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Oscillations of a string on an elastic foundation with space and time-varying rigidity

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Abstract The dynamics of a string on an elastic foundation with time- and coordinate-dependent coefficients have been studied. Asymptotic solutions have been constructed for the following cases: for an arbitrary value of the elastic foundation coefficient at small and large time values, and for small and large coefficients of the elastic foundation at arbitrary times. Also a special case originated from an ageing process has been studied. The ageing process is described by an expression approximating some well-known experimental data. The existence of localized modes along the x coordinate is shown. The existence of these localized modes can lead to a spatial resonance phenomenon under certain conditions. For the case of an arbitrary elastic foundation coefficient value at small and at large times, the spatial resonance phenomenon is observed at small, special frequencies. This effect depends also on a special phase and mode number. For large mode numbers, this special resonance seems to be not possible.

Keywords Time-varying rigidity · Ageing · Elastic foundation · Localized modes · Slender structure · Thin film

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1 Introduction

A lot of natural and engineering constructions consist of multi-layered structures. An upper coating of such a structure is, as a rule, a thin layer. Thus, a top layer of human/animal blood vessels is a slender structure resting on an interlayer between it and a vessel shell. An other example of these structures is a thin film (including protective films for windshields) connected to the main material through an adhesive layer [1]. Investigation of the upper structure behaviour, in case the whole multi-layer structure is subjected to an external load, is of practical interest. In some problems, the structure dynamic study can be reduced to the investigation of the dynamics of a thin film connected to an elastic foundation. In this case, the foundation simulates elastic characteristics of both the adhesive layer and the main material. The elastic characteristics can vary in time and can be non-uniform in space [5]. With the change of the elastic foundation rigidity, the slender structure dynamics also changes [2,3]. For example, when a thin-film substrate material fails, the rigidity of its elastic foundation decreases unevenly in time and space [4,6,7]. An ageing process of a foundation material is another example of a process causing a change of the foundation rigidity. The ageing affects elastic properties of a material depending on the type of this material [8–10]. In some cases, the rigidity increases and the Young's modulus increases in time. In other cases the above-mentioned characteristics decrease in time. For example, medicinal agents added in human/animal vessels can cause a change of the blood vessels rigidity and they age for shorter time than it would be without those drugs [8]. This fact should be taken into consideration when the dynamics of blood vessels is studied. This paper is mainly devoted to the investigation of oscillations of slender structures connected to a foundation altering its elastic characteristics due to an ageing process. Currently, the analysis of dynamic structures with ageing foundations has revealed that the modelling of the foundation characteristics lead to additional integral terms in the differential equations describing the body motions. A viscous-elastic model based on fundamental Boltzmann and Volterra concepts, and a theory for rheological models date back to Maxwell, Voigt and Thomson [2,11]. According to these models, a strain in any point of a body depends on a deformation history of the material in this particular point. The deformation /strain relation leads to a Volterra–Voigt linear equation. Most authors studied beams and plates on a viscous-elastic foundation with a bilateral contact. In the above-cited references [2,11], the coupling problem was stated according to the structural mechanics laws in an analytical formulation. However, only a few authors found an analytical solution in a closed form for some special cases. Often, the solution is numerical and it is obtained by discretizing the equation governing the problem. Unlike that approach, the current paper proposes to simulate and to describe a foundation based on available experimental data of the Young's modulus change. Experimental data given in [8–10] allows presenting the foundation coefficient in the form of a function depending on the space coordinates and time. The present paper proposes to find an approximation of the solution of the problem for a thin film on a foundation with a coefficient changing due to the ageing process. The corresponding equations are complicated; nonetheless, we are capable to prove that the problem is well posed. We obtain an estimate of the energy of the system. To analyse the system of governing equations in more detail,it is assumed that ageing is a slow process. It allows us to use asymptotic methods. We derive a system for the Fourier coefficients of the displacement and obtain estimates for averaged amplitudes of the Fourier coefficients, i.e. for energies of the linear modes. The main results are as follows. The problem under investigation has two types of eigenfunctions. The first type is given by non-localized eigenfunctions and the second type is described by localized ones. The second type includes new eigenfuctions, which exist only in the case of non-uniform space ageing. We also find other space depending effects in ageing. For large times there is possibly a "space resonance", when the frequencies of the string are influenced by the space non-uniform ageing term.

2 Statement of the problem

In this paper the oscillations of a string connected to an elastic foundation are studied. The coefficient of the elastic foundation is related to mechanical characteristics of the material. Such mechanical characteristics can be non-uniform in space and/or temporally non-uniform depending on the process in the material, for example, a process of material ageing or a process of damage growth. In [12] the authors mention that the hyperbolic equation they use, has a term which describes in time the change of the elastic foundation. However, the term is not specified as a particular function of time, and the dynamics of the system is not investigated. We consider the following governing equation:

$$F_0 u_{xx} - k_0 \left(1 - \gamma \exp\left(-\frac{t}{\tau_0}\left(p_0 + b_0 \cos(\omega_* x + \phi)\right)\right)\right) u - m u_{tt} = f(u, u_t), \quad (1)$$

where $u(x, t)$ is the string displacement, t is time, $t \geq 0$, $x \in [0, L]$, L is the length of the string, F_0 is a constant stretching force, k_0 is a coefficient of the elastic foundation when its material is not subjected to the ageing process, $k_0 > 0$, m is a string mass per unit length, and $f(u, u_t)$ is a smooth function that defines a dissipative term.The simplest choice for $f(u, u_t)$ is

$$f = c_1 u_t, \quad (2)$$

which describes dissipative effects in the elastic system when the dissipative coefficient $c_1 > 0$. The term

$k_0\left(1-\gamma\exp\left(-\frac{t}{\tau_0}\left(p_0+b_0\cos\left(\omega_* x+\phi\right)\right)\right)\right)u$ describes the ageing effect. In this expression $\gamma\in(0,1]$ is a non-dimensional constant, τ_0 is a constant of the elastic foundation material which determines the changing of the elastic foundation coefficient in time. Let us introduce the function $p(x)$ as follows:

$$p(x)=p_0+b_0\cos\left(\omega_* x+\phi\right),\quad p_0>b_0>0. \tag{3}$$

The function $p(x)$is a given smooth, positive function which corresponds to a possible non-uniform ageing process. Here b_0 is a non-dimensional amplitude of the variable in space part of the function $p(x)$ ($b_0>0$), p_0 is a non-dimensional constant, ω_* is a "space frequency",and ϕ is a "space phase". In the process of material ageing, the Young's modulus, in particular, is varying in time [4,8–10], which leads to the rigidity growth and increase in the elastic foundation coefficient. Based on experimental data given in [4,8–10] the function $p(x)$ for the elastic foundation behaviour was chosen in the presented form (3). If $b_0=0$, we have a spatially uniform ageing effect (later on refered to as case a), in all other cases we deal with the spatially non-uniform case b). It is assumed that the ageing process is progressing monotonically in time, and therefore $p(x)>0$. When the $p(x)$ distribution is known, this function can be expanded into a Fourier series, and be approximated by the first few terms. This is the motivation for (3). Also it is known [14] that some functionally graded materials can have varying characteristics in space according to this chosen form (3).

We take the following initial conditions

$$u\left(x,0\right)=v_0(x),\quad u_t\left(x,t\right)|_{t=0}=v_1(x), \tag{4}$$

where $||v_{0xx}||+||v_1||<\infty$. Here we use the standard notation $||v||$ for the norm, $||f||^2=\langle f,f\rangle$ and $\langle f,g\rangle$ is the scalar product in $L_2[0,L]$: $\langle f,g\rangle=\int_0^L f(x)g(x)\mathrm{d}x$. The boundary conditions are assumed to be given by:

$$u(0,t)=u(L,t)=0. \tag{5}$$

Notice that Eq. (1) can be transformed to a dimensionless form when we rescale the variables. For the rescaling, the following relations are used: $x=\bar{x}L$, $u=\bar{u}L$, $t=\bar{t}\frac{L}{c_0}$, $c_0^2=\frac{F_0}{m}$, $\omega_*=\bar{\omega}_*\frac{1}{L}$, $\alpha=\frac{L^2k_0}{c_0^2m}$, $\varepsilon=\frac{c_1L}{c_0m}$, $\varepsilon_a=\frac{L}{c_0\tau_0}$.

To simplify notations, we omit now the bars and obtain the final equation:

$$u_{xx}-\alpha(1-\gamma\exp\left(-\varepsilon_a tp\left(x\right)\right)u-u_{tt}=\varepsilon u_t, \tag{6}$$

The initial conditions have the same form as before, but we should take into account that they are now written for non-dimensional variables. The boundary conditions are as follows:

$$u\left(0,t\right)=u\left(1,t\right)=0, \tag{7}$$

In the next sections we will assume that ε_a and ε are small parameters.

3 Well-posedness of the problem

In this section we show that the problem is well posed for a large class of perturbations f and not only for the case when $f=\varepsilon u_t$.

Let us introduce a functional E associated with (6) when in the right-hand side of equation we have $\varepsilon f(u,u_t)$:

$$E[u(\cdot,t)]=\frac{1}{2}\Big(||u_x||^2+||u_t||^2 \tag{8}$$

$$+\alpha\int_0^1(1-\gamma\exp(-\varepsilon_a tp(x)))u^2(x,t)\mathrm{d}x\Big). \tag{9}$$

This functional can be interpreted as an energy. Let us derive an estimate for E. If $\varepsilon=\varepsilon_a=0$ one has $\mathrm{d}E[u(\cdot,t)]/\mathrm{d}t=0$ for solutions of (6), i.e. the energy is conserved. For $\varepsilon>0$ and $\varepsilon_a>0$ we multiply the right-hand and the left-hand sides of Eq. (6) by u_t. Then, by integrating with respect to x from $x=0$ to 1, one finds

$$\frac{\mathrm{d}E\left[u\left(\cdot,t\right)\right]}{\mathrm{d}t}=D, \tag{10}$$

and where

$$D=-\varepsilon\int_0^1\Big(f\left(u,u_t\right)u_t-\frac{\alpha\gamma\mu}{2}p(x)$$
$$\exp\left(-\varepsilon_a tp(x)\right)u^2\Big)\mathrm{d}x, \tag{11}$$

where μ is a parameter defined by

$$\mu=\varepsilon^{-1}\varepsilon_a \tag{12}$$

The functional D can be considered to be a ”dissipative function”. Under general assumptions on f, one can show that the initial boundary value problem (6), (7) and (4) is well posed. Below we will use the brief notation $E(t) = E[u(\cdot, t)]$.

Lemma 1 *Assume*

$$f(u, u_t)\, u_t \geq \beta u_t^2, \quad p(x) \geq p_{\min} = \min_x p(x) > 0, \tag{13}$$

where $\beta > 0$ is a constant. Then the energy $E(t)$ satisfies

$$E(t) \leq E(0) \exp\Big(-2\beta\varepsilon t + \frac{\gamma p_{\min}}{(1-\gamma) p_{\max}}\Big), \quad t \in (0, +\infty), \tag{14}$$

where $p_{\max} = \max_x p(x)$.

Proof Assumption (13) and (10) imply that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\beta\varepsilon \int_0^1 u_t^2 \mathrm{d}x + \frac{\alpha\gamma\varepsilon_a}{2} \int_0^1 p(x) \exp(-\varepsilon_a t p(x)) u^2(x, t) \mathrm{d}x.$$

One has

$$p(x) \exp\left(-\varepsilon_a t p(x)\right) \leq p_{\max} \exp\left(-\varepsilon_a t p_{\min}\right),$$

thus

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\beta\varepsilon \int_0^1 u_t^2 \mathrm{d}x + \frac{\alpha\gamma\varepsilon_a}{2} p_{\max} \exp\left(-\varepsilon_a t p_{\min}\right) \int_0^1 u^2(x, t) \mathrm{d}x.$$

Note that $\int_0^1 u_t^2 dx \leq 2E$ and $\int_0^1 u^2 dx \leq 2(\alpha(1-\gamma))^{-1}E$ by definition (8) for E. These inequalities imply that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq (-2\varepsilon\beta + \varepsilon_a\gamma(1-\gamma)^{-1} p_{\max} \exp\left(-\varepsilon_a t p_{\min}\right))E. \tag{15}$$

Therefore,

$$E(t) \leq \exp(-2\varepsilon\beta t + \gamma(1-\gamma)^{-1} p_{\max} p_{\min}^{-1} (1 - \exp\left(-\varepsilon_a t\right)) E(0)$$

which proves (14).

Estimate (14) shows that the energy is bounded. Therefore, the norms $||u_t||^2$, $||u_x||^2$ and $||u||^2$ are bounded. And so, a solution of the initial boundary value problem (1), (4) and (5) exists for $t \in [0, +\infty)$, lies in the corresponding Sobolev space, and is unique, i.e. the initial boundary value problem for (4), (6), and (7) is well posed. □

The assertion of this lemma has a consequence that admits a transparent physical interpretation. The energy is bounded and our system is stabilized for purely dissipative perturbations when f is defined by (2) with $c_1 > 0$.

4 Eigenfunctions of the linearized problem

4.1 Asymptotics for eigenfunctions and eigenvalues

Our first step is to consider the unperturbed Eq. (6) for $\varepsilon = 0$. Since $0 < \varepsilon_a \ll 1$, it is natural to introduce a slow time $T = \varepsilon_a t$. Then for a fixed ”frozen” T the unperturbed Eq. (6) can be solved by the Fourier method, i.e. by substitution of $u(x, t, T) = \psi(x, T) \exp(i\omega(T)t)$ into (6). Then we obtain the following linear operator $\mathcal{H}$:

$$\mathcal{H}\psi = \psi_{xx} + W(x, T)\, \psi. \tag{16}$$

This operator is defined for functions $\psi(x) \in C^2[0, 1]$ such that $\psi(0) = \psi(1) = 0$. Here,

$$W(x, T) = -\alpha\left(1 - \gamma \exp\left(-T p(x)\right)\right),$$

where the slow time T is a parameter in the potential. Note that $\mathcal{H}$ is a Schrödinger operator, which can be extended to a self-adjoint operator defined on the corresponding Sobolev space. Let us consider the spectral problem

$$\mathcal{H}\psi_n = \lambda_n(T)\psi_n, \tag{17}$$

where $\psi_n(x, T)$ is an eigenfunction depending on the slow time T, which can be considered to be a parameter, and

$$\omega_n(T) = \sqrt{-\lambda_n(T)} \tag{18}$$

is the corresponding frequency.

The following lemma describes general properties of the frequencies $\omega_n(T)$.

Lemma 2 *The frequencies $\omega_n(T)$ satisfy the estimate*

$$\omega_n^2(T) \geq \pi^2 + \alpha - \alpha\gamma \exp(-T p_{\min}). \tag{19}$$

Moreover, if $p_{\min} > 0$, they increase in T.

Proof We can assume that $||\psi_n|| = 1$. Then Eq. (17) implies

$$||\partial\psi_n/\partial x||^2 + \alpha \left(1 - \gamma \int_0^1 \exp\left(-Tp(x)\right) \psi_n^2(x)\mathrm{d}x\right) = \omega_n(T)^2.$$

Note that

$$\min_{\psi_n} ||\partial\psi_n/\partial x||^2 = \pi^2$$

under the conditions $\psi_n(0) = \psi_n(1) = 0$ and $||\psi_n|| = 1$. Moreover,

$$\int_0^1 \exp(-Tp(x))\psi_n^2(x)\mathrm{d}x \leq \exp\left(-T p_{\min}\right).$$

Combining all these estimates, we obtain (19). The second statement in Lemma 4.1 follows from the variational principle for the eigenvalues of the operator $\mathcal{H}$. In some cases we can obtain the asymptotics for the eigenvalues $\lambda_n(T)$. Actually, we have only two main methods: the Born approximation and the WKB method. The first approach allows us to find the asymptotics for small T or large T. □

Lemma 3 *(A) For small T the orthonormal eigenfunctions of the operator $\mathcal{H}$ can be represented by the following asymptotic relation:*

$$\psi_n(x,T) = \sqrt{2}\sin\left(n\pi x\right) + \tilde{\psi}_n(x,T), \quad \text{for} \quad n \in \mathbb{N}, \tag{20}$$

where the small correction $\tilde{\psi}_n$ satisfies the estimate

$$|\tilde{\psi}_n(x,T)| < C_n T, \quad \text{for} \quad n \in \mathbb{N}. \tag{21}$$

and the corresponding eigenvalues have as asymptotic expansions

$$\lambda_n(T) = -\bar{\omega}_n^2 + \tilde{\lambda}_n(T) + O(T^2), \tag{22}$$

where

$$\bar{\omega}_n = \sqrt{\pi^2 n^2 + \alpha(1-\gamma)} \tag{23}$$

and

$$\tilde{\lambda}_n(T) = -2\alpha\gamma T \int_0^1 p(x)\sin^2(n\pi x)\mathrm{d}x. \tag{24}$$

(B) For large T the eigenfunctions of $\mathcal{H}$ can be represented by the following asymptotic expansion:

$$\psi_n(x,T) = \sqrt{2}\sin(n\pi x) + \tilde{\psi}_n(x,T), \quad \text{for } n \in \mathbb{N}, \tag{25}$$

where

$$|\tilde{\psi}_n(x,T)| < C_n \exp(-T p_{\min}),$$

where $p_{\min}$ is the minimum of the positive function $p(x)$ on $[0, 1]$. The corresponding eigenvalues are defined by

$$\lambda_n(T) = -\bar{\omega}_n^2 + \tilde{\lambda}_n(T) + O(\tilde{\lambda}_n^2), \tag{26}$$

where

$$\bar{\omega}_n = \sqrt{\pi^2 n^2 + \alpha} \tag{27}$$

and

$$\tilde{\lambda}_n(T) = 2\alpha\gamma \int_0^1 \exp\left(-Tp(x)\right)\sin^2(n\pi x)\mathrm{d}x. \tag{28}$$

For the frequency ω_n in both cases A and B one has the asymptotic expansions

$$\omega_n(T) = \bar{\omega}_n - \frac{\tilde{\lambda}_n(T)}{2\bar{\omega}_n^2}, \tag{29}$$

where $\bar{\omega}_n$ and $\tilde{\lambda}_n$ are defined by (23) and (24) for case A, and by (27) and (28) in case B, respectively.

Proof These statements have been proved in the well-known perturbation theory; see [13]. □

Note that the eigenfunctions $\psi_n(x,T)$ depend on the slow time T. However, under the assumptions of this Lemma, this dependence leads to weak effects, which are not essential. The main contribution in ψ_n is independent of T. Moreover, we notice that both asymptotic expansions are consistent with Lemma 2:

$\tilde{\lambda}_n(T)$ decreases in T and the corresponding asymptotic expansion for the frequencies increases in T.

For small α the following asymptotic expansion for λ_n can be obtained :

$$\lambda_n = -\bar{\omega}_n^2 + \tilde{\lambda}_n(T) + O\left(\tilde{\lambda}_n^2\right), \tag{30}$$

where

$$\bar{\omega}_n = \pi n,$$
$$\tilde{\lambda}_n(T) = -2\alpha \int_0^1 (1 - \gamma \exp(-Tp(x))) \sin^2(n\pi x)\mathrm{d}x. \tag{31}$$

Note that for small T the correction $\tilde{\psi}_n$ can be computed by the perturbation approach used in quantum theory, that gives the following representation (see [15])

$$\tilde{\psi}_n(x, T) = 2\alpha\gamma T \sum_{m=1,2\ldots,m\neq n} \frac{B_{mn}\sin(m\pi x)}{\bar{\omega}_m^2 - \bar{\omega}_n^2} + O(T^2). \tag{32}$$

where

$$B_{mn} = \int_0^1 p(x)\sin(\pi m x)\sin(\pi n x)\,\mathrm{d}x.$$

We see that the main term in ψ_n does not depend on T, and that its correction is a linear function in T.

Therefore, we have two different cases, for which an asymptotic expansion for ω_n can be found: (I) arbitrary α but small T or large T, see the previous Lemma and (II) small α and arbitrary T. To conclude this section, let us make an important remark: for all continuous $p(x)$ such that $\min_{x\in[0,1]} p(x) > 0$ and for each n the eigenvalues $\lambda_n(T)$ are decreasing functions of T. It follows from this observation that the potential W is a decreasing function in T. This implies that the frequencies $\omega_n(T)$ increase in T.

To conclude this section, let us notice that the relations for $\tilde{\lambda}_n$ can be simplified for $n \gg 1$ (the case of high frequency modes). For $T \ll 1$ one has

$$\tilde{\lambda}_n(T) = -\alpha\gamma T \int_0^1 p(x)\mathrm{d}x. \tag{33}$$

For $T \gg 1$ one obtains

$$\tilde{\lambda}_n(T) \approx \alpha\gamma \int_0^1 \exp(-Tp(x))\,\mathrm{d}x. \tag{34}$$

5 The WKB method and localized modes

The WKB method allows us to find the modes and the asymptotic expansion for $\lambda_n(T)$ for large α. This asymptotic approach is valid for all T. We obtain two kinds of eigenfunctions.The first class of eigenfunctions consists of modes similar to the ones studied above. The second class includes new eigenfuctions, which exist only in the case of non-uniform ageing when $p(x)$ is not constant.

5.1 Non-localized eigenfunctions

Let $\alpha \gg 1$ and $\gamma \in (0, 1)$. We introduce the large parameter $h = \sqrt{\alpha}$ and look for eigenfunctions in the form

$$\psi_n(x, T) = a_n(x)\sin(h\Phi_n(x, T)), \tag{35}$$

where a_n and Φ_n are new unknown functions, which define the amplitude and the phase of the eigenfunctions, respectively. Then from (35) one obtains:

$$\frac{\partial^2\psi_n}{\partial x^2} = -h^2\frac{\partial\Phi_n}{\partial x}\sin(h\Phi_n(x)) + h\left(2\frac{\partial a_n}{\partial x}\frac{\partial\Phi_n}{\partial x} + a_n\frac{\partial^2\Phi_n}{\partial x^2}\right) + O(1).$$

Then we see that the terms of the order h^2 give the following eikonal equation for Φ_n:

$$\left(\frac{\partial\Phi_n}{\partial x}\right)^2 = -\lambda_n/h^2 - 1 + \gamma\exp(-Tp(x)). \tag{36}$$

The terms of order $O(h)$ give linear equations for a_n, yielding

$$a_n = C_n\left(\frac{\partial\Phi_n}{\partial x}\right)^{-1/2}, \tag{37}$$

where C_n are constants. We can set $\Phi_n(0) = 0$. Then, the boundary condition at $x = 1$ gives

$$h\int_0^1 \frac{\partial\Phi_n}{\partial x}\,\mathrm{d}x = n\pi, \tag{38}$$

where n is an integer. Now Eqs. (36) and (39) define the amplitude $a_n(x)$:

$$a_n = C_n \left(\omega_*^2/h^2 - 1 + \gamma \exp\left(-Tp(x)\right)\right)^{-1/4}. \quad (39)$$

Let us define the frequency ω_n by $\lambda_n = -\omega_n^2$. Then, by (38) and (36) we find the following equation for $\omega_n(T)$:

$$\int_0^1 \sqrt{\omega_n^2 - \alpha(1 - \gamma \exp(-Tp(x)))}\, \mathrm{d}x = n\pi. \quad (40)$$

The obtained eigenfunctions are not localized in x and define high frequency modes since $\omega_n = O(h)$. They exist both in space uniform and non-uniform cases.

Note that for the high frequency modes the WKB relation (40) gives an asymptotic expansion which is uniform in T. One can show that for small T this relation reduces to relation (29), where $\tilde{\lambda}_n$ is defined by (33), and that for large T we obtain relation (29) with $\tilde{\lambda}_n$ defined by (34).

5.2 Localized eigenfunctions

Localized eigenfunctions (modes) are concentrated at points x_* where $p(x)$ has local minima, i.e. $p^{'}(x_*) = 0$ and $p^{''}(x_*) = r^2 > 0$. Let us introduce a new variable $\xi = h^{1/2}(x - x_*)$. Then for $\xi = O(1)$, $T = O(1)$ and $x = O(h^{-1/2})$ we obtain

$$\exp(-Tp(x)) = \exp(-Tp(x_*)) - R(T)^2\xi^2/h + O(h^{-3/2}),$$

where

$$R(T)^2 = \frac{Tr^2}{2}\exp(-Tp(x_*)).$$

Therefore, for modes ψ_n^{loc} localized at $x = x_*$, we obtain the following relation:

$$\frac{d^2\psi_n^{\mathrm{loc}}}{d\xi^2} - (a_0(T) + R(T)^2\xi^2)\psi_n^{loc} = h^{-1}(-\Omega_n^2 + O(h^{-1/2}))\psi_n^{loc}, \quad (41)$$

where Ω_n is a frequency of the nth localized mode and

$$a_0(T) = h^{-1}(\alpha - \alpha\gamma \exp(-Tp(x_*))) = \sqrt{\alpha}(1 - \gamma \exp(-Tp(x_*)) > 0.$$

Up to small corrections of the order $h^{-1/2}$, Eq. (41) coincides with the Shrödinger equation for a quantum harmonic oscillator. The corresponding eigenfunctions can be expressed via Hermit's polynomials and have the form

$$\psi_n^{loc}(x, T) = R(T)^{-1/4}\Psi_n(R(T)^{1/2}\xi), \quad (42)$$

where

$$\Psi_n(X) = (2^n n!\sqrt{\pi})^{-1/2}\exp(X^2/2)\frac{d^n}{dX^n}\exp(-X^2),$$

and

$$\Omega_n(T)^2 = h(a_0(T) + (2n+1)R(T)). \quad (43)$$

For $n \ll 1/h$ one has the asymptotic expansion

$$\Omega_n(T)^2 \approx \alpha(1 - \gamma \exp(-Tp(x_*))). \quad (44)$$

For bounded values of n the boundary conditions at $x = 0, 1$ are satisfied with exponential accuracy $O(\exp(-c_0(R(T)h)^{1/2})$. The main mode for $n = 0$ has the form

$$\psi_0^{loc}(x, T) = (\pi R(T))^{-1/4}\exp(-R(T)h(x - x_*)^2/2).$$

Note that these localized modes exist only for T such that $R(T)h \gg 1$. For large T the functions $\psi_n^{loc}(x, T)$ are not asymptotic solutions of Eq. (41).

5.3 Comparison of frequencies of localized and non-localized modes

We can formulate the following general results for the frequencies of localized and non-localized modes. Let us assume that the function $p(x)$ reaches its global minimum at an interior point $x_* \in (0, 1)$ and $p^{''}(x_*) > 0$. Then the modes localized at x_* have the frequencies defined by relation (44). On the other hand let us consider relation (40) that defines the frequencies of the non-localized modes. Since $p(x) \geq p(x_*)$ this relation implies that

$$\omega_n^2 \geq \alpha(1 - \gamma \exp(-Tp(x_*))) + (n\pi)^2 > \Omega_n^2, \quad (45)$$

i.e. for large α the frequencies of the non-localized modes always are higher than the frequencies of the modes localized at the global minimum of p. In particular, in case (3) one has that the minimal frequency is determined by the localized modes and equals

$$\min_n \Omega_n(T) \approx \alpha(1 - \gamma \exp(-T(p_0 - b_0))).$$

6 Fourier decomposition

Solutions of (6) can be expressed through the eigenfunctions ψ_n as

$$u(x,t) = \sum_{n\in\mathbb{Z}, n\neq 0} X_n(t)\psi_n(x), \tag{46}$$

where $X_n(t)$ are unknown functions, which determine complex amplitudes of the oscillations such that $X_{-n} = X_n^*$ (the star denotes complex conjugation). We set $\psi_n(x) = \psi_{-n}^*(x)$. Then $u(x,t)$ is a real-valued function. Note that the sum in the right-hand side of (46) should contain all possible modes, localized and non-localized.

Let us consider the case with f given by (2). Then Eq. (6) is linear and we obtain the following equations for X_n:

$$\frac{\mathrm{d}^2 X_n(t)}{\mathrm{d}t^2} + \omega_n^2(T)X_n = -\varepsilon\frac{\mathrm{d}X_n(t)}{\mathrm{d}t} \tag{47}$$

where $\omega_n(T)$ is defined by (18), and $T = \varepsilon_a t$ is a slow time associated with the ageing process. Initial data for $X_n(t)$ and $V_n = \mathrm{d}X_n/\mathrm{d}t$ can be obtained by (4). We obtain

$$X_n(0) = ||\psi_n||^{-2}\langle v_0, \psi_n\rangle,$$
$$V_n(0) = ||\psi_n||^{-2}\langle v_1, \psi_n\rangle. \tag{48}$$

Note that the Galerkin approximations of $u(x,t)$ use a finite number N of the modes, i.e. $n = \pm 1, \pm 2, \ldots \pm N$. The truncation number N can be defined by energetic arguments. For $t = 0$ the energy $E(t)$ equals $E = \sum_{n\in\mathbf{Z}} \omega_n^2 |X_n(0)|^2 + |V_n(0)|^2$. The truncated energy is $E_N = \sum_{n=-N}^{N} \omega_n^2 |X_n(0)|^2 + |V_n(0)|^2$. Let us choose a sufficiently small $\delta > 0$. Note that if $v_0(x)$ and $v_1(x)$ are smooth functions of x, then the n-th term $\omega_n^2|X_n(0)|^2 + |V_n(0)|^2$ is a decreasing function of the mode number n. Then, due to this decreasing property, we can take N such that $E - E_N < \delta E$. Equation (47) involves two small parameters, ε and ε_a. This fact allows us to obtain an asymptotic solution of Eq. (47) , which have a transparent physical meaning.

6.1 Spatial resonance in the initial boundary value problem

In this section we describe an interesting effect connected with the existence of localized eigenfuctions. Consider the Cauchy problem for Eq. (47) in the WKB case, when $h = \sqrt{\alpha} \gg 1$, and there is a point x_* which yields a non-degenerate local minimum of $p(x)$. The Cauchy data $X_n(0), V_n(0) = \mathrm{d}X_n/\mathrm{d}t(0)$ for the amplitudes of the non-localized modes can be obtained from the initial conditions (4) by the relations

$$X_n(0) = \int_0^1 v_0(x)\psi_n(x)\mathrm{d}x,$$
$$V_n(0) = \int_0^1 v_1(x)\psi_n(x)\mathrm{d}x, \tag{49}$$

and similarly for the amplitudes of the localized modes:

$$X_n(0)^{\mathrm{loc}} = \int_0^1 v_0(x)\psi_n^{\mathrm{loc}}(x)\mathrm{d}x,$$
$$V_n(0) = \int_0^1 v_1(x)\psi_n^{\mathrm{loc}}(x)\mathrm{d}x. \tag{50}$$

Consider the following two cases. In the first case a, the initial data $v_0(x)$ and $v_1(x)$ are non-localized, for example, $v_i(x) = a_i(x)\sin(S_i(x))$, where a_i, S_i are smooth functions. In the second case b, these data are localized at a point x_0, for example, $v_i(x) = b_i(x)\exp(-(x-x_0)^2/2\sigma^2)$, where $\sigma > 0$ is a small parameter.

In case a the coefficients $X_n(0)^{\mathrm{loc}}$ and $V_n(0)^{\mathrm{loc}}$ can be computed by the standard asymptotics [16], see formula (42). They are small and have the order $O(h^{-1/4})$. Therefore, in this case the localized mode does not make an essential contribution in the Fourier decomposition.

For case b, if x_* and x_0 are separated, for example, $|x_* - x_0| \gg h^{-1/2}$, then again we have not an essential contribution in the Fourier decomposition of the localized mode. However, if these points are close, say, $|x_* - x_0| < h^{1/2}$, then we obtain an opposite picture. The contributions of the localized modes can be estimated as follows:

$$|X_n(0)|, |V_n(0)| = O(\sigma),$$
$$|X_n(0)^{\mathrm{loc}}|, |V_n(0)^{\mathrm{loc}}| = O(h^{1/4}(h+\sigma^2)^{-1/2}). \tag{51}$$

The contributions of the localized modes are larger under the condition $\sigma^2 \ll h^{-1/2}$. Therefore, we can conclude that for large rigidity α the spatial resonance effect arises, which is induced by the localized modes.

Consider this effect in the case (3). Then the points x_* are defined by

$$x_* = \frac{\pi(2k+1) - \phi}{\omega_*}, \quad 0 < x_* < 1,$$

where $k = 1, 2, \ldots$. Such points exist only for large space frequencies $\omega_* > \pi$. Below we will show that for non-localized modes spatial resonances are possible only for sufficiently small ω_*.

7 Asymptotic solution

It is natural, following the standard methods, to look for asymptotic approximations of the solutions of Eq. (47) in the following form:

$$X_n(t, \varepsilon, \varepsilon_a) = Y_n(T) \exp(i\varepsilon_a^{-1} S_n(T)) + O(\varepsilon_a + \varepsilon),$$
$$i = \sqrt{-1}, \quad (52)$$

where Y_n is an unknown complex function of order 1, and S_n is an unknown real-valued function of order 1. These functions determine a slowly evolving magnitude and a fast oscillating phase $\varepsilon_a^{-1} S_n$ of the n-th mode respectively, where $n > 0$. For $n < 0$ we set formally $X_n = X^*_{-n}$, then the displacement $u(t, x)$ takes real values. We substitute X_n into (47) and obtain that the principal terms of order 1 vanish under the condition

$$\mathrm{d}S_n(T)/\mathrm{d}T = \omega_n(T) \quad (53)$$

which gives us an eikonal equation to for S_n. If this equation is satisfied, then in Eq. (47) the main terms become $O(\varepsilon + \varepsilon_a)$. Removing terms of higher orders in ε and ε_a we obtain

$$2\frac{\mathrm{d}Y_n(T)}{\mathrm{d}T} = -(\mu c_1 + \omega_n(T)^{-1}\frac{\mathrm{d}\omega_n(T)}{\mathrm{d}T})Y_n, \quad (54)$$

where the parameter μ is defined by (12). This equation is correct under the restriction $\varepsilon_a \gg \varepsilon^2$, i.e. $\mu \gg \varepsilon$. Otherwise we must take into account the higher-order correction terms in (47) and the equation for Y_n becomes rather complicated.

From (54) we obtain

$$Y_n(T) = Y_n(0)\omega_n(T)^{-1/2} \exp(-\mu c_1 T/2). \quad (55)$$

Substituting this relation into the formula (52) for X_n and taking real values one obtains the following asymptotic approximation of the solution of the problem:

$$\sum_{n \in \mathbf{Z}} \omega_n(T)^{-1/2} \exp(-\mu c_1 T/2)$$
$$\times \left(a_n \sin(\varepsilon_a^{-1} S_n(T)) + b_n \cos(\varepsilon_a^{-1} S_n(T))\right) \psi_n(x, T), \quad (56)$$

where $\psi_n(x, T)$ are the asymptotic eigenfunctions found in Sects. 4.1 and 5, and a_n, b_n are constants. For large rigidities $\alpha >> 1$ this solution is a result of a "double" WKB method: we apply the WKB approach to find the time dependence of the amplitude for X_n, and to find the spatial form of the modes ψ_n.

8 Estimates of the amplitudes $|X_n|$

In this section, we derive energy estimates for the amplitudes X_n. Let us introduce the unperturbed energy of the nth mode X_n by

$$E_n(t) = \frac{1}{2}\left(P_n(t)^2 + \omega_n^2(T)X_n(t)^2\right),$$
$$P_n = \frac{\mathrm{d}X_n(t)}{\mathrm{d}t}. \quad (57)$$

The first term in the right-hand side is the kinetic energy associated with the n-th mode and the second term is the potential energy. Let us multiply the right-hand and the left-hand sides of (47) by $dX_n/\mathrm{d}t$. Then (47) and (57) give the following relation for the energy

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = -\varepsilon c_1 \frac{\mathrm{d}X_n(t)}{\mathrm{d}t}^2 + \varepsilon_a \omega_n(T)\frac{\mathrm{d}\omega_n(T)}{\mathrm{d}T} X_n^2. \quad (58)$$

We see that the energy slowly evolves in time, whereas $X_n(t)$ also depends on the fast time t. In fact, let us note that for $\varepsilon = \varepsilon_a = 0$ we have $X_n = Y_n \sin(\omega_n t + \phi_n)$, where ω_n, Y_n and ϕ_n are independent of t. For small ε and ε_a, these functions are slow functions of t. This allows us to average relation (58) over the interval $I_n(T) = [T, T + \tau_n]$, where $\tau_n = 2\pi\omega_n^{-1}(T)$ is a period depending on T. Note that, according to (52), we have the following result

$$2\tau_n^{-1} \int_T^{T+\tau_n} X_n(t)^2 \mathrm{d}t = |X_n(T)|^2 + O(\varepsilon + \varepsilon_a). \quad (59)$$

$$2\tau_n^{-1} \int_T^{T+\tau_n} X_n(t)^2 \mathrm{d}t = \omega_n^2 |Y_n(T)|^2 + O(\varepsilon + \varepsilon_a). \quad (60)$$

Let us denote the averaged energy $\bar{E}_n$. Then (59) and (60) imply that

$$\tau_n{}^{-1}\int_T^{T+\tau_n}(\frac{dX_n(t)}{dt})^2 dt = \bar{E}_n(T) + O(\varepsilon + \varepsilon_a), \tag{61}$$

$$\tau_n{}^{-1}\omega_n^2\int_T^{T+\tau_n} X_n(t)^2 dt = \bar{E}_n(T) + O(\varepsilon + \varepsilon_a). \tag{62}$$

Note that the averaged kinetic and the potential energies of the weakly perturbed linear oscillator are equal. Using the relations (61) and (62) we obtain from (58) the main evolution equation for the energies $\bar{E}_n$ of the slow modes:

$$\frac{d\bar{E}_n(T)}{dT} = \varepsilon_a^{-1}\kappa_n\bar{E}_n, \tag{63}$$

where

$$\kappa_n = -\varepsilon c_1 + \varepsilon_a\omega_n(T)^{-1}\frac{d\omega_n(T)}{dT} \tag{64}$$

determines the time evolution of the energy part associated with the nth mode.

9 Ageing and spatial resonances

The simplest case is when ageing is defined by $p(x) = a_0$. Then

$$\lambda_n(T) = -(n\pi)^2 - \alpha(1 - \gamma\exp(-a_0 T)). \tag{65}$$

In the non-uniform case we observe an interesting effect, which can be named "spatial resonance". Let $p(x)$ be defined by (3).

Case 1, small T.

For small T one has

$$\frac{d\tilde{\lambda}_n(T)}{dT} = -2\alpha\gamma\int_0^1 p(x)\sin^2(n\pi x)dx, \tag{66}$$

and thus

$$\kappa_n = -\varepsilon + \frac{\varepsilon_a\alpha\gamma}{\bar{\omega}_n^2}\int_0^1 p(x)\sin^2(n\pi x)dx + O\left(\varepsilon_a^2\right). \tag{67}$$

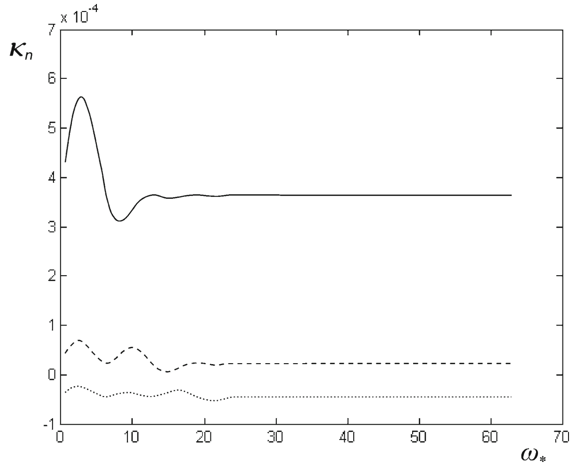

Fig. 1 The quantities $\kappa_n(\omega_*)$ for $n = 1$ (*solid curve*) $n = 2$ (*dashed curve*) and $n = 3$ (*dotted curve*). The ω_* ranges in the interval [0, 70] on the horizontal axis. The parameters are $\alpha = 1$, $\gamma = 0.1$, $\phi = \pi/2$, $\varepsilon_a = 0.1$, $\varepsilon = 0.001$, $p_0 = 1$ and $b_0 = 0.5$

Let us consider the dependence of $\kappa_n(\omega)$ on ω_n for different n, which is illustrated in Fig. 1.

We observe that κ_n is positive only for small n (in our case for $n = 1, 2$). So, the ageing helps to increase the magnitudes of the lower modes only. Moreover, κ_1 oscillates in ω_* with a larger amplitude than κ_2 and κ_3 in amplitude; we observe a sharp peak in $\kappa_1(\omega_*)$ values, i.e. a spatial resonance effect is present. For large ω_*: oscillations of $\kappa_n(\omega_*)$ damp and this resonance effect is stronger for small n.

Consider the case when $p(x)$ is defined by (3). Then,

$$\kappa_n = -\varepsilon + \frac{\varepsilon_a\gamma}{2\omega_n^2}(p_0 + b_0 R(\omega_*, \phi, n)), \tag{68}$$

where

$$R(\omega_*, \phi, n)) = \frac{1}{2}\Big[\Big(2\omega_*^{-1} - (\omega_* + 2\pi n)^{-1}\Big)(\sin(\omega_* + \phi) - \sin(\phi)) - (\omega_* - 2\pi n)^{-1}(\sin(\omega_* - \phi) - \sin(\phi))\Big].$$

These relations indicate that there is a possible *spatial resonance* when for the space non-uniform case $b_0 = 0$ all amplitudes $X_n(t)$ are exponentially decreasing in t (i.e. all $\kappa_n < 0$), but for sufficiently large b_0 the amplitudes $X_n(t)$ increase for some $n < N_0$. The sufficient conditions that the spatial resonance arises for the n-th mode are as follows:

$$\omega_* \approx 2\pi n, \quad -\varepsilon + \frac{\varepsilon_a \gamma a_0}{2\omega_n^2} < 0, \tag{69}$$

$$-\varepsilon + \frac{\varepsilon_a \gamma}{2\omega_n^2}(p_0 - b_0 \cos\phi) > 0. \tag{70}$$

Note that this effect depends on the phase ϕ and n. For large n or for large ω_* the spatial resonance is impossible. We can compare this conclusion with the result of Sect. 5.2, where it is shown that for large rigidities $\alpha >> 1$ the spatial resonance occurs only if the space frequency is sufficiently large.

Case 2, large T.

For large T one finds from (24) that

$$\frac{\mathrm{d}\tilde{\lambda}_n(T)}{\mathrm{d}T} = -2\alpha\gamma J_{\omega_*}(T),$$
$$J_{\omega_*} = \int_0^1 p(x)\exp(-Tp(x))\sin^2(n\pi x)\mathrm{d}x. \tag{71}$$

Consider the integral in the right-hand side of this relation. Using the well-known asymptotic estimate [16], for large T, it follows that:

$$J_{\omega_*}(T) = \tilde{c}_0\alpha\gamma(b_0T)^{-1/2}\omega^{-1}(p_0 - b_0)$$
$$\times \exp(-(p_0 - b_0)T)$$
$$\times \sum_{k\in I(\phi,\omega_*)} \sin^2(n\pi x_k) + O(T^{-3/2}), \tag{72}$$

where $\tilde{c}_0 > 0$ is a constant. Here $x_k = \omega_*^{-1}((2k+1)\pi - \phi)$ and $I(\phi,\omega_*)$ is the set of all integers k such that $x_k \in (0,1)$. Note that for some ω_* the main term in J_ω equals zero whereas for other ω_* values this term is not small. We observe here an effect of " a spatial resonance", when spatially non-uniform ageing influences the linear modes in a different way. In the next sections, we investigate this effect in more details.

9.1 The case of large rigidity

Consider the case $\alpha >> 1$. To describe the time evolution of the amplitudes X_n for all times T, we apply the WKB approximation (see Sect. 5) and the relation (40). By differentiating this relation with respect T, one obtains

$$\frac{\mathrm{d}\omega(T)}{\mathrm{d}T} = \frac{\alpha\gamma\int_0^1 p(x)\exp(-Tp(x))\Delta(x)\mathrm{d}x}{\omega_n\int_0^1 \Delta(x)\mathrm{d}x},$$
$$\Delta(x) = (\omega_n^2 - \alpha(1 - \gamma\exp(-Tp(x))))^{-1/2}. \tag{73}$$

As a result, Eq. (64) gives

$$\kappa_n = -\varepsilon + \frac{\varepsilon_a\alpha\gamma\int_0^1 p(x)\exp(-Tp(x))\Delta(x)\mathrm{d}x}{\omega_n^3\int_0^1 \Delta(x)\mathrm{d}x}. \tag{74}$$

We see that for sufficiently large n the κ_n are negative. Let us estimate the number of modes N_c such that the corresponding E_n increases. Using (74) one has as a rough estimate:

$$N_c^3(T) \approx C_0\varepsilon^{-1}\varepsilon_a\alpha\gamma\int_0^1 \exp(-Tp(x))p(x)\mathrm{d}x, \tag{75}$$

where C_0 is a positive constant. We see that $N_c(T)$ decreases in T.

For small T the number $N_c(T)$ depends on the average $\int_0^1 p(x)\mathrm{d}x$ only. For large T the number is defined by the value $p_{\min}$, i.e. the space points where ageing is minimal.

10 Effects of ageing

The uniform and non-uniform ageing processes are essentially different. To show this, let us consider the relation (74) in the cases a and b. In the first case $p(x) \equiv a_0 = \text{const}$ and $\frac{d\tilde{\lambda}_n(T)}{\mathrm{d}T} = \alpha\gamma a_0$ does not depend on n. We obtain

$$\kappa_n = -\varepsilon + \rho_n, \quad \rho_n = \frac{\varepsilon_a\alpha\gamma a_0}{\bar{\omega}_n^3} + O\left(\varepsilon_a^2\right). \tag{76}$$

In this case the ageing contributes to the energy evolution. This contribution is always positive. Therefore, the amplitudes of some modes with small n can increase in some time interval t. The energy of the modes with large n always decreases for all t since the term ρ_n is proportional to n^{-3} for large n. Moreover, the energies of all the modes are exponentially decreasing functions in t for large times t, since for large T the quantities ρ_n are exponentially small.

A general picture for the uniform case is as follows. During some time period $[0, T_0]$ the energies E_n of some modes with indices $n = 1, \ldots, n_c$ can increase, whereas all the other modes are exponentially decreasing functions in time for all times. Roughly speaking

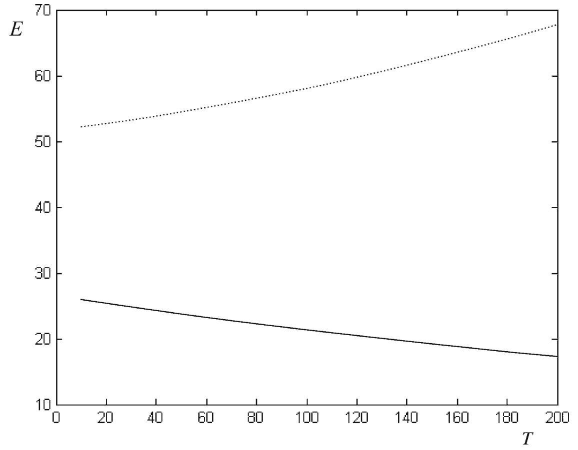

Fig. 2 Energies of the solutions for the uniform (*dotted curve*) and the non-uniform case (solid curve) as functions of time T (*horizontal axis*), $T \in [0, 200]$. The parameters are $\alpha = 1$, $\gamma = 0.5$, $\phi = \pi$, $\omega_* = 2\pi$, $\varepsilon_a = 0.2704$, $n = 1$ and $\varepsilon = 0.01$

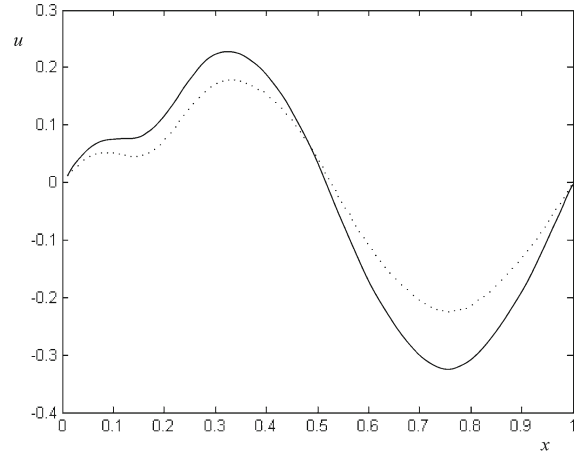

Fig. 3 Displacement $u(x, t)$ for the uniform (*dotted curve*) and the non-uniform case (*solid curve*). The parameters are taken as in Fig. 2. $t = 200$

in the uniform case, the modes with lowest frequencies give the main contribution to the energy.

In case b we have the same relation (76); however, now ρ_n is defined by the integral

$$\rho_n = \frac{2\varepsilon_a \alpha \gamma}{\bar{\omega}_n^3} \int_0^1 p(x) \sin^2(n\pi x) \mathrm{d}x.$$

This term is not necessarily a monotone function in n. Therefore, for large (but not too large) times t the energy can be defined by a single resonance mode, which gives a maximal term ρ_n.

The main effect produced by the non-uniform ageing is as follows. For some special parameter values when the relation

$$\varepsilon = \alpha \gamma a_0 \bar{\omega}_1^{-2}$$

holds, the non-uniform term can produce an exponential growth of the magnitude of the main mode with $n = 1$, whereas for the uniform ageing this magnitude decreases. The energy can also increase in the non-uniform case and decrease in the uniform case. This effect is illustrated in Figs. 2 and 3.

11 An external load

Let us consider the case where Eq. (6) contains a harmonic external load:

$$u_{xx} - \alpha(1 - \gamma \exp(-\varepsilon_a t p(x))u - u_{tt} \\ = \varepsilon u_t + b\theta(x) \sin(\Omega t), \tag{77}$$

where Ω is a frequency, and $b > 0$ is a non-dimensional amplitude of an external load, and θ is a function in x which shows the distribution of an external load in space. In this case we also can find an asymptotic approximation of the solution. We write the solution u as a sum $u = \tilde{u} + \bar{u}$, where $\tilde{u}$ satisfies Eq. (6) for $b = 0$, and $\bar{u}$ is a special, particular solution of (77). The function $\tilde{u}$ can be found by the asymptotic construction as presented in the previous section. To find $\bar{u}$, we set

$$\bar{u}(x, t) = \sum_{n \in \mathbb{Z}, n \neq 0} Z_n(t) \psi_n(x, T). \tag{78}$$

The unknown coefficients $Z_n(t)$ in the eigenfunction expansion (78) satisfy

$$\frac{\mathrm{d}^2 Z_n(t)}{\mathrm{d}t^2} + \omega_n^2(T) Z_n = b\theta_n(T) \sin(\Omega t) + O(\varepsilon + \varepsilon_a), \tag{79}$$

where

$$\theta_n(T) = \frac{\langle \theta, \psi_n \rangle}{\langle \psi_n, \psi_n \rangle}.$$

Note that due to the asymptotic results as obtained in Sect. 4 (see lemma 4.1 and relation (32) for small α or small T) the function θ_n weakly depends on T and (for small T)

$$\omega_n(T) = \sqrt{2} \int_0^1 \theta(x) \sin(\pi n x) \mathrm{d}x + O(T), \quad T \to 0.$$

We can look for a particular solution of (79) in the following form:

$$Z_n(t) = U_n(T)\sin(\Omega t), \quad (80)$$

where $U_n(T)$ is an unknown function. Substituting (80) into (79), and neglecting small terms of the orders ε and ε_a, we obtain

$$U_n(T) = b\theta_n(T)(\omega_n(T)^2 - \Omega^2)^{-1}. \quad (81)$$

For times T such that $\omega_n(T) \approx \Omega$ we obtain a resonance. For these T values we should take into account the small terms of the orders $O(\varepsilon)$ and ε_a, and apply a more sophisticated asymptotic method to find a particular solution of (79). Nonetheless, we can obtain general results in these resonance cases. The remark at the end of Sect. 4 shows that for each n the frequency $\omega_n(T)$ is a decreasing function in T. Therefore, we conclude that for each fixed mode number n we have either a single resonance for some T, or the resonance is absent. The general number N_r of the resonances for different n depends on the parameters α, γ, and Ω, but N is independent of the form of the ageing function $p(x)$. In fact, for each n we have the resonance uniqueness property, and thus the resonance exists if and only if $\omega_n(0) < \Omega$ and $\omega_n(+\infty) > \Omega$. The frequencies $\omega_n(0)$ and $\omega_n(+\infty) = \lim_{T\to+\infty}\omega_n(T)$ can be easily computed which implies the following.

(a) If $\alpha(1-\gamma) > \Omega^2$, then the resonances are absent and $N_r = 0$;
(b) If $\alpha(1-\gamma) < \Omega^2$ and $\alpha > \Omega^2$ then the resonance number is defined by the relation

$$N_r = \left[\sqrt{\Omega^2 - \alpha(1-\gamma)}/\pi\right], \quad (82)$$

where $[x]$ is the maximal integer, which is less than x (the floor of x);
(c) If $\alpha(1-\gamma) < \Omega^2$ and $\alpha < \Omega^2$, then the resonance number is defined by the relation

$$N_r = \left[\left(\sqrt{\Omega^2 - \alpha(1-\gamma)} - \sqrt{\Omega^2 - \alpha}\right)/\pi\right]. \quad (83)$$

Note that the resonance number increases in γ. The resonance is absent for small Ω and too large Ω. The properties of N_r can be illustrated in the following Fig. 4.

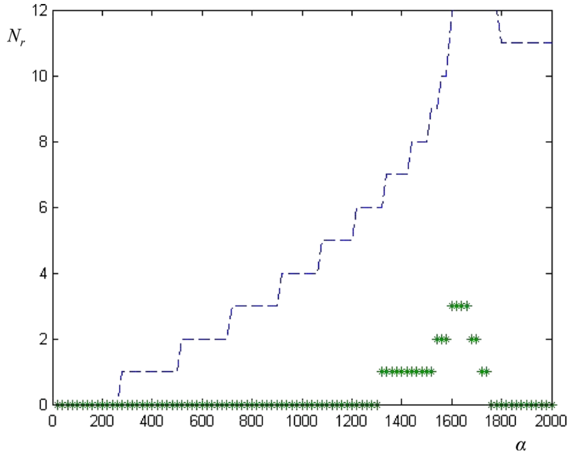

Fig. 4 The number of resonances for $\gamma = 0.9$ (*dotted curve*) and $\gamma = 0.09$ (*star curve*) for different values of α. The parameter $\Omega = 40$

12 Conclusion

The dynamics of a string on an elastic foundation with variable time- and space-coordinate coefficients has been studied. Asymptotic approximations of the solutions have been constructed in the following cases: 1. for an arbitrary value of the elastic foundation coefficient α at small and at large time values; 2. for small and large values of the coefficient α at arbitrary times. A special case for the elastic foundation coefficient function (originating from an ageing process) has been studied. The ageing process is described by an expression approximating some well-known experimental data. The obtained approximations of the solutions are accurate.

In case of a large coefficient α, and a non-uniform distribution of the ageing process in space, in addition to the uniform ageing modes, the existence of localized modes in local minima of the ageing function distributions $p(x)$ has been proved. It has been shown that when $p(x)$ has a global minimum the frequency of the relevant localized mode is located below the non-localized frequencies.

For large rigidity values of α, the existence of localized modes allows a "spatial resonance" phenomenon under certain conditions as described in this paper. In case the $p(x)$ function is given by the first two terms in a Fourier cosine series, the "spatial resonance" is only possible for large values of the frequencies $\omega(\omega > \pi)$. It can be confirmed, that in case an external force on the string is expressed by a function (which is non-localized along the coordinates), then the resonance with the localized mode is weaker than the resonance with the non-localized mode. But if the external force is localized, then it is vice versa, the resonance with the localized mode is stronger then with the non-localized mode. In case of an arbitrary elastic foundation coefficient α for small or large times, the "spatial resonance" phenomenon is observed at small "special frequencies". This effect also depends on the phase and the mode number n. Thus, for large n this resonance is not possible. This conclusion is opposite to the case for large elastic foundation rigidity coefficients, where the effect was found at small non-"special frequencies". The difference between the non-uniform ageing and the uniform ageing can be described as follows: for certain parameter values the non-uniform ageing causes an exponential growth of the main mode frequency amplitude for $n = 1$, when in case of a uniform ageing this mode is influenced.

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