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Multiple Markov properties for fractional parabolic SPDEs

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"Multiple Markov properties for fractional parabolic SPDEs"

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preface

During this year I have written my master thesis, which is the final part left of my master Applied Mathematics in Delft. This was not always easy for me, but there were a lot of people supporting me during this journey, and I am happy that I was able to finish this thesis in the end.

I want to start by thanking Kristin Kirchner, my daily supervisor, and Joshua Willems, her PhD student, for all their help. More than once have my meetings with either of them led to new insights and motivation to keep going. I don't think every supervisor would have put in the energy and commitment that you did, and I am really grateful for this. I would also like to thank Richard Kraaij, as well as Jan van Neerven, for taking part in my defense committee. Next I would like to thank my study friends, who helped me stay motivated by studying together at the campus. I always looked forward to those study sessions, and for me, those were the most enjoyable parts of my last year. Lastly, I would like to thank my family for supporting me emotionally, also when things got more difficult, and even proof reading my master thesis that most likely makes no sense to them.

I don't think I would have been able to finish my thesis without either of the people I just mentioned, and I am all of you incredibly thankful for that. With this, I am finally finished with my study here at Delft, which all in all was an amazing experience where I truly had the opportunity to grow. Now I am ready to face the next challenge ahead: my first job and what I want to do with my life in the foreseeable future.

Björn Titulaer
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1
Introduction

1.1. Background

Spatial and spatiotemporal SPDEs are used in a wide number of applications in a variety of fields, including epidemiology [2], neuroimaging [16], seismology [21], ecology [17] and meteorology. A common assumption here is that their distribution is Gaussian [11], mostly because those distributions are fully determined by their mean and covariance function. An important class of covariance functions for this is the Matérn covariance [13]. The Matérn covariance function is given by

$$\rho(s, s') = \frac{\sigma^2}{\Gamma(\nu)2^{\nu-1}}(\kappa\|s - s'\|)^{\nu} K_{\nu}(\kappa\|s - s'\|).$$

Here, K_ν denotes the modified Bessel function of the second kind, $\nu > 0$ is an index for smoothness, $\kappa > 0$ determines the correlation length and $\sigma^2 > 0$ is the variance. Whittle [19] showed that a stationary process $(X(x))_{x\in\mathbb{R}^d}$ that solves the SPDE

$$\tau(\kappa^2 - \Delta)^{\beta} X(x) = \mathscr{W}(x), \quad x \in \mathbb{R}^d, \tag{1.1}$$

has a Matérn function with $\nu = 2\beta - d/2$ and $\sigma^2 = \Gamma(\nu)\left[\Gamma(2\beta)(4\pi)^{d/2}\kappa^{2\nu}\tau^2\right]^{-1}$. In (1.1), Δ is the Laplace operator on $\mathbb{R}^d$ and $\mathscr{W}$ is Gaussian white noise. Because of Mercer's theorem, this covariance function gives rise to a covariance operator. Recently the viewpoint has shifted from the covariance function to the covariance operator, because of the many tools available to numerically approach linear operators. The covariance operator associated to the process X solving (1.1) is given by $\left(\tau(\kappa^2 - \Delta)\right)^{-2\beta}$.

Lindgren, Rue, and Lindström [10] considered the problem as in (1.1) on bounded domains, while imposing Dirichlet or Neumann boundary conditions. Their approach also allows for a spacially varying κ, which replaces the operator $\kappa^2 - \Delta$ with a more general strongly elliptic differential operator L, and it allows us to pose the problem on more general domains such as the sphere or manifolds. They also briefly mention the possibility to extend this to the stochastic space-time problem

$$\begin{aligned}(\partial_t + L)X(t, x) &= \mathscr{W}(t, x), \quad t \in [0, T]\\ X(0, x) &= X_0(x).\end{aligned}$$

This approach has gained a lot of attention in recent years due to computational benefits available to discretize the (possibly fractional-order) strongly elliptic differential operator. In [8], this problem was further extended to allow to control both spatial and temporal regularity. They considered the fractional SPDE

$$\begin{aligned}(\partial_t + L^{\beta})^{\gamma} X(t, x) &= \mathscr{W}(t, x), \quad t \in [0, T]\\ X(0, x) &= X_0(x).\end{aligned}$$

Here, the two parameters $\beta > 0$ and $\gamma > 0$ determine the spatial and temporal smoothness. They define a mild solution, show existence and uniqueness of this mild solution as well as spatial and temporal regularity. The proofs are mainly based on semigroup theory, which we will also use in this thesis.

1.2. Motivation

We will focus on the fractional parabolic SPDE

$$\begin{aligned}(\partial_t + A)^\gamma X_\gamma(t) &= W(t), \quad t \in [s,T], \quad \gamma > 0,\\ X_\gamma(s) &= \xi.\end{aligned} \tag{1.2}$$

Here, $X_\gamma(t)$ takes its values in a separable Hilbert space H and the operator $-A : D(A) \subseteq H \to H$ generates an exponentially stable C_0-semigroup. Finally, $W(t)$ is an H-valued Q-Wiener process and ξ is some random initial condition. See also Chapter 2 for more information on these definitions. In [8], it was shown that with initial condition $X_\gamma(0) = 0$, its weak solution satisfies a mild solution formula given by

$$X_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} S(t-r)\mathrm{d}W(r), \quad t \in [0,T].$$

In [20], an attempt has been made to extend this solution to arbitrary initial conditions ξ at time $s = 0$, by considering $Y_\gamma(t) := X_\gamma(t) - \xi$. They show that this process satisfies the problem

$$\begin{aligned}(\partial_t + A)^\gamma Y_\gamma(t) &= W(t) - A^\gamma \xi,\\ Y_\gamma(0) &= 0.\end{aligned}$$

Then after deriving the weak formulation of this problem, they ultimately find $X_\gamma(t;\xi)$ as the process satisfying (1.2), given by

$$X_\gamma(t;\xi) := \frac{1}{\Gamma(\gamma)} A^\gamma \int_t^\infty r^{\gamma-1} S(r)\xi \mathrm{d}r + \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} S(t-r)\mathrm{d}W(r). \tag{1.3}$$

There are two problems with this process. First, for integer $n \geq 2$, the process X_n as defined in (1.3) is in general only consistent with solving n Cauchy problems iteratively if all derivatives from order 1 up to order $n-1$ are set to 0. It would be desirable to incorporate nonzero initial conditions for the derivatives as well.

The second problem is that, in general, (1.3) is not restartable. As notation, $X_\gamma(t;s,\xi)$ will denote the solution process at time t starting with a (random) initial condition ξ at time s. By restartable, we mean that the following equation holds for all $t > s > u$ and initial conditions ξ:

$$X_\gamma(t;s,X(s;u,\xi)) = X_\gamma(t;u,\xi). \tag{1.4}$$

This property is crucial in the proof of the Markov property as done in [4]. A Markov property is very desirable numerically, because the resulting covariance matrix would be much more sparse, reducing computation times. To simplify calculations, we take $H = \mathbb{R}$, $S(t) = e^{-\lambda t}$ (so $A = \lambda$), and $\xi = y$ for some deterministic y. The mild solution formula for our process starting at time s instead of 0 is

$$X_\gamma(t;s,y) = \frac{\lambda^\gamma}{\Gamma(\gamma)} \int_{t-s}^\infty r^{\gamma-1} e^{-\lambda r}\mathrm{d}r\ y + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} e^{-\lambda(t-r)}\mathrm{d}W(r). \tag{1.5}$$

It is possible to substitute $u = \lambda r$ in the first deterministic integral. This gives

$$\frac{\lambda^\gamma}{\Gamma(\gamma)} \int_{t-s}^\infty r^{\gamma-1} e^{-\lambda r}\mathrm{d}r = \frac{1}{\Gamma(\gamma)} \int_{\lambda(t-s)}^\infty u^{\gamma-1} e^{-u}\mathrm{d}u = \frac{\Gamma(\gamma, \lambda(t-s))}{\Gamma(\gamma)}. \tag{1.6}$$

Here, $\Gamma(\gamma, x)$ denotes the upper incomplete gamma function, given by

$$\Gamma(\gamma,x) = \int_x^\infty r^{\gamma-1} e^{-r}\mathrm{d}r, \quad x \in [0,\infty).$$

Substituting (1.6) into (1.5) gives

$$X_\gamma(t;s,y) = \frac{\Gamma(\gamma,\lambda(t-s))}{\Gamma(\gamma)} y + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} e^{-\lambda(t-r)}\mathrm{d}W(r).$$

Now taking expectations here leaves us with

$$\mathbb{E}(X_\gamma(t;s,y)) = \frac{\Gamma(\gamma,\lambda(t-s))}{\Gamma(\gamma)} \mathbb{E}(y),$$

where we use that the expectation of a stochastic integral of a deterministic process is always 0 with respect to a Wiener process. In order for the restarted process to be the same as the original one, it is necessary to have equality in expectation. Taking expectations in (1.4) gives

$$\frac{\Gamma(\gamma,\lambda(t-s))}{\Gamma(\gamma)}\mathbb{E}(X_\gamma(s,u,y)) = \frac{\Gamma(\gamma,\lambda(t-u))}{\Gamma(\gamma)}y,$$

$$\frac{1}{\Gamma(\gamma)^2}\Gamma(\gamma,\lambda(t-s))\Gamma(\gamma,\lambda(s-u))y = \frac{\Gamma(\gamma,\lambda(t-u))}{\Gamma(\gamma)}y.$$

This implies that

$$\Gamma(\gamma,\lambda(t-s))\Gamma(\gamma,\lambda(s-u))y = \Gamma(\gamma,\lambda(t-u))\Gamma(\gamma)y.$$

Assuming $y \neq 0$, this only holds if

$$\Gamma(\gamma,\lambda(t-s))\Gamma(\gamma,\lambda(s-u)) = \Gamma(\gamma,\lambda(t-u))\Gamma(\gamma). \tag{1.7}$$

Now note that the right-hand side is independent of s. As a result, this can only hold if the left-hand side is independent of s as well. Taking the derivative to s on both sides, we see that Equation (1.7) requires for all $u \leq s \leq t$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(\gamma,\lambda(t-s))\Gamma(\gamma,\lambda(s-u)) = 0.$$

using that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Gamma(\gamma,x) = -x^{\gamma-1}e^{-x},$$

this reduces to

$$\lambda(\lambda(t-s))^{\gamma-1}e^{-\lambda(t-s)}\Gamma(\gamma,\lambda(s-u)) - \lambda(\lambda(s-u))^{\gamma-1}e^{-\lambda(s-u)}\Gamma(\gamma,\lambda(t-s)) = 0,$$

or equivalently,

$$(s-u)^{\gamma-1}e^{-\lambda(s-u)}\Gamma(\gamma,\lambda(t-s)) = (t-s)^{\gamma-1}e^{-\lambda(t-s)}\Gamma(\gamma,\lambda(s-u)).$$

Now taking the limit $t \downarrow s$, we see that the left-hand side is always well-defined and non-zero as long as $t > u$ ($\Gamma(\gamma,0)$ is positive and well defined for any strictly positive γ). The right-hand side, however, vanishes for $\gamma > 1$ and approaches infinity for $0 < \gamma < 1$. As a result, we obtain that (1.7) does not hold in general for $\gamma \neq 1$, so this process is not restartable in general.

2
Preliminaries

2.1. Linear operators on a Hilbert space

In this section some general results about linear operators will be presented. To do this, we first need to establish some basic notation regarding Hilbert and Banach spaces. If H is a Hilbert space, then its inner product will be denoted by $\langle\cdot,\cdot\rangle_H$, and the norm on H by $\|\cdot\|_H$. Next, if H is a Hilbert space, then the direct product space $H \times H := \{(h,g) : h, g \in H\}$ is again a Hilbert space if we consider the inner product

$$\langle (h_1, g_1), (h_2, g_2)\rangle_{H\times H} = \langle h_1, h_2\rangle_H + \langle g_1, g_2\rangle_H.$$

In this case, we will also write H^2 to denote the product Hilbert space. In the same way it is possible to define a Hilbert space H^n on $H \times \cdots \times H$.

For the remainder of this section, let U and H be separable complex Hilbert spaces. In what follows we will give the definitions and theorems used in regards to operators acting from U to H.

Definition 2.1.1 (Linear operator)**.** An operator $A : U \to H$ is called linear if

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$$

for all $x, y \in U$ and $\alpha, \beta \in \mathbb{R}$.

Unless otherwise indicated, an operator will always denote a linear operator from now on. A special kind of linear operators are the bounded (linear) operators. Instead of $A(x)$ we will often also just write Ax to indicate the operator A acting on $x \in U$.

Definition 2.1.2 (Bounded operator)**.** A linear operator $A : U \to H$ is called bounded if it satisfies

$$\sup_{\|x\|_U = 1} \|Ax\|_H < \infty.$$

Note that because of the linearity it is equivalent to take the supremum over $\|x\|_U \le 1$, which is sometimes done instead in the literature. The space of all bounded operators from U to H is denoted $\mathscr{L}(U,H)$. In the case $U = H$, we will write $\mathscr{L}(H)$ instead.

Now if we define $\|A\|_{\mathscr{L}(U,H)} := \sup_{\|x\|_U=1} \|Ax\|_H$, then this turns $\mathscr{L}(U,H)$ into a Banach space.

Definition 2.1.3 (Adjoint)**.** If $A \in \mathscr{L}(H)$ is a bounded operator, then we define the adjoint of A to be the operator $A^* \in \mathscr{L}(H)$ that for all $x, y \in H$ satisfies

$$\langle Ax, y\rangle_H = \langle x, A^* y\rangle_H.$$

It can be shown that this operator A^* both exists and is unique, which makes this well defined. Some operator classes with particularly nice properties will be given next.

Definition 2.1.4 (Self-adjoint operator)**.** An operator A is called self-adjoint if $A = A^*$.

Definition 2.1.5 (Nonnegative definite operator)**.** An operator $A \in \mathscr{L}(H)$ is called nonnegative definite if for all x in H we have that

$$\langle Ax, x\rangle_H \geq 0.$$

If instead we have for all $x \neq 0$ a strict inequality, then we call A positive definite instead.

Now in complex Hilbert spaces it holds that every nonnegative definite operator is also self-adjoint by the polarization identity, but in real Hilbert spaces this is not true in general.

For those positive operators, it is possible to define the trace [18, Section 14.2]:

Definition 2.1.6 (Trace of a nonnegative definite operator)**.** If $A \in \mathscr{L}(H)$ is a nonnegative definite operator and $(e_k)_{k\geq 1}$ is an orthonormal basis for H, then its trace is given by

$$\mathrm{tr}(A) = \sum_{k=1}^{\infty} \langle Ae_k, e_k\rangle_H.$$

This definition does not depend on the choice of $(e_k)_{k\geq 1}$, and the sum is well-defined (though possibly infinite) since we are only adding nonnegative numbers (A was chosen nonnegative definite). It is also possible to extend this definition to a more general class of operators. For this we first need to define the modulus of an operator.

Definition 2.1.7 (Modulus of an operator)**.** Let H be a complex Hilbert space, and let $A \in \mathscr{L}(H)$. Then its modulus $|A|$ is given by

$$|A| := (A^* A)^{\frac{1}{2}},$$

that is, the unique nonnegative definite operator such that $|A|^2 = A$ [18, Proposition 8.27].

If A is nonnegative in a complex Hilbert space, then it is also self-adjoint. It thus follows that in this case $A = |A|$, which leads us to the more general definition of the trace for complex Hilbert spaces.

Definition 2.1.8 (Trace for linear operators)**.** Let H be a complex Hilbert space, $A \in \mathscr{L}(H)$ and $(e_k)_{k\geq 1}$ an orthonormal basis for H. Then we define the trace of A as

$$\mathrm{tr}(A) := \sum_{k=1}^{\infty} \langle |A| e_k, e_k\rangle_H.$$

We write $\mathscr{L}_1(H)$ for all bounded operators with finite trace. This is a Banach space with respect to the norm

$$\|A\|_{\mathscr{L}_1(H)} := \mathrm{tr}(A).$$

The last type of operators that will be needed are the Hilbert–Schmidt operators.

Definition 2.1.9 (Hilbert–Schmidt)**.** Let $(e_k)_{k\geq 1}$ be an orthornormal basis for H. An operator $A \in \mathscr{L}(H)$ is called Hilbert–Schmidt if

$$\left(\sum_{k=1}^{\infty} \|Ae_k\|_H^2\right)^{\frac{1}{2}} < \infty.$$

The Hilbert–Schmidt operators will be denoted by $\mathscr{L}_2(H)$. Together with the inner product

$$\langle A, B\rangle_{\mathscr{L}_2(H)} := \sum_{k=1}^{\infty} \langle Ae_k, Be_k\rangle_H,$$

these operators become a Hilbert space.

Finally, we will introduce some notation regarding unbounded operators on H. Often these are only defined on a part of H instead of the entirety of H.

Definition 2.1.10 (Unbounded operators)**.** Let $D(A)$ be a linear subspace. An unbounded operator $A : D(A) \subseteq H \to H$ is a linear operator defined on $D(A)$. $D(A)$ will also be called the domain of A.

For such an unbounded operator A, we can define the graph of A as the set

$$G(A) := \{(x, Ax) : x \in D(A)\}.$$

On the graph of A we can define the norm $\|(x, Ax)\|_{G(A)} = (\|x\|_H^2 + \|Ax\|_H^2)^{\frac{1}{2}}$. We say that A is closed if the graph of A is closed with respect to the graph norm. Similarly, we can define a norm on H as $\|x\|_{D(A)} := (\|x\|_H^2 + \|Ax\|_H^2)^{\frac{1}{2}}$. If A is closed, then $D(A)$ is a Banach space with respect to this norm [18, Section 1.2], and even a Hilbert space when considering the associated inner product $\langle x, y\rangle_{D(A)} := \langle x, y\rangle_H + \langle Ax, Ay\rangle_H$.

2.2. Bochner spaces and Sobolev spaces

In this section we will define integrals for functions $f : J \to H$, where $J \subseteq \mathbb{R}$ is an arbitrary interval and H is a separable Hilbert space (though a Banach space would also suffice here). This construction is very similar to the construction of the Lebesgue integral for functions taking values in $\mathbb{R}$. We first define the integral for simple functions:

Definition 2.2.1. Let $\lambda : \mathscr{B}(\mathbb{R}) \to [0,\infty]$ denote the Lebesgue measure, let $J \subseteq \mathbb{R}$ be some (possibly unbounded) interval and let H be a Hilbert space. Then $f : J \to H$ is called a simple function if there exists a finite integer k such that

$$f(t) = \sum_{n=1}^{k} \mathbb{1}_{A_n}(t) x_n,$$

with $x_n \in H$ and $A_n \in \mathscr{B}(\mathbb{R})$ such that $\lambda(A_n) < \infty$, for all $n \leq k$.

In a general Hilbert space, a measurable function (defined in the sense of pre-images) is no longer always the limit of simple functions. In a separable Hilbert space, however, we do not run into this problem, and a measurable function is always the limit of a sequence of simple functions [6, Remark 3.2]. For a simple function, we define the integral as follows:

Definition 2.2.2 (Bochner integral for simple functions)**.** Let $f : J \to H$ be a simple function of the form $f = \sum_{n=1}^{k} \mathbb{1}_{A_n} x_n$. Then we define the Bochner integral of f as

$$\int_J f(t)\mathrm{d}t := \sum_{n=1}^{k} \lambda(A_n) x_n.$$

It can be shown that this definition does not depend on the choice of $(A_n)_{n\leq k}$ and $(x_n)_{n\leq k}$. From here we can define general integrable functions.

Definition 2.2.3 (Bochner integrable)**.** Let $f : J \to H$ be a measurable function. Then we say that f is Bochner integrable if there exists a sequence of simple functions $(f_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \int_J \|f(t) - f_n(t)\|_H \mathrm{d}t = 0,$$

where the integral is taken to be the Lebesgue integral. In this case, we define the Bochner integral of f as

$$\int_J f(t)\mathrm{d}t := \lim_{n\to\infty} \int_J f_n(t)\mathrm{d}t.$$

Again, this definition does not depend on the choice of $(f_n)_{n\geq 1}$, and the limit is well defined [18, Section 1.2.a]. The following theorem provides an easier way to check Bochner integrability.

Theorem 2.2.1. Let $f : J \to H$ be a measurable function. Then f is Bochner integrable if and only if

$$\int_J \|f(t)\|_H \mathrm{d}t < \infty,$$

where the integral is again interpreted as a Lebesgue integral. In this case we also have the inequality

$$\left\| \int_J f(t)\mathrm{d}t \right\|_H \leq \int_J \|f(t)\|_H \mathrm{d}t.$$

Proof. See [18, Proposition 1.2.2]. □

Now define the Bochner space $L^p(J;H)$ for $p \in [1,\infty)$ as the space of all Borel measurable functions $f : J \to H$ such that

$$\|f\|_{L^p(J;H)} := \left(\int_J \|f(t)\|_H^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Note that all functions that are equal almost everywhere are equivalent with this norm, so more precisely the Bochner space contains equivalence classes of functions mapping J to H. $L^p(J;H)$ is a Banach space for every p, and for $p = 2$ it is a Hilbert space if we take as inner product

$$\langle f, g \rangle_{L^2(J;H)} := \int_J \langle f(t), g(t) \rangle_H \mathrm{d}t.$$

Since we identify functions that are equal almost everywhere, pointwise evaluation no longer makes sense, so we will need a different way to introduce differentiabilty. For this, let $C_c^\infty(J;\mathbb{R})$ denote the set of all smooth (infinitely often differentiable) functions $f : J \to \mathbb{R}$ with compact support inside J.

Definition 2.2.4 (Weak derivative)**.** Let $f \in L^1(J;H)$. Then we say that $g \in L^1(J;H)$ is the weak derivative of f if for all $\phi \in C_c^\infty(J;\mathbb{R})$:

$$\int_J f(t)\phi'(t)\mathrm{d}t = -\int_J g(t)\phi(t)\mathrm{d}t.$$

If f is differentiable with derivative f' in the classical sense, then integration by parts together with the compact support of ϕ shows that f' is also the weak derivative of f. Moreover, if f is weakly differentiable, then the weak derivative is unique [18, Proposition 2.5.2]. Now define the Sobolev space $W^{n,p}(J;H)$ as the space of all functions $f \in L^p(J;H)$ that are n times weakly differentiable, with weak derivatives again in $L^p(J;H)$. Together with the norm

$$\|f\|_{W^{n,p}(J;H)} := \left(\sum_{k=0}^{n} \|f^{(k)}\|^p_{L^p(J;H)}\right)^{\frac{1}{p}},$$

this forms a Banach space. For $p = 2$ we get again a Hilbert space. The space $W^{n,2}(J;H)$ is also denoted by $H^n(J;H)$ to emphasize this. It turns out that for these spaces, we can once again define pointwise evaluation in a certain way.

Theorem 2.2.2. Let $f \in W^{1,p}(J;H)$. Then there exists a continuous function $\tilde{f} \in C(\bar{J};H)$ with $f = \tilde{f}$ almost everywhere, and for all $s \le t \in J$ we have

$$\tilde{f}(t) - \tilde{f}(s) = \int_s^t f'(r)\mathrm{d}r.$$

Proof. See [18, Proposition 2.5.9]. □

2.3. Semigroups

For bounded operators $A \in \mathscr{L}(H)$, it is possible to define its exponential operator by

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \quad t \ge 0,$$

which converges since for bounded A it converges absolutely in the $\|\cdot\|_{\mathscr{L}(H)}$-norm. This operator is important when solving infinite-dimensional initial value Cauchy problems of the form

$$\begin{aligned} u'(t) &= Au(t) \text{ for all } t > 0, \\ u(0) &= u_0. \end{aligned}$$

In this case, the solution of this initial value problem would then be given by $u(t) = u_0 e^{tA}$ [14, Section 4.1].

For general Hilbert spaces and unbounded operators A, however, this method fails, since the series defining e^{tA} no longer has to converge. In order to deal with unbounded A, we thus have to generalize the properties of the exponential operator to unbounded operators, which leads us to the following definition.

Definition 2.3.1 (Strongly continuous semigroup)**.** A family $(S(t))_{t\ge 0}$ of bounded linear operators on H is called a strongly continuous semigroup if:

- $S(0) = I$,
- $S(t+s) = S(t)S(s)$ for all $t, s \ge 0$,
- $\lim_{t\downarrow 0} S(t)x = x$ for every $x \in H$.

In general we will let C_0 denote the family of all strongly continuous semigroups. It is easy to verify that for A bounded, $(e^{tA})_{t\ge 0}$ is indeed a strongly continuous semigroup: the first two properties follow immediately, and for the last property it is possible to write

$$\lim_{t\downarrow 0} \|S(t)x - x\|_H = \lim_{t\downarrow 0} \|(S(t) - I)x\|_H \le \|x\|_H \lim_{t\downarrow 0} \|S(t) - I\|_{\mathscr{L}(H)}.$$

Now for the last part we find

$$\lim_{t\downarrow 0}\|S(t)-I\|_{\mathscr{L}(H)}=\lim_{t\downarrow 0}\left\|\sum_{k=1}^{\infty}\frac{t^k}{k!}A^k\right\|_{\mathscr{L}(H)}\le\lim_{t\downarrow 0}\sum_{k=1}^{\infty}\frac{t^k}{k!}\|A^k\|_{\mathscr{L}(H)}\le\lim_{t\downarrow 0}\sum_{k=1}^{\infty}\frac{t^k}{k!}\|A\|^k_{\mathscr{L}(H)}=\lim_{t\downarrow 0}e^{t\|A\|_{\mathscr{L}(H)}}-1=0.$$

Note that in this case we even found the stronger uniform convergence $\lim_{t\downarrow 0}\|S(t)-I\|_{\mathscr{L}(H)}=0$. A semigroup (a family of operators that has only the first two properties of Definition 2.3.1) satisfying this stronger property is called a uniformly continuous semigroup. Every uniformly continuous semigroup is also a strongly continuous semigroup, and in fact it can be shown that every uniformly continuous semigroup is of the form $(e^{tA})_{t\ge 0}$ with $A\in\mathscr{L}(H)$ [14, Chapter 1, Theorem 1.2].

For C_0-semigroups this is not the case, but it turns out that there exists an operator, possibly unbounded, that is similarly related to the C_0-semigroup as A is to a uniformly bounded semigroup $(e^{tA})_{t\ge 0}$.

Definition 2.3.2 (Infinitesimal generator). If $(S(t))_{t\ge 0}$ is a strongly continuous semigroup, then we define its infinitesimal generator as a possibly unbounded operator $A: D(A)\subseteq H\to H$ by

$$D(A):=\left\{x\in H:\lim_{t\downarrow 0}\frac{S(t)x-x}{t}\text{ exists}\right\},$$
$$Ax:=\lim_{t\downarrow 0}\frac{S(t)x-x}{t},\quad x\in D(A).$$

It can be verified that if A is bounded, then A is indeed the infinitesimal generator of $(e^{tA})_{t\ge 0}$, as one would expect.

A useful bound that again links the semigroup to the exponential function is the following.

Theorem 2.3.1. If $(S(t))_{t\ge 0}$ is a C_0-semigroup, then there are constants $M\ge 1$ and $\omega\in\mathbb{R}$ such that for every $0\le t<\infty$,

$$\|S(t)\|_{\mathscr{L}(H)}\le Me^{\omega t}.$$

Proof. See [14, Chapter 1, Theorem 2.2]. □

In some cases, we will require $\omega<0$, which we will refer to as exponentially stable.

So far we have only considered semigroups indexed by the positive real line. It is possible to extend this to a sector in the complex space, which is needed to define analytic semigroups.

Definition 2.3.3 (Analytic semigroups). Let H be a complex Hilbert space. Let $\phi>0$ and define the region in the complex plane

$$\Sigma:=\{\lambda\in\mathbb{C}:|\arg\lambda|<\phi\}.$$

Now an H-valued family $(S(z))_{z\in\Sigma}$ is called analytic if the following hold:

- $z\mapsto S(z)$ is analytic as a function mapping Σ to $\mathscr{L}(H)$,
- $S(0)=I$,
- $S(z+w)=S(z)S(w)$ for all $z,w\in\Sigma$,
- for every $x\in H$ we have that $\lim_{z\to 0}S(z)x=x$.

By only considering the nonnegative real line, an analytic semigroup naturally gives rise to a C_0-semigroup. Sometimes when working with a C_0-semigroup we will want the stronger continuity properties given by the analytic semigroups. If a C_0-semigroup defined on a Hilbert space can be extended to an analytic semigroup on some complex domain, then we will say that this C_0-semigroup is analytic, even if we only consider the nonnegative real axis.

The following bound specifically requires analycity.

Theorem 2.3.2 (Bound for analytic semigroups). Suppose $-A$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t\ge 0}$. If A is boundedly invertible, then for all $a\in\mathbb{N}$ and all $t\ge 0$ we have that $A^aS(t)$ is bounded and there exists constants $C\ge 0$ and $\delta>0$ such that we have

$$\|A^aS(t)\|_{\mathscr{L}(H)}\le Ct^{-a}e^{-\delta t}.$$

Proof. See [14, Chapter 2, Theorem 6.13]. We restricted the theorem to just integer values for a, but the bound still holds for other $a > 0$, though you would have to define A^a for non-integer a first. □

Now if we have a real Hilbert space we would still like to have a condition similar to analyticity. For this we will consider the complexification of a Hilbert space. Just like it is possible to identify $\mathbb{C}$ with $\mathbb{R} \times \mathbb{R}$, it is possible for a real Hilbert space H to turn $H \times H$ into a complex Hilbert space, which we will denote with $H_{\mathbb{C}}$. We will suggestively write $h + gi$ for an element of $H_{\mathbb{C}}$.

Now define

$$(h + gi) + (v + wi) := (h + v) + (g + w)i,$$

and define the inner product as

$$\langle h + gi, v + wi \rangle_{H_{\mathbb{C}}} := \langle h, v \rangle_H + \langle g, w \rangle_H - \langle h, w \rangle_H i + \langle g, v \rangle_H i.$$

So the norm is then defined as $\|h + gi\|^2_{H_{\mathbb{C}}} = \|h\|^2_H + \|g\|^2_H$. For a complex scalar $a + bi$, we then define scalar multiplication as

$$(a + bi)(h + gi) := (ah - bg) + (ag + bh)i.$$

It is not difficult to check that this indeed defines a complex Hilbert space. Now if we have a strongly continuous semigroup $(S(t))_{t\geq 0}$ acting on our real Hilbert space H, then we can define a C_0-semigroup $(S_{\mathbb{C}}(t))_{t\geq 0}$ on $H_{\mathbb{C}}$ by setting

$$S_{\mathbb{C}}(t)(h + gi) := S(t)h + S(t)gi, \; t \geq 0.$$

All properties of the strongly continuous semigroup can easily be verified here by the definition of $S_{\mathbb{C}}$ and the fact that $(S(t))_{t\geq 0}$ is a C_0-semigroup. We will call $(S_{\mathbb{C}}(t))_{t\geq 0}$ the complexification of $(S(t))_{t\geq 0}$. Now if the complexification of our semigroup can be analytically extended, we again find the same bound as before.

Corollary 2.3.1. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup acting on a real Hilbert space and let $-A$ be its infinitesimal generator. If the complexification of $(S(t))_{t\geq 0}$ can analytically be extended to some sector Σ, then we maintain the bound from Theorem 2.3.2, that is, for all $a \in \mathbb{N}$ there exist constants $C \geq 0$ and $\delta > 0$ such that

$$\|A^a S(t)\|_{\mathscr{L}(H)} \leq C t^{-a} e^{-\delta t}, \quad t \geq 0.$$

Proof. Let $(S_{\mathbb{C}}(t))_{t\geq 0}$ be the analytic extension of $(S(t))_{t\geq 0}$. Then we claim that its infinitesimal generator acting on $H_{\mathbb{C}}$ is given by $-A_{\mathbb{C}}$, where

$$D(A_{\mathbb{C}}) := \{h + gi : h \in D(A), g \in D(A)\},$$
$$A_{\mathbb{C}}(h + gi) := Ah + Agi, \quad (h + gi) \in D(A_{\mathbb{C}}).$$

To see this, we write out the definition and obtain, for $g, h \in D(A)$, that

$$\lim_{t\downarrow 0} \frac{S_{\mathbb{C}}(t)(g + hi) - (g + hi)}{t} = \lim_{t\downarrow 0} \frac{(S(t)g - g) + (S(t)h - h)i}{t} = \lim_{t\downarrow 0} \left(\frac{S(t)g - g}{t} + \frac{S(t)h - h}{t} i \right).$$

Now $\lim_{t\downarrow 0} \left\| \frac{S(t)g-g}{t} + \frac{S(t)h-h}{t} i \right\|^2_{H_{\mathbb{C}}} = \lim_{t\downarrow 0} \left\| \frac{S(t)g-g}{t} \right\|^2_H + \left\| \frac{S(t)h-h}{t} \right\|^2_H = 0$. Furthermore, for h or g outside $D(A)$ we could in the same way show that $\frac{S_{\mathbb{C}}(t)(h+gi)-(h+gi)}{t}$ would not converge. Now using Theorem 2.3.2, we get constants $C \geq 0$ and $\delta > 0$ such that

$$\|A^a_{\mathbb{C}} S_{\mathbb{C}}(t)\|_{\mathscr{L}(H_{\mathbb{C}})} \leq C t^{-a} e^{-\delta t}.$$

But writing out the definition of the operator norm, we see that

$$\|A^a_{\mathbb{C}} S_{\mathbb{C}}(t)\|_{\mathscr{L}(H_{\mathbb{C}})} = \sup_{\|h+gi\|_{H_{\mathbb{C}}}\leq 1} \|A^a_{\mathbb{C}} S_{\mathbb{C}}(t)(h + gi)\|_{H_{\mathbb{C}}} \geq \sup_{\|h+0i\|_{H_{\mathbb{C}}}\leq 1} \|A^a_{\mathbb{C}} S_{\mathbb{C}}(t)h\|_{H_{\mathbb{C}}} = \sup_{\|h\|_H\leq 1} \|A^a S(t)h\|_H,$$

which gives the required bound. □

2.4. Hilbert space valued random variables

In this section some basic principles with regards to probabilistic results for random variables on Hilbert or Banach spaces will be presented.

In general, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let H be a separable Hilbert space equipped with $\mathscr{B}(H)$, the σ-algebra generated by all open sets of H. Then if $X : \Omega \to H$ is a function, we say that X is a random variable if it is measurable. For a family of sets $\mathbf{F}$ in $\mathscr{F}$ we define the σ-algebra generated by $\mathbf{F}$ as the smallest σ-algebra that contains $\mathbf{F}$, notation $\sigma(\mathbf{F})$. Similarly, for a collection of random variables $(X_i)_{i\in I}$ we define $\sigma((X_i)_{i\in I})$ as the smallest σ-algebra for which all X_i are measurable.

Slightly weaker than a σ-algebra is a monotone class:

Definition 2.4.1 (Monotone class)**.** Let Ω be a set. Let 2^Ω denote the family of all subsets of Ω. Then $\mathscr{G} \subseteq 2^\Omega$ is called a monotone class if

- $\Omega \in \mathscr{G}$,
- If $A, B \in \mathscr{G}$ and $A \subseteq B$, then $B \setminus A \in \mathscr{G}$,
- If $(A_n)_{n\geq 1}$ is such that for all $n \geq 1$, we have $A_n \in \mathscr{G}$ and $A_n \subseteq A_{n+1}$, then $\bigcup_{n\geq 1} A_n \in \mathscr{G}$.

It is easy to check that every σ-algebra is also a monotone class. Just like with σ-algebras, for a family $\mathscr{C}$ we write $\mathscr{M}(\mathscr{C})$ to denote the smallest monotone class containing $\mathscr{C}$. In certain cases there is a direct relation between $\mathscr{M}(\mathscr{C})$ and $\sigma(\mathscr{C})$, which is given by the monotone class theorem:

Theorem 2.4.1 (Monotone class theorem)**.** If $\mathscr{C} \subseteq 2^\Omega$ and if for all finite collections of sets $A_1, \cdots, A_n$ in $\mathscr{C}$ we have $\bigcap_{k=1}^n A_k \in \mathscr{C}$, then

$$\mathscr{M}(\mathscr{C}) = \sigma(\mathscr{C}).$$

Proof. See [9, Appendix A1]. □

Now for a random variable $X : \Omega \to H$ which is Bochner integrable, that is

$$\mathbb{E}(\|X\|_H) < \infty,$$

we can define its expectation as

$$\mathbb{E}(X) = \int_\Omega X(\omega)\mathrm{d}\mathbb{P}(\omega).$$

We let $L^1(\Omega; H)$ be the family of all integrable random variables, which is a Banach space if we let

$$\|X\|_{L^1(\Omega;H)} = \int_\Omega \|X(\omega)\|_H \mathrm{d}\mathbb{P}(\omega).$$

Given a random variable X we can define its distribution as a probability measure on μ on $\mathscr{B}(H)$ given by

$$\mu(B) := \mathbb{P}(X^{-1}(B)), \quad B \in \mathscr{B}(H).$$

For a real-valued random variable f, we say that its distribution is Gaussian if there exists constants $m \in \mathbb{R}$ and $\sigma > 0$ such that its distribution ν is given by

$$\nu(A) := \int_A \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \mathrm{d}x, \quad A \in \mathbb{R}.$$

We will sometimes also refer to this as a Gaussian measure. Now for $\sigma = 0$ we say that f is Gaussian if its distribution equals the Dirac measure centered at m, that is, $\nu(A) = 1$ if $m \in A$ and 0 otherwise.

Similarly, for an H-valued random variable X we say that X is a Gaussian random variable if $\langle h, X\rangle_H$ is a real-valued Gaussian random variable for every $h \in H$. It is important to note here that m and σ generally will depend on h for an H-valued Gaussian random variable. The following gives a representation of a Gaussian measure.

Theorem 2.4.2 (Representation Gaussian measure on H)**.** A measure μ on $\mathscr{B}(H)$ is Gaussian if and only if there exists an $m \in H$ and a $Q \in \mathscr{L}(H)$ nonnegative, self-adjoint with finite trace such that for all $h \in H$,

$$\int_H e^{i\langle h, v\rangle_H} \mu(\mathrm{d}v) = e^{i\langle m, h\rangle_H - \frac{1}{2}\langle Qh, h\rangle_H}.$$

Moreover, Q and m are unique in this case.

Proof. See [3, Theorem 2.2.4]. □

We will refer to m as the mean and Q as the covariance (operator). As a result it follows that the distribution of any Gaussian random variable can thus be fully characterized by such an m and Q. We will write $N(m,Q)$ for a Gaussian distribution with mean m and covariance Q. For non-Gaussian random variables $X \in L^2(\Omega;H)$, we can still assign a mean and covariance operator. In this case, the mean m is given by the expectation, and the covariance as the operator Q on H such that for all $h_1,h_2 \in H$ we have

$$\mathbb{E}(\langle X-m,h_1\rangle_H \langle X-m,h_2\rangle_H) = \langle Qh_1,h_2\rangle_H,$$

see [4, Section 1.2]. The next theorem gives some properties of H-valued random variables, which are similar to what you would expect from the real-valued case. It also states that the covariance for Gaussian random variables, as defined earlier, is the same as the usual definition of the covariance operator.

Theorem 2.4.3. If X is an H-valued Gaussian random variable with mean m and covariance Q, then for all $h,g \in H$ the following holds:

- $\mathbb{E}(\langle X,h\rangle_H) = \langle m,h\rangle_H$,
- $\mathbb{E}(\langle X-m,h\rangle_H \langle X-m,g\rangle_H) = \langle Qh,g\rangle_H$,
- $\mathbb{E}(\|X-m\|_H^2) = \operatorname{tr}(Q)$.

Proof. See [12, Theorem 2.1.4]. □

If in addition to an integrable random variable X we are also given another σ-algebra $\mathscr{G}$, we can define the conditional expectation.

Definition 2.4.2 (Conditional expectation)**.** If X is a random variable from $(\Omega,\mathscr{F},\mathbb{P})$ to H and $\mathscr{G}$ is a sub-σ-algebra of $\mathscr{F}$, then we define the conditional expectation of X given $\mathscr{G}$, notation $\mathbb{E}(X|\mathscr{G})$, as the unique random variable $Y:\Omega \to H$ satisfying:

- Y is $\mathscr{G}$-measurable;
- Y is integrable;
- If $G \in \mathscr{G}$, then $\int_G Y \mathrm{d}\mathbb{P} = \int_G X \mathrm{d}\mathbb{P}$.

We will not show existence or uniqueness of Y here, see [4, Section 1.3] for this. Note that it is also possible to condition on a different random variable Z instead, which is defined as conditioning on $\sigma(Z)$ (the smallest σ-algebra for which Z is measurable).

Next we will present some properties of the conditional expectation:

Theorem 2.4.4 (Properties of the conditional expectation)**.** Let X,Y be random variables mapping from $(\Omega,\mathscr{F},\mathbb{P})$ to H and let $\mathscr{G}$ be a sub-σ-algebra of $\mathscr{F}$, then the following hold:

- $\mathbb{E}(\mathbb{E}(X|\mathscr{G})) = \mathbb{E}(X)$,
- For $a,b \in \mathbb{R}$ we have that $\mathbb{E}(aX+bY|\mathscr{G}) = a\mathbb{E}(X|\mathscr{G}) + b\mathbb{E}(Y|\mathscr{G})$, $\mathbb{P}$-a.s.
- If X is $\mathscr{G}$-measurable, then $\mathbb{E}(X|\mathscr{G}) = X$, $\mathbb{P}$-a.s.
- If X and $\mathscr{G}$ are independent, then $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}(X)$, $\mathbb{P}$ -a.s.
- If $\mathscr{J}$ is a further sub-σ algebra of $\mathscr{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathscr{G})|\mathscr{J}) = \mathbb{E}(X|\mathscr{J})$, $\mathbb{P}$-a.s.
- $\|\mathbb{E}(X|\mathscr{G})\|_H \le \mathbb{E}(\|X\|_H|\mathscr{G})$.

We are mostly interested in families of random variables, usually indexed by the (nonnegative) real line.

Definition 2.4.3 (H-valued stochastic process)**.** Let I be an index set and H a Hilbert space. Then we call $(X(i))_{i\in I}$ an H-valued stochastic process if $X(i)$ is an H-valued random variable for all $i \in I$. If moreover $X(i)$ is integrable for all $i \in I$, then we say that $(X(i))_{i\in I}$ is an integrable H-valued stochastic process.

In the same way we can define square integrable H-valued stochastic processes. Next we will consider a notion of measurability for these stochastic processes.

Definition 2.4.4 (Filtrations and adaptedness)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $(X(t))_{t\geq 0}$ be an H-valued stochastic process. A family $(\mathscr{F}_t)_{t\geq 0}$ of σ-algebras is a filtration if for $s < t$ we have $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$. We say that $(X(t))_{t\geq 0}$ is adapted to a filtration $(\mathscr{F}_t)_{t\geq 0}$ if $X(t)$ is $\mathscr{F}_t$-measurable for all t. A process is always adapted to its natural filtration, given by

$$\mathscr{F}_t^X := \sigma(X_s, s \leq t), \quad t \in [0, \infty).$$

For certain stochastic processes we can define differentiability as follows:

Definition 2.4.5 (Mean square differentiability)**.** Let $J \subseteq \mathbb{R}$ be a possibly unbounded interval and let $(X(t))_{t\in J}$ be a square integrable H-valued stochastic process. Then we say that X is mean square differentiable in a point $t \in J$ if there exists a square integrable H-valued stochastic process $(Y(t))_{t\in J}$ such that

$$\lim_{h\to 0} \mathbb{E}\left(\left\| \frac{X(t+h) - X(t)}{h} - Y(t) \right\|_H^2 \right) = 0,$$

where in the limit we only consider h such that $t + h \in J$. In this case we call the process Y the mean square derivative of X.

We will now consider Gaussian processes, which are very important for the stochastic integral defined later.

Definition 2.4.6 (Gaussian process)**.** Let $(X(t))_{t\geq 0}$ be an H-valued stochastic process. Then X is called Gaussian if for all finite collections of times $t_1, \cdots, t_n$ the vector $(X(t_1), \cdots, X(t_n))$ is jointly Gaussian, that is, a Gaussian random variable on H^n.

Definition 2.4.7 (Martingale)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $(X(t))_{t\geq 0}$ be an H-valued process adapted to a filtration $(\mathscr{F}_t)_{t\geq 0}$. Then we say that $(X(t))_{t\geq 0}$ is a martingale if it is integrable for every t and for all $s < t$ we have the martingale property:

$$\mathbb{E}(X(t)|\mathscr{F}_s) = X(s), \quad \mathbb{P}\text{-a.s.}$$

The intuitive interpretation of a martingale is a process for which the current value is always the best predictor for the future values.

Conditional probabilities given a σ-algebra are defined as follows.

Definition 2.4.8 (Conditional probability)**.** Given a σ-algebra $\mathscr{G}$ and an event $A \in \mathscr{F}$, the conditional probability of A given $\mathscr{G}$ is defined up to $\mathbb{P}$-null sets as

$$\mathbb{P}(A|\mathscr{G}) := \mathbb{E}(\mathbb{1}_A|\mathscr{G}).$$

In general, conditional probabilities are not measures despite their name: although we have by the monotone convergence theorem for disjoint events $(A_n)_{n\geq 0}$ the equality

$$\mathbb{P}(\cup_{n\geq 1} A_n|\mathscr{G}) = \sum_{n\geq 1} \mathbb{P}(A_n|\mathscr{G}) \quad \mathbb{P}\text{-a.s.},$$

this almost sure set will in general depend on the chosen A_n.

Next it is possible to define conditional independence. Recall that two sets A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. The following definition extends this principle.

Definition 2.4.9 (Conditional independence)**.** Suppose $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{G}$ are σ-algebras. We say that $\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent given $\mathscr{G}$ if

$$\mathbb{P}(A \cap B|\mathscr{G}) = \mathbb{P}(A|\mathscr{G})\mathbb{P}(B|\mathscr{G}) \quad \mathbb{P}\text{-a.s.}, \quad A \in \mathscr{F}_1,\ B \in \mathscr{F}_2.$$

Just like with conditional expectations, it is also possible to have probabilities conditioned on a random variable X, which is defined as taking the conditioning on $\sigma(X)$. Note that if for some fixed sets A and B we let $\mathscr{F}_1 = \{\emptyset, A, A^c, \Omega\}$ and $\mathscr{F}_2 = \{\emptyset, B, B^c, \Omega\}$ and condition on the trivial σ-algebra, we get that $\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent iff A and B are independent in the classical setting.

Next we will look at the simple Markov property. Intuitively, a process is simple Markov if the future is independent of the past given the present.

Definition 2.4.10 (Simple Markov property)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $(X(t))_{t\geq 0}$ be a stochastic process taking values in a separable Hilbert space H. We say that X is simple Markov with respect to a filtration $(\mathscr{G}_t)_{t\geq 0}$ or has the simple Markov property if $(X(t))_{t\geq 0}$ is adapted to $(\mathscr{G}_t)_{t\geq 0}$ and for all $s < t$, $A \in \mathscr{B}(H)$ we have

$$\mathbb{P}(X(t) \in A|\mathscr{G}_s) = \mathbb{P}(X(t) \in A|X(s)), \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

If the filtration is not specified, then simple Markov will be with respect to the natural filtration.

Sometimes a process is almost Markov, where we need to condition on a bit more than just $X(s)$ in order for Equation (2.1) to hold.

Definition 2.4.11 (Multiple Markov property)**.** An H-valued process $(X(t))_{t\geq 0}$ is called n-ple Markov with respect to a filtration $(\mathscr{G}_t)_{t\geq 0}$ if it is adapted to $(\mathscr{G}_t)_{t\geq 0}$, it is $n-1$ times differentiable (in mean square sense) and for all $0 \leq s < t$, $A \in \mathscr{B}(H)$ we have

$$\mathbb{P}(X(t) \in A|\mathscr{F}_s) = \mathbb{P}(X(t) \in A|X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s)).$$

While intuitively clear, the definition for multiple Markov can sometimes be difficult to use in practice. The next theorem gives two equivalent statements that can sometimes be easier to work with. We define $\mathbb{B}_b(H;\mathbb{R})$ as all measurable and bounded functions from H to $\mathbb{R}$.

Theorem 2.4.5. Let $(X(t))_{t\geq 0}$ be an H-valued stochastic process adapted to a filtration $(\mathscr{G}_t)_{t\geq 0}$. Then the following are equivalent:

1. $(X(t))_{t\geq 0}$ is an n-ple Markov process with respect to a filtration $(\mathscr{G}_t)_{t\geq 0}$.
2. For all $\phi \in \mathbb{B}_b(H;\mathbb{R})$ and $0 \leq s \leq t$ it holds that
$$\mathbb{E}(\phi(X(t))|\mathscr{G}_s) = \mathbb{E}(\phi(X(t))|X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s)). \tag{2.2}$$
3. For all $0 \leq s \leq t$, $\mathscr{G}_s$ is conditionally independent of $X(t)$ given $(X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s))$.

Proof. $2 \to 1$ follows immediately by taking $\phi = \mathbb{1}_A$ for any $A \in \mathscr{B}(H)$. Conversely, suppose X is n-ple Markov. Note that if $\phi = \mathbb{1}_A$ for some $A \in \mathscr{B}(H)$, then (2.2) is clearly satisfied. If ϕ is a simple function, so $\phi = \sum_{i=1}^n c_i 1_{A_i}$ with $A_i \in \mathscr{B}(H)$ and $c_i \in \mathbb{R}$, then (2.2) follows from linearity of the conditional expectation (Theorem 2.3.3). Now let ϕ be any bounded measurable function. Then there exists a sequence of simple functions $(\phi_n)_{n\geq 1}$ that increase to ϕ. But then, using the monotone convergence theorem for conditional expectations, we get that

$$\begin{aligned}\mathbb{E}(\phi(X(t))|\mathscr{G}_s) &= \lim_{n\to\infty} \mathbb{E}(\phi_n(X(t))|\mathscr{G}_s) = \lim_{n\to\infty} \mathbb{E}(\phi_n(X(t))|X(s)) \\ &= \mathbb{E}(\phi(X(t))|X(s)).\end{aligned}$$

Lastly, the equivalence of 1 and 3 follows from [7, Theorem 8.9]. For the choice $\mathscr{F} = \mathscr{G}_s, \mathscr{G} = \sigma(X(s))$ and $\mathscr{H} = \sigma(X(t))$, this gives us that $\mathscr{G}_s$ is conditionally independent of $X(t)$ given $X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s)$ if and only if for all $A \in \sigma(X(t))$ we have

$$\mathbb{P}(A|X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s)) = \mathbb{P}(A|\mathscr{F}_s^X).$$

Note that every set in $\sigma(X(t))$ can be written in the form $\{X(t) \in B\}$ for some $B \in \mathscr{B}(H)$. So we indeed find the required equivalence. □

The same equalities also hold for simple Markov processes (by considering 1-ple multiple Markov processes).

Theorem 2.4.6. Let $(X(t))_{t\geq 0}$ be an H-valued stochastic process. Let $Y : \Omega \to \mathbb{R}$ be an integrable, $\mathscr{F}_s^X$-measurable random variable and let $\phi \in \mathbb{B}_b(H;\mathbb{R})$. If for all $n \in \mathbb{N}$ and all finitely many times $s_1 < \cdots < s_n = s$ we have

$$\mathbb{E}(\phi(X(t))|X(s_1), \cdots, X(s_n)) = Y \quad \mathbb{P}\text{-a.s.}, \tag{2.3}$$

then it holds that

$$\mathbb{E}(\phi(X(t))|\mathscr{F}_s^X) = Y \quad \mathbb{P}\text{-a.s.} \tag{2.4}$$

'

Proof. Assume that for all $s_1 < \cdots < s_n = s$, (2.3) holds. Since Y is already $\mathscr{F}_s^X$-measurable and integrable, by definition of the conditional expectation it is enough to show that

$$\mathbb{E}(\mathbb{1}_A \phi(X(t))) = \mathbb{E}(\mathbb{1}_A Y), \quad \forall A \in \mathscr{F}_s^X.$$

Let $\mathscr{C} := \{\bigcap_{i=1}^n A_i : n \in \mathbb{N}, s_1 < \cdots < s_n = s, A_i \in \sigma(X(s_i))\}$. $\mathscr{C}$ is clearly closed under taking finite intersections, so by the monotone class theorem we find

$$\mathscr{M}(\mathscr{C}) = \sigma(\mathscr{C}).$$

Let $\mathscr{G} = \{A \in \mathscr{F} : \mathbb{E}(\mathbb{1}_A \phi(X(t))) = \mathbb{E}(\mathbb{1}_A Y)\}$. Observe that $\mathscr{G}$ is a monotone class:

$\Omega \in \mathscr{G}$ is trivial by taking expectations on both sides of Equation (2.3). If $A, B \in \mathscr{G}$ and $A \subseteq B$, then using linearity

$$\mathbb{E}(\mathbb{1}_{B\setminus A}\phi(X(t))) = \mathbb{E}((\mathbb{1}_B - \mathbb{1}_A)\phi(X(t))) = \mathbb{E}((\mathbb{1}_B - \mathbb{1}_A)Y) = \mathbb{E}(\mathbb{1}_{B\setminus A}Y),$$

hence $B \setminus A \in \mathscr{G}$.

Lastly, if $(A_n)_{n\geq 1}$ is an increasing sequence of sets in $\mathscr{G}$, set $A = \bigcup_{i=1}^\infty A_i$. Now since the A_n are increasing, we use monotone convergence (ϕ is bounded) to find

$$\mathbb{E}(\mathbb{1}_A \phi(X(t))) = \mathbb{E}(\lim_{n\to\infty} \mathbb{1}_{A_n}\phi(X(t))) = \lim_{n\to\infty} \mathbb{E}(\mathbb{1}_{A_n}\phi(X(t))) = \lim_{n\to\infty} \mathbb{E}(\mathbb{1}_{A_n} Y) = \mathbb{E}(\lim_{n\to\infty} \mathbb{1}_{A_n} Y) = \mathbb{E}(\mathbb{1}_A Y),$$

so $A \in \mathscr{G}$.

Next, for $A \in \mathscr{C}$ there exists $s_1, \cdots, s_n$ such that $A = \bigcap_{i=1}^n A_i$, $A_i \in \sigma(X(s_i))$. But then $A \in \sigma(X(s_1), \cdots, X(s_n))$, so by Equation (2.3) and the definition of the conditional expectation we have

$$\mathbb{E}(\phi(X(t))\mathbb{1}_A) = \mathbb{E}(Y\mathbb{1}_A).$$

So from this it follows that $A \in \mathscr{G}$, hence $\mathscr{C} \subseteq \mathscr{G}$. Since $\mathscr{G}$ is a monotone class, we thus find $\sigma(\mathscr{C}) = \mathscr{M}(\mathscr{C}) \subseteq \mathscr{G}$. So if we can now show that $\sigma(\mathscr{C}) = \mathscr{F}_s^X$, we obtain

$$\mathscr{F}_s^X \subseteq \mathscr{G},$$

so we would be done. Indeed, $\mathscr{C} \subseteq \mathscr{F}_s^X$, hence $\sigma(\mathscr{C}) \subseteq \mathscr{F}_s^X$ since $\mathscr{F}_s^X$ is a σ-algebra.

On the other hand, $\mathscr{F}_s^X = \sigma(\bigcup_{u\leq s}\sigma(X(u)))$. Now $\sigma(X(u)) \subseteq \mathscr{C}$ for all $u \leq s$, hence $\bigcup_{u\leq s}\sigma(X(u)) \subseteq \mathscr{C}$, so

$$\mathscr{F}_s^X = \sigma\left(\bigcup_{u\leq s}\sigma(X(u))\right) \subseteq \sigma(\mathscr{C}).$$

So $\mathscr{F}_s^X \subseteq \mathscr{G}$, from which the claim follows. □

Using this theorem, we can now state and show a sufficient condition for a Gaussian process to be multiple Markov.

Theorem 2.4.7 (Multiple Markov for Gaussian processes)**.** Let $(X(t))_{t\geq 0}$ be a Gaussian process on H that is $n-1$ times mean square differentiable. If for all $s < t$,

$$\mathbb{E}(X(t)|\mathscr{F}_s^X) = \mathbb{E}(X(t)|X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s)), \quad \mathbb{P}\text{-a.s.}, \tag{2.5}$$

then X is n-ple Markov.

Proof. Fix $\phi \in \mathbb{B}_b(H,\mathbb{R})$. Note that $\mathbb{E}(X(t)|X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s))$ is integrable and $\mathscr{F}_s^X$-measurable, so by Theorem 2.4.6 and Theorem 2.4.5 it is sufficient to show for arbitrary $s_1 < \cdots < s_n = s < t$ that

$$\mathbb{E}(\phi(X(t))|X(s_1), \cdots, X(s_n)) = \mathbb{E}(\phi(X(t))|X(s), X^{(1)}(s), \cdots, X^{(n-1)}(s)), \quad \mathbb{P}\text{-a.s.}$$

Now define $Z_1 = \mathbb{E}(X(t)|X(s_1), \cdots, X(s_n))$ and let $Z_2 = X(t) - Z_1$. Then since X is Gaussian, we know that the vector $(X(t), X(s_1), \cdots, X(s_n))$ is jointly Gaussian, hence Z_2 is independent of $(X(s_1), \cdots, X(s_n))$ [3, Theorem 3.10.1]. So we obtain that

$$\mathbb{E}(\phi(X(t))|X(s_1), \cdots, X(s_n)) = \mathbb{E}(\phi(Z_1 + Z_2)|X(s_1), \cdots, X(s_n)), \quad \mathbb{P}\text{-a.s.}$$

Now Z_1 is $\sigma(X(s_1),\cdots,X(s_n))$-measurable and Z_2 is independent of $\sigma(X(s_1),\cdots,X(s_n))$. If we set $\psi:\Omega\to H$ as $\psi(\omega):=\mathbb{E}(\phi(Z_1(\omega)+Z_2))$, then using the freezing lemma [1, Lemma 4.1] we obtain (for almost all ω) that

$$\mathbb{E}(\phi(Z_1+Z_2)|X(s_1),\cdots,X(s_n))(\omega)=\psi(\omega).$$

Now by the assumption, $Z_1=\mathbb{E}(X(t)|X(s),X^{(1)}(s),\cdots,X^{(n-1)}(s))$ $\mathbb{P}$-a.s., so using the same argument we find that Z_1 is $\sigma(X(s),X(s)^{(1)},\cdots,X(s)^{(n)})$-measurable and Z_2 is independent of $\sigma(X(s))$. But then, again using the freezing lemma, we obtain

$$\mathbb{E}(\phi(X(t))|X(s))(\omega)=\mathbb{E}(\phi(Z_1+Z_2)|X(s))(\omega)=\psi(\omega),\quad \mathbb{P}\text{-a.s.}$$

So we indeed obtain $\mathbb{E}(\phi(X(t))|X(s_1),\cdots,X(s_n))=\mathbb{E}(\phi(X(t))|X(s),X^{(1)}(s),\cdots,X^{(n-1)}(s))$, $\mathbb{P}$-a.s, which completes the proof. □

2.5. Two-sided Wiener process

In this section we will define the two-sided H-valued Wiener process which will be needed later to construct the stochastic integral on $\mathbb{R}$. Let $Q\in\mathscr{L}(H)$ be a nonnegative, self-adjoint operator with finite trace. Now let W_1 and W_2 be two independent H-valued Q-Wiener processes on $[0,\infty)$. Now define

$$W(t)=\begin{cases} W_1(t) & t\geq 0,\\ W_2(-t) & t<0.\end{cases}$$

We will now show that this motion satisfies the properties you would expect from a Wiener process:

1. For all $t>s\geq t'>s'$ we have that $W(t)-W(s)$ and $W(t')-W(s')$ are independent,
2. $W(t)-W(s)$ is $N(0,Q(t-s))$ for all $t>s$,
3. Continuous sample paths $\mathbb{P}$-a.s.,
4. $W(0)=0$ $\mathbb{P}$-a.s.

The last property is trivial from the definition. Continuous sample paths follows from the continuity of the original Wiener processes, combined with the fact that both vanish at $t=0$.

Now take $s<t$. If both $s\leq 0$ and $t\leq 0$ or $s\geq 0$ and $t\geq 0$, the distribution of $W(t)-W(s)$ follows from the properties of W_1 and W_2. If $s<0<t$, then $W(t)-W(s)=W(t)-W(0)+W(0)-W(s)$. Now $W(t)-W(0)$ and $W(0)-W(s)$ are both independent because W_1 and W_2 are independent, and respectively $N(0,Qt)$ and $N(0,Q(-s))$ distributed. So as a sum of two independent normal distributed random variables, it follows that $W(t)-W(s)$ is $N(0,Q(t-s))$ distributed.

For the independence, let $t>s\geq u>v$. We want to show that $W(t)-W(s)$ and $W(u)-W(v)$ are independent random variables. Now if $s\geq 0\geq u$, the result is trivial since W_1 and W_2 are independent. If $v\geq 0$ or $0\geq t$, the result also follows immediately from the independent increments of W_1 and W_2 respectively. Lastly, we will show the case $t\geq 0\geq s$ and note that $u\geq 0\geq v$ works very similarly.

In this case we can write $W(t)-W(s)=(W(t)-W(0))+(W(0)-W(s))$. Now $W(t)-W(0)$ and $W(u)-W(v)$ are independent since W_1 and W_2 are independent. Next, $W(0)-W(s)$ and $W(u)-W(v)$ are independent because W_2 as a Wiener process has independent increments. As a result, we conclude that $W(t)-W(s)$ and $W(u)-W(v)$ are independent.

The main downside of this two-sided Wiener process is that it is no longer a martingale: If we consider $\mathbb{E}(W_0|\mathscr{F}_t^W)$, for $t<0$, then this will almost surely be equal to 0 because of the restriction of $W(0)=0$ a.s. But the martingale property would require $\mathbb{E}(W_0|\mathscr{F}_t^W)=W_t$ a.s., which would give $W_t=0$ a.s., which is clearly not the case for nontrivial Q. In fact, we can show that any process that satisfies the first two properties of the Wiener process, namely the independent increments and the normal distribution of the increments, is not a martingale with respect to any filtration:

Theorem 2.5.1. Suppose $(X(t))_{t\in\mathbb{R}}$ is a process with independent increments. Suppose for all $s<t$ its increments $X(t)-X(s)$ are $N(0,(t-s)Q)$ distributed with nonzero covariance Q. Then $X(t)$ is not a martingale with respect to any filtration.

The proof relies on backward martingales, which can be thought of as some sort of reversed martingale.

Definition 2.5.1 (Backward martingale)**.** Let I either be $\mathbb{N}$ or $[0,\infty)$. An H-valued stochastic process $(X(t))_{t\in I}$ is called a backward martingale with respect to a decreasing family of σ-algebras $(\mathscr{F}_t)_{t\in I}$, that is, for $s<t$ we have $\mathscr{F}_t \subseteq \mathscr{F}_s$, if it satisfies

1. $(X(t))_{t\in I}$ is $(\mathscr{F}_t)_{t\in I}$ adapted,
2. $X(t)$ is integrable for every t,
3. $\mathbb{E}(X(s)|\mathscr{F}_t) = X(t)$ for all $s<t$.

Note that a backward martingale $(X(t))_{t\geq 0}$ with respect to a filtration $(\mathscr{F}_t)_{t\geq 0}$ also gives a backward martingale indexed by $\mathbb{N}$ by only considering $(X(t))_{t\in\mathbb{N}}$ and $(\mathscr{F}_t)_{t\in\mathbb{N}}$. We will also need the following theorem.

Theorem 2.5.2 (Backward martingale theorem)**.** Let $(X(n))_{n\in\mathbb{N}}$ be a backward martingale with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$. Then $\lim_{n\to\infty} X(n)$ exists and the convergence is both almost surely and in $L^1(\Omega;H)$.

Proof. See [5, Chapter 12.7, Theorem 4]. □

There is a direct relation between backward martingales and martingales on the entire real line. If $(X(t))_{t\in\mathbb{R}}$ is a $\mathscr{F}_t$-martingale indexed by $\mathbb{R}$, then for $t\geq 0$ define $Y(t) = X(-t)$ and let $\mathscr{G}_t := \mathscr{F}_{-t}$. Then it is easy to see that $(Y(t))_{t\geq 0}$ is a backward martingale with respect to $(\mathscr{G}_t)_{t\geq 0}$. Using this we get the following result.

Theorem 2.5.3. Any real-valued process indexed by $\mathbb{R}$ with independent, stationary, and nonzero increments is not a martingale.

Proof. Let $(X(t))_{t\in\mathbb{R}}$ be a process with stationary, independent nontrivial increments and suppose it is a martingale with respect to some filtration $(\mathscr{F}_t)_{t\in\mathbb{R}}$. Then as explained earlier, we can define a backward martingale $(Y(n))_{n\in\mathbb{N}}$ with respect to $(\mathscr{G}_n)_{n\in\mathbb{N}}$ by defining $Y(n) := X(-n)$ and $\mathscr{G}_n := \mathscr{F}_{-n}$ for $n\in\mathbb{N}$. Together with the backward martingale convergence theorem, this gives that $\lim_{n\to\infty} Y(n) = \lim_{n\to-\infty} X(-n)$ exists in $L^1(\Omega;\mathbb{R})$. But this gives a contradiction since $(X(-n))_{n\in\mathbb{N}}$ is not even Cauchy: For any N, for example, we can consider

$$\|X_{-N} - X_{-(N+1)}\|_{L^1(\Omega;\mathbb{R})} = \|X_{-1} - X_0\|_{L^1(\Omega;\mathbb{R})} \neq 0,$$

since we assumed the increments were nontrivial. So we can never have convergence, which is a contradiction, hence $(X(t))_{t\in\mathbb{R}}$ can not be a martingale. □

With this result we are in a position to prove Theorem 2.5.1. The idea is to reduce the H-valued process to an $\mathbb{R}$-valued process and use Theorem 2.5.3.

Proof of Theorem 2.5.1. We prove by contradiction. Suppose $(X(t))_{t\in\mathbb{R}}$ is a martingale with respect to some filtration $(\mathscr{G}_t)_{t\in\mathbb{R}}$. Let h be an eigenvector of Q corresponding to a nonzero (positive) eigenvalue λ. Now consider

$$Z(t) := \langle X(t), h\rangle_H.$$

Now it immediately follows that $(Z(t))_{t\in\mathbb{R}}$ is also a real-valued martingale. We will show that $(Z(t))_{t\in\mathbb{R}}$ also has independent normally distributed increments, which would imply that it is not a martingale, which is a contradiction. Independence of the increments follows from $(X(t))_{t\in\mathbb{R}}$ having independent increments. For the distribution we find that

$$Z(t) - Z(s) = \langle X(t) - X(s), h\rangle_H, \quad s<t.$$

Now by Theorem 2.4.3, $Z(t) - Z(s)$ is indeed normally distributed with mean 0 and variance

$$(t-s)\langle Qh, h\rangle_H = (t-s)\lambda\|h\|_H^2.$$

So we see that $(Z(t))_{t\in\mathbb{R}}$ is a process with stationary, independent and nontrivial increments, so by Theorem 2.5.3 we find that $(Z(t))_{t\in\mathbb{R}}$ is not a martingale. So we obtain a contradiction, so $(X(t))_{t\in\mathbb{R}}$ is not a martingale. □

Note that we did not need an eigenvector h: any h for which $\langle Qh, h\rangle > 0$ would have also worked.

3

Matérn type process on the real line

The mild solution process considered in Equation (1.3) has the disadvantage of not being restartable. To circumvent this we will consider the modified process

$$X_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{t} (t-r)^{\gamma-1} S(t-r) \mathrm{d}W(r), t \in (-\infty, T]. \tag{3.1}$$

It is not clear on first glance how to interpret this stochastic integral starting at $-\infty$. Making sense of this in a meaningful way will be the first focus of this section. Afterwards the differentiability of the process in a mean square sense will be investigated.

3.1. Stochastic integration on the real line

Now we will define the stochastic integral for simple deterministic functions $\Phi : (-\infty, T] \to \mathscr{L}_2(H)$ of the form

$$\Phi(t) = \sum_{i=1}^{n} \Phi_i \mathbb{1}_{[t_{i-1}, t_i)}(t), \quad t \in (-\infty, T].$$

where Φ_i is an element of $\mathscr{L}_2(H)$ and $t_0, t_1, \cdots, t_n \in \mathbb{R}$ are increasing values between $-\infty$ and T. Now for this class of functions we define

$$\int_{-\infty}^{t} \Phi(r) \mathrm{d}W(r) := \sum_{i=1}^{n} \Phi_i (W(t_i \wedge t) - W(t_{i-1} \wedge t)).$$

We will sometimes shorten this to $I_\Phi(t)$. However, if we consider the standard filtration $(\mathscr{F}_t^W)_{t\geq 0}$, we run into a problem. Take for example $\Phi = 1_{[-1,0)}$. Then for $t \in [-1, 0]$, we find that $\int_{-\infty}^{t} \Phi(r) \mathrm{d}W(r) = W(t) - W(-1)$. Now for any s satisfying $-1 \leq s < 0$ we obtain

$$\mathbb{E}(I_\phi(0)|\mathscr{F}_s) = \mathbb{E}\left(\int_{-\infty}^{0} \Phi(r) \mathrm{d}W(r) | \mathscr{F}_s \right) = \mathbb{E}(-W(-1)|\mathscr{F}_s) = -W(-1) \neq W(s) - W(-1) = I_\phi(s).$$

So $I_\Phi(t)$ is now no longer a martingale.

In order to obtain a martingale again, we will consider the increment filtration $(\widetilde{\mathscr{F}}_t^W)_{t\geq 0}$, which will be defined as $\widetilde{\mathscr{F}}_t^W := \sigma(W(t) - W(s) : s \leq t)$. We claim that $(I_\Phi(t))_{t\geq 0}$ is in fact a martingale with respect to $(\widetilde{\mathscr{F}}_t^W)_{t\geq 0}$: Adaptedness follows since $W(t_i) - W(t_{i-1})$ is $\widetilde{\mathscr{F}}_t^W$-measurable for $t \geq t_i$. For the martingale property we will show that

$$\mathbb{E}((W(t_i \wedge t) - W(t_{i-1} \wedge t))|\widetilde{\mathscr{F}}_s^W) = W(t_i \wedge s) - W(t_{i-1} \wedge s).$$

Now since the conditional expectation is linear, this is enough to show that $\int_{-\infty}^{t} \Phi(r) \mathrm{d}W(r)$ is a martingale. Fix $s < t$. There are multiple cases: either $s > t_i$, $t_i \geq s \geq t_{i-1}$ or $t_{i-1} > s$.

In the first case, we get

$$\mathbb{E}(W(t_i \wedge t) - W(t_{i-1} \wedge t)|\widetilde{\mathscr{F}}_s^W) = W(t_i \wedge t) - W(t_{i-1} \wedge t) = W(t_i \wedge s) - W(t_{i-1} \wedge s),$$

where we used that $W(t_i \wedge t) - W(t_{i-1} \wedge t)$ is $\widetilde{\mathscr{F}}_s^W$-measurable. If $t_i \geq s \geq t_{i-1}$, then we find, since because of independence $\mathbb{E}(W(t_i \wedge t) - W(s)|\widetilde{\mathscr{F}}_s^W) = \mathbb{E}(W(t_i \wedge t) - W(s)) = 0$, that

$$\begin{aligned}\mathbb{E}(W(t_i \wedge t) - W(t_{i-1} \wedge t)|\widetilde{\mathscr{F}}_s^W) &= E((W(t_i \wedge t) - W(s) + W(s) - W(t_{i-1} \wedge t))|\widetilde{\mathscr{F}}_s^W) \\ &= W(s) - W(t_{i-1} \wedge t) \\ &= W(t_i \wedge s) - W(t_{i-1} \wedge s).\end{aligned}$$

Lastly, if $t_{i-1} > s$, then we immediately get

$$\mathbb{E}((W(t_i \wedge t) - W(t_{i-1} \wedge t))|\widetilde{\mathscr{F}}_s^W) = 0 = W(t_i \wedge s) - W(t_{i-1} \wedge s).$$

So in every case the martingale property holds, so by linearity the entire integral has the martingale property.

Note that for $t \geq 0$ we have $\mathscr{F}_t^W = \widetilde{\mathscr{F}}_t^W$:
For any $s < t$ we have that W_t and W_s are $\mathscr{F}_t^W$-measurable, so $W_t - W_s$ is also $\mathscr{F}_t^W$-measurable, which gives $\widetilde{\mathscr{F}}_t^W \subseteq \mathscr{F}_t^W$. On the other hand we have that $W_t = W_t - W_0$, so we get that W_t is $\widetilde{\mathscr{F}}_t^W$-measurable, so $\mathscr{F}_t^W \subseteq \widetilde{\mathscr{F}}_t^W$ also follows. Now for $t < 0$, this second part fails, so we only have $\widetilde{\mathscr{F}}_t^W \subseteq \mathscr{F}_t^W$.

Moreover, $I_\Phi(t)$ is square integrable for every t, that is,

$$\mathbb{E}(\|I_\Phi(t)\|_H^2) < \infty.$$

This follows since

$$\|\Phi_i(W(t_i \wedge t) - W(t_{i-1} \wedge t))\|_H \leq \|\Phi_i\|_{\mathscr{L}(H)} \|(W(t_i \wedge t) - W(t_{i-1} \wedge t))\|_H.$$

Now $W(t)$ is square integrable, and since a finite sum of square integrable functions is again square integrable, it follows that $I_\Phi(t)$ is also square integrable.

Theorem 3.1.1. Let M_T^2 denote all continuous square integrable martingales with respect to $(\widetilde{\mathscr{F}}_t^W)_{t\in(-\infty;T]}$. If we take as norm

$$\|M\|_{M_T^2} := \sup_{t\in(-\infty,T]} (\mathbb{E}(\|M(t)\|_H^2))^{\frac{1}{2}},$$

then M_T^2 is a Banach space.

Proof. A proof in the case of a bounded time domain can be found in [12, Proposition 2.2.9], but we will specifically need it in the case of an unbounded time domain.

It turns out that this norm considerably simplifies since we are only considering martingales. For this, note that if $(M(t))_{t\leq T}$ is a martingale, then we can use Theorem 2.4.4 and Jensen's inequality (which also holds for conditional expectations) to find for all $t \leq T$

$$\|M(t)\|_H^2 = \|\mathbb{E}(M(T)|\widetilde{\mathscr{F}}_t^W)\|_H^2 \leq \mathbb{E}(\|M(T)\|_H|\widetilde{\mathscr{F}}_t^W)^2 \leq \mathbb{E}(\|M(T)\|_H^2|\widetilde{\mathscr{F}}_t^W).$$

Now taking expectations on both sides results in

$$\mathbb{E}(\|M(t)\|_H^2) \leq \mathbb{E}(\mathbb{E}(\|M(T)\|_H^2|\widetilde{\mathscr{F}}_t^W)) = \mathbb{E}(\|M(T)\|_H^2).$$

Together this gives us thus

$$\|M\|_{M_T^2} = (\mathbb{E}(\|M(T)\|_H^2))^{\frac{1}{2}},$$

which also shows that the norm is finite. Now $L^2(\Omega; L^\infty(-\infty, T; H))$ is a Banach space with respect to $\|\cdot\|_{M_T^2}$, so we only need to show that M_T^2 is a closed subspace of $L^2(\Omega; L^\infty(-\infty, T; H))$. For this, let $(X_n)_{n\geq 1}$ be a sequence in M_T^2 converging in $\|\cdot\|_{M_T^2}$ to $X \in L^2(\Omega; L^\infty(-\infty, T; H))$. We will show $X \in M_T^2$. Clearly X is square integrable. For continuity, recall that if $(X_n)_{n\geq 1}$ converges to X in $L^2(\Omega; L^\infty(-\infty, T; H))$, then there exists a subsequence $(X_{n_k})_{k\geq 1}$ that converges $\mathbb{P}$-a.s. to X. So for almost all ω we obtain that $X_{n_k}(\omega) \to X(\omega)$ in $L^\infty(-\infty, T; H)$. As the uniform limit of continuous functions, we thus find that $X(\omega)$ is continuous for almost all ω. Lastly, to show that X is a martingale, fix $s \leq t \leq T$. we first show that $\mathbb{E}(X_n(t)|\widetilde{\mathscr{F}}_s^W) \to \mathbb{E}(X(t)|\widetilde{\mathscr{F}}_s^W)$ in $L^1(\Omega; H)$, which implies $\mathbb{P}$-a.s. convergence of a subsequence. For this, note that

$$\begin{aligned}\|\mathbb{E}(X(t)|\widetilde{\mathscr{F}}_s^W) - \mathbb{E}(X_n(t)|\widetilde{\mathscr{F}}_s^W)\|_{L^1(\Omega;H)} &= \mathbb{E}(\|\mathbb{E}(X(t) - X_n(t)|\widetilde{\mathscr{F}}_s^W)\|_H) \leq \mathbb{E}(\mathbb{E}(\|X(t) - X_n(t)\|_H|\widetilde{\mathscr{F}}_s^W)) \\ &= \mathbb{E}(\|X(t) - X_n(t)\|_H) \leq \mathbb{E}(\|X(t) - X_n(t)\|_H^2)^{\frac{1}{2}} \leq \|X - X_n\|_{M_T^2},\end{aligned}$$

which converges to 0 by assumption. So there exists a subsequence $(X_{n_k})_{k\geq 0}$ for which for almost all ω we have that $\mathbb{E}(X_{n_k}(t)|\widetilde{\mathscr{F}}_s^W)(\omega)$ converges to $\mathbb{E}(X(t)|\widetilde{\mathscr{F}}_s^W)(\omega)$ as k goes to infinity. By possibly passing to a further subsequence, we can assume that $(X_{n_k})_{k\geq 0}$ also converges to X almost everywhere. But then, again for almost all ω, we have

$$\mathbb{E}(X(t)|\widetilde{\mathscr{F}}_s^W)(\omega) = \lim_{k\to\infty} \mathbb{E}(X_{n_k}(t)|\widetilde{\mathscr{F}}_s^W)(\omega) = \lim_{k\to\infty} (X_{n_k}(s))(\omega) = (X(s))(\omega).$$

So $X \in M_T^2$, which shows that M_T^2 is closed in $L^2(\Omega; L^\infty(-\infty; T; H))$, and hence a Banach space. □

Now if we calculate $\| \int_{-\infty}^t \Phi(r)\mathrm{d}W(r)\|^2_{M_T^2}$, still for $\Phi(t) = \sum_{i=1}^n \Phi_i \mathbb{1}_{[t_{i-1},t_i)}$, we obtain

$$\begin{aligned}
&\mathbb{E}\left(\left\|\int_{-\infty}^T \Phi(r)\mathrm{d}W(r)\right\|_H^2\right) = \mathbb{E}\left(\left\|\sum_{i=1}^n \Phi_i(W(t_i) - W(t_{i-1}))\right\|_H^2\right)\\
=&\mathbb{E}\left(\sum_{i=1}^n\sum_{j=1}^n \langle \Phi_i(W(t_i) - W(t_{i-1})), \Phi_j(W(t_j) - W(t_{j-1}))\rangle_H\right)\\
=&\mathbb{E}\left(\sum_{i=1}^n \|\Phi_i(W(t_i) - W(t_{i-1}))\|_H^2\right) + \mathbb{E}\left(\sum_{i=1}^n\sum_{j\neq i}^n \langle \Phi_i(W_{t_i} - W_{t_{i-1}}), \Phi_j(W_{t_j} - W_{t_{j-1}})\rangle_H\right).
\end{aligned}$$

Now the latter sum equals zero because of the independence of the increments. Let $(e_k)_{k\geq 1}$ be any orthonormal basis of H. Setting $W(t_{i+1}) - W(t_i) := \Delta_i$, we get

$$\mathbb{E}(\|\Phi_i\Delta_i\|_H^2) = \sum_{k\in\mathbb{N}} \mathbb{E}(\langle \Phi_i\Delta_i, e_k\rangle_H)^2 = \sum_{k\in\mathbb{N}} \mathbb{E}(\langle \Delta_i, \Phi_i^* e_k\rangle_H)^2.$$

Here we can use the definition of the covariance. Then using the fact that Q is nonnegative definite, we can split the Q into $Q^{\frac{1}{2}}Q^{\frac{1}{2}}$ and use that $Q^{\frac{1}{2}}$ is also self-adjoint [15, Theorem VI.9] to obtain

$$\begin{aligned}
\mathbb{E}(\|\Phi_i\Delta_i\|_H^2) &= \sum_{k\in\mathbb{N}} (t_{i+1} - t_i)\langle Q\Phi_i^* e_k, \Phi_i^* e_k\rangle_H\\
&= \sum_{k\in\mathbb{N}} (t_{i+1} - t_i)\|Q^{\frac{1}{2}}\Phi_i^* e_k\|_H^2\\
&= (t_{i+1} - t_i)\|Q^{\frac{1}{2}}\Phi_i^*\|^2_{\mathscr{L}_2(H)}\\
&= (t_{i+1} - t_i)\|\Phi_i Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)},
\end{aligned}$$

where we used that $\|Q^{\frac{1}{2}}\Phi_i^*\|_{\mathscr{L}_2(H)} = \|(Q^{\frac{1}{2}}\Phi_i^*)^*\|_{\mathscr{L}_2(H)}$. Taking the sum over i gives

$$\|I_\phi\|^2_{M_T^2} = \mathbb{E}\left(\left\|\int_{-\infty}^T \Phi(r)\mathrm{d}W(r)\right\|_H^2\right) = \sum_{i=1}^n (t_i - t_{i-1})\|\Phi_i Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} = \int_{-\infty}^T \|\Phi(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}s.$$

So if for simple deterministic functions we define

$$\|\Phi\|_T := \left(\int_{-\infty}^T \|\Phi(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}s\right)^{\frac{1}{2}},$$

then the Itô integral is indeed an isometry from the simple deterministic functions equipped with $\|\cdot\|_T$ to the continuous square integrable martingales equipped with $\|\cdot\|_{M_T^2}$.

Now for general processes $\Phi : (-\infty, T] \to \mathscr{L}_2(H)$ that satisfy $\int_{-\infty}^T \|\Phi(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}s < \infty$, we can define the integral as a limit of integrals of simple functions. Then if $(\Phi_n)_{n\geq 1}$ is a sequence of simple functions converging to Φ in $\|\cdot\|_T$, we define the integral as

$$\int_{-\infty}^T \Phi(r)\mathrm{d}W(r) := \lim_{n\to\infty}\int_{-\infty}^T \Phi_n(r)\mathrm{d}W(r).$$

What is left is to show that this representation does not depend on the choice of Φ_n and that this limit actually exists in M_T^2. We start with the latter. Note that $(\Phi_n)_{n\geq 1}$ is Cauchy, so we obtain by the Itô isometry and the linearity of the stochastic integral that

$$\left\|I_{\phi_n} - I_{\phi_m}\right\|_{M_T^2} = \|\Phi_n - \Phi_m\|_T,$$

so the integral is Cauchy as well in M_T^2. Now since M_T^2 is complete, it follows that this integral indeed has a limit in M_T^2. The independence of representation follows in the same way: if $(\Phi_n)_{n\geq 1}$ and $(\Psi_n)_{n\geq 1}$ are two different sequences of simple functions converging to Φ, then

$$\left\| I_{\phi_n} - I_{\psi_n} \right\|_{M_T^2} = \|\Phi_n - \Psi_n\|_T \leq \|\Phi_n - \Phi\|_T + \|\Psi_n - \Phi\|_T,$$

which converges to 0 as $n \to \infty$. So the integral is indeed well defined this way. It is also possible to relax the finite trace in the definition of Q. A detailed construction of this in the case with stochastic integrands on a bounded domain $[0, T]$ can be found in [12, Chapter 2.5]. The unbounded case follows in the same manner.

3.2. Matérn type process

Now we can proceed with the existence of our process X_γ, as defined in Equation (3.1).

Theorem 3.2.1. Suppose that $\int_0^\infty \|r^{\gamma-1}S(r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}r < \infty$. Then X_γ is mean square continuous, that is, $X_\gamma \in C((-\infty, T]; L^2(\Omega; H))$.

For the proof we will need the following result from [8], which we will restate here.

Lemma 3.2.1. Suppose U is a Hilbert space and $u \in L^2(0,\infty; U)$. For $h \in \mathbb{R}$, let $J_h := ((-h) \vee 0, \infty)$ and define $u_h : J_h \to U$ by

$$u_h(t) := u(t+h), \quad t \in J_h.$$

Then $\lim_{h\to 0} \|u_h - u\|_{L^2(J_h;U)} = 0$.

Proof. See [8, Lemma A.4]. □

Proof of Theorem 3.2.1. We first show that X_γ indeed takes values in $L^2(\Omega; H)$. Applying the Itô isometry gives

$$\|X_\gamma(t)\|^2_{L^2(\Omega;H)} = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^t \|(t-r)^{\gamma-1}S(t-r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}r. \tag{3.2}$$

Performing the change of variables $s = t - r$ indeed gives

$$\|X_\gamma(t)\|^2_{L^2(\Omega;H)} = \frac{1}{\Gamma(\gamma)} \int_0^\infty \|s^{\gamma-1}S(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}s < \infty, \tag{3.3}$$

by assumption.

Next note that, for $t \leq T$ and $0 \leq h \leq T - t$,

$$\begin{aligned}
&\|X_\gamma(t+h) - X_\gamma(t)\|_{L^2(\Omega;H)} \\
&\leq \frac{1}{\Gamma(\gamma)} \left(\left\| \int_t^{t+h} (t+h-r)^{\gamma-1}S(t+h-r)\mathrm{d}W(r) \right\|_{L^2(\Omega;H)} \right. \\
&\quad \left. + \left\| \int_{-\infty}^t [(t+h-r)^{\gamma-1}S(t+h-r) - (t-r)^{\gamma-1}S(t-r)]\mathrm{d}W(r) \right\|_{L^2(\Omega;H)} \right).
\end{aligned}$$

We can rewrite the (square of the) first part using the Itô Isometry and a change of variables to find

$$\begin{aligned}
&\left\| \int_t^{t+h} (t+h-r)^{\gamma-1}S(t+h-r)\mathrm{d}W(r) \right\|^2_{L^2(\Omega;H)} = \int_t^{t+h} \|(t+h-r)^{\gamma-1}S(t+h-r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}r \\
&= \int_0^h \|r^{\gamma-1}S(r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}r.
\end{aligned}$$

Now since $\int_0^\infty \|r^{\gamma-1}S(r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}r < \infty$, this converges to 0 as $h \downarrow 0$.

For the second part we first use the Itô isometry and then perform the change of variables $t - r = s$ to obtain

$$\begin{aligned}
&\left\| \int_{-\infty}^t [(t+h-r)^{\gamma-1}S(t+h-r) - (t-r)^{\gamma-1}S(t-r)]\mathrm{d}W(r) \right\|^2_{L^2(\Omega;H)} \\
&= \int_{-\infty}^t \|(t+h-r)^{\gamma-1}S(t+h-r)Q^{\frac{1}{2}} - (t-r)^{\gamma-1}S(t-r)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}r \\
&= \int_0^\infty \|(s+h)^{\gamma-1}S(s+h)Q^{\frac{1}{2}} - s^{\gamma-1}S(s)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}s.
\end{aligned}$$

Now since by assumption $s^{\gamma-1}S(s)Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$, Lemma 3.2.1 gives that this converges to 0 as $h \downarrow 0$.

Now if $h \le 0$, then we instead find

$$\begin{aligned}
&\|X_\gamma(t+h) - X_\gamma(t)\|_{L^2(\Omega;H)} \\
&\le \frac{1}{\Gamma(\gamma)}\left(\left\|\int_{t+h}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\right\|_{L^2(\Omega;H)}\right. \\
&\quad + \left.\left\|\int_{-\infty}^{t+h}[(t+h-r)^{\gamma-1}S(t+h-r)-(t-r)^{\gamma-1}S(t-r)]\mathrm{d}W(r)\right\|_{L^2(\Omega;H)}\right).
\end{aligned}$$

For the second part we again obtain

$$\begin{aligned}
&\left\|\int_{-\infty}^{t+h}[(t+h-r)^{\gamma-1}S(t+h-r)-(t-r)^{\gamma-1}S(t-r)]\mathrm{d}W(r)\right\|_{L^2(\Omega;H)}^2 \\
&= \int_{-\infty}^{t+h}\|(t+h-r)^{\gamma-1}S(t+h-r)Q^{\frac{1}{2}} - (t-r)^{\gamma-1}S(t-r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\mathrm{d}r \\
&= \int_{-h}^{\infty}\|(s+h)^{\gamma-1}S(s+h)Q^{\frac{1}{2}} - s^{\gamma-1}S(s)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\mathrm{d}s.
\end{aligned}$$

Just like before, this goes to 0 as $h \uparrow 0$ by Lemma 3.2.1.

The first part now gives

$$\begin{aligned}
&\left\|\int_{t+h}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\right\|_{L^2(\Omega;H)}^2 = \int_{t+h}^{t}\|(t-r)^{\gamma-1}S(t-r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\mathrm{d}r \\
&= \int_0^{-h}\|r^{\gamma-1}S(r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\mathrm{d}r,
\end{aligned}$$

which we have already shown goes to 0 as $h \uparrow 0$. Taking everything together we thus find

$$\lim_{h\to 0}\|X_\gamma(t+h) - X_\gamma(t)\|_{L^2(\Omega;H)} = 0,$$

which is what we had to show. □

3.3. Restarting property

In this section we will show that the new process $X_\gamma(t)$ is restartable. For this it is important to realize what imposing an initial condition actually means here. In order to see this, we rewrite

$$\begin{aligned}
X_\gamma(t) &= \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r) \\
&= \frac{1}{\Gamma(\gamma)}\left(\int_{-\infty}^{s}(s-r)^{\gamma-1}S(s-r)\mathrm{d}W(r) - \int_{-\infty}^{s}(s-r)^{\gamma-1}S(s-r)\mathrm{d}W(r)\right. \\
&\quad + \left.\int_{-\infty}^{s}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r) + \int_{s}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\right) \\
&= \frac{1}{\Gamma(\gamma)}\left(\int_{-\infty}^{s}(s-r)^{\gamma-1}S(s-r)\mathrm{d}W(r) + \int_{-\infty}^{s}\left[(t-r)^{\gamma-1}S(t-s)-(s-r)^{\gamma-1}\right]S(s-r)\mathrm{d}W(r)\right. \\
&\quad + \left.\int_{s}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\right) \\
&= X_\gamma(s) + \frac{1}{\Gamma(\gamma)}\left(\int_{-\infty}^{s}\left[(t-r)^{\gamma-1}S(t-s)-(s-r)^{\gamma-1}\right]S(s-r)\mathrm{d}W(r)\right. \\
&\quad + \left.\int_{s}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\right).
\end{aligned}$$

This motivates the following definition for $X_\gamma(t;s,\xi)$:

Definition 3.3.1. We define the process at time t starting at s with initial condition ξ as

$$X_\gamma(t;s,\xi) = \xi + \frac{1}{\Gamma(\gamma)}\left(\int_{-\infty}^{s}\left[(t-r)^{\gamma-1}S(t-s)-(s-r)^{\gamma-1}\right]S(s-r)\mathrm{d}W(r) + \int_{s}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\right).$$

From here, it is possible to show that this modified process is in fact restartable.

Theorem 3.3.1. X_γ is a restartable process. That is, for any $u \le s \le t$ and any ξ,

$$X_\gamma(t;s,X_\gamma(s;u,\xi)) = X_\gamma(t;u,\xi). \tag{3.4}$$

Proof. Writing out the left-hand side, we get

$$\begin{aligned}X_\gamma(t;s,X_\gamma(s;u,\xi)) = \xi + \frac{1}{\Gamma(\gamma)}\bigg(&\int_{-\infty}^{u}\left[(s-r)^{\gamma-1}S(s-u)-(u-r)^{\gamma-1}\right]S(u-r)\mathrm{d}W(r)\\ &+\int_u^s (s-r)^{\gamma-1}S(s-r)\mathrm{d}W(r)+\int_{-\infty}^{s}\left[(t-r)^{\gamma-1}S(t-s)-(s-r)^{\gamma-1}\right]S(s-r)\mathrm{d}W(r)\\ &+\int_s^t (t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\bigg).\end{aligned}$$

Rewriting the part from u to s gives

$$\begin{aligned}X_\gamma(t;s,X_\gamma(s;u,\xi)) = \xi + \frac{1}{\Gamma(\gamma)}\bigg(&\int_{-\infty}^{u}\left[(s-r)^{\gamma-1}S(s-u)-(u-r)^{\gamma-1}\right]S(u-r)\mathrm{d}W(r)\\ &+\int_u^s (t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)+\int_{-\infty}^{u}\left[(t-r)^{\gamma-1}S(t-s)-(s-r)^{\gamma-1}\right]S(s-r)\mathrm{d}W(r)\\ &+\int_s^t (t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\bigg).\end{aligned}$$

Now taking the first and third integral together and the second and fourth integral ultimately gives us

$$\begin{aligned}X_\gamma(t;s,X_\gamma(s;u,\xi)) = \xi + \frac{1}{\Gamma(\gamma)}\bigg(&\int_{-\infty}^{u}[(t-r)^{\gamma-1}S(t-u)-(u-r)^{\gamma-1}]S(u-r)\mathrm{d}W(r)\\ &+\int_u^t (t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)\bigg) = X_\gamma(t;u,\xi),\end{aligned}$$

which proves the theorem. □

3.4. Mean square differentiability

In this section the mean square differentiability of the process will be investigated and a closed form for this will be given. For this we need the following theorem.

Theorem 3.4.1. Suppose $\psi \in H^n(0,\infty;\mathscr{L}(H))$, where $H^n(0,\infty;\mathscr{L}(H))$ is the Sobolev space as defined in Section 2.2, is such that for all $k \in \{0,1,\cdots,n\}$ we have $\psi^{(k)}Q^{\frac{1}{2}}$ in $L^2(0,\infty;\mathscr{L}_2(H))$. Suppose furthermore for $k \in \{0,1,\cdots,n-1\}$ that $\psi^{(k)}$ vanishes continuously at zero, that is, for the continuous version of $\psi^{(k)}$ we have

$$\lim_{t\downarrow 0}\psi^{(k)}(t) = 0. \tag{3.5}$$

Then the function $t \mapsto \int_{-\infty}^{t}\psi(t-s)\mathrm{d}W(s)$ mapping $(-\infty,T]$ to $L^2(\Omega;H)$ is n times differentiable in time and

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^{t}\psi(t-s)\mathrm{d}W(s) = \int_{-\infty}^{t}\psi^{(n)}(t-s)\mathrm{d}W(s), \quad t \in (-\infty,T].$$

Recall that a function in $W^{1,p}(0,\infty;\mathscr{L}(H))$ always has a continuous version, hence the limit in Equation (3.5) is well defined. We will require a theorem from [8] for the proof, which we will restate here.

Theorem 3.4.2. Let U be a Hilbert space and suppose $\psi \in H^1(J;U)$. Define J_h and ψ_h as in Lemma 3.2.1. Then

$$\lim_{h\to 0}\left\|\frac{\psi_h-\psi}{h}-\psi'\right\|_{L^2(J_h;U)} = 0.$$

Proof. See [8, Proposition A.8] □

Proof of Theorem 3.4.1. The proof uses induction. First consider the case $n = 1$ and fix $t \in (-\infty, T]$. Suppose $0 \le h \le T - t$. We will consider

$$\frac{1}{h}\left(\int_{-\infty}^{t+h} \psi(t+h-s)\mathrm{d}W(s) - \int_{-\infty}^{t} \psi(t-s)\mathrm{d}W(s)\right) - \int_{-\infty}^{t} \psi'(t-s)\mathrm{d}W(s)$$
$$= \int_{-\infty}^{t} \left[\frac{1}{h}(\psi(t+h-s) - \psi(t-s)) - \psi'(t-s)\right] \mathrm{d}W(s) + \frac{1}{h}\int_{t}^{t+h} \psi(t+h-s)\mathrm{d}W(s).$$

We will start with the last term. Using the Itô isometry and Theorem 2.2.2, the last term becomes

$$\mathbb{E}\left(\left\|\frac{1}{h}\int_{t}^{t+h} \psi(t+h-s)\mathrm{d}W(s)\right\|_H^2\right) = \frac{1}{h^2}\int_0^h \|\psi(s)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \mathrm{d}s = \frac{1}{h^2}\int_0^h \left\|\int_0^s \psi'(r)\mathrm{d}r\, Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}s.$$
$$\le \frac{1}{h^2}\int_0^h \left(\int_0^s \|\psi'(r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}\mathrm{d}r\right)^2 \mathrm{d}s.$$

Using Cauchy–Schwarz then ultimately gives us

$$\frac{1}{h^2}\int_0^h \left(\int_0^s \|\psi'(r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}\mathrm{d}r\right)^2 \mathrm{d}s \le \frac{1}{h^2}\int_0^h s\int_0^s \|\psi'(r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \mathrm{d}r\mathrm{d}s$$
$$\le \frac{1}{h}\int_0^h \|\psi' Q^{\frac{1}{2}}\|_{L^2(0,s;\mathscr{L}_2(H))}^2 \mathrm{d}s \le \|\psi' Q^{\frac{1}{2}}\|_{L^2(0,h;\mathscr{L}_2(H))}^2.$$

Since $\psi' Q^{\frac{1}{2}}$ is integrable, using dominated convergence shows that this term goes to 0 as $h \downarrow 0$.

For the remaining part, we can first apply the Itô isometry and then the change of variables $r = t - s$ to obtain

$$\mathbb{E}\left(\left\|\int_{-\infty}^{t} \left[\frac{1}{h}(\psi(t+h-s) - \psi(t-s)) - \psi'(t-s)\right] \mathrm{d}W(s)\right\|_H^2\right)$$
$$= \int_{-\infty}^{t} \left\|\left[\frac{1}{h}(\psi(t+h-s) - \psi(t-s)) - \psi'(t-s)\right] Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}s$$
$$= \int_0^{\infty} \left\|\left[\frac{1}{h}(\psi(r+h) - \psi(r)) - \psi'(r)\right] Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}r.$$

Now by Theorem 3.4.2, this goes to 0 as h goes to 0.

Now if we take $h < 0$, we instead find

$$\frac{1}{h}\left(\int_{-\infty}^{t+h} \psi(t+h-s)\mathrm{d}W(s) - \int_{-\infty}^{t} \psi(t-s)\mathrm{d}W(s)\right) - \int_{-\infty}^{t} \psi'(t-s)\mathrm{d}W(s)$$
$$= \int_{-\infty}^{t+h} \left[\frac{1}{h}(\psi(t+h-s) - \psi(t-s)) - \psi'(t-s)\right] \mathrm{d}W(s) - \int_{t+h}^{t} \frac{1}{h}\psi(t-s)\mathrm{d}W(s) - \int_{t+h}^{t} \psi'(t-s)\mathrm{d}W(s).$$

Rewriting the first term gives

$$\mathbb{E}\left(\left\|\int_{-\infty}^{t+h} \left[\frac{1}{h}(\psi(t+h-s) - \psi(t-s)) - \psi'(t-s)\right] \mathrm{d}W(s)\right\|_H^2\right)$$
$$= \int_{-\infty}^{t+h} \left\|\left[\frac{1}{h}(\psi(t+h-s) - \psi(t-s)) - \psi'(t-s)\right] Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}s$$
$$= \int_{-h}^{\infty} \left\|\left[\frac{1}{h}(\psi(r+h) - \psi(r)) - \psi'(r)\right] Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}r,$$

where in the last term we did the substitution $r = t - s$. Just like before, this goes to 0 as $h \uparrow 0$ by Theorem 3.4.2. The second term also reduces to

$$\mathbb{E}\left(\left\|\int_{t+h}^{t} \frac{1}{h}\psi(t-s)\mathrm{d}W(s)\right\|_H^2\right) = \frac{1}{h^2}\int_0^{-h} \|\psi(r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \mathrm{d}r,$$

which we have already shown goes to 0 as $h \uparrow 0$. For the last term, we find

$$\mathbb{E}\left(\left\|\int_{t+h}^{t} \psi'(t-s)\mathrm{d}W(s)\right\|_H^2\right) = \int_{t+h}^{t} \|\psi'(t-s)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \mathrm{d}s = \int_0^{-h} \|\psi'(r)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 \mathrm{d}r.$$

Since $\psi' Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$, this goes to 0 as $h \uparrow 0$, which proves the case $n = 1$.

Now suppose the statement holds for a certain n, and let ψ satisfy the conditions of the statement for $n+1$, that is, for $k \in \{0,\cdots,n+1\}$ the operator-valued function $\psi^{(k)}Q^{\frac{1}{2}}$ belongs to $L^2(0,\infty;\mathscr{L}_2(H))$ and for $k \in \{0,\cdots,n\}$ we have (the continuous version of) $\psi^{(k)}$ vanishing continuously at 0. Then

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^n}\int_{-\infty}^{t} \psi(t-s)\mathrm{d}W(s) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^{t} \psi(t-s)\mathrm{d}W(s) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} \psi^{(n)}(t-s)\mathrm{d}W(s),$$

where the last step follows from the induction hypothesis. Now $\psi^{(n)}$ vanishes at 0, has a mean square derivative $\psi^{(n+1)}$ and we have both $\psi^{(n)}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$ and $\frac{\mathrm{d}}{\mathrm{d}t}\psi^{(n)}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$. Hence we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t} \psi^{(n)}(t-s)\mathrm{d}W(s) = \int_{-\infty}^{t} \psi^{(n+1)}(t-s)\mathrm{d}W(s),$$

which is what we wanted to show. □

Now define for $a \in \mathbb{R}$ and $b \in \mathbb{N}$ the function $\phi_{a,b} : (0,\infty) \to \mathscr{L}(H)$ by

$$\phi_{a,b}(t) := t^a A^b S(t), \quad t \in (0,\infty).$$

Now [8, Lemma 3.20] states that their (classical) derivatives are given by

$$\phi_{a,b}^{(k)}(t) := \frac{\mathrm{d}^k}{\mathrm{d}t^k}\phi_{a,b}(t) = \sum_{j=0}^{k} C_{a,j,k}\phi_{a-(k-j),b+j}(t), \quad t \in (0,\infty), \tag{3.6}$$

where $C_{a,j,k} = (-1)^j \binom{k}{j}\prod_{i=1}^{k-j}(a-(k-j)+i)$. Note that

$$X_\gamma(t) = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t} \phi_{\gamma-1,0}(t-s)\mathrm{d}W(s), \quad t \in (-\infty, T].$$

We are now in a position to prove the differentiability.

Theorem 3.4.3. Let $\gamma > n$. Suppose for $k \in \{0,1,\cdots,n\}$ that $\phi_{\gamma-1-k,0}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$. Then the stochastic process $X_\gamma : (-\infty, T] \to L^2(\Omega; H)$ is n times mean square differentiable with derivatives

$$X_\gamma^{(k)}(t) = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t} \phi_{\gamma-1,0}^{(k)}(t-s)\mathrm{d}W(s), \quad k \in \{0,\cdots,n\}. \tag{3.7}$$

Proof. We want to use Theorem 3.4.1. This requires for $k \in \{0,\cdots,n\}$ that $\phi_{\gamma-1,0}^{(k)}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$. Now by Equation (3.6) we have

$$\phi_{\gamma-1,0}^{(k)}(t) = \sum_{j=0}^{k} C_{\gamma-1,j,k}\phi_{\gamma-1-(k-j),j}(t), \quad t \in (0,\infty). \tag{3.8}$$

So $\phi_{\gamma-1,0}^{(k)}$ is a linear combination of $\phi_{\gamma-1-(k-j),j}$, for $j \in \{0,\cdots,k\}$. So it suffices to show for $j \in \{0,\cdots,k\}$ that $\phi_{\gamma-1-(k-j),j} \in L^2(0,\infty;\mathscr{L}_2(H))$ holds. Moreover, if $(e_l)_{l\geq 1}$ is an orthonormal basis for H, then we can make use of Theorem 2.3.2 to obtain

$$\begin{aligned}\|t^{a+j}A^jS(t)Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2 &= \sum_{l=1}^{\infty}\|t^{a+j}A^jS(t)Q^{\frac{1}{2}}e_l\|_H^2 \leq \sum_{l=1}^{\infty}\left\|A^jS(t/2)\right\|_{\mathscr{L}(H)}^2\left\|t^{a+j}S(t/2)Q^{\frac{1}{2}}e_l\right\|_H^2 \\ &\leq Ce^{-\delta t}(t/2)^{-2j}\sum_{l=1}^{\infty}t^{2j}\left\|t^aS(t/2)Q^{\frac{1}{2}}e_l\right\|_H^2.\end{aligned}$$

Now using the estimate $e^{-\delta t} \leq 1$, we ultimately find

$$Ce^{-\delta t}(t/2)^{-2j}\sum_{l=1}^{\infty} t^{2j}\left\|t^a S(t/2)Q^{\frac{1}{2}}e_l\right\|_H^2 = \tilde{C}\sum_{l=1}^{\infty}\left\|(t/2)^a S(t/2)Q^{\frac{1}{2}}e_l\right\|_H^2 = \tilde{C}\|(t/2)^a S(t/2)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}.$$

Hence

$$\begin{aligned}\int_0^\infty \left\|t^{a+j}A^j S(t)Q^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(H)}\mathrm{d}t &\le \tilde{C}\int_0^\infty \left\|(t/2)^a S(t/2)Q^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(H)}\mathrm{d}t\\ =&2\tilde{C}\int_0^\infty \left\|r^a S(r)Q^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(H)}\mathrm{d}r,\end{aligned}$$

where in the last step we used the substitution $2r = t$. From this it follows that we only need, for $k \in \{0,\cdots,n\}$, that $\phi_{\gamma-1-k,0}Q^{\frac{1}{2}}$ belongs to $L^2(0,\infty;\mathscr{L}_2(H))$, which was the assumption.

To show for $k \in \{0,\cdots,n-1\}$ that $\psi^{(k)}Q^{\frac{1}{2}}$ vanishes at 0, we can use a similar argument. First note, using Theorem 2.3.2, that

$$\lim_{t\downarrow 0}\|t^a A^b S(t)\|_{\mathscr{L}(H)} \le \lim_{t\downarrow 0} Ct^{a-b},$$

which goes to 0 as long as $a-b>0$. Now the lowest value of $a-b$ in the expansion of Equation (3.8), only considering the first $n-1$ derivatives, is given by $\gamma-1-(n-1)$. Since by assumption $\gamma-1-(n-1)>0$, the first $n-1$ derivatives vanish at 0. Now applying Theorem 3.4.1 gives the result. □

After taking derivatives, the question arises whether or not we are allowed to pull the powers of A outside the integral. The following Theorem is proven in [4, Proposition 4.30] for a bounded time domain, but we will need it in greater generality. The proof uses a similar argument as given there.

Theorem 3.4.4. If A is closed and both

$$\int_0^\infty \|t^a S(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}t < \infty$$

as well as

$$\int_0^\infty \|t^a AS(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}t < \infty,$$

then $\mathbb{P}\left(\int_{-\infty}^t (t-s)^a S(t-s)\mathrm{d}W(s) \in D(A)\right) = 1$, and we have

$$A\int_{-\infty}^t (t-s)^a S(t-s)\mathrm{d}W(s) = \int_{-\infty}^t (t-s)^a AS(t-s)\mathrm{d}W(s), \quad \mathbb{P}\text{-a.s.}$$

Proof. Recall from Section 2.1 that if A is closed, then $D(A)$ is a Hilbert space when endowed with the inner product $\langle x,y\rangle_{D(A)} := \langle x,y\rangle_H + \langle Ax,Ay\rangle_H$. Now let $(e_k)_{k\ge1}$ be an orthonormal basis for H. Then by definition,

$$\begin{aligned}&\int_0^\infty \|t^a S(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(D(A))}\mathrm{d}t = \int_0^\infty \sum_{k=1}^\infty \|t^a S(t)Q^{\frac{1}{2}}e_k\|^2_{D(A)}\mathrm{d}t\\ =&\int_0^\infty \sum_{k=1}^\infty \left[\|t^a S(t)Q^{\frac{1}{2}}e_k\|^2_H + \|t^a AS(t)Q^{\frac{1}{2}}e_k\|^2_H\right]\mathrm{d}t = \int_0^\infty \|t^a S(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}t + \int_0^\infty \|t^a AS(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)}\mathrm{d}t,\end{aligned}$$

which is finite by assumption. Now since $D(A)$ is a Hilbert space, we can find simple functions $(f_n)_{n\ge1}$ of the form $f_n : (0,\infty) \to \mathscr{L}_2(D(A))$ such that

$$\lim_{n\to\infty}\int_0^\infty \left\|\left[t^a S(t) - f_n(t)\right]Q^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(D(A))}\mathrm{d}t = 0.$$

By the same argument as before, this implies both

$$\lim_{n\to\infty}\int_0^\infty \left\|\left[t^a S(t) - f_n(t)\right]Q^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(H)}\mathrm{d}t = 0 \tag{3.9}$$

as well as

$$\lim_{n\to\infty}\int_0^\infty \left\|\left[At^a S(t) - Af_n(t)\right]Q^{\frac{1}{2}}\right\|^2_{\mathscr{L}_2(H)}\mathrm{d}t = 0. \tag{3.10}$$

Note that for a simple function $g := \sum_{k=1}^N \phi_k \mathbb{1}_{[t_{k-1},t_k)}$, with $\phi_i \in \mathscr{L}_2(D_A)$ and $t_i \geq 0$ for all $i \leq N$, we have

$$\int_{-\infty}^t Ag(t-s)\mathrm{d}W(s) = \sum_{k=1}^N A\phi_k(W(t-t_k) - W(t-t_{k+1}))$$
$$= A\sum_{k=1}^N \phi_k(W(t-t_k) - W(t-t_{k+1})) = A\int_{-\infty}^t g(t-s)\mathrm{d}W(s).$$

By definition of the stochastic integral, Equation (3.9) implies

$$\int_{-\infty}^t (t-s)^a S(t-s)\mathrm{d}W(s) = \lim_{n\to\infty}\int_{-\infty}^t f_n(t-s)\mathrm{d}W(s)$$

in $L^2(\Omega; H)$. Similarly we find, from Equation (3.10),

$$\int_{-\infty}^t A(t-s)^a S(t-s)\mathrm{d}W(s) = \lim_{n\to\infty}\int_{-\infty}^t Af_n(t-s)\mathrm{d}W(t)$$
$$= \lim_{n\to\infty} A\int_{-\infty}^t f_n(t-s)\mathrm{d}W(s),$$

again in $L^2(\Omega; H)$, where the last equality follows because f_n is simple for all n, hence we can take A out of the integral. By passing to a subsequence $(f_{n_k})_{k\geq 1}$, we obtain a subsequence that also converges $\mathbb{P}$-a.s. If we now for $k \geq 1$ set $x_k = \int_{-\infty}^t f_{n_k}(t-s)\mathrm{d}W(s)$ and $x = \int_{-\infty}^t (t-s)^a S(t-s)\mathrm{d}W(s)$, then for almost all $\omega \in \Omega$, x_k converges to x in the H-norm and Ax_k converges to $\int_{\infty}^t A(t-s)^a S(t-s)\mathrm{d}W(s)$ (again in the H-norm). Now since A is closed, we thus obtain that $\int_{-\infty}^t (t-s)^a S(t-s)\mathrm{d}W(s)$ is in $D(A)$ for almost all ω, and thus

$$A\int_{-\infty}^t (t-s)^a S(t-s)\mathrm{d}W(s) = Ax = \lim_{k\to\infty} Ax_k = \int_{-\infty}^t A(t-s)^a S(t-s)\mathrm{d}W(t), \quad \mathbb{P}\text{-a.s},$$

as required. □

4

Multiple Markov property

In this section the dependence on the initial value will be investigated when γ is an integer. It turns out that it is possible to split the process in something depending on the random initial conditions and a random variable independent of it.

4.1. Incorporating initial data

We split the integral as follows:

$$X_\gamma(t) = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r) \tag{4.1}$$

$$= \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{s}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r) + \frac{1}{\Gamma(\gamma)}\int_{s}^{t}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r). \tag{4.2}$$

In what follows we will write $v_\gamma(t) := \int_{-\infty}^{s}(t-r)^{\gamma-1}S(t-r)\mathrm{d}W(r)$. We first need to establish a general identity.

Theorem 4.1.1. Suppose that both $\phi_{\gamma,0}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$ and $\phi_{\gamma-1,0}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$. Then we have $(\partial_t + A)X_{\gamma+1} = X_\gamma$, $\mathbb{P}$-a.s.

Proof. From Theorem 3.4.3 the mean square derivative in $L^2(\Omega;H)$ is given by

$$\begin{aligned}\partial_t X_{\gamma+1}(t) &= \frac{1}{\Gamma(\gamma+1)}\int_{-\infty}^{t}\phi_{\gamma,0}^{(1)}(t-s)\mathrm{d}W(s)\\ &= \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t}(t-s)^{\gamma-1}S(t-s)\mathrm{d}W(s) - \frac{1}{\Gamma(\gamma+1)}\int_{-\infty}^{t}(t-s)^{\gamma}AS(t-s)\mathrm{d}W(s), \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where in the last step we just wrote out $\phi_{\gamma,0}^{(1)}$. Now we can use Theorem 3.4.4 to pull A outside the integral, which is allowed, since we assumed directly that $\phi_{\gamma,0}Q^{\frac{1}{2}}$ is in $L^2(0,\infty;\mathscr{L}_2(H))$, while $\phi_{\gamma,1}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$ follows from $\phi_{\gamma-1,0}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$, as also seen in the proof of Theorem 3.4.3. □

In exactly the same way it follows that $(\partial_t + A)v_{\gamma+1}(t) = v_\gamma(t)$, $\mathbb{P}$-a.s. It turns out that v_γ also has a much nicer form as long as γ is an integer.

Theorem 4.1.2. Define, for $n \in \mathbb{N}$ and $t \geq s$,

$$D_n(t) := \sum_{k=0}^{n-1}\frac{(t-s)^k}{k!}C_{n-k}(t-s)X_n^{(k)}(s), \quad \mathbb{P}\text{-a.s.},$$

with $C_l(t) = \sum_{i=0}^{l-1}\frac{t^iA^i}{i!}S(t)$. Let $n \geq 1$. If for $k \in \{0,1,\cdots,n\}$ we have $\phi_{n-k,0}Q^{\frac{1}{2}} \in L^2(0,\infty;\mathscr{L}_2(H))$, then

$$(\partial_t + A)D_{n+1}(t) = D_n(t), \quad \mathbb{P}\text{-a.s.}$$

Proof. Note that, by Theorem 3.4.3, X_n is at least $n-1$ times differentiable, and X_{n+1} is at least n times differentiable. So both $D_n(t)$ and $D_{n+1}(t)$ are well defined. If we only look at the derivative of C_l, we find that for every integer $l \geq 2$ and all $t \geq s$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}C_l(t-s) &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{l-1}\frac{(t-s)^i A^i}{i!}S(t-s) = \sum_{i=1}^{l-1}\frac{(t-s)^{i-1}A^i}{(i-1)!}S(t-s) - \sum_{i=0}^{l-1}\frac{(t-s)^i A^{i+1}}{i!}S(t-s)\\ &= A\sum_{i=0}^{l-2}\frac{A^i(t-s)^i}{i!}S(t-s) - A\sum_{i=0}^{l-1}\frac{(t-s)^i A^i}{i!}S(t-s) = AC_{l-1}(t-s) - AC_l(t-s).\end{aligned}$$

For $C_1(t-s) = S(t-s)$, we just get $\frac{\mathrm{d}}{\mathrm{d}t}C_1(t-s) = -AS(t-s) = -AC_1(t-s)$ for all $t \geq 0$, since $-A$ is the generator of $(S(t))_{t\geq 0}$. Writing out the product rule we now obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D_{n+1}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{n}\frac{(t-s)^k}{k!}C_{n+1-k}(t-s)X_{n+1}^{(k)}(s)\\ &= \sum_{k=1}^{n}\frac{(t-s)^{k-1}}{(k-1)!}C_{n+1-k}(t-s)X_{n+1}^{(k)}(s) \quad + A\sum_{k=0}^{n-1}\frac{(t-s)^k}{k!}C_{n-k}(t-s)X_{n+1}^{(k)}(s)\\ &\qquad - A\sum_{k=0}^{n}\frac{(t-s)^k}{k!}C_{n+1-k}(t-s)X_{n+1}^{(k)}(s), \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

So ultimately we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}D_{n+1}(t) &= \sum_{k=0}^{n-1}\left[\frac{(t-s)^k}{k!}C_{n-k}(t-s)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(X_{n+1}^{(1)}(s) + AX_{n+1}(s)\right)\right] - AD_{n+1}(t)\\ &= \sum_{k=0}^{n-1}\frac{(t-s)^k}{k!}C_{n-k}(t-s)X_n^{(k)}(s) - AD_{n+1}(t) = D_n(t) - AD_{n+1}(t), \quad \mathbb{P}\text{-a.s.},\end{aligned}$$

where in the last equation we used Theorem 4.1.1. □

Now both $D_n(t)$ and $v_n(t)$ satisfy the same Cauchy initial value problem with initial condition $D_n(s) = v_n(s) = X_n(s)$. In order to show that the two processes coincide, it is hence useful to look at uniqueness of the equation. We restate the following result from [14].

Theorem 4.1.3. Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be a linear operator. If $-A$ is the infinitesimal generator of a C_0-semigroup, then for every $x \in X$ the Cauchy intial value problem

$$\begin{aligned}(\partial_t + A)u(t) &= u(t), \quad t > 0,\\ u(0) &= x\end{aligned} \tag{4.3}$$

has a unique solution.

Proof. See [14, Chapter 4, Theorem 1.3]. □

In order to apply this, we need to view A as an operator acting on $L^2(\Omega; H)$ instead of acting on H, which is done with the following Lemma.

Lemma 4.1.1. Suppose $A : D(A) \subseteq H \to H$ is a possibly unbounded linear operator and suppose $-A$ generates a C_0-semigroup on H. Define $\hat{A} : D(\hat{A}) \subseteq L^2(\Omega; H) \to L^2(\Omega; H)$ by

$$\begin{aligned}D(\hat{A}) &:= \{Y \in L^2(\Omega; H) : \mathbb{P}(Y \in D(A)) = 1, \|\hat{A}Y\|_{L^2(\Omega;H)} < \infty\},\\ [\hat{A}Y](\omega) &:= A[Y(\omega)], \quad Y \in D(\hat{A}), \quad \text{almost all } \omega \in \Omega.\end{aligned}$$

Then $-\hat{A}$ generates a C_0-semigroup acting on $L^2(\Omega; H)$ given by $(\hat{S}(t))_{t\geq 0}$, where $\hat{S}$ is defined on $L^2(\Omega; H)$ in the same way as $\hat{A}$.

Proof. We will first show that $\|\hat{S}(t)\|_{\mathscr{L}(L^2(\Omega;H))} = \|S(t)\|_{\mathscr{L}(H)}$, for all $t \geq 0$, after which we will prove that $(\hat{S}(t))_{t\geq 0}$ is indeed a C_0-semigroup. To show $\|\hat{S}(t)\|_{\mathscr{L}(L^2(\Omega;H))} \geq \|S(t)\|_{\mathscr{L}(H)}$, note that, for any $x \in H$ with $\|x\|_H \leq 1$, we can define a function $Y_x \in L^2(\Omega; H)$ by $Y_x(\omega) = x$. Then clearly, $\|Y_x\|_{L^2(\Omega;H)} = \|x\|_H \leq 1$, and

$$\|\hat{S}(t)Y_x\|^2_{L^2(\Omega;H)} = \int_\Omega \|[\hat{S}(t)Y_x](\omega)\|^2_H \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \|S(t)x\|^2_H \mathrm{d}\mathbb{P}(\omega) = \|S(t)x\|^2_H.$$

So it follows that

$$\|\hat{S}(t)\|_{\mathscr{L}(L^2(\Omega;H))} = \sup_{\|Y\|_{L^2(\Omega;H)}\leq 1} \|\hat{S}(t)Y\|_{L^2(\Omega;H)} \geq \sup_{Y_x, \|x\|_H\leq 1} \|\hat{S}(t)Y_x\|_{L^2(\Omega;H)} = \|S(t)\|_{\mathscr{L}(H)}.$$

For the other direction, let Y be any function in $L^2(\Omega, H)$ with $\|Y\|_{L^2(\Omega;H)} \leq 1$. Then

$$\begin{aligned}\|\hat{S}(t)Y\|^2_{L^2(\Omega;H)} &= \int_\Omega \|[\hat{S}(t)Y](\omega)\|_H^2 \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \|S(t)[Y(\omega)]\|_H^2 \mathrm{d}\mathbb{P}(\omega) \\ &\leq \|S(t)\|^2_{\mathscr{L}(H)} \int_\Omega \|Y(\omega)\|_H^2 \mathrm{d}\mathbb{P}(\omega) = \|S(t)\|^2_{\mathscr{L}(H)} \|Y\|^2_{L^2(\Omega;H)} \leq \|S(t)\|^2_{\mathscr{L}(H)}.\end{aligned}$$

Hence it follows that $\|S(t)\|_{\mathscr{L}(H)} = \|\hat{S}(t)\|_{\mathscr{L}(L^2(\Omega;H))}$, so $\hat{S}(t)$ is a bounded operator for all $t \geq 0$. We proceed with showing that $(\hat{S}(t))_{t\geq 0}$ is a C_0-semigroup. Note that the first two properties are trivial by the definition of $(\hat{S}(t))_{t\geq 0}$. What is left is to show that, for all Y, $\lim_{h\downarrow 0} \hat{S}(h)Y = Y$. Let $Y \in L^2(\Omega; H)$. But this follows immediately by dominated convergence (which is allowed because by Theorem 2.3.2, as long as $h \leq 1$, we have $\|\hat{S}(h)\|_{\mathscr{L}(L^2(\Omega;H))} \leq Me^{ch} \leq Me^c$ if c is positive and $\|\hat{S}(h)\|_{\mathscr{L}(L^2(\Omega;H))} \leq M$ if c is negative) together with the strong continuity of $(S(t))_{t\geq 0}$, so we obtain

$$\begin{aligned}\lim_{h\downarrow 0} \|\hat{S}(h)Y - Y\|^2_{L^2(\Omega;H)} &= \lim_{h\downarrow 0} \int_\Omega \|[\hat{S}(h)Y](\omega) - Y(\omega)\|_H^2 \mathrm{d}\mathbb{P}(\omega)^2 = \lim_{h\downarrow 0} \int_\Omega \|S(h)[Y(\omega)] - Y(\omega)\|_H^2 \mathrm{d}\mathbb{P}(\omega)^2 \\ &= \int_\Omega \lim_{h\downarrow 0} \|S(h)[Y(\omega)] - Y(\omega)\|_H^2 \mathrm{d}\mathbb{P}(\omega)^2 = 0.\end{aligned}$$

Now we wish to show that $-\hat{A}$ is the infinitesimal generator of $\hat{S}(t)$. Note that, for $x \in H$,

$$\left\| \frac{1}{h}(S(h)x - x) + Ax \right\|_H^2 \leq C \left\| \frac{1}{h}(S(h)x - x) \right\|_H^2 + C\|Ax\|_H^2.$$

Now for the first term we have

$$\left\| \frac{1}{h}(S(h)x - x) \right\|_H^2 = \left\| \frac{1}{h} \int_0^h -S(s)Ax\mathrm{d}s \right\|_H^2 \leq \left(\frac{1}{h^2} \int_0^h \| - S(s)Ax\|_H \mathrm{d}s \right)^2 \leq M^2 \|Ax\|_H^2.$$

From this it follows that, for $Y \in L^2(\Omega; H)$,

$$\begin{aligned}\left\| \frac{1}{h}(\hat{S}(h)Y - Y) + \hat{A}Y \right\|^2_{L^2(\Omega;H)} &= \int_\Omega \left\| \frac{1}{h}(S(h)Y(\omega) - Y(\omega)) + AY(\omega) \right\|_H^2 \mathrm{d}\mathbb{P}(\omega) \\ &\leq C(1 + M^2) \int_\Omega \|[AY(\omega)]\|_H^2 \mathrm{d}\mathbb{P}(\omega) = C(1 + M^2)\|\hat{A}Y\|^2_{L^2(\Omega;H)},\end{aligned}$$

So by dominated convergence,

$$\begin{aligned}\lim_{h\downarrow 0} \left\| \frac{1}{h}(\hat{S}(h)Y - Y) + \hat{A}Y \right\|^2_{L^2(\Omega;H)} &= \lim_{h\downarrow 0} \int_\Omega \left\| \frac{1}{h}([\hat{S}(h)]Y(\omega) - Y(\omega)) + [\hat{A}Y](\omega) \right\|_H^2 \mathrm{d}\mathbb{P}(\omega) \\ &= \lim_{h\downarrow 0} \int_\Omega \left\| \frac{1}{h}(S(h)[Y(\omega)] - Y(\omega)) + A[Y(\omega)] \right\|_H^2 \mathrm{d}\mathbb{P}(\omega) = 0.\end{aligned}$$

So it follows that $-\hat{A}$ is indeed the infinitesimal generator of $(\hat{S}(t))_{t\geq 0}$. □

Theorem 4.1.4. Let $n \geq 1$. If for $k \in \{0, 1, \cdots, n-1\}$ we have $\phi_{n-k-1,0} Q^{\frac{1}{2}} \in L^2(0, \infty; \mathscr{L}_2(H))$, then $D_n(t) = v_n(t)$ $\mathbb{P}$-a.s. for all $t \leq T$.

Proof. The proof goes by induction. For $n = 1$, we have

$$D_1(t) = S(t-s)X_1(s) = S(t-s) \int_{-\infty}^s S(s-r)\mathrm{d}W(r) = \int_{-\infty}^s S(t-r)\mathrm{d}W(r) = v_1(t), \quad \mathbb{P}\text{-a.s.}$$

So suppose the theorem holds up to a certain k. Then by Theorem 4.1.1 and Theorem 4.1.2, both D_{k+1} and v_{k+1} are solutions to $(\partial_t + A)D_{k+1}(t) = D_k(t)$ respectively $(\partial_t + A)v_{k+1}(t) = v_k(t)$, $\mathbb{P}$-a.s., for all $s \leq t \leq T$. Using the definition of $\hat{A}$ above, it follows that both also satisfy $(\partial_t + \hat{A})D_{k+1}(t) = D_k(t)$ and $(\partial_t + \hat{A})v_{k+1}(t) = v_k(t)$ in $L^2(\Omega; H)$. Note that from the induction hypothesis we know $D_k(t) = v_k(t)$, $\mathbb{P}$-a.s., for all $s \le t \le T$. Moreover we have $v_{k+1}(s) = D_{k+1}(s) = X_{k+1}(s)$, $\mathbb{P}$-a.s. Now consider $u_{k+1}(t) = v_{k+1}(t) - D_{k+1}(t)$. Then it follows that u_{k+1} satisfies the Cauchy initial value problem

$$\begin{aligned}&(\partial_t + \hat{A})u_{k+1}(t) = v_k(t) - D_k(t) = 0, \quad \mathbb{P}\text{-a.s.}\\ &u_{k+1}(s) = 0, \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

By Lemma 4.1.1, $-\hat{A}$ generates a c_0-semigroup on $L^2(\Omega; H)$. Applying Theorem 4.1.3 thus gives that $u_k + 1(t) = 0$ in $L^2(\Omega; H)$ for all $s \le t \le T$. So $u_{k+1}(t) = 0$, $\mathbb{P}$-a.s., for all $s \le t \le T$ follows. Hence $v_{k+1}(t) = D_{k+1}(t)$, $\mathbb{P}$-a.s., for all $s \le t \le T$, and by induction the claim follows. □

4.2. Proof of the multiple Markov property

Using the initial value identity from Theorem 4.1.4, we are now in a position to state and prove the multiple Markov property for this Matérn type process. For this we would like to use Theorem 2.4.7, but for this we first need to show that our stochastic integral is indeed a Gaussian process.

Theorem 4.2.1. Let $f : [0,\infty) \to \mathscr{L}(H)$ be such that $\int_0^\infty \|f(t)Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} \mathrm{d}t < \infty$. Then $\int_{-\infty}^t f(t-s)\mathrm{d}W(s)$ is a Gaussian process.

Proof. First assume that f is a simple process of the form

$$f(t) = \sum_{k=1}^n \mathbb{1}_{[t_k, t_{k+1})}(t)\phi_k,$$

with $t_k \ge 0$ and $\phi_k \in \mathscr{L}_2(H)$ for all $k \le n$. In this case, by definition

$$\int_{-\infty}^t f(t-s)\mathrm{d}W(s) = \int_0^\infty f(t')\mathrm{d}W(t-t') = \sum_{k=1}^n \phi_k(W(t-t_k) - W(t-t_{k+1})).$$

For convenience we define $J_f(t) := \int_{-\infty}^t f(t-s)\mathrm{d}W(s)$. Now suppose that we have a finite collection of times $s_1, \cdots, s_N$. We show that the vector $(J_f(s_1), \cdots, J_f(s_N))$ is a Gaussian random variable on H^N. That is, for all $(h_1, \cdots, h_N)$ the real valued random variable

$$Y := \sum_{i=1}^N \langle J_f(s_i), h_i\rangle_H = \sum_{i=1}^N \sum_{k=1}^n \langle \phi_k(W(s_i - t_k) - W(s_i - t_{k+1})), h_i\rangle_H.$$

is Gaussian. Note that we can always rewrite this summation with disjoint increments, by adding parts of overlapping increments together (though notation for this becomes cumbersome here). In doing this, we obtain a sum of independent (since the increments would be disjoint) Gaussian random variables, which is thus again Gaussian.

Now if f is not a simple function, then we can find a sequence of simple functions $(f_n)_{n\ge1}$ that converge to f in $\|\cdot\|_T$. But then $(J_{f_n}(t))_{n\ge1}$ converges to $J_f(t)$ in $L^2(\Omega; H)$, hence in distribution. Since $J_{f_n}(t)$ is Gaussian distributed for all n, it thus follows that $J_f(t)$ is also Gaussian, since it is a limit of Gaussian random variables. □

This implies in particular that the Matérn process X_γ is Gaussian (if the stochastic integral is well defined).

Theorem 4.2.2. Let $n \ge 1$. If for $k \in \{0, 1, \cdots, n-1\}$ we have $\phi_{n-1-k,0}Q^{\frac{1}{2}} \in L^2(0,\infty; \mathscr{L}_2(H))$, then X_n is n-ple Markov.

Proof. We want to use Theorem 2.4.7 to show that X_n is n-ple Markov. For this let $s < t$. At this point we have shown that $X_n(t) = D_n(t) + \frac{1}{\Gamma(n)}\int_s^t (t-r)^{n-1}S(t-r)\mathrm{d}W(r)$, $\mathbb{P}$-a.s., with

$$D_n(t) = \sum_{k=0}^{n-1} \frac{(t-s)^k}{k!} C_{n-k}(t-s)S(t-s)X_n^{(k)}(s), \quad \mathbb{P}\text{-a.s.}$$

Now $D_n(t)$ is $\sigma(X_n(s), X_n^{(1)}(s), \cdots, X_n^{(n-1)}(s))$-measurable. Moreover, $X_n(s)$ is $\widetilde{\mathscr{F}}_s^W$-measurable. Now since $X_n^{(1)}(s)$ exists, we can take the limit from the left, so we find

$$X_n^{(1)}(s) = \lim_{h \downarrow 0} \frac{X_n(s-h) - X_n(s)}{h}.$$

This is a limit of $\widetilde{\mathscr{F}}_s^W$-measurable functions, hence $X_n^{(1)}(s)$ is $\widetilde{\mathscr{F}}_s^W$-measurable. By continuing this we see that $X_n^{(k)}$ is $\widetilde{\mathscr{F}}_s^W$-measurable for $k \in \{0, \cdots, n-1\}$.

Now $\widetilde{\mathscr{F}}_s^W$ is independent of $\int_s^t (t-r)^{n-1} S(t-r) \mathrm{d}W(r)$, so as a result, $\sigma(X_n(s), X_n^{(1)}(s), \cdots, X_n^{(n-1)}(s))$ is independent of $\int_s^t (t-r)^{n-1} S(t-r) \mathrm{d}W(r)$. So taking conditional expectations gives us

$$\mathbb{E}\left(D_n(t) + \int_s^t (t-r)^{n-1} S(t-r) \mathrm{d}W(r) | X_n(s), X_n^{(1)}(s), \cdots, X_n^{(n-1)}(s)\right) = D_n(t), \quad \mathbb{P}\text{-a.s.}$$

where we used that the expectation of the stochastic integral is 0.

Similarly we can calculate

$$\mathbb{E}\left(D_n(t) + \int_s^t (t-r)^{n-1} S(t-r) \mathrm{d}W(r) | \widetilde{\mathscr{F}}_s^W\right) = D_n(t), \quad \mathbb{P}\text{-a.s.}$$

So by Theorem 2.4.7 we find that X_n is indeed n-ple Markov. □

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