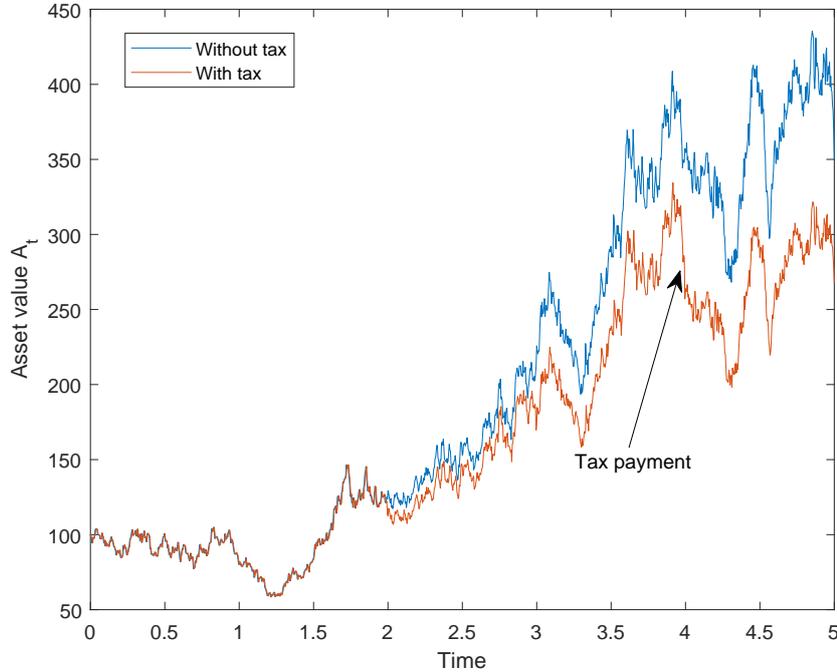
$$(3.7) \quad C^{BS}(K, T, A_t, \sigma, r, t) = A_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad \text{where}$$

$$d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \log \left( \frac{A_t}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T-t) \right]$$

$$d_2 = d_1 - \sigma \sqrt{T-t}.$$

In this formula  $A_t$  is the starting value of the undertaking,  $K$  is the strike price,  $T$  is the exercise date,  $\sigma$  is the volatility of the assets,  $r$  is the risk-free interest rate and  $\Phi(\cdot)$  is the cumulative distribution function of the standard

<sup>4</sup>The settlement term of DTL's is not always at the mercy of an undertaking, e.g. when untaxed profit is realized on fixed contracts with some maturity date  $T$ . For reasons of parsimony, this case is ignored.



**Figure 3.1.** Typical asset path of geometric Brownian motion with and without tax (no deferred taxes). The asset path with tax jumps down if at the end of year  $t$  profit is made. The geometric Brownian motion restarts from the point  $\tilde{A}_t$ , whereas the asset path without tax just continues from  $A_t$ .

normal. In Chapter 4, we interpret the strike  $K$  in (3.7) as the threshold from which an undertaking pays taxes. For example, in the 1-period model without fiscal history (see (1.1)),  $K = A_0$ , since an undertaking pays taxes whenever  $A_1 > A_0$ . As such, the quantity  $\log(A_t/K)$  for  $0 < t \leq 1$  in (3.7) represents the return on the assets when substituting  $K = A_0$ . Similarly, if an undertaking has carry forward, (1.2) reveals that the strike will be  $K = A_0 + CF_1$ .

In addition, throughout the rest of the thesis we make some important conventions. The following parameters are **fixed**:  $\sigma, t, r, A_0$ . Because we are only interested in the value of deferred taxes now we always take  $t = 0$ . Moreover, we assume the Black-Scholes assumptions described above to hold for the asset and the underlying market. To emphasize the dependency on two varying parameters  $K$  and  $T$ , we write  $C^{\text{BS}}(K, T) \triangleq C^{\text{BS}}(K, T, A_t, \sigma, r, t)$ .

Similarly,  $V^{\text{BS}}, V_{\text{cb}}^{\text{BS}}, V_{\text{cf}}^{\text{BS}}, V_{\text{L}}^{\text{BS}}$  denote the value of a firm without fiscal history, with carry back, with carry forward and with a DTL respectively under the Black-Scholes assumptions. In this case, there is no subscript to explicate the  $T$ -period model, as this can always be understood from the context.

### 3.2. Modeling levered firms

The value of a levered undertaking depends on the additional parameter debt ( $D$ ).<sup>5</sup> Coupon payments ( $\mathcal{C}$ ) are modeled as a function of debt. In analogy to Black and Cox (1976), we assume that the firm issues a single homogeneous type of debt with face value  $D$  which it promises to repay at future time  $t = T$  in the  $T$ -period model. In addition, the firm makes yearly coupon payments  $\mathcal{C}$ , which for simplicity occurs at the same time tax payments are due (this is similar to Brennan and Schwartz (1978)). Bondholders are endowed with *safety covenants*, which means they have the right to force the firm into bankruptcy and take hold of the assets whenever the post-tax asset value drops below some critical threshold  $K$ .<sup>6</sup> Debt holders can execute the safety covenants at discrete time points, corresponding to the moment taxes are levied at the end of each year. In particular, this means that bankruptcy is triggered after coupon payments are made. At maturity, when debt payment  $D$  is due, a firm can also go bankrupt if the post-tax asset value is less than the face value of outstanding debt.<sup>7</sup> When a firm goes bankrupt, a fraction  $\alpha$  of the remaining asset value is lost due to bankruptcy cost. This time, post-tax asset values are calculated by

$$(3.8) \quad \begin{aligned} \tilde{A}_{t+1} &= A_{T+1} - \mathcal{C} - \tau(A_{t+1} - A_t - \gamma\mathcal{C})^+ \text{ (No deferred tax)} \\ \tilde{A}_{t+1} &= A_{T+1} - \mathcal{C}_{\text{cf}} - \tau(A_{t+1} - A_t - CF_{t+1} - \gamma\mathcal{C}_{\text{cf}})^+ \text{ (Carry forward)} \\ \tilde{A}_{t+1} &= A_{T+1} + \tau CB_{t+1} - \mathcal{C}_{\text{cb}} - \tau(A_{t+1} - A_t + CB_{t+1} - \gamma\mathcal{C}_{\text{cb}})^+ \text{ (Carry back)}, \end{aligned}$$

as long as the firm is solvent, otherwise the undertaking is liquidated prior to maturity and the DTA/DTL value equals the difference in firm value at the point of bankruptcy. In these equations, we tacitly assume the following order

- (i) First, the coupon is paid to creditors and subtracted from the assets, since this money flows out of the company.
- (ii) Secondly,  $\gamma\mathcal{C}$  is subtracted from taxable income, which creates the so-called *tax shield*.

<sup>5</sup>As in the unlevered model, dividend policies are not taken into account.

<sup>6</sup>This should not be confused with the strike of a call option in (3.7).

<sup>7</sup>Hence, the bankruptcy trigger at maturity is not  $K$  but  $D$ .

- (iii) Finally, taxes are levied over the remaining profit  $(A_{t+1} - A_t - \gamma C)^+$ , or any of the last terms in the second and third line of (3.8), depending on the available deferred taxes.

This set-up creates a trade-off between a loss of potential, arising from coupon payments resulting in lower asset values and the gain of shielding part of taxable income. We investigate these two opposing forces in Section 4.6.

**Remark 3.3.** In the literature, one often makes the assumption that the stochastic process of the underlying assets (3.2) is *unaffected* by the financial structure of a firm (Brennan and Schwartz (1978), Leland (1994) and Leland and Toft (1996)). Consequentially, cash outflows associated with the choice of leverage, like coupon payments, must be financed by selling additional equity. However, it is hard to believe that equity value is unaffected by the sale of stock. Therefore we depart from this assumption, which has repercussions for the valuation of levered firms in Chapter 4. Adapting the assumption of stochastic invariance to the financial structure would *eliminate* the loss of potential due to debt financing.

The different coupon notations  $C, C_{cf}, C_{cb}$  will become clear in Chapter 4. For now it suffices to say that undertakings with DTA's at period  $t = 0$  are more likely to repay debt and coupon. The reason for this is the *absolute priority rule*, which means that claims by the tax authority rank above those by general creditors and equity holders (Brouwer, 2006).<sup>8</sup> To illustrate this mechanism, we consider an undertaking in the 2-period model.

**Example 3.4.** Suppose a firm has the following characteristics:  $A_0 = 100, D = 95, C = 20, CF_1 = 10$  and tax rate  $\tau = 0.5$ . Suppose  $A_1 - C = 105$ ; the undertaking can use carry forward to offset all tax expenses in period one and is left with  $CF_2 = 10 - 5 = 5$ . However, a reference undertaking without carry forward pays  $\tau \cdot 5$  and is left with asset value  $\tilde{A}_1 = 102.5$ , creating  $CF_2 = 5$ . If the asset before tax and coupon payments increase with 11%,  $A_2 = 1.11 \cdot 105 = 116.78$  and after giving effect to concomitant coupon payments, the firm is left with  $\tilde{A}_2 = 96.78$ , which is just enough to repay debt. The assets of the reference undertaking before tax and coupon rise to  $1.11 \cdot 102.5 = 114$ . The post-coupon (and tax) asset value equals  $\tilde{A}_2 = 94$ , which is not enough to repay the entire debt and triggers bankruptcy. Hence, undertakings starting with carry forward are more likely to avoid bankruptcy than a reference undertaking without tax benefits. The same holds for firms having carry back.

<sup>8</sup>In reality there are many more priority rules, depending on the specific type of claim on the company, (see <https://taxmap.irs.gov/taxmap/pubs/p908-006.htm>). Since our main concern is it to analyze the tax effect on coupon payments, we ignore the myriad of different priority rules, as this would render the model infeasible to work with.

Following [Leland and Toft \(1996\)](#), we assume that the coupon is set such that debt sells at *par value*, i.e. debt at time zero ( $D_0$ ) equals the face value of debt at maturity ( $D$ ). In [Chapter 4](#), we outline how to find the coupon payments using numerical methods. In addition, we assume that  $D_0 \leq A_0$ , i.e. a firm does not have negative equity value at  $t = 0$ . The risk-neutral pricing approach yields the firm value at time  $T$  when bankruptcy is avoided

$$(3.9) \quad \begin{aligned} e^{-rT} \mathbb{E}^Q \left( \tilde{A}_T + \sum_{i=0}^{T-1} e^{ri} \mathcal{C} \right) & \quad (\text{Undertaking without DTA}) \\ e^{-rT} \mathbb{E}^Q \left( \tilde{A}_T + \sum_{i=0}^{T-1} e^{ri} \mathcal{C}_{cf} \right) & \quad (\text{Undertaking with carry forward}) \\ e^{-rT} \mathbb{E}^Q \left( \tilde{A}_T + \sum_{i=0}^{T-1} e^{ri} \mathcal{C}_{cb} \right) & \quad (\text{Undertaking with carry back}), \end{aligned}$$

where  $\tilde{A}_t$  is calculated according to equations [\(3.8\)](#) at each time instant. In case bankruptcy occurs at time  $t$  at or prior to maturity, firm values are calculated by

$$\begin{aligned} e^{-rt} \mathbb{E}^Q \left( (1 - \alpha) \tilde{A}_t + \sum_{i=0}^{t-1} e^{ri} \mathcal{C} \right) & \quad (\text{Undertaking without DTA}) \\ e^{-rt} \mathbb{E}^Q \left( (1 - \alpha) \tilde{A}_t + \sum_{i=0}^{t-1} e^{ri} \mathcal{C}_{cf} \right) & \quad (\text{Undertaking with carry forward}) \\ e^{-rt} \mathbb{E}^Q \left( (1 - \alpha) \tilde{A}_t + \sum_{i=0}^{t-1} e^{ri} \mathcal{C}_{cb} \right) & \quad (\text{Undertaking with carry back}), \end{aligned}$$

In the one-year model, this means that firms can only go bankrupt if they incur a loss in period one. As such, taxes have no influence on the bankruptcy condition for undertakings without deferred taxes nor for firms with carry forward (when  $\gamma = 1$ ). For these firms,  $\tilde{A}_1$  can be replaced with  $A_1$  in [\(3.8\)](#), as taxes only play a role once you make a profit in period one. In contrast, a firm with carry back has the possibility of a tax refund when incurring a loss in period one and this might be enough to avoid bankruptcy. Similarly, taxes play a role for undertakings with a DTL, since tax obligations persist even in case of a loss (as measured by  $A_1 - A_0$ ). This suggests that firms with a DTL ought to pay a higher coupon than firms without tax liabilities.

# Results

## 4.1. One-period model unlevered firms

Since we are dealing with a one year time horizon, we fix the parameter  $T$  (so  $T = 1$ ). Hence, for ease of notation, we write  $C^{\text{BS}}(K) \triangleq C^{\text{BS}}(K, T, A_t, \sigma, r, t)$  whenever we are concerned with the one-period model.

**4.1.1. Carry forward.** Let us now turn to the original quest of determining a market consistent valuation of deferred taxes. First, we take a firm without fiscal history, whose asset value after tax at time one is given by (1.1). This is a contingent  $T$ -claim (Definition A.2), whose value at time zero is given by

$$(4.1) \quad V^{\text{BS}} = e^{-r} \mathbb{E}^Q(A_1 - \tau(A_1 - A_0)^+ | \mathcal{F}_0) = A_0 - \tau C^{\text{BS}}(K = A_0),$$

where  $C^{\text{BS}}$  is the Black-Scholes price of a European at-the-money call option. In (4.1) we have  $K = A_0$ , so that by virtue of (3.7) we get  $d_1 = \frac{1}{\sigma}(r + \frac{\sigma^2}{2})$  and similarly  $d_2 = \frac{1}{\sigma}(r - \frac{\sigma^2}{2})$ . It is well known that the Black-Scholes call option value is greater than or equal to the payoff received at expiry. Hence,  $V^{\text{BS}}$  is always less than the actual asset value minus tax at expiry in (1.1).

Similar analysis allows us to find the market consistent value of a company with carry forward. Again, by martingale pricing, the no-arbitrage value of a company with carry forward is found by discounting (1.2)

$$(4.2) \quad \begin{aligned} V_{\text{cf}}^{\text{BS}} &= e^{-r} \mathbb{E}^Q(A_1 - \tau(A_1 - A_0 - CF_1)^+ | \mathcal{F}_0) \\ &= A_0 - \tau C^{\text{BS}}(K = A_0 + CF_1). \end{aligned}$$

Notice that  $V_{\text{cf}}^{\text{BS}}$  in (4.2) is always greater than or equal to  $V^{\text{BS}}$  appearing in (4.1). This makes sense, because a company having future tax deduction possibilities should be more valuable than a company that doesn't have these possibilities. By Definition 3.1, the market consistent DTA value of carry forward follows by comparing (4.2) with (4.1), which yields

$$(4.3) \quad \xi_{\text{cf}}^{\text{BS}} \triangleq V_{\text{cf}}^{\text{BS}} - V^{\text{BS}} = \tau \left( C^{\text{BS}}(A_0) - C^{\text{BS}}(A_0 + CF_1) \right) \\ \stackrel{(3.7)}{=} \tau \left[ A_0 \left( \Phi(d_1) - \Phi(d_1^{\text{cf}}) \right) - e^{-r} \left( A_0 \Phi(d_2) - (A_0 + CF_1) \Phi(d_2^{\text{cf}}) \right) \right].$$

Equation (4.3) is monotonically increasing in  $CF_1$  and always bigger than zero, however for large values of  $CF_1$  the additional benefit of extra carry forward is rather limited. This can also be seen from Figure 4.1, which shows the diminishing marginal returns of carry forward for various tax rates. The reason is that expected profits are insufficient to utilize additional carry forward.

It is instructive to analyze the sensitivity of the DTA (or DTL) value with respect to the variable from which the DTA arises (such as  $CF_1$ ). The following proposition facilitates these computations.

**Proposition 4.1.** *For a standard European call option with constant interest rate we have the following expression for the first and second derivative with respect to the call price*

$$\frac{\partial}{\partial K} C(K, T, A_t, \sigma, r, t) = -e^{-r(T-t)}(1 - F(K)) \\ \frac{\partial^2}{\partial K^2} C(K, T, A_t, \sigma, r, t) = e^{-r(T-t)} f(K),$$

where  $F(K)$  and  $f(K)$  are the risk-neutral CDF and PDF of the underlying asset respectively. These expressions are independent of the Black-Scholes model and can be derived irrespective of modeling issues.

**Proof.** The proof essentially follows from (A.3) in the Appendix. Accordingly we can set

$$(4.4) \quad C(K, T, A_t, \sigma, r, t) = e^{-r(T-t)} \mathbb{E}^Q((A_T - K)^+ | \mathcal{F}_t) \\ = e^{-r(T-t)} \int_K^\infty (z - K) f(z) dz.$$

Straightforward differentiation of (4.4) (in conjunction with Leibniz' rule) yields the result. ■

By [Proposition 4.1](#), the derivative of [\(4.3\)](#) is given by

$$(4.5) \quad \frac{\partial}{\partial CF_1} \xi_{\text{cf}}^{\text{BS}} = \tau e^{-r} (1 - F(A_0 + CF_1)).$$

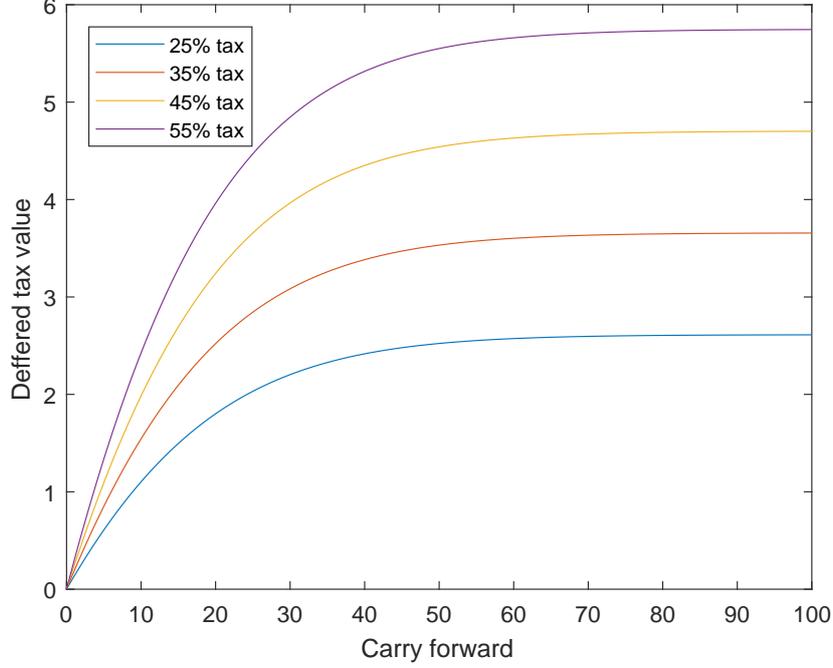
Equation [\(4.5\)](#) leads to an interesting interpretation. Because  $F(x)$  is the risk-neutral probability that assets at time one are less than  $x$ , we can rewrite [\(4.5\)](#) to  $\tau \exp(-r) Q(A_1 > A_0 + CF_1)$ , where  $Q$  is the risk-neutral measure.<sup>1</sup> In other words, the sensitivity w.r.t.  $CF_1$  is equal to the probability of  $\{\omega \in \Omega : A_1(\omega) > A_0 + CF_1\}$  under the risk-neutral measure  $Q$ , weighted by a discount factor consisting of the tax rate  $\tau$  and the risk-free rate. The event  $\{\omega \in \Omega : A_1(\omega) > A_0 + CF_1\}$  corresponds to the risk-neutral probability that the entire carry forward will be used. This bears some resemblance to current valuation methods, which are discussed in more detail in [Section 4.1.5](#). The variable  $\tau$  works as a kind of amplification factor; higher tax rates increase the sensitivity of the DTA to  $CF_1$  because a change in  $CF_1$  has a more pronounced effect on firm value.

Since distribution functions are always bounded by one (and non decreasing), it follows that [\(4.5\)](#) is always positive. Taking the (formal) limit  $CF_1 \rightarrow \infty$  renders that [\(4.5\)](#) goes to zero. Initially, if  $CF_1$  is small, there is a high probability that the entire carry forward can be used for tax deduction, so a small change leads to a relatively big change in DTA value. On the contrary, if  $CF_1$  is high, it is not likely that the entire carry forward will be used for tax deduction (since future profits are unlikely to settle the complete carry forward), so that a change in  $CF_1$  does not have a considerable effect on the DTA value. [Figure 4.2](#) illustrates the behavior of [\(4.5\)](#) for different tax rates.

[Figure 4.3](#) shows the value of  $CF_1$  according to [\(4.3\)](#) together with the value of  $CF_1$  at maturity. Initially the market consistent value of carry forward is worth more than the final value at maturity. This is because there still exists a probability that some of the carry forward will be used. At some point, however, the upward potential is not enough to offset the guaranteed carry forward value at maturity, causing the graphs to intersect so that the market consistent value is worth less than the guaranteed payoff for large values of  $A_1$ . The structure of the payoff has the same form as that of a *bull call spread*, which corresponds to the option trading strategy used by investors to profit from the limited rise of an underlying security.

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<sup>1</sup>In fact, we know that in the Black-Scholes model  $F(x)$  is the CDF of the Log-normal distribution.



**Figure 4.1.** DTA value given from carry forward (4.3) as function of carry forward ( $CF_1$ ) for different tax rates. Parameters:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ .

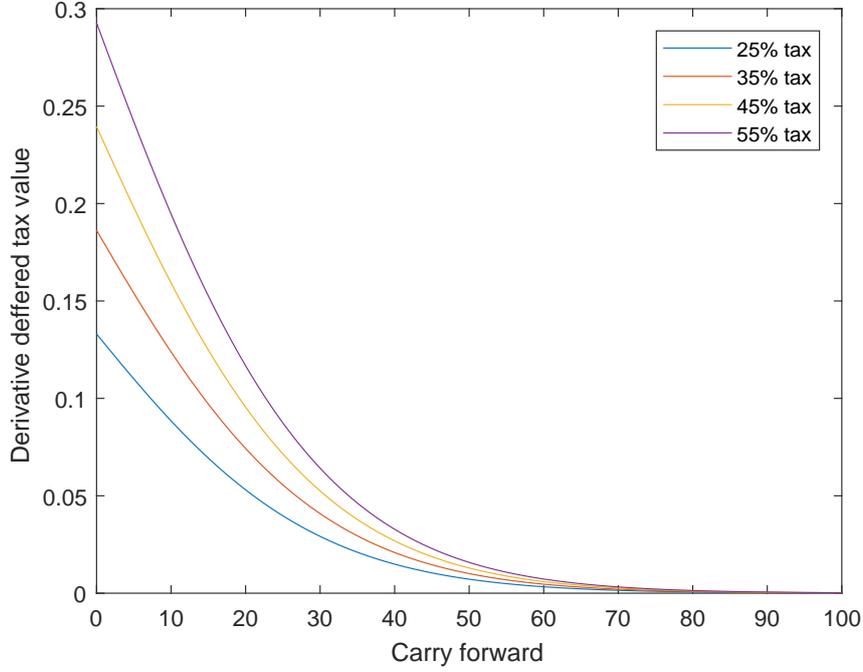
**4.1.2. Carry back.** The market consistent value of a company with carry back follows by discounting (1.3), which gives

$$\begin{aligned}
 V_{cb}^{BS} &= e^{-r} \mathbb{E}^Q(A_1 + \tau CB_1 - \tau(A_1 - A_0 + CB_1)^+ | \mathcal{F}_0) \\
 (4.6) \quad &= A_0 + e^{-r} \tau CB_1 - \tau C^{BS}(K = A_0 - CB_1).
 \end{aligned}$$

In a similar vein, we obtain the market consistent DTA value of carry back by comparing the difference (4.6) and (4.1)

$$\begin{aligned}
 (4.7) \quad \xi_{cb}^{BS} &\triangleq V_{cb}^{BS} - V^{BS} = e^{-r} \tau CB_1 - \tau \left( C^{BS}(A_0 - CB_1) - C^{BS}(A_0) \right) \\
 &\stackrel{(3.7)}{=} \tau e^{-r} CB_1 - \tau \left[ A_0 \left( \Phi(d_1^{cb}) - \Phi(d_1) \right) - e^{-r} \left( (A_0 - CB_1) \Phi(d_2^{cb}) - A_0 \Phi(d_2) \right) \right].
 \end{aligned}$$

Equation (4.7) expresses the DTA value of carry back as a linear combination of two factors. The first one corresponds to the value of carry back today if no settlement risk were involved. However, because it is not guaranteed that the (entire) carry back will be materialized at period one, the second term is subtracted to take this risk into account. The sensitivity of carry



**Figure 4.2.** Derivative of DTA from carry forward (4.5) as a function of carry forward ( $CF_1$ ) for different tax rates. Parameters:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ .

back to its DTA value is expressed by

$$(4.8) \quad \frac{\partial}{\partial CB_1} \xi_{cb}^{BS} = \tau e^{-r} - \tau \left[ \frac{A_0 \Phi'(d_1^{cb})}{\sigma(A_0 - CB_1)} + e^{-r} \left( \Phi(d_2^{cb}) - \frac{\Phi'(d_2^{cb})}{\sigma} \right) \right].$$

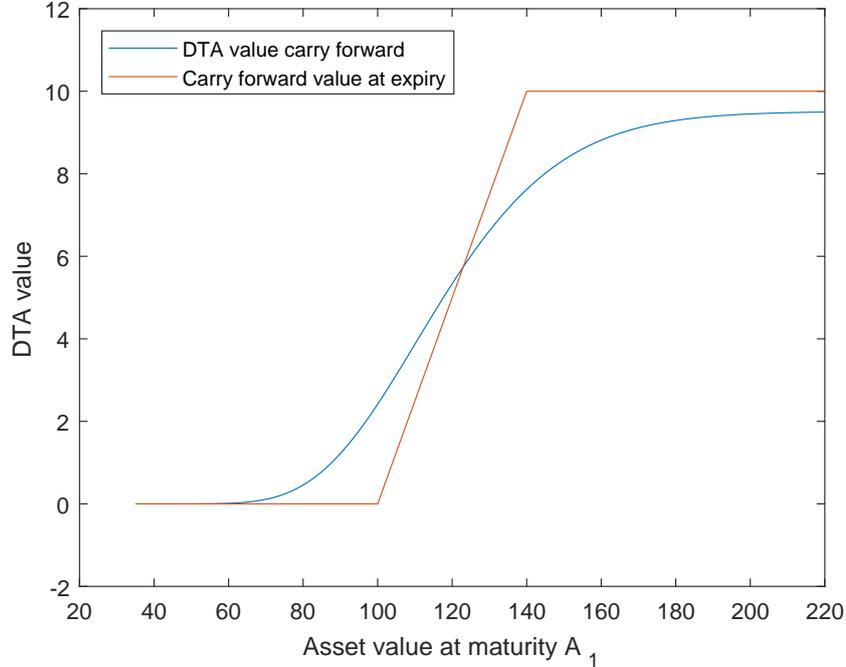
Once more we appeal to [Proposition 4.1](#), which gives the alternative representation of (4.8), namely

$$(4.9) \quad \begin{aligned} \frac{\partial}{\partial CB_1} \xi_{cb}^{BS} &= \tau e^{-r} \left( 1 - (1 - F(A_0 - CB_1)) \right) \\ &= \tau e^{-r} F(A_0 - CB_1). \end{aligned}$$

The factor after the tax rate is the probability that  $A_1$  exceeds the asset value at time zero minus the carry back under the risk neutral measure  $Q$ , i.e.  $Q(A_1 < A_0 - CB_1)$ . So alternatively we may write<sup>2</sup>

$$\frac{\partial}{\partial CB_1} \xi_{cb}^{BS} = \tau e^{-r} Q(A_1 < A_0 - CB_1).$$

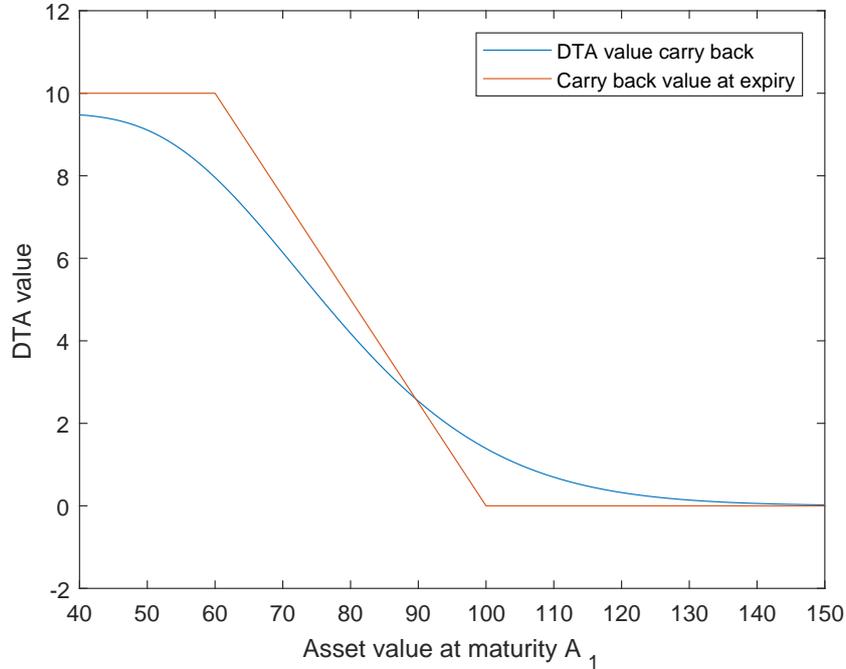
<sup>2</sup>The event  $Q(A_1 < A_0 - CB_1)$  can be written down in explicit terms. Let  $X \sim N(0, 1)$ , then  $Q(A_1 < A_0 - CB_1) = P(A_0 \exp(r - \sigma^2/2 + \sigma X) < A_0 - CB_1) = P(X \leq \frac{1}{\sigma} (\log(\frac{A_0 - CB_1}{A_0}) + \sigma^2/2 - r)) = \Phi(\frac{1}{\sigma} (\log(\frac{A_0 - CB_1}{A_0}) + \sigma^2/2 - r))$ .



**Figure 4.3.** Value of  $CF_1$  under Black-Scholes vs. payoff at maturity. Parameters:  $r = 0.05$ ,  $\sigma = 0.2$ ,  $A_0 = 100$ ,  $CF_1 = 40$ ,  $\tau = 0.25$ .

The probability of the event  $\{\omega \in \Omega : A_1 < A_0 - CB_1\} = \{\omega \in \Omega : A_0 - A_1 \geq CB_1\}$  is the probability (under risk-neutrality) that the loss in period one is sufficient to materialize the complete carry back. Also, from here it follows that (4.8) is always greater than zero and decreasing in  $CB_1$  until  $CB_1 \leq A_0$  after which it vanishes since  $\{\omega \in \Omega : A_1(\omega) \leq 0\}$  has (risk-neutral) probability measure zero. The latter observation holds since we always have the constraint  $CB_1 \leq A_0$ . If this condition is not satisfied, the asset value prior to time  $t = 0$  (say  $t = -1$ ) would be less than zero, i.e.  $A_{-1} < 0$ . This cannot happen with probability one since the asset value is always bigger than zero by definition. Figure 4.4 shows the value of  $CB_1$  in the market consistent model together with its value at expiry. In contrast to carry forward, the DTA value (under the Black-Scholes model) is less than the payoff at maturity when  $A_1$  is small. The “payoff” structure of carry back is similar to the *bear spread strategy*, which is used by option traders to profit from the limited decrease of an underlying security.

**4.1.3. The trade-off between carry forward and carry back.** Having established the market consistent DTA value of carry forward (4.3) and carry back (4.7), we investigate which deferred tax asset is more valuable



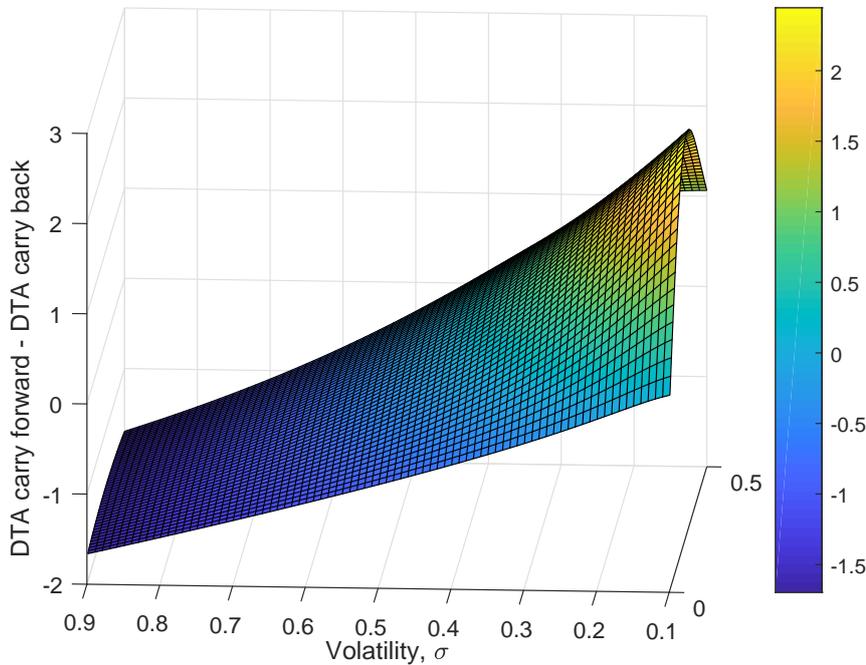
**Figure 4.4.** Value of  $CB_1$  under Black-Scholes vs. payoff at maturity. Parameters:  $r = 0.05$ ,  $\sigma = 0.2$ ,  $A_0 = 100$ ,  $CB_1 = 40$ .

and under what circumstances. To clarify this, we set  $CF_1 = CB_1 = 20$  and consider the difference between (4.3) and (4.7). Figure 4.5 shows the difference as a function of  $\sigma$  and  $r$ . For fixed values of  $r$ , the function in Figure 4.5 is monotonically decreasing in  $\sigma$ . Hence, we conclude that carry back is *more* valuable for firms with high volatility relative to the interest rate  $r$ . The precise extent to which the valuation difference depends on the interplay between  $r$  and  $\sigma$  remains unclear, but some insight is gained by looking at the asset process before taxes. The assumption on the asset process (3.2) implies that the probability of a loss in year one equals

$$P(A_1 < A_0) = P\left(X \leq \frac{1}{\sigma}\left(\frac{\sigma^2}{2} - r\right)\right) \quad X \sim N(0, 1),$$

so that the sign of  $\sigma^2/2 - r$  provides a link between the valuation difference of carry back and carry forward. Carry back is only beneficial in period one when the firm incurs a loss. The opposite holds for carry forward. From this one might be tempted to hypothesize that the DTA arising from carry back is more valuable than the DTA coming from carry forward if  $\sigma^2/2 > r$ . However, this is not entirely true, due to the skewness of the (risk-neutral) lognormal distribution. Even though the probability that you lose or gain

is equal, the probability that you gain ten is bigger than the probability that you lose ten.<sup>3</sup> This makes the call option arising from carry forward more valuable, so that equality between (4.3) and (4.7) is achieved for higher values of  $\sigma$  than what one would obtain by solving  $\sigma^2/2 = r$ . Even though the analytical threshold that determines the equality between carry back and carry forward remains out of reach, Figure 4.5 and extensive simulation suggest that high volatility relative to  $r$  implies that carry back is more valuable than carry forward.



**Figure 4.5.** The difference between (4.3) and (4.7) as a function of  $r$  and  $\sigma$ , for the fixed deferred tax values  $CF_1 = CB_1 = 20$ . The domain  $r \times \sigma = \{(r, \sigma) : r \in [0.01, 0.5], \sigma \in [0.1, 0.9]\}$  and  $A_0 = 100, \tau = 0.25$ .

**4.1.4. DTL temporary differences.** Finally, the firm value of an undertaking with a deferred tax liability follows by discounting (1.4) under

<sup>3</sup>Assume that  $\sigma^2/2 = r$ , and let  $c > 0$ . The probability of losing  $c$  equals  $P(A_1 - A_0 < -c) = \Phi(\frac{1}{\sigma}(\log \frac{A_0 - c}{A_0}))$ . In contrast, the probability of gaining  $c$  is  $P(A_1 - A_0 > c) = \Phi(-\frac{1}{\sigma} \log(\frac{A_0 + c}{A_0}))$ . The latter probability is *always* bigger than the probability of losing  $c$ , due to the properties of the  $\log(\cdot)$  function.

martingale measure, which yields

$$(4.10) \quad \begin{aligned} V_L^{\text{BS}} &= e^{-r} \mathbb{E}(A_1 - \tau(A_1 - A_0 + \text{gain}_1)^+ | \mathcal{F}_0) \\ &= A_0 - \tau C^{\text{BS}}(A_0 - \text{gain}_1). \end{aligned}$$

We stick to the convention of modeling the difference between normal tax conditions and a DTL by a negative quantity. Thus the market consistent DTL value is obtained by subtracting (4.10) from (4.1), which gives

$$(4.11) \quad \xi_L^{\text{BS}} \triangleq V_L^{\text{BS}} - V^{\text{BS}} = \tau \left( C^{\text{BS}}(A_0) - C^{\text{BS}}(A_0 - \text{gain}_1) \right).$$

The shape of (4.11) as a function of  $A_1$  is similar to that of carry back, which can be seen from Figure 4.6. However, this time the values are negative, since the DTL is a tax obligation and hence a future liability. For large values of  $A_1$ , the untaxed profit (as measured by  $\text{gain}_1$ ) has to be paid in its entirety to the tax authority, by the amount of  $\tau \cdot \text{gain}_1$ .

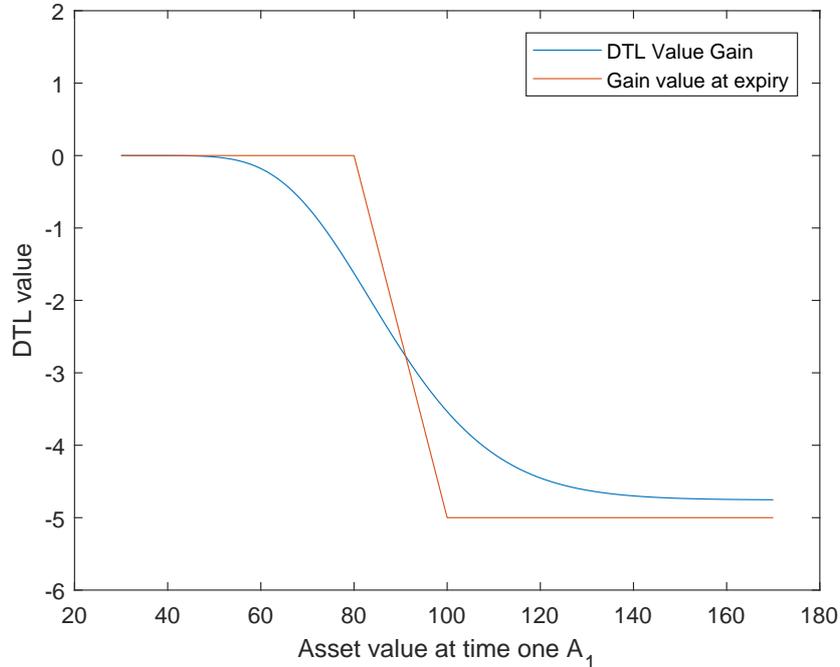
At last we turn to the sensitivity of the DTL to the untaxed profit  $\text{gain}_1$ . Straightforward differentiation of (4.11) in conjunction with Proposition 4.1 gives

$$(4.12) \quad \frac{\partial}{\partial \text{gain}_1} \xi_L^{\text{BS}} = -\tau e^{-r} (1 - F(A_0 - \text{gain}_1)) = -\tau e^{-r} Q(A_1 > A_0 - \text{gain}_1).$$

Hence, higher values of  $\text{gain}_1$  increase the likelihood of  $\{\omega \in \Omega : A_1(\omega) > A_0 - \text{gain}_1\}$ , which means that (4.12) is decreasing in  $\text{gain}_1$ , but stabilizes when  $\text{gain}_1 \geq A_0$ . The latter condition is excluded, since  $\text{gain}_1 < A_0$  by construction.<sup>4</sup> A company starts to lose value from the moment  $A_1 > A_0 - \text{gain}_1$ , so higher values of  $\text{gain}_1$  lead to a more severe reduction in asset value compared to a firm with lower  $\text{gain}_1$ . Hence, the slope of (4.11) ought to be decreasing.

**4.1.5. Comparison to current valuation methods.** Having the precise formulas for the DTA's/DTL at hand, we can now analyze the difference between current valuation methods and our market consistent approach. In Section 1.1 we explained that extant valuation procedures acknowledge the underlying value of the different type of DTA/DTL as the nominal amount appearing on the balance sheet. We compare our method to two frequently employed accounting principles: GAAP and IAS12. Most other accounting guidelines use techniques for valuing deferred taxes by methods either based on GAAP or IAS12. However, both of these guidelines neglect carry back possibilities and the value creation as a result of this possibility. Hence, we

<sup>4</sup>Remember that  $\text{gain}_1$  is the untaxed profit made in period  $t = 0$ , so that  $\text{gain}_1 + A_{-1} = A_0 \implies \text{gain}_1 < A_0$ .



**Figure 4.6.** Value of *DTL* under Black-Scholes vs. payoff at maturity. Parameters:  $r = 0.05$ ,  $\sigma = 0.2$ ,  $A_0 = 100$ ,  $gain_1 = 20$ .

only draw comparison to carry forward valuation.

Let us first analyze the way under which GAAP computes the value of carry forward. Under GAAP principles, carry forward is recognized whenever there is a more than 50% chance that future profit settles the complete carry forward. In particular, [The Financial Accounting Standards Board \(1992\)](#) states the following in **Paragraph 98**:

*“The Board acknowledges that future realization of a tax benefit sometimes will be expected for a portion but not all of a deferred tax asset, and that the dividing line between the two portions may be unclear. In those circumstances, application of judgment based on a careful assessment of all available evidence is required to determine the portion of a deferred tax asset for which it is more likely than not a tax benefit will not be realized.”*

In other words, the DTA arising from carry forward is recognized completely if and only if the probability of materializing the entire carry forward has a more than 50% chance. If this is not the case, a valuation allowance (VA) is issued, which reduces the overall DTA value. In our model, this

translates to the condition

$$P(A_1 - A_0 \leq A_1^*) = \frac{1}{2},$$

where  $A_1^*$  is the median profit and  $P(\cdot)$  the physical probability measure. The geometric Brownian motion assumption of the asset process (3.1) renders the explicit expression  $A_1^* = A_0 e^{\mu - \sigma^2/2} - A_0$ . Hence, for carry forward, a valuation allowance is issued whenever

$$(4.13) \quad CF_1 > A_1^* \iff CF_1 > A_0(e^{\mu - \sigma^2/2} - 1).$$

This expression reveals that sufficiently large values of volatility always lead to the issuance of a valuation allowance, i.e. when  $\sigma^2/2 \gg \mu$ . Secondly, it is less likely that a valuation allowance is issued for large values of the starting value  $A_0$  when  $\mu - \sigma^2/2 > 0$ . This is because the geometric Brownian motion assumption on the asset process is concerned with relative profits. So for higher values of  $A_0$ , a small percentage change leads to a more pronounced difference in absolute asset values, which makes it more likely that the complete carry forward will be settled. In analogy to De Waegenare et al. (2003), a valuation allowance (VA) is issued if (4.13) holds and the balance sheet value is given by

$$VA = \left( \tau(CF_1 - A_1^*) \right)^+.$$

The DTA value under the GAAP approach thus takes the form

$$(4.14) \quad \begin{aligned} \xi_{cf,GAAP} &\triangleq DTA - VA \\ &= \tau \left( CF_1 - (CF_1 - A_1^*)^+ \right). \end{aligned}$$

It follows from (4.14) that the DTA value arising from carry forward stabilizes when it hits the threshold level for which it becomes more likely than not to settle the complete carry forward. Interestingly, in Figure 4.7 we see that the GAAP approach is higher for small values of  $CF_1$  but is lower for high values of  $CF_1$ . However, this relation is ambiguous as  $A_1^*$  depends on the growth parameter  $\mu$ . To see this, for large values of  $CF_1$ , the DTA value of carry forward  $\xi_{cf}^{BS}$  in (4.3) goes to  $\tau C^{BS}(A_0)$ . In contrast,  $\xi_{cf,GAAP} \rightarrow \tau A_1^*$  for large values of  $CF_1$ . Therefore, the relation boils down to comparing  $C^{BS}(A_0)$  with  $A_1^*$ . However, the relation between these two quantities is inconclusive because  $A_1^*$  depends on  $\mu$ . For the particular case shown in Figure 4.7,  $\mu$  is chosen small enough such that the market consistent value is eventually higher than the GAAP value. But we might equally well take  $\mu$  so large that the relationship breaks down eventually.<sup>5</sup> On the other hand, it always holds true that the GAAP approach renders higher values for the

<sup>5</sup>In fact,  $\mu = 0.12$  is already sufficient in this example.

DTA when  $CF_1 < A_1^*$ .

Valuation guidelines of deferred taxes under IAS12 are less flexible. According to these accounting principles, deferred taxes are recognized only if there is a more than 50% chance that the complete DTA will be materialized (Deloitte, 2017). Otherwise, the deferred tax asset is not recognized. Hence, we have the following value of the DTA arising from carry forward under IAS12

$$\xi_{cf,IAS12} = \tau 1_{\{CF_1 \leq A_1^*\}} CF_1.$$

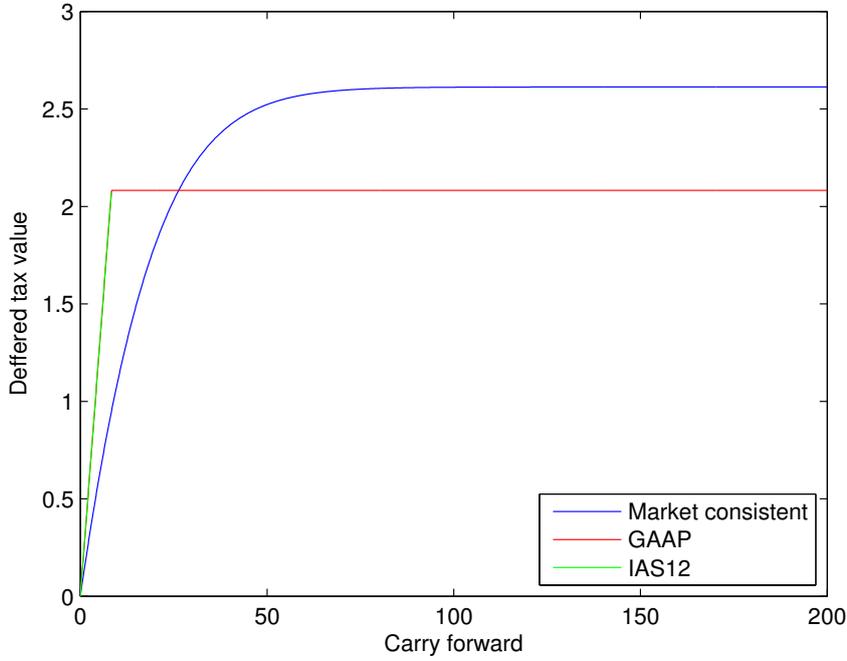
The carry forward value under IAS12 in Figure 4.7 (green line) concurs with the GAAP value, but vanishes as soon as carry forward exceeds the median profit. In some sense, our model contains the GAAP and IAS12 approach as a special case, namely if we take  $\lim_{\sigma \rightarrow 0^+} \xi_{cf}^{BS} = \tau e^{-r} CF_1$ , provided that  $(e^r - 1)A_0 > CF_1$ . This is almost equal to the GAAP and IAS12 value, apart from the discounting term. Our model not only encompasses the GAAP and IAS12 approach as special cases, but is preferred in certain other aspects:

- (i) The market consistent approach does not depend on the subjective substantiation of future profit. In our model, the uncertainty of future profit is implicitly measured by the volatility of the assets  $\sigma$ , which determines the likelihood of materializing the entire carry forward under all future scenarios.
- (ii) If the probability of realizing the entire carry forward is huge, the GAAP and IAS12 approach are not in line with conventional economic theory, which suggests that the nominal carry forward value should at least be discounted to reflect time preferences. However, the precise discounting value is somewhat diffuse and depends on parameters difficult to measure, such as likelihood and timing of the settlement (Givoly and Hayn, 1992). Other researchers even find evidence against discounting of deferred taxes (e.g. Amir et al. (1997)), which can be explained by assuming a skewed income distribution (De Waegenaere et al., 2003).

As in the previous section(s), we analyze the sensitivity of the conventional accounting valuation principles to carry forward. The GAAP and IAS12 valuation methods are not classically differentiable. At least the GAAP approach gives rise to a function that is weakly differentiable, where the weak derivative is given by<sup>6</sup>

$$(4.15) \quad \frac{\partial}{\partial CF_1} \xi_{cf,GAAP} = \begin{cases} \tau, & \text{for } CF_1 \leq A_1^* \\ 0, & \text{else.} \end{cases}$$

<sup>6</sup>Recall that a function  $f$  has weak derivative  $g$  if  $\int_{\mathbb{R}} f(x)\varphi(x)'dx = -\int_{\mathbb{R}} g(x)\varphi(x)dx$  for all  $\varphi \in C_c^\infty(\mathbb{R})$ , which is the space of smooth functions with compact support.



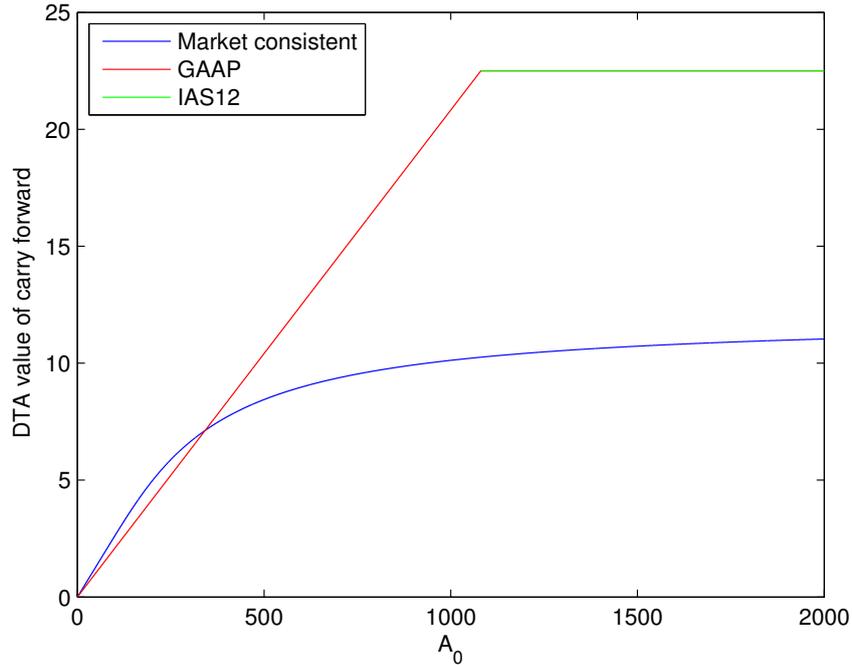
**Figure 4.7.** DTA value of carry forward as a function of  $CF_1$  under different valuation approaches. Parameters:  $A_0 = 100$ ,  $\tau = 0.25$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\mu = 0.1$ .

The sensitivity is larger compared to the market consistent sensitivity in (4.5) at each point on the support of (4.15).

Another illuminating quantity is the dependence on the initial start value  $A_0$ , which is shown in Figure 4.8. Initially, the asset value is so small that materializing the complete carry forward is unlikely. Hence, under GAAP principles, a valuation allowance is issued, which reduces the nominal value of  $CF_1$ . However, at some point, the initial asset value is large enough so that the probability of materializing the complete carry forward is larger than 50%. At this point,  $\xi_{cf,GAAP}$  stabilizes and becomes constant. This is also the point where  $\xi_{cf,IAS12}$  gets positive and concurs with IAS12 (green line in Figure 4.8). Both  $\xi_{cf,GAAP}$  and  $\xi_{cf,IAS12}$  converge to  $\tau CF_1$  for large values of  $A_0$ .

**Remark 4.2.** The size of  $A_0$  relative to  $CF_1$  plays an important role in the loss absorbing capacity of deferred taxes. After a negative shock in the asset value, a loss is incurred, which can be used as carry forward. Hence,  $CF_1$  increases, but the asset value decreases. The asset value after shock can be

so low that the market consistent carry forward is actually worth less than it was before, even though the nominal value increased. We analyze this in more detail in [Section 5.1](#).



**Figure 4.8.** DTA value of carry forward as a function of  $A_0$  under different valuation approaches. Parameters:  $\tau = 0.25$ ,  $CF_1 = 90$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\mu = 0.1$ .

## 4.2. Valuation of debt

Subsequent valuation methods of deferred taxes of levered firms requires knowledge of the coupon payment  $\mathcal{C}$ . These are obtained from debt  $D$  with maturity  $T$ . Recall our assumption that bankruptcy can be triggered at discrete times (each year after taxes are levied), when the post-tax asset value  $\tilde{A}_t$  drops below some critical threshold  $K$  prior to maturity, or if  $\tilde{A}_T < D$  at maturity. The time of bankruptcy is thus given by the *stopping time* (see [Definition A.1](#))

$$\tau_D = \inf \left\{ t \in \{1, 2, \dots, T-1\} \mid \tilde{A}_t \leq K \right\},$$

where the stopping time is interpreted as  $T$  whenever the set above is empty. Default at maturity happens when  $\tau_D = T$  and  $\tilde{A}_T < D$ . This framework

thus only considers *time independent* default barriers. When default occurs, the firm loses a fraction  $\alpha \in [0, 1]$  of the asset value  $\tilde{A}_{\tau_D}$  due to bankruptcy cost. Using risk neutral valuation gives the value of the issued debt at time zero

$$(4.16) \quad D_0 = \mathbb{E}^Q \left( e^{-r\tau_D} \sum_{t=0}^{\tau_D-1} e^{rt} \mathcal{C} \right) + (1 - \alpha) \mathbb{E}^Q \left( e^{-r\tau_D} \tilde{A}_{\tau_D} 1_{\tau_D < T} \right) \\ + e^{-rT} \mathbb{E}^Q \left( (D 1_{\tilde{A}_T \geq D} + (1 - \alpha) \tilde{A}_T 1_{\tilde{A}_T < D}) 1_{\tau_D = T} \right).$$

The first term on the right is the discounted payoff of the coupon payments as long as the firm is not bankrupt. The second term is the expected amount that bondholders receive when bankruptcy is triggered prior to maturity multiplied by the recovery rate  $1 - \alpha$ . Taxes and coupon payments are already deducted from the assets in this term. Finally, the last term takes into consideration the value that is left to bondholders at maturity after taxes have been levied, in case bankruptcy is avoided before maturity. However, the firm can still default at maturity when assets after tax and coupon are less than the face value of debt, in which case one recovers the fraction  $1 - \alpha$  of the remaining asset value.<sup>7</sup> We henceforth assume that the default boundary  $K$  is determined exogenously. The next example serves to illustrate the various scenarios that may occur.

**Example 4.3.** Assume that  $A_0 = 100, r = 0.05, \alpha = 0.5, \mathcal{C} = 5$  and  $K = D = 80$  in the 2-period model.

- (i) If the firm loses 50% of the assets in period one, so that  $A_1 = 50$ , then the asset value after tax and coupon equals  $\tilde{A}_1 = 50 - \mathcal{C} - \tau(50 - A_0 - \mathcal{C})^+ = 45$ . Since  $\tilde{A}_1 < K$ , bondholders declare bankruptcy ( $\tau_D = 1$ ) and they receive the remaining asset value  $\alpha \tilde{A}_1 = 22.5$  due to bankruptcy cost. As a result, the total share received by bondholders discounted back to period zero equals  $e^{-r\tau_D}(\mathcal{C} + \alpha \tilde{A}_1) = 26.16$ . Limited liability ensures that equity holders receive nothing.
- (ii) Suppose now that the firm loses 10% in period one and 20% in period two. This implies  $\tilde{A}_1 = 0.9 \cdot A_0 - \mathcal{C} = 85 > K$ , so bondholders do not declare bankruptcy in period one. The asset value after coupon and tax in period two equals  $\tilde{A}_2 = 0.8 \cdot \tilde{A}_1 - \mathcal{C} = 63 < D$ . In this case, the undertaking goes bankrupt in period two ( $\tau_D = 2$ ) and the

<sup>7</sup>Strictly speaking, we should include a fourth term in (4.16) to account for the possibility that the asset value before tax ( $A_{\tau_D}$ ) is less than the coupon payment. In most reasonable calibrations, the probability that the remaining asset value is insufficient to pay the coupon is small and the risk-neutral expectation of this event contributes just little to debt value at time zero. However,  $K$  and  $\mathcal{C}$  could be close in crisis periods. This is because  $\mathcal{C}$  reflects credit spreads, which is a measure of riskiness of a company. Therefore, for highly leveraged companies and in a period of turbulence  $\mathcal{C}$  could be close to  $K$ .

total share received by bondholders discounted back to period zero is  $e^{-r\tau D}(\mathcal{C} + e^r\mathcal{C} + \alpha\tilde{A}_2) = 37.78$ . Again, equity holders receive nothing.

- (iii) Finally, assume that a firm makes 5% profit in period one and two. Hence, asset values are given by  $\tilde{A}_1 = A_0 \cdot 1.05 - \mathcal{C} - \tau(A_0 \cdot 1.05 - A_0 - \mathcal{C})^+ = 100$  and  $\tilde{A}_2 = \tilde{A}_1 \cdot 1.05 - \mathcal{C} - \tau(1.05 \cdot \tilde{A}_1 - \tilde{A}_1 - \mathcal{C})^+ = 100$ . No bankruptcy occurs since  $\tilde{A}_1 > K$  and  $\tilde{A}_2 > D$ . Thus, the total share received by bondholders discounted back is given by  $e^{-2r}(5 + e^r 5 + D) = 81.67$ . Equity holders receive the remaining amount  $\tilde{A}_2 - D = 20$ .

As in [Leland and Toft \(1996\)](#), coupon payments  $\mathcal{C}$  are determined such that debt sells at par value, i.e.  $D = D_0$  in [\(4.16\)](#). The next section presents closed form solutions for such debt payments. For general  $T$ -period models  $T \geq 2$  we have to resort to simulation in order to obtain  $\mathcal{C}$ . The framework thus described operates in between the credit models of [Merton \(1974\)](#) and [Black and Cox \(1976\)](#). The [Merton \(1974\)](#) model allows debt holders to file for bankruptcy when the coupon expires, whereas [Black and Cox \(1976\)](#) permit debt holders to file for bankruptcy at every time point prior to maturity when the assets drop below some critical threshold  $K$ . The first approach leads to interpret debt value as a put option on the assets, whereas the second approach values debt as the difference between two *barrier call options*. Our framework [\(4.16\)](#) could, although not precisely, be interpreted as modeling debt as a *Bermudan barrier call option*. This is because bankruptcy dates are discrete (*Bermudan*) and are exercised whenever the post-tax assets drop below some critical threshold (*barrier*).

For the valuation of DTA's/DTL's we always assume that bankruptcy costs are zero, i.e.  $\alpha = 0$  in [\(4.16\)](#). Later on, when studying optimal capital structures and agency problems, bankruptcy cost play an important role and we will work with  $\alpha > 0$ .

### 4.3. One-period model levered firms

In the one-period model, we are able to find closed-form formulas for debt at time zero, so that  $\mathcal{C}$  can be obtained without simulation. The one year model significantly simplifies [\(4.16\)](#). Namely, bankruptcy cannot be triggered before maturity, so that the stopping time is deterministic ( $\tau_D = 1$ ) and we do not have to specify the exogenous bankruptcy trigger  $K$ . Moreover, bankruptcy cost are assumed to be zero for DTA valuations. Hence, [\(4.16\)](#) becomes

$$(4.17) \quad D_0 = e^{-r}\mathcal{C} + e^{-r}\mathbb{E}^Q(\min(D, \tilde{A}_1)),$$

which is recognized as Merton's debt value (Merton, 1974) with coupon payments. The precise form of  $\tilde{A}_1$  depends on the availability of deferred taxes.

**4.3.1. Carry forward.** Initially, we suppose that  $\gamma = 1$ , so that an undertaking can subtract all interest payments from taxable income. In this case, carry forward has no influence on the bankruptcy condition, since bankruptcy only occurs if the firm incurs a loss. This is because we assume that  $D_0 \leq A_0$ . Hence, by writing  $\min(D, \tilde{A}_1) = D - (D - \tilde{A}_1)^+$ , (4.17) equals

$$\begin{aligned} D_0 &= e^{-r}(\mathcal{C} + D) - e^{-r}\mathbb{E}^Q((D - \tilde{A}_1)^+) \\ &= e^{-r}(\mathcal{C} + D) - e^{-r}\mathbb{E}^Q((D + \mathcal{C} - A_1)^+) \\ (4.18) \quad &= e^{-r}(D + \mathcal{C}) - P^{BS}(D + \mathcal{C}). \end{aligned}$$

The second line follows from  $\tilde{A}_1 < D \iff A_1 - \mathcal{C} < D$ , since no taxes have to be paid in case of a loss. The notation  $P^{BS}(K)$  denotes the Black-Scholes price of a European call option with strike  $K$ . We choose  $\mathcal{C}$  such that  $D_0 = D$ . The solution can be obtained by numerical methods, e.g. Newton-Raphson iteration. The resulting coupon payment is shown in Figure 4.9 as a function of  $D$ . For sufficiently high levels of debt, the coupon  $\mathcal{C}$  is seen to rise exponentially as a consequence of the imminence of bankruptcy.

The more general case corresponding to  $\gamma \in [0, 1]$  yields a more complicated form than (4.18). In particular,  $D_0$  for fixed  $\mathcal{C}$  corresponding to a firm without deferred taxes follows from the next theorem.

**Theorem 4.4.** *The debt value at time zero for an undertaking without deferred taxes is given by*

$$D_0 = e^{-r}(D + \mathcal{C}) - e^{-r}(D + \mathcal{C})\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

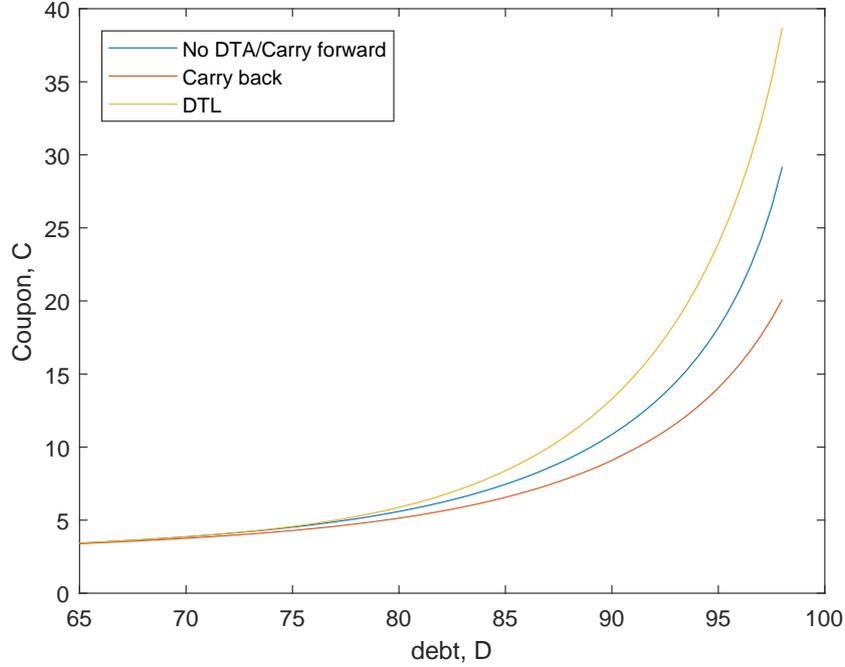
if  $D + \mathcal{C} \leq A_0 + \gamma\mathcal{C}$ . Otherwise

$$\begin{aligned} D_0 &= e^{-r}(D + \mathcal{C}) - e^{-r}(D + \mathcal{C})\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) \\ &\quad - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)] + e^{-r}\tau(A_0 + \gamma\mathcal{C})[\Phi(\theta_2) - \Phi(\theta_1)]. \end{aligned}$$

In these expressions

$$\begin{aligned} \theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 + \gamma\mathcal{C}}{A_0} \right) - r + \sigma^2/2 \right] \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C} - \tau(A_0 + \gamma\mathcal{C})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

**Proof.** See Appendix C.1. ■



**Figure 4.9.** Coupon payment  $C$  on risky debt as a function of debt  $D$  for undertakings with different deferred taxes. Parameters:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\tau = 0.25$ ,  $CB_1 = 10$ ,  $gain_1 = 30$ ,  $\gamma = 1$ .

Likewise, debt at time zero for undertakings with carry forward follows from

**Theorem 4.5.** *The debt value at time zero for an undertaking having  $CF_1$  is given by<sup>8</sup>*

$$D_0 = e^{-r}(D + C_{cf}) - e^{-r}(D + C_{cf})\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

if  $D + C_{cf} \leq A_0 + CF_1 + \gamma C_{cf}$ . Otherwise

$$D_0 = e^{-r}(D + C_{cf}) - e^{-r}(D + C_{cf})\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) \\ - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)] + e^{-r}\tau(A_0 + CF_1 + \gamma C_{cf})[\Phi(\theta_2) - \Phi(\theta_1)].$$

<sup>8</sup>The subscript  $a$  in  $C_a$  refers to the specific type of deferred tax. No subscript indicates an undertaking without deferred taxes.

In these expressions

$$\begin{aligned}\theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 + CF_1 + \gamma C_{cf}}{A_0} \right) - r + \sigma^2/2 \right] \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + C_{cf} - \tau(A_0 + CF_1 + \gamma C_{cf})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + C_{cf}}{A_0} \right) - r + \sigma^2/2 \right].\end{aligned}$$

**Proof.** See [Appendix C.2](#). ■

From [Theorem 4.4](#) and [Theorem 4.5](#), the Newton-Raphson method can be employed to find  $\mathcal{C}$  (or  $\mathcal{C}_{cf}$ ) such that  $D_0 = D$ . In this case, there is a difference between the coupon paid by undertakings without deferred taxes and one that has carry forward. The disparity arises if  $0 \leq \gamma < 1$ . To see this, note that if  $D + \mathcal{C} > A_0$ , a firm can go bankrupt even if it makes a profit in period one. When interest payments are fully deductible ( $\gamma = 1$ ), the firm never has to pay taxes over that profit. However, when  $\gamma < 1$ , some profit will be taxed, and this tax deduction might be enough to trigger the bankruptcy condition.<sup>9</sup>

Having established the coupon  $\mathcal{C}$ , we now turn to the valuation of levered undertakings and their deferred taxes. The value of a levered undertaking without deferred taxes is given by discounting the assets at time one (2.1), plus the coupon payment to creditors

$$(4.19) \quad \mathcal{V}^{BS} = e^{-r} \mathbb{E}(\tilde{A}_1 + \mathcal{C} | \mathcal{F}_0) = A_0 - \tau C^{BS} (K = A_0 + \gamma \mathcal{C}).$$

We henceforth denote the value of levered firms by  $\mathcal{V}_a^{BS}$ , to distinguish it from unlevered firms. Consistent with previous notation, subscript  $a$  depicts the deferred tax available at the starting period. The strike value in (4.19) is higher compared to unlevered firms, since interest deductions lead to a tax advantage. The interest deduction also contains an option component, as it is not certain that the entire interest payment can be deducted from taxable income, e.g. when a firm incurs a loss, so there is no taxable income to offset the interest payment.

The market consistent pricing approach gives the value of a levered firm having some carry forward by discounting (2.3) and adding the coupon payment

$$(4.20) \quad \mathcal{V}_{cf}^{BS} = e^{-r} \mathbb{E}^Q(\tilde{A}_1 + C_{cf} | \mathcal{F}_0) = A_0 - \tau C^{BS} (K = A_0 + CF_1 + \gamma C_{cf}).$$

<sup>9</sup>The absolute priority rule ensures that taxes are levied first, before bondholders can file for bankruptcy.

**Remark 4.6.** Since our approach to valuing DTA's/DTL's has been to compare firm values with an otherwise identical firm, which does not have deferred taxes (Definition 3.1), we must assume that coupon payments for the reference undertaking are the same to avoid circular reasoning. To facilitate subsequent sensitivity calculations, we therefore chose to set  $C_{cf} = C$  when valuing carry forward.

As a result, the DTA value arising from carry forward for levered undertakings is given by the difference (4.20) and (4.19)

$$(4.21) \quad \xi_{cf}^{BS} = \tau \left( C^{BS}(A_0 + \gamma C) - C^{BS}(A_0 + CF_1 + \gamma C) \right).$$

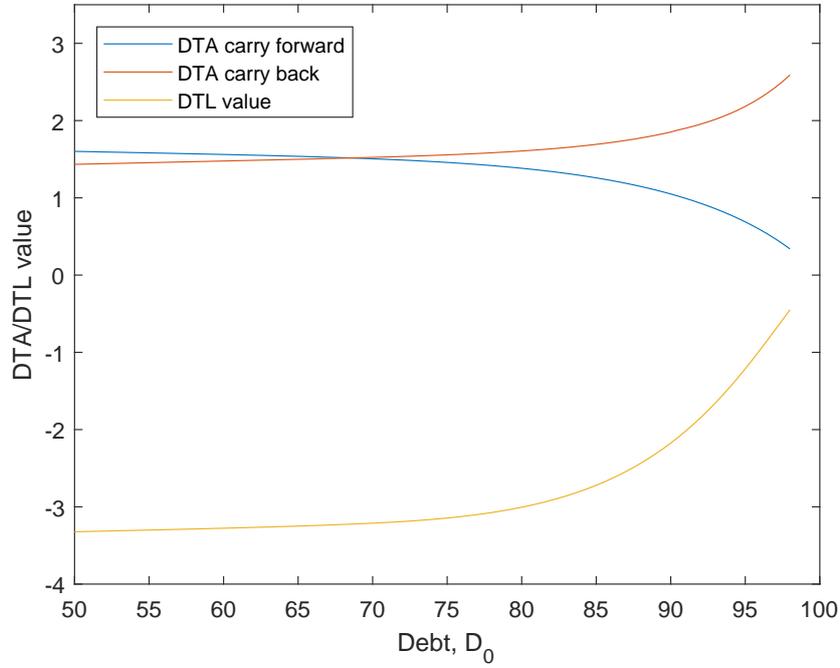
Equation (4.21) contains the unlevered DTA value of carry forward (4.3) as a special case when  $\gamma = 0$  or  $D = 0$ . The value of carry forward for levered firms in (4.21) is smaller in comparison to the value of carry forward for unlevered firms in (4.3). Mathematically, this is evident from Proposition 4.1, as the derivative of a European call option to the strike is decreasing in absolute value. The slope of the call option seen as a function of the strike is steeper, so the difference between two call options is greater compared to the difference between two call options further in the tail. There is also some economic rationale behind this result. The value of a DTA coming from carry forward is positively dependent on the amount of tax payments. In case a firm is levered, less tax is paid due to the interest tax shield. Hence, the overall DTA is reduced in value compared to unlevered firms. The relation between debt and the DTA value of carry forward is shown in Figure 4.10. The negative relationship between the DTA value arising from carry forward and debt is clearly visible. The DTA value even tends to zero when a firm is extremely leveraged, because coupon payments are excessive in those cases.

The sensitivity of the DTA value for levered firms is given by

$$\frac{\partial}{\partial CF_1} \xi_{cf}^{BS} = \tau e^{-r} (1 - F(A_0 + CF_1 + \gamma C)) = \tau e^{-r} Q(A_1 > A_0 + CF_1 + \gamma C).$$

This is lower than the sensitivity for unlevered firms, since the probability that  $A_1$  exceeds the term on the right is lower when interest payments are included. Ceteris paribus, a levered firm is less likely to profit from the full carry forward than an unlevered firm, so a small change in carry forward has less impact on the overall value for levered firms.

**4.3.2. Carry back.** In case a firm has  $CB_1$  at year one, bankruptcy can be avoided in the event of a severe loss, by reclaiming previous tax expenses. This suggests that the coupon payment for companies with carry back should



**Figure 4.10.** DTA/DTL value of levered firms seen as a function of debt. The coupon payment corresponding to debt level  $D_0$  is determined via numerical methods and depends on the deferred tax. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, CF_1 = CB_1 = \text{gain}_1 = 20$ .

be lower compared to firms without deferred taxes (or firms with carry forward when  $\gamma = 1$ ). In this case, the debt value at  $t = 0$  follows from

**Theorem 4.7.** *The debt value at time zero for an undertaking having  $CB_1$  is given by*

$$D_0 = e^{-r}(D + \mathcal{C}_{cb}) - e^{-r}(D + \mathcal{C}_{cb} - \tau CB_1)\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

if  $D + \mathcal{C}_{cb} \leq A_0 - CB_1 + \gamma\mathcal{C}_{cb}$ . Otherwise

$$D_0 = e^{-r}(D + \mathcal{C}_{cb}) - e^{-r}(D + \mathcal{C}_{cb} - \tau CB_1)\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) \\ - e^{-r}\tau(CB_1 - A_0 - \gamma\mathcal{C}_{cb})[\Phi(\theta_2) - \Phi(\theta_1)] - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)].$$

In these expressions

$$\begin{aligned}\theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 - CB_1 + \gamma \mathcal{C}_{cb}}{A_0} \right) - r + \sigma^2/2 \right] \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cb} - \tau(A_0 + \gamma \mathcal{C}_{cb})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cb} - \tau CB_1}{A_0} \right) - r + \sigma^2/2 \right].\end{aligned}$$

**Proof.** See [Appendix C.3](#). ■

Again, Newton-Raphson can be used to find  $\mathcal{C}_{cb}$  such that  $D_0 = D$ . Even if  $\gamma = 1$ , the coupon payments for undertakings with carry back are lower compared to those of the reference undertaking (see [Figure 4.9](#)). As a result, the value of a firm having some carry back is calculated by discounting (2.4) and adding the coupon

$$(4.22) \quad \mathcal{V}_{cb}^{BS} = e^{-r} \mathbb{E}(\tilde{A}_1 + \mathcal{C}_{cb} | \mathcal{F}_0) = A_0 + e^{-r} \tau CB_1 - \tau C^{BS}(K = A_0 + \gamma \mathcal{C}_{cb} - CB_1).$$

To avoid circular reasoning, we assume once more that  $\mathcal{C}_{cb} = \mathcal{C}$ . The DTA value for carry back is then given by the difference between (4.22) and (4.19)

$$(4.23) \quad \xi_{cb}^{BS} = \tau e^{-r} CB_1 - \tau \left( C^{BS}(A_0 + \gamma \mathcal{C} - CB_1) - C^{BS}(A_0 + \gamma \mathcal{C}) \right).$$

In contrast to carry forward, the DTA value arising from carry back is actually more valuable when a firm is increasingly leveraged. The last two terms in (4.23) are smaller in difference compared to the last two terms appearing in (4.7) for unlevered firms. This is due to the higher strike value of the call option, which is also visible in [Figure 4.10](#). The economic reason behind this phenomenon comes from the coupon payments, which decreases fiscal loss even further. Hence, in case of a loss, it is more likely that a higher part of the carry back will be materialized, which increases the value of the DTA. The derivative of the DTA value to the carry back is given by

$$\frac{\partial}{\partial CB_1} \xi_{cb}^{BS} = \tau e^{-r} F(A_0 + \gamma \mathcal{C} - CB_1) = \tau e^{-r} Q(A_1 < A_0 + \gamma \mathcal{C} - CB_1).$$

This is higher compared to unlevered firms. The same carry back value has higher probability of being realized, so that a small change in the carry back has more influence on the DTA value when a firm is levered.

**4.3.3. DTL.** A firm having a deferred tax liability is more likely to go bankrupt, since even in case of a loss the firm might be obliged to pay taxes. Thus, it can potentially happen that  $A_1 > D + \mathcal{C}_L$ , but after taxes  $\tilde{A}_1 < D + \mathcal{C}_L$ . This should be taken into account when calculating the

coupon  $\mathcal{C}_L$  and implies that  $\mathcal{C}_L$  is generally higher for firms having a DTL in comparison to undertakings without deferred tax obligations.

**Theorem 4.8.** *The debt value at time zero for an undertaking having  $gain_1$  is given by*

$$D_0 = e^{-r}(D + \mathcal{C}_L) - e^{-r}(D + \mathcal{C}_L)\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

if  $D + \mathcal{C}_L \leq A_0 - gain_1 + \gamma\mathcal{C}_L$ . Otherwise

$$D_0 = e^{-r}(D + \mathcal{C}_L) - e^{-r}(D + \mathcal{C}_L)\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) \\ - e^{-r}\tau(gain_1 - A_0 - \gamma\mathcal{C}_L)[\Phi(\theta_2) - \Phi(\theta_1)] - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)].$$

In these expressions

$$\theta_1 = \frac{1}{\sigma} \left[ \log \left( \frac{A_0 - gain_1 + \gamma\mathcal{C}_L}{A_0} \right) - r + \sigma^2/2 \right] \\ \theta_2 = \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_L - \tau(A_0 - gain_1 + \gamma\mathcal{C}_L)}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\ \theta_3 = \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_L}{A_0} \right) - r + \sigma^2/2 \right].$$

**Proof.** See [Appendix C.4](#). ■

The resulting coupon payments are higher compared to those of the reference undertaking, which can be seen from [Figure 4.9](#). The market consistent firm value follows from discounting (2.5) and adding the coupon payment

$$(4.24) \quad \mathcal{V}_L^{\text{BS}} = e^{-r} \mathbb{E}^Q(\tilde{A}_1 + \mathcal{C}_L | \mathcal{F}_0) = A_0 - \tau C^{\text{BS}}(K = A_0 + \gamma\mathcal{C}_L - gain_1).$$

Once again we impose that  $\mathcal{C}_L = \mathcal{C}$  to avoid circular reasoning. The DTL value is given by comparing (4.24) and (4.19)

$$(4.25) \quad \xi_L^{\text{BS}} = \mathcal{V}_L^{\text{BS}} - \mathcal{V}^{\text{BS}} = \tau \left( C^{\text{BS}}(A_0 + \gamma\mathcal{C}) - C^{\text{BS}}(A_0 + \gamma\mathcal{C} - gain_1) \right).$$

The DTL value arising from temporary differences for levered firms is greater compared to unlevered firms due to higher strike values. Economically, this holds since coupon payments reduce fiscal profits, which makes it less likely that untaxed profit remains in period one.

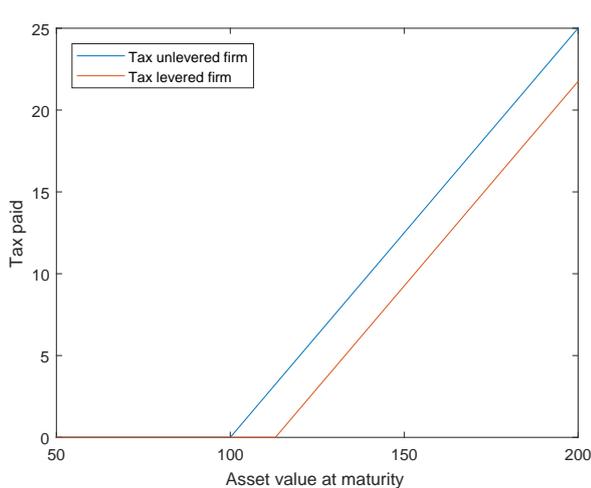
The sensitivity of the DTL value (4.25) to a change in untaxed profit equals

$$\frac{\partial}{\partial gain_1} \xi_L^{\text{BS}} = \tau e^{-r} (1 - F(A_0 + \gamma\mathcal{C}_L - gain_1)) = \tau e^{-r} Q(A_1 > A_0 + \gamma\mathcal{C}_L - gain_1).$$

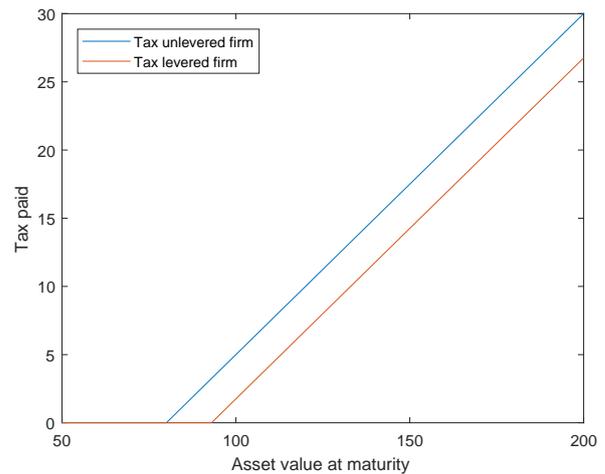
Hence, the DTL value arising from temporary differences is less sensitive to a change in the untaxed profit when a firm is levered. A change in  $gain_1$  has

less influence on tax payments as coupon payments make it less likely that such tax payments are materialized.

**4.3.4. Valuation interest tax shield.** We can obtain a market consistent value of the tax shield under the Black-Scholes assumptions by adapting the valuation formulas for levered firms. The interest tax shield derives its value from the uncertainty related to the fact that not all of the interest tax shield will be materialized. However, the extent to which a firm is able to profit from the tax shield depends on the fiscal history. Figure 4.11 shows the difference in tax payments for levered and unlevered firms for two type of undertakings; a reference undertaking without fiscal history and an undertaking having a DTL. The difference between each of the graphs is what determines the tax shield value.



(a) Reference undertaking without fiscal history,  $C = 12$ .



(b) Undertaking with a DTL ( $gain_1 = 20$ ). The coupon  $C_L$  equals 13.

**Figure 4.11.** Tax payments for levered and unlevered firms with different fiscal history. Parameters:  $A_0 = 100$ ,  $\tau = 0.25$ .

There is no general consensus in the literature about tax shield valuation. This topic started off with the classical article of Modigliani and Miller (1963), which suffers from some serious drawbacks such as risk-free debt and the tacit assumption that the tax shield will be completely materialized each year. Numerous investigations have tried to improve upon this work, such as Kemsley and Nissim (2002) who assess the impact of debt financing by cross sectional regression, concluding that firm value is a strongly positive

function of debt. [Arzac and Glosten \(2005\)](#) take a more theoretical approach, in which future cash flows arising from tax payments are discounted by a pricing kernel. The value of the tax shield is subsequently obtained as the difference in tax payments for levered and unlevered firms. [Arzac and Glosten \(2005\)](#) show that their framework contains the Modigliani-Miller theorem as a special case, by making specific assumptions about the dynamics of the free cash flow process. In line with [Arzac and Glosten \(2005\)](#), we calculate the value of the tax shield by subtracting the enterprise value of an unlevered firm from a levered firm. In particular, the following theorem presents the valuation of the two type of interest tax shields that might arise in our model.

**Theorem 4.9.** *The interest tax shield corresponding to firms with different fiscal history is given by*

(i) *For firms with no fiscal history, carry forward or carry back*

$$(4.26) \quad R^{BS} \triangleq \mathcal{V}^{BS} - V^{BS} = \tau \left( C^{BS}(A_0) - C^{BS}(A_0 + \gamma\mathcal{C}) \right).$$

(ii) *For firms with a DTL arising from temporary differences*

$$(4.27) \quad R_L^{BS} \triangleq \mathcal{V}_L^{BS} - V_L^{BS} = \tau \left( C^{BS}(A_0 - \text{gain}_1) - C^{BS}(A_0 - \text{gain}_1 + \gamma\mathcal{C}) \right).$$

**Proof.** For (i), simply subtract (4.1) from (4.19). In [Section 3.2](#) we assumed a specific order for the coupon payments. After accounting for coupon payments, carry forward/back can be used for the remaining profit/loss. Hence, under this assumption, deferred tax assets like carry forward/back are immaterial for tax shield valuations. They only have an influence on the coupon payments, but to avoid circular reasoning coupons are taken to be the same as those of the reference undertaking. Finally, (ii) takes into consideration that, *ex-ante*, it is known that additional taxes are levied over  $\text{gain}_1$  in period one. Therefore, a DTL makes it more likely that part of the tax shield is materialized and the value is obtained by subtracting (4.10) from (4.24). ■

**Remark 4.10.** This theorem is markedly different than the conventional Modigliani-Miller theorem, which (in our notation) states that

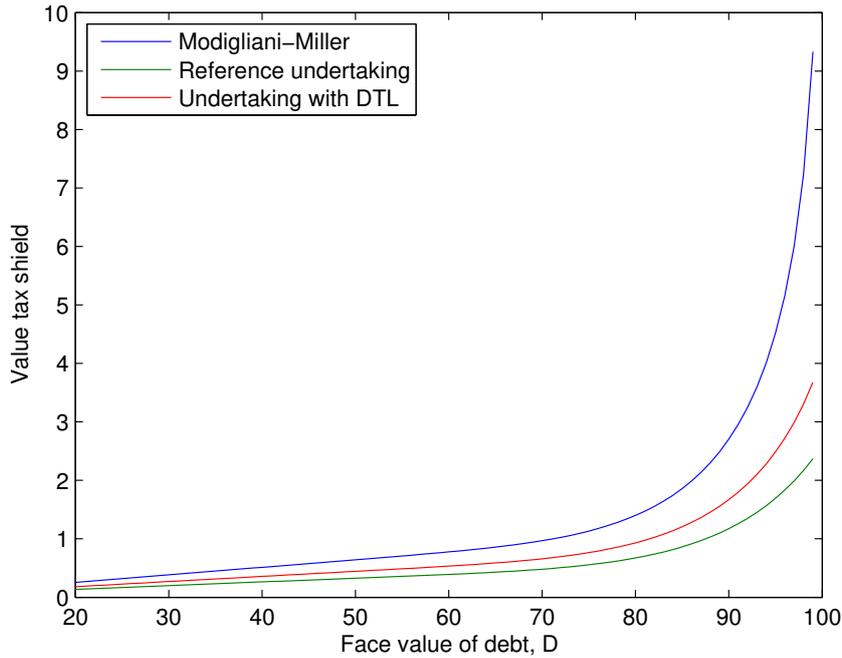
$$(4.28) \quad R = e^{-r}\tau\mathcal{C}.$$

[Modigliani and Miller \(1963\)](#) tacitly assume that the full interest tax shield can be deducted each year, which is not generally valid since taxable income might not be sufficient to compensate the entire tax advantage. Our model does account for this risk, which is reflected by the option price formulas in the previous theorem. The model for firms without fiscal history (4.26)

contains the Modigliani-Miller theorem as a special case, which can be seen by taking  $\sigma \rightarrow 0^+$  in (4.26).<sup>10</sup> Using the Black-Scholes formula (3.7) gives that (4.26) converges to

$$\lim_{\sigma \rightarrow 0^+} R^{\text{BS}} = \tau e^{-r} \gamma \mathcal{C},$$

provided  $(e^r - 1)A_0 > \gamma \mathcal{C}$ . Mathematically, this condition derives from the fact that  $d_2$  in (3.7) goes to infinity if and only if  $\log(A_0/(A_0 + \gamma \mathcal{C})) + r > 0$  when  $\sigma \rightarrow 0^+$ .<sup>11</sup> In our model, the condition expresses that taxable income (as measured by  $(e^r - 1)A_0$ ) should be greater than  $\gamma \mathcal{C}$  in order to benefit completely from the tax shield. If all interest payments can be deducted from taxable income (i.e.  $\gamma = 1$ ), this is precisely the (continuously) discounted interest tax shield.



**Figure 4.12.** Interest tax shield seen as a function of debt. The Modigliani-Miller value is given by (4.28), whereas the value of the Reference undertaking and the Undertaking with DTL are given by  $R^{\text{BS}}$  and  $R_L^{\text{BS}}$  respectively. Parameters:  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\tau = 0.25$ ,  $A_0 = 100$ ,  $\text{gain}_1 = 10$ ,  $\gamma = 1$ . Bankruptcy cost are zero ( $\alpha = 0$ ).

<sup>10</sup>There is no explicit  $\sigma$  in (4.26) since we assumed it was constant throughout our analysis. However, the DTA/DTL values do depend on this quantity.

<sup>11</sup>The strike value of the option is  $K = A_0 + \gamma \mathcal{C}$ .

Figure 4.12 shows the value of the tax shield as a function of debt. The option interpretation of the tax shield renders values lower than the Modigliani-Miller approach, since this model takes into account that part of the tax shield may not be settled in period one. The valuation difference is most pronounced when a firm is highly leveraged, due to the exponential rise in coupon payments. Figure 4.12 also shows that the tax shield is more valuable for firms with a DTL, although the difference with a reference undertaking is relatively moderate.

#### 4.4. Multi-period model unlevered firms

**4.4.1. Simulation set-up.** In this section, we outline how the DTA/DTL values are obtained in the  $T$ -period model for  $T \geq 2$  by simulation. We generate 10,000 asset paths (5,000 asset paths plus 5,000 *antithetic* variates) according to Equations (3.4). Moreover, the simulation depends on the availability of DTA's at time zero ( $CF_1$  or  $CB_1$ ), duration of carry forward/back, tax rate ( $\tau$ ), starting value of the assets ( $A_0$ ), risk-free rate  $r$ , whether carry back is allowed or not ( $1_{\text{Carry back allowed}}$ ) and the geometric Brownian motion (3.2). Based on these input parameters, the asset paths are generated according to Equations (3.4). Subsequently, the firm value (3.3) is estimated by

$$\begin{aligned}\widehat{V}^{BS} &= e^{-rT} \left( \frac{1}{10,000} \sum_{n=1}^{10,000} \widetilde{A}_T^{(n)} \right) \text{ if } CF_1 = CB_1 = 0 \\ \widehat{V}_{\text{cf}}^{\text{BS}} &= e^{-rT} \left( \frac{1}{10,000} \sum_{n=1}^{10,000} \widetilde{A}_T^{(n)} \right) \text{ if } CF_1 > 0 \\ \widehat{V}_{\text{cb}}^{\text{BS}} &= e^{-rT} \left( \frac{1}{10,000} \sum_{n=1}^{10,000} \widetilde{A}_T^{(n)} \right) \text{ if } CB_1 > 0.\end{aligned}$$

As a result, the DTA value of carry forward and carry back respectively are estimated by

$$\begin{aligned}\hat{\xi}_{\text{cf}}^{\text{BS}} &\triangleq \widehat{V}_{\text{cf}}^{\text{BS}} - \widehat{V}^{BS} \\ \hat{\xi}_{\text{cb}}^{\text{BS}} &\triangleq \widehat{V}_{\text{cb}}^{\text{BS}} - \widehat{V}^{BS}.\end{aligned}$$

The DTL value in the  $T$ -period model corresponding to the **base case** (i) can be obtained analytically, hence it need not be simulated. The DTL value in the **more realistic case** (ii) is obtained in the same way the DTA values are simulated, but this time carry back is either not allowed or otherwise has a duration of one year. The untaxed profit is decreased each

year according to (3.6). The estimated DTL value then follows from

$$\begin{aligned}\hat{\xi}_L^{\text{BS}} &\triangleq \hat{V}_L^{\text{BS}} - \hat{V}^{\text{BS}} \\ \hat{V}_L^{\text{BS}} &= e^{-rT} \left( \frac{1}{10,000} \sum_{n=1}^{10,000} \tilde{A}_T^{(n)} \right)\end{aligned}$$

In the last equation,  $\tilde{A}_T^{(n)}$  is given by (3.5).

**4.4.2. Carry forward.** It is still possible to derive closed form formulas in the two-period model, but the expressions tend to be rather cumbersome (see Appendix B). One might be able to extend the results of Appendix B to concoct formulas for longer time periods, but these will necessarily be extremely tedious. Moreover, the longer the time period, the more parameter combinations are available.<sup>12</sup> Hence, DTA values for time periods  $t \geq 3$  are obtained by Monte-Carlo simulation. The results of Appendix B yield the following DTA values for carry forward in the 2-period model with a duration of one year.

**Proposition 4.11.** *The value of a company without fiscal history, excluding carry back and assuming a one year duration of carry forward is given by*

$$\begin{aligned}V^{\text{BS}} &= A_0 - \tau C^{\text{BS}}(A_0) - \tau A_0 \left( \Phi^{(\rho_1)}(g_1, g_2) - e^{-2r} \Phi^{(\rho_1)} \left( \zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}} \right) \right. \\ &\quad \left. + (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) [(1 - \tau) \Phi(\sigma - \zeta_1) + e^{-r} \tau \Phi(-\zeta_1)] \right).\end{aligned}$$

Here  $\Phi^{(\rho)}(x, y)$  is the distribution function of a bivariate standard normal distribution with correlation coefficient  $\rho$  and

$$\begin{aligned}\rho_1 &\triangleq \frac{-1}{\sqrt{(2)}}, \quad \zeta_1 \triangleq \frac{\sigma}{2} - \frac{r}{\sigma}, \quad g_1 \triangleq \zeta_1 - \sigma, \quad g_2 \triangleq \frac{2r/\sigma + \sigma}{\sqrt{(2)}} \\ d_{1,2} &= \frac{1}{\sigma} [r + \sigma^2/2], \quad d_{2,2} = d_{1,2} - \sigma.\end{aligned}$$

**Proof.** See Proposition B.2. ■

**Theorem 4.12.** *The value of a firm having  $CF_1$  available at time zero equals*

$$V_{cf}^{\text{BS}} = A_0 - \tau C^{\text{BS}}(A_0 + CF_1) - \tau(I_1 + I_{2,1} + I_3),$$

<sup>12</sup>Given the  $t$ -period model, we have  $t - 1$  combinations corresponding to the duration of carry forward. In addition, we can choose to allow carry back or not, and if we allow carry back, then one has another  $t - 2$  choices for the duration of carry back. This means there are  $(t - 1) + (t - 1)(t - 2)$  different models starting with carry forward. The like can be said starting with carry back.

where

$$\begin{aligned} I_1 &= A_0 \Phi^{(\rho_1)}(g_1, g_2) - e^{-2r} A_0 \Phi^{(\rho_1)}\left(\zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}}\right) \\ I_{2,1} &= (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2}))(A_0(1 - \tau)\Phi(\sigma - \zeta_1 - \delta) + e^{-r} \tau(A_0 + CF_1)\Phi(-\zeta_1 - \delta)) \\ I_3 &= A_0(\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2}))(\Phi(\zeta_1 + \delta - \sigma) - \Phi(\zeta_1 - \sigma)). \end{aligned}$$

All constants are the same as in [Proposition 4.11](#) and  $\delta = \frac{1}{\sigma} \log\left(\frac{CF_1}{A_0} + 1\right)$ .

**Proof.** See [Appendix B.2](#). ■

**Corollary 4.13.** *Assuming one year duration for carry forward and excluding carry back, the DTA value is given by*

$$(4.29) \quad \xi_{cf}^{BS} = \underbrace{\tau(C^{BS}(A_0) - C^{BS}(A_0 + CF_1))}_{\text{Carry forward value in one period model.}} - \underbrace{\tau(I_{2,1} - I_2 + I_3)}_{>0}.$$

In this expression, all constants are the same as in [Theorem 4.12](#) and

$$I_2 = A_0(\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2}))[(1 - \tau)\Phi(\sigma - \zeta_1) + e^{-r} \tau\Phi(-\zeta_1)].$$

**Proof.** This follows by subtracting  $V^{BS}$  from  $V_{cf}^{BS}$ . The fact that the second term in (4.29) is positive follows from the proof in [Appendix B.2](#). ■

This result might be somewhat vexing, since the carry forward value is worth less in the two year model than in the one year model. However, this can be explained by observing that the asset value of a firm with carry forward in period one is higher in comparison to a firm without the tax advantage. This has repercussions for period two. A relative change in period two leads to a more pronounced difference in absolute asset value, so that a firm with carry forward has to pay more taxes in period two. A careful inspection of the proof in [Appendix B.2](#) reveals that the loss of value comes from integrating two at-the-money call options over the region  $\{\omega \in \Omega : A_1(\omega) > A_0\}$ , i.e. where the undertaking makes a profit. The tax authority holds an option on a firm expiring in period two, which is exercised as soon as the undertaking makes a profit. The call option is at the money for both undertakings (since no DTA's are available in period two), so the value deduction is a consequence of  $C_{\text{at-the-money}}^{BS}(K_1) > C_{\text{at-the-money}}^{BS}(K_2)$  if  $K_1 > K_2$ , i.e. the tax authority expects to levy more taxes from firms with higher asset value in period one.

**Example 4.14.** In a static environment, together with the risk-neutral assumption, we could for simplicity assume that assets grow with rate  $1 + r$  each year. Let us also assume that  $rA_0 \leq CF_1$ , so that an undertaking with

carry forward does not pay taxes in year one. A firm without deferred taxes results in the following asset value after tax in periods one and two

$$\begin{aligned}\tilde{A}_1 &= (1+r)A_0 - \tau((1+r)A_0 - A_0) = A_0(1+r(1-\tau)) \\ \tilde{A}_2 &= A_1(1+r(1-\tau)) = A_0(1+r(1-\tau))^2.\end{aligned}$$

In contrast, a firm having  $CF_1$  results in the following after-tax asset value

$$\begin{aligned}\tilde{A}_1^{(1)} &= A_0(1+r) \quad (\text{Since } rA_0 \leq CF_1) \\ \tilde{A}_2^{(1)} &= A_1(1+r(1-\tau)) = A_0(1+r)(1+r(1-\tau)).\end{aligned}$$

By definition, this results in the following DTA values for carry forward in the one and two year model respectively

$$\begin{aligned}\xi_1 &\triangleq \frac{\tilde{A}_1^{(1)} - \tilde{A}_1}{1+r} = \frac{A_0 r \tau}{1+r} \\ \xi_2 &\triangleq \frac{\tilde{A}_2^{(1)} - \tilde{A}_2}{(1+r)^2} = \frac{A_0 r \tau (1+r-r\tau)}{(1+r)^2} = \frac{A_0 r \tau}{1+r} \frac{1+r-r\tau}{1+r} < \xi_1,\end{aligned}$$

since  $1+r-r\tau < 1+r$ . Indeed, in this simple example carry forward is more valuable in the one year model, since part of the tax benefit in period one is offset by the payment of extra taxes in period two.

The analytic results of [Appendix B.3](#) permit us to express the DTA value of carry forward in the two year model assuming a two year duration of carry forward.

**Theorem 4.15.** *In the two year model without carry back and assuming a two year duration of carry forward, firm value is given by*

$$(4.30) \quad V_{cf}^{BS} = A_0 - \tau C^{BS}(A_0 + CF_1) - \tau(I_{1,2} + I_{2,1}).$$

**Proof.** See [Appendix B.3](#). ■

**Corollary 4.16.** *In the two year model*

$$(4.31) \quad \xi_{cf}^{BS} = \tau(C^{BS}(A_0) - C^{BS}(A_0 + CF_1)) - \tau(I_{1,2} + I_{2,1} - I_1 - I_2),$$

where

$$\begin{aligned}I_{1,2} &= A_0 \Phi^{(\rho_1)}(\zeta_1 + \delta - \sigma, g_1) - e^{-2r}(A_0 + CF_1) \Phi^{(\rho_1)}(\zeta_1, g_2), \\ g_1 &\triangleq \frac{\frac{2r}{\sigma} + \sigma + \frac{1}{\sigma} \log(\frac{A_0}{A_0 + CF_1})}{\sqrt{2}}, \quad g_2 \triangleq \frac{\frac{2r}{\sigma} + \frac{1}{\sigma} \log(\frac{A_0}{A_0 + CF_1}) - \sigma}{\sqrt{2}}.\end{aligned}$$

The rightmost term contains four integrals, for which we have the following inequalities

$$(4.32) \quad I_{1,2} < I_1, \quad I_{2,1} < I_2.$$

**Proof.** This follows by subtracting  $V^{BS}$  in [Proposition 4.11](#) from  $V_{cf}^{BS}$  in [Theorem 4.15](#). The inequalities (4.32) are proved in [Appendix B.3](#). ■

In contrast to [Corollary 4.13](#), the DTA value of carry forward corresponding to a two year duration does increase compared to the one year model. The trade-off between higher taxes in period two is offset by the extra use of carry forward in that period.

**Example 4.17.** As in [Example 4.14](#), assume that assets grow with rate  $1 + r$  and that  $(1 + r)^2 A_0 \leq CF_1$ , so that in periods one and two no tax needs to be paid for a firm having  $CF_1$  that can be settled in two years. The post-tax asset value for the undertaking without carry forward is the same as in [Example 4.14](#) for both periods, the same holds for the undertaking with a DTA in period one. However, since the undertaking with carry forward is also exempted from tax payments in period two, we now have

$$\tilde{A}_2^{(1)} = A_0(1 + r)^2.$$

Working out the differences yields the DTA value of carry forward in the one and two year model

$$\begin{aligned} \xi_1 &\triangleq \frac{A_0 r \tau}{1 + r} \quad (\text{Same as in } \a href="#">\text{Example 4.14}) \\ \xi_2 &\triangleq \frac{A_0(1 + r)^2 - A_0(1 + r(1 - \tau))^2}{(1 + r)^2} = \frac{A_0 r \tau(2 - r\tau + 2r)}{(1 + r)^2} \\ &= \frac{A_0 r \tau}{1 + r} \frac{2 - r\tau + 2r}{1 + r} = \xi_1 \left( \frac{1 + r + (1 + r - r\tau)}{1 + r} \right) > \xi_1, \end{aligned}$$

since  $1 + r - r\tau > 1$ . Hence, in the deterministic scenario with constant growth, the DTA arising from carry forward is indeed more valuable in the two year model.

**Proposition 4.18.** *The added value of going from one year to two years carry forward in the two year model is given by*

$$(4.33) \quad \tau(I_1 + I_3 - I_{1,2})$$

**Proof.** By definition this is the difference between (4.31) and (4.29). ■

The analytical solution (4.33) is not easily interpreted, but the proofs in [Appendix B.2](#) and [Appendix B.3](#) give some insight where these quantities are derived from. Once an undertaking's profit in period one exceeds the nominal carry forward value, it doesn't matter whether  $CF_1$  is one or two years valid as it is completely settled in period one and no other deferred tax assets are taken to period two. The difference comes from considering all scenarios in which taxable income is not sufficient to settle the entire carry forward in period one. The integrals  $I_1 + I_3$  quantify the extent to

which a firm is expected to lose value due to taxes in period two, given it has incurred a loss in period one and the duration of carry forward is only one year. The integral  $I_{1,2}$  quantifies the expected tax payments in period two, given a loss incurred in period one and a two year duration of carry forward. Hence, the difference (4.33) gauges the expected tax savings resulting from extending the duration of carry forward with one year.

Table 4.1 summarizes the DTA value of carry forward corresponding to different time models and settlement terms, assuming no carry back possibilities. The values are quite subtle; when carry forward has a one year duration, one has the upper bound on the DTA value,  $e^{-r}\tau CF_1$ , which is the value you would acknowledge if the entire carry forward would be settled with certainty. For time periods  $T \geq 2$ , we no longer have the upper bound  $e^{-rT}\tau CF_1$ , since part of (or the entire) carry forward might be settled before maturity. On the other hand, the probability of materializing the complete carry forward can be small if  $CF_1$  is high relative to  $A_0$  and this can offset the value creation arising from intermediate tax deduction. To illustrate this, Table 4.1 yields that the DTA value of  $CF_1 = 10$  in the 10-period model, with settlement term ten years equals 1.67, which is more than  $e^{-r \times 10}\tau CF_1 = 1.53$ . In contrast, the DTA value of  $CF_1 = 50$  in the same model renders  $6.86 < 7.58 = e^{-r \times 10}\tau CF_1$  since  $CF_1$  is high relative to  $A_0$  so that the probability of complete settlement is low.

The values in Table 4.2 also concern the DTA value of carry forward, this time, however, allowing carry back possibilities. Apart from the one year model (in which both models concur), the values are uniformly lower compared to Table 4.1. Some intuition is gained by looking at the 2-period model. Consider the event  $\mathcal{A} \triangleq \{\omega \in \Omega : A_1(\omega) > A_0 + CF_1\}$ , which is where both models diverge. Excluding carry back refrains undertakings from using DTA's in period two under scenario  $\mathcal{A}$ . Allowing carry back yields the undertaking without carry forward with  $CB_2 = A_1 - A_0$ , whereas the undertaking with carry forward only has  $CB_2 = A_1 - A_0 - CF_2$ . In case of a large (negative) shock in period two, the asset values of both firms come closer (in absolute value) and the carry back possibility might then suffice to offset the tax disadvantage which arose in period one. See Example 4.19 below for a simple illustration. In general  $T$ -period models, the same forces play a role.

**Example 4.19.** Consider first the two year model in which carry back possibilities are excluded. Suppose the relative increments are given by 1.5 and 0.5 in period one and two respectively. In addition, the initial asset value  $A_0 = 100$  and tax  $\tau = 0.25$ . This yields the asset values for an undertaking

without deferred taxes

$$\begin{aligned}\tilde{A}_1 &= 100 \cdot 1.5 - \tau(100 \cdot 1.5 - 100)^+ = 137.5 \\ \tilde{A}_2 &= 137.5 \cdot 0.5 - \tau(137.5 \cdot 0.5 - 137.5)^+ = 68.75.\end{aligned}$$

In a similar vein, the undertaking having  $CF_1 = 20$  is left with

$$\begin{aligned}\tilde{A}_1 &= 100 \cdot 1.5 - \tau(100 \cdot 1.5 - 100 - 20)^+ = 142.5 \\ \tilde{A}_2 &= 142.5 \cdot 0.5 - \tau(142.5 \cdot 0.5 - 142.5)^+ = 71.25.\end{aligned}$$

In this scenario, the DTA value at the end of period two equals  $71.25 - 68.75 = 2.5$ . Similar computations lead to the DTA value of carry forward when carry back is allowed

$$\begin{aligned}\tilde{A}_1 &= 100 \cdot 1.5 - \tau(100 \cdot 1.5 - 100)^+ = 137.5 \text{ (50 carry back is created)} \\ \tilde{A}_2 &= 137.5 \cdot 0.5 + 50 \cdot \tau = 81.25\end{aligned}$$

the last equality follows, since  $CB_2 = 50$ , which can be used in its entirety since  $\tilde{A}_1 - A_2 > 50$ . Subsequently, a firm having  $CF_1 = 20$  gives rise to the asset values

$$\begin{aligned}\tilde{A}_1 &= 100 \cdot 1.5 - \tau(100 \cdot 1.5 - 100 - 20)^+ = 142.5 \text{ (30 carry back is created)} \\ \tilde{A}_2 &= 142.5 \cdot 0.5 + 30 \cdot \tau = 78.75.\end{aligned}$$

The DTA value arising from carry forward in this example is negative ( $78.75 - 81.25$  in period two). This illustrates that if carry back is allowed, a firm may actually *lose* value despite having the tax advantage  $CF_1$  at the beginning of period zero. A firm expecting an anomalous negative shock in period two may thus be better off settling only part of the carry forward in period one (or nothing in the most extreme case). In the general case, an undertaking with a long time horizon should take all future scenarios into account to determine the *optimal* way in which the DTA's should be settled at each time period. An interesting research question would be to investigate whether such an optimal strategy exists and how it influences firm value. The domain of mathematics that is best suited to attack these type of problems belongs to the realm of *stochastic optimal control*, see Øksendal (2003) Chapters 10 & 11 for a thorough discussion.

Even though only one scenario is considered in Example 4.19, it serves to illustrate that there are forces having a negative influence on carry forward value, which do not arise when carry back is excluded. This explains why the values in Table 4.2 are lower compared to those in Table 4.1. In general  $T$ -period models, a myriad of such scenarios having a negative influence on carry forward might occur. However, this still does not weigh up against all scenarios in which carry forward has a positive influence on firm value, as

**Table 4.1.** Monte-Carlo simulation of DTA value corresponding to different nominal carry forward values (header). Left most column concerns the duration of the model. Two cases are distinguished, one where duration of carry forward is one year, other where duration is ten years. Carry back is not allowed and the maximum *standard error* during all simulations equals 0.0194. The following parameters are used:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$  and 10.000 Monte-Carlo paths.

$CF_1$	10	20	30	40	50	10	20	30	40	50
	Carry forward 1 year					Carry forward 10 years				
$T = 1$	1.10	1.80	2.19	2.43	2.53	1.10	1.80	2.18	2.40	2.51
$T = 2$	1.07	1.75	2.15	2.34	2.46	1.42	2.47	3.25	3.75	4.08
$T = 3$	1.05	1.72	2.09	2.30	2.42	1.56	2.81	3.79	4.51	5.07
$T = 4$	1.03	1.67	2.05	2.27	2.35	1.64	2.99	4.09	4.99	5.70
$T = 5$	1.01	1.65	1.99	2.18	2.32	1.67	3.09	4.30	5.30	6.12
$T = 6$	0.98	1.60	1.95	2.15	2.26	1.69	3.16	4.42	5.50	6.40
$T = 7$	0.96	1.56	1.92	2.10	2.20	1.70	3.19	4.49	5.63	6.59
$T = 8$	0.94	1.55	1.88	2.05	2.15	1.70	3.21	4.54	5.71	6.75
$T = 9$	0.92	1.49	1.84	2.02	2.10	1.69	3.20	4.54	5.76	6.84
$T = 10$	0.91	1.48	1.80	1.99	2.07	1.67	3.20	4.54	5.76	6.86

all values in [Table 4.2](#) are positive.

In each  $T$ -period model, the DTA arising from carry forward will eventually decrease, as can be seen from [Figure 4.13](#) panel **(a)** and **(b)**. In panel **(a)**, the duration period is varied over 1,3 and 8 years. Increasing the duration time always leads to an increase in DTA value, since settlement of the entire carry forward is more probable. However, panel **(b)** shows that over a very long time horizon, the DTA value starts to decrease. The point of inflection is the same for the 10 and 100 year duration, namely in the 9-period model. Once more, the reason for this coming from the trade-off described in [Example 4.14](#) and [Example 4.17](#). Initially, the benefit of one extra time period increases the likelihood of materializing the entire carry forward, but at some time point this doesn't weigh up anymore against the increased tax expenses in subsequent time periods. Panel **(c)** in [Figure 4.13](#) shows the DTA value as a function of carry forward in the 10-period model. A settlement term of one year causes the DTA value to stabilize already for

**Table 4.2.** Monte-Carlo simulation of DTA value corresponding to different nominal carry forward values (header). Left most column concerns the duration of the model. Two cases are distinguished, one where duration of carry forward is one year, other where duration is ten years. Carry back is allowed for one and ten years in the respective models and the maximum *standard error* in all simulations equals 0.0214. The following parameters are used:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$  and 10.000 Monte-Carlo paths.

$CF_1$	10	20	30	40	50	10	20	30	40	50
	Carry forward 1 year <sup>*</sup>					Carry forward 10 years <sup>**</sup>				
$T = 1$	1.10	1.80	2.20	2.41	2.51	1.10	1.81	2.21	2.40	2.51
$T = 2$	0.86	1.38	1.68	1.84	1.92	1.20	2.10	2.76	3.23	3.56
$T = 3$	0.92	1.48	1.80	1.96	2.06	1.25	2.26	3.06	3.66	4.14
$T = 4$	0.89	1.43	1.73	1.88	1.97	1.29	2.35	3.22	3.93	4.54
$T = 5$	0.88	1.42	1.72	1.87	1.95	1.31	2.41	3.37	4.12	4.81
$T = 6$	0.86	1.37	1.69	1.83	1.92	1.32	2.47	3.45	4.28	5.01
$T = 7$	0.84	1.35	1.65	1.82	1.88	1.33	2.50	3.50	4.36	5.11
$T = 8$	0.82	1.34	1.63	1.78	1.85	1.33	2.50	3.56	4.47	5.28
$T = 9$	0.81	1.32	1.60	1.77	1.80	1.34	2.53	3.58	4.52	5.36
$T = 10$	0.80	1.29	1.55	1.71	1.78	1.34	2.53	3.64	4.54	5.40

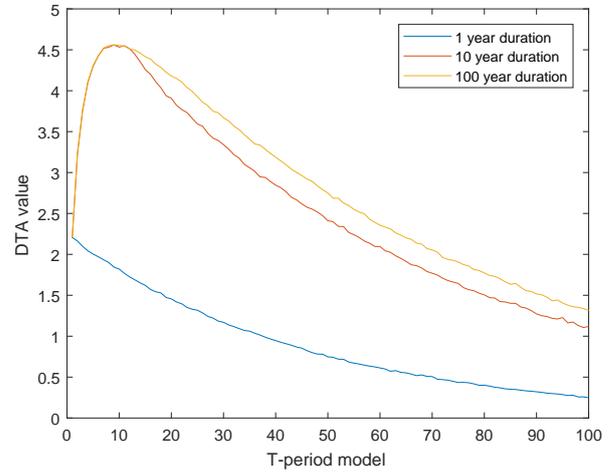
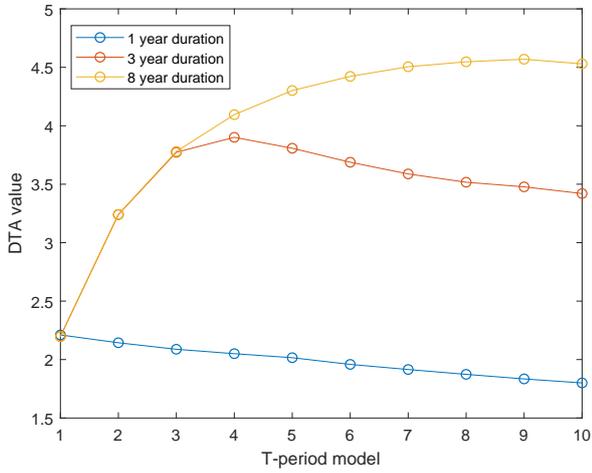
\* Carry back has a one year duration

\*\* Carry back has a ten year duration

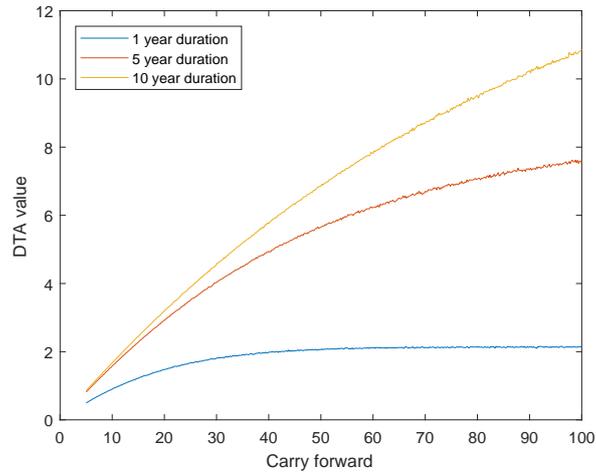
(relatively) small values of  $CF_1$ , since the probability of materializing the entire carry forward is small. An increase in the settlement term results in a concomitant rise in DTA value and the stabilization occurs for much larger value of  $CF_1$ .

**4.4.3. Carry Back.** The DTA values arising from carry back are shown in Table 4.3. Both models considered in Table 4.3 are simulated under the assumption that carry forward is allowed, since virtually all tax regimes worldwide also acknowledge carry forward possibilities when carry back is allowed. Following the discussion in Section 4.1.3, the fact that the values in Table 4.3 are uniformly lower compared to the DTA values of carry forward in Table 4.2 is a consequence of  $r$  being large relative to  $\sigma^2/2$ .<sup>13</sup> To see

<sup>13</sup>By simulation, we find that  $\sigma = 0.5$  and keeping all other parameters fixed, yields that carry back is more valuable than carry forward.



(a) DTA value of  $CF_1 = 30$  for different duration periods over the  $T$ -period model. (b) DTA value of  $CF_1 = 30$  for different duration periods over the  $T$ -period model.



(c) DTA value as function of carry forward in the 10-period model. The settlement term in the models is 1,5 and 10 years.

**Figure 4.13.** Monte-Carlo simulation of DTA value arising from carry forward in multi-period models. In all sub figures carry back is not allowed. Parameters:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\tau = 0.25$ .

why this condition also plays a role in the multi-period model, we consider the dynamics of the assets without taxes (3.2) over long periods of time. In particular, the following can be proved about the asset process when  $t \rightarrow \infty$

(Karatzas and Shreve, 2012)

$$\lim_{t \rightarrow \infty} A_t = \begin{cases} \infty \text{ a.s.}, & \text{if } r - \frac{1}{2}\sigma^2 > 0 \\ 0 \text{ a.s.}, & \text{if } r - \frac{1}{2}\sigma^2 < 0 \\ \text{does not exist a.s.}, & \text{if } r = \frac{1}{2}\sigma^2. \end{cases}$$

The first two statements follow from the *strong law of large numbers*, whereas the latter statement is a consequence of the *law of the iterated logarithm*.<sup>14</sup> Hence, if  $\sigma^2$  is large relative to  $r$ , this result suggests that an undertaking is more likely to incur losses, which makes  $CB_1$  more valuable than  $CF_1$ . Panel (c) in Figure 4.14 shows the DTA value of carry back in different  $T$ -period models for varying values of  $\sigma$  and  $r$ . The DTA values are fairly close to each other when  $\sigma = r$ . Dropping the latter constraint causes further divergence between the DTA values. Panel (d) in Figure 4.14 shows that the shape of the DTA value as a function of  $A_0$  is similar in different  $T$ -period models. Even though the payoff of carry back in multi-period models can no longer be expressed as an explicit call option, its behaviour is similar to the one-period model where the explicit call option interpretation is available.

There is a second factor that contributes to differences in value between  $CB_1$  and  $CF_1$ . Firstly, Example 4.19 shows there are scenarios under which it is not beneficial to have carry forward. Such examples *cannot* be found for undertakings having carry back, i.e. there are no scenarios for which it is harmful to have carry back in comparison with the reference undertaking, assuming a one year settlement term of carry back.<sup>15</sup> Example 4.19 emanates from scenarios under which the reference undertaking creates more carry back in period one than the firm starting with carry forward. Panel (b) in Figure 4.14 shows that longer settlement terms of carry forward engender lower DTA values of carry back, when profits can be settled for 10 years. In such cases, the same logic applies as in Example 4.19; there are scenarios for which it is not beneficial to use carry back, which has a downward pushing effect on the DTA value. Finally, it is visible from panel (a) that varying the settlement term of carry back/forward at the same time is beneficial for longer settlement terms in the various  $T$ -period models. The DTA value in panel (a) reaches a maximum when profit/loss can be carried back/forward for multiple years. This is because firms with initial tax benefits are expected to pay more taxes after the DTA is settled, whose effect is more

<sup>14</sup>Interestingly, one also has  $\lim_{t \rightarrow \infty} A_t = \infty$  in  $L^1$  when  $r > 0$ . Hence, the convergence does not depend on  $\sigma$ . Off course, this is not a contradiction, since convergence in  $L^1$  and *almost sure* convergence are two different notions. The failure of agreement in the two limits comes from the fact that geometric Brownian motion is not *uniformly integrable*.

<sup>15</sup>It is indeed possible to find such examples when carry back has a longer settlement term, but this is not realistic given that no country allows to carry back losses for more than one year. See also ??.

**Table 4.3.** Monte-Carlo simulation of DTA value corresponding to different nominal carry back values (header). Left most column concerns the duration of the model. Two cases are distinguished, one where duration of carry back is one year, other where duration is ten years. Carry forward is allowed for one and ten years in the respective models and the maximum *standard error* in all simulations equals 0.0081. The following parameters were used:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\tau = 0.25$  and 10.000 Monte-Carlo paths.

$CB_1$	10	20	30	40	50	10	20	30	40	50
	Carry back 1 year*					Carry back 10 years**				
$T = 1$	0.81	1.22	1.36	1.38	1.39	0.81	1.22	1.36	1.39	1.40
$T = 2$	0.54	0.79	0.88	0.90	0.89	0.80	1.30	1.53	1.67	1.68
$T = 3$	0.48	0.70	0.77	0.79	0.78	0.77	1.29	1.60	1.74	1.80
$T = 4$	0.42	0.61	0.66	0.68	0.68	0.74	1.26	1.60	1.79	1.87
$T = 5$	0.36	0.54	0.60	0.61	0.60	0.71	1.23	1.56	1.76	1.89
$T = 6$	0.33	0.48	0.54	0.54	0.55	0.69	1.20	1.53	1.74	1.88
$T = 7$	0.31	0.44	0.49	0.50	0.50	0.66	1.16	1.50	1.73	1.86
$T = 8$	0.28	0.41	0.45	0.47	0.46	0.64	1.12	1.47	1.69	1.84
$T = 9$	0.26	0.38	0.42	0.43	0.42	0.62	1.10	1.43	1.65	1.80
$T = 10$	0.25	0.36	0.40	0.40	0.40	0.60	1.06	1.39	1.63	1.75

pronounced over longer periods of time. This is similar to [Example 4.14](#).

If we consider a firm starting with carry back, it is certainly possible to tweak [Example 4.19](#) in such a way that the reference undertaking creates more carry forward for period two than a firm with the initial carry back.<sup>16</sup> However, the following example shows that such scenarios still do not lead to higher firm values for the reference undertaking.

**Example 4.20.** In contrast to [Example 4.19](#), assume now that the relative shocks are given by 0.5 and 1.5. In addition suppose that a firm has  $CB_1 = 20$ . The other parameters are still the same, i.e.  $A_0 = 100$  and  $\tau = 0.25$ . The post-tax asset values in period one and two for the reference undertaking

<sup>16</sup>Just consider all scenarios which lead to a loss in period one. The firm with carry back will use some of the DTA to get a tax refund, but as a result less carry forward is available in period two.

are given by

$$\tilde{A}_1 = 100 \cdot 0.5 - \tau(100 \cdot 0.5 - 100)^+ = 50 \text{ (50 carry forward is created)}$$

$$\tilde{A}_2 = 50 \cdot 1.5 - \tau(50 \cdot 1.5 - 50 - 50)^+ = 75.$$

The post-tax asset values for a firm with carry back are given by

$$\tilde{A}_1 = 100 \cdot 0.5 + \tau CB_1 - \tau(100 \cdot 0.5 - 100 + CB_1)^+ = 55 \text{ (45 carry forward is created)}$$

$$\tilde{A}_2 = 55 \cdot 1.5 - \tau(55 \cdot 1.5 - 55 - 45)^+ = 82.5$$

Hence, in this particular example, it is still beneficial to use carry back, even though that reduces carry forward in subsequent periods. A straightforward generalization of this example (replacing the shocks by arbitrary numbers  $\psi_1$  and  $\psi_2$ ) shows that there are no scenarios for which the reference undertaking has a higher post-tax asset value in period two. The same conclusion holds for arbitrary  $T$ -period models for  $T \geq 3$  due to the one year settlement term of carry back.

**4.4.4. DTL.** In the **base case** (i), we assume that no intermediate tax payments are made until final time period  $T$ . In year  $T$ , taxes are levied over the untaxed profit  $gain_1$  and  $A_T - A_0$ . This yields the post-tax asset value

$$\tilde{A}_T = A_T - \tau(A_T - A_0 + gain_1)^+.$$

Hence, the firm value is given by

$$\begin{aligned} V_L^{BS} &= e^{-rT} \mathbb{E}(A_T - \tau(A_T - A_0 + gain_1)^+ | \mathcal{F}_0) \\ &= A_0 - \tau C^{BS}(A_0 - gain_1, T). \end{aligned}$$

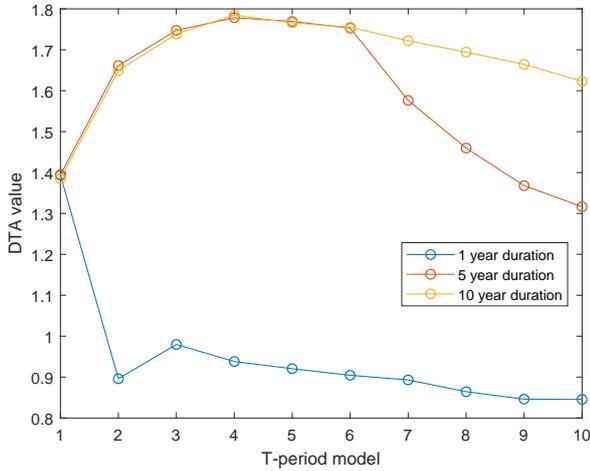
In this base case scenario, the firm value of an undertaking without deferred taxes is given by

$$\begin{aligned} V^{BS} &= e^{-rT} \mathbb{E}^Q(A_T - \tau(A_T - A_0)^+ | \mathcal{F}_0) \\ &= A_0 - \tau C^{BS}(A_0, T) \end{aligned}$$

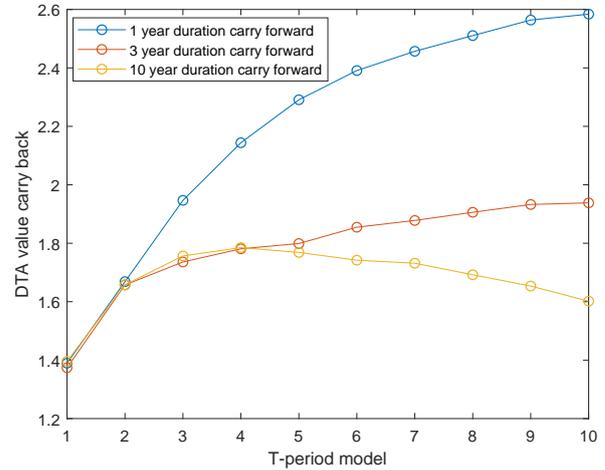
As a result, the DTL value equals

$$(4.34) \quad \xi_L^{BS} \triangleq V_L^{BS} - V^{BS} = \tau(C^{BS}(A_0, T) - C^{BS}(A_0 - gain_1, T)).$$

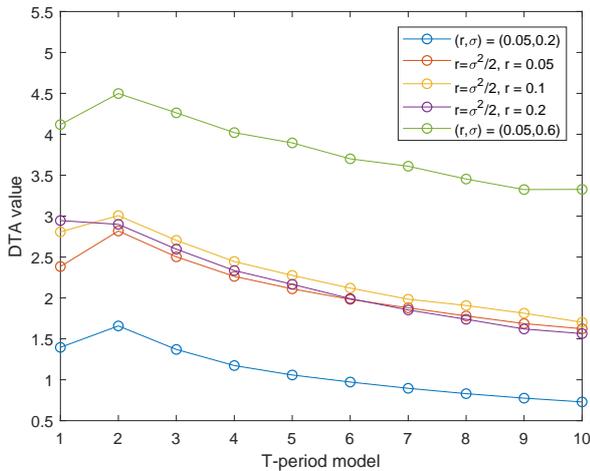
Panel (a) in Figure 4.15 shows the DTL value (4.34) as a function of the  $T$ -period model. The function is strictly increasing, because on average, the difference between a firm with a DTL and a reference undertaking is the same after taxes are levied at the end of year  $T$  for  $T \in \{1, 2, \dots\}$ . This difference is worth less for higher values of  $T$  due to discounting. The value of the **more realistic case** (ii) is obtained by simulation, as illustrated in panel (b) of Figure 4.15. The DTL value is monotonically increasing in the 2-period model, since carry back has no influence on the DTL value.



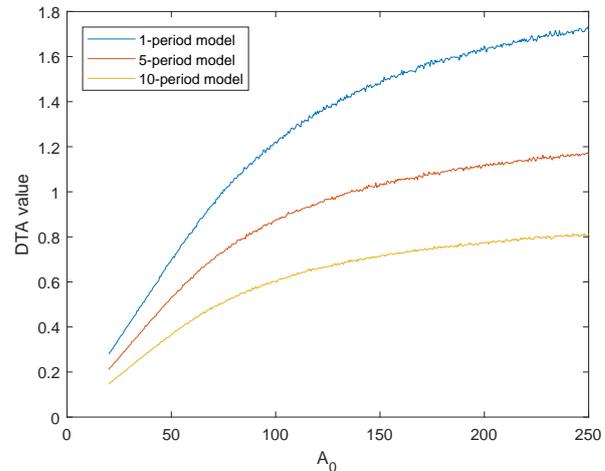
(a) DTA value of  $CB_1 = 40$  for different duration periods. Duration of carry forward is the same as duration of carry back in each respective model.



(b) DTA value of  $CB_1 = 40$  keeping the duration time of carry back fixed (10 years) and varying the carry forward duration.



(c) DTA value of  $CB_1 = 40$  for different ordered pairs  $(r, \sigma)$ . Duration carry forward is 10 years, duration carry back is 2 a function of  $A_0$ . Duration carry forward is 10 years, duration carry back is 2 years.

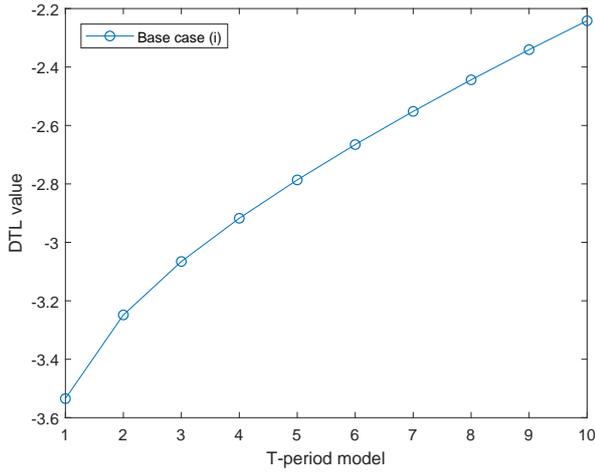


(d) DTA value of  $CB_1 = 20$  for different  $T$ -period models as a function of  $A_0$ . Duration carry forward is 10 years, duration carry back is 2 years.

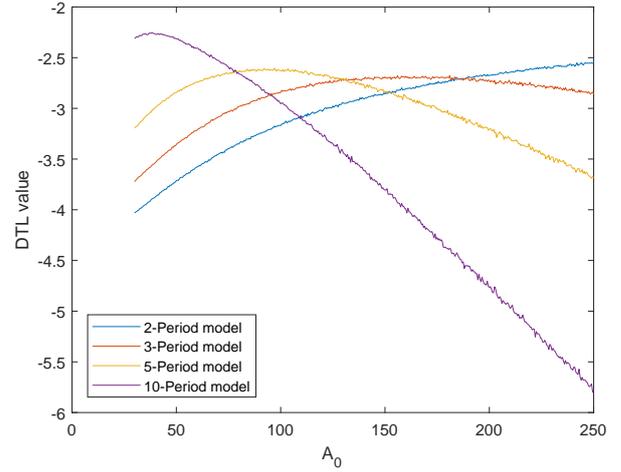
**Figure 4.14.** Monte-Carlo simulation of DTA value arising from carry back in multi-period models. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$ .

However, for time-periods  $T \geq 3$ , carry back plays a role. We see that the DTL value is rising initially for small values of  $A_0$ . But, at some point,  $A_0$  is so large that a positive shock creates enough carry back for future time-periods. As a result, it is increasingly unlikely that the DTL is diminished

at future time periods (since carry back is used first), which means that the DTL is bigger at the end of the settlement period.



(a) DTL value of (4.34) for different  $T$ -period models.



(b) DTL value arising from  $gain_1 = 20$  as a function of  $A_0$  in different  $T$ -period models.

**Figure 4.15.** Monte-Carlo simulation of DTL value arising from temporary differences in multi-period models. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$ .

## 4.5. Multi-period model levered firms

**4.5.1. Simulation set-up.** In this section we outline how the coupon payments are obtained in  $T$ -period models for  $T \geq 2$  such that debt sells at par value. Since there are no analytical formulas available for  $D_0$  when  $T \geq 2$ , the asset processes in (3.8) are generated taking  $\mathcal{C}, \mathcal{C}_{cf}$  and  $\mathcal{C}_{cb}$  from the 1-period model as starting values. The set-up in (4.16) means that firms fall into one of three classes

- (i) The firm does *not go* bankrupt over the lifespan of the  $T$ -period model. This means that at the end of each year  $t < T$ ,  $\tilde{A}_t > K$  and  $\tilde{A}_T > D$  at maturity. The number of firms belonging to this class is referred to as  $N_a$ .
- (ii) The firm defaults *at* maturity. This means that  $\tilde{A}_t > K$  for  $t < T$ , but  $\tilde{A}_T < D$ . The number of firms belonging to this class is referred to as  $N_b$ .
- (iii) The firm defaults *before* maturity. This means that the stopping rule  $\tau_D < T$  and  $\tilde{A}_{\tau_D} < K$ . The number of firms belonging to this class is referred to as  $N_c$ .

When simulating the 10.000 asset paths we thus have  $N_a + N_b + N_c = 10.000$ . For simplicity, we assume that the bankruptcy trigger prior to maturity equals the face value of the outstanding bond, i.e.  $K = D$ . Henceforth, the debt value at time zero (4.16) is estimated by

$$(4.35) \quad \left[ \underbrace{N_a \left( e^{-rT} \left( \sum_{t=0}^{T-1} e^{rt} \mathcal{C} + D \right) \right)}_{=I_1} + \underbrace{N_b \left( e^{-rT} \sum_{t=0}^{T-1} e^{rt} \mathcal{C} + e^{-rT} (1 - \alpha) \sum_{n=1}^{N_b} \tilde{A}_T^{(n)} \right)}_{=I_2} + \underbrace{N_c \left( (1 - \alpha) \sum_{n=1}^{N_c} e^{-r\tau_{D,(n)}} \tilde{A}_{\tau_D}^{(n)} + \sum_{n=1}^{N_c} e^{-r\tau_{D,(n)}} \sum_{t=0}^{\tau_{D,(n)}} \mathcal{C}^{(n)} e^{rt} \right)}_{=I_3} \right] / (N_a + N_b + N_c).$$

The first term  $I_1$  corresponds to class (i) when a firm does not go bankrupt. In those cases, debt at time zero is the discounted sum of all coupon payments plus the discounted face value of debt at maturity. The second term  $I_2$  corresponds to class (ii) when a firm defaults at maturity. In that case, all necessary coupon payments are made and discounted, which is the first term. The second term is the discounted asset value at maturity, multiplied by the recovery fraction  $1 - \alpha$  due to bankruptcy cost. The final term  $I_3$  corresponds to class (iii) when the firm defaults prior to maturity. The first term is the discounted asset value at the time of default. The notation  $\tau_{D,(n)}$  indicates the time of default before maturity of firm  $n$  belonging to class (iii). The second term in  $I_3$  is the discounted value of the coupon stream, where coupons are paid out up to and including the time of bankruptcy.

The weighted average in (4.35) gives an estimate of  $D_0$ . Subsequently, numerical methods are employed to find  $\mathcal{C}$  such that (4.35) equals  $D$ . The same is done to find  $\mathcal{C}_{cf}$  and  $\mathcal{C}_{cb}$  in case the firm has carry forward or carry back respectively. The values  $\mathcal{C}, \mathcal{C}_{cf}, \mathcal{C}_{cb}$  obtained in this way are used as input to compute the firm values according to (3.9).

**4.5.2. Carry forward.** We argue in Section 4.3 that firms with carry forward should pay less coupon than firms without a DTA, since the probability of bankruptcy is higher for these firms when taking taxes into account. The same conclusion holds for general  $T$ -period models for  $T \geq 2$  without carry back. Table 4.4 shows the resulting coupon payments for the reference undertaking and a firm having carry forward in the 5-period model. Indeed,  $\mathcal{C}_{cf}$  is slightly smaller than  $\mathcal{C}$ , although both values do not differ substantially. When allowing carry back, all coupon payments are lower than in Table 4.4, as can be seen from Table 4.5. Again, this holds since bankruptcy is less

likely when allowing more tax deduction possibilities. However, [Table 4.5](#) shows that the difference is rather limited. The disparity between  $\mathcal{C}$  and  $\mathcal{C}_{cf}$  is more pronounced in [Table 4.5](#). This can be explained by our earlier finding that there are scenarios in which carry forward leads to a reduction in firm value in comparison with the reference undertaking (see [Example 4.19](#)). Taking such scenarios into account forces the two coupon payments  $\mathcal{C}$  and  $\mathcal{C}_{cf}$  to be closer in absolute value. Finally, [Table 4.5](#) also shows that  $\mathcal{C}_{cb} < \mathcal{C}_{cf}$  for each fixed face value of debt. This is because carry back is positively influenced by leverage, as opposed to carry forward. Moreover, assuming a one year settlement term of carry back excludes scenarios under which a reference undertaking is better off. Therefore, downward pushing forces on firm value that play a role when starting with carry forward cannot occur. These observations combined cause lower coupon payments for firms starting with carry back.

The DTA value arising from carry forward still bears a negative relation with leverage in multi-period models, as can be seen from [Figure 4.16](#). The reason for this is the same as in the 1-period model, namely that coupon payments reduce taxable income. As a result, fewer carry forward can be used to offset tax expenses. This effect is more pronounced over longer time periods, since coupon payments rise as a result of increased likelihood of bankruptcy. However, due to coupon payments, more carry forward accrues in each period compared to firms which are unlevered.

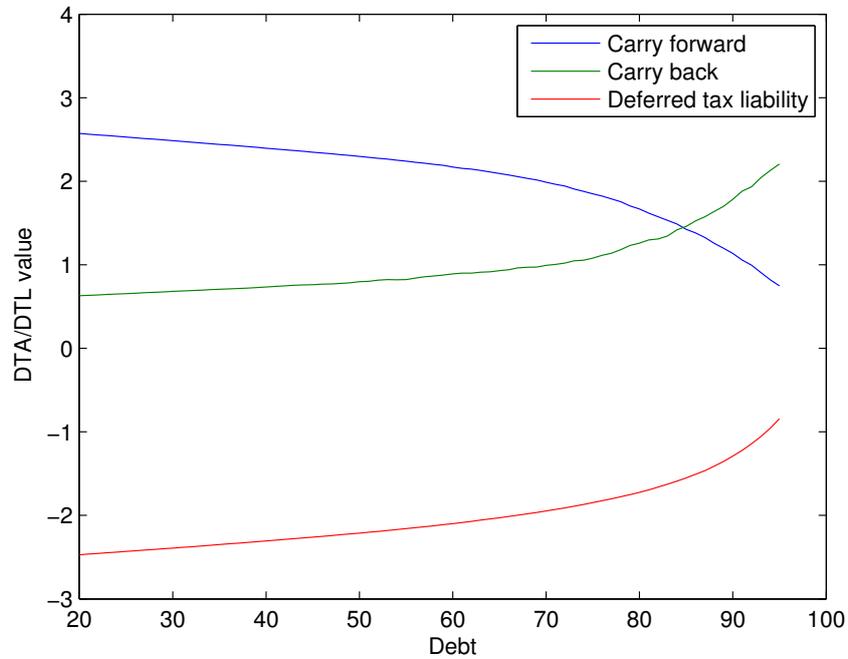
**Table 4.4.** Coupon payments in the 5-period model, when carry back is not allowed. The coupon  $\mathcal{C}_{cf}$  corresponds to the coupon payment of a firm having  $CF_1$  and the settlement term of carry forward is five years. Parameters:  $A_0 = 100, CF_1 = 20, r = 0.05, \sigma = 0.2, \tau = 0.25$ .

$D$	30	40	50	60	70	80
$\mathcal{C}$	1.5405	2.0746	2.6678	3.3897	4.3452	5.8083
$\mathcal{C}_{cf}$	1.5405	2.0742	2.6652	3.3806	4.3198	5.7395

**4.5.3. Carry back.** The coupon payments  $\mathcal{C}_{cb}$  for firms with carry back are lower than  $\mathcal{C}$  and  $\mathcal{C}_{cf}$ , as can be seen from [Table 4.5](#). By extensive simulation for a large number of different parameter settings we find that this is always the case. There are two findings that shed light on this phenomenon. [Example 4.20](#) illustrates that there are no scenarios that lead to negative carry back values, assuming a one year duration of carry back. Secondly, [\(4.23\)](#) shows that the DTA value of carry back increases when a firm is levered.

**Table 4.5.** Coupon payments for different levels of debt with face value  $D$  when carry back is allowed for one year and losses can be carried forward five years. For firms with carry forward/back we assume  $CF_1 = CB_1 = 20$ . Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.25, \tau = 0.2$

$D$	30	40	50	60	70	80
$C$	1.5406	2.0733	2.6617	3.3700	4.3003	5.7042
$C_{cf}$	1.5406	2.0734	2.6617	3.3693	4.2955	5.6816
$C_{cb}$	1.5397	2.0680	2.6436	3.3265	4.2076	5.4745



**Figure 4.16.** DTA/DTL value of carry forward, carry back and untaxed profit  $gain_1$  for levered undertakings as a function of debt  $D$  in the 5-period model. The deferred taxes are set such that  $CF_1 = CB_1 = gain_1 = 20$  and coupons  $C_{cf}, C_{cb}, C_L$  are determined such that debt sells at par value. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, \gamma = 1$ . Duration of carry forward/back is 5/1 year(s) respectively.

These observations induce lower coupon payments for firms with carry back.

As in the 1-period model, leverage tends to increase the DTA value of carry back. This is also visible in Figure 4.16, which shows that carry back is

positively related to leverage. Higher leverage ratios result in more coupon payments. Consequentially, a firm is more likely to incur a loss which makes it more probable that the initial carry back ( $CB_1$ ) is settled during the next period. For low levels of leverage, the DTA value of  $CB_1$  is rather small and less than the DTA value of  $CF_1 = 20$  (see [Figure 4.16](#)). However, a vast increase in leverage may give the DTA value of carry back such a boost that it will eventually be more valuable than carry forward.

**4.5.4. DTL.** We only examine the effect of leverage on the DTL value in the **more realistic case** (ii). In this model, leverage has a damping effect on the DTL values in between, since the DTL is reduced each time the firm incurs a loss. With leverage, it is more likely that the firm incurs a loss in each period. This loss mitigates the overall DTL value (in absolute terms) before maturity, whence less tax liabilities are due at the end of the  $T$ -period model. This is also visible from [Figure 4.16](#), which shows that the DTL goes faster to zero for higher levels of debt, resulting from the exponential increase in coupon payments.

## 4.6. Optimal capital structures

The previous sections on DTA/DTL values with leverage necessitate the calculation of firm values. The simulation set-up for those computations introduces a *trade-off* between the tax shield (arising from leverage) and bankruptcy, which can be triggered if a firm maintains unsustainable high levels of debt. Bankruptcy occurs if the firm fails to repay  $D$  at maturity, or if the post-tax asset value drops below the bankruptcy trigger  $K$  prior to maturity. The tax shield increases in value when a firm raises higher levels of debt, but these tax benefits are lost in case of bankruptcy.

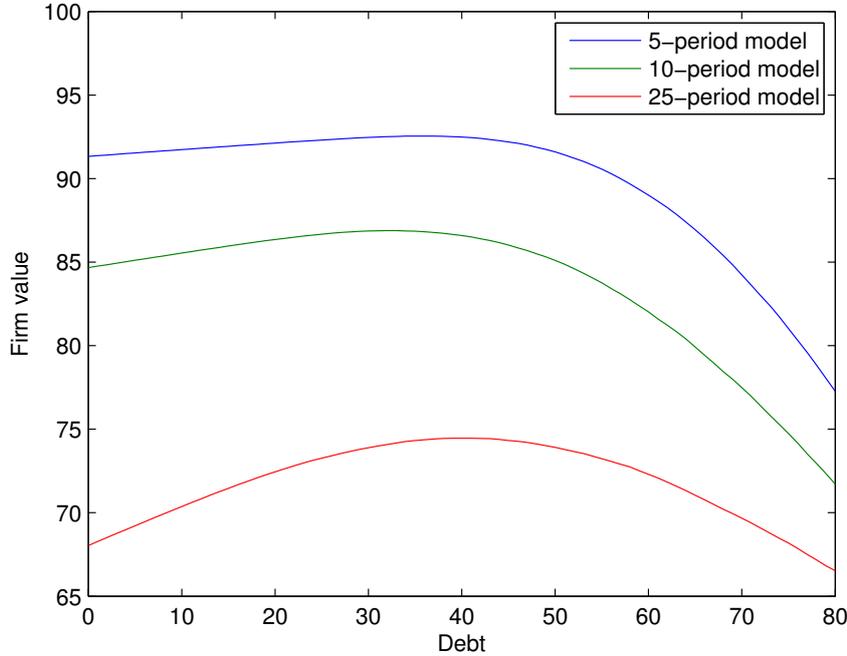
The trade-off arising in this framework suggests there is an optimal level of debt which maximizes firm value. As in the previous sections, the coupon payments are chosen such that debt sells at par value. In addition, the bankruptcy trigger  $K$  prior to maturity (see [\(4.16\)](#)) equals the face value of debt, i.e.  $K = D$ . Finally, as in [Leland and Toft \(1996\)](#), bankruptcy cost amount to  $\alpha = 0.5$ . [Figure 4.17](#) shows the resulting firm value as a function of debt in different  $T$ -period models. In each of these models, there is an optimal level of debt  $D$  that maximizes firm value. The coupon payments  $\mathcal{C}$  corresponding to these levels of debt are presented in the upper half of [Table 4.6](#), together with the maximum premium the firm can get by debt financing (*maximum leverage premium*). The leverage premium is increasing for higher  $T$ -period models, since tax savings arising from debt financing accrue over time. Bankruptcy cost engender lower optimal leverage levels,

as the influence on firm value is more pronounced when bankruptcy is triggered.

Table 4.6 shows that optimal capital structures are not monotonically increasing over time, as debt in the 10-period model is less compared to the 5-period model. This contrasts the empirical findings of [Barclay and Smith \(1995\)](#), who find that optimal leverage ratios increase over longer periods of debt issuance. The main factor responsible for this disparity comes from our assumption that coupon payments *reduce* the asset value, which affects bankruptcy conditions before maturity. This effect is more pronounced over longer periods of time, as coupon payments in between increase the probability of default. Some influential papers on capital structuring ([Brennan and Schwartz \(1978\)](#), [Leland \(1994\)](#) and [Leland and Toft \(1996\)](#)) bypass this assumption by assuming that any cash outflow related to leverage is financed by selling additional equity (see also [Remark 3.3](#)). Incorporating this assumption in our model would significantly alter optimal capital structures.

**Table 4.6.** Optimal debt levels in various  $T$ -period models. Coupon payment  $\mathcal{C}$  is determined such that debt sells at par value. The *maximum leverage premium* is the difference between columns four and three. The *exogenous* bankruptcy case takes  $K = D$ . The *endogenous* bankruptcy case finds the optimal  $K^*$  implied by the model. Parameters:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $\tau = 0.25$ ,  $\alpha = 0.5$  and the settlement term of carry forward/back is 10/1 year(s) respectively. The firm has no initial deferred taxes.

	Optimal debt	Coupon ( $\mathcal{C}$ )	Unlevered firm value	Optimal firm value	Maximum leverage premium	$K^*$
Exogenous bankruptcy						
5-period model	35.00	1.88	91.33	92.55	1.22	-
10-period model	33.00	1.74	84.67	86.89	2.22	-
25-period model	42.00	2.25	68.04	74.46	6.42	-
Endogenous bankruptcy						
5-period model	37.56	1.98	91.33	92.93	1.60	25.52
10-period model	39.57	2.16	84.67	89.00	4.33	15.70
25-period model	28.64	1.52	68.04	75.31	7.09	12.98



**Figure 4.17.** Firm value of undertaking without deferred taxes in different  $T$ -period models. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, \gamma = 1$ . The firm has no initial deferred taxes, i.e.  $CF_1 = CB_1 = gain_1 = 0$ . Settlement term of carry forward/back is 10/1 year(s) respectively.

Instead of imposing the exogenous bankruptcy condition  $K = D$  determined by a covenant, we consider the case when  $K$  is chosen by the firm. Following Leland (1994), we assume the company chooses  $K$  so to maximize firm value, under the additional constraint of staying in a particular rating class. For example, we could assume a *probability of default* (PD) threshold of 10%. This could be motivated by the desire of firms to stay in a certain credit rating class after issuing new debt. In the 5-period model, a threshold of 10% roughly corresponds to a firm which is Ba rated according to Moody's.<sup>17</sup> In the 10- and 25-period model, this corresponds to an undertaking which is Baa and A rated respectively. The resulting (multivariate) constrained optimization problem is thus to find  $D$  and  $K$  to maximize firm value, such that  $PD \leq 0.1$ . The optimal  $K$  (denoted by  $K^*$ ) and  $D$  are reported in the second half of Table 4.6. The firm values compare favorably to those when  $K$  is determined exogenously. The additional freedom to chose

<sup>17</sup>See Table IV in [http://pages.stern.nyu.edu/~eelton/working\\_papers/corp%20bonds/all%20tables%20and%20figures%201.pdf](http://pages.stern.nyu.edu/~eelton/working_papers/corp%20bonds/all%20tables%20and%20figures%201.pdf)

$K$  outweighs the confinement that  $\text{PD} \leq 0.1$ , which is not required in the exogenous bankruptcy case.

The firm values in [Table 4.6](#) for different  $T$ -period models should not be compared literally. The firm value corresponding to the 5-period model is obtained by simulating asset paths over five years, whereas the firm value in the 25-period model results from simulating asset paths over 25 years. Without tax payments, the discounted asset path is a martingale, but including tax payments means that the asset process is a *supermartingale*. Hence, the firm value tends to go down over longer periods of time. If we assume that a firm issues new debt with maturity  $T$  whenever it manages to repay outstanding debt with maturity  $T$  and simulate the model for 25 years, then we find that firm values are monotonically increasing with debt maturity. This is consistent with [Leland and Toft \(1996\)](#).

# Empirical applications and agency problems

## 5.1. Loss absorbing capacity of deferred taxes for European insurers

As a practical implementation of the theory developed in previous chapters, we analyze the impact of our new valuation approach to the *loss absorbing capacity* of deferred taxes on European insurance undertakings. The recently established *Solvency II* regulations (analogue of Basel III for insurers) dictate that European insurers should maintain a *Solvency ratio* ratio greater than one, where

$$\text{Solvency ratio} = \frac{\text{Eligible own funds}}{\text{Solvency capital requirements}}.$$

The *Solvency capital requirements* (SCR) are calculated such that an insurer can withstand a shock that occurs once every 200 years (this is essentially the 99.5% VaR). The *standard formula* used to calculate the SCR makes use of a modular approach (EIOPA, 2014). This means that the overall risk is subdivided into sub risks and sub-sub risks. For each sub risk (or sub-sub risk) one calculates the capital requirements (corresponding to a 99.5% VaR over a one year period). All these capital requirements are aggregated using correlation matrices, which results in the Solvency capital requirements (EIOPA, 2014). We showed in Section 1.1 that part of a shock is absorbed by deferred taxes, as such anomalies mitigate DTL's or create additional carry forward when the net DTA position is positive. Basically, undertakings transfer part of the loss to the tax authority, as it reduces future taxable income. The Solvency II guidelines take this *loss absorbing capacity*

of deferred taxes (LAC DT) into account by subtracting this from the Solvency capital requirements. Hence, when taking LAC DT into account, the Solvency ratio follows from

$$(5.1) \quad \text{Solvency ratio} = \frac{\text{Eligible own funds}}{\text{Solvency capital requirements} - \text{LAC DT}}.$$

Suppose that LAC DT is 25% of the Solvency capital requirements, then incorporating LAC DT can increase the Solvency ratio from 100% to 133%. By definition, LAC DT is the difference in net DTA position *post* and *ex-ante* shock, i.e.  $\text{LAC DT} = \text{post shock net DTA} - \text{ex-ante shock net DTA}$ . The maximum LAC DT is equal to the tax rate multiplied by the magnitude of the shock, but in reality these values are often lower since insurers cannot substantiate enough future profits to prove that the deferred tax asset will be settled completely.

**Example 5.1.** Suppose  $A_0 = 100$ ,  $CF_1 = 10$  and  $SCR = 40$ , i.e. the undertaking loses maximally 40 following a shock that is bound to occur once every 200 years. The loss of 40 immediately raises carry forward to  $CF_1 = 50$ . If the undertaking can substantiate enough future profits to prove that the additional carry forward will be settled completely, then  $\text{LAC DT} = \tau(50 - 10) = \tau \cdot 40$ , which is the entire shock loss times the corporate tax rate. If the undertaking expects to settle only 20 of the shock loss of 40 (because future profits are not sufficient), then  $\text{LAC DT} = \tau(30 - 10) = \tau \cdot 20$ .

**5.1.1. Market consistent approach.** This section is deleted, because confidential data are used.

**5.1.2. Implication for policymakers.** This section is deleted, because confidential data are used.

## 5.2. BP tax loss estimates

The methodology developed in previous chapters is sufficiently flexible to cover a wide range of problems. In this section, we analyze the effect of the U.S. tax overhaul on deferred tax assets. In particular, we investigate the influence of tax cuts on the DTA value of BP, following the widely covered news reports about their estimated losses arising from tax reforms.<sup>1</sup> Among others, tax reforms in the U.S. constitute a change in the corporate tax rate from 35% to 21%. Naturally, such a tax cut positively influences firm value, but extant deferred tax assets are adversely affected by such revisions, as the value positively depends on the tax rate  $\tau$ . In BP's *fourth quarter 2017 results announcement*, it is estimated that tax reforms induce a one-off charge of \$859 million (BP, 2018). We adapt our simulation of the  $T$ -period model

<sup>1</sup>See for example <https://www.ft.com/content/5d477db2-efac-11e7-ac08-07c3086a2625>

with leverage to give a market consistent estimate of the deferred tax loss arising from the U.S. tax overhaul. Since it is not clear-cut how to calibrate all parameters from the model, we use the [Merton \(1974\)](#) model on credit risk to obtain proxies for some of the input variables.

In particular, daily stock prices ( $S_{t_i}$ ) of BP over the period August 31<sup>st</sup>, 2016 - December 30, 2016 are used to obtain a time series of market capitalizations ( $E_{t_i}$ ) over the same period by multiplying the stock price with the number of outstanding shares. [Table 5.1](#) lists the number of shares outstanding, as well as other market data needed for the simulation. The value of debt due at maturity ( $D$ ) is taken to be a fraction of the market capitalization at time  $t_0$ . The fraction comes from the book values of debt and equity reported on page 23 of [BP \(2018\)](#) over the year 2016. This leads to the debt value

$$(5.2) \quad D = \frac{\text{Gross debt}}{\text{Equity}} \cdot E_{t_0} = \frac{58.3}{96.843} \cdot E_{t_0} = 383,559.1269 \cdot 10^6.$$

Subsequently, the time series of market capitalizations are used to estimate the implied asset volatility  $\hat{\sigma}_A$  using the maximum likelihood (MLE) method of [Duan \(1994\)](#). The [Merton \(1974\)](#) model on credit risk gives that

$$E_{t_i} = C^{\text{BS}}(D, T, A_{t_i}, \sigma, r, t_i),$$

where maturity date  $T$  is usually interpreted as the duration of debt. The optimization problem to find the MLE can be written down explicitly

$$(5.3) \quad (\hat{\sigma}_A, \hat{\mu}_A) = \underset{\sigma, \mu}{\operatorname{argmax}} \mathcal{L}_E(\hat{A}_{t_1}(\sigma), \dots, \hat{A}_{t_N}(\sigma); \sigma, \mu) - \sum_{i=1}^N \log(\Phi(d_i(\sigma))).$$

Let  $\Delta t_i$  be the time between two consecutive observations, i.e.  $\Delta t_i = t_{i+1} - t_i$ , then [Duan \(1994\)](#) finds that the log-likelihood function is

$$\begin{aligned} \mathcal{L}_E(E_{t_1}, \dots, E_{t_N}; \sigma, \mu) &= -\frac{N}{2} \log(2\pi\sigma^2\Delta t_i) - \frac{1}{2} \sum_{i=1}^N \frac{(R_i - (\mu - \sigma^2/2)\Delta t_i)^2}{\sigma^2\Delta t_i} - \sum_{i=1}^N \log(E_i) \\ R_i &= \log \frac{E_{t_i}}{E_{t_{i-1}}} \\ \hat{A}_{t_i} &= (C^{\text{BS}}(E_{t_i}; \sigma))^{-1} \text{ (Inverse of Black-Scholes call option)} \\ d_i &= \frac{\log(\hat{A}_{t_i}/D) + (r + \sigma^2/2)(T - i\Delta t_i)}{\sigma\sqrt{T - i\Delta t_i}}. \end{aligned}$$

The resulting solution gives an estimated time series of asset values  $\hat{A}_{t_i}$  and an estimate of the asset volatility  $\hat{\sigma}_A$ .<sup>2</sup> The duration of debt  $T$  is required as input for this model. The book value of debt  $D$  usually combines a number

<sup>2</sup>One also obtains an estimate  $\hat{\mu}_A$ , but this is immaterial for subsequent valuation purposes.

of debt issuance with different maturities. To resolve this, the optimization problem (5.3) is iterated over discrete debt maturities  $T \in [1, 30]$  and we choose the optimal  $T$  (denoted by  $T_{\text{optimal}}$ ), corresponding to the model that attains the highest log-likelihood. This is somewhat similar to the approach of Engle and Siriwardane (2017). For each discrete  $T \in [1, 30]$ , we set  $r = r_{\text{forward},T}$ , where  $r_{\text{forward},T}$  is the nominal yield on average U.S. treasury coupon issues  $T$ -years from December 31, 2016.<sup>3</sup> The parameter values thus obtained are summarized in Table 5.2.

**Table 5.1.** Input variables taken from the market. *Shares outstanding* is the basic weighted average number of shares outstanding over 2016, see page 22 of BP (2018). *Carry forward* ( $= CF_1$ ) is the DTA value over 2016 on page 15 of BP (2018), converted to nominal scale (assuming the tax rate  $\tau = 0.35$ ). Hence, we tacitly assume that all DTA's are coming from loss carry forward. The single debt issuance  $D$  comes from (5.2). All numbers are reported in millions ( $10^6$ ).

Shares outstanding	Carry forward	$D$
18,744,800	13,545.7143	383,559.1269

**Table 5.2.** Parameter values resulting from the optimization problem (5.3).

$\hat{\sigma}_A$	$\hat{A}_{t_N}$	$T_{\text{optimal}}$	Log-likelihood
0.2656	$834,395.7324 \cdot 10^6$	20	-2122.4376

Coupon payments  $\mathcal{C}$  are determined *exogenously*, meaning that we take the coupon rate ( $r_{\text{coupon}}$ ) from the market and determine the coupon payment using  $\mathcal{C} = (\exp(r_{\text{coupon}}) - 1) \cdot D$ . We have not been able to find data on coupons issued by BP with maturity  $T_{\text{optimal}} = 20$ . The best we could find is a coupon issued by BP with face value \$1000, maturing at September 19, 2027.<sup>4</sup> The rate on this coupon and the forward rate corresponding to  $T_{\text{optimal}}$  are respectively given by

$$r_{\text{coupon}} = 0.0328, \quad r_{\text{forward},20} = 0.0268.$$

As in Section 5.1.1, we set the bankruptcy trigger  $K = 0$  and bankruptcy cost  $\alpha = 0$ , since no reliable data were available. The parameters described above remain constant throughout the simulation that is needed to assess

<sup>3</sup>These data are coming from <https://www.treasury.gov/resource-center/economic-policy/corp-bond-yield/Pages/TNC-YC.aspx>. Accessed: 2018-03-12.

<sup>4</sup>These data are coming from <http://cbonds.com/emissions/issue/357247>. Accessed: 2018-03-14.

the impact of the U.S. tax overhaul on BP's deferred tax assets. The most important changes in the U.S. tax regime are shown in Table 5.3. To estimate the effect on DTA's following such a regime change, we reassess the DTA value of carry forward in Table 5.1 using the parameters under *current law* from Table 5.3. This renders the firm values  $\widehat{V}$ ,  $\widehat{V}_{cf}^{BS}$  and the market consistent DTA under current law follows from

$$(5.4) \quad \hat{\xi}_{cf} = \widehat{V}_{cf}^{BS} - \widehat{V}.$$

In a second step, we simulate the asset paths over one year using the current law. From year two, corresponding to time January 1, 2018, the parameters of the G.O.P. Bill are used. This includes the cap of 80% carry forward on taxable income. Once again this gives estimates of the firm values  $\widehat{V}$ ,  $\widehat{V}_{cf}^{BS}$  and the new DTA value

$$(5.5) \quad \hat{\xi}_{cf} = \widehat{V}_{cf}^{BS} - \widehat{V}.$$

The impact of the tax overhaul on deferred tax assets of BP is then simply the difference between (5.5) and (5.4). In this way, we find that the loss is only \$203.122 million dollars, significantly less than BP's own estimate of \$859 million dollars. However, this disparity should be interpreted with some care, since we have not been able to inspect BP's own calculations leading to their estimated loss.

**Table 5.3.** Most important legislative transitions in the U.S. corporate tax overhaul. The *current law* column describes the extant tax laws, whereas the *G.O.P. Bill* column describes the proposed tax changes of the Trump administration.

	Current law	G.O.P Bill
Corporate tax rate ( $\tau$ )	35%	21%
Business interest-deduction	Generally fully deductible ( $\gamma = 1$ )	Cap deduction at 30% of income ( $\gamma = 0.3$ )
Carry forward	20 years settlement; can potentially offset all taxable income	Infinite duration; limit deduction to 80% of taxable income
Carry back	2 year settlement	No carry back allowed

**5.2.1. Simulation with random tax regimes.** In the previous section we analyzed the loss of deferred tax assets following the transition in corporate taxes. However, those results were obtained using a deterministic change after year one. In this section we consider random changes in the corporate tax rate and assess its effect on deferred taxes. To this extent, all variables during the simulation are kept fixed, except for the corporate tax rate. The corporate tax rate is assumed to be in one of two states, i.e.  $\tau \in \{\tau_1, \tau_2\}$ , where  $(\tau_1)$  is the low tax rate and  $(\tau_2)$  represents the high tax regime. In line with [Hassett and Metcalf \(1999\)](#), we assume that the corporate tax rate follows a Poisson process randomly switching between  $\tau_1$  and  $\tau_2$ . In particular, the dynamics of the corporate tax rate is described by

$$d\tau_t = \begin{cases} \Delta\tau & \lambda_{1,t}dt \\ 0 & (1 - \lambda_{1,t})dt & \tau = \tau_0 \\ -\Delta\tau & \lambda_{0,t}dt \\ 0 & (1 - \lambda_{0,t})dt & \tau = \tau_1, \end{cases}$$

where  $\Delta\tau = \tau_1 - \tau_0 > 0$ . This means that the corporate tax rate randomly switches between  $\tau_0$  and  $\tau_1$  with (Poisson) transition parameters  $\lambda_{1,t}$  and  $\lambda_{0,t}$ . The inverse of the transition parameters  $\lambda_{0,t}^{-1}, \lambda_{1,t}^{-1}$  represents the expected time in each tax regime before a change occurs. The average time spend in the high tax regime equals  $\lambda_{0,t}^{-1}$  and the fraction of time spend in this regime is given by  $\lambda_{1,t}/(\lambda_{0,t} + \lambda_{1,t})$ . [Table 5.4](#) shows that a negative change in the tax rate does not necessarily imply that carry forward is also negatively affected. Over moderate time horizons, the DTA value is indeed inversely related to a tax cut, since the influence of carry forward is less pronounced when tax rates are lower. However, there is an opposing effect which is not prominent over short time horizons, but plays an important role on firm values over extended time horizons. Namely, lower tax rates cause more divergence between the value of a reference undertaking and the value of a firm with carry forward. The example below illustrates this.

**Example 5.2.** Consider a 2-period model, with  $A_0 = 1000$ ,  $r = 0.05$  and  $CF_1 = 30$  and suppose the tax rate jumps after year one from  $\tau_1 = 0.4$  to  $\tau_2 = 0.2$ . In a risk neutral world, the assets grow with rate  $\exp(r) \approx 1 + r$ . Consider the DTA value in a world without tax jumps. The post-tax asset values of the reference undertaking are given by

$$\begin{aligned} \tilde{A}_1 &= (1 + r)A_0 - \tau_2((1 + r)A_0 - A_0)^+ = 1030 \\ \tilde{A}_2 &= (1 + r)\tilde{A}_1 - \tau_2((1 + r)\tilde{A}_1 - \tilde{A}_1)^+ = 1060.9. \end{aligned}$$

The post-tax asset value of a firm with carry forward follows from

$$\begin{aligned}\tilde{A}_1 &= (1+r)A_0 - \tau_2((1+r)A_0 - A_0 - CF_1)^+ = 1042 \\ \tilde{A}_2 &= (1+r)\tilde{A}_1 - \tau_2((1+r)\tilde{A}_1 - \tilde{A}_1)^+ = 1073.26.\end{aligned}$$

Thus the DTA value in period two equals  $\xi_{cf} = 1073.26 - 1060.9 = 12.36$ . Now, suppose a transition in the corporate tax rate is imposed after year one, so that in year two the corporate tax rate is given by  $\tau_2$ . In this case, post-tax asset values for the reference undertaking are given by

$$\begin{aligned}\tilde{A}_1 &= (1+r)A_0 - \tau_2((1+r)A_0 - A_0)^+ = 1030 \\ \tilde{A}_2 &= (1+r)\tilde{A}_1 - \tau_1((1+r)\tilde{A}_1 - \tilde{A}_1)^+ = 1071.2.\end{aligned}$$

The post-tax asset value for a firm with carry forward is given by

$$\begin{aligned}\tilde{A}_1 &= (1+r)A_0 - \tau_2((1+r)A_0 - A_0 - CF_1)^+ = 1042 \\ \tilde{A}_2 &= (1+r)\tilde{A}_1 - \tau_1((1+r)\tilde{A}_1 - \tilde{A}_1)^+ = 1083.68.\end{aligned}$$

Hence, the DTA value in period two is  $\xi_{cf}^{(1)} = 1083.68 - 1071.2 = 12.48 > \xi_{cf}$ . In this example, firm values are further apart following a tax transition since the DTA from carry forward is completely settled before the tax overhaul. In higher  $T$ -period models, this disparity might still persist even if part of (or the entire) carry forward is settled in low tax regimes. The reason for this is similar to the example given above, namely that lower tax rates cause a greater discrepancy in firm value between a reference undertaking and the firm with carry forward. This can eventually crowd-out the tax transition effect. These dynamics can also provoke an increase in DTA value when moving from a low to high tax regime.

**Table 5.4.** DTA value of carry forward with and without tax change in different  $T$ -period models, using  $\lambda_{1,t} = \lambda_{0,t} = 0.35$ . The low and high tax rates are respectively given by  $\tau_1 = 0.2$  and  $\tau_2 = 0.45$ . Both models start with the high tax rate  $\tau_2$ . Additional parameters:  $A_0 = 1000$ ,  $r = 0.05$ ,  $CF_1 = 40$ ,  $C = 5$ ,  $\gamma = 1$ . Settlement term of carry forward is ten years and carry back can be settled for one year.

	5-Period model	15-Period	30-Period	50-Period
Without tax change	10.43	9.24	6.19	3.65
With tax change	9.69	9.08	6.38	4.82

### 5.3. Tax smoothing effects and agency problems

Deferred tax assets like carry forward and carry back give an undertaking the possibility to *smooth* their tax expenses. In this section we quantify how much firms gain by deferred tax possibilities. As a *base case*, we consider a tax regime in which no carry forward/back possibilities are allowed. [Table 5.5](#) shows the relative gain of introducing deferred taxes with different settlement terms. In the most extreme case, when profit and losses can be carried back and forward indefinitely, a firm spends 40% less on taxes. By extensive simulation, we find that these ratios are independent of the initial asset value ( $A_0$ ). The difference for distinct volatility values ( $\sigma$ ) is neither pronounced, although higher volatility values tend to increase the ratios by a few percent. For small  $T$ -period models, the ratios differ significantly from those in [Table 5.5](#), however for  $T \geq 5$  we find ratios that are commensurate with the ones in [Table 5.5](#); the difference being only several percentage points.

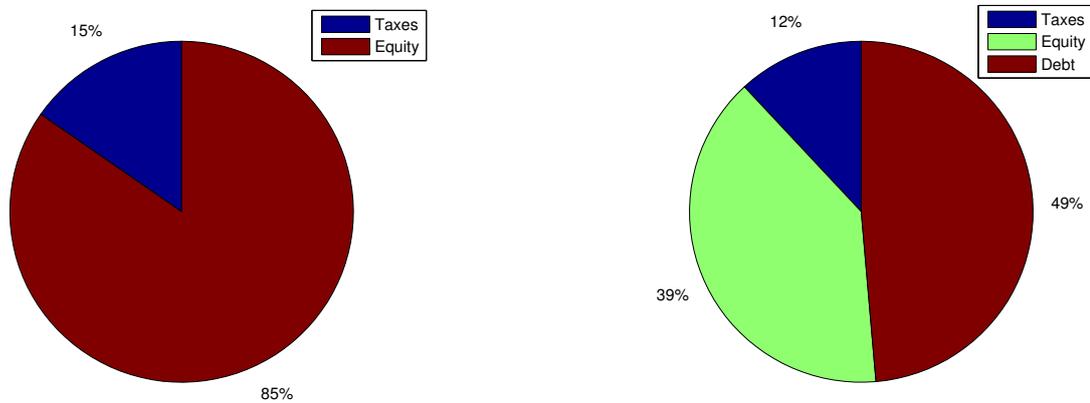
**Table 5.5.** Tax savings expressed as a fraction of the *base case* without deferred taxes in the 30-period model without leverage for different settlement terms. In particular: Tax savings =  $1 - (\text{Tax expense with DTA})/(\text{Tax expense without DTA possibility})$ . In the notation  $[a; b]$ ,  $a$  is the duration of carry forward and  $b$  concerns the duration of carry back. Parameters:  $CF_1 = CB_1 = 0$ ,  $A_0 = 100$ ,  $r = 0.05$ .

Duration	[1;0]	[1;1]	[2;1]	[3;1]	[10;1]	[30;1]	[30;30]
Tax savings	0.11	0.21	0.25	0.27	0.32	0.33	0.40

Even though the ratios in [Table 5.5](#) are roughly unaffected by volatility, the absolute tax expenses of firms are affected by volatility and this creates so-called *agency problems*. Agency problems arise when there are conflicting interests between stakeholders. [Jensen and Meckling \(1976\)](#) observe that equity holders want to increase the riskiness of a firm's activities after debt has been issued, so that value is transferred from debt to equity holders. This is the so-called *asset substitution* problem. Ignoring leverage for a moment, we see from [\(4.1\)](#) that even in the basic one-period model there is a conflict between equity holders and the tax authority. The tax authority holds a call option on the profit stream of an undertaking, which increases in value when the firm's volatility increases. This at the expense of equity holders, who find themselves in the same situation as debt holders in the original model of [Merton \(1974\)](#). By simulation we find the same conflicting interests for multi-period models with and without deferred taxes. Essentially, the same logic applies as in the one-period model, due to the

asymmetric payoff of the tax authority. At the end of each year, the tax authority does not lose anything if a firm incurs a loss, but gains substantially when profits are large. Higher volatility values make such scenarios more likely, only the precise influence of  $\sigma$  on the tax authority can no longer be quantified with analytical formulas.

The conflicting interests described above changes when a firm is levered, as a third stakeholder enters the stage; debt holders. The conflicting interests between equity holders, debt holders and the tax authority are subtle. On one hand, the tax authority may benefit from an initial increase in  $\sigma$  whenever debt has been issued, but this is accompanied with a higher likelihood of bankruptcy, causing the tax authority to lose all future tax payments. The like holds for equity holders, who may benefit from an initial increase in  $\sigma$  but in case of bankruptcy equity holders lose all money as well. To analyze the matter at hand, we include bankruptcy cost  $\alpha = 0.5$  whenever a firm defaults at or prior to maturity.



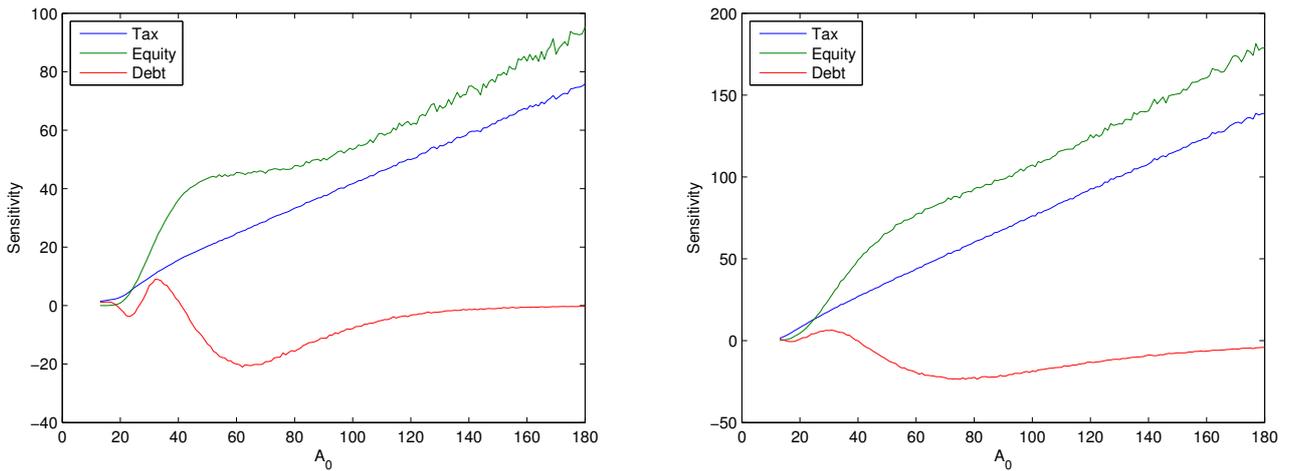
(a) Relative payout to stakeholders of unlevered firm.

(b) Relative payout to stakeholders of levered firm.

**Figure 5.1.** Pie chart of payout to stakeholders for levered and unlevered firm in the 10-period model. All payouts are discounted back to period zero. For the levered firm, parameters are chosen to match the optimal capital structure in the *endogenous bankruptcy case* in Table 4.6. This means that  $D = 39.57$ ,  $K = 15.70$ ,  $C = 2.16$ ,  $\gamma = 1$  and  $\alpha = 0.5$ . Other parameters:  $A_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $CF_1 = CB_1 = 0$  and the settlement term of carry forward/back is 10/1 year(s) respectively.

Figure 5.1 details how the cash flows of the levered and unlevered firm are divided among stakeholders in the 10-period model. The firm issues debt

$D = 39.57$  with bankruptcy trigger  $K = 15.70$  corresponding to the optimal capital structure when bankruptcy is determined endogenously in Table 4.6. All intermediate and final cash flows are discounted back to period zero, depending on the moment of reception. Panel (a) shows that the unlevered firm spends a total of 15% on taxes. The remaining part goes to equity holders (85%). The relative cash flows to stakeholders in case the firm is levered are shown in panel (b). This time, the undertaking only spends 12% on taxes and the remaining part is paid to equity holders (39%) and bondholders (49%). In this simulation example, we find that the unlevered firm pays 84.79 to equity holders, whereas the levered firm pays a total of 87.53 to equity and bondholders. Hence, the tax shield arising from debt financing allows the levered firm to pay out more to investors.



(a) Sensitivity in the 5-period model. Optimal capital parameters:  $D_0 = 37.56$ ,  $K = 25.52$ ,  $C = 1.98$ ,  $\gamma = 1$  and  $\alpha = 0.5$ . (b) Sensitivity in the 10-period model. Optimal capital parameters:  $D_0 = 39.57$ ,  $K = 15.70$ ,  $C = 2.16$ ,  $\gamma = 1$  and  $\alpha = 0.5$ .

**Figure 5.2.** Effect of an increase in risk ( $\sigma$ ) as measured by the partial derivatives w.r.t.  $\sigma$  on tax, equity and debt value in the 5-period model (panel (a)) and 10-period model (panel (b)) as a function of  $A_0$ . Parameters are chosen to match the optimal capital structure in the *endogenous bankruptcy case* when  $A_0 = 100$  in Table 4.6. Parameters:  $r = 0.05$ ,  $\sigma = 0.2$ ,  $CF_1 = CB_1 = 0$  and the settlement term of carry forward/back is 10/1 year(s) respectively.

Finally, we quantify the extent of agency cost related to the asset substitution problem when firms are levered in the 5- and 10-period model. Panel (a) in Figure 5.2 illustrates the partial derivative of tax, equity and debt value with respect to risk,  $\sigma$ , as a function of the underlying asset value in the 5-period model. For each asset value,  $\partial E/\partial\sigma > 0$ , where  $E$  is the equity

value. This means that equity holders want to increase the riskiness of the firm's activities at any level of  $A_0$ , consistent with Merton (1974). In the 5-period model, bankruptcy is triggered if  $K = 25.52$  (see Table 4.6). The sign of  $\partial D/\partial\sigma$  is somewhat ambiguous when  $A_0$  is close to bankruptcy. In particular, we observe that

- (i) In the range  $A_0 \in [13, 18] \cup [27, 40]$ ,  $\partial D/\partial\sigma > 0$ , i.e. debt holders also want to increase the riskiness of the firm's investments. In this case, equity and debt holders do *not* have conflicting interests.
- (ii) When  $A_0 \in [19, 26] \cup [41, 180]$ ,  $\partial D/\partial\sigma < 0$ . In this range, agency problems emerge due to conflicting interest between debt and equity holders.

Moreover, panel (a) also reveals that for each asset value  $A_0$ , the tax authority benefits from an increase in risk of the firm's investment. The loss of tax payments in case of bankruptcy do not weigh up against the increase in tax payments when a firm engages in more risky business. Even though the option analogy in multi-period models is no longer exact, the conclusion that equity holders and the tax authority want to increase risk at all times is similar to that of the 1-period model. Panel (b) of Figure 5.2 gives an analogous conclusion for the 10-period model. In both panels,  $\partial D/\partial\sigma$  approaches zero from below as  $A_0 \rightarrow \infty$ , since debt becomes essentially risk-free in those cases. Following Leland and Toft (1996), one could measure the asset-substitution agency costs as the *Lebesgue*-measure of all asset values  $A_0$  for which  $\text{sign}(\partial D/\partial\sigma) \neq \text{sign}(\partial E/\partial\sigma)$ . Using this definition, agency costs in this model are lower compared to those of Merton's model, since our model yields a range of values (case (i)) for which  $\text{sign}(\partial D/\partial\sigma) = \text{sign}(\partial E/\partial\sigma)$ . Contrarily, in the Merton (1974) model, there do not exist asset values which lead to similar signs in equity and debt value sensitivities w.r.t  $\sigma$ , since bankruptcy and tax benefits are not taken into account. In our model, the tax shield benefit and its loss in case of bankruptcy implies that equity and debt holders do not split a claim that only depends on the underlying asset value. This gives rise to a range of values where equity and bondholders have similar objectives. However, by letting  $\tau, \alpha \rightarrow 0^+$ , one recovers the split claim and the game between debt and equity holders approaches a *zero-sum game*. In this case, conflicting interests between the two stakeholders are most pronounced, akin to the findings of Leland and Toft (1996).

**Remark 5.3.** One should be careful by concluding that the tax authority wants to increase the riskiness of a firm's activities at all times. For example, the bankruptcy of a systemically important undertaking has repercussions for many other actors in the economy. We have not taken into consideration the amplified loss of tax payments when the bankruptcy of one

company triggers bankruptcy of another undertaking. The interconnectedness of bankruptcy events would alter the incentives of the tax authority. It would be an interesting research question to investigate tax payments on aggregate level and how this influences the tax authorities' incentives in light of the asset substitution problem.

## Conclusion

We propose a market consistent valuation of deferred taxes based on the option interpretation of tax payments. This leads us to express DTA/DTL values as the difference between Black-Scholes call option formulas over a one year time horizon. These formulas offer clear insight in the contingent nature of deferred taxes and avoid the necessity of subjective profit forecasts that are needed for extant accounting valuation techniques. Moreover, the market consistent model acknowledges value creation due to loss carry back, which is not recognized by applicable accounting standards.

Over multiple time periods, valuation results for deferred taxes are obtained by simulation, since path dependency eschews the tractability of analytical formulæ. The option interpretation is no longer exact, but the shape of the pricing formulas for different parameter values bears strong resemblance to the one-period model. In this case, the valuation also depends on the settlement term of carry forward/back and whether carry back is allowed or not. Extending the  $T$ -period model by an extra year negatively influences DTA value, since an undertaking with the initial tax advantage is expected to pay more taxes after the DTA has been settled, compared to a reference undertaking that does not have the tax benefit.

Moreover, we make a clear distinction between levered and unlevered firms. Coupon payments resulting from debt financing mitigates taxable income, thereby negatively influencing carry forward values, but igniting a positive effect on carry back value. Leverage also reduces tax liabilities (in absolute terms) arising from temporary differences. The option interpretation of deferred taxes can also be applied to tax shield valuations, which

leads us to an alternative version of the Modigliani-Miller theorem (Modigliani and Miller, 1963), essentially containing the latter as a special case. In light of these results, optimal capital structures emerge as a consequence of the trade-off between tax benefits due to coupon payments and bankruptcy scenarios. Optimal leverage ratios are maximal in the 25-period model when bankruptcy is determined exogenously. In contrast, a firm should maintain lower leverage ratios when the bankruptcy trigger is determined endogenously. The extra freedom for the firm to choose the bankruptcy trigger themselves positively influences firm value. Finally, we show that this model renders lower agency cost in comparison to Merton (1974) when considering the asset substitution problem.

Lastly, the model is flexible enough to cover a wide range of practical applications. The *loss absorbing capacity* of deferred taxes of European insurance undertakings is reassessed using the market consistent approach. Hereby, we find that the loss absorbing capacity is less than anticipated, but the overall effect is relatively moderate. Secondly, the loss of deferred tax assets of BP following the U.S. tax overhaul is examined. We find that the loss is only 25% of BP's own tax loss estimate. In conclusion, we offer an omnibus framework that offers clear insight in the contingent nature of deferred taxes relevant to virtually all undertakings with deferred taxes on the balance sheet.

### Shortcomings of approach

The valuation approach developed in this thesis derives many of the assumptions from Black and Scholes (1973). It has been shown that these assumptions are valid only under restrictive circumstances, thus our model inherits the essential weaknesses of the Black-Scholes model. This is best illustrated by the empirical applications, which necessitate the input of forward rates varying over time. Under the Black and Scholes (1973) assumptions, there is only one risk-free asset, which is constant over time. Neither does our model allow for time varying volatility, which is a ubiquitous phenomenon in financial time series data.

The empirical applications also reveal that the model supported in this thesis is not easily calibrated. The model relies on parameters observed in the market, but quantities like total asset value or asset volatility are usually unobserved. It is not clear how to derive those quantities endogenously, relying solely on the market consistent valuation approach. In BP's case, we resolve this by relying on proxies inspired by the Merton (1974) model on

credit risk, which is quite different from the credit model developed in this thesis. Such proxies may entail wrong estimates of the factual market values and consequentially lead to wrong inferences about deferred tax values.

Moreover, we assume that taxable income consists of the difference in asset value over two consecutive periods. This assumption could be defended well for particular industries like insurance undertakings, but in general there can be a great disparity between applicable taxable income and the difference in asset value. Such situations engender considerable dissonance amongst estimates resulting from our model. One could dispose this assumption and directly model the profit stream as a stochastic process. However, the option interpretation of deferred taxes would still be valid.

### Suggestions for further research

The model in this thesis uses martingale pricing to eliminate the average growth rate  $\mu$  in (3.1). It is interesting to investigate the implication of pricing (3.1) under physical measure  $P$ , instead of martingale measure  $Q$ . In particular, this implies that all pricing formulas will depend on  $\mu$ , instead of the risk-free rate  $r$ . The assumption  $\mu > r$  is embedded in any reasonable pricing model, otherwise it would not make sense to invest in such a company. Since  $\mu > r$ , the substitution  $r \rightarrow \mu$  engenders higher carry forward values, because under physical measure it is more likely that an undertaking makes profit. In contrast, carry back and the DTL will reduce in value. The precise extent of the resulting valuation difference, especially over longer periods of time, remains unclear.

Secondly, as mentioned in [Example 4.19](#), it might not always be beneficial for undertakings to use all carry forward in a certain time period. An interesting research question is to investigate the optimal way in which carry forward should be settled, so to maximize firm value. The machinery of stochastic optimal control, in particular the *Itô-Bellman* equation, could be employed to approach such a problem. However, the path dependency embedded in our model makes it unlikely that the problem can be solved analytically.



# Elements of financial mathematics

This appendix serves as an overview of some of the core parts of financial mathematics that underlies the valuation approach in this thesis. Our overview is necessarily somewhat parsimonious, additional background information can be found in some of the references we include.

## A.1. Martingale pricing

The useful concept of a *stopping time* is used to model stopping rules available up to time  $t$ . Moreover they provide a useful generalization of martingales. In formal terms we have

**Definition A.1.** A stopping time  $\tau$  is a random variable  $\tau : \Omega \rightarrow [0, \infty]$  such that  $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$  for each  $0 \leq t < \infty$ .

One of the core business of mathematical finance is to come up with a “correct” price of a contingent claims, which informally is a claim that can be made when certain specified outcomes occur.

**Definition A.2.** A *contingent  $T$ -claim* is a lower bounded  $\mathcal{F}_T$ -measurable random variable  $F(\omega) \in L^2(\Omega)$ .

When we have a market consisting of  $d+1$  traded securities with strictly positive prices  $P_0(t), \dots, P_d(t)$ , the following

$$\hat{P}_i(t) := \frac{P_i(t)}{P_0(t)},$$

is called the discounted price process and  $P_0(t)$  is called the *numéraire*.

**Definition A.3.** A *portfolio* is an  $\mathbb{R}^d$ -valued progressively measurable process with respect to  $\{\mathcal{F}_t\}_{t \in [0, T]}$

$$\theta(t, \omega) = (\theta_0(t, \omega), \theta_1(t, \omega), \dots, \theta_n(t, \omega))$$

satisfying

$$\int_0^T |\theta_0(t)| dt < \infty \text{ a.s. } P,$$

$$\sum_{j=1}^d \int_0^T (\theta_j(t) P_j(t))^2 dt < \infty \text{ a.s. } P \text{ for } j = 1, \dots, d.$$

The value at time  $t$  of a portfolio  $\theta(t)$  is defined by

$$V(t, \omega) = V^\theta(t, \omega) = \sum_{i=0}^n \theta_i(t) P_i(t).$$

**Definition A.4.** A portfolio  $\theta(t)$  which is self-financing is called *admissible* if the corresponding value process  $V^\theta(t)$  is  $(t, \omega)$  a.s. lower bounded, i.e. there exists  $K = K(\theta) < \infty$  such that

$$V^\theta(t, \omega) \geq -K \quad \text{for a.a. } (t, \omega) \in [0, T] \times \Omega$$

and  $V^\theta(t, \omega) \in L^2(\mathcal{F}_t, \Omega, P)$ .

**Definition A.5.** The claim  $F(\omega)$  is *attainable* if there exists an admissible portfolio  $\theta(t)$  and a real number  $z$  such that

$$F(\omega) = z + \int_0^T \theta(t) dP(t) \quad \text{a.s.}$$

**Definition A.6.** The market  $\{P(t)\}_{t \in [0, T]}$  is called *complete* if every  $T$ -claim is attainable.

The modern theory of pricing contingent claims often proceeds by finding the price that precludes *arbitrage*.

**Definition A.7.** An *arbitrage* is a portfolio with value  $V_t$  such that  $V_0 = 0$  and, for some  $T > 0$ ,

$$P(V_T \geq 0) = 1 \text{ and } P(V_T > 0) > 0.$$

A price of a contingent claim that leads to arbitrage in the sense above is obviously not stable, as people will acquire this portfolio en masse since it is guaranteed that one does not lose money by investing in this portfolio. The key idea to prevent arbitrage is by switching to another probability measure under which the discounted price process is a martingale. For this to be accomplished we need some additional definitions.

**Definition A.8.** Two (probability) measures  $P$  and  $Q$  on a measure space  $(\Omega, \mathcal{F})$  are called *equivalent* if  $P \ll Q$  and  $Q \ll P$ , i.e. they are mutually absolutely continuous. That means they agree on those set of  $\mathcal{F}$  that have probability measure zero.

**Theorem A.9** (Girsanov). *Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space with a complete filtration  $\mathcal{F}_t$  and  $B(t) = (B_1(t), \dots, B_d(t))$  a  $d$ -dimensional Brownian motion with respect to  $\mathcal{F}_t$ . Fix  $0 < T < \infty$  and let  $H(t) = (H_1(t), \dots, H_d(t))$  be an  $\mathbb{R}^d$ -valued adapted, measurable process such that*

$$(A.1) \quad \int_0^T |H(t, \omega)|^2 dt < \infty \quad \forall T < \infty, \text{ for } P\text{-almost every } \omega.$$

Define the stochastic exponential

$$Z_t := \exp \left( \int_0^t H(s) dB(s) - \frac{1}{2} \int_0^t |H(s)|^2 ds \right),$$

and a new probability measure  $Q_T$  on  $\mathcal{F}_T$  by

$$Q_T(A) = \mathbb{E}^P(1_A Z_T) \quad \forall A \in \mathcal{F}_T.$$

Then on the probability space  $(\Omega, \mathcal{F}_T, Q_T)$ , the process  $\{W(t) : t \in [0, T]\}$  is a  $d$ -dimensional Brownian motion relative to  $\{\mathcal{F}_t\}_{t \in [0, T]}$ .

**Proof.** See [Seppäläinen \(2012\)](#) Theorem 8.10. ■

**Definition A.10.** A probability measure  $Q$  defined on  $(\Omega, \mathcal{F}_T)$  which is equivalent to  $P$  is called an *equivalent martingale measure* for  $(P_0(t), \dots, P_d(t))$  if the discounted prices

$$\hat{P}_i(t) = \frac{P_i(t)}{P_0(t)}, i = 1, \dots, d \quad t \in [0, T]$$

are martingales with respect to  $Q$ .  $P_0(t)$  is the risk-free asset.

**Proposition A.11.** *If there exists an equivalent martingale measure then the market given by the price processes  $(P_0(t), \dots, P_d(t))$  contains no arbitrage opportunity.*

**Proof.** Let  $V(t)$  be a portfolio at time  $t$  such that  $V(0) = 0$ . The martingale property implies

$$(A.2) \quad \mathbb{E}^Q \left( \frac{V(T)}{P_0(T)} \middle| \mathcal{F}_0 \right) = 0.$$

Suppose  $P(V(T) < 0) = 0$  where  $P$  is the physical measure. Equivalence then gives  $Q(V(T) < 0) = 0$ . In conjunction with (A.2), standard measure theory arguments yield  $Q(V(T) > 0) = 0$ . Invoking equivalence once more in the other direction renders  $P(V(T) > 0) = 0$ . So altogether no portfolio can be an arbitrage, since any portfolio with  $V(0) = 0$  cannot be an arbitrage. ■

The foregoing suggests an effective way of finding an arbitrage free price of a contingent claim. Simply take

$$(A.3) \quad X_B^Q(t) \triangleq \mathbb{E}^Q \left( \frac{P_0(t)}{P_0(T)} B \mid \mathcal{F}_t \right).$$

However, there are some technical subtleties that we have overlooked. Most notably the existence and uniqueness of equivalent martingale measures. These issues are taken up in [Korn and Korn \(2001\)](#). It turns out that under some (unrealistic) assumptions on the underlying market you can indeed ensure existence and uniqueness. In the literature these theorems are known as the Fundamental theorems of Asset pricing.

# Valuation of carry forward in 2-period model

We present pricing formulas of carry forward in the two year model, distinguishing several cases.

## B.1. No fiscal history

Throughout the following derivations, we repeatedly appeal to this lemma (see also [Korn and Korn \(2001\)](#) Lemma 4.2)

**Lemma B.1.** *If  $X$  and  $Y$  are independent random variables with*

$$X \sim N(\mu, \sigma^2), Y \sim N(0, 1),$$

*then for  $\tilde{x}, \alpha, \beta \in \mathbb{R}, \alpha > 0$  we have*

$$\int_{\tilde{x}}^{\infty} \varphi_{\mu, \sigma^2}(x) \Phi(\alpha x + \beta) dx = P(X \geq \tilde{x}, Y \leq \alpha X + \beta) = P(X \geq \tilde{x}, Z \leq \beta),$$

*where*

$$(X, Z) \sim N\left(\begin{pmatrix} \mu \\ -\alpha\mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & -\alpha\sigma^2 \\ -\alpha\sigma^2 & 1 + \alpha^2\sigma^2 \end{pmatrix}\right)$$

*Here  $\varphi_{\mu, \sigma^2}$  is the density of the normal distribution with mean  $\mu$  and variance  $\sigma^2$ .*

**Proof.** The first equality is obvious. Write  $Z = Y - \alpha X$ , then  $\mathbb{E}(Z) = -\alpha\mu$ ,  $\text{Var}(Z) = 1 + \alpha^2\sigma^2$  and  $\text{Cov}(X, Z) = -\alpha\sigma^2$  by independence of  $Y$  and  $X$ . ■

Firstly, we focus on the pricing formula of a company without fiscal history, excluding carry back and assuming one year duration of carry forward. In particular, the value of such a firm is given in the following proposition.

**Proposition B.2.** *The value of a company without fiscal history, excluding carry back and assuming a one year duration of carry forward is given by*

$$V^{BS} = A_0 - \tau C^{BS}(A_0) - \tau A_0 \left( \Phi^{(\rho_1)}(g_1, g_2) - e^{-2r} \Phi^{(\rho_1)} \left( \zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}} \right) + (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) [(1 - \tau) \Phi(\sigma - \zeta_1) + e^{-r} \tau \Phi(-\zeta_1)] \right).$$

Here  $\Phi^{(\rho)}(x, y)$  is the distribution function of a bivariate standard normal distribution with correlation coefficient  $\rho$  and where

$$\rho_1 \triangleq \frac{-1}{\sqrt{(2)}}, \quad \zeta_1 \triangleq \frac{\sigma}{2} - \frac{r}{\sigma}, \quad g_1 \triangleq \zeta_1 - \sigma, \quad g_2 \triangleq \frac{2r/\sigma + \sigma}{\sqrt{(2)}} \\ d_{1,2} = \frac{1}{\sigma} [r + \sigma^2/2], \quad d_{2,2} = d_{1,2} - \sigma.$$

**Proof.** Our assumptions on the deferred tax possibilities imply that the asset value of such a company is worth

$$\tilde{A}_2 = A_2 - \tau(A_2 - \tilde{A}_1 - 1_{A_1 < A_0}(A_0 - A_1))^+,$$

after paying taxes in period two. By risk neutral arguments, the firm value at time zero is given by

$$\begin{aligned} V^{BS} &= \mathbb{E} \left( e^{-2r} \tilde{A}_2 | \mathcal{F}_0 \right) \\ &= \mathbb{E} \left( \mathbb{E} (e^{-2r} \tilde{A}_2 | \mathcal{F}_1) | \mathcal{F}_0 \right) \\ &= e^{-r} \mathbb{E} \left( \tilde{A}_1 - \tau C^{BS}(\tilde{A}_1 + 1_{A_1 < A_0}(A_0 - A_1)) | \mathcal{F}_0 \right) \\ \text{(B.1)} \quad &= A_0 - \tau C^{BS}(A_0) - e^{-r} \tau \mathbb{E} \left( C^{BS}(\tilde{A}_1 + 1_{A_1 < A_0}(A_0 - A_1)) | \mathcal{F}_0 \right). \end{aligned}$$

At this point set  $A_1 = A_0 \exp((r - \sigma^2/2) + \sigma X)$ , where  $X \sim N(0, 1)$ . Note that  $A_1 < A_0$  (carry forward situation) if  $X < \sigma/2 - r/\sigma \triangleq \zeta_1$ . In this situation  $\tilde{A}_1 = A_0 \exp((r - \sigma^2/2) + \sigma X)$ , this corresponds to the option

price

$$\begin{aligned}
C_1^{BS} &= A_1 \Phi(d_{1,1}) - e^{-r}(K = A_1 + A_0 - A_1 = A_0) \Phi(d_{2,1}) \\
d_{1,1} &= \frac{1}{\sigma} \left[ \ln \left( \frac{A_0 \exp(r - \sigma^2/2 + \sigma x)}{A_0} \right) + (r + \sigma^2/2) \right] \\
&= \frac{1}{\sigma} [\sigma x + r - \sigma^2/2 + r + \sigma^2/2] = 2r/\sigma + x \\
d_{2,1} &= d_{1,1} - \sigma.
\end{aligned}$$

Now consider  $X(\omega) > \zeta_1$ , in this case profit is made in period one, so that no carry forward is available in period 2. Consequentially,  $\tilde{A}_1 = A_1 - \tau(A_1 - A_0) = (1 - \tau)A_1 + \tau A_0$ . In this case the strike is equal to  $A_1$  and the Black-Scholes price reduces to

$$\begin{aligned}
\text{(B.2)} \quad C_2^{BS} &= \tilde{A}_1 \Phi(d_{1,2}) - e^{-r}(K = \tilde{A}_1) \Phi(d_{2,2}) \\
d_{1,2} &= \frac{1}{\sigma} \left[ \ln \left( \frac{(1 - \tau)A_1 + \tau A_0}{(1 - \tau)A_1 + \tau A_0} \right) + r + \sigma^2/2 \right] \text{ (At the money call)} \\
&= \frac{1}{\sigma} [r + \sigma^2/2] \\
d_{2,2} &= d_{1,2} - \sigma.
\end{aligned}$$

The latter expectation in (B.1) can be rewritten as

$$\begin{aligned}
&e^{-r} \mathbb{E} \left( C^{BS}(\tilde{A}_1 - 1_{A_1 < A_0}(A_0 - A_1)) | \mathcal{F}_0 \right) \\
&= e^{-r} \int_{\mathbb{R}} C^{BS} dQ = e^{-r} \underbrace{\int_{-\infty}^{\zeta_1} C_1^{BS} dQ}_{=I_1} + e^{-r} \underbrace{\int_{\zeta_1}^{\infty} C_2^{BS} dQ}_{=I_2}.
\end{aligned}$$

We now focus on each of the integrals separately. By the previous results this gives

$$\begin{aligned}
I_1 &= e^{-r} A_0 \int_{-\infty}^{\zeta_1} e^{r - \sigma^2/2 + \sigma x} \Phi(2r/\sigma + x) \varphi(x) dx \\
&\quad - e^{-2r} A_0 \int_{-\infty}^{\zeta_1} \Phi(2r/\sigma - \sigma + x) \varphi(x) dx.
\end{aligned}$$

Here  $\varphi(x)$  is the density of a standard normal distribution. The two integrals appearing above are amenable to analytical expressions, which follows from [Lemma B.1](#). Let us turn attention to the first integral in  $I_1$ . Using the same

notation as in Lemma B.1, we get

$$\begin{aligned} & e^{-r} A_0 \int_{-\infty}^{\zeta_1} e^{r-1/2\sigma^2+\sigma x} \Phi(2r/\sigma + x) \varphi(x) dx \\ &= A_0 \int_{-\infty}^{\zeta_1} \varphi_{\sigma,1}(x) \Phi(2r/\sigma + x) dx \\ &= A_0 P(X \leq \zeta_1, Z \leq 2r/\sigma), \end{aligned}$$

where by Lemma B.1 the distribution of  $(X, Z)$  is given by

$$(X, Z) \sim N\left(\begin{pmatrix} \sigma \\ -\sigma \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\right).$$

Usual standardization arguments imply that the first integral is equal to:

$$A_0 \Phi^{(\rho_1)}(g_1, g_2),$$

where  $\Phi^{(\rho)}(x, y)$  is the cumulative distribution function of a bivariate standard normal distribution with correlation coefficient  $\rho$  and

$$\rho_1 \triangleq \frac{-1}{\sqrt{2}}, \quad g_1 \triangleq \zeta_1 - \sigma, \quad g_2 \triangleq \frac{2r/\sigma + \sigma}{\sqrt{2}}.$$

In a similar vein, the second integral is equal to

$$e^{-2r} A_0 \Phi^{(\rho_1)}\left(\zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}}\right).$$

As a result

$$I_1 = A_0 \Phi^{(\rho_1)}(g_1, g_2) - e^{-2r} A_0 \Phi^{(\rho_1)}\left(\zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}}\right).$$

Consider now  $I_2$

$$\begin{aligned} I_2 &= e^{-r} (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) \int_{\zeta_1}^{\infty} \tilde{A}_1 d\mu \\ &= A_0 e^{-r} (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) \int_{\zeta_1}^{\infty} ((1-\tau) \exp(r - \sigma^2/2 + \sigma x) + \tau) \varphi(x) dx \\ &= A_0 (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) [(1-\tau) \Phi(\sigma - \zeta_1) + e^{-r} \tau \Phi(-\zeta_1)]. \end{aligned}$$

This means that the total value is given by

$$\begin{aligned} \text{(B.3)} \quad V^{BS} &= A_0 - \tau C^{BS}(A_0) - \tau(I_1 + I_2) \\ &= A_0 - \tau C^{BS}(A_0) - \tau A_0 \left( \Phi^{(\rho_1)}(g_1, g_2) - e^{-2r} \Phi^{(\rho_1)}\left(\zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}}\right) \right. \\ &\quad \left. + (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) [(1-\tau) \Phi(\sigma - \zeta_1) + e^{-r} \tau \Phi(-\zeta_1)] \right). \end{aligned}$$

■

## B.2. One year duration carry forward

Here we derive the value of carry forward under the standing assumption that no carry back is allowed and the duration time of carry forward is one year. This leads to the following.

**Theorem B.3.** *The value of a firm having  $CF_1$  available at time zero equals*

$$V_{cf}^{BS} = A_0 - \tau C^{BS}(A_0 + CF_1) - \tau(I_1 + I_{2,1} + I_3),$$

where

$$\begin{aligned} I_1 &= A_0 \Phi^{(\rho_1)}(g_1, g_2) - e^{-2r} A_0 \Phi^{(\rho_1)}\left(\zeta_1, \frac{2r/\sigma - \sigma}{\sqrt{2}}\right) \\ I_{2,1} &= (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2}))(A_0(1 - \tau)\Phi(\sigma - \zeta_1 - \delta) + e^{-r} \tau(A_0 + CF_1)\Phi(-\zeta_1 - \delta)) \\ I_3 &= A_0(\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2}))(\Phi(\zeta_1 + \delta - \sigma) - \Phi(\zeta_1 - \sigma)). \end{aligned}$$

The parameters are the same as in [Proposition B.2](#), i.e.

$$\begin{aligned} \rho_1 &\triangleq \frac{-1}{\sqrt{(2)}}, \quad \zeta_1 \triangleq \frac{\sigma}{2} - \frac{r}{\sigma}, \quad g_1 \triangleq \zeta_1 - \sigma, \quad g_2 \triangleq \frac{2r/\sigma + \sigma}{\sqrt{(2)}} \\ \delta &= \frac{1}{\sigma} \ln\left(\frac{CF_1}{A_0} + 1\right), \quad d_{1,2} = \frac{1}{\sigma}[r + \sigma^2/2], \quad d_{2,2} = d_{1,2} - \sigma. \end{aligned}$$

As a result, we have the following no-arbitrage value for carry forward

$$\xi_{cf}^{BS} = \underbrace{\tau(C^{BS}(A_0) - C^{BS}(A_0 + CF_1))}_{\text{Carry forward value in one period model.}} - \underbrace{\tau(I_{2,1} - I_2 + I_3)}_{>0}.$$

**Proof.** Let  $CF_1$  be carry forward available at time  $t = 1$ . If  $X < \zeta_1$ , we are in the same situation as [Appendix B.1](#). However, a firm only pays taxes in period one if  $A_1 > A_0 + CF_1$ . Again, by writing  $A_1 = A_0 \exp(r - \sigma^2/2 + \sigma X)$  we solve the equation  $A_1 = A_0 + CF_1$  and denote its solution by  $x \triangleq \zeta_1 + \delta$ , where  $\delta > 0$  and  $\zeta_1$  defined as in the previous paragraph. In fact, a straightforward calculation gives  $x = \frac{1}{\sigma}(\log(\frac{CF_1}{A_0} + 1) + \sigma^2/2 - r)$ , so that  $\delta = \frac{1}{\sigma} \log(\frac{CF_1}{A_0} + 1)$ . Three cases can occur:

- (i)  $X < \zeta_1$ , this case was already solved before. (No tax but carry forward). Notice that  $CF_1$  expires worthless, because duration is only one year.
- (ii)  $\zeta_1 < X < \zeta_1 + \delta$ . In this case a firm makes profit, but it's not taxable due to carry forward. However, no additional carry forward is created which can be used in period 2.
- (iii)  $X > \zeta_1 + \delta$ , this case was already solved before. However, some minor things change due to carry forward from previous period.

Since cases (i) and (iii) are already solved in [Appendix B.1](#), we focus on (ii). In this case

$$(B.4) \quad C_3^{BS} = A_1\Phi(d_{1,2}) - e^{-r}(K = A_1)\Phi(d_{2,2}),$$

where  $d_{2,1}, d_{2,2}$  are the same as in [\(B.2\)](#), because it is an at-the-money call option. The tedious computations leading to the total firm value will not be repeated, as they are very similar to the ones in the previous section. The integral  $I_1$  will be completely identical. Let us define a third integral corresponding to the call option that arises from case (ii)

$$\begin{aligned} I_3 &\triangleq e^{-r} \int_{\zeta_1}^{\zeta_1+\delta} C_3^{BS} dQ \\ &= A_0 e^{-r} (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) \int_{\zeta_1}^{\zeta_1+\delta} e^{r-\sigma^2/2+\sigma x} \varphi(x) dx \\ &= A_0 (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) (\Phi(\zeta_1 + \delta - \sigma) - \Phi(\zeta_1 - \sigma)). \end{aligned}$$

Consider  $I_{2,1}$ , corresponding to the integral of a Black-Scholes call option after taxes are levied. The firm value after tax payments equals  $\tilde{A}_1 = A_1 - \tau(A_1 - A_0 - CF_1)$ . The integral is therefore

$$(B.5) \quad \begin{aligned} I_{2,1} &\triangleq e^{-r} (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) \int_{\zeta_1+\delta}^{\infty} \tilde{A}_1 dQ = \\ &e^{-r} (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) \int_{\zeta_1+\delta}^{\infty} (A_0(1-\tau) \exp(r - \sigma^2/2 + \sigma x) + \tau(A_0 + CF_1)) \varphi(x) dx \\ &= (\Phi(d_{1,2}) - e^{-r} \Phi(d_{2,2})) (A_0(1-\tau) \Phi(\sigma - \zeta_1 - \delta) + e^{-r} \tau(A_0 + CF_1) \Phi(-\zeta_1 - \delta)). \end{aligned}$$

This leads to the total firm value for a company having some carry forward

$$(B.6) \quad V_{cf}^{BS} = A_0 - \tau C^{BS}(A_0 + CF_1) - \tau(I_1 + I_{2,1} + I_3).$$

The resulting carry forward just follows by subtracting [\(B.3\)](#) from [\(B.6\)](#)

$$\xi_{cf}^{BS} = \underbrace{\tau(C^{BS}(A_0) - C^{BS}(A_0 + CF_1))}_{\text{Carry forward value in one period model.}} - \underbrace{\tau(I_{2,1} - I_2 + I_3)}_{>0}.$$

■

The latter term on the right-hand side is greater than zero, which essentially follows from comparing the integral of two at-the-money call options over the region  $\{\omega \in \Omega : X(\omega) > \zeta_1\}$ , i.e. where  $A_1 > A_0$ . Let  $C(K)$  be the call option value with strike  $K$ . Whenever  $A_1 > A_0$  we have  $\int_{\{A_1 > A_0\}} C_1(A_1) > \int_{\{A_1 > A_0\}} C_1(\tilde{A}_1)$ .

### B.3. Two year duration carry forward

The value of a firm without fiscal history remains the same in this case. The value of a firm with carry forward is obtained by splitting the integral corresponding to the expectation into pieces. Consider a firm having  $CF_1$  created at period  $t = 0$ , which is two year valid.

- (i) If  $X < \zeta_1$  then none of the carry forward is used in year 1, due to loss in that period. That means that the total carry forward available in year 2 is equal to  $CF_1 + A_0 - A_1$ .
- (ii) If  $\zeta_1 \leq X < \zeta_1 + \delta$ , some of the carry forward is used in period 1, so that the total carry forward in period two equals  $CF_1 + A_0 - A_1$ . No taxes are paid in this case, so the option is the same as in (i).
- (iii)  $X > \zeta_1 + \delta$  implies that all carry forward is used in period 1 and no carry forward is available in period two. In this case taxes are paid.

We consider (i) first. In this case the call option value is given by

$$\begin{aligned} C_1^{BS} &= A_1 \Phi(d_{1,1}) - e^{-r}(K = A_0 + CF_1) \Phi(d_{2,1}) \\ d_{1,1} &= \frac{2r}{\sigma} + \frac{\log(A_0/(A_0 + CF_1))}{\sigma} + x \\ d_{2,1} &= d_{1,1} - \sigma. \end{aligned}$$

Secondly, item (iii) renders a call option value that is equivalent to (B.5), so we ignore this case. The calculations leading to the integral over  $C_1$  are very similar to the ones before, so we only state the final result

$$\begin{aligned} I_{1,2} &\triangleq e^{-r} \int_{-\infty}^{\zeta_1 + \delta} C_1^{BS} dQ \\ &= A_0 \Phi^{(\rho_1)}(\zeta_1 + \delta - \sigma, g_1) - e^{-2r}(A_0 + CF_1) \Phi^{(\rho_1)}(\zeta_1, g_2), \\ g_1 &\triangleq \frac{\frac{2r}{\sigma} + \sigma + \frac{1}{\sigma} \log(\frac{A_0}{A_0 + CF_1})}{\sqrt{2}}, \quad g_2 \triangleq \frac{\frac{2r}{\sigma} + \frac{1}{\sigma} \log(\frac{A_0}{A_0 + CF_1}) - \sigma}{\sqrt{2}}. \end{aligned}$$

Thus, the value of a company having some carry forward that can be used for two consecutive years is

$$(B.7) \quad V_{cf}^{BS} = A_0 - \tau C^{BS}(A_0 + CF_1) - \tau(I_{1,2} + I_{2,1}).$$

As a result, the DTA value of carry forward follows by subtracting (B.3) from (B.7)

$$\xi_{cf}^{BS} = \tau(C^{BS}(A_0) - C^{BS}(A_0 + CF_1)) - \tau(I_{1,2} + I_{2,1} - I_1 - I_2).$$

The rightmost term contains four integrals, for which we have the following inequalities

$$I_{1,2} < I_1, \quad I_{2,1} < I_2,$$

since on each domain of integration the call option is worth less when having carry forward available. This implies that carry forward is always more valuable when it is two year valid compared to a one year duration.

# Proof coupon payments

## C.1. No deferred taxes

**Theorem C.1.** *The debt value at time zero for an undertaking without deferred taxes is given by*

$$D_0 = e^{-r}(D + C) - e^{-r}(D + C)\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

if  $D + C \leq A_0 + \gamma C$ . Otherwise

$$D_0 = e^{-r}(D + C) - e^{-r}(D + C)\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)] + e^{-r}\tau(A_0 + \gamma C)[\Phi(\theta_2) - \Phi(\theta_1)].$$

In these expressions

$$\begin{aligned} \theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 + \gamma C}{A_0} \right) - r + \sigma^2/2 \right] \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + C - \tau(A_0 + \gamma C)}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + C}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

**Proof.** A firm without deferred taxes gives rise to the following debt value at time zero

$$D_0 = e^{-r}(D + C) - e^{-r}\mathbb{E}^Q((D + C - [A_1 - \tau(A_1 - A_0 - \gamma C)^+])^+).$$

To evaluate this expression, we assume that  $C$  is fixed. Introduce  $\theta_1$  as the solution to  $A_1 = A_0 + \gamma C$  (point from which the undertaking pays taxes).

As in [Appendix B.1](#), write  $A_1 = A_0 \exp(r - \sigma^2/2 + \sigma X)$ , where  $X \sim N(0, 1)$ . The analytical solution for  $\theta_1$  follows from

$$\begin{aligned} A_0 \exp(r - \sigma^2/2 + \sigma\theta_1) &= A_0 + \gamma\mathcal{C} \implies \\ \theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 + \gamma\mathcal{C}}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Let  $\theta_2$  be the solution to  $D + \mathcal{C} = A_1 - \tau(A_1 - A_0 - \gamma\mathcal{C})$ . Rearranging leads to

$$\begin{aligned} (1 - \tau)A_1 &= D + \mathcal{C} - \tau(A_0 + \gamma\mathcal{C}) \implies \\ (1 - \tau)A_0 \exp(r - \sigma^2/2 + \sigma\theta_2) &= D + \mathcal{C} - \tau(A_0 + \gamma\mathcal{C}) \implies \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C} - \tau(A_0 + \gamma\mathcal{C})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Finally,  $\theta_3$  is defined to be the solution to  $D + \mathcal{C} = A_1$ . This gives

$$\begin{aligned} A_0 \exp(r - \sigma^2/2 + \sigma\theta_3) &= D + \mathcal{C} \implies \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

We now distinguish the following two cases

- (i)  $D + \mathcal{C} \leq A_0 + \gamma\mathcal{C}$ .
- (ii)  $D + \mathcal{C} > A_0 + \gamma\mathcal{C}$ .

In case (i), the expectation equals

$$\begin{aligned} &e^{-r} \mathbb{E}^Q((D + \mathcal{C} - [A_1 - \tau(A_1 - A_0 - \gamma\mathcal{C})^+])^+) \\ &= e^{-r} \int_{-\infty}^{\theta_3} (D + \mathcal{C} - A_1) dQ \\ &= e^{-r} \int_{-\infty}^{\theta_3} (D + \mathcal{C} - A_0 e^{r - \sigma^2/2 + \sigma x}) \varphi(x) dx \\ &= e^{-r} (D + \mathcal{C}) \Phi(\theta_3) - A_0 \Phi(\theta_3 - \sigma). \end{aligned}$$

Case (ii) yields

$$\begin{aligned} &e^{-r} \mathbb{E}^Q((D + \mathcal{C} - [A_1 - \tau(A_1 - A_0 - \gamma\mathcal{C})^+])^+) \\ &= e^{-r} \int_{-\infty}^{\theta_1} (D + \mathcal{C} - A_1) dQ \\ &\quad + e^{-r} \int_{\theta_1}^{\theta_2} (D + \mathcal{C} - A_1 + \tau(A_1 - A_0 - \gamma\mathcal{C})) dQ \\ &= e^{-r} (D + \mathcal{C}) \Phi(\theta_2) - A_0 \Phi(\theta_2 - \sigma) \\ &\quad + \tau A_0 [\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)] - e^{-r} \tau (A_0 + \gamma\mathcal{C}) [\Phi(\theta_2) - \Phi(\theta_1)]. \end{aligned}$$

■

## C.2. Carry forward

**Theorem C.2.** *The debt value at time zero for an undertaking having  $CF_1$  is given by*

$$D_0 = e^{-r}(D + \mathcal{C}_{cf}) - e^{-r}(D + \mathcal{C}_{cf})\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

if  $D + \mathcal{C}_{cf} \leq A_0 + CF_1 + \gamma\mathcal{C}_{cf}$ . Otherwise

$$D_0 = e^{-r}(D + \mathcal{C}_{cf}) - e^{-r}(D + \mathcal{C}_{cf})\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) \\ - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)] + e^{-r}\tau(A_0 + CF_1 + \gamma\mathcal{C}_{cf})[\Phi(\theta_2) - \Phi(\theta_1)].$$

In these expressions

$$\theta_1 = \frac{1}{\sigma} \left[ \log \left( \frac{A_0 + CF_1 + \gamma\mathcal{C}_{cf}}{A_0} \right) - r + \sigma^2/2 \right] \\ \theta_2 = \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cf} - \tau(A_0 + CF_1 + \gamma\mathcal{C}_{cf})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\ \theta_3 = \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cf}}{A_0} \right) - r + \sigma^2/2 \right].$$

**Proof.** For an undertaking with carry forward, debt at time zero equals

$$D_0 = (D + \mathcal{C}_{cf}) - e^{-r}\mathbb{E}^Q((D + \mathcal{C}_{cf} - [A_1 - \tau(A_1 - A_0 - CF_1 - \gamma\mathcal{C}_{cf})]^+)^+).$$

To compute the expectation explicitly, we proceed as in the previous section. Let  $\theta_1$  be the solution to  $A_1 = A_0 + CF_1 + \gamma\mathcal{C}_{cf}$  (moment from which firm pays taxes). In particular

$$A_0 \exp(r - \sigma^2/2 + \sigma\theta_1) = A_0 + CF_1 + \gamma\mathcal{C}_{cf} \implies \\ \theta_1 = \frac{1}{\sigma} \left[ \log \left( \frac{A_0 + CF_1 + \gamma\mathcal{C}_{cf}}{A_0} \right) - r + \sigma^2/2 \right].$$

In addition  $\theta_2$  is the solution to  $D + \mathcal{C}_{cf} = A_1 - \tau(A_1 - A_0 - CF_1 - \gamma\mathcal{C}_{cf})$ . Rearranging terms yields

$$(1 - \tau)A_0 = D + \mathcal{C}_{cf} - \tau(A_0 + CF_1 + \gamma\mathcal{C}_{cf}) \implies \\ \theta_2 = \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cf} - \tau(A_0 + CF_1 + \gamma\mathcal{C}_{cf})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right].$$

Finally,  $\theta_3$  is the solution to  $D + \mathcal{C}_{cf} = A_1$ .

$$A_0 \exp(r - \sigma^2/2 + \sigma\theta_3) = D + \mathcal{C}_{cf} \implies \\ \theta_3 = \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cf}}{A_0} \right) - r + \sigma^2/2 \right].$$

This time we distinguish between

- (i)  $D + \mathcal{C}_{cf} \leq A_0 + CF_1 + \gamma\mathcal{C}_{cf}$ .
- (ii)  $D + \mathcal{C}_{cf} > A_0 + CF_1 + \gamma\mathcal{C}_{cf}$ .

Case (i) renders the following solution to the expectation

$$\begin{aligned}
& e^{-r} \mathbb{E}^Q((D + \mathcal{C}_{cf} - [A_1 - \tau(A_1 - A_0 - CF_1 - \gamma\mathcal{C}_{cf})^+]^+)) \\
&= e^{-r} \int_{-\infty}^{\theta_3} (D + \mathcal{C}_{cf} - A_1) dQ \\
&= e^{-r} (D + \mathcal{C}_{cf}) \Phi(\theta_3) - A_0 \Phi(\theta_3 - \sigma).
\end{aligned}$$

Case (ii) yields

$$\begin{aligned}
& e^{-r} \mathbb{E}^Q((D + \mathcal{C}_{cf} - [A_1 - \tau(A_1 - A_0 - CF_1 - \gamma\mathcal{C}_{cf})^+]^+)) \\
&= e^{-r} \int_{-\infty}^{\theta_1} (D + \mathcal{C}_{cf} - A_1) dQ \\
&+ e^{-r} \int_{\theta_1}^{\theta_2} (D + \mathcal{C}_{cf} - A_1 + \tau(A_1 - A_0 - CF_1 - \gamma\mathcal{C}_{cf})) dQ \\
&= e^{-r} (D + \mathcal{C}_{cf}) \Phi(\theta_2) - A_0 \Phi(\theta_2 - \sigma) \\
&+ \tau A_0 [\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)] - e^{-r} \tau (A_0 + CF_1 + \gamma\mathcal{C}_{cf}) [\Phi(\theta_2) - \Phi(\theta_1)].
\end{aligned}$$

■

### C.3. Carry back

**Theorem C.3.** *The debt value at time zero for an undertaking having  $CB_1$  is given by*

$$D_0 = e^{-r} (D + \mathcal{C}_{cb}) - e^{-r} (D + \mathcal{C}_{cb} - \tau CB_1) \Phi(\theta_3) + A_0 \Phi(\theta_3 - \sigma),$$

if  $D + \mathcal{C}_{cb} \leq A_0 - CB_1 + \gamma\mathcal{C}_{cb}$ . Otherwise

$$\begin{aligned}
D_0 &= e^{-r} (D + \mathcal{C}_{cb}) - e^{-r} (D + \mathcal{C}_{cb} - \tau CB_1) \Phi(\theta_2) + A_0 \Phi(\theta_2 - \sigma) \\
&- e^{-r} \tau (CB_1 - A_0 - \gamma\mathcal{C}_{cb}) [\Phi(\theta_2) - \Phi(\theta_1)] - \tau A_0 [\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)].
\end{aligned}$$

In these expressions

$$\begin{aligned}
\theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 - CB_1 + \gamma\mathcal{C}_{cb}}{A_0} \right) - r + \sigma^2/2 \right] \\
\theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cb} - \tau(A_0 + \gamma\mathcal{C}_{cb})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\
\theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cb} - \tau CB_1}{A_0} \right) - r + \sigma^2/2 \right].
\end{aligned}$$

**Proof.** We know that debt at time zero is given by

(C.1)

$$D_0 = e^{-r} (D + \mathcal{C}_{cb}) - e^{-r} \mathbb{E}^Q((D + \mathcal{C}_{cb} - [A_1 + \tau CB_1 - \tau(A_1 - A_0 + CB_1 - \gamma\mathcal{C}_{cb})^+]^+)).$$

Let us denote the solution to  $A_1 = A_0 - CB_1 + \gamma\mathcal{C}_{cb}$  by  $\theta_1$ .

$$\begin{aligned} A_0 \exp(r - \sigma^2/2 + \sigma\theta_1) &= A_0 - CB_1 + \gamma\mathcal{C}_{cb} \implies \\ \theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 - CB_1 + \gamma\mathcal{C}_{cb}}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Also, set  $\theta_2$  to be the solution to  $D + \mathcal{C}_{cb} = A_1 + \tau CB_1 - \tau(A_1 - A_0 + CB_1 - \gamma\mathcal{C}_{cb})$ . Rearranging leads to

$$\begin{aligned} (1 - \tau)A_1 &= D + \mathcal{C}_{cb} - \tau(A_0 + \gamma\mathcal{C}_{cb}) \implies \\ (1 - \tau)A_0 \exp(r - \sigma^2/2 + \sigma\theta_2) &= D + \mathcal{C}_{cb} - \tau(A_0 + \gamma\mathcal{C}_{cb}) \implies \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cb} - \tau(A_0 + \gamma\mathcal{C}_{cb})}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Finally, define  $\theta_3$  to be the solution to  $D + \mathcal{C}_{cb} = A_1 + \tau CB_1$ , which gives

$$\begin{aligned} A_0 \exp(r - \sigma^2/2 + \sigma\theta_3) &= D + \mathcal{C}_{cb} - \tau CB_1 \implies \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_{cb} - \tau CB_1}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Distinguish the following cases

- (i)  $D + \mathcal{C}_{cb} \leq A_0 - CB_1 + \gamma\mathcal{C}_{cb}$ .
- (ii)  $D + \mathcal{C}_{cb} > A_0 - CB_1 + \gamma\mathcal{C}_{cb}$ .

In case (i), the expectation in (C.1) is easily evaluated, as the firm can only go bankrupt if  $A_1 \leq A_0 + CB_1 - \gamma\mathcal{C}_{cb}$ , which means debt is so low compared to  $A_0$  that if bankruptcy is triggered, a firm can reclaim the complete carry back. The expectation then follows from

$$\begin{aligned} &e^{-r} \mathbb{E}^Q \left( (D + \mathcal{C}_{cb} - [A_1 + \tau CB_1 - \tau(A_1 - A_0 + CB_1 - \gamma\mathcal{C}_{cb})]^+)^+ \right) \\ &= e^{-r} \int_{-\infty}^{\theta_3} D + \mathcal{C}_{cb} - A_1 - \tau CB_1 dQ \\ &= e^{-r} \int_{-\infty}^{\theta_3} (D + \mathcal{C}_{cb} - A_0 e^{r - \sigma^2/2 + \sigma x} - \tau CB_1) \varphi(x) dx \\ &= e^{-r} (D + \mathcal{C}_{cb} - \tau CB_1) \Phi(\theta_3) - A_0 \Phi(\theta_3 - \sigma). \end{aligned}$$

In case (ii), bankruptcy is triggered already when a firm can only reclaim part of the carry back from the tax authority. In this case the expectation

in (C.1) is found by splitting the integral

$$\begin{aligned}
& e^{-r} \mathbb{E}^Q((D + \mathcal{C}_{cb} - [A_1 + \tau CB_1 - \tau(A_1 - A_0 + CB_1 - \gamma \mathcal{C}_{cb})^+])^+) \\
&= e^{-r} \int_{-\infty}^{\theta_1} D + \mathcal{C}_{cb} - A_1 - \tau CB_1 dQ \\
&+ e^{-r} \int_{\theta_1}^{\theta_2} D + \mathcal{C}_{cb} - A_1 - \tau CB_1 + \tau(A_1 - A_0 + CB_1 - \gamma \mathcal{C}_{cb}) dQ \\
&= e^{-r}(D + \mathcal{C}_{cb} - \tau CB_1)\Phi(\theta_2) - A_0\Phi(\theta_2 - \sigma) + e^{-r}\tau \int_{\theta_1}^{\theta_2} (A_1 - A_0 + CB_1 - \gamma \mathcal{C}_{cb}) dQ \\
&= e^{-r}(D + \mathcal{C}_{cb} - \tau CB_1)\Phi(\theta_2) - A_0\Phi(\theta_2 - \sigma) \\
&+ e^{-r}\tau(CB_1 - A_0 - \gamma \mathcal{C}_{cb})[\Phi(\theta_2) - \Phi(\theta_1)] + \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)].
\end{aligned}$$

■

#### C.4. DTL

**Theorem C.4.** *The debt value at time zero for an undertaking having gain<sub>1</sub> is given by*

$$D_0 = e^{-r}(D + \mathcal{C}_L) - e^{-r}(D + \mathcal{C}_L)\Phi(\theta_3) + A_0\Phi(\theta_3 - \sigma),$$

if  $D + \mathcal{C}_L \leq A_0 - \text{gain}_1 + \gamma \mathcal{C}_L$ . Otherwise

$$\begin{aligned}
D_0 &= e^{-r}(D + \mathcal{C}_L) - e^{-r}(D + \mathcal{C}_L)\Phi(\theta_2) + A_0\Phi(\theta_2 - \sigma) \\
&- e^{-r}\tau(\text{gain}_1 - A_0 - \gamma \mathcal{C}_L)[\Phi(\theta_2) - \Phi(\theta_1)] - \tau A_0[\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)].
\end{aligned}$$

In these expressions

$$\begin{aligned}
\theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 - \text{gain}_1 + \gamma \mathcal{C}_L}{A_0} \right) - r + \sigma^2/2 \right] \\
\theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_L - \tau(A_0 - \text{gain}_1 + \gamma \mathcal{C}_L)}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right] \\
\theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_L}{A_0} \right) - r + \sigma^2/2 \right].
\end{aligned}$$

**Proof.** The DTL increases the probability of bankruptcy and thus influences coupon payments. To see this, we write

(C.2)

$$D_0 = e^{-r}(D + \mathcal{C}_L) - e^{-r} \mathbb{E}^Q((D + \mathcal{C}_L - [A_1 - \tau(A_1 - A_0 + \text{gain}_1 - \gamma \mathcal{C}_L)^+])^+).$$

Now we define  $\theta_1$  to be the solution to  $A_1 = A_0 - \text{gain}_1 + \gamma\mathcal{C}_L$  (moment from which levered firm with DTL pays taxes). Solving gives

$$\begin{aligned} A_0 \exp(r - \sigma^2/2 + \sigma\theta_1) &= A_0 - \text{gain}_1 + \gamma\mathcal{C}_L \implies \\ \theta_1 &= \frac{1}{\sigma} \left[ \log \left( \frac{A_0 - \text{gain}_1 + \gamma\mathcal{C}_L}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Similarly,  $\theta_2$  is the solution to  $D + \mathcal{C}_L = A_1 - \tau(A_1 - A_0 + \text{gain}_1 - \gamma\mathcal{C}_L)$ . Rearranging leads to

$$\begin{aligned} (1 - \tau)A_1 &= D + \mathcal{C}_L - \tau(A_0 - \text{gain}_1 + \gamma\mathcal{C}_L) \implies \\ (1 - \tau)A_0 \exp(r - \sigma^2/2 + \sigma\theta_2) &= D + \mathcal{C}_L - \tau(A_0 - \text{gain}_1 + \gamma\mathcal{C}_L) \implies \\ \theta_2 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_L - \tau(A_0 - \text{gain}_1 + \gamma\mathcal{C}_L)}{(1 - \tau)A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

Finally,  $\theta_3$  is the solution to  $D + \mathcal{C}_L = A_1$ . Solving renders

$$\begin{aligned} A_0 \exp(r - \sigma^2/2 + \sigma\theta_3) &= D + \mathcal{C}_L \implies \\ \theta_3 &= \frac{1}{\sigma} \left[ \log \left( \frac{D + \mathcal{C}_L}{A_0} \right) - r + \sigma^2/2 \right]. \end{aligned}$$

The expectation in (C.2) follows by distinguishing two cases

- (i)  $D + \mathcal{C}_L \leq A_0 - \text{gain}_1 + \gamma\mathcal{C}_L$ .
- (ii)  $D + \mathcal{C}_L > A_0 - \text{gain}_1 + \gamma\mathcal{C}_L$ .

Case (i) renders the solution

$$\begin{aligned} &e^{-r} \mathbb{E}^Q \left( (D + \mathcal{C}_L - [A_1 - \tau(A_1 - A_0 + \text{gain}_1 - \gamma\mathcal{C}_L)]^+)^+ \right) \\ &= e^{-r} \int_{-\infty}^{\theta_3} (D + \mathcal{C}_L - A_1) dQ \\ &= e^{-r} (D + \mathcal{C}_L) \Phi(\theta_3) - A_0 \Phi(\theta_3 - \sigma). \end{aligned}$$

In case (ii), we have

$$\begin{aligned} &e^{-r} \mathbb{E}^Q \left( (D + \mathcal{C}_L - [A_1 - \tau(A_1 - A_0 + \text{gain}_1 - \gamma\mathcal{C}_L)]^+)^+ \right) \\ &= e^{-r} \int_{-\infty}^{\theta_1} D + \mathcal{C}_L - A_1 dQ \\ &+ e^{-r} \int_{\theta_1}^{\theta_2} D + \mathcal{C}_L - A_1 + \tau(A_1 - A_0 + \text{gain}_1 - \gamma\mathcal{C}_L) dQ \\ &= e^{-r} (D + \mathcal{C}_L) \Phi(\theta_2) - A_0 \Phi(\theta_2 - \sigma) \\ &+ e^{-r} \tau(\text{gain}_1 - A_0 - \gamma\mathcal{C}_L) [\Phi(\theta_2) - \Phi(\theta_1)] + \tau A_0 [\Phi(\theta_2 - \sigma) - \Phi(\theta_1 - \sigma)]. \end{aligned}$$

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# List of Figures

- 1.1 Asset value after tax. Blue line denotes asset without fiscal history with  $A_0 = 100$ , while orange line depicts asset with 40 carry forward. 4
- 1.2 Asset value after tax. Blue line denotes asset without fiscal history with  $A_0 = 100$ , while orange line depicts asset with 40 carry back. 5
- 1.3 Asset value after tax. Blue line denotes asset without fiscal history with  $A_0 = 100$ , while orange line depicts asset with 20 tax liabilities. 7
- 3.1 Typical asset path of geometric Brownian motion with and without tax (no deferred taxes). The asset path with tax jumps down if at the end of year  $t$  profit is made. The geometric Brownian motion restarts from the point  $\tilde{A}_t$ , whereas the asset path without tax just continues from  $A_t$ . 19
- 4.1 DTA value given from carry forward (4.3) as function of carry forward ( $CF_1$ ) for different tax rates. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2$ . 26
- 4.2 Derivative of DTA from carry forward (4.5) as a function of carry forward ( $CF_1$ ) for different tax rates. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2$ . 27
- 4.3 Value of  $CF_1$  under Black-Scholes vs. payoff at maturity. Parameters:  $r = 0.05, \sigma = 0.2, A_0 = 100, CF_1 = 40, \tau = 0.25$ . 28
- 4.4 Value of  $CB_1$  under Black-Scholes vs. payoff at maturity. Parameters:  $r = 0.05, \sigma = 0.2, A_0 = 100, CB_1 = 40$ . 29

- 
- 4.5 The difference between (4.3) and (4.7) as a function of  $r$  and  $\sigma$ , for the fixed deferred tax values  $CF_1 = CB_1 = 20$ . The domain  $r \times \sigma = \{(r, \sigma) : r \in [0.01, 0.5], \sigma \in [0.1, 0.9]\}$  and  $A_0 = 100, \tau = 0.25$ . 30
- 4.6 Value of  $DTL$  under Black-Scholes vs. payoff at maturity. Parameters:  $r = 0.05, \sigma = 0.2, A_0 = 100, gain_1 = 20$ . 32
- 4.7 DTA value of carry forward as a function of  $CF_1$  under different valuation approaches. Parameters:  $A_0 = 100, \tau = 0.25, r = 0.05, \sigma = 0.2, \mu = 0.1$ . 35
- 4.8 DTA value of carry forward as a function of  $A_0$  under different valuation approaches. Parameters:  $\tau = 0.25, CF_1 = 90, r = 0.05, \sigma = 0.2, \mu = 0.1$ . 36
- 4.9 Coupon payment  $\mathcal{C}$  on risky debt as a function of debt  $D$  for undertakings with different deferred taxes. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, CB_1 = 10, gain_1 = 30, \gamma = 1$ . 40
- 4.10 DTA/DTL value of levered firms seen as a function of debt. The coupon payment corresponding to debt level  $D_0$  is determined via numerical methods and depends on the deferred tax. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, CF_1 = CB_1 = gain_1 = 20$ . 43
- 4.11 Tax payments for levered and unlevered firms with different fiscal history. Parameters:  $A_0 = 100, \tau = 0.25$ . 46
- 4.12 Interest tax shield seen as a function of debt. The Modigliani-Miller value is given by (4.28), whereas the value of the Reference undertaking and the Undertaking with DTL are given by  $R^{BS}$  and  $R_L^{BS}$  respectively. Parameters:  $r = 0.05, \sigma = 0.2, \tau = 0.25, A_0 = 100, gain_1 = 10, \gamma = 1$ . Bankruptcy cost are zero ( $\alpha = 0$ ). 48
- 4.13 Monte-Carlo simulation of DTA value arising from carry forward in multi-period models. In all sub figures carry back is not allowed. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$ . 58
- 4.14 Monte-Carlo simulation of DTA value arising from carry back in multi-period models. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$ . 62
- 4.15 Monte-Carlo simulation of DTL value arising from temporary differences in multi-period models. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$ . 63
- 4.16 DTA/DTL value of carry forward, carry back and untaxed profit  $gain_1$  for levered undertakings as a function of debt  $D$  in the 5-period model. The deferred taxes are set such that  $CF_1 = CB_1 = gain_1 = 20$  and coupons  $\mathcal{C}_{cf}, \mathcal{C}_{cb}, \mathcal{C}_L$  are determined such that debt sells at par value. Parameters:

- 
- $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, \gamma = 1$ . Duration of carry forward/back is 5/1 year(s) respectively. 66
- 4.17 Firm value of undertaking without deferred taxes in different  $T$ -period models. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, \gamma = 1$ . The firm has no initial deferred taxes, i.e.  $CF_1 = CB_1 = gain_1 = 0$ . Settlement term of carry forward/back is 10/1 year(s) respectively. 69
- 5.1 Pie chart of payout to stakeholders for levered and unlevered firm in the 10-period model. All payouts are discounted back to period zero. For the levered firm, parameters are chosen to match the optimal capital structure in the *endogenous bankruptcy case* in Table 4.6. This means that  $D = 39.57, K = 15.70, C = 2.16, \gamma = 1$  and  $\alpha = 0.5$ . Other parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, CF_1 = CB_1 = 0$  and the settlement term of carry forward/back is 10/1 year(s) respectively. 79
- 5.2 Effect of an increase in risk ( $\sigma$ ) as measured by the partial derivatives w.r.t.  $\sigma$  on tax, equity and debt value in the 5-period model (panel **(a)**) and 10-period model (panel **(b)**) as a function of  $A_0$ . Parameters are chosen to match the optimal capital structure in the *endogenous bankruptcy case* when  $A_0 = 100$  in Table 4.6. Parameters:  $r = 0.05, \sigma = 0.2, CF_1 = CB_1 = 0$  and the settlement term of carry forward/back is 10/1 year(s) respectively. 80



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# List of Tables

- 4.1 Monte-Carlo simulation of DTA value corresponding to different nominal carry forward values (header). Left most column concerns the duration of the model. Two cases are distinguished, one where duration of carry forward is one year, other where duration is ten years. Carry back is notallowed and the maximum *standard error* during all simulations equals 0.0194. The following parameters are used:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$  and 10.000 Monte-Carlo paths. 56
- 4.2 Monte-Carlo simulation of DTA value corresponding to different nominal carry forward values (header). Left most column concerns the duration of the model. Two cases are distinguished, one where duration of carry forward is one year, other where duration is ten years. Carry back is allowed for one and ten years in the respective models and the maximum *standard error* in all simulations equals 0.0214. The following parameters are used:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$  and 10.000 Monte-Carlo paths. 57
- 4.3 Monte-Carlo simulation of DTA value corresponding to different nominal carry back values (header). Left most column concerns the duration of the model. Two cases are distinguished, one where duration of carry back is one year, other where duration is ten years. Carry forward is allowed for one and ten years in the respective models and the maximum *standard error* in all simulations equals 0.0081. The following parameters were used:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25$  and 10.000 Monte-Carlo paths. 60
- 4.4 Coupon payments in the 5-period model, when carry back is not allowed. The coupon  $C_{cf}$  corresponds to the coupon payment of a firm having  $CF_1$  and the settlement term of carry forward is five years. Parameters:  $A_0 = 100, CF_1 = 20, r = 0.05, \sigma = 0.2, \tau = 0.25$ . 65
- 4.5 Coupon payments for different levels of debt with face value  $D$  when carry back is allowed for one year and losses can be carried forward five years. For firms with carry forward/back we assume  $CF_1 = CB_1 = 20$ . Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.25, \tau = 0.2$  66

- 4.6 Optimal debt levels in various  $T$ -period models. Coupon payment  $C$  is determined such that debt sells at par value. The *maximum leverage premium* is the difference between columns four and three. The *exogenous* bankruptcy case takes  $K = D$ . The *endogenous* bankruptcy case finds the optimal  $K^*$  implied by the model. Parameters:  $A_0 = 100, r = 0.05, \sigma = 0.2, \tau = 0.25, \alpha = 0.5$  and the settlement term of carry forward/back is 10/1 year(s) respectively. The firm has no initial deferred taxes. 68
- 5.1 Input variables taken from the market. *Shares outstanding* is the basic weighted average number of shares outstanding over 2016, see page 22 of BP (2018). *Carry forward* ( $= CF_1$ ) is the DTA value over 2016 on page 15 of BP (2018), converted to nominal scale (assuming the tax rate  $\tau = 0.35$ ). Hence, we tacitly assume that all DTA's are coming from loss carry forward. The single debt issuance  $D$  comes from (5.2). All numbers are reported in millions ( $10^6$ ). 74
- 5.2 Parameter values resulting from the optimization problem (5.3). 74
- 5.3 Most important legislative transitions in the U.S. corporate tax overhaul. The *current law* column describes the extant tax laws, whereas the *G.O.P. Bill* column describes the proposed tax changes of the Trump administration. 75
- 5.4 DTA value of carry forward with and without tax change in different  $T$ -period models, using  $\lambda_{1,t} = \lambda_{0,t} = 0.35$ . The low and high tax rates are respectively given by  $\tau_1 = 0.2$  and  $\tau_2 = 0.45$ . Both models start with the high tax rate  $\tau_2$ . Additional parameters:  $A_0 = 1000, r = 0.05, CF_1 = 40, C = 5, \gamma = 1$ . Settlement term of carry forward is ten years and carry back can be settled for one year. 77
- 5.5 Tax savings expressed as a fraction of the *base case* without deferred taxes in the 30-period model withoutleverage for different settlement terms. In particular: Tax savings =  $1 - (\text{Tax expense with DTA})/(\text{Tax expense without DTA possibility})$ . In the notation  $[a; b]$ ,  $a$  is the duration of carry forward and  $b$  concerns the duration of carry back. Parameters:  $CF_1 = CB_1 = 0, A_0 = 100, r = 0.05$ . 78