# Intention Aware Routing System with Variable Station Pricing

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by

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# Summary

Intention aware routing system is a route-planning algorithm for electric vehicles that minimizes overall travel time by taking into consideration congestion at charging stations. This thesis extends this algorithm to allow choices to be made based on prices at charging stations. The goal of this thesis is to find a way to minimize maximum congestion while maximizing overall profit across the stations. To achieve this an optimal price has to be calculated. To this end, a formula is devised and applied to several graphs.

# Contents

# 1 Introduction

There seems to be a common consensus in the scientific world and the vehicle industry: The future of vehicles is going to be electric. This is great, as Electric Vehicles (EVs) play a significant role in tackling climate change, which is one of the biggest problems of our time. EVs do not emit any greenhouse gases, as opposed to petrol vehicles which do.
While the time has not yet arrived at which driving electric is the standard, companies like Tesla, Volkswagen and Renault are pushing electric vehicles to their limits, and are steadily advancing this market. But while advances are being made in charging EVs, they are not universal. Many vehicles cannot make use of superchargers. And even superchargers, which are supposed to be the fastest way to charge EVs, still cannot compare to petrol vehicles. The quickest an EV can get the charge equivalent to 600km is about 30 minutes, compared to a petrol vehicle where this takes about 5 minutes. If EVs want to charge en route, any queue at the charging stations increases waiting times linearly with the size of the queue. Considering the case where each vehicle charges for 30 minutes, on average this will increase waiting times with 15 minutes per vehicle [2]. As such, the waiting times at charging stations have to be taken into account while scheduling. This is not required for petrol vehicles, as the time to refill a tank is significantly quicker.
De Weerdt et al. [2] has already attempted to tackle this issue. Their paper suggests an Intention Aware Routing System (IARS). The basic idea is that individual vehicles share their intentions with a central system. The system then updates the traffic information and as a result, the vehicles can then choose a better route. One thing that is missing in this model is the pricing at the charging stations as different stations might have different prices, and this might affect if people may want to charge or not. Various authors suggest different ways to deal with pricing of electricity for electric vehicles, like [1], [3], and [4]. Ban et al. [1] discuss a way to balance EVs over charging stations by setting certain prices. Gerding et al. [3] suggest having a bidding system for charging spots. And finally, Malandrino et al. [4] describe the game-theoretic nature of buying and selling electricity for EVs. In this thesis, aspects from [1] and [4] are used to extend the model developed by de Weerdt et al. [2]
Adding pricing to the model then begs the question:

*Assuming stations will cooperate, how can prices be set across the stations to minimize maximum congestion while maximizing overall profit across the stations?*

To aid in answering this question, we answer the following subquestions:

1. How can pricing be included in the model of de Weerdt et al. [2]?
2. How can decision policies of a distribution of EVs across the system be modelled?
3. What strategies can stations use to affect the flow of vehicles?
4. What is the optimal strategy to reduce maximum congestion?

This thesis is structured as follows. Section 2 introduces the background of the problem, and the model from [2] which we extend. Section 3 extends the model to include pricing and draws parallels between our model and a congestion game. Section 4 introduces different solution methods to solve the various scenarios. Section 5 demonstrates how the derived formula should be applied on a number of graphs. In Section 6, conclusions are made based on the results, and a brief discussion on possible improvements and future research is given on the topic.

# 2

# Literature

This chapter gives a summary of the background literature including interesting results obtained from it. Section 2.2 details the model, introduced in [2], which the model in this thesis uses as a base.

## 2.1. Game theory

The problem this thesis discusses is very closely related to a problem in game theory. This problem is known as a congestion game. This section discusses the general idea of a congestion game, and highlights the differences with the problem described in this thesis. A basic congestion game contains the following components:

- $N$ players
- A set $E$ of congestible elements $e$
- For each player $j \in N$, a set of strategies $S_j \subseteq \mathcal{P}(E)$ is given, where a strategy represents a possible combination of elements the player can choose. Here, $\mathcal{P}(E)$ is the power set of $E$. The strategy each player chooses is represented by $s_j$.
- Each congestible element $e$ has a delay function $C$ associated with it. $C_e$ is a function of $n_e$, which is the number of players that have $e_i$ in their strategy.

Each player experiences a total delay equal to the sum of the delays of the elements in their chosen strategy. In other words

$$\sum_{e \in s_j} C_e(n_e).$$

The goal for each player is to minimize their total delay. If no player can lower their delay, by exclusively changing their own strategy, we speak of a Nash equilibrium. The existence of a Nash equilibrium, and how to find it, is one of the main focuses of game theory. As such, congestion games as described above have been studied, and it has been found that for these congestion games a Nash equilibrium can always be found. Important to note however, is that a Nash equilibrium in this case is a local minimum.

## 2.2. Base model

The model used for modelling the EV routing problem was introduced in [2]. This model works on a domain given by $\langle V, E, T, P, S, C \rangle$, where $e = (v_i, v_j) \in E$ are edges, with $v_i \in V$ vertices. Both roads and charging stations are represented by these edges, but charging stations are represented by loops, so edges where $v_i = v_j$. We use the notation $E_{station} \subset E$ and $E_{road} \subset E$ for roads and stations, respectively.

For each edge, there is a probabilistic distribution, which models possible waiting times. This distribution is time-dependent for a finite set of time points represented by $T = \{1, 2, ..., t_{max}\}$. This represents

different moments throughout the day. To be precise, the distribution represents a probability mass function, which for edge $e = (v_i, v_j)$ gives the probability P$(\Delta t = t_j - t_i | e, t_i)$ on taking $\Delta t$ time units to traverse edge $e$ when you leave vertex $v_i$ at time $t_i$.
Since vehicles have a finite amount of charge, the model includes $S$ which represents the state of charge (SoC) of a vehicle. Each $S_{i,t} \in S$ corresponds to vehicle $i$ having SoC $S_{i,t}$ at time $t$. The domain of $S$ is finite and corresponds to $\{0, 1, 2, ..., s_{max}\}$, where $0$ and $s_{max}$ represent an empty battery, and a fully charged battery, respectively. The values in between represent to what proportion the battery is charged.
Finally, each edge depletes a certain amount of charge based on cost function $C(e)$, which gives the cost of charge for each edge in the graph. For charging stations, $C(e)$ is negative to represent charging the vehicle. Charging stations always fully charge EVs in this model.
To plan a route, this model uses a policy as opposed to a simple route. This policy is a function $\pi : V \times T \times S \to V$, which for each state, consisting of a vertex $v_c$, current time at the vertex $t_c$, and state of charge at that vertex $s_c$, gives the next vertex $w$. Using this, the next edge for a current state $(v_c, t_c, s_c)$ and a policy $\pi$ is given by $e = (v_c, \pi(v_c, t_c, s_c) = w)$. Each vehicle calculates their own policy. The policy used by IARS is calculated by finding a policy which maximizes the expected utility function given by this recursive definition:

$$\text{EU}(e_c = (v_c, w), t_c, s_c | \pi) = \begin{cases} -\infty, & \text{if } s_c < 0, \\ \sum_{\Delta t \in T} P(\Delta t | e_c, t_c) \cdot U(t_c + \Delta t, s'), & \text{if } w = v_{dest}, \\ \sum_{\Delta t \in T} P(\Delta t | e_c, t_c) & \\ \quad \cdot \text{EU}((w, \pi(w, t_c + \Delta t, s')), t_c + \Delta t, s' | \pi), & \text{otherwise.} \end{cases}$$

Here $s'$ is the new state of charge after taking an edge. $U(t_c, s)$ is the utility function, which is defined as follows:

$$U(t_c, s) = \begin{cases} -\infty, & \text{if } s_c < 0, \\ -t_c, & \text{otherwise.} \end{cases}$$

As such maximizing the utility function, is equivalent to minimizing arrival time.
Since the vehicles using IARS share their intentions with the rest of the vehicles in the system, this can then be used to update the traffic information. Based on this, all vehicles recalculate their policy. How this happens in the simulation, is that one by one each vehicle calculates a best policy for themselves in order of arrival. After a vehicle calculated their policy, it updates the system. Then, the same thing happens with the next vehicle. Once all vehicles have calculated their policy, the cycle restarts. Each vehicle recalculates their policy and decides if they want to change their policy or not. A policy which originally had seemed best, might become worse as a result of the other vehicles increasing the queue length on their route. Based on this the vehicle may choose to adjust its policy. This process of updating policies continues until none of the vehicles in the system decide on a different policy. This is equivalent to a Nash equilibrium, and may be a local optimum.

# 3
# Model

In this chapter we introduce the final model used in this thesis. Some analysis is done afterwards to compare the model with the congestion game setting. Section 3.1 extends the model introduced in Section 2.2 to include pricing. Section 3.2 discusses the differences between our model and a basic congestion game and formalizes the problem of this thesis.

## 3.1. Pricing extension

To influence the traffic in the model based on pricing, first the model needs to be equipped to deal with pricing. To this end, this thesis extends the IARS model to include pricing. This leads to a model which works on a domain defined by $\langle V, E, T, P, S, C, M\rangle$. This introduces money to the model, where the amount of money spent is represented by a value in a finite set $M = \{0, 1, ..., m_{max}\}$, where $m_{max}$ is the maximum price charged in the system. This then also affects the states of the individual EVs, which changes from $(v_c, t_c, s_c) \in (V \times T \times S)$ to $(v_c, t_c, s_c, m_c) \in (V \times T \times S \times M)$. The cost of charging at a charging station is defined by

$$\Pi(e) = \begin{cases} 0, & \forall e \in E_{roads}, \\ p_e, & \forall e \in E_{stations}, \end{cases}$$

where $p_e$ is a fixed station price. It is possible to make $p_e$ time dependent, but in this thesis, we consider the impact on the period with highest possible congestion, i.e. rush hour. As such, making the price time dependent is not necessary.

The original model worked with a utility function dependent on charge and arrival time, where the utility function $U$ is given by

$$U(t_c, s_c) = \begin{cases} -\infty, & \text{if } s_c < 0, \\ -t_c, & \text{otherwise.} \end{cases}$$

To extend IARS to handle pricing, multiple different methods can be used. One possible example is as given in [3], where $U$ is given by:

$$U(t_c, s_c, m_c) = \begin{cases} -\infty, & \text{if } s_c < 0, \\ -t_c - \gamma \cdot m_c, & \text{otherwise.} \end{cases} \tag{3.1}$$

with $\gamma > 0$ representing a time/ money trade-off. Here $\gamma = 10$ could represent that 10 minutes of detour is worth 1 euro of discount. In concept, the final utility function is similar, but it uses normalizing factors both in terms of decision parameters, and based on the values in the domain.

The final utility function decided upon is:

$$U(t_c, s_c, m_c) = \begin{cases} -\infty, & \text{if } s_c < 0, \\ \gamma\left(\frac{T_{max}-t_c}{T_{max}-T_{min}}\right) & \\ \quad +(1-\gamma)\left(\frac{M_{max}-m_c}{M_{max}-M_{min}}\right), & \text{otherwise.} \end{cases} \tag{3.2}$$

Here, $\gamma \in [0, 1]$ is a normalized decision parameter, used to represent the time/ money trade-off. As such, $\gamma = 1$ represents a pure focus on arriving early, and $\gamma = 0$ represents only caring about getting the cheapest price. $T_{min}, T_{max}, M_{max},$ and $M_{min}$ are route dependent constants, that can be set in different ways to suit the range of values expected from the utility function. They act as normalizing factors, and are included since decisions regarding prices are usually based on relative discount.

## 3.2. Adjusted congestion game

For the most part, our model is quite similar to a basic congestion game, but there are some key differences.

One of the differences is the fact that we use a utility function instead of a cost function. However, an equivalent formulation can be obtained by minimizing cost $C$, which is equal to minus utility $U$. What is not immediately clear is that the utility function is in fact the sum of all the utilities until the end. The final utility function can be reformulated to a form where this is more obvious:

$$U = \gamma \left(\frac{T_{max} - t_c}{T_{max} - T_{min}}\right) + (1 - \gamma)\left(\frac{M_{max} - m_c}{M_{max} - M_{min}}\right) \tag{3.3}$$

$$= \gamma\, T_{const} + (1 - \gamma)M_{const} - \gamma\left(\frac{t_c}{T_{max} - T_{min}}\right) - (1 - \gamma)\left(\frac{m_c}{M_{max} - M_{min}}\right). \tag{3.4}$$

Here $T_{const}$ and $M_{const}$ are constants defined by $T_{max}$ and $T_{min}$, and $M_{max}$ and $M_{min}$, respectively. The variable $t_c$ is defined as the total time it takes to get to the final node, which is equivalent to the sum of all the edges the vehicle took. Variable $m_c$ is the total money spent, which can be added as the cost of a station node. As such, our formulation is equivalent to a congestion game with player-specific cost functions.

Next to that, we use player-specific utility functions, instead of a globally defined utility function. This is the case because of two different reasons. Firstly, the utility function depends on the $\gamma$ of the respective vehicle owner. Secondly, there is an inherent ordering to the vehicles meaning that certain vehicles will experience less delay than others at the same station. This ordering is assigned by an ID given to each vehicle. In the case of two vehicles arriving at the same time in the simulation, ties are broken by which vehicle has the lower ID. In the basic congestion game, all vehicles would experience the same congestion at the same station.

Our goal is to find the right station prices to minimize maximum congestion. Assuming all vehicles leave at the same time, this is equivalent to finding the station prices such that in a Nash equilibrium each station receives the same number of vehicles. To achieve this, we first have to know that a Nash equilibrium actually exists for this congestion game. If there were no Nash equilibrium, we could not argue about one specific solution. Normal congestion games have been proven to have a Nash-equilibrium due to the finite improvement property. Congestion games with player-specific cost functions do not have this property, as such, the proof for this is not as straight-forward. However, Milchtaich [5] has tackled this proof, and has proven that there is a pure strategy Nash equilibrium for congestion games with player-specific cost functions.

Next, we need to find a way to calculate the station prices necessary for the vehicles to distribute themselves equally across the stations. This is explored in the next chapter.

# 4

# Solution methods

In this chapter, we analyze the problem and give the methods to answer the questions central to this thesis. Section 4.1 gives an initial equation to solve this problem, and gives examples of how to apply it to bottleneck graphs. Section 4.2 shows how to adjust this equation to apply it to grid graphs.

## 4.1. Bottleneck graph

For certain restricted scenarios, it is possible to directly calculate for what station prices the vehicles split evenly across all stations. This will be referenced as an even split. This chapter introduces a formula which does exactly that. To do this this thesis first considers the simplest of these scenarios, the bottleneck graph. A formal definition of this graph is given in Section 4.1.1. The derived formula is then applied in Chapter 5 on various graphs.

To derive the formula, we need to define the constants in Equation 3.2. $M_{max}$ represents the highest price available at a charging station in the system. $M_{min}$ is set to $0$ as the highest utility should be received from not charging at all. $T_{max}$ is the maximum time the vehicle is willing to arrive at the destination, and $T_{min}$ is the minimum possible time to reach the destination excluding charging time. $T_{max}$ is set to $3 \cdot T_{min}$ as this is a reasonable upper bound on the maximum willingness to make a detour.

In Subsection 4.1.1, the case where there are $2$ stations and just one class of vehicles is discussed. Subsection 4.1.2 considers the case where there are multiple classes of vehicles. And finally, Subsection 4.1.3 generalises the $2$-station scenario to the $R$-station scenario.

### 4.1.1. $2$-Stations

The formula to be used is defined on a so-called bottleneck graph. This is defined as a graph with one origin node and one destination node. Between this origin and destination node, there is one column of stations, with connections only to the origin node and the destination node.

To determine the formula, we consider a graph with two stations, Station $1$ and Station $2$. Such a graph can be seen in Figure 4.1.

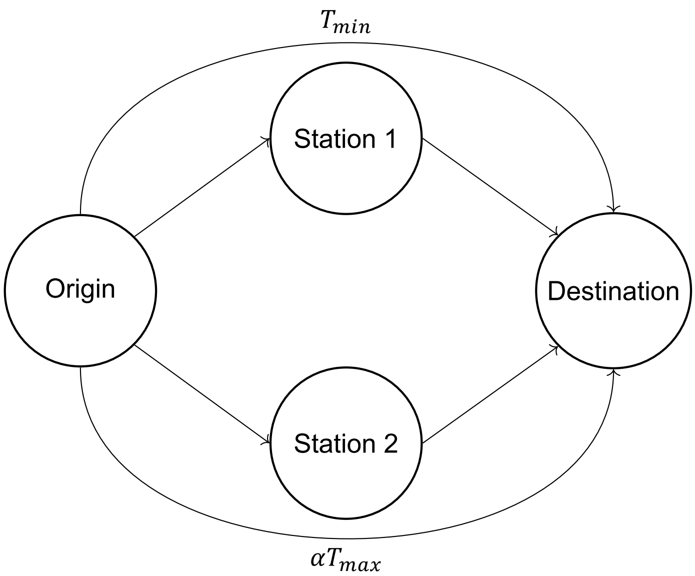


Figure 4.1: Bottleneck graph with 2 stations. Edge lengths represent travel time.

While within the simulation the lengths of the edges between the origin node and the stations are defined, they are not individually relevant to the solution. Only the sum of the length of two subsequent edges on a route are relevant. Without loss of generality, the route via Station 1 is shorter than the route via Station 2, and is denoted by $T_1 = T_{min}$. The time it takes to travel to the destination node via Station 2 is considered to be a fraction of $T_{max} = 3T_{min}$. This time is denoted by $T_2 = \alpha T_{max}$, with $\alpha \in [\frac{1}{3}, 1]$. This domain is to ensure that $T_{min} \leq T_2 \leq T_{max}$. To ensure that results can be calculated, we assume that all vehicles leave at $T = 0$. We consider one class of vehicles with $\gamma \in [0, 1]$. Since $T_1 \leq T_2$, the price at Station 2 needs to be lower than the price at Station 1. As such, we set the price at Station 1 equal to $p_1 = M_{max}$. The price at Station 2 is a fraction of $M_{max}$ and is given by $p_2 = \beta M_{max}$, with $\beta \in [0, 1]$. This setup allows for the formulation of a direct formula to calculate what $\beta$ should be.

A Nash equilibrium occurs when the utility of switching to another option is lower than the current option for all players. This means that for an even split to be achieved, the utility of having an even split needs to be higher than not having an even split, for vehicles at both stations. This translates to

$$\begin{aligned} U_{1,\frac{N}{2}} &\geq U_{2,\frac{N}{2}+1} \\ U_{2,\frac{N}{2}} &\geq U_{1,\frac{N}{2}+1}, \end{aligned} \tag{4.1}$$

where $U_{i,j}$ is the utility of taking the route via station $i$ when it has $j$ vehicles. $N$ refers to the total number of vehicles in the system.
These conditions turn out to be sufficient for the nash equilibrium to correspond to the even split in the 2-station bottleneck case. In particular, for convergence to occur, we need that for every scenario that is not an even split, the best option for any individual vehicle is to switch in favor of an even split. This means that the utility for switching from the station with more vehicles should be higher than for staying. This leads to what needs to be proved being:

$$\begin{aligned} U_{1,\frac{N}{2}-k} &\geq U_{2,\frac{N}{2}+k+1} \quad \forall k \in \mathbb{N}_0 \\ U_{2,\frac{N}{2}-k} &\geq U_{1,\frac{N}{2}+k+1} \quad \forall k \in \mathbb{N}_0. \end{aligned} \tag{4.2}$$

It is trivial that considering the same route, a longer queue means a lower utility. Expressed in formula this gives:

$$U_{i,k} \geq U_{i,j} \quad \forall i, \forall j \geq k \tag{4.3}$$

Now assume the conditions set at (4.1). Applying (4.3) to this assumption, (4.2) follows immediately.

Now that we know that the system of inequalities in Equation (4.1) lead to an even split, we can solve them for $\beta$. For this, we introduce variable $T_{q,m}$, which represents the waiting time caused by the queue for $m$ vehicles. Using that, we get that the utility function for the path via Station 1 is

$$U_{1,m} = \gamma\frac{T_{max} - T_1 - T_{q,m}}{T_{max} - T_{min}} + (1-\gamma)\frac{M_{max} - p_1}{M_{max} - M_{min}} \tag{4.4}$$

$$= \gamma\frac{T_{max} - T_{min}}{T_{max} - T_{min}} - \gamma\frac{T_{q,m}}{T_{max} - T_{min}} + (1-\gamma)\frac{M_{max} - M_{max}}{M_{max} - M_{min}} \tag{4.5}$$

$$= \gamma - \gamma\frac{T_{q,m}}{T_{max} - T_{min}}. \tag{4.6}$$

Following the same logic, the utility function for the path via Station 2 is

$$U_{2,m} = \gamma\frac{T_{max} - T_2 - T_{q,m}}{T_{max} - T_{min}} + (1-\gamma)\frac{M_{max} - p_2}{M_{max} - M_{min}} \tag{4.7}$$

$$= \gamma\frac{T_{max} - \alpha T_{max}}{T_{max} - \frac{1}{3}T_{max}} + (1-\gamma)\frac{M_{max} - \beta M_{max}}{M_{max}} - \gamma\frac{T_{q,m}}{T_{max} - T_{min}} \tag{4.8}$$

$$= \gamma\frac{3(1-\alpha)}{2} + (1-\gamma)(1-\beta) - \gamma\frac{T_{q,m}}{T_{max} - T_{min}}. \tag{4.9}$$

Transforming the domain of $\alpha$ from $[\frac{1}{3}, 1]$ to $[0, 1]$ using

$$\alpha = \left(\frac{2}{3}\alpha_{new} + \frac{1}{3}\right)$$

gives

$$U_{2,m} = \gamma\frac{3 - 3\left(\frac{2}{3}\alpha_{new} + \frac{1}{3}\right)}{2} + (1-\gamma)(1-\beta) - \gamma\frac{T_{q,m}}{T_{max} - T_{min}} \tag{4.10}$$

$$= \gamma\frac{2 - 2\alpha_{new}}{2} + (1-\gamma)(1-\beta) - \gamma\frac{T_{q,m}}{T_{max} - T_{min}} \tag{4.11}$$

$$= \gamma(1 - \alpha_{new}) + (1-\gamma)(1-\beta) - \gamma\frac{T_{q,m}}{T_{max} - T_{min}}. \tag{4.12}$$

This $\alpha_{new}$ is defined in such a way, that $\alpha_{new} = 0$ corresponds to $T_{min}$ and $\alpha_{new} = 1$ to $T_{max}$. So we get

$$\alpha_{new} = \frac{T_2 - T_{min}}{T_{max} - T_{min}}.$$

$\alpha_{new}$ is referred to as $\alpha$ from now on.
Now we attempt to solve the inequalities of (4.1) for $\beta$. Working out the first of the two gives:

$$U_{1,\frac{N}{2}} \geq U_{2,\frac{N}{2}+1} \tag{4.13}$$

$$\gamma - \gamma\frac{T_{q,\frac{N}{2}}}{T_{max} - T_{min}} \geq \gamma(1-\alpha) + (1-\gamma)(1-\beta) - \gamma\frac{T_{q,\frac{N}{2}+1}}{T_{max} - T_{min}} \tag{4.14}$$

$$\alpha\gamma + \gamma\frac{T_{q,\frac{N}{2}+1} - T_{q,\frac{N}{2}}}{T_{max} - T_{min}} \geq (1-\gamma)(1-\beta) \tag{4.15}$$

This $\frac{T_{q,\frac{N}{2}+1} - T_{q,\frac{N}{2}}}{T_{max} - T_{min}}$ term will appear more often. It is useful to define explicitly:

$$\frac{T_{q,\frac{N}{2}+1} - T_{q,\frac{N}{2}}}{T_{max} - T_{min}} = \epsilon_{\frac{N}{2}}. \tag{4.16}$$

Replacing then gives:

$$\gamma(\alpha + \epsilon_{\frac{N}{2}}) \geq (1-\gamma)(1-\beta) \tag{4.17}$$

$$\beta \geq 1 - \frac{\gamma\left(\alpha + \epsilon_{\frac{N}{2}}\right)}{1-\gamma} \tag{4.18}$$

Similarly the second inequality gives:

$$U_{2,\frac{N}{2}} \geq U_{1,\frac{N}{2}+1} \tag{4.19}$$

$$\gamma(1-\alpha) + (1-\gamma)(1-\beta) - \gamma\frac{T_{q,\frac{N}{2}}}{T_{max} - T_{min}} \geq \gamma - \gamma\frac{T_{q,\frac{N}{2}+1}}{T_{max} - T_{min}} \tag{4.20}$$

$$(1-\gamma)(1-\beta) \geq \alpha\gamma + \gamma\frac{T_{q,\frac{N}{2}} - T_{q,\frac{N}{2}+1}}{T_{max} - T_{min}} \tag{4.21}$$

$$(1-\gamma)(1-\beta) \geq \gamma(\alpha - \epsilon_{\frac{N}{2}}) \tag{4.22}$$

$$\beta \leq 1 - \frac{\gamma\left(\alpha - \epsilon_{\frac{N}{2}}\right)}{1-\gamma} \tag{4.23}$$

Combining these gives that:

$$1 - \frac{\gamma\left(\alpha + \epsilon_{\frac{N}{2}}\right)}{1-\gamma} \leq \beta \leq 1 - \frac{\gamma\left(\alpha - \epsilon_{\frac{N}{2}}\right)}{1-\gamma}. \tag{4.24}$$

This means that there is a range of $\beta$ for which the conditions are right for there to be an even split. As any value within this range should lead to an even split, we choose the maximum value, as this corresponds to the highest price. As such we define $\beta^* = \min(\max(\beta), 1)$, or explicitly:

$$\beta^* = \min\left(1 - \frac{\gamma\left(\alpha - \epsilon_{\frac{N}{2}}\right)}{1-\gamma}, 1\right) \tag{4.25}$$

The $\epsilon_{\frac{N}{2}}$ in this equation represents the difference between the expected waiting times at the two queues. The expected waiting time is equal to the average waiting time at a queue, since the vehicles have no way of knowing where in the queue they arrive if they arrive at the same time. As such:

$$EW_n = \frac{1}{n}\sum_{i=1}^{n} T_i,$$

where $EW_n$ is the expected waiting time for any vehicle in an $n$ vehicle queue, $T_i$ is the waiting time for the $i^{th}$ vehicle in the queue and $n$ is the total number of vehicles in the queue. $T_i$ increases as $i$ increases, in fact $T_i$ can be seen as $Q$ times an arithmetic series, where $Q$ represents the capacity at a charging station. For the first $Q$ vehicles, there is no waiting time, for the next $Q$ vehicles the waiting time is equal to the charging time $T_c$ in minutes, the next $Q$ vehicles have waiting time equal to $2T_c$ etc. If $n \bmod Q = 0$, this is exactly equivalent to $Q$ times an arithmetic series. However, if $n \bmod Q \neq 0$, we have $n \bmod Q$ vehicles which we have to consider separately. These vehicles each contribute $\left\lfloor\frac{n}{Q}\right\rfloor T_c$

to the waiting time. Combining all of these terms gives

$$
\begin{aligned}
EW_n &= Q\frac{\left\lfloor \frac{n}{Q} \right\rfloor \left( \left\lfloor \frac{n}{Q} \right\rfloor - 1 \right) T_c}{2n} + \frac{(n \bmod Q) \left\lfloor \frac{n}{Q} \right\rfloor T_c}{n} \\
&= \frac{T_c \left\lfloor \frac{n}{Q} \right\rfloor}{n} \left( \frac{Q \left( \left\lfloor \frac{n}{Q} \right\rfloor - 1 \right)}{2} + n \bmod Q \right).
\end{aligned}
$$

Using $n \bmod Q = n - Q\left\lfloor \frac{n}{Q} \right\rfloor$, we can rewrite $EW_n$ to:

$$
EW_n = \frac{T_c \left\lfloor \frac{n}{Q} \right\rfloor}{n} \left( n - \frac{Q \left( \left\lfloor \frac{n}{Q} \right\rfloor + 1 \right)}{2} \right).
$$

The expected waiting time for a vehicle in an $n + 1$ queue is:

$$
EW_{n+1} = \frac{T_c \left\lfloor \frac{n+1}{Q} \right\rfloor}{n+1} \left( (n+1) - \frac{Q \left( \left\lfloor \frac{n+1}{Q} \right\rfloor + 1 \right)}{2} \right).
$$

This leads to the expected increase in waiting time for an extra vehicle added to an $n$ vehicle queue being :

$$EW_{n+1} - EW_n = \frac{T_c \left\lfloor \frac{n+1}{Q} \right\rfloor}{n+1}\left((n+1) - \frac{Q\left(\left\lfloor \frac{n+1}{Q} \right\rfloor + 1\right)}{2}\right) - \frac{T_c \left\lfloor \frac{n}{Q} \right\rfloor}{n}\left(n - \frac{Q\left(\left\lfloor \frac{n}{Q} \right\rfloor + 1\right)}{2}\right)$$

$$= T_c\left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right) - \frac{QT_c}{2}\left(\frac{\left\lfloor \frac{n+1}{Q} \right\rfloor\left(\left\lfloor \frac{n+1}{Q} \right\rfloor + 1\right)}{n+1} - \frac{\left\lfloor \frac{n}{Q} \right\rfloor\left(\left\lfloor \frac{n}{Q} \right\rfloor + 1\right)}{n}\right)$$

$$= T_c\left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right) - \frac{QT_c}{2}\left(\frac{\left\lfloor \frac{n+1}{Q} \right\rfloor^2 + \left\lfloor \frac{n+1}{Q} \right\rfloor}{n+1} - \frac{\left\lfloor \frac{n}{Q} \right\rfloor^2 + \left\lfloor \frac{n}{Q} \right\rfloor}{n}\right)$$

$$= T_c\left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right) - \frac{QT_c}{2}\left(\frac{n\left(\left\lfloor \frac{n+1}{Q} \right\rfloor^2 + \left\lfloor \frac{n+1}{Q} \right\rfloor\right) - (n+1)\left(\left\lfloor \frac{n}{Q} \right\rfloor^2 + \left\lfloor \frac{n}{Q} \right\rfloor\right)}{n(n+1)}\right)$$

$$= T_c\left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right) - \frac{QT_c}{2}\left(\frac{\left(\left\lfloor \frac{n+1}{Q} \right\rfloor^2 + \left\lfloor \frac{n+1}{Q} \right\rfloor\right) - \left(\left\lfloor \frac{n}{Q} \right\rfloor^2 + \left\lfloor \frac{n}{Q} \right\rfloor\right)}{(n+1)}\right)$$

$$+ \left(\frac{QT_c}{2}\frac{\left\lfloor \frac{n}{Q} \right\rfloor^2 + \left\lfloor \frac{n}{Q} \right\rfloor}{n(n+1)}\right)$$

$$= T_c\left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right) - \frac{QT_c}{2}\left(\frac{\left(\left\lfloor \frac{n+1}{Q} \right\rfloor^2 - \left\lfloor \frac{n}{Q} \right\rfloor^2\right) + \left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right)}{(n+1)}\right)$$

$$+ \left(\frac{QT_c}{2}\frac{\left\lfloor \frac{n}{Q} \right\rfloor^2 + \left\lfloor \frac{n}{Q} \right\rfloor}{n(n+1)}\right).$$

This expression can be simplified by proving that the first part is equal to 0 for all cases. So we prove that:

$$T_c\left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right) - \frac{QT_c}{2}\left(\frac{\left(\left\lfloor \frac{n+1}{Q} \right\rfloor^2 - \left\lfloor \frac{n}{Q} \right\rfloor^2\right) + \left(\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor\right)}{(n+1)}\right) = 0.$$

This can be proven by looking at the expression in two cases:

1. $(n+1) \bmod Q \neq 0$,
2. $(n+1) \bmod Q = 0$.

Useful for this is seeing that

$$\left\lfloor \frac{n+1}{Q} \right\rfloor - \left\lfloor \frac{n}{Q} \right\rfloor = \begin{cases} 0 & (n+1) \bmod Q \neq 0, \\ 1 & (n+1) \bmod Q = 0, \end{cases}$$

assuming $n$ and $Q$ are integer. This is the case for our system, since we do not deal with fractional vehicles or fractional capacity charging stations. Now we consider the expression in both cases.

$$T_c\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)-\frac{QT_c}{2}\left(\frac{\left(\left\lfloor\frac{n+1}{Q}\right\rfloor^2-\left\lfloor\frac{n}{Q}\right\rfloor^2\right)+\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)}{(n+1)}\right)=$$

$$T_c\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)-\frac{QT_c}{2}\left(\frac{\left(\left\lfloor\frac{n+1}{Q}\right\rfloor+\left\lfloor\frac{n}{Q}\right\rfloor\right)\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)+\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)}{(n+1)}\right)$$

For the first case, it is trivial that this expression is 0, since all expressions in the numerator have a term $\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)$.

The second case requires some closer inspection. We know that all $\left(\left\lfloor\frac{n+1}{Q}\right\rfloor-\left\lfloor\frac{n}{Q}\right\rfloor\right)$ terms will be equal to $1$. This means that we can simplify the expression to this:

$$T_c-\frac{QT_c}{2}\left(\frac{\left\lfloor\frac{n+1}{Q}\right\rfloor+\left\lfloor\frac{n}{Q}\right\rfloor+1}{(n+1)}\right).$$

Rewriting shows that for $(n+1) \bmod Q = 0$ we have that $\left\lfloor\frac{n+1}{Q}\right\rfloor=\left\lfloor\frac{n}{Q}\right\rfloor+1$. This means we can simplify again:

$$T_c-\frac{QT_c}{2}\left(\frac{2\left\lfloor\frac{n+1}{Q}\right\rfloor}{(n+1)}\right).$$

Because $(n+1) \bmod Q = 0$, we have that $\left\lfloor\frac{n+1}{Q}\right\rfloor=\frac{n+1}{Q}$ so:

$$T_c-\frac{QT_c}{2}\left(\frac{2\left\lfloor\frac{n+1}{Q}\right\rfloor}{(n+1)}\right)=T_c-\frac{QT_c}{2}\left(\frac{2\frac{n+1}{Q}}{(n+1)}\right)=0.$$

Since this expression is always $0$, we have that

$$EW_{n+1}-EW_n=\frac{QT_c}{2}\left(\frac{\left\lfloor\frac{n}{Q}\right\rfloor^2+\left\lfloor\frac{n}{Q}\right\rfloor}{n(n+1)}\right).$$

Since the above expression represents a difference in time, it still needs to be normalized by dividing by $T_{max}-T_{min}$. So the final expression for $\epsilon_n$ is:

$$\epsilon_n=\frac{QT_c}{2}\left(\frac{\left\lfloor\frac{n}{Q}\right\rfloor^2+\left\lfloor\frac{n}{Q}\right\rfloor}{n(n+1)}\right)\frac{1}{T_{max}-T_{min}}. \tag{4.26}$$

Filling in $\epsilon_{\frac{N}{2}}$ in Equation (4.25), gives:

$$\beta^*=\min\left(1-\frac{\gamma\left(\alpha-\frac{QT_c}{2}\left(\frac{\left\lfloor\frac{N}{2Q}\right\rfloor^2+\left\lfloor\frac{N}{2Q}\right\rfloor}{\frac{N}{2}\left(\frac{N}{2}+1\right)}\right)\frac{1}{T_{max}-T_{min}}\right)}{1-\gamma},1\right). \tag{4.27}$$

This formula gives some interesting results. When the two routes have the same length, corresponding to $\alpha = 0$, the function gives $\beta = 1$, meaning that the price should be the same at both stations. This makes sense, since the distances are the same, the algorithm would evenly split vehicles across the station based on the time aspect. Making one of the two stations cheaper would increase the utility function for that route, skewing the distribution towards that station. For $\gamma = 0$, once again we have $\beta = 1$. This is due to the fact that for $\gamma = 0$, the algorithm only looks at the price to determine the route. If there were even a small difference in price, all vehicles would go to one station.

Next to that, the formula shows the cases where it is impossible to create an even split. Setting $\gamma = 0.6$, we find that for $\alpha - \epsilon_{\frac{N}{2}} > \frac{2}{3}$, $\beta < 0$. From the perspective of drivers, it make sense that certain scenarios would not be solvable. There is a maximal detour which drivers would find acceptable, though how large that detour is depends on the individual driver. This shows that even considering the most simplistic scenario, there are cases which cannot give an even split.

### 4.1.2. Multiclass bottleneck

Extending this scenario to two classes of vehicles adds another dimension to this problem. If we generalize the problem, we have two classes of vehicles $V_1$ and $V_2$ with $\gamma_1$ and $\gamma_2$, respectively, where $\gamma_1 \geq \gamma_2$ without loss of generality. Assuming there are the same number of vehicles of both classes in the system, it is possible to do the following. As we are trying to reach an even split, the easiest way to solve the problem would be to have all vehicles of one class go to one station, and all of the other class to the other station. For this, define $U_{i,j,m}$ as the utility function for class $i$, for the route via station $j$ with $m$ vehicles.
To get all vehicles of one class to one station, and all vehicles of the other class to the other station, we want

$$\begin{aligned} U_{1,1,\frac{N}{2}} &\geq U_{1,2,\frac{N}{2}+1} \\ U_{2,2,\frac{N}{2}} &\geq U_{2,1,\frac{N}{2}+1}. \end{aligned} \tag{4.28}$$

This was done because by assumption $V_1$ has a preference for the shorter route, so it is easiest to let them keep choosing that route. These inequalities are basically equivalent to the ones used to formulate Equation (4.25), except here the gamma changes. From the first inequality we get:

$$\beta \geq 1 - \frac{\left(\alpha + \epsilon_{\frac{N}{2}}\right)\gamma_1}{1 - \gamma_1}.$$

The second then gives

$$\beta \leq 1 - \frac{\left(\alpha - \epsilon_{\frac{N}{2}}\right)\gamma_2}{1 - \gamma_2}.$$

These inequalities for $\beta$ give the conditions for all the vehicles from class $V_1$ to choose the route via station $1$, and all the vehicles from class $V_2$ to choose the route via station $2$. Combining this gives:

$$1 - \frac{\left(\alpha + \epsilon_{\frac{N}{2}}\right)\gamma_1}{1 - \gamma_1} \leq \beta \leq 1 - \frac{\left(\alpha - \epsilon_{\frac{N}{2}}\right)\gamma_2}{1 - \gamma_2}.$$

This means that for all $\beta$ in this range, the vehicles from class $V_1$ choose route $1$, and the vehicles from class $V_2$ choose route $2$. Since both classes have the same number of vehicles, this leads to an even split. Seeing as we want to find the highest such $\beta$, we set $\beta^* = \min\left(1 - \frac{(\alpha-\epsilon)\gamma_2}{1-\gamma_2}, 1\right)$. This is equivalent to applying (4.25), using the lower $\gamma_2$.

While this works for situations where there are two classes, and both classes have an equal number of vehicles in the system, this is a too restricted scenario. Also, this cannot directly be applied to graphs with more than $2$ stations, as at some point it becomes infeasible to calculate a $\beta$ which assigns one class to each station. As such, we should find an alternative for an approximate solution, but this was considered out of scope for this thesis.

### 4.1.3. $R$-Stations

Another extension to the bottleneck scenario, is adding more stations to choose. Intuitively, we call a bottleneck graph with $R$ stations, an $R$-station bottleneck graph. This changes the number of vehicles that each station needs to have from $\frac{N}{2}$ to $\frac{N}{R}$. This has as added condition that $N \bmod R = 0$. Once again we fix the price of one station, the most popular station if all prices were equal. Without loss of generality, we define this station to be station 1. Moreover, we define a $\beta_i$ for every other station. This means that we have to solve the system to find the $\beta_i$ value for $R-1$ stations. The conditions for this scenario are a generalisation of (4.1), considering more stations. This gives:

$$U_{i,\frac{N}{R}} \geq U_{j,\frac{N}{R}+1} \quad \forall i,j. \tag{4.29}$$

Solving these inequalities gives $R-1$ ranges for each $\beta_i$. However, all $\beta_i$ will be bounded using the utility of station 1, since $U_{1,\frac{N}{R}+1}$ does not depend on any variable. As a result, for every route, we can use Equation (4.25) which gives:

$$\beta_i^* = \min\left(1 - \frac{\left(\alpha_i - \epsilon_{\frac{N}{R}}\right)\gamma}{1-\gamma}, 1\right) \tag{4.30}$$

Here, $\alpha_i$ is calculated using the relevant edge lengths for its respective route.

To prove that $\beta_i^*$ suffices, we have to prove that filling in $\beta_i^*$ in the utility functions in inequalities (4.29), leads to it being satisfied. The proof of this can be found in Appendix A.

## 4.2. 2-Stations grid

The other scenario considered is the 2-stations grid. This graph is defined by having two origin nodes, two stations and two destination nodes. A generic example of such a graph can be seen in Figure 4.2 below.

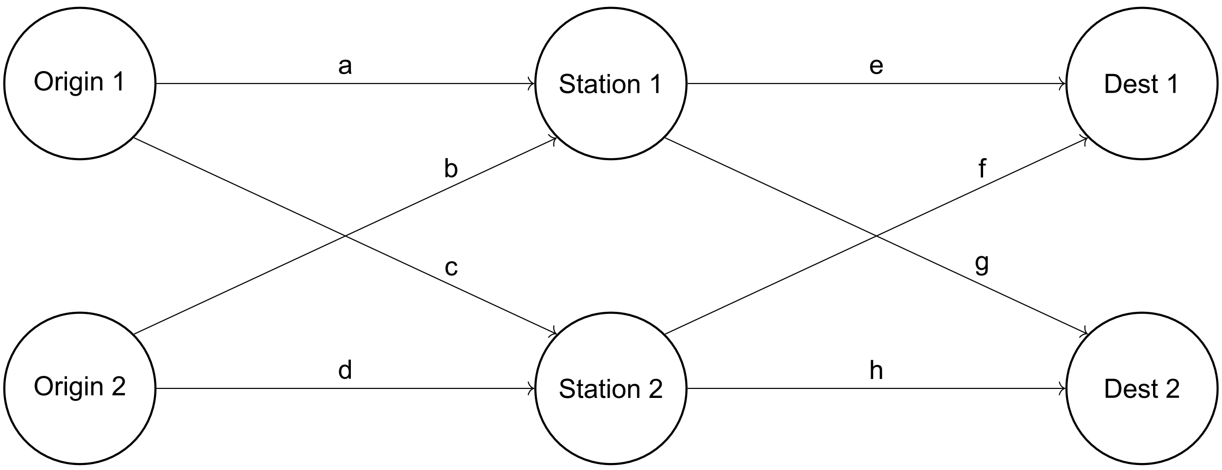


Figure 4.2: 2-stations grid graph with 2 stations. Edge lengths represent travel time.

Vehicles are equally divided over origin and destination nodes. This means that there are four unique origin node, destination node (O-D) pairs. Each origin node is connected to both Station 1 and Station 2, and so are the destination nodes. As such, for every O-D pair, a vehicle can either choose to take the route via Station 1 or the route via Station 2. The fact that each unique O-D pair has different routes available, means that each O-D pair also has its own unique $T_{min}$. For the previous scenarios, only one situation had to be considered. For this problem, however, 4 unique cases have to be considered. To be as concise as possible, graphs are represented by $[a, b, c, d, e, f, g, h]$, where the index in the list represents the respective edge in the graph.

- The first case has $a = b$, $c = d$, and $e - g$ arbitrary. In this case, the destination decides the relevant travel times.
- The second case has $a < b$ and $c > d$, or has $a > b$ and $c < d$, with $e - g$ arbitrary. In this case, each origin node has a station at which the vehicles arrive earliest.
- The third case has $a = b$ or $c = d$ with the remaining edges arbitrary. So at one of the stations, vehicles from both origin nodes can arrive at the same time, but at the other station, vehicles from one of the origin nodes arrives earlier.
- The fourth and final case has $a < b$ and $c < d$, or has $a > b$ and $c > d$, with $e - g$ arbitrary. This means that vehicles starting at one of the origin nodes arrive earlier at both stations.

This covers all the different types of graphs. More distinctions could be made, but these would be subcategories of the given 4 cases. In this thesis, we only handle the first case. This is because the asymmetry introduced in the other cases complicates the solution.

In this first case, vehicles from both origin nodes arrive at Station $1$ at time $a$, and arrive at Station $2$ at time $c$. This means that in practice there are only $2$ unique O-D pairs: One which goes to Destination $1$, and the other which goes to Destination $2$. The routes going to Destination $1$ have travel times $(a + e, c + f)$ and the routes going to Destination $2$ have travel times $(a + g, c + h)$, where the first entry uses Station $1$ and the second entry uses Station $2$. Note that if $e = g$ and $f = h$, both route pairs have exactly the same travel times, which means this situation would be equivalent to a bottleneck graph. If $a + e < c + f$ and $a + g > c + h$ or the other way around, then each O-D pair has a preference for a separate station, and the price at both stations can be the same. For the remaining sub-cases, we can assume without loss of generality that $a + e < c + f$ and $a + g < c + h$. This means that both unique O-D pairs then prefer Station $1$. As a result, the lower price should be at Station $2$. You could consider this problem as two bottleneck graphs which are linked, one for each O-D pair. Using this approach, both O-D pairs can be used to calculate a range of $\beta_i$ for an even split, where $\beta_i$ is the range of $\beta$ considering the O-D pair with destination $i$. These $\beta_i$'s come from taking the inequalities which ensure an even split for an individual O-D pair. Taking these together, for an entire 2-stations grid system we have:

$$U_{1,1,\frac{N}{2}} \geq U_{1,2,\frac{N}{2}+1} \tag{4.31}$$

$$U_{1,2,\frac{N}{2}} \geq U_{1,1,\frac{N}{2}+1} \tag{4.32}$$

$$U_{2,1,\frac{N}{2}} \geq U_{2,2,\frac{N}{2}+1} \tag{4.33}$$

$$U_{2,2,\frac{N}{2}} \geq U_{2,1,\frac{N}{2}+1} \tag{4.34}$$

These inequalities are exactly the same as inequalities (4.1), repeated for each destination. As such, looking at the set of inequalities for each station individually, we know that the first inequalities, so inequalities (4.31) and (4.33), represents the lower bound for the respective $\beta_i$. The second inequalities, (4.32) and (4.34), then represent the upper bounds for the respective $\beta_i$. If Station $2$ were preferable for both O-D pairs, these roles would be flipped. We refer to the lower and upper bound for $\beta_i$ as $\beta_i^{min}$ and $\beta_i^{max}$, respectively. As a result, we know that $\beta_i \in [\beta_i^{min}, \beta_i^{max}]$. This means that we can consider the following cases, visually represented in Figure 4.3.

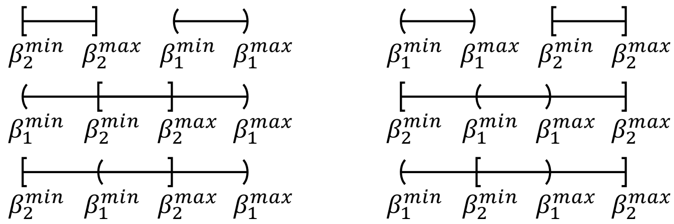

Figure 4.3: Visual representation of the bounds given by inequalities (4.31), (4.32) and (4.33), (4.34). Parentheses represent $\beta_1^{min}$ and $\beta_1^{max}$, square brackets represent $\beta_2^{min}$ and $\beta_2^{max}$.

What Figure 4.3 represents is all possible ways the bounds can fit on the number line. For the sake of clarity, we represent $\beta_1^{min}$ and $\beta_1^{max}$ with the parentheses, and $\beta_2^{min}$ and $\beta_2^{max}$ with the square brackets. The right cases are equivalent to the left cases by symmetry. As such without loss of generality, we can consider only the left cases, or $\beta_1^{max} > \beta_2^{max}$. In words, these cases can be explained in the following way. In the first case, there is no overlap between the ranges. In the second case, one range is a subset of the other range. And in the final case, there is overlap, but neither range is a subset of the other.

If a $\beta$ is chosen in the range where there is overlap between the two ranges, we know that inequalities (4.31)-(4.34) all hold. By definition, then we have that there will be an even split. Both O-D pairs have an incentive to swap stations in favor of an even split. As a result, as long as there is an even split, vehicles from either O-D pair have no further incentive to swap.
The interesting cases lie where $\beta$ is chosen in only one of the ranges. What happens then is that some of the inequalities do not hold. Assume we are in the following case:

$$\beta_1^{min} \leq \beta \leq \beta_2^{min}. \tag{4.35}$$

Then, all the inequalities hold except for (4.33). Instead, we get the following:

$$U_{1,1,\frac{N}{2}} \geq U_{1,2,\frac{N}{2}+1}, \tag{4.36}$$

$$U_{1,2,\frac{N}{2}} \geq U_{1,1,\frac{N}{2}+1}, \tag{4.37}$$

$$U_{2,1,\frac{N}{2}} < U_{2,2,\frac{N}{2}+1}, \tag{4.38}$$

$$U_{2,2,\frac{N}{2}} \geq U_{2,1,\frac{N}{2}+1}. \tag{4.39}$$

Looking at Inequalities (4.38) and (4.39), this shows a clear preference for vehicles heading to Destination 2 to favor Station 2. Since, even in an even split, it is beneficial to switch from Station 1 to Station 2 for these vehicles. However, as Inequalities (4.36) and (4.37) still hold, considering the fact that the vehicles are distributed equally across O-D pairs, there still will be an even split. This is because it remains beneficial for vehicles heading to Destination 1, to switch stations to create an even split.
If we find ourselves in the inverse case,

$$\max\left(\beta_1^{min}, \beta_2^{max}\right) \leq \beta \leq \beta_1^{max}, \tag{4.40}$$

then instead Inequality (4.34) does not hold. Taken together we get:

$$U_{1,1,\frac{N}{2}} \geq U_{1,2,\frac{N}{2}+1}, \tag{4.41}$$

$$U_{1,2,\frac{N}{2}} \geq U_{1,1,\frac{N}{2}+1}, \tag{4.42}$$

$$U_{2,1,\frac{N}{2}} \geq U_{2,2,\frac{N}{2}+1}, \tag{4.43}$$

$$U_{2,2,\frac{N}{2}} < U_{2,1,\frac{N}{2}+1}. \tag{4.44}$$

In this case by a similar argument as before, there will be an even split. However, this time the vehicles heading to Destination 2 prefer Station 1.

Considering the inequalities that are satisfied, based on where the $\beta$ is chosen, we can group the cases in the way shown in Figure 4.4.

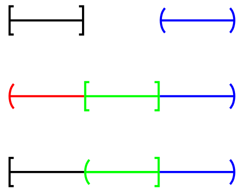

Figure 4.4: Visual representation of the bounds given by Inequalities (4.31)-(4.34). The colors represent areas in which the same inequalities are satisfied.

The case not discussed here, represented by black in Figure 4.4, is where Inequality (4.31) is not satisfied, but (4.32)-(4.34) are. This gives:

$$U_{1,1,\frac{N}{2}} < U_{1,2,\frac{N}{2}+1}, \tag{4.45}$$

$$U_{1,2,\frac{N}{2}} \geq U_{1,1,\frac{N}{2}+1}, \tag{4.46}$$

$$U_{2,1,\frac{N}{2}} \geq U_{2,2,\frac{N}{2}+1}, \tag{4.47}$$

$$U_{2,2,\frac{N}{2}} \geq U_{2,1,\frac{N}{2}+1}. \tag{4.48}$$

This indicates that the vehicles heading to Destination $1$ prefer taking Station $2$. However, once again, as the vehicles are distributed evenly over the O-D pairs, and the vehicles heading to Destination $2$ benefit from an even split, there will be an even split. The end result is all vehicles heading to Destination $1$ taking station $2$ and all the vehicles heading to Destination $2$ taking station $1$.
The final case to consider, is when we are between the two ranges in the first case. Here, we have that two of the four inequalities hold, namely:

$$\begin{aligned} U_{1,1,\frac{N}{2}} &< U_{1,2,\frac{N}{2}+1}, \\ U_{1,2,\frac{N}{2}} &\geq U_{1,1,\frac{N}{2}+1}, \\ U_{2,1,\frac{N}{2}} &\geq U_{2,2,\frac{N}{2}+1}, \\ U_{2,2,\frac{N}{2}} &< U_{2,1,\frac{N}{2}+1}. \end{aligned} \tag{4.49}$$

In this case, the vehicles heading to Destination $1$ prefer Station $2$, and the vehicles heading to Destination $2$ prefer Station $1$. As the vehicles are equally distributed across O-D pairs, this ends up balancing out, meaning there is an even split once again.
Having handled all cases, this shows that any $\beta \in [\beta^{min}, \beta^{max}]$ suffices, where $\beta^{min} = \min(\beta_1^{min}, \beta_2^{min})$ and $\beta^{max} = \max(\beta_1^{max}, \beta_2^{max})$. Seeing as we want to maximize profits, we can use $\beta^{max}$. Meaning that as a final result we get that:

$$\beta^* = \min(1, \max(\beta_1^{max}, \beta_2^{max})). \tag{4.50}$$

# 5

# Results

In this chapter, results from Chapter 4 are verified with the help of a simulation of the model. Since the model worked on all examples tried, for each case a representative graph was chosen to demonstrate the hypothesized behavior. Section 5.1 discusses the bottleneck graph. Next, Section 5.2 discusses the grid graphs. Section 5.3 discusses the validity of the results. Finally, Section 5.4 gives a general conclusion of the results.

## 5.1. Bottleneck graph

In Subsection 5.1.1 an example is given of the case where there are 2 stations and just one class of vehicles. Subsection 5.1.2, handles the case where there are multiple classes of vehicles. And finally, Subsection 5.1.3 discusses the $R$-station scenario.

### 5.1.1. 2-stations

To verify the method used in Section 4.1, we demonstrate how it works on a concrete example. Figure 5.1 shows an instance of the 2-stations bottleneck graph. As the shortest route goes through Station 2, we instead have to calculate the optimal price at Station 1. First, it is necessary to calculate the $\alpha$ from the values in Figure 5.1.

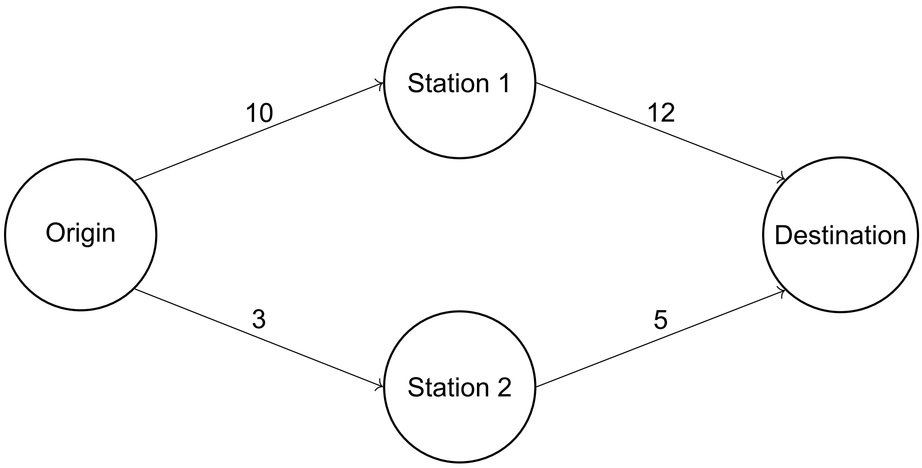


Figure 5.1: Specific instance of a bottleneck graph with 2 stations. Edge lengths represent travel time.

For this example we use parameters, the total number of vehicles $N = 10$, the capacity at the stations $Q = 2$, the time it takes to charge $T_c = 3$ and the choice parameter $\gamma = 0.6$. From the graph, we have $T_{min} = 8$ resulting in $T_{max} = 3T_{min} = 24$, and the route via Station 1 takes a total of $22$ minutes. This gives $\alpha = \frac{22-8}{24-8} = \frac{14}{16} = 0.875$. The vehicles used in this example was 10. Using Equation (4.26) for $\frac{N}{2} = 5$ vehicles, gives $\epsilon_5 = \frac{2 \cdot 3}{2}\left(\frac{\lfloor \frac{5}{2} \rfloor^2 + \lfloor \frac{5}{2} \rfloor}{5 \cdot 6}\right)\frac{1}{16} = \frac{3}{80}$. Filling this in, in Equation (4.25) gives

$$\beta = 1 - \frac{\left(\frac{7}{8} - \frac{3}{80}\right) \cdot 0.6}{1 - 0.6} = -\frac{41}{160}.$$

The formula gives $\beta < 0$, which means it is impossible to get an even split for this graph, if all vehicles have $\gamma = 0.6$. Doing the same however for $\gamma = 0.4$ gives

$$\beta = 1 - \frac{\left(\frac{7}{8} - \frac{3}{80}\right) \cdot 0.4}{1 - 0.4} = \frac{53}{120} \approx 0.442\ .$$

With $p_2 = M_{max} = 10$, this gives $p_1 = 4$ as the model uses integer values for prices.
Figure 5.2 shows a bar graph of how the 10 vehicles, with $\gamma = 0.6$ distribute themselves for different prices at Station 1.

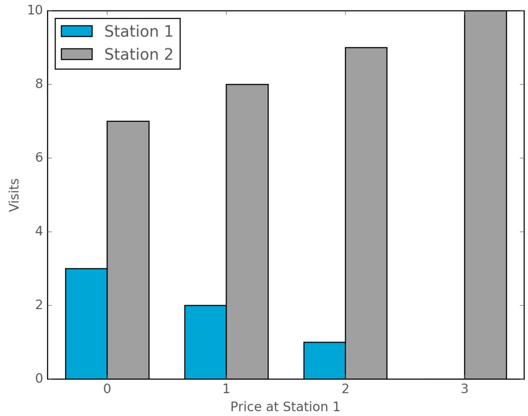


Figure 5.2: Bar Graph showing different numbers of visits across the stations for various $p_1$ with $\gamma = 0.6$

The distribution for $\beta = 0$ is also shown, even though the price is unrealistic. This was done to show that even with the lowest price possible, vehicles still prefer Station 2 because of the shorter travel time. The distributions for $p_1 \geq 4$ is not shown as all vehicles are already choosing Station 2. Raising the price further would only lower the utility, making it less likely for vehicles to choose this route.
Figure 5.3 shows a bar graph of how the vehicles distribute themselves for different prices at Station 1, this time the vehicles have $\gamma = 0.4$.

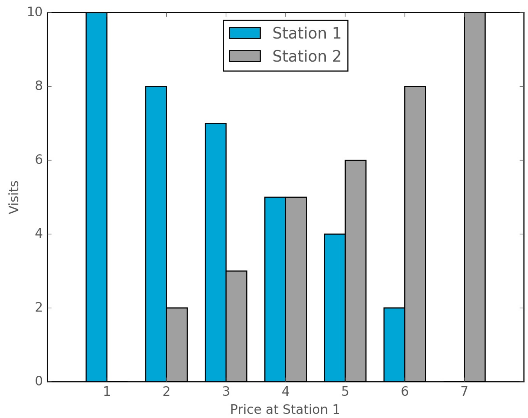


Figure 5.3: Bar Graph showing different numbers of visits across the stations for various $p_1$ with $\gamma = 0.4$

As can be seen, if $p_1 = 1$, all vehicles choose for Station 1. As $p_1$ increases, more vehicles start choosing for Station 2, until for $p_1 \geq 7$ all vehicles choose Station 2. The even split occurs when $p_1 = 4$, just as hypothesized.

### 5.1.2. Multiclass bottleneck

The calculations for the Multiclass bottleneck case are the exact same as for the case in Section 5.1.1. The only difference is that you choose the lowest $\gamma$ to calculate $\beta^*$. As a result the behavior is very similar to the behavior shown in 5.1.1.

### 5.1.3. $R$-Stations

The calculations for the $R$-stations bottleneck graph are the exact same as for the 2-stations bottleneck graph. An example is shown for the 3-stations case, based on Figure 5.4. As parameters we use, the total number of vehicles $N = 15$, the capacity at the stations $Q = 2$, the time it takes to charge $T_c = 3$ and the choice parameter $\gamma = 0.6 = \frac{3}{5}$ which leads to $\frac{\gamma}{1-\gamma} = \frac{3}{2}$.

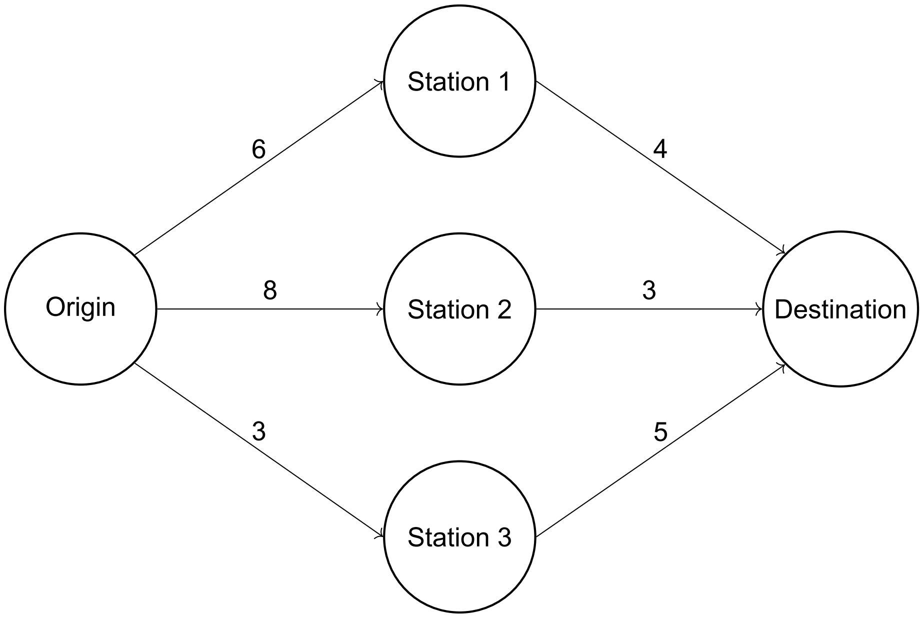


Figure 5.4: Specific instance of a bottleneck graph with 3 stations. Edge lengths represent travel time.

First we identify the shortest route from the origin to the destination, this is the route through Station 3, which gives $T_{min} = 8$. Calculating the $\alpha_i$ gives:

$$\alpha_1 = \frac{10-8}{24-8} = \frac{1}{8}$$

$$\alpha_2 = \frac{11-8}{24-8} = \frac{3}{16}$$

In this case we need $\epsilon_{\frac{N}{3}}$. However, since $N = 15$, we use $\epsilon_5 = \frac{2 \cdot 3}{2}\left(\frac{\left\lfloor\frac{5}{2}\right\rfloor^2 + \left\lfloor\frac{5}{2}\right\rfloor}{5 \cdot 6}\right)\frac{1}{16} = \frac{3}{80}$

Which gives:

$$\beta_1^* = 1 - \frac{\left(\frac{1}{8} - \frac{3}{80}\right) \cdot 0.6}{1 - 0.6} = \frac{139}{160},$$

and,

$$\beta_2^* = 1 - \frac{\left(\frac{3}{16} - \frac{3}{80}\right) \cdot 0.6}{1 - 0.6} = \frac{59}{80}$$

## 5.2. 2-Stations grid

The final scenario, while seemingly very similar to the two station bottleneck problem, immediately becomes significantly more complex. Firstly, the utility functions for vehicles with different O-D pairs is different, even for vehicles within the same class. This is due to the fact that $T_{min}$ can differ for each unique O-D pair. And secondly, instead of considering two possible routes, we have to consider two routes for each O-D pair. This leads to a total of 8 routes to consider.

The 2-stations grid scenario is split into 4 scenarios to describe how to solve for an even split. These scenarios are described in Section 4.2. However, all but the first of these were considered too complex for this thesis. As a result, only results will be shown for the first scenario: $a = b, c = d$.

While certain parameters for calculation are dependent on the graph, a number of them were set globally for these graphs. These are, the total number of vehicles $N = 120$, the capacity at the stations $Q = 2$, the time it takes to charge $T_c = 3$ and the choice parameter $\gamma = 0.6 = \frac{3}{5}$ which leads to $\frac{\gamma}{1-\gamma} = \frac{3}{2}$

Following the steps described Section 4.2, a worked example is given using values shown in Figure 5.5.

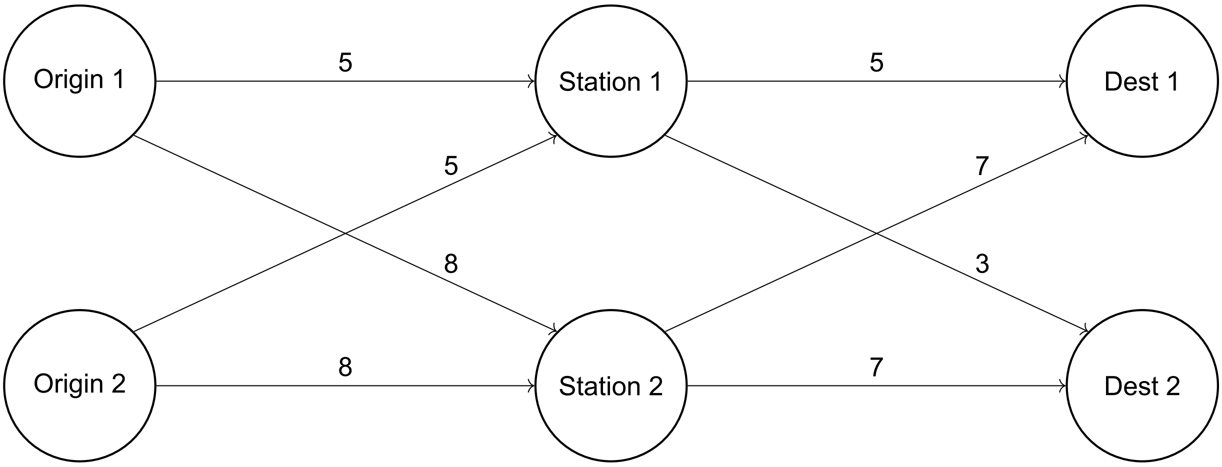


Figure 5.5: 2-stations grid graph with 2 stations. Edge lengths represent travel time. Shortened representation: [5,5,8,8,5,7,3,7]

In this example, all O-D pairs have a smaller distance for the route via Station 1. As such, the price at Station 2 must be lower to counteract this. For this, we calculate the $\beta$ for the O-D pair to Dest 1 ($\beta_1$), and the O-D pair to Dest 2 ($\beta_2$). This can be calculated using (4.25).

$$\epsilon_{60,1} = \frac{2 \cdot 3}{2} \frac{\left\lfloor \frac{60}{2} \right\rfloor^2 + \left\lfloor \frac{60}{2} \right\rfloor}{60 \cdot 61} \frac{1}{2 \cdot 10} = \frac{93}{2440}$$
$$\alpha_1 = \frac{15 - 10}{2 \cdot 10} = \frac{1}{4}$$
$$\beta_1 = 1 - \left( \frac{1}{4} - \frac{93}{2440} \right) \frac{3}{2} = \frac{3329}{4880}$$

$$\epsilon_{60,2} = \frac{2 \cdot 3}{2} \frac{\left\lfloor \frac{60}{2} \right\rfloor^2 + \left\lfloor \frac{60}{2} \right\rfloor}{60 \cdot 61} \frac{1}{2 \cdot 8} = \frac{93}{1952}$$
$$\alpha_2 = \frac{15 - 8}{2 \cdot 8} = \frac{7}{16}$$
$$\beta_2 = 1 - \left( \frac{7}{16} - \frac{93}{1952} \right) \frac{3}{2} = \frac{1621}{3904}$$

Of these $\beta$s we find that the maximum is $\beta_1^* = \frac{3329}{4880}$. Setting the price at Station 2 to $\beta_1^* \cdot p_1$, gives an even split.

To test the hypothesis that the preferred station changes as the $\beta$ changes as described in Section 4.2, we look at the graph given in Figure 5.6. The parameters used are the same as the ones used for Figure 5.5

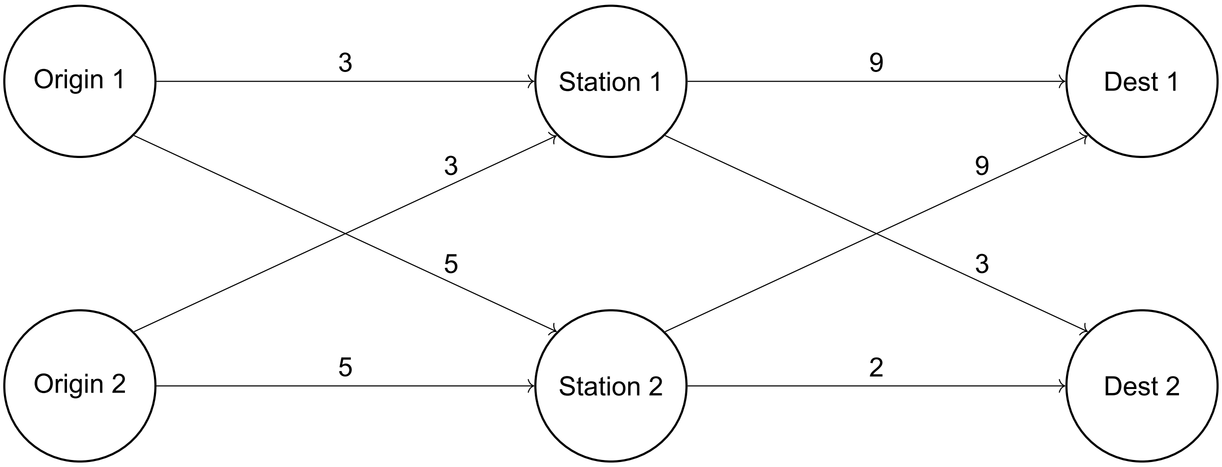


Figure 5.6: 2-stations grid graph with 2 stations. Edge lengths represent travel time. Shortened representation: [3,3,5,5,9,9,3,2]

The preferred station is Station 1 for both O-D pairs, so Station 2 should receive the lower price. Using the utility functions and equation (4.25), we can find $\beta_1^{min}, \beta_1^{max}$ and $\beta_2^{min}, \beta_2^{max}$. In doing so we find:

$$\beta_1^{min} \approx 0.827,$$
$$\beta_1^{max} \approx 0.923,$$
$$\beta_2^{min} \approx 0.780,$$
$$\beta_2^{max} \approx 0.970.$$

Thus, $[\beta_1^{min}, \beta_1^{max}] \subset [\beta_2^{min}, \beta_2^{max}]$. Because of this what we expect to happen, is that when $\beta < \beta_2^{min}$, there is no even split. When $\beta_2^{min} \leq \beta \leq \beta_1^{min}$, we expect all vehicles heading to Destination 1 to prefer Station 2, and all vehicles heading to Destination 2 to prefer Station 1. When $\beta_1^{min} \leq \beta \leq \beta_1^{max}$, there will be an even split, but no conclusion can be drawn about the distribution at the stations. When $\beta_1^{max} \leq \beta \leq \beta_2^{max}$, there will once again be an even split, but this time vehicles heading to Destination 1 prefer Station 1, and vehicles heading to Destination 2 prefer Station 2. And finally, for $\beta_2^{max} < \beta$ there will not be an even split. Running the simulation for $\beta \in (0.7, 0.8, 0.9, 0.95, 1.0)$ showcases these separate cases. The result of those runs can be seen in Figure 5.7.

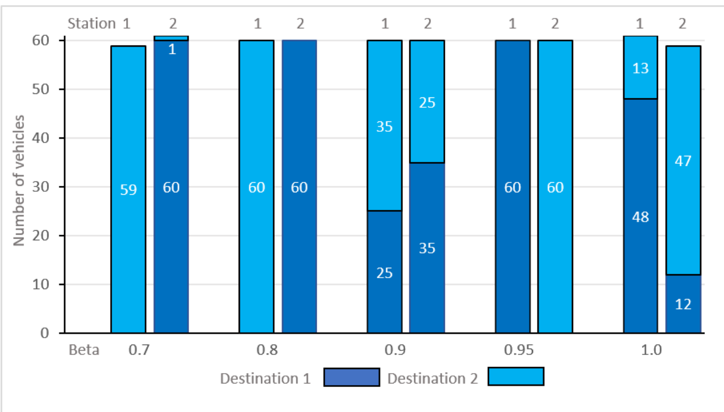


Figure 5.7: Bar Graph showing different numbers of visits across the stations for various $\beta$. For each beta, the left bar shows number of visits to Station 1, while the right bar shows visits to Station 2. Dark blue represents vehicles heading to Destination 1, while light blue represents vehicles heading to Destination 2.

Figure 5.7 is evidence that the model works the way we hypothesized in Section 4.2.

## 5.3. Precision

In the previous sections of this chapter, the behavior expected was verified. However, no real argument was made about the validity of the results in numerical terms. While overall the simulation for the model performs well, there is a certain flaw in it. The simulation only takes fractional values for the price, and the runtime and memory usage depends on the magnitude of the denominator of the price. As a result, the only way to check the validity of the calculated $\beta^*$ is to run the model twice. One run with the internal $\beta$ such that $\beta < \beta^*$, and one run with internal $\beta$ such that $\beta^* < \beta$. By choosing these $\beta$'s close to $\beta^*$ we can be fairly confident that $\beta^*$ is correct. The plot shown in Figure 5.8 shows the size of the range within which $\beta^*$ could lie. The smaller the number, the more confidence we have in the result being correct.

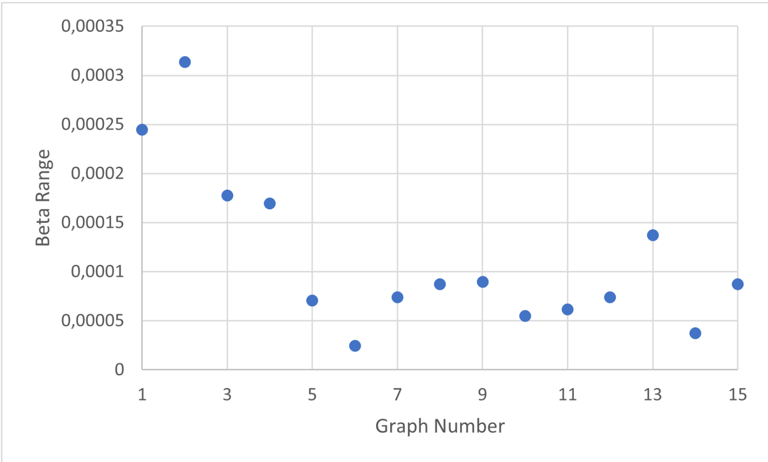


Figure 5.8: Scatter graph showing the maximal range of the beta for the respective graph.

These results are also shown in a table present in Appendix B. The smallest range was achieved for the sixth graph, for which the range was only $2.4 * 10^{-5}$. Seeing as the influence $\epsilon$ on $\beta$ is usually in the order of magnitude $10^{-2}$, we can be fairly confident that the calculated $\beta^*$ is correct.

## 5.4. General conclusions

Having seen these examples, a few conclusions can be made. For bottleneck scenarios, Equation (4.25) (see Chapter 4.1) can be used to calculate even splits. This formula can be further adjusted to handle multiple classes, and an increase in stations. However, finding a direct solution for the grid scenario proved more difficult. Both due to an increase in O-D pairs, as the added difficulty of dealing with different arrival times. However, limiting ourselves to the "$a = b, c = d$" case, makes it possible to once again solve the relevant equations.

# 6 Conclusion and discussion

In this thesis, the IARS model introduced in [2] was extended to adjust to station prices. This was done to find a way to calculate prices to minimize maximum queue size across the stations, while maximising profit. Minimizing queue size is achieved by splitting the vehicles evenly across all stations. To simultaneously maximize profits, Equation (4.25) was derived by setting the Nash Equilibrium at an even split. This Equation was subsequently applied to different graphs. Further research will have to be done to see if a feasible solution can be found for other graphs.

For future work, more research could be done regarding the grid scenario. Expanding the number of O-D pairs, and researching the influence of adding more stations to the graphs. However, the complexity of the problem grows exponentially based on adding more nodes. Next to that, the current model makes assumptions which need to be relaxed to make it more realistic. The two main ones are that both charge time and money spent at a charging station are not charge dependent. In reality, you have to pay for the amount of charge, and charging an battery to $80\%$ capacity, goes a lot quicker than charging to full.

The most interesting result would come from generalizing the test cases to allow for vehicles to leave at different start times. However, for this to be done in a reasonable amount of time, some changes have to be made to the code. The main fix that has to be made is that the runtime is not dependent on the magnitude of the price, but instead be dependent on the number of different prices at the stations. Finally, research could be done into approximations for optimal pricing. Since it is unlikely that a direct equation for the price can be found which takes into account all the relevant parameters, an approximation could be a good alternative.

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# A

# R-Stations proof

In this appendix, we give a supplementary proof to Section 4.1.3.

$$U_{i,\frac{N}{R}} \geq U_{j,\frac{N}{R}+1}$$

gives

$$\gamma\,(1-\alpha_i) + (1-\gamma)\,(1-\beta_i) - \gamma\frac{T_{q,\frac{N}{R}}}{T_{max}-T_{min}} \geq \gamma\,(1-\alpha_j) + (1-\gamma)\,(1-\beta_j) - \gamma\frac{T_{q,\frac{N}{R}+1}}{T_{max}-T_{min}}$$

$$(1-\gamma)\,(1-\beta_i) - (1-\gamma)\,(1-\beta_j) \geq \gamma\,(1-\alpha_j) - \gamma\,(1-\alpha_i) - \left(\gamma\frac{T_{q,\frac{N}{R}+1}}{T_{max}-T_{min}} - \gamma\frac{T_{q,\frac{N}{R}}}{T_{max}-T_{min}}\right)$$

$$(1-\gamma)(\beta_j-\beta_i) \geq \gamma\left(\alpha_i-\alpha_j-\epsilon_{\frac{N}{R}}\right)$$

$$(\beta_j-\beta_i) \geq \frac{\gamma\left(\alpha_i-\alpha_j-\epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

By using Equations (4.30), we get four cases:

$$\beta_j^* = 1, \beta_i^* = 1$$

$$\beta_j^* < 1, \beta_i^* < 1$$

$$\beta_j^* < 1, \beta_i^* = 1$$

$$\beta_j^* = 1, \beta_i^* < 1$$

The first ($\beta_j = 1, \beta_i = 1$) gives:

$$(\beta_j-\beta_i) = 0 \geq \frac{\gamma\left(\alpha_i-\alpha_j-\epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

Since $\beta_i^* = \min\left(1-\frac{\left(\alpha_i-\epsilon_{\frac{N}{R}}\right)\gamma}{1-\gamma}, 1\right)$, $\beta_i^* = 1$ implies that

$$1-\frac{\gamma\left(\alpha_i-\epsilon_{\frac{N}{R}}\right)}{1-\gamma} \geq 1$$

which gives

$$\frac{\gamma\left(\alpha_i - \epsilon_{\frac{N}{R}}\right)}{1-\gamma} \leq 0$$

so

$$\alpha_i \leq \epsilon_{\frac{N}{R}}.$$

From this, the inequality follows directly.
The second ($\beta_j < 1, \beta_i < 1$) gives:

$$\left(\beta_j - \beta_i\right) = 1 - \frac{\gamma\left(\alpha_j - \epsilon_{\frac{N}{R}}\right)}{1-\gamma} - \left(1 - \frac{\gamma\left(\alpha_i - \epsilon_{\frac{N}{R}}\right)}{1-\gamma}\right) = \frac{\gamma\left(\alpha_i - \alpha_j\right)}{1-\gamma} \geq \frac{\gamma\left(\alpha_i - \alpha_j - \epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

Since $\epsilon_{\frac{N}{R}} > 0$, this follows directly.
The third ($\beta_j < 1, \beta_i = 1$) gives:

$$\left(\beta_j - \beta_i\right) = 1 - \frac{\gamma\left(\alpha_j - \epsilon_{\frac{N}{R}}\right)}{1-\gamma} - 1 = \frac{\gamma\left(-\alpha_j + \epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

Since $\beta_i = 1$, $\alpha_i \leq \epsilon_{\frac{N}{R}}$ and because of this $\alpha_i - \epsilon_{\frac{N}{R}} \leq 0$ and also $\alpha_i - 2 \cdot \epsilon_{\frac{N}{R}} \leq 0$ .

$$\frac{\gamma\left(-\alpha_j + \epsilon_{\frac{N}{R}}\right)}{1-\gamma} \geq \frac{\gamma\left(-\alpha_j + \epsilon_{\frac{N}{R}} + \left(\alpha_i - 2 \cdot \epsilon_{\frac{N}{R}}\right)\right)}{1-\gamma} = \frac{\gamma\left(\alpha_i - \alpha_j - \epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

So the third case also works correctly.
The final case ($\beta_j = 1, \beta_i < 1$) gives:

$$\left(\beta_j - \beta_i\right) = 1 - \left(1 - \frac{\gamma\left(\alpha_i - \epsilon_{\frac{N}{R}}\right)}{1-\gamma}\right) = \frac{\gamma\left(\alpha_i - \epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

Since $\alpha_j \geq 0$ (by definition), we have the following:

$$\left(\beta_j - \beta_i\right) = \frac{\gamma\left(\alpha_i - \epsilon_{\frac{N}{R}}\right)}{1-\gamma} \geq \frac{\gamma\left(\alpha_i - \alpha_j - \epsilon_{\frac{N}{R}}\right)}{1-\gamma}$$

Which was what we wanted to prove.

# B

# Precision results

This appendix contains a table with results used for Figure 5.8. The number corresponds to the number in the figure. Graph shows the shortened representation of the 2-stations grid graph used in the simulation. The $\beta^*$ column shows the exact calculated $\beta^*$ for the respective graph. The internal parameters used were $N = 120$, $Q = 2$, $T_c = 3$ and $\gamma = \frac{3}{5}$. The upper and lower columns represent the internal upper and lower bound beta's used to verify $\beta^*$. For the upper case, the resulting distribution then would not be an even split, and for the lower case, the resulting distribution would be an even split. The final column was calculated by subtracting the lower column from upper. This is what was plotted against the graph number in Figure 5.8.

| # | graph | $\beta^*$ | upper | lower | range |
|---|---|---|---|---|---|
| 1 | [3,3,10,10,2,8,8,8] | $\frac{3085}{5368}$ | $\frac{50}{87}$ | $\frac{27}{47}$ | 0.000244559 |
| 2 | [10,10,2,2,8,10,3,6] | $\frac{1313}{1952}$ | $\frac{37}{55}$ | $\frac{39}{58}$ | 0.00031348 |
| 3 | [10,10,8,8,10,5,10,10] | $\frac{2777}{2928}$ | $\frac{92}{97}$ | $\frac{55}{58}$ | 0.00017746 |
| 4 | [4,4,2,2,9,9,9,5] | $\frac{4915}{5368}$ | $\frac{76}{83}$ | $\frac{65}{71}$ | 0.000169693 |
| 5 | [2,2,2,2,3,2,7,6] | $\frac{3817}{3904}$ | $\frac{44}{45}$ | $\frac{307}{314}$ | 0.000070771 |
| 6 | [10,10,9,9,9,3,7,3] | $\frac{1435}{1952}$ | $\frac{161}{219}$ | $\frac{136}{185}$ | 0.000024682 |
| 7 | [3,3,3,3,10,9,10,5] | $\frac{1923}{1952}$ | $\frac{199}{202}$ | $\frac{66}{67}$ | 0.000073888 |
| 8 | [4,4,6,6,8,4,8,5] | $\frac{5281}{5368}$ | $\frac{61}{62}$ | $\frac{182}{185}$ | 0.000087184 |
| 9 | [5,5,10,10,3,2,6,5] | $\frac{4183}{5368}$ | $\frac{113}{145}$ | $\frac{60}{77}$ | 0.000089566 |
| 10 | [4,4,4,4,3,10,6,8] | $\frac{4427}{4880}$ | $\frac{88}{97}$ | $\frac{342}{377}$ | 0.000054691 |
| 11 | [4,4,3,3,6,4,9,8] | $\frac{4915}{5368}$ | $\frac{76}{83}$ | $\frac{358}{391}$ | 0.000061628 |
| 12 | [2,2,3,3,10,10,6,7] | $\frac{1923}{1952}$ | $\frac{199}{202}$ | $\frac{66}{67}$ | 0.000073888 |
| 13 | [7,7,9,9,6,9,7,8] | $\frac{859}{976}$ | $\frac{257}{292}$ | $\frac{22}{25}$ | 0.000136986 |
| 14 | [6,6,2,2,8,8,10,3] | $\frac{739}{976}$ | $\frac{290}{383}$ | $\frac{53}{70}$ | 0.0000373 |
| 15 | [9,9,6,6,9,8,3,5] | $\frac{5281}{5368}$ | $\frac{61}{62}$ | $\frac{182}{185}$ | 0.000087184 |

Table B.1: Table containing results for 5.8