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Non-commutative differentiation and estimates on operator integrals

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Abstract

In 2017 Martijn Caspers, Fedor Sukochev and Dmitriy Zanin published a paper which generalises the proof of Davies' 1988 paper, and thus resolves the Nazarov-Peller conjecture. The proofs of these papers have been presented in this thesis. They have been expanded with a proof that generalises the conjecture to arbitrary Schatten classes. The optimality of the estimates in the conjecture is also studied, following the example of the 2016 paper by Coine et al. In addition, the quantum mechanical context is provided to interpret the presented results.

Contents

1 Introduction

Quantum mechanics is the field of research in physics that concerns itself with describing the natural world at the atomic and sub-atomic scales [14]. When measurements and energy levels reach this scale, systems behave unlike anything we observe with the naked eye, and our intuition about the natural world fails us. Experiments at the start of the twentieth century showed that not just our intuition, but also the entire body of research on physics up to that point was unable to explain newly observed phenomena, such as black body radiation [19] and the photo-electric effect [12]. Clearly, a new framework for describing particles and energy systems was needed.

Quantum formalism

This introduction to the formalism of quantum mechanics is based on [14]. In order to define quantum systems in a mathematically rigorous way, the theory of Hilbert spaces is used. Specifically, any quantum system is described as a vector in a Hilbert space. Which space is used depends on the type of system we wish to describe. Consider, for example, a system which classically has two possible states: the bit. It can be described fully in binary language: at any time t, the bit is either in state 0 or state 1. Now we will attempt to make such a system at the atomic level.

The spin of an electron is an observable with two possible values. In any axis of measurement, we always measure either a clockwise or a counter-clockwise spin: any measurement outcome is always binary. However, in order to accurately describe the spin-state of an electron, a simple binary does not suffice. Instead, we can describe the spin-state of an electron as an operator in the complex two-dimensional Hilbert space spanned by $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, corresponding to the bit being 0, and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, corresponding to the bit being 1. Classically, there are only four possibilities for such an operator: we can map using the identity, we can swap the bit, or we can force the bit onto one of the basis states no matter the input. However, in quantum mechanics, any trace-one, positive semi-definite operator represents a quantum state.

Note that the basis states presented above are not operators. Writing states as vectors is a very common short-hand notation for a subset of states, called pure states. More formally, we would write the state corresponding to the bit being in 0 as $|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}^\dagger = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$. Note that this matrix indeed has trace equal to one, and is positive semi-definite:

$$x^\dagger \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} x = x_1^* x_1 = |x_1|^2 \geq 0$$

for any $x \in \mathbb{C}^2$.

An interesting quantity to consider when studying quantum mechanics is the trace distance between two states, ρ and σ, defined as follows: $T(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$. Since both ρ and σ are trace-one operators, this quantity is always well-defined and we have $0 \leq T(\rho, \sigma) \leq 1$. It is related to the overlap between two states, in the following sense: the optimal probability of distinguishing two states ρ, σ is $\frac{1}{2} + \frac{1}{2}T(\rho, \sigma)$. If both states are identical, they can not be distinguished, so the best we can do is flip a coin. Indeed $T(\rho, \sigma) = 0$, so we get an optimal probability of $\frac{1}{2}$. However, if there is no overlap between the states (for example, $\rho = |0\rangle\langle 0|$ and $\sigma = |1\rangle\langle 1|$), we can distinguish them simply by measuring the bit. Indeed, $T(\rho, \sigma) = 1$, so we should be able to distinguish perfectly.

Spectral theory

It should be clear now that studying the norm of the difference of operators on a Hilbert space is relevant to quantum mechanics. Specifically, in this thesis, we will study how these norms change when applying functions to bounded Hilbert space operators. To do this, we make use of spectral theory. We will introduce this framework step-by-step. Consider first how we apply a continuous function to a Hermitian matrix $A \in M^n$. We can decompose A into projections onto its eigenvectors, with its eigenvalues as coefficients:

$$A = \lambda_1 P_{\xi_1} + \lambda_2 P_{\xi_2} + ... + \lambda_n P_{\xi_n}.$$

Now we can see what we expect to get when we apply an integer power or a scalar multiplication to A. αA clearly has the same eigenvectors as A, with eigenvalues multiplied by α:

$$A\xi_i = \lambda_i \xi_i \implies (\alpha A)\xi_i = \alpha\lambda_i\xi_i, \quad 1 \leq i \leq n.$$

Now consider A^2:

$$A\xi_i = \lambda_i \xi_i \implies A^2\xi_i = A\lambda_i\xi_i = \lambda_i A\xi_i = \lambda_i^2\xi_i.$$

We extend this calculation to find that for any $n \geq 1$ we have $A\xi_i = \lambda_i\xi_i \implies A^n\xi_i = \lambda_i^n\xi_i$. So we can write $\alpha A^n = \alpha\lambda_1^n P_{\xi_1} + \alpha\lambda_2^n P_{\xi_2} + ... + \alpha\lambda_n^n P_{\xi_n}$. By linearity we can expand this result to the polynomials, and by Stone-Weierstrass we can approximate any continuous function using polynomials. So to apply a function to a Hermitian matrix, we do the following:

$$f(A) = \sum_{i=1}^{n} f(\lambda_i) P_{\xi_i}.$$

We can equivalently write this as a sum over the unique eigenvalues, by defining $\{\widetilde{\lambda}_i\}_{i=1}^{n_0}$ to be the set of unique eigenvalues, and taking $E_A(\{\widetilde{\lambda}_i\}) = \sum_{j=1}^{n} \delta_{\widetilde{\lambda}_i, \lambda_j} P_{\xi_j}$. Then

$$f(A) = \sum_{i=1}^{n_0} f(\tilde{\lambda}_i) E_A(\{\widetilde{\lambda}_i\}).$$

It turns out we can expand this notion for any self-adjoint operator $A \in B(H)$, for any Hilbert space H. To do so, we first need to define the spectrum of operator A, denoted $\sigma(A)$. It is defined as the set of $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ does not have a two-sided inverse. Note that, for operators defined by a matrix multiplication on $\mathbb{C}^n$, $\sigma(A)$ is simply the set of eigenvalues of that matrix. We denote the spectral valued measure for $\lambda \in \sigma(A)$ by E_λ (see [8]). We can then write:

$$A = \int_{\sigma(A)} f(\lambda) E_\lambda$$

Outline of the main results

Now armed with this knowledge, we can introduce the main goal of this thesis, which is to prove the following theorem. It used to be known as the Nazarov-Peller conjecture before its resolution.

Theorem 1.1. ([6], Conjecture 1.1) *Let $f : \mathbb{R} \to \mathbb{C}$ be a Lipschitz continuous function. Whenever $A, B \in B(H)$ are self-adjoint operators such that $A - B \in \mathcal{S}^1(H)$ we have that $f(A) - f(B) \in \mathcal{S}^{1,\infty}(H)$ and, for some absolute constant c_{abs},*

$$\|f(A) - f(B)\|_{1,\infty} \leq c_{abs} \|f'\|_\infty \|A - B\|_1.$$

In this theorem and throughout this chapter, $\mathcal{S}^{1,\infty}$ refers to the non-commutative equivalent of the weak-L_1 space: the space consisting of all compact operators for which the sequence of singular values $(\tau_k)_{k\geq 0}$ satisfies $\tau_k = \mathcal{O}(\frac{1}{k+1})$. The details of this will be introduced in chapter 2. $\mathcal{S}^1$ is the Schatten class with $p = 1$, the non-commutative equivalent of L^1, which we will examine soon.

When resolving conjectures, a distinction can be made between positive and negative results. Positive results confirm (part of) a conjecture, while a negative result disproves it. Thus, in the case of the Nazarov-Peller conjecture, an upper bound on $\|f(A) - f(B)\|_{1,\infty}$ is a positive result, while a diverging lower bound is a negative result. In order to prove Theorem 1.1, we actually prove a stronger result for double operator integrals $T^{A,A}_{f^{[1]}}$. After proving this, Theorem 1.1 is a corollary. $T^{A,A}_{f^{[1]}}$ is quite a complicated operator, which will be formally introduced in chapter 3. For now it suffices to know that we can use this linear operator to prove results such as Theorem 1.1. A definition for matrices is given in this introduction, see equation (1.1).

Theorem 1.2. ([6], Theorem 1.2) *If $A \in B(H)$ is self-adjoint, and if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then for $V \in (\mathcal{S}^1 \cap \mathcal{S}^2)(H)$ we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|V\|_1.$$

In order to prove these results, we need a lot of preliminary theory. In chapter 2, we introduce singular values as follows:

Definition 1.3. For any bounded operator on a Hilbert space $A \in B(H)$, we definte the n-th singular value by

$$a_n(u) = \inf\{\|u - v\| : \dim \mathrm{R}(v) < n\}.$$

Using this definition, we can also introduce the Schatten classes.

Definition 1.4. For a Hilbert space H and $p \in [1, \infty)$, the Schatten class $\mathcal{S}^p$ is defined as

$$\mathcal{S}^p(H) := \{u \in K(H) : \|u\|_p := \|(a_n(u))_{n=1}^\infty\|_{l^p} < \infty\}.$$

Here $K(H)$ is the space of all compact operators.

In chapter 2, we also introduce the trace on $\mathcal{S}^1$, which is an important quantity for quantum mechanics as discussed earlier.

Then, in chapter 3, we will discuss what exactly a double operator integral is. To introduce it rigorously, we study linear and bilinear Schur multipliers: operators on Schatten classes analogous to two- and three-dimensional entry-wise matrix multiplications, respectively.

Definition 1.5. A matrix M, often written as $\{m_{ij}\}_{i,j\geq 1}$, with $m_{ij} \in \mathbb{C}$ for all i, j, is a linear Schur multiplier on $\mathcal{S}$ if the action

$$M(A) = \sum_{i,j\geq 1} m_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1} \in \mathcal{S}^p$$

defines a bounded linear operator on $\mathcal{S}^p$. Here E_{ij} are the standard matrix units.

Note that for matrices of finite dimension, this action always defines a bounded linear operator. We can then specifically introduce the Schur multipliers of the type seen in Theorem 1.2.

Definition 1.6. The linear Schur multiplier associated with a bounded Borel function $\phi : \mathbb{R}^2 \to \mathbb{C}$ is defined by the following formula, where we sum over the spectral projections:

$$T^{A_0,A_1}_\phi(V) = \sum_{i=1}^{n_0}\sum_{j=1}^{n_1} \phi(\lambda^{(0)}_i, \lambda^{(1)}_j) E_{A_0}(\{\lambda^{(0)}_i\}) V E_{A_1}(\{\lambda^{(1)}_j\}), \quad V \in B(\mathbb{C}^n). \tag{1.1}$$

Furthermore, in that same chapter, we introduce several properties necessary for proofs in later chapters. Chapter 4 introduces the final preliminary results needed to start proving the main results of this thesis. Then in chapters 5 and 6 we prove Theorems 1.2 and 1.1 respectively.

In the final chapters, we take a look at two negative results. In chapter 7, we show a result which was originally published in [9]. We prove that, for $n \geq 1$ there are self-adjoint $2n \times 2n$ matrices A and C s.t. $A \neq C$ and

$$\||A| - |C|\|_1 > \frac{1}{2} k \log n \|A - C\|_1.$$

This shows that the bound we have found is optimal for a very specific case. In chapter 8, we prove the following result, showing that for our most general positive result, there is a corresponding negative result. This implies that we have found the best possible bound on the norm of the operator integrals.

Lemma 1.7. ([7], Lemma 28) *There is a C^2-function g, with g'' bounded, and an $N \in \mathbb{N}$, such that for any sequence $\{\alpha_n\}_{n\geq N}$ with $\alpha_n \in \mathbb{R}^+$ there is a sequence of operators $\tilde{B}_n \in B(\mathbb{C}^{8n+4})$ such that, for $n \geq N$, $\|\tilde{B}_n\|_2 \leq 4\alpha_n$ and*

$$\|T_{g^{[2]}}^{\tilde{A}_n+\tilde{B}_n,\tilde{A}_n,\tilde{A}_n}(\tilde{B}_n,\tilde{B}_n)\|_1 \geq const\ \alpha_n^2 \log n.$$

2 Singular values and Schatten classes

In the results discussed in this thesis, the norms with a numerical subscript p denote the Schatten p-norms of the operators. A Schatten class $\mathcal{S}^p$ is the space of all operators on a Hilbert space with finite Schatten p-norm. In order to introduce these norms, we need first need to introduce the singular values. This chapter is based on the book Analysis in Banach Spaces, see [15].

Definition 2.1. Take a bounded operator u on Hilbert space H. Then the nth singular value of u is

$$a_n(u) = \inf\{\|u - v\| : \dim \mathrm{R}(v) < n\}.$$

Note that the infimum is taken over all $v \in B(H)$ with $\dim \mathrm{R}(v) < n$. $\mathrm{R}(v)$ is the range of v.

We know that compact operators have a singular value decomposition, as follows: if u is a compact operator, we can find a sequence $\{\tau_k\}_{k\geq 1}, \tau_k \in \mathbb{R}^+$, decreasing to zero, and two sequences of orthonormal elements of H, e_k and f_k, such that

$$u = \sum_{k=1}^{\infty} \tau_k \langle e_k, \cdot \rangle f_k.$$

By allowing a slightly expanded definition of an orthonormal sequence, we can also cover the finite dimensional case (when normally infinite orthonormal sequences are not possible). We can do this by allowing e_k and f_k to be zero, but only when τ_k is zero.These τ_k in a singular value decomposition are also referred to as singular values, which would not make sense unless $\tau_k = a_k$. Fortunately, this is indeed the case.

Lemma 2.2. *If u has a singular value decomposition, $\tau_k = a_k(u)$ for any $k \in \mathbb{N}$.*

Proof. We show that $a_k(u) \leq \tau_k$ and $\tau_k \leq a_k(u)$. Define

$$u_n := \sum_{k=1}^{n-1} \tau_k \langle e_k, \cdot \rangle f_k,$$

the truncation of the singular value decomposition of u at the nth element of the sum. Then $a_n(u) \leq \|u - u_n\| = \tau_n$. For the other inequality: if an operator v has $\dim \mathrm{R}(v) < n$, we can find a unit vector $\xi = \sum_{k=1}^{n} \xi_k e_k$ in the null space of v. Then we get

$$\|u - v\| \geq \|(u - v)\xi\| = \|u\xi\| = (\sum_{k=1}^{n} \tau_k^2 |\xi_k|^2)^{1/2} \geq \tau_n$$

Taking the infimum as in definition 2.1 on the left hand side, we get $a_n(u) \geq \tau_n$ as required. We conclude $\tau_k = a_k(u)$ for any $k \in \mathbb{N}$. □

Some trivial but important properties of the singular value decomposition are

$$u^* = \sum_{k=1}^{\infty} \tau_k \langle f_k, \cdot \rangle e_k \text{ and } u^*u = \sum_{k=1}^{\infty} \tau_k^2 \langle e_k, \cdot \rangle e_k. \tag{2.1}$$

It follows that $a_n(u^*) = a_n(u)$ and $a_n(u^*u) = a_n(u)^2$ by the previous lemma. We also have $a_n(vuw) \leq \|v\| a_n(u) \|w\|$ and $a_n(u) \leq a_n(w) + \|u - w\|$ for any w. All of these are easy results of the definition, see [15] for details.

Lemma 2.3. *For any compact operator u, we have*

$$\sum_{j=1}^{n} a_j(u) = \max |\sum_{j=1}^{n} \langle h_j, ug_j \rangle| = \max \sum_{j=1}^{n} |\langle h_j, ug_j \rangle|,$$

where the maxima are taken over all orthonormal sequences $\{g_j\}_{j\geq 1}$ and $\{h_j\}_{j\geq 1}$

Proof. See [15]. □

Using the above, we can prove the most important result up to this point, which is the sublinearity of the singular values:

Proposition 2.4. *Let u and v be compact operators. Then*

$$\sum_{k=1}^{n} a_k(u+v) \leq \sum_{k=1}^{n} (a_k(u) + a_k(v)).$$

Proof. This follows quickly from Lemma 2.3:

$$\sum_{k=1}^{n} a_k(u+v) = \max |\sum_{j=1}^{n} \langle h_j, (u+v)g_j\rangle| \leq \max |\sum_{j=1}^{n} \langle h_j, ug_j\rangle| + \max |\sum_{j=1}^{n} \langle h_j, vg_j\rangle| = \sum_{k=1}^{n} (a_k(u) + a_k(v)).$$

□

For some of the theorems in future chapters, it is necessary to work with a more general definition of singular values. To this end, we first introduce what a von Neumann algebra is. Since this is only necessary for a small part of this thesis, we will skip many details, by assuming prior knowledge of spectral theory and $*$-algebras. We recommend [16] for interested readers.

Definition 2.5. A von Neumann algebra $\mathcal{M}$ is an algebra of bounded operators on a Hilbert space with an involution *, that is closed in the weak operator topology and contains the identity operator.

An example of a von Neumann algebra is simply the space of all bounded operators on a Hilbert space. The involution is the Hilbert space adjoint. Now let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful (injective), normal, semifinite trace τ. A closed and densely defined operator x on $\mathcal{M}$ is, by definition, τ-measurable if $\tau(E_{|x|}(s,\infty)) < \infty$ for large enough s, where $E_{|x|}$ is an indicator function. The set of all τ-measurable operators is denoted by $S(\mathcal{M},\tau)$. For each of these operators we can define the singular value function, $\mu(x)$, by

$$\mu(t,x) = \inf\{\|x(1-p)\|_\infty : \tau(p) \leq t\}.$$

Here p is any projection, and we take the infimum over all projections p with $\tau(p) \leq t$. When we write $\mu(x)$ instead of $\mu(t,x)$, the statement holds for all t. Since p is a projection, we can equivalently define μ as

$$\mu(t,x) = \inf\{\|x\chi_{A^c}\|_\infty : m(A) \leq t\},$$

where $m(A)$ is the Lebesgue measure of A, and χ_{A^c} is the indicator function of the complement of A. With $\mathcal{M} = B(H)$ and $\tau(p) := \dim \mathrm{R}(p)$, this definition is equivalent to the earlier definition of singular values on $B(H)$.

Now we are ready to introduce the Schatten classes.

Definition 2.6. For a Hilbert space H and $p \in [1,\infty)$, the Schatten class $\mathcal{S}^p(H)$ is defined as

$$\mathcal{S}^p(H) := \{u \in K(H) : \|u\|_p := \|\{a_n(u)\}_{n\geq 1}\|_{l^p} < \infty\}.$$

Here $K(H)$ is the space of all compact operators. When the choice of Hilbert space is clear, the notation is abbreviated to $\mathcal{S}^p$. $\|\cdot\|_p$ as defined here is the Schatten p-norm. For $p = \infty$, we define $S^\infty = K(H)$.

Schatten classes are considered the non-commutative equivalent of L_p-spaces. There is also an equivalent for the weak L_p spaces. We introduce these as follows.

Definition 2.7. For a Hilbert space H and $p \in [1,\infty)$, the weak Schatten class $\mathcal{S}^{p,\infty}(H)$ is defined as

$$\mathcal{S}^{p,\infty}(H) := \{u \in K(H) : \|u\|_{p,\infty} := \sup_{n\geq 1} n(a_n(u))^{1/p} < \infty\}.$$

Next are some important properties of Schatten classes. First we show that, for any p, $\mathcal{S}^p$ is a Banach space.

Theorem 2.8. *For any Hilbert space H and $p \in [1, \infty)$, the Schatten class $\mathcal{S}^p(H)$ is a complete space.*

Proof. Note that we have

$$(\sum_{k=1}^{n} a_k(u+v)^p)^{1/p} \leq (\sum_{k=1}^{n}(a_k(u)+a_k(v))^p)^{1/p}$$

$$\leq (\sum_{k=1}^{n}(a_k(u)^p)^{1/p} + (\sum_{k=1}^{n}(a_k(v)^p)^{1/p},$$

using proposition 2.4. We have $\|u\|_p \geq a_1(u) = \|u\|$. This means that the Cauchy condition $\|u_m - u_n\|_p \to 0$ implies $\|u_m - u_n\| \to 0$, so $u_n \to u$ in operator norm for some compact operator u. Now, let us write out the Cauchy condition: for every $\epsilon > 0$, there exists an $N \in N$ such that for all $m, n > N$ and all $k \in N$ we have

$$\sum_{j=1}^{k} a_j(u_n - u_m)^p \leq \|u_n - u_m\|_p^p \leq \epsilon.$$

Since we know the limit of u_m in operator norm is u, we can, by fixing k and n and taking the limit $m \to \infty$, pass to the following expression:

$$\sum_{j=1}^{k} a_j(u_n - u)^p \leq \epsilon.$$

With $k \to \infty$ we find that $\|u_n - u\|_p^p \leq \epsilon$ so $u \in \mathcal{S}^p$. This proves the completeness of the Schatten classes. □

Now let us show some properties of Schatten norms. Since $a_n(vuw) \leq \|v\| a_n(u) \|w\|$ we also have

$$\|vuw\|_p \leq \|v\| \|u\|_p \|w\|$$

and since we have $a_n(u) = a_n(u^*) = a_n(u^*u)^{1/2}$ we get

$$\|u\|_p = \|u^*\|_p = \|u^*u\|_{p/2}^{1/2}$$

For the case of $p = 2$, we have an interesting consequence: on $\mathcal{S}^2$ we can introduce an inner product that induces the Schatten norm, and results in a Hilbert space. We define this inner product as

$$\langle v, u \rangle_{\mathcal{S}^2} := \sum_{i \in I} \langle vh_i, uh_i \rangle$$

where $\{h_i\}_{i \in I}$ is any orthonormal basis of H. Each basis yields the same value for the inner product. This can be proven as follows. If u has a singular value decomposition, using the identity above we get $\|u\|_2^2 = \|u^*u\|_1 = \sum_{k=1}^{\infty} \tau_k^2$. For any orthonormal basis we get

$$\sum_{i \in I} \langle uh_i, uh_i \rangle = \sum_{i \in I} \langle h_i, u^*uh_i \rangle = \sum_{i \in I} \sum_{k=1}^{\infty} \tau_k^2 |\langle e_k, h_i \rangle|^2 = \sum_{k=1}^{\infty} \tau_k^2$$

where the second equality follows from the polar decomposition of u^*u. This shows that the inner product induces the norm on the Schatten class and is a well-defined inner product, independent of the chosen orthonormal basis.

We will now work towards the Hölder inequality for Schatten classes, by introducing three lemmas.

Lemma 2.9. *For any finite dimensional Hilbert space H, we have, for any operator u on H,*

$$|\det(u)| = \prod_{k=1}^{n} a_k(u).$$

Proof. Note that

$$|\det(u)|^2 = \overline{\det(u)}\det(u) = \det(u^*)\det(u) = \det(u^*u).$$

Recall that

$$u^*u = \sum_{k=1}^{\infty} \tau_k^2 \langle e_k, \cdot \rangle e_k,$$

as in equation (2.1). When represented in the basis $\{e_k\}_{k=1}^{\infty}$, this is a diagonal matrix. Hence

$$|\det(u)|^2 = \prod_{k=1}^{n} \tau_k^2.$$

After taking the square root, and noting that $\tau_k = a_k(u)$, the proof is completed. □

Lemma 2.10. *If u is compact, we have for arbitrary vectors $\phi_1, ...\phi_n \in H$:*

$$det(\langle u\phi_i, u\phi_j \rangle_{i,j=1}^n) \le \prod_{k=1}^{n} a_k(u)^2 \, det(\langle \phi_i, \phi_j \rangle_{i,j=1}^n).$$

Equality holds when the vectors ϕ_k form a singular value decomposition of u.

Proof. The case of equality is clear from $\langle ue_i, ue_j \rangle_{i,j=1}^n = \{\delta_{ij} a_j(u)^2\}_{i,j=1}^n$ and $\det(\{\langle e_i, e_j \rangle_{i,j=1}^n\}_{i,j=1}^n) = 1$. Take arbitrary vectors $\phi_1, ...\phi_n \in H$, and let $h_1, ...h_n \in H$ be orthonormal vectors that span a space that contains $\phi_1, ...\phi_n$, so we get $\text{span}\{\phi_1, ...\phi_n\} \subseteq \text{span}\{h_1, ...h_n\}$. Then

$$\langle u\phi_i, u\phi_j \rangle = \sum_{l,m=1}^{n} \langle \phi_i, h_l \rangle \langle uh_l, uh_m \rangle \langle h_m, \phi_j \rangle.$$

We can interpret this as the product of three matrices:

$$\langle \phi_i, h_l \rangle_{i,l=1}^n, \quad \langle uh_l, uh_m \rangle_{l,m=1}^n, \quad \langle h_m, \phi_j \rangle_{m,j=1}^n.$$

For its determinant we get

$$\det(\langle u\phi_i, u\phi_j \rangle_{i,j=1}^n) = \det(\langle \phi_i, h_l \rangle_{i,l=1}^n)\det(\langle uh_l, uh_m \rangle_{l,m=1}^n)\det(\langle h_m, \phi_j \rangle_{m,j=1}^n). \tag{2.2}$$

Now note that $\det(\langle \phi_i, h_l \rangle_{i,l=1}^n)\det(\langle h_m, \phi_j \rangle_{m,j=1}^n)$ (the first term times the third) can be expressed as the determinant of the product of these two matrices, which is $\langle \phi_i, \phi_j \rangle_{i,j=1}^n$. To examine the final part of the product, $\det(\langle uh_l, uh_m \rangle_{l,m=1}^n)$, let π be the orthogonal projection onto $\text{span}\{h_1, ...h_n\}$. Then

$$\det(\langle uh_l, uh_m \rangle_{l,m=1}^n) = \det(\pi u^* u \pi^*) = \prod_{k=1}^{n} a_k(\pi u^* u \pi^*)$$

$$\le \prod_{k=1}^{n} a_k(u^*u) = \prod_{k=1}^{n} a_k(u)^2.$$

Substituting these results back into 2.2 yields the required result. □

Lemma 2.11. *For two compact operators u and v we have*

$$\prod_{k=1}^{n} a_k(uv) \leq \prod_{k=1}^{n} a_k(u)a_k(v).$$

Proof. From Lemma 2.10, we have $(\prod_{k=1}^{n} a_k(uv))^2 = \max \det(\langle uv\phi_i, uv\phi_j\rangle_{i,j=1}^{n})$, where he maximum is taken over the possible sequences $\phi_1, ...\phi_n \in H$ with $\det\langle\phi_i, \phi_j\rangle_{i,j=1}^{n} \leq 1$. Then we can use Lemma 2.10 twice to show this to be less than or equal to $\prod_{k=1}^{n} a_k(u)^2 a_k(v)^2 \max \det\langle\phi_i, \phi_j\rangle_{i,j=1}^{n} = \prod_{k=1}^{n} a_k(u)^2 a_k(v)^2$ as required. □

Now we can get to Hölder's inequality for Schatten classes:

Corollary 2.12. *Let $p, q, r \in [1, \infty)$ with $1/p = 1/q + 1/r$. For $u \in \mathcal{S}^q$, $v \in \mathcal{S}^r$ we get $uv \in \mathcal{S}^p$, and $\|uv\|_p \leq \|u\|_q \|v\|_r$. Conversely, every $w \in \mathcal{S}^p$ can be factored $w = uv$ with equality: $\|w\|_p = \|u\|_q \|v\|_r$.*

Proof. By taking the logarithm of the previous proposition, we get an inequality for sums, which also holds after applying a convex function e^{pt}. By doing so we get the following:

$$(\sum_{j=1}^{n} a_j(uv)^p)^{1/p} \leq (\sum_{j=1}^{n} a_j(u)^p a_j(v)^p)^{1/p}.$$

Then we can use Hölder's inequality to get

$$(\sum_{j=1}^{n} a_j(u)^p a_j(v)^p)^{1/p} \leq (\sum_{j=1}^{n} a_j(u)^q)^{1/q} (\sum_{j=1}^{n} a_j(v)^r)^{1/r}.$$

For the factorisation of w, take the singular value decomposition $w = \sum_{k=1}^{\infty} \tau_k \langle e_k, \cdot\rangle f_k$. Take any orthonormal sequence $\{g_k\}_{k\geq 1}$ and set $u = \sum_{k=1}^{\infty} \tau_k^{p/q} \langle g_k, \cdot\rangle f_k$ and $v = \sum_{k=1}^{\infty} \tau_k^{p/r} \langle e_k, \cdot\rangle g_k$ to get the required result. □

With this proof completed, we can now define the trace on $\mathcal{S}^1$, a very important quantity for the interpretation of the results of this thesis.

Proposition 2.13. *The trace on $\mathcal{S}^1(H)$, defined as the functional*

$$tr : \mathcal{S}^1(H) \to \mathbb{C}, u \mapsto tr(u) := \sum_{i\in I} \langle h_i, uh_i\rangle$$

is well-defined and contractive. Here $\{h_i\}_{i\in I}$ can be any orthonormal basis; each yields the same value.

Proof. Given u, we can use the previous corollary to get a factorisation $u = v^*w$, with $v, w \in \mathcal{S}^2$ and $\|u\|_1 = \|v\|_2 \|w\|_2$. Then $\text{tr}(u)$ is just the inner product of v and w as defined previously for $\mathcal{S}^2$. There we already showed that it was independent of the chosen orthonormal basis. In addition, since $\langle vh_i, wh_i\rangle = \langle h_i, uh_i\rangle$ it is also independent of the chosen factorisation. From the Cauchy-Schwarz inequality, we get that it is a contraction:

$$|\text{tr}(u)| = |\langle v, w\rangle_{\mathcal{S}^2}| \leq \|v\|_2 \|w\|_2 = \|u\|_1.$$

□

3 Double operator integrals and Schur Multipliers

In this chapter, the mathematical framework, which is necessary to interpret and prove the results of this thesis, will be laid out. First, linear and bilinear Schur multipliers will be introduced in sections 3.1 and 3.2. This chapter is based largely on [7], section 3.

3.1 Linear Schur multipliers

In this section, we define what a linear Schur multiplier is. Denote the standard matrix units with E_{ij}. In finite dimensions, a linear Schur multiplier is an entry-wise matrix multiplication of a matrix. Formally, we use the following definition, which is necessary for infinite dimensional matrices.

Definition 3.1. Let $1 \leq p \leq \infty$. A matrix M, often written as $\{m_{ij}\}_{i,j\geq 1}$, with $m_{ij} \in \mathbb{C}$ for all i, j, is a linear Schur multiplier if the action

$$M(A) = \sum_{i,j\geq 1} m_{ij} a_{ij} E_{ij}, \quad A = \{a_{ij}\}_{i,j\geq 1} \in \mathcal{S}^p$$

defines a bounded linear operator on $\mathcal{S}^p$.

Note that if matrix M is finite-dimensional, the action in the definition is bounded on any $\mathcal{S}^p, 1 \leq p \leq \infty$, since any matrix with finite coëfficients is in $\mathcal{S}^p$. Hence, for the finite-dimensional case, a Schur multiplier is just an interpretation of any matrix as an operator on $\mathcal{S}^p$, defined by the action in 3.1.

By duality, we can show that any matrix that is a linear Schur multiplier on $\mathcal{S}^p$ is also a linear Schur multiplier on $\mathcal{S}^{p'}$, where p' is the Hölder conjugate of p:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Importantly, the norm of both operators is the same:

$$\|M : \mathcal{S}^p \to \mathcal{S}^p\| = \|M : \mathcal{S}^{p'} \to \mathcal{S}^{p'}\|.$$

Schur multipliers on $\mathcal{S}^1$ (or equivalently $\mathcal{S}^\infty$) can be described as follows, as proven in [18]:

Theorem 3.2. ([7], Theorem 1) *A matrix M is a linear Schur multiplier on $\mathcal{S}^1$ (or equivalently $\mathcal{S}^\infty$) if and only if there is a Hilbert space E, and there are two bounded sequences $\{\xi_i\}_{i\geq 1}$ and $\{\eta_j\}_{j\geq 1}$ in this Hilbert space, such that*

$$m_{ij} = \langle \xi_i, \eta_j \rangle, \quad i, j \geq 1.$$

In this case,

$$\|M : \mathcal{S}^\infty \to \mathcal{S}^\infty\| = \inf\{\sup_i \|\xi_i\| \sup_j \|\eta_j\|\},$$

where the infimum is taken over all possible sequences $\{\xi_i\}_{i\geq 1}$ and $\{\eta_j\}_{j\geq 1}$.

For $p = 2$, the description is far easier: a matrix is a linear Schur multiplier on $\mathcal{S}^2$ if and only if $\sup_{i,j\geq 1} |m_{ij}| \leq \infty$, and indeed $\|M : \mathcal{S}^2 \to \mathcal{S}^2\|_2 = \sup_{i,j\geq 1} |m_{ij}|$. No descriptions of Schur multipliers on $\mathcal{S}^p$ are known for other values of p.

3.2 Bilinear Schur multipliers

This entire section is based on [7], section 2. As mentioned in the previous section, a linear Schur multiplier is intuitively an entry-wise matrix multiplication with a two-dimensional matrix. A bilinear Schur multiplier is the three-dimensional equivalent.

Definition 3.3. ([7], Definition 2) For $1 \leq r \leq \infty$, a three dimensional matrix $M = \{m_{ijk}\}_{i,j,k\geq 1}$ with complex entries is Schur multiplier into $\mathcal{S}^r$ if the action

$$M(A,B) = \sum_{i,j,k\geq 1} m_{ijk} a_{ij} b_{jk} E_{ik}, \quad A = \{a_{ij}\}_{i,j\geq 1}, B = \{b_{ij}\}_{i,j\geq 1} \in \mathcal{S}^2$$

defines a bounded, bilinear operator $M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^r$.

Note that in this definition, three dimensional refers to the amount of indices. Each index can be infinite. There is a more general notion of a bilinear Schur multiplier, where we allow the pre-image of the operator to be $\mathcal{S}^p \times \mathcal{S}^q$ for $1 \leq p, q \leq \infty$. For the results proven in this thesis, the definition above suffices. As in the previous section, we can describe the Schatten classes explicitly using relatively simple conditions for certain values of r. We will first consider $r = 2$.

Lemma 3.4. ([7], Lemma 3) *A matrix $M = \{m_{ijk}\}_{i,j,k\geq 1}$ is a bilinear Schur multiplier into $\mathcal{S}^2$ if and only if*

$$\sup_{i,j,k\geq 1} |m_{ijk}| \leq \infty.$$

In this case,

$$\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| = \sup_{i,j,k\geq 1} |m_{ijk}|$$

Proof. First we prove $\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \leq \sup_{i,j,k\geq 1} |m_{ijk}|$. To this end, take $A = \{a_{ij}\}_{i,j\geq 1} \in \mathcal{S}^2$ and $B = \{b_{jk}\}_{j,k\geq 1} \in \mathcal{S}^2$. We use Definition 3.3 to calculate

$$\|M(A,B)\|_2^2 = \|\sum_{i,j,k\geq 1} m_{ijk} a_{ij} b_{jk} E_{ik}\|_2^2 = \sum_{i,k\geq 1} |\sum_{j\geq 1} m_{ijk} a_{ij} b_{jk}|^2,$$

by definition of the $\|\cdot\|_2$-norm. Using elementary techniques and the Cauchy-Schwarz inequality, we get

$$\sum_{i,k\geq 1} |\sum_{j\geq 1} m_{ijk} a_{ij} b_{jk}|^2 \leq \sup_{i,j,k} |m_{ijk}|^2 \sum_{i,k\geq 1} (\sum_{j\geq 1} |a_{ij} b_{jk}|)^2$$

$$\leq \sup_{i,j,k} |m_{ijk}|^2 \sum_{i,k\geq 1} \sum_{j\geq 1} |a_{ij}|^2 \sum_{j\geq 1} |b_{jk}|^2 \leq \sup_{i,j,k} |m_{ijk}|^2 \|A\|_2^2 \|B\|_2^2.$$

Now we still need to prove the converse inequality. We can do the following:

$$\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \geq \|M(E_{ij}, E_{jk})\|_2 = |m_{ijk}|.$$

By taking the supremum left and right, we prove that $\|M : \mathcal{S}^2 \times \mathcal{S}^2 \to \mathcal{S}^2\| \geq \sup_{i,j,k\geq 1} |m_{ijk}|$. This also proves the first statement of the lemma: if the matrix M has bounded entries, the action $M(A,B)$ is bounded, and vice-versa. □

From this point onward, we will only consider finite rank operators, corresponding to finite matrices. The results below also hold for infinite dimensions, but the proofs involve various subtleties, particularly with regards to completion of tensor products. For details, see [7]. Since we do not need these results, we present the finite dimensional versions.

Now we turn to $r = 1$: bilinear Schur multipliers into $\mathcal{S}^1$. In order to prove the important results about these, we need some auxilliary lemmas. For any two Banach spaces X, Y let $X \otimes Y$ denote their algebraic tensor product. We can define the projective tensor norm by the following formula:

$$\pi(u) := \inf\{\sum_{i=1}^{m} \|x_i\|\|y_i\| : u = \sum_{i=1}^{m} x_i \otimes y_i, \quad m \in \mathbb{N}\}.$$

Note that the infimum is taken over all possible m-tuples x_i and y_i, of which the tensor product sums to u. We can take the completion of the algebraic tensor product $X \otimes Y$ under this norm. We call this completion the projective tensor product, denoted $X \widehat{\otimes} Y$. In the finite-dimensional case, while we do not need a completion, we still use this notation to signify which tensor norm we are using. Let Z be another Banach space. Denote the space of all bilinear operators $X \times Y \to Z$, equipped with the supremum norm, by $B^2(X \times Y, Z)$. Note that we also use the supremum norm to define the norm of $X \times Y$: for $(x, y) \in X \times Y$, we have $\|(x, y)\| = \sup\{\|x\|, \|y\|\}$. Then the norm of B^2 is defined by the usual formula for the operator norm:

$$\|B^2\| = \sup \frac{\|B(x, y)\|}{\sup(\{\|x\|, \|y\|\})}.$$

In addition, let $B(X \widehat{\otimes} Y, Z)$ denote the Banach space of all bounded linear operators $X \widehat{\otimes} Y \to Z$, again with the supremum norm. Then we have an isometric isomorphism:

$$B^2(X \times Y, Z) = B(X \widehat{\otimes} Y, Z). \tag{3.1}$$

The isometric isomorphism is given by $T \mapsto \widetilde{T}$, where $\widetilde{T}(x \otimes y) = T(x, y)$ (see [21], theorem 2.9). Let H denote a Hilbert space, and let $\overline{H}$ denote its conjugate space (the space with the same elements and addition, but where scalar multiplication is involves complex conjugation of the scalar). For $h_1, h_2 \in H$, we can identify $\overline{h_1} \otimes h_2$ with an operator:

$$h : H \to H \text{ with } h \mapsto \langle h, h_1 \rangle h_2.$$

By applying this operator, we get an identification between the space $\overline{H} \otimes H$ and the space of finite-rank operators on H, since all finite-rank operators on a Hilbert space can we written as a linear combination of operators of the form $h \mapsto \langle h, h_1 \rangle h_2$. Since we took our Hilbert space H to be finite dimensional, all operators are finite rank, and we find an isomorphism between $\overline{H} \widehat{\otimes} H$ and $\mathcal{S}^1(H)$. We can even find an isometric isomorphism, as shown in, for example, [21], theorem 2.9. Hence,

$$\overline{H} \widehat{\otimes} H = \mathcal{S}^1(H). \tag{3.2}$$

We will now consider an unfamiliar type of matrices. We write M_{n^2} to denote the space of matrices of which the columns and rows are indexed by $\{1, ..., n\}^2$. The standard matrix units of this space are thus denoted by $E_{(i,j),(k,l)}$ with $(i, j) \in \{1, ..., n\}^2$ and $(k, l) \in \{1, ..., n\}^2$, meaning $1 \leq i, j, k, l \leq n$. We now need a third type of tensor product: the minimal tensor product, notation $\otimes_{\min}$. In some literature, it is referred to as the spatial tensor product.

Now take $M_n \otimes_{\min} M_n$, the minimal tensor product of two copies of M_n. By definition of the minimal tensor product, the isomorphism $J_0 : M_n \otimes_{\min} M_n \to M_{n^2}$ with

$$J_0(E_{ik} \otimes E_{jl}) = E_{(i,j),(k,l)} \tag{3.3}$$

is an isometry. Thus, using our previous notation,

$$M_n \otimes_{\min} M_n = M_{n^2}$$

Now we move on to some duality results. With our definition of a trace on $\mathcal{S}^1$, as introduced in Proposition 2.13, we can show that $(S_n^1)^*$ is isometrically isomorphic to M_n, using the following duality pairing:

$$\mathcal{S}_n^1 \times M_n \to \mathbb{C}, \quad (A, B) \mapsto \mathrm{Tr}(A^T B).$$

Due to the transposition, the dual basis of $(E_{ij})_{1 \leq i,j \leq n}$ is simply $(E_{ij})_{1 \leq i,j \leq n}$. Let γ denote the cross norm on $\mathcal{S}_n^1 \otimes \mathcal{S}_n^1$:

$$\|A \otimes B\|_\gamma = \|A\|_1 \|B\|_1, \quad A, B \in \mathcal{S}_n^1.$$

Denote the closure of $\mathcal{S}_n^1 \otimes \mathcal{S}_n^1$ under this norm by $\mathcal{S}_n^1 \otimes_\gamma \mathcal{S}_n^1$. We now have

$$(\mathcal{S}_n^1 \otimes_\gamma \mathcal{S}_n^1)^* = M_n \otimes_{\min} M_n \tag{3.4}$$

We can use this to explicitly write an expression for $\|\cdot\|_\gamma$, the norm on $\mathcal{S}_n^1 \otimes \mathcal{S}_n^1$. For any $(t_{ijkl})_{1\le i,j,k,l\le n}$ with $t_{ijkl} \in \mathbb{C}$, we have the following:

$$\|\sum_{i,j,k,l=1}^{n} t_{ijkl} E_{ij} \otimes E_{kl}\|_\gamma =$$

$$\sup\{|\sum_{i,j,k,l=1}^{n} t_{ijkl}s_{ijkl}| : \|\sum_{i,j,k,l=1}^{n} s_{ijkl}E_{ij} \otimes E_{kl}\|_{M_n\otimes_{\min}M_n} \le 1\}$$

Now we can prove the first auxiliary lemma.

Lemma 3.5. ([7], Lemma 4) *The isomorphism* $J : \mathcal{S}_n^2 \widehat{\otimes} \mathcal{S}_n^2 \to \mathcal{S}_n^1 \otimes_\gamma \mathcal{S}_n^1$, *defined by the formula*

$$J(E_{ij} \otimes E_{kl}) = E_{ik} \otimes E_{jl}$$

is an isometry.

Proof. We need the equality

$$\|\sum_{i,j} c_{ij}E_{ij}\|_2 = (\sum_{i,j} |c_{ij}|^2)^{\frac{1}{2}},$$

which shows that we can naturally identify $\mathcal{S}_n^2$ with $l_{n^2}^2$ (the Hilbert space of square summable tuples indexed by $(i, j) \in \{1, ..., n\}^2$), or its conjugate space. Since this is a Hilbert space, we can use equation (3.2), showing that $\overline{l_{n^2}^2}\widehat{\otimes} l_{n^2}^2$ and $\mathcal{S}_{n^2}^1$ are isometrically isomporphic. Consequently, the mapping $J : \mathcal{S}_n^2\widehat{\otimes}\mathcal{S}_n^2 \to \mathcal{S}_{n^2}^1$ given by

$$J_1(E_{ij} \otimes E_{kl}) = E_{(i,j),(k,l)}, \quad 1 \le i, j, k, l \le n$$

is an isometry. We can define another isomorphism: $J_2 : \mathcal{S}_n^1 \otimes_\gamma \mathcal{S}_n^1 \to \mathcal{S}_{n^2}^1$ by the formula

$$J_2(E_{ik} \otimes E_{jl}) = E_{(i,j),(k,l)}, \quad 1 \le i, j, k, l \le n.$$

Combining (3.3) and (3.4), we see that $J_2^{-1} = J_0$, which is an isometry, hence J_2 is an isometry. $J = J_2^{-1}J_1$, so J is a composition of two isometric isomorphisms, which proves the lemma. □

We will now consider a specific subspace of $M_n \otimes_{\min} M_n$, spanned by $E_{rk} \otimes E_{ks}$ with $1 \le r, k, s \le n$. We first provide a lemma describing the subspace. Let $(e_1, ..., e_n)$ denote the standard basis of l_n^∞.

Lemma 3.6. ([7], Lemma 5) *The linear mapping* $\theta : l_n^\infty(M_n) \to M_n \otimes_{\min} M_n$ *with*

$$\theta(e_k \otimes E_{rs}) = E_{rk} \otimes E_{ks}$$

is isometric.

Proof. Let $y = \sum_{k=1}^n e_k \otimes y_k \in l_n^\infty(M_n)$, where y_k is defined by $y_k = \sum_{r,s=1}^n y_k(r, s)E_{rs}$. By definition of θ, we have

$$\theta(y) = \sum_{r,s,k=1}^{n} y_k(r, s)E_{rk} \otimes E_{ks}.$$

We use the isometric isomophism J_0 as defined in (3.3). Using this, we can calculate

$$J_0\theta(y) = \sum_{r,s,k=1}^{n} y_k(r, s)E_{(r,k),(k,s)}.$$

Now take a, b as follows:

$$a = \{a_{rk}\}_{r,k=1}^n, b = \{b_{ls}\}_{l,s=1}^n \in l_{n^2}^2.$$

We then have

$$\langle J_0\theta(y)b, a\rangle = \sum_{r,s,k=1}^n y_k(r,s)\langle E_{(r,k),(k,s)}b, a\rangle = \sum_{r,s,k=1}^n y_k(r,s)a_{rk}b_{ks}.$$

We can now use the Cauchy-Schwarz inequality:

$$|\langle J_0\theta(y)b, a\rangle| = |\sum_{r,s,k=1}^n y_k(r,s)a_{rk}b_{ks}|$$

$$\leq \sum_{k=1}^n |\sum_{r,s}^n y_k(r,s)a_{rk}b_{ks}|$$

$$\leq \sum_{k=1}^n \|y_k\|(\sum_{r=1}^n |a_{rk}|^2)^{\frac{1}{2}}(\sum_{s=1}^n |b_{ks}|^2)^{\frac{1}{2}}$$

$$\leq \max_{1\leq k\leq n} \|y_k\| \sum_{k=1}^n (\sum_{r=1}^n |a_{rk}|^2)^{\frac{1}{2}}(\sum_{s=1}^n |b_{ks}|^2)^{\frac{1}{2}}$$

$$\leq \max_{1\leq k\leq n} \|y_k\|(\sum_{k,r=1}^n |a_{rk}|^2)^{\frac{1}{2}}(\sum_{s,k=1}^n |b_{ks}|^2)^{\frac{1}{2}}$$

$$\leq \max_{1\leq k\leq n} \|y_k\|\|a\|_2\|b\|_2.$$

Hence, $\|\theta(y)\| \leq \max_{1\leq k\leq n}\|y_k\|$. Now to prove the converse inequality. Fix some k_0 such that $1 \leq k_0 \leq n$. Now take α, β arbitrarily in l_n^2 as follows:

$$\alpha = \{\alpha_r\}_{r=1}^n, \beta = \{\beta_s\}_{s=1}^n \in l_n^2.$$

Define $\{a_{rk}\}_{r,k=1}^n, \{b_{ls}\}_{l,s=1}^n \in l_{n^2}^2$:

$$a_{rk} := \begin{cases} \alpha_r & \text{if } k = k_0 \\ 0 & \text{otherwise} \end{cases} \quad b_{ls} := \begin{cases} \beta_s & \text{if } l = k_0 \\ 0 & \text{otherwise} \end{cases}.$$

Using this definition, we can easily calculate

$$\langle J_0\theta(y)b, a\rangle = \langle y_{k_0}(\beta), \alpha\rangle.$$

Since both pairs (a, α and b, β) have the same non-zero elements, $\|a\|_2 = \|\alpha\|_2, \|b\|_2 = \|\beta\|_2$. Thus we find $\|y_{k_0}\| \leq \|\theta(y)\|$ and since k_0 was arbitrarily fixed we get the converse inequality we were looking for, $\|\theta(y)\| \geq \max_{1\leq k\leq n}\|y_k\|$, thus proving the lemma. □

Now we turn to the main result, which is the key theorem in one of the results discussed later in this thesis.

Theorem 3.7. ([7], Theorem 6) *Let $n \in \mathbb{N}$. Let $M = \{m_{ijk}\}_{i,j,k=1}^n$ be a three-dimensional matrix. For any $j \in \{1, ..., n\}$, define the two-dimensional matrix $M(j) = \{m_{ijk}\}_{i,k=1}^n$. Then*

$$\|M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\| = \sup_{1\leq j\leq n} \|M(j) : M_n \to M_n\|.$$

Proof. According to the isometric isomorphism in equation 3.1, the bilinear map $M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1$ induces a linear map $\tilde{M} : \mathcal{S}_n^2 \widehat{\otimes} \mathcal{S}_n^2 \to \mathcal{S}_n^1$ isometrically, meaning $\|M\| = \|\tilde{M}\|$. Now consider an operator

$$T_m = (\tilde{M}J^{-1})^* : M_n \to M_n \otimes_{\min} M_n.$$

Here J is defined as in Lemma 3.5. Since J is an isometry, we have

$$\|T_m\| = \|\tilde{M}\| = \|M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\|. \tag{3.5}$$

Now we take a closer look at the action of T_m. For $1 \le r, s \le n$, we have, for all $1 \le i, j, k, l \le n$:

$$\begin{aligned}
\langle T_M(E_{rs}), E_{ij} \otimes E_{kl} \rangle &= \langle E_{rs}, \tilde{M}J^{-1}(E_{ij} \otimes E_{kl}) \rangle \\
&= \langle E_{rs}, \tilde{M}(E_{ik} \otimes E_{jl}) \rangle \\
&= \begin{cases} m_{ikl}\langle E_{rs}, E_{il} \rangle & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} m_{ikl} & \text{if } k = j, r = i, s = l \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Thus

$$T_M(E_{rs}) = \sum_{k=1}^{n} m_{rks} E_{rk} \otimes E_{ks}.$$

Since the second index of the first matrix unit coincides with the first index of the second matrix unit for each element of the sum, T_M maps into the range of the operator θ, as seen in Lemma 3.6. Using this operator, we can write

$$T_M(E_{rs}) = \sum_{k=1}^{n} m_{rks} \theta(e_k \otimes E_{rs}).$$

Now we can use the linearity of θ:

$$T_M(E_{rs}) = \theta(\sum_{k=1}^{n} m_{rks} e_k \otimes E_{rs}).$$

Now we can extend to $C \in M_n$:

$$T_M(C) = \theta(\sum_{k=1}^{n} e_k \otimes [M(k)](C)).$$

Now using Lemma 3.6, we can deduce

$$\|T_M(C)\| = \max_k \|[M(k)](C)\|,$$

for any $C \in M_n$. Hence

$$\|T_M\| = \max_k \|M(k)\|.$$

Combining with equation (3.5) we have the desired result:

$$\|M : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\| = \max_k \|M(k)\| = \sup_{1 \le k \le n} \|M(k) : M_n \to M_n\|.$$

□

3.3 Schur multipliers associated with functions and operators

In this section and the next, we will take a look at the Schur multipliers we need to prove our results. Both are based on [7], section 3. Here specifically we will introduce the operators of the form $T_\phi^{A_0,A_1}$ as seen in Theorem 1.2.

Let $A_0, A_1 \in B(\mathbb{C}^n)$ be diagonisable and self-adjoint. Let $\xi_m = \{\xi_i^{(m)}\}_{i=1}^n$ denote an orthonormal basis of eigenvectors of A_m. Let $\{\lambda_i^{(m)}\}_{i=1}^n$ denote the associated eigenvalues of A_m, i.e. $A_m\xi_i^{(m)} = \lambda_i^{(m)}\xi_i^{(m)}$. We can write $\{\lambda_i^{(m)}\}_{i=1}^{n_m}$ for the tuple of distinct eigenvalues. Then the spectral projections are, by definition for any $i \in \{1, ..., n_m\}$,

$$E_{A_m}(\{\lambda_i^{(m)}\}) = \sum_{k=1}^n \delta_{\lambda_i^{(m)},\lambda_k^{(m)}} P_{\xi_k^{(m)}}.$$

Here $P_x(y) = \langle y, x\rangle x$ as usual. In words: the spectral projection for any eigenvalue is the sum of all projections onto an eigenvector that corresponds to that eigenvalue. Then we can define the linear Schur multiplier associated with a bounded Borel function $\phi : \mathbb{R}^2 \to \mathbb{C}$ by the following formula, where we sum over the spectral projections:

$$T_\phi^{A_0,A_1}(V) = \sum_{i=1}^{n_0}\sum_{j=1}^{n_1} \phi(\lambda_i^{(0)}, \lambda_j^{(1)}) E_{A_0}(\{\lambda_i^{(0)}\}) V E_{A_1}(\{\lambda_j^{(1)}\}), \quad V \in B(\mathbb{C}^n), \tag{3.6}$$

or equivalently, this formula, where we sum over the eigenvector projections:

$$T_\phi^{A_0,A_1}(V) = \sum_{i,j=1}^n \phi(\lambda_i^{(0)}, \lambda_j^{(1)}) P_{\xi_i^{(0)}} V P_{\xi_j^{(1)}}, \quad V \in B(\mathbb{C}^n). \tag{3.7}$$

We can associate the operator V with a matrix, as follows. Let $\xi_m = \{\xi_i^{(m)}\}_{i=1}^n$ as before. We can define the matrix M_n by $\{v_{ij}^{\xi_0,\xi_1}\}_{i,j=1}^n$, where $v_{ij}^{\xi_0,\xi_1} = \langle V(\xi_j^{(1)}), \xi_i^{(0)}\rangle$. Using this association, $T_\phi^{A_0,A_1}$ acts a a linear Schur multiplier $\{\phi(\lambda_i^{(0)}, \lambda_j^{(1)})\}_{i,j=1}^n$. In other words, we can interpret the operator $T_\phi^{A_0,A_1}$ as an entry-wise matrix multiplication. We can show this by looking at the entries of the sum in equation (3.7), as follows:

$$\langle (P_{\xi_i^{(0)}} V P_{\xi_j^{(1)}})(\xi_s^{(1)}), \xi_r^{(0)}\rangle = \langle (V P_{\xi_j^{(1)}})(\xi_s^{(1)}), P_{\xi_i^{(0)}}\xi_r^{(0)}\rangle = \delta_{s,j}\delta_{r,i}\langle V(\xi_s^{(1)}), \xi_r^{(0)}\rangle = \delta_{s,j}\delta_{r,i} v_{ij}^{\xi_0,\xi_1}.$$

By linearity in the first argument of the inner product, we get

$$\langle (T_\phi^{A_0,A_1}(\xi_j^{(1)}), \xi_i^{(0)}\rangle = \phi(\lambda_i^{(0)}, \lambda_j^{(1)}) v_{ij}^{\xi_0,\xi_1},$$

thus showing that $T_\phi^{A_0,A_1} \sim \{\phi(\lambda_i^{(0)}, \lambda_j^{(1)})\}_{i,j=1}^n : M_n \to M_n$. Clearly, this identification is isometric. We get

$$\|T_\phi^{A_0,A_1} : \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\| = \|\{\phi(\lambda_i^{(0)}, \lambda_j^{(1)})\}_{i,j=1}^n : \mathcal{S}_n^\infty \to \mathcal{S}_n^\infty\|. \tag{3.8}$$

Equation 3.6, when $A_0 = A_1 = A$, is a special case of a double operator integral. A more general formula for the double operator will now be introduced, see [6]. Let $A \in B(H)$ be a self-adjoint operator. For any Borel subset of $\mathcal{B} \subseteq \mathbb{R}$, consider the spectral projection of A on $\mathcal{B}$ acting on $\mathcal{S}^2(H)$: $P_A^1(x) = E_A(\mathcal{B})x$ and $P_A^2(x) = xE_A(\mathcal{B})$. The associated spectral measures act on the Hilbert space $\mathcal{S}^2(H)$ by $\nu_1(\mathcal{B}_1) : x \mapsto E_A(\mathcal{B}_1)x$ and $\nu_2(\mathcal{B}_2) : x \mapsto xE_A(\mathcal{B}_2)$. Since these spectral measures commute, by [2], theorem V.2.6, there exists a countably additive projection valued measure ν on $\mathbb{R}^2$ acting on the Hilbert space $\mathcal{S}^2(H)$, defined by the formula

$$\nu(\mathcal{B}_1 \otimes \mathcal{B}_2) : x \mapsto E_A(\mathcal{B}_1)xE_A(\mathcal{B}_2), \quad x \in \mathcal{S}^2(H). \tag{3.9}$$

To formally define the double operator integral, we integrate over this measure:

$$T_{\xi}^{A,A}(V) = \int_{\mathbb{R}^2} \xi(\lambda,\mu) d(E_A(\lambda) V E_A(\mu)), \quad V \in \mathcal{S}^2(H). \tag{3.10}$$

Integrating a bounded Borel function ξ as shown in the equation yields a bounded operator $T_{\xi}^{A,A}(V)$ acting on H. If A is bounded, and has a spectrum contained in the integers, we have

$$T_{\xi}^{A,A}(V) = \sum_{k,l\in\mathbb{Z}} \xi(k,l) E_A(\{k\}) V E_A(\{l\}).$$

3.4 Bilinear Schur Multipliers associated with functions and operators

In this section, we have so far studied the linear Schur multipliers, associated with two operators A_0 and A_1. We will now introduce the bilinear version, associated with three operators. Take three diagonizable, bounded, self-adjoint operators: A_0, A_1, A_2. For $m = 0,1,2$, take an orthonormal basis of eigenvectors of A_m: $\xi_m = \{\xi_i^{(m)}\}_{i=1}^n$. Let $\{\lambda_i^{(m)}\}_{i=1}^n$ be the corresponding eigenvalues. Now, for any bounded Borel function $\psi : \mathbb{R}^3 \to \mathbb{C}$ we can define a bilinear operator $T_{\psi}^{A_0,A_1,A_2}$:

$$T_{\psi}^{A_0,A_1,A_2}(V,W) = \sum_{i=1}^{n_0}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)}) E_{A_0}(\{\lambda_i^{(0)}\}) V E_{A_1}(\{\lambda_j^{(1)}\}) W E_{A_2}(\{\lambda_k^{(2)}\}), \quad V,W \in B(\mathbb{C}^n).$$

Note that we are summing over the distinct eigenvalues, hence we sum to n_0, n_1, n_2. There is again an equivalent definition in terms of projections on the eigenvectors:

$$T_{\psi}^{A_0,A_1,A_2}(V,W) = \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)}) P_{\xi_i^{(0)}} V P_{\xi_j^{(1)}} W P_{\xi_k^{(2)}}, \quad V \in B(\mathbb{C}^n). \tag{3.11}$$

We can once again identify the operators V, W with matrices, as before, using the appropriate basis for each:

$$M_n^V = \{v_{ij}^{\xi_0,\xi_1}\}_{i,j=1}^n, \quad v_{ij}^{\xi_0,\xi_1} = \langle V(\xi_j^{(1)}), \xi_i^{(0)}\rangle,$$

$$M_n^W = \{w_{ij}^{\xi_1,\xi_2}\}_{i,j=1}^n, \quad w_{ij}^{\xi_1,\xi_2} = \langle W(\xi_j^{(2)}), \xi_i^{(1)}\rangle.$$

Under these identifications, we can show that $T_{\psi}^{A_0,A_1,A_2}$ acts as a bilinear Schur multiplier associated with a three dimensional matrix: $M = \{\psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)})\}_{i,j,k=1}^n$. This can be shown as follows.

$$\langle (P_{\xi_i^{(0)}} V P_{\xi_j^{(1)}} W P_{\xi_k^{(2)}})(\xi_s^{(2)}), \xi_r^{(0)}\rangle = \langle (V P_{\xi_j^{(1)}} W)(P_{\xi_k^{(2)}}\xi_s^{(2)}), P_{\xi_i^{(0)}}\xi_r^{(0)}\rangle$$

This is equal to zero, unless $r = i, s = k$. Assume $r = i, s = k$ and use the definition of a projection to continue.

$$\langle (V P_{\xi_j^{(1)}} W)(\xi_s^{(2)}), \xi_r^{(0)}\rangle = \langle V(P_{\xi_j^{(1)}}(W(\xi_s^{(2)}))), \xi_r^{(0)}\rangle = \langle V(\langle W(\xi_s^{(2)}), \xi_j^{(1)}\rangle \xi_j^{(1)}), \xi_r^{(0)}\rangle =$$

$$\langle W(\xi_s^{(2)}), \xi_j^{(1)}\rangle \langle V(\xi_j^{(1)}), \xi_r^{(0)}\rangle = w_{js}^{\xi_1,\xi_2} v_{rj}^{\xi_0,\xi_1}.$$

In conclusion:

$$\langle (P_{\xi_i^{(0)}} V P_{\xi_j^{(1)}} W P_{\xi_k^{(2)}})\xi_s^{(2)}, \xi_r^{(0)}\rangle = \begin{cases} w_{js}^{\xi_1,\xi_2} v_{rj}^{\xi_0,\xi_1}, & \text{if } r = i, s = k, \\ 0, & \text{otherwise.} \end{cases}$$

Now we can use the linearity of the inner product to show that

$$\langle T_\psi^{A_0,A_1,A_2}(V,W)(\xi_s^{(2)}),\xi_r^{(0)}\rangle = \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)})\langle (P_{\xi_i^{(0)}} V P_{\xi_j^{(1)}} W P_{\xi_k^{(2)}})(\xi_s^{(2)}),\xi_r^{(0)}\rangle =$$

$$\sum_{j=1}^{n} \psi(\lambda_r^{(0)},\lambda_j^{(1)},\lambda_s^{(2)})\langle (P_{\xi_r^{(0)}} V P_{\xi_j^{(1)}} W P_{\xi_s^{(2)}})(\xi_s^{(2)}),\xi_r^{(0)}\rangle = \sum_{j=1}^{n} \psi(\lambda_r^{(0)},\lambda_j^{(1)},\lambda_s^{(2)}) w_{js}^{\xi_1,\xi_2} v_{rj}^{\xi_0,\xi_1}.$$

From this we can conclude that the operator $T_\psi^{A_0,A_1,A_2}(V,W)$ can be expressed as follows:

$$T_\psi^{A_0,A_1,A_2}(V,W) = \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)}) v_{ij}^{\xi_0,\xi_1} w_{jk}^{\xi_1,\xi_2} E_{ik}.$$

The above identifications are again isometric, for all Schatten classes. As such, we can deduce the following formula:

$$\|T_\psi^{A_0,A_1,A_2} : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\| = \|\{\psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)})\}_{i,j,k=1}^n : \mathcal{S}_n^2 \times \mathcal{S}_n^2 \to \mathcal{S}_n^1\|. \tag{3.12}$$

The operator $T_\psi^{A_0,A_1,A_2}$ is formally known as a bilinear Schur multiplier associated with function ψ and operators A_0, A_1, A_2. It is a special case of what is known as a multiple operator integral.

3.5 Relevant properties

Some specific properties of Schur multipliers are required for proofs in later chapters. For the entirety of this section, let $\phi : \mathbb{R}^2 \to \mathbb{C}$ and $\psi : \mathbb{R}^3 \to \mathbb{C}$ be arbitrary bounded Borel functions, and keep $n \in \mathbb{N}$ fixed.

Lemma 3.8. ([7], Lemma 9) *Let $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ be self-adjoint operators, and denote the identity of $B(\mathbb{C}^n)$ by I^n. For $j = 0, 1$, define*

$$\psi_j(x_0,x_1,x_2) = x_j \psi(x_0,x_1,x_2), \quad x_0,x_1,x_2 \in \mathbb{R}$$

and

$$\tilde{\psi}_j(x_0,x_1,x_2) = \phi(x_j,x_2), \quad x_0,x_1,x_2 \in \mathbb{R}.$$

Then for any $X \in B(\mathbb{C}^n)$ we have

$$T_{\psi_j}^{A_0,A_1,A_2}(I_n,X) = T_\psi^{A_0,A_1,A_2}(A_j,X) \tag{3.13}$$

and

$$T_{\tilde{\psi}_j}^{A_0,A_1,A_2}(I_n,X) = T_\phi^{A_j,A_2}(X) \tag{3.14}$$

Proof. Take an arbitrary $X \in B(\mathbb{C}^n)$. The proof of each equation is simply a calculation. Starting with equation 3.13, with $j = 0$:

$$T_\psi^{A_0,A_1,A_2}(A_0,X) = \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)}) P_{\xi_i^{(0)}} A_0 P_{\xi_j^{(1)}} X P_{\xi_k^{(2)}}$$

$$= \sum_{i,j,k=1}^{n} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)}) P_{\xi_i^{(0)}} (\sum_{l=1}^{n} \lambda_l^{(0)} P_{\xi_l^{(0)}}) P_{\xi_j^{(1)}} X P_{\xi_k^{(2)}}$$

$$= \sum_{i,j,k=1}^{n} \lambda_i^{(0)} \psi(\lambda_i^{(0)},\lambda_j^{(1)},\lambda_k^{(2)}) P_{\xi_i^{(0)}} I_n P_{\xi_j^{(1)}} X P_{\xi_k^{(2)}}$$

$$= T_{\psi_0}^{A_0,A_1,A_2}(I_n,X).$$

The case $j = 1$ can be proven with a similar calculation: we use the decomposition of A_1 in terms of the projections onto the its eigenvectors $\{\xi_l^{(1)}\}_{l=1}^n$ and multiply it on the right-hand side with $P_{\xi_j^{(1)}}$ to get $T_{\psi_1}^{A_0,A_1,A_2}(I_n, X)$.

Now for equation 3.14, and again $j = 0$:

$$
\begin{aligned}
T_{\tilde{\psi}_0}^{A_0,A_1,A_2}(I_n, X) &= \sum_{i,j,k=1}^{n} \tilde{\psi}_0(\lambda_i^{(0)}, \lambda_j^{(1)}, \lambda_k^{(2)}) P_{\xi_i^{(0)}} I_n P_{\xi_j^{(1)}} X P_{\xi_k^{(2)}} \\
&= \sum_{i,j,k=1}^{n} \phi(\lambda_i^{(0)}, \lambda_k^{(2)}) P_{\xi_i^{(0)}} I_n P_{\xi_j^{(1)}} X P_{\xi_k^{(2)}} \\
&= \sum_{i,k=1}^{n} \phi(\lambda_i^{(0)}, \lambda_k^{(2)}) P_{\xi_i^{(0)}} (\sum_{j=1}^{n} P_{\xi_j^{(1)}}) X P_{\xi_k^{(2)}} \\
&= \sum_{i,k=1}^{n} \phi(\lambda_i^{(0)}, \lambda_k^{(2)}) P_{\xi_i^{(0)}} X P_{\xi_k^{(2)}} \\
&= T_{\phi}^{A_0,A_2}(X).
\end{aligned}
$$

For $j = 1$, a similar same procedure works: $\tilde{\psi}_1$ is independent of $\lambda_i^{(0)}$, allowing us to move the sum over i to the relevant projection $P_{\xi_i^{(0)}}$, giving another identity, and resulting in $T_{\phi}^{A_1,A_2}(X)$.

□

Lemma 3.9. ([7], Lemma 10) *Let $A, X, Y \in B(\mathbb{C}^n)$, with A self-adjoint. Define*

$$
\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \; \tilde{X} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}.
$$

We then have

$$
T_{\psi}^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X}, \tilde{X}) = \begin{pmatrix} T_{\psi}^{A,A,A}(X, Y) & 0 \\ 0 & T_{\psi}^{A,A,A}(Y, X) \end{pmatrix}.
$$

Proof. Denote the set of distinct eigenvalues of A by $\{\lambda_i\}_{i=1}^{n_0}$. Note that $n_0 \leq n$. For any $i \in \{1, ..., n_0\}$, denote the spectral projection of A associated with λ_i by $E_A(\{\lambda_i\})$. By construction $\tilde{A}$ has the same set of distinct eigenvalues as A. The corresponding spectral projections are, for any $i \in \{1, ..., n_0\}$,

$$
E_{\tilde{A}}(\{\lambda_i\}) = \begin{pmatrix} E_A(\{\lambda_i\}) & 0 \\ 0 & E_A(\{\lambda_i\}) \end{pmatrix}.
$$

Using this, we can calculate

$$
\begin{aligned}
T_{\psi}^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X}, \tilde{X}) &= \sum_{i,j,k=1}^{n_0} \psi(\lambda_i, \lambda_j, \lambda_k) E_{\tilde{A}}(\{\lambda_i\}) \tilde{X} E_{\tilde{A}}(\{\lambda_j\}) \tilde{X} E_{\tilde{A}}(\{\lambda_k\}) \\
&= \sum_{i,j,k=1}^{n_0} \psi(\lambda_i, \lambda_j, \lambda_k) \begin{pmatrix} E_A(\{\lambda_i\}) & 0 \\ 0 & E_A(\{\lambda_i\}) \end{pmatrix} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \times \\
&\quad \begin{pmatrix} E_A(\{\lambda_j\}) & 0 \\ 0 & E_A(\{\lambda_j\}) \end{pmatrix} \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \begin{pmatrix} E_A(\{\lambda_k\}) & 0 \\ 0 & E_A(\{\lambda_k\}) \end{pmatrix} \\
&= \sum_{i,j,k=1}^{n_0} \psi(\lambda_i, \lambda_j, \lambda_k) \begin{pmatrix} E_A(\{\lambda_i\}) X E_A(\{\lambda_j\}) Y E_A(\{\lambda_j\}) & 0 \\ 0 & E_A(\{\lambda_i\}) Y E_A(\{\lambda_j\}) X E_A(\{\lambda_j\}) \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} T_\psi^{A,A,A}(X,Y) & 0 \\ 0 & T_\psi^{A,A,A}(Y,X) \end{pmatrix}.$$

□

Lemma 3.10. ([7], Lemma 11) *Take self-adjoint operators $A, B \in B(\mathbb{C}^n)$ such that they have the same set of distinct eigenvalues. Take $X, Y \in B(\mathbb{C}^n)$ arbitrarily. Define the following operators:*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tilde{X} = \begin{pmatrix} 0 & X \\ 0^n & 0 \end{pmatrix}, \tilde{Y} = \begin{pmatrix} 0^n & 0 \\ 0 & Y \end{pmatrix}.$$

Then

$$T_\psi^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X},\tilde{Y}) = \begin{pmatrix} 0 & T_\psi^{A,B,B}(X,Y) \\ 0^n & 0 \end{pmatrix}.$$

Proof. Take $\{\lambda_i\}_{i=1}^{n_0}$ the set of distinct eigenvalues of A as in Lemma 3.9, noting that this is also the set of distinct eigenvalues for B and $\tilde{A}$. For any $i \in \{1, ..., n_0\}$ we have

$$E_{\tilde{A}}(\{\lambda_i\}) = \begin{pmatrix} E_A(\{\lambda_i\}) & 0 \\ 0 & E_B(\{\lambda_i\}) \end{pmatrix}.$$

Now we can calculate

$$T_\psi^{\tilde{A},\tilde{A},\tilde{A}}(\tilde{X},\tilde{Y}) = \sum_{i,j,k=1}^{n_0} \psi(\lambda_i,\lambda_j,\lambda_k) E_{\tilde{A}}(\{\lambda_i\})\tilde{X} E_{\tilde{A}}(\{\lambda_j\})\tilde{Y} E_{\tilde{A}}(\{\lambda_k\})$$

$$= \sum_{i,j,k=1}^{n_0} \psi(\lambda_i,\lambda_j,\lambda_k) \begin{pmatrix} E_A(\{\lambda_i\}) & 0 \\ 0 & E_B(\{\lambda_i\}) \end{pmatrix} \begin{pmatrix} 0 & X \\ 0^n & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} E_A(\{\lambda_j\}) & 0 \\ 0 & E_B(\{\lambda_j\}) \end{pmatrix} \begin{pmatrix} 0^n & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} E_A(\{\lambda_j\}) & 0 \\ 0 & E_B(\{\lambda_j\}) \end{pmatrix}$$

$$= \sum_{i,j,k=1}^{n_0} \psi(\lambda_i,\lambda_j,\lambda_k) \begin{pmatrix} 0 & E_A(\{\lambda_i\})XE_B(\{\lambda_j\})YE_B(\{\lambda_k\}) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & T_\psi^{A,B,B}(X,Y) \\ 0^n & 0 \end{pmatrix}.$$

□

Lemma 3.11. ([7], Lemma 12) *For any $A_0, A_1, A_2 \in B(\mathbb{C}^n)$ and $a \in \mathbb{R}\backslash\{0\}$ we have*

$$T_\psi^{aA_0,aA_1,aA_2} = T_{\psi_a}^{A_0,A_1,A_2}$$

if we define

$$\psi_a(x_0,x_1,x_2) := \psi(ax_0,ax_1,ax_2), \quad x_0,x_1,x_2 \in \mathbb{R}.$$

Proof. For any $j \in \{0,1,2\}$, denote the distinct eigenvalues of A_j by $\{\lambda_i^{(j)}\}_{i=1}^{n_j}$. Fix $a \in \mathbb{R}\backslash\{0\}$. By the definition of eigenvalues it is easy to see that the distinct eigenvalues of aA_j are $\{a\lambda_i^{(j)}\}_{i=1}^{n_j}$ for any j: if we have $A_j\xi_i^{(j)} = \lambda_i^{(j)}\xi_i^{(j)}$ we also have $(aA_j)\xi_i^{(j)} = (a\lambda_i^{(j)})\xi_i^{(j)}$. This also shows that the spectral projections coincide: $E_{A_j}(\{\lambda_i^{(j)}\}) = E_{aA_j}(\{a\lambda_i^{(j)}\})$ for any j and $i \in \{1, ..., n_j\}$. Thus we have, for any $X, Y \in B(\mathbb{C}^n)$,

$$T_\psi^{aA_0,aA_1,aA_2}(X,Y) = \sum_{i=1}^{n_0}\sum_{j=1}^{n_1}\sum_{k=1}^{n_2} \tilde{\psi}_0(a\lambda_i^{(0)}, a\lambda_j^{(1)}, a\lambda_k^{(2)}) E_{A_0}(\{\lambda_i^{(0)}\})XE_{A_1}(\{\lambda_j^{(1)}\})YE_{A_2}(\{\lambda_k^{(2)}\})$$

by definition. It is clear to see that this coincides with $T_{\psi_a}^{A_0,A_1,A_2}$ by writing out its definition too. □

Lemma 3.12. ([7], Lemma 13) *Take self-adjoint operators $A, B \in B(\mathbb{C}^n)$. Take a sequence $\{U_m\}_{m\geq 1}$ of unitary operators in $B(\mathbb{C}^n)$, with $\lim_{m\to\infty} U_m = I_n$. Additionally take $X, Y \in B(\mathbb{C}^n)$ and sequences $\{X_m\}_{m\geq 1}, \{Y_m\}_{m\geq 1}$ of operators in $B(\mathbb{C}^n)$, with $\lim_{m\to\infty} X_m = X$ and $\lim_{m\to\infty} Y_m = Y$. Lastly, take $\psi, \psi_m : \mathbb{R}^3 \to \mathbb{C}$ bounded Borel functions, such that $\lim_{m\to\infty} \psi_m = \psi$ pointwise. Then we have*

$$\lim_{m\to\infty} T_{\psi_m}^{U_m A U_m^*, B, B}(X_m, Y_m) = T_\psi^{A,B,B}(X, Y).$$

Proof. Denote the sets of distinct eigenvalues of A and B by $\{\lambda_i\}_{i=1}^{m_0}$ and $\{\mu_j\}_{j=1}^{m_1}$ respectively. Note the distinct eigenvalues of $U_m A U_m^*$ are also $\{\lambda_i\}_{i=1}^{m_0}$. Also note that for any $1 \leq i \leq m_0$,

$$E_{U_m A U_m^*}(\{\lambda_i\}) = U_m E_A U_m^*(\{\lambda_i\}).$$

Now we can start calculations.

$$T_{\psi_m}^{U_m A U_m^*, B, B}(X_m, Y_m) = \sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} \psi_m(\lambda_i, \mu_j, \mu_k) E_{U_m A U_m^*}(\{\lambda_i\}) X_m E_B(\{\mu_j\}) Y_m E_B(\{\mu_j\})$$

$$= \sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} \psi_m(\lambda_i, \mu_j, \mu_k) U_m E_A(\{\lambda_i\}) U_m^* X_m E_B(\{\mu_j\}) Y_m E_B(\{\mu_j\})$$

$$= U_m \sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} \psi_m(\lambda_i, \mu_j, \mu_k) E_A(\{\lambda_i\})(U_m^* X_m) E_B(\{\mu_j\}) Y_m E_B(\{\mu_j\})$$

$$= U_m T_{\psi_m}^{A,B,B}(U_m^* X_m, Y_m).$$

Now we will show that $\lim_{m\to\infty} T_{\psi_m}^{A,B,B}(U_m^* X, Y) = T_\psi^{A,B,B}(X_m, Y_m)$, by finding an upper estimate to the norm of the difference, which converges to zero.

$$\|T_{\psi_m}^{A,B,B}(U_m^* X_m, Y_m) - T_\psi^{A,B,B}(X_m, Y_m)\|_\infty$$

$$\leq \|T_{\psi_m}^{A,B,B}(U_m^* X_m, Y_m) - T_{\psi_m}^{A,B,B}(X_m, Y_m)\|_\infty + \|T_{\psi_m}^{A,B,B}(X_m, Y_m) - T_\psi^{A,B,B}(X_m, Y_m)\|_\infty$$

$$= \|T_{\psi_m}^{A,B,B}(U_m^* X_m - X_m, Y_m)\|_\infty + \|T_{\psi_m - \psi}^{A,B,B}(X_m, Y_m)\|_\infty$$

$$= \|\sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} \psi_m(\lambda_i, \mu_j, \mu_k) E_A(\{\lambda_i\})(U_m^* X_m - X_m) E_B(\{\mu_j\}) Y_m E_B(\{\mu_j\})\|_\infty$$

$$+\|\sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} (\psi_m - \psi)(\lambda_i, \mu_j, \mu_k) E_A(\{\lambda_i\})(U_m^* X_m) E_B(\{\mu_j\}) Y_m E_B(\{\mu_j\})\|_\infty$$

$$\leq \sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} |\psi_m(\lambda_i, \mu_j, \mu_k)| \|(U_m^* - I)\|_\infty \|X_m\|_\infty \|Y_m\|_\infty$$

$$+ \sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} |(\psi_m - \psi)(\lambda_i, \mu_j, \mu_k)| \|X_m\|_\infty \|Y_m\|_\infty.$$

Both of these final terms converge to zero by assumption. Additionally,

$$\|T_\psi^{A,B,B}(X_m, Y_m) - T_\psi^{A,B,B}(X, Y)\|_\infty = \|T_\psi^{A,B,B}(X_m - X, Y_m - Y)\|_\infty$$

$$= \|\sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} \psi(\lambda_i, \mu_j, \mu_k) E_A(\{\lambda_i\})(X_m - X) E_B(\{\mu_j\})(Y_m - Y) E_B(\{\mu_j\})\|_\infty$$

$$\leq \sum_{i=1}^{m_0}\sum_{j=1}^{m_1}\sum_{k=1}^{m_1} |\psi(\lambda_i, \mu_j, \mu_k)| \|X_m - X\|_\infty \|Y_m - Y\|_\infty$$

which also converges to zero by assumption. By again applying $\lim_{m\to\infty} U_m = I_n$ and the previous two estimates we find

$$\lim_{m\to\infty} T_{\psi_m}^{U_m A U_m^*, B, B}(X_m, Y_m) = \lim_{m\to\infty} U_m T_{\psi_m}^{A,B,B}(U_m^* X_m, Y_m) = T_{\psi}^{A,B,B}(X, Y)$$

□

Lemma 3.13. ([7], Lemma 14) *Take $A \in B(\mathbb{C}^n)$ self-adjoint, and $X \in B(\mathbb{C}^n)$ such that A commutes with X: $[A, X] = 0$. Take any bounded Borel function $\psi : \mathbb{R}^3 \to \mathbb{C}$, and define $\widehat{\psi} : \mathbb{R} \to \mathbb{C}$ by $\widehat{\psi}(x) = \psi(x, x, x)$ for any $x \in \mathbb{R}$. Also define $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathbb{C}$ by $\phi_1(x_0, x_1) = \psi(x_0, x_1, x_1)$ and $\phi_2(x_0, x_1) = \psi(x_0, x_0, x_1)$ for any $x_0, x_1 \in \mathbb{R}$. Then the following equations hold:*

$$T_{\psi}^{A,A,A}(X, X) = \widehat{\psi}(A) \times X^2 \tag{3.15}$$

$$T_{\psi}^{A,A,A}(Y, X) = T_{\phi_1}^{A,A}(Y) \times X \tag{3.16}$$

$$T_{\psi}^{A,A,A}(X, Y) = X \times T_{\phi_2}^{A,A}(Y) \tag{3.17}$$

Proof. Denote the sets of eigenvectors of A and the corresponding eigenvalues by $\{\xi_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ respectively. Since A commutes with X, P_{ξ_i} commutes with X for any $i \in \{1, ..., n\}$, allowing us to swap P_{ξ_i} and X in our upcoming calculations. We can now show that (3.15) holds:

$$T_{\psi}^{A,A,A}(X, X) = \sum_{i,j,k=1}^{n} \psi(\lambda_i, \lambda_j, \lambda_k) P_{\xi_i} X P_{\xi_j} X P_{\xi_k} = \sum_{i,j,k=1}^{n} \psi(\lambda_i, \lambda_j, \lambda_k) P_{\xi_i} P_{\xi_j} P_{\xi_k} X X$$

$$= \sum_{i=1}^{n} \psi(\lambda_i, \lambda_i, \lambda_i) P_{\xi_i} \times X^2 = \sum_{i=1}^{n} \widehat{\psi}(\lambda_i) P_{\xi_i} \times X^2 = \widehat{\psi}(A) \times X^2.$$

We can also show that (3.16) holds:

$$T_{\psi}^{A,A,A}(Y, X) = \sum_{i,j,k=1}^{n} \psi(\lambda_i, \lambda_j, \lambda_k) P_{\xi_i} Y P_{\xi_j} X P_{\xi_k} = \sum_{i,j,k=1}^{n} \psi(\lambda_i, \lambda_j, \lambda_k) P_{\xi_i} Y P_{\xi_j} P_{\xi_k} X$$

$$= \sum_{i,j=1}^{n} \psi(\lambda_i, \lambda_j, \lambda_j) P_{\xi_i} Y P_{\xi_j} \times X = \sum_{i,j=1}^{n} \phi_1(\lambda_i, \lambda_j) P_{\xi_i} Y P_{\xi_j} \times X = T_{\phi_1}^{A,A}(Y) \times X.$$

The proof of (3.17) is the same. □

4 Additional preliminary results

In this chapter, some additional preliminary results will be presented, which are necessary to prove the positive results in the coming chapters. We assume the reader has prior knowledge of real harmonic analysis, specifically Fourier multipliers and Calderón-Zygmund theory. See for example [1].

4.1 Weak type inequalities for Calderón-Zygmund operators

The results of this section were first proven in [17]. The proofs have since been simplified in [3] and [4]. Let K be a tempered distribution, which we will name the convolution kernel. Let W_K be the Calderón-Zygmund operator associated with K: $f \mapsto K * f$, the convolution with K. We will only consider situations where K can be identified with a measurable function $K : \mathbb{R}^d \to \mathbb{C}$. Let $B(H)$ be a the bounded operators on a Hilbert space H. Then we can define the operator $1 \otimes W_K$ under suitable conditions, as a noncommutative Calderón-Zygmund operator that acts only on the second tensor leg of $\mathcal{S}^1(H) \otimes \mathcal{S}^1(\mathbb{R}^d)$ We can set conditions such that this operator maps $\mathcal{S}^1$ to $\mathcal{S}^{1,\infty}$.

Theorem 4.1. ([17]) *Let $K : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ be a kernel satisfying*

$$|K|(t) \leq \frac{c_1}{|t|^d}, \quad |\nabla K|(t) \leq \frac{c_2}{|t|^{d+1}},$$

where c_1 and c_2 can be any constants. If $W_K \in B(\mathcal{S}^2(\mathbb{R}^d))$, then the operator $1 \otimes W_K$ defines a bounded map from $\mathcal{S}^1(B(H) \otimes \mathcal{S}^\infty(\mathbb{R}^d))$ to $\mathcal{S}^{1,\infty}(B(H) \otimes \mathcal{S}^\infty(\mathbb{R}^d))$.

4.2 Approximate intertwining properties of Fourier multipliers

In this section, we will prove some properties of Fourier multipliers. These are necessary to prove the main result of chapter 5. The results are first presented in [6], section 3, and based on [10], section 2.

We first introduce G_l as a probability density for a Gaussian random variable:

$$G_l(s) := \frac{1}{l\sqrt{\pi}} e^{-(\frac{s}{l})^2}.$$

For higher dimensions, we introduce $G_l^{\otimes d}$ as the tensor product of d instances of G_l.

Lemma 4.2. ([6], Lemma 3.1) *For every $f \in \mathcal{S}^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} f(s)ds = 0$, we have $f * G_l \to 0$ in $\mathcal{S}^1(\mathbb{R})$ as $l \to \infty$*

Proof. Let f be a step function: $f = \sum_{k=1}^m \alpha_k \chi_{I_k}$, where I_k are disjoint intervals $[a_k, b_k]$, and $f = \sum_{k=1}^m \alpha_k m(I_k) = 0$, so it has integral 0. Then

$$(f * G_l)(t) = \sum_{k=1}^m \alpha_k \int_{a_k}^{b_k} G_l(t-s)ds = \sum_{k=1}^m \alpha_k \int_{t-a_k}^{t-b_k} G_l(u)du =$$

$$\sum_{k=1}^m \alpha_k \int_{\frac{t-a_k}{l}}^{\frac{t-b_k}{l}} G_1(s)ds = \sum_{k=1}^m \alpha_k (F(\frac{t-a_k}{l}) - F(\frac{t-b_k}{l})),$$

where F can be any primitive of G_1. We take $F(t) = \int_{-\infty}^t G_1(s)ds$. To prove this lemma for the step functions f, we show that

$$l \int_{-\infty}^{\infty} |\sum_{k=1}^m \alpha_k (F(t - \frac{a_k}{l}) - F(t - \frac{b_k}{l}))|dt \to 0. \tag{4.1}$$

Note that we have substituted $\frac{t}{l}$ for t, hence the extra factor l. By Taylor expansion we have

$$|F(t-\frac{a_k}{l})-F(t)+\frac{a_k}{l}F'(t)|\leq \frac{a_k^2}{2l^2}\max_{s\in[t-a_k/l,t]}|F''(s)|.$$

We can use this formula to approximate 4.1. If l is larger than the maximum absolute values of all a_k and of all b_k, we get

$$|\sum_{k=1}^m \alpha_k(F(t-\frac{a_k}{l})-F(t-\frac{b_k}{l}))|\leq \frac{1}{2l^2}(\sum_{k=1}^m|\alpha_k|(a_k^2+b_k^2))\max_{s\in[t-1,t+1]}|F''(s)|.$$

So we have proven the lemma for step functions with zero integral. Now we can approximate arbitrary functions by step functions. Fix $(f_m)_{m\geq 1}$ such that $f_m\to f$ in $\mathcal{S}^1(\mathbb{R})$. Since $\|G_l\|_1=1$ (it is a probability distribution), from Young's inequality we get

$$\|f*G_l\|_1\leq\|(f-f_m)*G_l\|_1+\|f_m*G_l\|_1\leq\|f-f_m\|_1+\|f_m*G_l\|_1.$$

Therefore we have

$$\limsup_{l\to\infty}\|f*G_l\|_1\leq\|f-f_m\|_1$$

which goes to zero as $m\to\infty$, which concludes the proof. □

We can now prove a multi-dimensional variant of this lemma.

Lemma 4.3. ([6], Lemma 3.2) *For every $f\in\mathcal{S}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}f(s)ds=0$, we have $f*G_l^{\otimes d}\to 0$ in $\mathcal{S}^1(\mathbb{R}^d)$ as $l\to\infty$.*

Proof. Let f first be a linear combination of elementary tensors:

$$f=\sum_{k=1}^m\bigotimes_{j=1}^d f_{jk},\quad f_{jk}\in\mathcal{S}^1(\mathbb{R}) \tag{4.2}$$

First, let's consider the case that for every $k, 1\leq k\leq m$, there is a $j, 1\leq j\leq d$, such that $\int_{\mathbb{R}}f_{jk}(s)ds=0$. Then by lemma 4.2

$$\|f*G_l^{\otimes d}\|_1\leq\sum_{k=1}^m\prod_{j=1}^d\|f_{jk}*G_l^{\otimes d}\|_1\to 0,$$

since at least one element of each product goes to zero (and the other elements are bounded) so the sum of the products goes to zero. Now consider an f as in 4.2 with

$$\sum_{k=1}^m\prod_{j=1}^d\int_{\mathbb{R}}f_{jk}(s)ds=0. \tag{4.3}$$

We will show that any of these can be reduced to the case that for every $k, 1\leq k\leq m$, there is a $j, 1\leq j\leq d$, such that $\int_{\mathbb{R}}f_{jk}(s)ds=0$. To do this, we set for any subset $\mathscr{A}\subset\{1,...,d\}$, define the following:

$$f_{j,k,\mathscr{A}}=\begin{cases}f_{jk}-(\int_{\mathbb{R}}f_{jk}(s)ds)\chi_{(0,1)}, & j\in\mathscr{A}\\ (\int_{\mathbb{R}}f_{jk}(s)ds)\chi_{(0,1)}, & j\notin\mathscr{A}.\end{cases}$$

By linearity, we can rewrite 4.2 to

$$f=\sum_{k=1}^m\sum_{\mathscr{A}\subset\{1,...,d\}}\bigotimes_{j=1}^d f_{j,k,\mathscr{A}}.$$

Note that when $\mathscr{A}$ is the empty set, we have

$$f=\sum_{k=1}^{m}\bigotimes_{j=1}^{d}f_{j,k,\emptyset}=(\sum_{k=1}^{m}\prod_{j=1}^{d}\int_{\mathbb{R}}f_{jk}(s)ds)\chi_{(0,1)}=0,$$

by choice of f. So we get

$$f=\sum_{k=1}^{m}\sum_{\emptyset\neq\mathscr{A}\subset\{1,\ldots,d\}}\bigotimes_{j=1}^{d}f_{j,k,\mathscr{A}}.$$

If $j\in\mathscr{A}$, we have $f_{j,k,\mathscr{A}}$ is mean zero. So we have made a representation of f such that for every $k, 1\leq k\leq m$, there is a $j, 1\leq j\leq d$, such that $\int_{\mathbb{R}}f_{jk}(s)ds=0$. So we can apply the result of 4.2 to 4.3. Now we still need to prove the general case. Fix $(f_m)_{m\geq1}$ to be a sequence of mean zero sums of elementary tensors such that $f_m\to f$ in $\mathcal{S}^1(\mathbb{R})$. Since $\|G_l^{\otimes d}\|_1=1$, from Young's inequality we get

$$\|f*G_l^{\otimes d}\|_1\leq\|(f-f_m)*G_l^{\otimes d}\|_1+\|f_m*G_l^{\otimes d}\|_1\leq\|f-f_m\|_1+\|f_m*G_l^{\otimes d}\|_1.$$

Therefore we have

$$\limsup_{l\to\infty}\|f*G_l^{\otimes d}\|_1\leq\|f-f_m\|_1$$

which goes to zero as $m\to\infty$, which concludes the proof. □

We again introduce some notation: $e_k(t):=e^{i\langle k,t\rangle}$, and $\mathcal{F}$ denotes the unitary version of the Fourier transform:

$$\mathcal{F}(g(t))=\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}e_{-k}(t)g(t)dt.$$

Lemma 4.4. ([6], Lemma 3.3) *If $g\in\mathcal{S}^\infty(\mathbb{R}^d)$ is such that $\mathcal{F}(g)\in\mathcal{S}^1(\mathbb{R}^d)$, then for every $k\in\mathbb{R}^d$,*

$$(g(\nabla))(G_l^{\otimes d})-g(k)G_l^{\otimes d}e_k\to0$$

in $\mathcal{S}^1(\mathbb{R}^d)$ as $l\to\infty$.

Proof. Fix $k\in\mathbb{R}^d$. Set $h_1(t)=g(k)e^{-|t-k|^2}$ and $h_0(t)=g(t)-h_1(t)$, $t\in\mathbb{R}^d$. Note that for every $t\in\mathbb{R}^d$ we have the following Fourier transform of $G_l^{\otimes d}$:

$$\mathcal{F}(G_l^{\otimes d})(t)=\pi^{-d/2}e^{-l^2|t|^2}.$$

Using the elementary Fourier transform result

$$\mathcal{F}(g(t)e_k(t))=G(t-k),$$

we get

$$\mathcal{F}(G_l^{\otimes d}e_k)(t)=\pi^{-d/2}e^{-l^2|t-k|^2}.$$

We can now introduce the Fourier multiplier of $h_1(\nabla)$, the operator that first applies the Fourier transform, then h_1, and then the inverse Fourier transform. Using this, we get

$$\mathcal{F}(h_1(\nabla)(G_l^{\otimes d}e_k))(t)=h_1(t)\mathcal{F}(G_l^{\otimes d}e_k)(t)=g(k)e^{-|t-k|^2}\pi^{-d/2}e^{-l^2|t-k|^2}=g(k)\pi^{-d/2}e^{-(l^2+1)|t-k|^2}.$$

Since $g(k)$ is just a constant with respect to the Fourier transform, we can apply our earlier results to calculate the inverse Fourier transform:

$$h_1(\nabla)(G_l^{\otimes d}e_k)=g(k)\mathcal{F}^{-1}(\pi^{-d/2}e^{-(l^2+1)|t-k|^2})=g(k)G_{\sqrt{l^2+1}}^{\otimes d}e_k.$$

Knowing that $\sqrt{l^2+1} \to l$ as $l \to \infty$ we can calculate that $G_{\sqrt{l^2+1}}^{\otimes d} \to G_l^{\otimes d}$ in $\mathcal{S}^1(\mathbb{R}^d)$ as $l \to \infty$. Applying this, we get

$$h_1(\nabla)(G_l^{\otimes d} e_k) - g(k) G_l^{\otimes d} e_k \to 0$$

in $\mathcal{S}^1(\mathbb{R}^d)$ as $l \to \infty$. Note that by the definition of h_0 this is equivalent to $h_0(\nabla)(G_l^{\otimes d} e_k) \to 0$ in $\mathcal{S}^1(\mathbb{R}^d)$ as $l \to \infty$, which is what we will prove. Define $f(t) = e_k(t)(\mathcal{F}h_0)(t)$ with $t \in \mathbb{R}^d$. Then $h_0(\nabla)(G_l^{\otimes d} e_k) = f * G_l^{\otimes d}$. For f we have

$$\int_{\mathbb{R}^d} f(s)ds = \int_{\mathbb{R}^d} e_k(s)(\mathcal{F}h_0)(s)ds$$

which is the inverse Fourier transform of $\mathcal{F}h_0$ (up to multiplication with a constant) so we get

$$\int_{\mathbb{R}^d} f(s)ds = h_0(k) = 0.$$

Now we get the desired result by applying lemma 4.3 □

This concludes the preliminary results of this thesis. In the next chapters, the main results will be presented.

5 Weak type estimates for operators with integer spectrum

In this chapter, Theorem 1.2 will be proven, but only for operators with integer spectrum. The generalisation to all bounded self-adjoint operators will be made in the next chapter.

Let σ_s denote the dilation operator, acting on the measurable functions on $\mathbb{R}$, which is defined by the formula $(\sigma_s x)(t) = x(t/s)$. Let μ denote the singular values as in the previous chapter.

Lemma 5.1. ([6], Lemma 4.1) *Let x, y be measurable and θ be integrable functions on $\mathbb{R}$. Define $z(t) := t^{-1}$ for $t > 0$ and $z(t) := 0$ for $t < 0$. Let $u > 0$. Then $x \otimes y$ and $\theta \otimes z$ are measurable on $\mathbb{R}^2$ and we have*

$$\mu(\sigma_u(x) \otimes y) = \sigma_u \mu(x \otimes y), \quad \mu(t, \theta \otimes z) = \|\theta\|_1 t^{-1}, \quad t > 0.$$

Proof. For all t, we have

$$\mu(t, \sigma_u(x) \otimes y) = \inf\{\|(\sigma_u(x) \otimes y)\chi_{A^c}|\|_\infty : m(A) \leq t\} = \inf\{\|(x \otimes y)\chi_{(\sigma_u \otimes I)A^c}|\|_\infty : m(A) \leq t\} =$$

$$\inf\{\|(x \otimes y)\chi_{(\sigma_u \otimes I)A^c}|\|_\infty : m((\sigma_u \otimes I)A) \leq u^{-1}t\}.$$

By defining $B = (\sigma_u \otimes I)A$ we get

$$\mu(t, \sigma_u(x) \otimes y) = \inf\{\|(x \otimes y)\chi_{B^c}|\|_\infty : m(B) \leq u^{-1}t\} = \mu(u^{-1}t, x \otimes y) = \sigma_u \mu(t, x \otimes y).$$

And since this holds for all t, we can conclude that $\mu(\sigma_u(x) \otimes y) = \sigma_u \mu(x \otimes y)$.

To prove the second equality, we first prove it for a simple function $x \in \mathcal{S}^1(\mathbb{R})$. If $x = \sum_k a_k \chi_{B_k}$, with B_k pairwise disjoint sets, then we have

$$\mu(x \otimes z) = \mu(\bigoplus_k (a_k \chi_{B_k} \otimes z)) = \mu(\bigoplus_k \mu(a_k \chi_{B_k} \otimes z)).$$

If B is a set of finite measure we have

$$\mu(\chi_B \otimes z) = \mu(\chi(0, m(B)) \otimes z) = m(B)z.$$

So, we get

$$\mu(x \otimes z) = \mu(\bigoplus_k (|a_k| m(B_k) z)) = (\sum_k a_k m(B_k))z.$$

Since we can approximate measurable functions using simple functions, the second inequality follows. □

Lemma 5.2. ([6], Lemma 4.2) *For every $X \in \mathcal{S}^{1,\infty}(H)$ and $l > 0$ we have*

$$e^{-d}\pi^{-\frac{d}{2}}\|X\|_{1,\infty} \leq \|X \otimes G_l^{\otimes d}\|_{1,\infty} \leq \|X\|_{1,\infty}$$

Proof. For every operator $A \in K(H)$, the compact operators on H and for every function $g \in \mathcal{S}^\infty(0, \infty)$, we have

$$\mu(A \otimes g) = \mu(|A| \otimes g) = \mu(\mu(A) \otimes g) = \mu(\mu(A) \otimes \mu(g)).$$

Let z be as before: $z(t) := t^{-1}$ for $t > 0$ and $z(t) := 0$ for $t < 0$. By definition of $\|\cdot\|_{1,\infty}$, $\mu(X) \leq \|X\|_{1,\infty} z$. We thus have by lemma 5.1

$$\mu(X \otimes G_l^{\otimes d}) = \mu(\mu(X) \otimes G_l^{\otimes d}) \leq \|X\|_{1,\infty}\mu(z \otimes G_l^{\otimes d}) = \|X\|_{1,\infty}\mu(z).$$

This proves the right hand side inequality of this lemma. For the other side, note that $\mu(G_l) = l^{-1}\sigma_l \mu(G_1)$. By lemma 5.1 we have $\mu(G_l^{\otimes d}) = l^{-d}\sigma_{l^d}\mu(G_1^{\otimes d})$. So we have

$$\mu(X \otimes G_l^{\otimes d}) = l^{-d}\sigma_{l^d}\mu(X \otimes G_1^{\otimes d}).$$

Then we can calculate its quasi-norm:

$$\|X\otimes G_l^{\otimes d}\|_{1,\infty}=\sup_{t>0}\frac{t}{l^d}\mu(\frac{t}{l^d},X\otimes G_l^{\otimes d})=\sup_{s>0}s\mu(s,X\otimes G_l^{\otimes d})=\|X\otimes G_1^{\otimes d}\|_{1,\infty}.$$

By the formula for G_1 it is clear to see that $\mu(G_1)\geq\frac{1}{e\sqrt{\pi}}\chi_{(0,1)}$ So we get

$$\|X\otimes G_1^{\otimes d}\|_{1,\infty}=\|X\otimes\mu(G_1)^{\otimes d}\|_{1,\infty}\geq\|X\otimes(\frac{1}{e\sqrt{\pi}}\chi_{(0,1)})^{\otimes d}\|_{1,\infty}=e^{-d}\pi^{-\frac{d}{2}}\|X\|_{1,\infty}.$$

Now the left-hand side of the inequality is also proven. □

Lemma 5.3. ([6], Lemma 4.3) *If g is a C^∞-function that is constant on any open ray starting at the origin (smooth and homogeneous) on $\mathbb{R}^d\backslash\{0\}$, then $\mathcal{F}(g)$, the Fourier transform of g, satisfies the conditions of Theorem 4.1:*

$$|K|(t)\leq\frac{c_1}{|t|^d},\quad|\nabla K|(t)\leq\frac{c_2}{|t|^{d+1}}.$$

Proof. Since g is smooth everywhere ($g\in C^\infty$), it is certainly smooth on the circle $\{|z|=1\}$, so

$$g(e^{i\theta})=\sum_{k\in\mathbb{Z}}\alpha_k e^{ik\theta}.$$

Fourier coefficients decrease faster than any polynomial. Differentiating in the time domain corresponds to multiplication with k in the Fourier domain, and since $g\in C^\infty$ we need all of the derivatives to be well-defined. If α_k does not decrease faster with k than any polynomial, we get a diverging sum. So Fourier coefficients decrease faster than any polynomial. We thus get

$$g=\sum_{k\in\mathbb{Z}}\alpha_k g_k,\quad g_k(z)=\frac{z^k}{|z|^k},\quad 0\neq z\in\mathbb{C}.$$

We can apply the Fourier transform to these g_k, and later use linearity to get $\mathcal{F}(g)$.

$$\mathcal{F}(g_k)(z)=\frac{|k|}{2\pi i^k}\frac{g_k(z)}{|z|^2},\quad 0\neq z\in\mathbb{C}.$$

$$\mathcal{F}(g)(z)=\alpha_0\delta+\frac{1}{|z|^2}h(e^{i\arg(z)}),$$

where h is defined by the formula

$$h(e^{i\theta})=\sum_{0\neq k\in\mathbb{Z}}\frac{|k|}{2\pi i^k}\alpha_k e^{ik\theta}.$$

So we conclude that $(\mathcal{F}(g)-\alpha_0\delta)(z)=\mathcal{O}(|z|^{-2})$, thus satisfying the first requirement. For the divergence we can use the product rule and the chain rule:

$$\nabla(\frac{1}{|z|^2}h(e^{i\arg(z)}))=h(e^{i\arg(z)})\nabla(\frac{1}{|z|^2})+\frac{1}{|z|^2}\frac{dh(e^{i\theta})}{d\theta}|_{\theta=\arg(z)}\nabla(\arg(z))=\mathcal{O}(\frac{1}{|z|^3}).$$

So both conditions are satisfied. □

For a Lipschitz function f, we define $f^{[1]}$ by

$$f^{[1]}(\lambda,\mu)=\begin{cases}\frac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \lambda\neq\mu,\\ 0, & \lambda=\mu.\end{cases}$$

This is equal to the divided difference function (which is usually denoted by $f^{[1]}$) when $\lambda\neq\mu$. However, the divided difference function is only well-defined on the line $\mu=\lambda$ if a left- and right-derivative of f exist, which is not necessarily the case for an arbitrary Lipschitz function.

Theorem 5.4. ([6], Theorem 4.4) *For every self-adjoint operator $A \in B(H)$ with integer spectrum and for every Lipschitz function f, $V \in \mathcal{S}^1(H)$, we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|V\|_1.$$

Proof. Fix a smooth, homogeneous function g on $\mathbb{R}^2$ such that $g(e^{i\theta}) = \tan(\theta)$ for $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \frac{5\pi}{4})$. Without loss of generality we may assume that g is mean zero on the unit circle. By lemma 5.3 $\mathcal{F}(g)$ satisfies the conditions of Theorem 4.1. Since g is bounded, $g(\nabla) \in B(\mathcal{S}^2(\mathbb{R}^2))$. Recall that $(g(\nabla))(x) = (\mathcal{F}(g)) * x$. By Theorem 4.1 we have

$$1 \otimes g(\nabla) : \mathcal{S}^1(B(H) \otimes \mathcal{S}^\infty(\mathbb{R}^2)) \to \mathcal{S}^{1,\infty}(B(H) \otimes \mathcal{S}^\infty(\mathbb{R}^2)).$$

Now consider Schwartz functions ϕ_m on $\mathbb{R}^2$ that vanish near 0, such that $\phi_m(t) = 1$ for $|t| \in (\frac{1}{m}, m)$ and such that $\|\mathcal{F}\phi_m\|_1 \leq c_{abs}$ for all $m \geq 1$. Then we get, for $m \geq 1$:

$$\|1 \otimes (g\phi_m)(\nabla)\|_{\mathcal{S}^1 \to \mathcal{S}^{1,\infty}} \leq \|1 \otimes g(\nabla)\|_{\mathcal{S}^1 \to \mathcal{S}^{1,\infty}} \|1 \otimes \phi_m(\nabla)\|_{\mathcal{S}^1 \to \mathcal{S}^{1,\infty}} \leq$$

$$\|1 \otimes g(\nabla)\|_{\mathcal{S}^1 \to \mathcal{S}^{1,\infty}} \|\mathcal{F}(\phi_m)\|_1 \leq \|1 \otimes g(\nabla)\|_{\mathcal{S}^1 \to \mathcal{S}^{1,\infty}} c_{abs} = c_{abs}.$$

This last equality holds since g is a fixed function. We assumed A with integer spectrum, so its spectral decomposition can be written as $A = \sum_{j \in \mathbb{Z}} jp_j$, where p_j are pairwise orthogonal projections that sum to the identity. A is a bounded operator, so p_j is non-zero for only finitely many $j \in \mathbb{Z}$, so this is a finite sum. Now consider a unitary operator $u = \sum_{j \in \mathbb{Z}} p_j \otimes e_{(j,f(j))}$. Without loss of generality we may assume that $\|f'\|_\infty \leq 1$, i.e. the Lipschitz constant is smaller than or equal to 1. For every $m \geq \|A\|_\infty$, we have $|i-j|$, $|f(i) - f(j)| \leq 2m$ for i, j in the spectrum of A. So for these i, j we get

$$(g\phi_m)(i-j, f(i) - f(j)) = g(i-j, f(i) - f(j)) = \frac{f(i) - f(j)}{i-j}$$

It follows from the previous paragraph, and from $\|G_l^{\otimes 2}\|_1 = 1$, that

$$\|(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*)\|_{1,\infty} \leq c_{abs}\|u(V \otimes G_l^{\otimes 2})u^*\|_1 = c_{abs}\|V\|_1. \tag{5.1}$$

We also have

$$(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*) = \sum_{i,j} p_i V p_j \otimes (g\phi_m(\nabla))(G_l^{\otimes 2} e_{(i-j, f(i)-f(j))}).$$

Since there is only a finite number of summands, we have by lemma 4.4 that

$$(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*) - \sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i-j}(G_l^{\otimes 2} e_{(i-j, f(i)-f(j))}) \to 0$$

in $\mathcal{S}^1(B(H) \otimes \mathcal{S}^\infty(\mathbb{R}^2))$ as $l \to \infty$. For the right-hand side term we have

$$\sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i-j}(G_l^{\otimes 2} e_{(i-j, f(i)-f(j))}) =$$

$$(\sum_{k \in \mathbb{Z}} p_k \otimes e_{(k, f(k))}) \cdot (\sum_{i \neq j} p_i V p_j \otimes \frac{f(i) - f(j)}{i-j} G_l^{\otimes 2}) \cdot (\sum_{l \in \mathbb{Z}} p_l \otimes e_{(-l, -f(l))}) = u(T^{A,A}_{f^{[1]}}(V) \otimes G_l^{\otimes 2})u^*$$

Therefore we get

$$(1 \otimes (g\phi_m)(\nabla))(u(V \otimes G_l^{\otimes 2})u^*) - u(T^{A,A}_{f^{[1]}}(V) \otimes G_l^{\otimes 2})u^* \to 0$$

in in $\mathcal{S}^1(B(H) \otimes \mathcal{S}^\infty(\mathbb{R}^2))$ as $l \to \infty$. Combining this with 5.1 we find

$$\limsup_{l\to\infty} \|u(T^{A,A}_{f^{[1]}}(V) \otimes G_l^{\otimes 2})u^*\| \leq c_{abs}\|V\|_1,$$

and since u is a unitary,

$$\limsup_{l\to\infty} \|T^{A,A}_{f^{[1]}}(V) \otimes G_l^{\otimes 2}\| \leq c_{abs}\|V\|_1.$$

The proof is concluded using lemma 5.2 □

6 Weak type estimates for operators with general spectrum

In this chapter, the results of the previous chapters are combined to prove the main theorems announced in the introduction.

Lemma 6.1. ([6], Lemma 5.1) *Let A be a self-adjoint operator in $B(H)$. If $(\xi)_{n\geq 1}$ is a uniformly bounded sequence of Borel functions on $\mathbb{R}^2$ that converges to some limit ξ everywhere, then*

$$T^{A,A}_{\xi_n}(V) \to T^{A,A}_{\xi}(V), \quad V \in \mathcal{S}^2(H)$$

in $\mathcal{S}^2(H)$ as $n \to \infty$.

Proof. Let ν be a projection valued measure on $\mathbb{R}^2$ as defined in (3.9). If $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a Borel measurable bijection, $\nu \circ \gamma$ is a projection valued measure on $\mathbb{R}$. So there is a bounded self-adjoint operator B acting on $\mathcal{S}^2(H)$ such that $E_B = \nu \circ \gamma$. Now let $\eta_n = \xi_n \circ \gamma$ and $\eta = \xi \circ \gamma$. We have $\eta_n \to \eta$ everywhere, so

$$T^{A,A}_{\xi_n} = \int_{\mathbb{R}^2} \xi_n d\nu = \int_{\mathbb{R}^2} \eta_n(\lambda) dE_b(\lambda) = \eta_n(B) \to \eta(B) = \int_{\mathbb{R}^2} \eta(\lambda) dE_b(\lambda) = \int_{\mathbb{R}^2} \xi d\nu = T^{A,A}_{\xi},$$

with convergence in the strong operator topology. Therefor convergence of $T^{A,A}_{\xi_n}(V) \to T^{A,A}_{\xi}(V), \quad V \in \mathcal{S}^2(H)$ follows. □

Now we have all the necessary tools to prove Theorem 1.2, which we will restate here.

Theorem 6.2. ([6], Theorem 1.2) *If $A \in B(H)$ is a self-adjoint operator, and if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then for $V \in (\mathcal{S}^1 \cap \mathcal{S}^2)(H)$ we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|V\|_1.$$

Proof. Let A be a bounded operator. Define, for every $n \geq 1$,

$$A_n := \sum_{k\in\mathbb{Z}} \frac{k}{n} E_A([\frac{k}{n}, \frac{k+1}{n}))$$

$$\xi_n(t,s) = f^{[1]}(\frac{k}{n}, \frac{l}{n})$$

where k and l have to be such that $t \in [\frac{k}{n}, \frac{k+1}{n})$ and $s \in [\frac{l}{n}, \frac{l+1}{n})$. Note that $\xi_n \to f^{[1]}$ everywhere. It is known that $T^{A,A}_{\xi_n} = T^{A_n,A_n}_{f^{[1]}} = T^{nA_n,nA_n}_{(n\sigma_n f)^{[1]}}$. Now we can use Theorem 5.4. We get

$$\|T^{A,A}_{\xi_n}(V)\|_{1,\infty} = \|T^{nA_n,nA_n}_{(n\sigma_n f)^{[1]}}(V)\|_{1,\infty} \leq c_{abs}\|(n\sigma_n f)'\|_\infty \|V\|_1 = c_{abs}\|f'\|_\infty \|V\|_1.$$

Since $\xi_n \to f^{[1]}$ everywhere, we have by Lemma 6.1, for $V \in \mathcal{S}^2(H)$:

$$T^{A,A}_{\xi_n}(V) \to T^{A,A}_{f^{[1]}}(V)$$

in $\mathcal{S}^2(H)$, and thus also in measure, as $n \to \infty$. The Fatou property holds for the $\mathcal{S}^{1,\infty}$-quasi-norm, meaning that since $T^{A,A}_{\xi_n}(V) \to T^{A,A}_{f^{[1]}}(V)$ and $T^{A,A}_{\xi_n}(V)$ bounded, we have

$$\|T^{A,A}_{f^{[1]}}(V)\|_{1,\infty} \leq \liminf_{n\to\infty} \|T^{A,A}_{\xi_n}(V)\|_{1,\infty} = c_{abs}\|f'\|_\infty \|V\|_1,$$

where we need $V \in \mathcal{S}^1(H) \cap \mathcal{S}^2(H)$: $V \in \mathcal{S}^1(H)$ to use Theorem 5.4 and $V \in \mathcal{S}^2(H)$ to use lemma 6.1. For the continuation of the proof, let A now be an arbitrary operator, and let $(A_n)_{n\geq 1}$ be a sequence of

bounded operators defined by $A_n = AE_A([-n,n])$. Since these are bounded we can apply our previous result of this proof:
$$\|T^{A_n,A_n}_{f^{[1]}}(V)\|_{1,\infty} \le c_{abs}\|f'\|_\infty\|V\|_1.$$
From the definition of the double operator integral, we get
$$T^{A_n,A_n}_{f^{[1]}}(V) = E_A([-n,n])T^{A,A}_{f^{[1]}}(V)E_A([-n,n]),$$
which clearly converges to $T^{A,A}_{f^{[1]}}(V)$ in $\mathcal{S}^2(H)$ (and consequentially measure) as $n \to \infty$. Now we can use the Fatou property again:
$$\|T^{A,A}_{f^{[1]}}(V)\|_{1,\infty} \le \liminf_{n\to\infty} T^{A_n,A_n}_{f^{[1]}}(V) \le c_{abs}\|f'\|_\infty\|V\|_1,$$
as required. □

We need one lemma to get Theorem 1.1 from Theorem 1.2.

Lemma 6.3. ([6], Lemma 5.2) *If $A, B \in H$ such that $[A,B] \in \mathcal{S}^2(H)$, then for every Lipschitz continuous function f we have*
$$T^{A,A}_{f^{[1]}}([A,B]) = [f(A),B].$$

Proof. By definition of the double operator integral,
$$T^{A,A}_{\xi_1}T^{A,A}_{\xi_2} = T^{A,A}_{\xi_1\xi_2} \tag{6.1}$$
Let $\xi_1 = f^{[1]}$ and
$$\xi_2 = \begin{cases} \lambda - \mu & |\lambda|, |\mu| \le \|A\|_\infty \\ 0 & \text{otherwise} \end{cases}$$
If p is a finite rank projection, then $pB \in \mathcal{S}^2(H)$, and
$$T^{A,A}_{\xi_1\xi_2}(pB) = f(A)pB - pBf(A), \quad T^{A,A}_{\xi_2}(pB) = ApB - pBA.$$
Applying (6.1) to pB, we get
$$T^{A,A}_{\xi_1}(ApB - pBA) = f(A)pB - pBf(A).$$
Now, using Propostion 6 of [11], we construct a sequence $(p_{n,k})_{n\ge1}$ of finite rank projections such that $p_{n,k} \to 1$ increasing as $k \to \infty$, and such that $\|[nA, p_{n.k}]\|_2 \le 1$. Let $(\eta_m)m \ge 1$ be an orthonormal basis in $\mathcal{S}^2(H)$. Fix k_n large enough that
$$\|(1 - p_{n,k_n})\eta_m\|_2 \le \frac{1}{n}, \quad 0 \le m < n,$$
and set $q_n = p_{n,k_n}$. Then it follows that $q_n \to 1$ in the strong operator topology as $n \to \infty$. We also have $[A, q_n] \to 0$ in $\mathcal{S}^2(H)$. From this construction we now have
$$Aq_nB - q_nBA = [A,q_n]B + q_n[A,B] \to [A,B]$$
as $n \to \infty$ in $\mathcal{S}^2(H)$. Since $T^{A,A}_{f^{[1]}}$ is bounded, we have
$$f(A)q_nB - q_nBf(A) = T^{A,A}_{f^{[1]}}(Aq_nB - q_nBA) \to T^{A,A}_{f^{[1]}}([A,B])$$
as $n \to \infty$ in $\mathcal{S}^2(H)$. We also have
$$f(A)q_nB - q_nBf(A) \to [f(A),B]$$
as $n \to \infty$ in the strong operator topology. So we have two limits of $f(A)q_nB - q_nBf(A)$, which must agree, so we get the desired result:
$$T^{A,A}_{f^{[1]}}([A,B]) = [f(A),B].$$
□

Now we can combine this lemma with theorem 6.1 to prove theorem 1.1.

Theorem 6.4. ([6], Theorem 5.3) *For all self-adjoint operators $A, B \in B(H)$ with $[A, B] \in \mathcal{S}^1(H)$ and for every Lipschitz continuous function f we have*

$$\|[f(A), B]\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|[A, B]\|_1.$$

Additionally, for self adjoint operators $X, Y \in H$ with $X - Y \in \mathcal{S}^1(H)$ and for every Lipschitz continuous function f we have

$$\|f(X) - f(Y)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|X - Y\|_1.$$

Proof. The first result is obtained by combining Theorem 1.2 and Lemma 6.3. The second result can be obtained by taking the first result and

$$A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

□

Interpolation

In this section, we will use interpolation to generalise the result of Theorem 6.2 to Schatten classes $\mathcal{S}^p$, $1 \leq p \leq \infty$. To do so, we prove the statement of Theorem 6.2 for $p = 2$ instead of $p = 1$, and use interpolation and duality to get the required result.

Theorem 6.5. *If $A \in B(H)$ is a self-adjoint operator, and if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then for $V \in \mathcal{S}^2(H)$ we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_{2,\infty} \leq c_{abs}\|f'\|_\infty \|V\|_2.$$

Proof. Note that if f is is a Lipschitz function, $f^{[1]}$ is defined by

$$f^{[1]}(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \lambda \neq \mu, \\ 0, & \lambda = \mu. \end{cases}$$

The Lipschitz constant is by definition the supremum of $(f(\lambda) - f(\mu))/(\lambda - \mu)$. Hence $f^{[1]}$ is bounded by the Lipschitz constant $\|f'\|_\infty$. By equation (3.7) we see that all entries of the matrix representation of $T^{A,A}_{f^{[1]}}$ are bounded by $\|f'\|_\infty$. This implies, by the final result of section 3.1, that $T^{A,A}_{f^{[1]}}$ acts as a linear Schur multiplier $\mathcal{S}^2 \to \mathcal{S}^2$, with operator norm $\|f'\|_\infty$. So now we have

$$\|T^{A,A}_{f^{[1]}}(V)\|_{2,\infty} = \|T^{A,A}_{f^{[1]}}(V)\|_2 \leq \|f'\|_\infty \|V\|_2.$$

This proves the theorem, and shows that $c_{abs} \leq 1$. □

Theorem 6.6. *If $A \in B(H)$ is a self-adjoint operator, and if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then for any $1 < p < \infty$, $V \in \mathcal{S}^p(H)$ we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_p \leq c_p\|f'\|_\infty \|V\|_p.$$

Note that c_p depends on p, but not on the dimension of H.

Proof. The Schatten class-equivalent of the Hadamard three lines lemma (see [1]), which is necessary for interpolation, was first proven in [13]. As shown in [15], Proposition D.3.1, this allows us to use Theorems 6.2 and 6.5 to generalise 6.5 for all $1 < p < 2$. Take $0 < \theta < 1$ such that $1/p = (1 - \theta)/1 + \theta/2$, so $\theta = 2 - 2/p$. Then we have, for the interpolation space,

$$[\mathcal{S}^1(H), \mathcal{S}^2(H)]_\theta = \mathcal{S}^p$$

isometrically. This concludes the proof for $1 < p < 2$. Recall that by duality, we can extend to the Schatten classes $\mathcal{S}^{p'}$, where $1/p + 1/p' = 1$, isometrically (see section 3.1), which concludes the proof. □

7 Optimality of the upper bounds for $\mathcal{S}^1$

This chapter presents an older result, first shown in [9], which shows that the upper bound we have found for $\mathcal{S}^1$ is optimal.

Lemma 7.1. ([9], Lemma 10) *Take $n \geq 1$. There are $\lambda_1, ..., \lambda_n > 0$ such that the matrix $A \in M^n$, defined by*

$$A_{ij} := \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j},$$

satisfies the inequality

$$\|A\|_1 \geq kn \log n.$$

Proof. Define $A^{(m)}$ by taking $\lambda_i = m^i$, so

$$A_{ij}^{(m)} = \frac{m^i - m^j}{m^i + m^j}.$$

Then $\lim_{m\to\infty} A_{ij}^{(m)}$ is dominated by the highest power in the fraction, so we can define matrix B by

$$B_{ij} := \lim_{m\to\infty} A_{ij}^{(m)} = \begin{cases} 1 & i > j \\ 0 & i = j \\ -1 & i < j \end{cases}.$$

We then have $\lim_{m\to\infty} \|A^{(M)}\|_1 = \|B\|_1$. So it remains to be shown that $\|B\|_1 \geq 2kn \log n$. Define $\omega_r := e^{\pi i(2r-1)/n}$. Then the eigenvalues of B are

$$\mu_r = \frac{1 + \omega_r}{1 - \omega_r},$$

and the corresponding eigenvectors are

$$x_r = (1, \omega_r, \omega_r^2, ..., \omega_r^{n-1}).$$

This gives singular values

$$\sigma_r = |\cot \frac{\pi(2r-1)}{2n}|,$$

so

$$\|B\|_1 = \sum_{r=1}^{n} |\cot \frac{\pi(2r-1)}{2n}| = \mathcal{O}(n \log n).$$

□

This shows that the norm of the Schur multiplier defined by A does not permit an upper bound independent of n. More generally, it shows that there is no possible bound of the form

$$\|T_{f^{[1]}}^{A,A}(V)\|_1 \leq c_{abs}\|f'\|_\infty \|V\|_1,$$

and we indeed must consider the weak results as presented in the previous chapter. Indeed, lower bounds can be found, showing a growth with $\log n$.

Theorem 7.2. ([9], Theorem 12) *For $n \geq 1$ there are self-adjoint $2n \times 2n$ matrices A and B such that $[A, B] \neq 0$ and*

$$\|[|A|, B]\|_1 \geq k \log n \|[A, B]\|_1,$$

where $k > 0$ is the constant from Lemma 7.1

Proof. Let C be an $n \times n$ matrix with

$$C_{ij} = \frac{1}{\lambda_i + \lambda_j}$$

for $1 < i, j < n$ and λ_i as in Lemma 7.1. Then

$$A'_{ij} = (\lambda_i - \lambda_j) C_{ij},$$

where A' is A from Lemma 7.1. Then for $\|(\lambda_i + \lambda_j)C_{ij}\|_1$, meaning the Schatten p-norm for p=1 of the matrix with entries $(\lambda_i + \lambda_j)C_{ij}$, we have

$$\|(\lambda_i + \lambda_j)C_{ij}\|_1 = \sum_{i=1}^{n} (\lambda_i + \lambda_i) C_{ii} = n.$$

But, using the same notation,

$$\|(\lambda_i - \lambda_j)C_{ij}\|_1 = \|A'\|_1 \geq kn \log n.$$

Let D be the matrix with entries

$$D_{ij} = \lambda_i \delta_{ij}.$$

Then $\|DC + CD\|_1 = \|\lambda_i C_{ij} + C_{ij}\lambda_j\|_1 = n$ and $\|DC - CD\|_1 = \|\lambda_i C_{ij} - C_{ij}\lambda_j\|_1 = \|A'\|_1 \geq kn \log n$. Now we can define A and B:

$$A = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \quad B = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}.$$

Then

$$[A, B] = \begin{pmatrix} 0 & DC + CD \\ -(DC + CD) & 0 \end{pmatrix},$$

so $\|[A, B]\|_1 = 2n$. Furthermore,

$$[|A|, B] = \begin{pmatrix} 0 & DC - CD \\ DC - CD & 0 \end{pmatrix}$$

so $\|[|A|, B]\|_1 = 2\|DC - CD\|_1 \geq 2kn \log n = k \log n \|[A, B]\|_1$. □

Theorem 7.3. ([9], Theorem 13) *For $n \geq 1$ there are self-adjoint $2n \times 2n$ matrices A and C s.t. $A \neq C$ and*

$$\||A| - |C|\|_1 > \frac{1}{2} k \log n \|A - C\|_1.$$

Proof. Take A, B as in Theorem 7.2. Define $C := e^{i\epsilon B} A e^{-i\epsilon B}$ with $\epsilon \in \mathbb{R}$. If ϵ is small enough we get

$$\|C - A\|_1 = \|i\epsilon[B, A] + \mathcal{O}(\epsilon^2)\|_1 \leq 2|\epsilon| \|[B, A]\|_1.$$

We similarly get

$$\||C| - |A|\|_1 = \|i\epsilon[B, |A|] + \mathcal{O}(\epsilon^2)\|_1 \geq \frac{1}{2}|\epsilon| \|[B, |A|]\|_1.$$

So using Theorem 7.2 we get

$$\||A| - |C|\|_1 > \frac{1}{2} k \log n \|C - A\|_1.$$

□

However, an upper bound is also possible, which also grows with $\log n$. This shows that we have found the best possible lower bound.

Theorem 7.4. ([9], Theorem 14) *There is an absolute constant $k_1 > 0$ such that, with A, C self-adjoint $n \times n$ matrices and $n \geq 2$,*

$$\||A| - |C|\|_1 \leq k_1 \log n \|A - C\|_1$$

Proof. We can use the following estimate: if $1 < p \leq 2$ and $p^{-1} + q^{-1} = 1$, we can use Hölder's inequality to show that

$$\||A| - |C|\|_1 \leq \||A| - |C|\|_p \|1\|_q.$$

Then by Theorem 8 of [9] (which is a corollary of the results of chapters 5 and 6),

$$\||A| - |C|\|_p \|1\|_q \leq 2(1 - \frac{cp}{p-1}) n^{1/q} \|A - C\|_p.$$

By our interpolation result, Theorem 6.6, we have

$$2(1 - \frac{cp}{p-1}) n^{1/q} \|A - C\|_p \leq k_2 (1 - p^{-1}) n^{1-1/p} \|A - C\|_1.$$

If, for $n \geq 8$, we let $p^{-1} = 1 - (\log n)^{-1}$, we get

$$\||A| - |C|\|_1 \leq k_1 \log n \|A - C\|_1.$$

For $2 \leq n < 8$ we can embed the matrices into M_8 and deduce the same result, although we will get a different constant k_1. □

8 Generalisation of the lower bound for $\mathcal{S}^1$

In this chapter we generalise the Theorem 7.3 to operator integrals. Define $f_0 : \mathbb{R} \to \mathbb{R}$ by $f_0(x) = |x|$. We need an equivalent statement to Theorem 7.3 to start working towards a more general statement.

Theorem 8.1. ([7], Theorem 17) *For all $n \in \mathbb{N}$ there are self-adjoint operators $A_n, B_n \in B(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide, 0 is an eigenvalue of A_n, and*

$$\|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq \mathrm{const}\log \mathrm{n}\|\mathrm{B_n}\|_1$$

Proof. Given the matrices $A, C \in \mathbb{C}^n$ from Theorem 7.3, we can construct $A_n, B_n \in \mathbb{C}^{2n+1}$ such that all claims of this theorem hold. Take $A'_n = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \in \mathbb{C}^{2n}$ and $B'_n = \begin{pmatrix} C-A & 0 \\ 0 & A-C \end{pmatrix} \in \mathbb{C}^{2n}$. Clearly the spectra of $A'_n + B'_n$ and A'_n coincide. For these matrices, we have

$$\|f_0(A'_n + B'_n) - f_0(A'_n)\|_1 = \|f_0(\begin{pmatrix} C & 0 \\ 0 & A \end{pmatrix}) - f_0(\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix})\|_1 = \| \begin{pmatrix} |C| & 0 \\ 0 & |A| \end{pmatrix} - \begin{pmatrix} |A| & 0 \\ 0 & |C| \end{pmatrix} \|_1 =$$

$$\||C| - |A|\|_1 + \||A| - |C|\|_1.$$

Now we can use Theorem 7.3:

$$\||C| - |A|\|_1 + \||A| - |C|\|_1 > k \log n \|A - C\|_1 = \mathrm{const}\log n \|B'_n\|_1.$$

To satisfy the last requirement, we need 0 to be an eigenvalue. By extending the dimension of our matrices to $2n+1$, and adding a row and column of only zeroes, we satisfy that requirement while all the above equations still hold. □

Here we introduce the first and second order divided-difference functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous functions that admits left- and right derivatives, $f'_l(x)$ and $f'_r(x)$ at all $x \in \mathbb{R}$. Assume these derivatives are bounded. Then we define

$$f^{[1]}(x_0, x_1) = \begin{cases} \frac{f(x_0) - f(x_1)}{x_0 - x_1}, & \text{if } x_0 \neq x_1 \\ \frac{f'_l(x_0) + f'_r(x_0)}{2}, & \text{if } x_0 = x_1 \end{cases}, \quad x_0, x_1 \in \mathbb{R}$$

The second order divided difference function can be defined using the following formula, but only for C^2 functions:

$$f^{[2]}(x_0, x_1, x_2) := \begin{cases} \frac{f^{[1]}(x_0, x_1) - f^{[1]}(x_1, x_2)}{x_0 - x_2}, & \text{if } x_0 \neq x_2 \\ \frac{d}{dx_0} f^{[1]}(x_0, x_1), & \text{if } x_0 = x_2 \end{cases} \tag{8.1}$$

For the operator integral of this function, we have the following result, as presented in [7]:

$$T^{A,B}_{f^{[1]}}(A - B) = f(A) - f(B) \tag{8.2}$$

Now we will show the general result for operator integrals.

Corollary 8.2. ([7], Corollary 19) *For all $n \geq 1$, there are self-adjoint operators $A_n, B_n \in B(\mathbb{C}^{2n+1})$ such that the spectra of $A_n + B_n$ and A_n coincide and*

$$\|T^{A_n+B_n, A_n}_{f_0^{[1]}} : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\| \geq \mathrm{const}\log n$$

Proof. Take A_n, B_n as in Theorem 8.1. By Equation (8.2): $T^{A_n+B_n,A_n}_{f_0^{[1]}}(B_n) = f_0(A_n + B_n) - f_0(A_n)$. By Theorem 8.1: $\|T^{A_n+B_n,A_n}_{f_0^{[1]}}(B_n)\|_1 = \|f_0(A_n + B_n) - f_0(A_n)\|_1 \geq \text{const} \log n \|B_n\|_1$, so $\|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^1_{2n+1} \to \mathcal{S}^1_{2n+1}\| \geq \text{const} \log n$. By duality we also have $\|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^\infty_{2n+1} \to \mathcal{S}^\infty_{2n+1}\| \geq \text{const} \log n$. □

Let $g_0 : \mathbb{R} \to \mathbb{R}$ be given by $g_0(x) = x|x|$. This is clearly not a C^2 function, but we can still use formula (8.1) when x_0, x_1 and x_2 have at least two distinct values. We can define the following bounded Borel function, $\psi_0 : \mathbb{R}^3 \to \mathbb{C}$ by

$$\psi_0(x_0, x_1, x_2) := \begin{cases} g_0^{[2]}(x_0, x_1, x_2) & \text{if } x_0 \neq x_1 \text{ or } x_1 \neq x_2 \\ 2 & \text{if } x_0 = x_1 = x_2 > 0 \\ -2 & \text{if } x_0 = x_1 = x_2 < 0 \\ 0 & \text{if } x_0 = x_1 = x_2 = 0 \end{cases}. \tag{8.3}$$

Now follows arguably the most important step in this chapter, which relates the double operator integral of $f_0^{[1]}$ to the triple operator integral of ψ_0.

Lemma 8.3. ([7], Lemmma 20) *For self-adjoint operators $A_n, B_n \in B(\mathbb{C}^n)$ with 0 in their spectrum, the following inequality holds:*

$$\|T^{A_n+B_n,A_n,A_n}_{\psi_0} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\| \geq \|T^{A_n+B_n,A_n}_{f_0^{[1]}} : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\| \tag{8.4}$$

Proof. Denote the sequence of eigenvalues of operator A_n by $\{\mu_k\}_{k=1}^n$. Without loss of generality, we assume $\mu_1 = 0$. Similarly, let $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of $A_n + B_n$. For this proof, we need the results from chapter 3. Combining equations (3.8) and (3.12) and Theorem 3.7, we have

$$\|T^{A_n+B_n,A_n,A_n}_{\psi_0} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\| = \|\{\psi_0(\lambda_i, \mu_j, \mu_k)\}_{i,j,k=1}^n : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\|$$
$$= \sup_{1\leq j\leq n} \|\{\psi_0(\lambda_i, \mu_j, \mu_k)\}_{i,k=1}^n : M_n \to M_n\| = \max_{1\leq j\leq n} \|\{\phi_j(\lambda_i, \mu_k)\}_{i,k=1}^n : M_n \to M_n\|$$
$$= \max_{1\leq j\leq n} \|T^{A_n+B_n,A_n}_{\phi_j} : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\|.$$

For the final step, recall that $(\mathcal{S}^1_n)^*$ is isometrically isomorphic to M_n and $\mathcal{S}^\infty_n$. Here ϕ_j is defined by the formula

$$\phi_j(x_0, x_1) := \psi_0(x_0, \mu_j, x_1), \quad x_0, x_1 \in \mathbb{R}, j \in \{1, ..., n\}.$$

Fixing $j = 1$ instead of taking the maximum on the right hand side, we get

$$\|T^{A_n+B_n,A_n,A_n}_{\psi_0} : \mathcal{S}^2_n \times \mathcal{S}^2_n \to \mathcal{S}^1_n\| \geq \|T^{A_n+B_n,A_n}_{\phi_1} : \mathcal{S}^\infty_n \to \mathcal{S}^\infty_n\|.$$

It remains to be shown that

$$\phi_1 = f_0^{[1]}.$$

Using the definitions in this chapter, we can check

$$\phi_1(0, 0) = \psi_0(0, \mu_1, 0) = \psi_0(0, 0, 0) = 0 = f_0^{[1]}(0, 0).$$

Now take $x_0, x_1 \in \mathbb{R}$ such that at least one is not equal to zero. Then

$$\phi_1(x_0, x_1) = \psi_0(x_0, 0, x_1) = g_0^{[2]}(x_0, 0, x_1).$$

If neither x_0 nor x_1 is equal to zero, we have

$$g_0^{[2]}(x_0, 0, x_1) = \frac{g_0^{[1]}(x_0, 0) - g_0^{[1]}(0, x_1)}{x_0 - x_1} = \frac{\frac{x_0 f_0(x_0) - 0}{x_0 - 0} - \frac{0 - x_1 f_0(x_1)}{0 - x_1}}{x_0 - x_1} =$$

$$\frac{f_0(x_0) - f_0(x_1)}{x_0 - x_1} = f_0^{[1]}(x_0, x_1).$$

If either x_0 or x_1 is equal to zero, we can use the following calculation. We assume $x_0 = 0$, but the calculation is similar when $x_1 = 0$. So, assuming $x_0 = 0$ and $x_1 \neq 0$:

$$g_0^{[2]}(0, 0, x_1) = \frac{g_0^{[1]}(0, 0) - g_0^{[1]}(0, x_1)}{0 - x_1} = \frac{g_0'(0) - \frac{0 - x_1 f_0(x_1)}{0 - x_1}}{0 - x_1} = \frac{f_0(x_1)}{x_1} = f_0^{[1]}(0, x_1).$$

The last case to consider is $x_0 = x_1 \neq 0$.

$$g_0^{[2]}(x_0, 0, x_0) = \frac{d}{dx} g_0^{[1]}(x, 0)|_{x=x_0} = \frac{d}{dx}(\frac{x f_0(x) - 0}{x - 0})|_{x=x_0} = f_0'(x_0) = f_0^{[1]}(x_0, x_0).$$

So indeed, we have $\phi_1 = f_0^{[1]}$, proving the lemma. □

Now we can combine Corollary 8.2 and Lemma 8.3 to get our first lower bound for the norm of the triple operator integral $T_{\psi_0}^{A_n+B_n,A_n,A_n}$.

Corollary 8.4. ([7], Corollary 21) *For any $n \geq 1$ one can find self-adjoint operators $A_n, B_n \in B(\mathbb{C}^{2n+1})$ such that $\sigma(A_n + B_n) = \sigma(A_n)$ (the spectra of $A_n + B_n$ and A_n coincide) and*

$$\|T_{\psi_0}^{A_n+B_n,A_n,A_n} : \mathcal{S}^2_{2n+1} \times \mathcal{S}^2_{2n+1} \to \mathcal{S}^1_{2n+1}\| \geq \mathit{const} \log n$$

Now follows a series of lemmas, the goal of which is to prove Lemma 8.11. For this entire series, take $n \geq 1$ and A_n, B_n fixed such that Corollary 8.4 holds. The first lemma is an immediate result of Corollary 8.4, and as such requires no proof.

Lemma 8.5. ([7], Lemma 22) *There are operators $X_n, Y_n \in B(\mathbb{C}^{2n+1})$, with $\|X_n\|_2 = \|Y_n\|_2 = 1$, such that*

$$\|T_{\psi_0}^{A_n+B_n,A_n,A_n}(X_n, Y_n)\|_1 \geq \mathit{const} \log n$$

We now define, using our already fixed A_n, B_n, an operator $H_n \in B(\mathbb{C}^{4n+2})$:

$$H_n := \begin{pmatrix} A_n + B_n & 0 \\ 0 & A_n \end{pmatrix} \in B(\mathbb{C}^{4n+2}). \tag{8.5}$$

Using H_n, we define the operator T_1:

$$T_1 := T_{\psi_0}^{H_n,H_n,H_n} : \mathcal{S}^2_{4n+2} \times \mathcal{S}^2_{4n+2} \to \mathcal{S}^1_{4n+2}.$$

The next lemma proves a result similar to that of Corollary 8.4 for the operator T_1.

Lemma 8.6. ([7], Lemma 23) *There are operators $\tilde{X}_n, \tilde{Y}_n \in B(\mathbb{C}^{4n+2})$, with $\|\tilde{X}_n\|_2 = \|\tilde{Y}_n\|_2 = 1$, such that*

$$\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 \geq \mathit{const} \log n$$

Proof. We can use Lemma 8.5 to find $\tilde{X}_n, \tilde{Y}_n$. Take $X_n, Y_n \in B(\mathbb{C}^{2n+1})$ as in Lemma 8.5, and define

$$\tilde{X}_n := \begin{pmatrix} 0 & X_n \\ 0^{2n+1} & 0 \end{pmatrix}, \tilde{Y}_n := \begin{pmatrix} 0^{2n+1} & 0 \\ 0 & Y_n \end{pmatrix},$$

with 0_{2n+1} the null-element of $B(\mathbb{C}^{2n+1})$. From the definition of the Schatten norm, it is clear that $\|\tilde{X}_n\| = \|X_n\| = 1$ and $\|\tilde{Y}_n\| = \|Y_n\| = 1$. Since $A_n + B_n$ and A_n have the same spectrum, we can apply Lemma 3.10:

$$T_1(\tilde{X}_n, \tilde{Y}_n) = \begin{pmatrix} 0 & T_{\psi_0}^{A_n+B_n,A_n,A_n}(X_n, Y_n) \\ 0^{2n+1} & 0 \end{pmatrix}.$$

We can conclude, using Lemma 8.5,

$$\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 = \|T_{\psi_0}^{A_n+B_n,A_n,A_n}(X_n, Y_n)\|_1 \geq \text{const} \log n.$$

□

The next lemma is very similar, but now we only need one operator and its adjoint. However, instead of this being a norm-one operator, we can only guarantee that it is a contraction under the Schatten 2-norm.

Lemma 8.7. ([7], Lemma 24) *There is an operator $S_n \in B(\mathbb{C}^{4n+2})$, with $\|S_n\|_2 \leq 1$, such that*

$$\|T_1(S_n, S_n^*)\|_1 \geq \mathit{const} \log n$$

Proof. Take $\tilde{X}, \tilde{Y} \in B(\mathbb{C}^{4n+2})$ as in 8.6. Since T_1 is a bilinear operator which maps into $\mathbb{C}$, we can use the following polarisation identity:

$$T_1(\tilde{X}_n, \tilde{Y}_n) = \frac{1}{4}\sum_{k=0}^{3} i^k T_1((\tilde{X}_n + i^k \tilde{Y}_n^*), (\tilde{X}_n + i^k \tilde{Y}_n^*)^*).$$

We can now set an upper bound for the trace norm:

$$\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 = \frac{1}{4}\|\sum_{k=0}^{3} i^k T_1((\tilde{X}_n + i^k \tilde{Y}_n^*), (\tilde{X}_n + i^k \tilde{Y}_n^*)^*)\|_1$$

$$\leq \frac{1}{4}\sum_{k=0}^{3} \|i^k T_1((\tilde{X}_n + i^k \tilde{Y}_n^*), (\tilde{X}_n + i^k \tilde{Y}_n^*)^*)\|_1$$

$$\leq \max_{0\leq k\leq 3} \|T_1((\tilde{X}_n + i^k \tilde{Y}_n^*), (\tilde{X}_n + i^k \tilde{Y}_n^*)^*)\|_1.$$

Take k_0 such that this maximum is achieved, and set

$$S_n = \frac{1}{2}(\tilde{X}_n + i^{k_0} \tilde{Y}_n^*).$$

Note that the first requirement of S_n is satisfied:

$$\|S_n\|_2 = \frac{1}{2}\|\tilde{X}_n + i^{k_0}\tilde{Y}_n^*\|_2 \leq \frac{1}{2}(\|\tilde{X}_n\|_2 + \|i^{k_0}\tilde{Y}_n^*\|_2) = \frac{1}{2}(\|\tilde{X}_n\|_2 + \|\tilde{Y}_n\|_2) = 1.$$

Now we prove the main inequality:

$$\|T_1(S_n, S_n^*)\|_1 = \frac{1}{4}\|T_1((\tilde{X}_n + i^{k_0}\tilde{Y}_n^*), (\tilde{X}_n + i^{k_0}\tilde{Y}_n^*)^*)\|_1 \geq \frac{1}{4}\|T_1(\tilde{X}_n, \tilde{Y}_n)\|_1 \geq \text{const} \log n.$$

□

Now we again double the dimension of our space, to $\mathbb{C}^{8n+4}$, which allows us to prove a similar result for bounded self-adjoint operators. Define $\tilde{H}_n \in B(\mathbb{C}^{8n+4})$ using H_n as in equation 8.5:

$$\tilde{H}_n := \begin{pmatrix} H_n & 0 \\ 0 & H_n \end{pmatrix} = \begin{pmatrix} A_n + B_n & 0 & 0 & 0 \\ 0 & A_n & 0 & 0 \\ 0 & 0 & A_n + B_n & 0 \\ 0 & 0 & 0 & A_n \end{pmatrix} \tag{8.6}$$

and define the operator

$$T_2 = T_{\psi_0}^{\tilde{H}_n, \tilde{H}_n, \tilde{H}_n} : \mathcal{S}^2_{8n+4} \times \mathcal{S}^2_{8n+4} \to \mathcal{S}^1_{8n+4}. \tag{8.7}$$

Now we can prove the next inequality.

Lemma 8.8. ([7], Lemma 25) *There is a self-adjoint operator $Z_n \in B(\mathbb{C}^{8n+4})$ such that $\|Z_n\|_2 \leq 1$ and*

$$\|T_2(Z_n, Z_n)\|_1 \geq \mathit{const} \log n.$$

Proof. An operator that satisfies the requirements is

$$Z_n := \frac{1}{2}\begin{pmatrix} 0 & S_n \\ S_n^* & 0 \end{pmatrix}.$$

Firstly, we have

$$\|Z_n\|_2 = \frac{1}{2}(\|S_n\|_2 + \|S_n^*\|_2) = \|S_n\|_2 \leq 1.$$

Additionally, by Lemma 3.9, we have

$$T_2(Z_n, Z_n) = \frac{1}{4}\begin{pmatrix} T_1(S_n, S_n^*) & 0 \\ 0 & T_1(S_n^*, S_n) \end{pmatrix}.$$

Now we can calculate the trace norm, by applying Lemma 8.7, concluding the proof:

$$\|T_2(Z_n, Z_n)\| = \frac{1}{4}(\|T_1(S_n, S_n^*)\|_1 + \|T_1(S_n^*, S_n)\|_1) \geq \frac{1}{4}\|T_1(S_n, S_n^*)\|_1 \geq \text{const} \log n.$$

□

Next, we need Lemma 8.9 which proves a decomposition that we need to continue. Afterwards, we continue with our series of lower bounds on norms of operator integrals. We use the following notation for the commutator of two operators H, F:

$$[H, F] := HF - FH$$

Lemma 8.9. ([7], Lemma 26) *For any pair of self-adjoint operators $Z, H \in B(\mathbb{C}^n)$ one can find a pair of self-adjoint operators $F, G \in B(\mathbb{C}^n)$ such that*

$$Z = G + i[H, F], \tag{8.8}$$

$$[G, H] = 0, \tag{8.9}$$

$$\|[H, F]\|_2 \leq 2\|Z\|_2, \tag{8.10}$$

and

$$\|G\|_2 \leq \|Z\|_2. \tag{8.11}$$

Proof. Denote the distinct eigenvalues of H by $\{h_j\}_{j=1}^{n_0}$. For h_j, denote the associated spectral projection by E_j, as a short-hand for $\{E_H(\{h_j\})\}$. By definition, we have

$$H = \sum_{j=1}^{n_0} h_j E_j, \quad HE_j = h_j E_j.$$

We can now define F and G:

$$G = \sum_{j=1}^{n_0} E_j Z E_j,$$

$$F = \sum_{j \in \{1,\ldots,n_0\} \setminus \{k\}} i(h_k - h_j)^{-1} E_j Z E_k,$$

and since these are clearly bounded and self-adjoint, it remains to be shown that equations (8.8)-(8.11) all hold. Since $HE_j = h_j E_j = E_j H$, we have

$$[H, E_j Z E_k] = HE_j Z E_k - E_j Z E_k H = h_j E_j Z E_k - E_j Z E_k h_k = (h_j - h_k) E_j Z E_k.$$

Using linearity, we calculate

$$i[H,F] = -\sum_{j\in\{1,\dots,n_0\}\setminus\{k\}} (h_k - h_j)^{-1}[H, E_j Z E_k]$$

$$= -\sum_{j\in\{1,\dots,n_0\}\setminus\{k\}} (h_k - h_j)^{-1}(h_j - h_k) E_j Z E_k = \sum_{j\in\{1,\dots,n_0\}\setminus\{k\}} E_j Z E_k.$$

Consequentially, equation (8.8) holds:

$$G + i[H,F] = \sum_{j=1}^{n_0} E_j Z E_j + \sum_{j\in\{1,\dots,n_0\}\setminus\{k\}} E_j Z E_k = \sum_{j=1}^{n_0}\sum_{k=1}^{n_0} E_j Z E_k = Z.$$

(8.9) is clear from a simple calculation:

$$GH = \sum_{j=1}^{n_0} E_j Z E_j \sum_{k=1}^{n_0} h_k E_k = \sum_{j=1}^{n_0} E_j Z E_j h_j = \sum_{j=1}^{n_0} h_j E_j Z E_j = \sum_{k=1}^{n_0} h_k E_k \sum_{j=1}^{n_0} E_j Z E_j = HG.$$

To prove (8.11) take

$$U_t = \sum_{j=1}^{n_0} e^{ijt} E_j, \quad -\pi \le t \le \pi.$$

Note that i is not an index, but the imaginary unit. Then

$$\int_{-\pi}^{\pi} U_t Z U_t^* \frac{\mathrm{d}t}{2\pi} = \int_{-\pi}^{\pi} (\sum_{j=1}^{n_0} e^{ijt} E_j) Z (\sum_{k=1}^{n_0} e^{ikt} E_k)^* \frac{\mathrm{d}t}{2\pi}$$

$$= \int_{-\pi}^{\pi} (\sum_{j=1}^{n_0} e^{ijt} E_j) Z (\sum_{k=1}^{n_0} e^{-ikt} E_k) \frac{\mathrm{d}t}{2\pi} = \sum_{j,k=1}^{n_0} E_j Z E_k \int_{-\pi}^{\pi} e^{i(j-k)t} \frac{\mathrm{d}t}{2\pi}.$$

Here we have used that projections are self-adjoint. Clearly, the integral is zero unless $j = k$, in which case it is equal to one. Hence we get

$$\int_{-\pi}^{\pi} U_t Z U_t^* \frac{\mathrm{d}t}{2\pi} = \sum_{j,k=1}^{n_0} E_j Z E_k \delta_{j,k} = \sum_{j=1}^{n_0} E_j Z E_j = G.$$

Now, using the fact that U_t is unitary, we can calculate

$$\|G\|_2 = \| \int_{-\pi}^{\pi} U_t Z U_t^* \frac{\mathrm{d}t}{2\pi} \|_2 \le \int_{-\pi}^{\pi} \|U_t Z U_t^*\|_2 \frac{\mathrm{d}t}{2\pi} = \int_{-\pi}^{\pi} \|Z\|_2 \frac{\mathrm{d}t}{2\pi} = \|Z\|_2,$$

proving equation (8.11). By (8.8) we have

$$i[H,F] = Z - G.$$

Now we can prove the last equation, (8.10):

$$\|[H,F]\|_2 = \|Z - G\|_2 \le \|Z\|_2 + \|G\|_2 \le 2\|Z\|_2.$$

□

For the next proof, we once again use $\tilde{H}_n \in B(\mathbb{C}^{8n+4})$ as defined in (8.6) and T_2 as defined in (8.7).

Lemma 8.10. ([7], Lemma 27) *One can find a self-adjoint operator $F_n \in B(\mathbb{C}^{8n+4})$ such that*

$$\|[\tilde{H}_n, F_n]\|_2 \leq 2 \quad \textit{and} \quad \|T_2(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n])\|_1 \geq \textit{const}\log n - 10.$$

Proof. Take $Z_n \in B(\mathbb{C}^{8n+4})$ as in Lemma 8.8. Note that Z_n is self-adjoint, as is $\tilde{H}_n$. As such, we can apply Lemma 8.9 to find operators $F_n, G_n \in B(\mathbb{C}^{8n+4})$ such that

$$Z_n = G_n + i[\tilde{H}_n, F_n], \tag{8.12}$$

$$[G_n, \tilde{H}_n] = 0, \tag{8.13}$$

$$\|[\tilde{H}_n, F_n]\|_2 \leq 2\|Z_n\|_2, \tag{8.14}$$

and

$$\|G_n\|_2 \leq \|Z_n\|_2. \tag{8.15}$$

Since $\|Z_n\|_2 \leq 1$ by Lemma 8.8, equation (8.14) shows the first claim of this lemma: $\|[\tilde{H}_n, F_n]\|_2 \leq 2\|Z_n\|_2 \leq 2$.

We continue by calculating $T_2(Z_n, Z_n)$. By using (8.12) and linearity, we split this expression into four terms, as follows:

$$\begin{aligned} T_2(Z_n, Z_n) &= T_2(G_n + i[\tilde{H}_n, F_n], G_n + i[\tilde{H}_n, F_n]) \\ &= T_2(G_n, G_n) \\ &+ T_2(G_n, i[\tilde{H}_n, F_n]) \\ &+ T_2(i[\tilde{H}_n, F_n], G_n) \\ &+ T_2(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n]). \end{aligned} \tag{8.16}$$

The last of these is the one we are interested in. As such, we shall find estimates for the first three to prove this lemma. First note that, since (8.13) holds, we can apply Lemma 3.13 to get

$$T_2(G_n, G_n) = \widetilde{\psi_0}(\tilde{H}_n) \times G_n^2.$$

By definition

$$\widehat{\psi_0}(x) = \psi_0(x, x, x) \in \{-2, 0, 2\}.$$

Hence

$$\|\widehat{\psi_0}(\tilde{H}_n)\|_\infty \leq 2,$$

which implies

$$\|T_2(G_n, G_n)\|_1 \leq \|\widehat{\psi_0}(\tilde{H}_n)\|_\infty \|G_n\|_2^2 \leq 2\|Z_n\|_2^2 \leq 2.$$

Now we can apply the second and third part of Lemma 3.13 to work towards the other necessary estimates. If we define

$$\phi_1(x_0, x_1) = \psi_0(x_0, x_1, x_1), \quad x_0, x_1 \in \mathbb{R}$$

and

$$\phi_2(x_0, x_1) = \psi_0(x_0, x_0, x_1), \quad x_0, x_1 \in \mathbb{R}$$

we have

$$T_2(i[\tilde{H}_n, F_n], G_n) = iT_{\phi_1}^{\tilde{H}_n, \tilde{H}_n}([\tilde{H}_n, F_n]) \times G_n,$$

$$T_2(G_n, i[\tilde{H}_n, F_n]) = iG_n \times T_{\phi_2}^{\tilde{H}_n, \tilde{H}_n}([\tilde{H}_n, F_n]).$$

By the Mean Value Theorem, $\|\psi_0\|_\infty \leq 2$, implying $\|\phi_1\|_\infty \leq 2$ and $\|\phi_2\|_\infty \leq 2$. Hence we get

$$\|T_{\phi_1}^{\tilde{H}_n, \tilde{H}_n}([\tilde{H}_n, F_n]) \times G_n\|_1 \leq \|T_{\phi_1}^{\tilde{H}_n, \tilde{H}_n}([\tilde{H}_n, F_n])\|_2 \|G_n\|_2$$

$$\leq \|\phi_1\|_\infty \|[\tilde{H}_n, F_n]\|_2 \|G_n\|_2 \leq 2\|\phi_1\|_\infty \|Z_n\|_2^2 \leq 4$$

by applying (8.14) and (8.15). Using a similar calculation, we get

$$\|G_n \times T_{\phi_2}^{\tilde{H}_n, \tilde{H}_n}([\tilde{H}_n, F_n])\|_1 \leq 4.$$

Now we can combine these estimates, and plug them into (8.16), to get

$$\|T_2(Z_n, Z_n)\|_1 \leq 4 + 4 + 2 + \|T_2(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n])\|_1.$$

Now we can conclude using Lemma 8.8:

$$\|T_2(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n])\|_1 \geq \|T_2(Z_n, Z_n)\|_1 - 10 \geq \text{const} \log n - 10$$

□

Now we can prove the final lemma of this chapter.

Lemma 8.11. ([7], Lemma 28) *There is a C^2-function g, with g'' bounded, and an $N \in \mathbb{N}$, such that for any sequence $\{\alpha_n\}_{n \geq N}$ with $\alpha_n \in \mathbb{R}^+$ there is a sequence of operators $\tilde{B}_n \in B(\mathbb{C}^{8n+4})$ such that, for $n \geq N$, $\|\tilde{B}_n\|_2 \leq 4\alpha_n$ and*

$$\|T_{g^{[2]}}^{\tilde{A}_n + \tilde{B}_n, \tilde{A}_n, \tilde{A}_n}(\tilde{B}_n, \tilde{B}_n)\|_1 \geq \textit{const } \alpha_n^2 \log n$$

Proof. Firstly, we need to slightly change the estimate of Lemma 8.10. By decreasing the value of 'const', and taking $N \in \mathbb{N}$ sufficiently large, we get

$$\|T_2(i[\tilde{H}_n, F_n], i[\tilde{H}_n, F_n])\|_1 \geq \text{const} \log n, \quad n \geq N. \tag{8.17}$$

Now take an arbitrary sequence $\{\alpha_n\}_{n \geq N}$ with $\alpha_n \in \mathbb{R}^+$. Using F_n as in Lemma 8.10, denote $\tilde{F}_n = \alpha_n F_n$. Now define, for any $t > 0$,

$$\gamma_t(\tilde{H}_n) := e^{it\tilde{F}_n} \tilde{H}_n e^{-it\tilde{F}_n}$$

and

$$V_{n,t} := \frac{\gamma_t(\tilde{H}_n) - \tilde{H}_n}{t}.$$

We will now consider two limits as $t \downarrow 0$. Firstly,

$$\lim_{t \downarrow 0} V_{n,t} = \lim_{t \downarrow 0} \frac{\gamma_t(\tilde{H}_n) - \tilde{H}_n}{t} = \lim_{t \downarrow 0} \frac{e^{it\tilde{F}_n} \tilde{H}_n e^{-it\tilde{F}_n} - \tilde{H}_n}{t} = \lim_{t \downarrow 0} \frac{e^{it\tilde{F}_n} \tilde{H}_n - \tilde{H}_n e^{it\tilde{F}_n}}{t e^{it\tilde{F}_n}}$$

$$= \lim_{t \downarrow 0} \frac{e^{it\tilde{F}_n} \tilde{H}_n - \tilde{H}_n e^{it\tilde{F}_n}}{t} \lim_{t \downarrow 0} \frac{1}{e^{it\tilde{F}_n}} = \frac{d}{dt}(e^{it\tilde{F}_n})|_{t=0} \tilde{H}_n - \tilde{H}_n \frac{d}{dt}(e^{it\tilde{F}_n})|_{t=0} \tilde{H}_n = i[\tilde{F}_n, \tilde{H}_n],$$

which, combined with Lemma 8.10, implies that there is a $t_1 > 0$ such that, for $t \leq t_1$,

$$\|V_{n,t}\|_2 \leq 2\|[\tilde{F}_n, \tilde{H}_n]\|_2 = 2\alpha_n \|[F_n, \tilde{H}_n]\|_2 \leq 4\alpha_n. \tag{8.18}$$

Secondly,

$$\lim_{t \downarrow 0} \tilde{H}_n + tV_{n,t} = \lim_{t \downarrow 0} \gamma_t(\tilde{H}_n) = \tilde{H}_n.$$

Now take a C^2-function g with $g(x) = g_0(x) = x|x|$ for any $x \in \mathbb{R} \backslash [-1, 1]$ and $g(0) = g'(0) = g''(0) = 0$. Define, for $t > 0$ and $x_0, x_1, x_2 \in \mathbb{R}$,

$$g_t(x_0, x_1, x_2) := g^{[2]}(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}),$$

the second order divided difference function of g contracted by a factor $1/t$ in each variable. Now we will show that, for any $x_0, x_1, x_2 \in \mathbb{R}$,

$$\lim_{t \downarrow 0} g_t(x_0, x_1, x_2) = \psi_0(x_0, x_1, x_2). \tag{8.19}$$

Here ψ_0 is the function as defined in equation (8.3). By a very lengthy but easy calculation, using the definitions of ψ_0 and g_0, it can be shown that

$$\psi_0(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}) = \psi_0(x_0, x_1, x_2). \tag{8.20}$$

Additionally, since the values and derivatives of $g(x)$ and $g_0(x)$ are the same for $x = 0$ and $|x| > 1$ by definition, for fixed $x \in \mathbb{R}$ we have

$$g(\frac{x}{t}) = g_0(\frac{x}{t}), \quad g'(\frac{x}{t}) = g_0'(\frac{x}{t}),$$

for small enough t. This is trivial when $x = 0$. When $x \neq 0$, take t_0 such that $|\frac{x}{t_0}| > 1$. Then equality holds by definition for $t < t_0$. Since $f^{[1]}(x_0, x_1)$ depends only on the values of f and f' at x_0 and x_1 for any function, we get

$$g^{[1]}(\frac{x_0}{t}, \frac{x_1}{t}) = g_0^{[1]}(\frac{x_0}{t}, \frac{x_1}{t})$$

for t small enough. By the same logic, if $x_0 \neq x_1$ and $x_1 \neq x_2$,

$$g^{[2]}(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}) = g_0^{[2]}(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t})$$

for small enough t. By definition we have, if $x_0 \neq x_1$ and $x_1 \neq x_2$,

$$g_0^{[2]}(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}) = \psi_0(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}).$$

Combining these last two equations with (8.20) we get

$$g^{[2]}(\frac{x_0}{t}, \frac{x_1}{t}, \frac{x_2}{t}) = \psi_0(x_0, x_1, x_2),$$

if $x_0 \neq x_1$ and $x_1 \neq x_2$. Now we should consider the case $x_0 = x_1 = x_2$. By definition of the second order divided difference function, we then have

$$g^{[2]}(\frac{x_0}{t}, \frac{x_0}{t}, \frac{x_0}{t}) = g''(\frac{x_0}{t}).$$

If $x_0 > 0$, we have, for small enough $t > 0$, $\frac{x_0}{t} > 1$, and hence

$$g''(\frac{x_0}{t}) = g_0''(\frac{x_0}{t}) = 2.$$

If $x_0 < 0$, we have, for small enough $t > 0$, $\frac{x_0}{t} < -1$, and hence

$$g''(\frac{x_0}{t}) = g_0''(\frac{x_0}{t}) = -2.$$

If $x_0 = 0$ we have $g''(0) = 0$ by definition of g. Having dealt with all cases, we conclude that equation (8.19) holds. Now we can apply Lemma 3.11, taking $a = 1/t$, to state the following:

$$T_{g^{[2]}}^{\frac{1}{t}\tilde{H}_n + V_{n,t}, \frac{1}{t}\tilde{H}_n, \frac{1}{t}\tilde{H}_n}(V_{n,t}, V_{n,t}) = T_{g_t}^{\tilde{H}_n + tV_{n,t}, \tilde{H}_n, \tilde{H}_n}(V_{n,t}, V_{n,t}).$$

We have shown that $\lim_{t\downarrow 0} \tilde{H}_n + tV_{n,t} = \lim_{t\downarrow 0} \gamma_t(\tilde{H}_n) = \lim_{t\downarrow 0} e^{it\tilde{F}_n}\tilde{H}_n e^{-it\tilde{F}_n} = \tilde{H}_n$, $\lim_{t\downarrow 0} g_t(x_0, x_1, x_2) = \psi_0(x_0, x_1, x_2)$, and $\lim_{t\downarrow 0} V_{n,t} = i[\tilde{F}_n, \tilde{H}_n]$. Hence we can apply Lemma 3.12 (taking $m = 1/t$ and chaning the limit accordingly) to get

$$\lim_{t\downarrow 0} T_{g^{[2]}}^{\frac{1}{t}\tilde{H}_n + V_{n,t}, \frac{1}{t}\tilde{H}_n, \frac{1}{t}\tilde{H}_n}(V_{n,t}, V_{n,t}) = \lim_{t\downarrow 0} T_{g_t}^{\tilde{H}_n + tV_{n,t}, \tilde{H}_n, \tilde{H}_n}(V_{n,t}, V_{n,t})$$

$$= T_{\psi_0}^{\tilde{H}_n, \tilde{H}_n, \tilde{H}_n}(i[\tilde{F}_n, \tilde{H}_n], i[\tilde{F}_n, \tilde{H}_n]) = T_2(i[\tilde{F}_n, \tilde{H}_n], i[\tilde{F}_n, \tilde{H}_n])$$

$$= \alpha_n^2 T_2(i[F_n, \tilde{H}_n], i[F_n, \tilde{H}_n]).$$

By equation (8.17) we thus get that there is a $t_2 > 0$ such that for any $0 < t \leq t_2$ we have

$$\|T_{g^{[2]}}^{\frac{1}{t}\tilde{H}_n + V_{n,t}, \frac{1}{t}\tilde{H}_n, \frac{1}{t}\tilde{H}_n}(V_{n,t}, V_{n,t})\|_1 \geq \text{const } \alpha_n^2 \log n.$$

Recall that, for equation (8.18) to hold, we needed $t \leq t_1$. Hence take $t_n = \min\{t_1, t_2\}$, such that both hold for t_n. Set $\tilde{A}_n := \frac{1}{t_n}\tilde{H}_n$, $\tilde{B}_n = V_{n,t_n}$. Now by (8.18) we have, for any $n \geq N$, $\|\tilde{B}_n\|_2 \leq 4\alpha_n$ and

$$\|T_{g^{[2]}}^{\tilde{A}_n + \tilde{B}_n, \tilde{A}_n, \tilde{A}_n}(\tilde{B}_n, \tilde{B}_n)\|_1 \geq \text{const } \alpha_n^2 \log n,$$

as required. □

9 Application in quantum mechanics

In the introduction, we introduced the fact that quantum states can be represented using positive semi-definite matrices, with trace equal to one. Knowing this introduces meaning to some of the results in this thesis. However, to truly grasp the significance of the presented results, it is important to understand the significance of commutators in quantum mechanics. One of the most important results of quantum mechanics is the uncertainty principle. To introduce this principle, we need some terminology. In quantum mechanics, an observable $A \in B(H)$ is any Hermitian, or self-adjoint, operator. The expectation value of A for a state Ψ with $|\Psi\rangle \in H$ is denoted $\langle A \rangle := \langle \Psi | A | \Psi \rangle$. The standard deviation of A is denoted by $\sigma_A^2 := \langle (A - \langle A \rangle)^2 \rangle$. Now we can state the uncertainty principle:

Theorem 9.1. ([14], 3.5.1) *For self-adjoint operators A, B, and for any state Ψ, the following inequality holds:*

$$\sigma_A^2 \sigma_B^2 \geq (\frac{1}{2i} \langle [A, B] \rangle)^2$$

Proof. Define $f := (A - \langle A \rangle)\Psi$ and $g := (B - \langle B \rangle)\Psi$. Then $\sigma_A^2 = \langle f|f \rangle$, $\sigma_B^2 = \langle g|g \rangle$. Recall that $\langle \cdot | \cdot \rangle$ denotes the inner product of two vectors. By the Cauchy-Schwarz inequality,

$$\sigma_A^2 \sigma_B^2 = \langle f|f \rangle \langle g|g \rangle \geq | \langle f|g \rangle |^2.$$

Noting that

$$|z|^2 \geq (\mathrm{Im}(z))^2 = (\frac{1}{2i}(z - z^*))^2$$

and letting $z = \langle f|g \rangle$ we get

$$\sigma_A^2 \sigma_B^2 \geq (\frac{1}{2i}(\langle f|g \rangle - \langle g|f \rangle))^2.$$

Additionally,

$$\langle f|g \rangle = \langle \Psi | AB\Psi \rangle - \langle A \rangle \langle \Psi | B\Psi \rangle - \langle B \rangle \langle \Psi | A\Psi \rangle + \langle A \rangle \langle B \rangle \langle \Psi | \Psi \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle$$

and similarly $\langle g|f \rangle = \langle BA \rangle - \langle A \rangle \langle B \rangle$. So

$$\sigma_A^2 \sigma_B^2 \geq (\frac{1}{2i}(\langle f|g \rangle - \langle g|f \rangle))^2 = (\frac{1}{2i}(\langle AB \rangle - \langle BA \rangle))^2 = (\frac{1}{2i} \langle [A, B] \rangle)^2.$$

□

How do we interpret this result? It shows that two observables that do not commute can never be precisely determined at one time. An example of two observables that do not commute is the position and momentum of a particle. Taking $A = x$ and $B = (\hbar/i)d/dx$, we get $[A, B] = i\hbar$. This is known as the canonical uncertainty principle. Note that our results do not apply to this canonical example, since the operators are not bounded. However, there are many interesting observables that are not bounded. Consider for example the spin state of an electron along a certain axis, z. There are only two eigenstates, up and down, and as such the spin state can be expressed on a two-dimensional Hilbert space. Hence any observable for a spin-state is a bounded operator, and our results apply.

Clearly, the value of a commutator of two self-adjoint operators is an important quantity in quantum mechanics. Now consider the meaning of Theorem 6.4, restated here.

Theorem 9.2. *For all self-adjoint operators $A, B \in B(H)$ with $[A,B] \in \mathcal{S}^1(H)$ and for every Lipschitz continuous function f we have*

$$\|[f(A), B]\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|[A,B]\|_1.$$

Additionally, for self adjoint operators $X, Y \in H$ with $X - Y \in \mathcal{S}^1(H)$ and for every Lipschitz continuous function f we have

$$\|f(X) - f(Y)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|X - Y\|_1.$$

The first result of this theorem can be interpreted as an upper bound to the change that the lower bound of the uncertainty principle undergoes as the result of a change by function f to one of the observables. The second result can be interpreted along the lines of the introduction, by considering X, Y as quantum states: this equation gives an upper bound to the change in overlap between states as a result of applying an identical function to both states.

10 Conclusion

Schatten classes, denoted by $\mathcal{S}^p$, $1 \leq p \leq \infty$, are the proper spaces to consider when attempting to tackle commutator estimates of non-commutative operators. To apply a function to one element of a commutator in a linear manner, we need to use the theory of Schur multipliers and operator integrals, denoted by $T_\phi^{A,A}$. Combining this theory with some results from harmonic analysis, we can prove a weak-type estimate for operator integrals applied to operators with integer spectrum:

Theorem 10.1. *For every self-adjoint operator $A \in B(H)$ with integer spectrum and for every Lipschitz function f, $V \in \mathcal{S}^1(H)$, we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|V\|_1.$$

Here $\|f'\|_\infty$ denotes the Lipschitz constant of f. Using careful approximation, we can generalise this result to operators with an arbitrary spectrum. This result can then be interpolated to any Schatten class:

Theorem 10.2. *If $A \in B(H)$ is a self-adjoint operator, and if $f : \mathbb{R} \to \mathbb{C}$ is Lipschitz, then for any $1 < p < \infty$, $V \in \mathcal{S}^p(H)$ we have*

$$\|T^{A,A}_{f^{[1]}}(V)\|_p \leq c_{abs}\|f'\|_\infty \|V\|_p.$$

We have also shown that these bounds are optimal, meaning that we can find lower bounds of the same order. By carefully choosing which function f to use, these results show a couple of estimates with applications in quantum mechanics:

Theorem 10.3. *For all self-adjoint operators $A, B \in B(H)$ with $[A, B] \in \mathcal{S}^1(H)$ and for every Lipschitz continuous function f we have*

$$\|[f(A), B]\|_{1,\infty} \leq c_p\|f'\|_\infty \|[A, B]\|_1.$$

Additionally, for self adjoint operators $X, Y \in H$ with $X - Y \in \mathcal{S}^1(H)$ and for every Lipschitz continuous function f we have

$$\|f(X) - f(Y)\|_{1,\infty} \leq c_{abs}\|f'\|_\infty \|X - Y\|_1.$$

Knowing the importance of the trace distance between states and the commutator of self-adjoint operators (known as observables), these results could be applied in the study of quantum mechanics. The estimates presented in this thesis have recently been expanded to multiple operators. Further study of [5] and [20] is recommended for anyone looking to expand these results. In these articles, the main positive results presented in this thesis have been expanded to multiple operator integrals. Possibly, some of the constants presented in these results can be sharpened, using more advanced non-commutative spaces. Using the theory of BMO- or Hardy spaces could be the key to unlocking these sharper and more general results.

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