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Robust Network-Wide Bus Scheduling With Transfer Synchronizations

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***Abstract*— Travel time and demand disturbances lead to unreliable bus operations and missed passenger transfers. This study formulates the multi-line synchronization problem as a robust min(i)max problem that considers the fluctuations of the travel and dwell times of bus trips. Given the infeasibility of the multi-line synchronization problem in extreme cases of travel/dwell time disturbances, we introduce a flexible problem formulation that incorporates the constraint violations into the objective function. To produce a robust schedule, the dispatching times of trips are our design variables and the travel and dwell time fluctuations are the environmental variables which have an adversarial role in our minimax problem. We validate our approach in the bus network of The Hague using 1 month of actual vehicle location and passenger counting data. There, we demonstrate the potential improvement in terms of service regularity and increased synchronizations in common case and extreme case conditions.**

***Index Terms*— Bus scheduling, minimax, regularity-based services, passenger transfers, transfer coordination.**

I. Introduction

SCHEDULING the dispatching times of bus trips is a sub-problem of the tactical planning phase. This problem follows the stages of frequency settings and vehicle allocation [1]–[4]. After setting the dispatching times of trips, service operators may apply control strategies in real time such as holding, stop-skipping or dispatching time adjustments [5]–[11].

The industry practice is to determine the scheduled dispatching times of each bus line in isolation [12]. In studies that attempt a network-wide synchronization, the variability of the bus travel times during the actual operations is not taken into consideration at the timetabling stage [13], [14]. Notwithstanding, the negative consequences of considering deterministic bus travel times when optimizing the passenger transfers were already identified in several experiments in the early 1990s [15].

In a more comprehensive study, [16] explored the waiting times of passengers at transfer stops in the case of rail synchronization. Reference [16] showed that synchronization attempts at the tactical planning stage are ineffective if the actual arrival times at the transfer stops fluctuate significantly from the planned ones.

This study contributes to the network-wide service scheduling problem by determining trip dispatching times that (i) favor the synchronization of different services at interchange locations and (ii) maintain scheduling robustness to travel and dwell time fluctuations during the daily operations. While solving the network-wide synchronized scheduling problem, we consider the regularity levels of different bus lines as an additional key performance indicator. The inclusion of the service regularity as a problem objective guarantees that we do not sacrifice the regularity of individual services in the pursue of improved passenger transfers [17]. Finally, we consider multiple regulatory constraints related to schedule sliding prevention, dispatching headway bounds and minimum layover times.

In the remainder of this section we review related studies and describe the features of this work. In Section 2, we formulate the bus synchronization problem based on the above considerations. Section 3 details our mathematical program of the robust, network-wide synchronization problem. A robust design is defined here as a design that performs best at the worst-case scenario imposed by the adversary (in our case, the adversary of our design is the travel and dwell time disturbance). In Section 4 we present the solution method. The numerical experiments for an idealized network (demonstration) and the bus network of The Hague (application) are presented in Section 5. Sections 6 and 7 discuss the results and draw the future research directions.

A. Related Studies

The problem of timetable synchronization has been addressed by [14], [17]–[20] with the objective of reducing the waiting time of passengers at the transfer stops while keeping the departure times of the daily trips evenly spaced. Reference [21], [22] and [23] tried a less complex approach by merely shifting the pre-existing timetables to find the optimal solution for the passenger transfers with the use of a Genetic Algorithm (GA). Most works in the literature decouple the timetabling synchronization from the other tactical planning problems. An exception is the work of [24] that tried to minimize also the total number of the required vehicles. Even at this case though, [24] solved each objective separately by using bi-level programming. In [24], the number of the required vehicles was determined at the upper-level and the minimization of the total transfer time of passengers was solved by a heuristic algorithm at the lower-level.

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[25] and [26] generated timetables that maximize the number of synchronizations at the transfer points of the network. In these works, the dispatching headways were considered as given and the objective was to maximize the simultaneous arrivals of buses at transfer stops. The problem was modeled as a mixed integer linear program and a heuristic algorithm was employed on an Israeli case study due to the computational intractability of the problem. The definition of bus synchronization of [25] was modified by [27] and then by [28]. In [27] and [28], interconnected bus trips were not required to arrive simultaneously at the transfer point, but rather within a small time window (time buffer).

[28] allowed only oriented synchronization where passengers can transfer from one line to the other but not necessarily vice-versa. They also tried to keep the dispatching times of the daily trips as evenly spaced as possible and developed a multi-start, local search heuristic given the NP-hardness of the problem at hand. Following a different approach, [29] used time-varying travel times and passenger demand for bus scheduling but did not consider the variability of the actual travel times and passenger demand from their pre-determined, time-varying values in the optimization process.

[30] conceptualized the synchronization problem as a demand-supply problem and optimized the timetables of public transport modes by matching the passenger demand expressed via journey planners with the public transport supply in order to reduce missed connections. Other works that expand the synchronization problem to mixed (rail-bus) operations such as [31]–[34] proposed multi-modal synchronization methods based on the so-called "feeder model" that adjusts the bus schedules to the less flexible rail schedules. Reference [35] focused solely on rail operations and minimized the total passenger waiting time at stations by computing and adjusting train timetables for a rail corridor with given time-varying origin-to-destination passenger demand matrices. Although [35] considered time-varying demand, the variability of travel times was not considered in the formulation of their nonlinear integer program.

B. Focus of This Work

In the works mentioned above, the variability of travel and dwell times from their expected values was not considered at the optimization stage. However, this may lead to significant discrepancies between the scheduled and the actual arrival times of buses at stops resulting in missed connections.

The most relevant previous work by [36] incorporated the travel time variability at the multi-line synchronization problem. Nevertheless, [36] addressed the real-time bus holding problem at transfer stops, where bus trips were held at the transfer stops in order to perform the transfer. In addition, [36] minimized the transfer times under stochastic travel time conditions by modeling the noise of the bus arrivals at transfer stops with the use of normal distributions.

Our work considers the potential variability in the travel times of daily trips at the tactical planning stage and has the following additional features:

(i) considers explicitly the variability of dwell times due to fluctuations in passenger demand;
(ii) considers a dual objective: maximizing the regularity of individual bus lines while ensuring the synchronization of trips at the transfer stops;
(iii) produces *robust* solutions that (a) do not require determining the *probability distributions* of travel/dwell times such as in [36] and (b) avoid designs that are good on average but unsatisfactory in low-probability regions of the estimated probability distributions;
(iv) satisfies operational regulatory constraints such as schedule sliding prevention and layover time limits.

II. Problem Formulation

The frequencies of the bus service lines and the respective numbers of daily trips are determined during the frequency settings (FS) stage that precedes our problem. Setting the frequencies using the well-known maximum loading point method [37] ensures that the number of bus trips can accommodate the passenger demand at peak-hours even at the stations with the highest volume. We consider that the number of daily trips is already determined by the FS stage when scheduling the dispatching times of the daily trips.

Bus trips from the same line are assumed to avoid overtaking one another (this is a common assumption in related literature [38], [39]). Before proceeding to the description of the multi-line synchronization problem, the following notation is introduced.

Nomenclature

Sets

L	$L = \{1, \ldots, l, \ldots\}$ are the different bus lines in the study area.
N_l	$N_l = \{1, \ldots, n, \ldots, \lvert N_l \rvert\}$ is the ordered set of all daily trips of each bus line $l \in L$.
S_l	$S_l = \{1 \ldots, s, \ldots, \lvert S_l \rvert\}$ is the ordered set of bus stops of each bus line $l \in L$.
B_{lj}	All stops that allow for transferring between lines l and j where the arrival times of trips that belong to line l need to be synchronized with the arrival times of trips that belong to line j.

Parameters

f_l	Number of trips for each line $l \in L$ which are needed to satisfy the demand within the planning period (note: the number of trips is already determined by the frequency settings stage).
T	The planning period (note: the suggested planning period is at most one day of operations).
$h_l^* = \frac{T}{f_l}$	The ideal headway of bus line $l \in L$ that should be maintained at all bus stops for attaining a perfectly regular service (sec).
$t_{l,n,s}$	The *expected* travel time of bus trip n of line l between stops $s-1$ and s (sec).
δ_l^{min}	The dispatching time of the first trip within the planning period (sec).

δ_l^{max} The latest possible time where all trips of line $l \in L$ must have completed their service to prevent schedule sliding (sec).

$k_{l,n,s}$ The *expected* dwell time of bus trip n of bus line l at stop s (sec).

ψ_l The required layover time for line l after completing each bus trip (sec).

h_l^{min}, h_l^{max} Lower and upper bounds of the dispatching time headway between two subsequent trips of line l for guaranteeing a certain level of service (sec).

Design Variables

$x_{l,n}$ The dispatching time of the n^{th} trip that belongs to line l (sec).

Environmental Variables (Adversaries)

$\xi_{l,n,s}$ Travel time "noise" between stops $s-1$ and s for trip n of line l (in sec). $\xi_{l,n,s} \in [\xi_{l,s}^{min}, \xi_{l,s}^{max}]$ and can take any value within the range $[\xi_{l,s}^{min}, \xi_{l,s}^{max}]$.

$\zeta_{l,n,s}$ The dwell time "noise" at stop s for trip n of line l (in sec). $\zeta_{l,n,s}$ can take any value within the range $[\zeta_{l,s}^{min}, \zeta_{l,s}^{max}]$.

In contrast to stochastic optimization approaches, we do not make any assumptions with respect to the probability distribution of the environmental variables $\xi_{l,n,s}$ and $\zeta_{l,n,s}$. Instead, we allow them to take any value within the uncertainty sets $[\xi_{l,s}^{min}, \xi_{l,s}^{max}]$ and $[\zeta_{l,s}^{min}, \zeta_{l,s}^{max}]$, respectively.

Following the above notation, we denote by $a_{l,n,s} = a_{l,n,s}(x, \xi, \zeta)$ the arrival time of any trip $n \in N_l$ that belongs to a bus line $l \in L$ at stop $s \in \widehat{S}_l = S_l \setminus \{1\}$. Formally,

$$a_{l,n,s}(x,\xi,\zeta) := x_{l,n} + \sum_{z=2}^{s} \left(t_{l,n,z} + \xi_{l,n,z}\right) + \sum_{z=2}^{s-1} \left(k_{l,n,z} + \zeta_{l,n,z}\right), \quad (1)$$

where $\xi_{l,n,z}$ is the travel time deviation from the expected travel time value $t_{l,n,z}$ for the road section defined by bus stops $z-1$ and z. $\zeta_{l,n,z}$ is the dwell time deviation from the expected dwell time $k_{l,n,z}$ at stop z. In Eq.(1), the arrival time of a trip n at stop s is set equal to the dispatching time of the trip, $x_{l,n}$, plus the sum of the travel time realizations until reaching stop s, $\sum_{z=2}^{s} \left(t_{l,n,z} + \xi_{l,n,z}\right)$, plus the dwell time realizations until reaching stop $s-1$, $\sum_{z=2}^{s-1} \left(k_{l,n,z} + \zeta_{l,n,z}\right)$. From Eq.(1) one should note that the arrival times of buses at stops vary based on the dispatching times of the trips and the travel time/dwell time noise.

A. Formulating the Objectives of the Network-Wide Synchronization Problem

To increase the regularity of bus services, the actual time headways at bus stops should be as close as possible to their scheduled values. The ideal headway $h_l^* = \frac{T}{f_l}$ of a bus line $l \in L$ is already defined at the frequency settings stage. In addition, the time headway $h_{l,n,s} = h_{l,n,s}(x, \xi, \zeta)$ between two consecutive services $n-1, n \in \widehat{N}_l = N_l \setminus \{1\}$ of line $l \in L$ at stop $s \in \widehat{S}_l$ is

$$h_{l,n,s}(x,\xi,\zeta) := a_{l,n,s} - a_{l,n-1,s} \quad (2)$$

The difference $(h_{l,n,s} - h_l^*)$ between the actual headways and the ideal headways at stops is the sole key performance indicator of regularity-based services and has been in use in London, Singapore, Barcelona and many other densely populated areas where the bus services operate in high frequencies [40], [41]. The main reason of its use in high-frequency services is that it indicates the excessive waiting times (EWTs) of passengers at stops. EWTs indicate the difference between the actual waiting times and the scheduled ones. Note that in high-frequency services, the waiting time of a passenger of trip n at stop s is half the headway between trip n and trip $n-1$, $\frac{h_{l,n,s}}{2}$, because passenger arrivals at stops can be considered as uniformly distributed (see [41], [42]).

To reduce the deviation between the actual waiting times of passengers at stops and the ideal ones for a bus line $l \in L$, we introduce $f_l(x, \xi, \zeta)$ which is the aggregated difference between the actual and the ideal half-headways:

$$f_l(x,\xi,\zeta) := \sum_{s \in \widehat{S}_l} \sum_{n \in \widehat{N}_l} \left(\frac{h_{l,n,s}}{2} - \frac{h_l^*}{2}\right)^2 \quad (3)$$

where $\left(\frac{h_{l,n,s}}{2} - \frac{h_l^*}{2}\right)$ is squared for over-penalizing extreme discrepancies from the ideal headway values. The squared value $\left(\frac{h_{l,n,s}}{2} - \frac{h_l^*}{2}\right)^2$ is commonly used in both past literature [43]–[45] and in practice [46] to monitor the service regularity with the use of the EWT indicator. Namely, the EWT indicator uses the squared difference between the actual and the ideal headways to penalize progressively the headway deviations from the ideal case (see [46]).

Remark 1: If some bus lines are considered more important than others, the *network-wide* regularity can be indicated by the weighted sum of the daily excessive waiting times for all bus lines:

$$f(x,\xi,\zeta) := \sum_{l \in L} \frac{w_l}{4} f_l(x,\xi,\zeta), \quad (4)$$

where w_l are weight factors that assign greater importance to the regularity of some bus lines in the expense of others. Note that $w_l \geq 0, \forall l \in L$, and $\sum_{l \in L} w_l = 1$. In addition, $f(x, \xi, \zeta)$ is the daily, network-wide excessive waiting time of passengers that is indicative of the service regularity.

Remark 2: In practice, optimizing function $f(x, \xi, \zeta)$ is a tedious task. It depends on the realized noise-pair (ξ, ζ), which is typically not known a-priori.

Now, let us consider the waiting times of passengers at transfer stops. Reckon that $B_{lj} \subseteq \left(S_l \cap S_j\right) \times N_l \times N_j$ is the set with all transfer stops between lines $l \in L$ and $j \in L$ where the arrival times of (some) trips that belong to line l need to be synchronized with the arrival times of a subset of the trips that belong to line j.

The set B_{lj} can be specified based on common stops which allow for interchanging. However, in large networks this may become prohibitive. Alternatively, the lines and locations can be determined using the clustering method proposed in [47] for prioritizing service synchronization. The seminal work of [25] denominated a perfect synchronization when trip $n \in N_l$ arrives at the transfer stop $b \in S_l \cap S_j$ exactly at the same time as trip $m \in N_j$, i.e., $(b, n, m) \in B_{lj}$. In their mathematical program, their objective is the maximization of the number of (perfect) synchronizations. Following the definition of [25], in order to ensure that all required transfers are synchronized, the arrival times $a_{l,n,b}$ and $a_{j,m,b}$ of each trip pair (n, m) at transfer stops b so that $(b, n, m) \in B_{lj}$ should be identical:

$$a_{l,n,b} - a_{j,m,b} = 0, \quad \forall (b, n, m) \in B_{lj} \tag{5}$$

If the constraints of Eq.(5) are met, all required transfers are (perfectly) synchronized.

B. Regulatory Constraints

1) Minimum Layover Times: This study considers layover constraints. The layover time of a bus that finishes a bus trip is the minimum required time before starting its next trip. Typically, this layover time is explicitly mentioned in the labor union contracts.

We introduce set $C_l \subseteq N_l \times N_l$, $l \in L$ of buses that are operated in sequence. The *minimum* required layover time for bus line $l \in L$ is ψ_l and it consists of: the required deadhead time for traveling from the last to the first stop, the resting time of the bus driver and the time needed for passenger boardings at the first stop. Considering $(n, n') \in C_l$, $l \in L$, the dispatching time, $x_{l,n'}$, of trip n' should satisfy the inequality:

$$\left(x_{l,n'} - \omega_{l,n}(x, \xi, \zeta)\right) \geq \psi_l, \quad \forall (n, n') \in C_l, \ \forall l \in L \tag{6}$$

where

$$\omega_{l,n}(x, \xi, \zeta) := x_{l,n} + \sum_{s \in \widehat{S}_l} (t_{l,n,s} + \xi_{l,n,s}) + \sum_{s \in \widehat{S}_l} (k_{l,n,s} + \zeta_{l,n,s}) \tag{7}$$

is the time when trip n of line l has been completed and all its passengers have disembarked.

2) Minimum and Maximum Dispatching Headways: To guarantee a certain level of service, the dispatching headways of subsequent trips of any line $l \in L$ should be within a predetermined range $[h_l^{min}, h_l^{max}]$ with $h_l^{max} \geq h_l^{min} \geq 0$. These dispatching headway bounds are determined at the frequency settings stage that precedes our problem [48] and impose the inequality constraints:

$$h_l^{min} \leq x_{l,n} - x_{l,n-1} \leq h_l^{max}, \quad \forall n \in \hat{N}_l, \ \forall l \in L \tag{8}$$

3) Schedule Sliding: Finally, to prevent schedule sliding and maintain the duration of the planned operations, all trips of a bus line $l \in L$ must have been completed before time $\delta_l^{max} \in \mathbb{R}_{\geq 0}$. The schedule sliding constraint ensures that the operations of the examined planning period are not prolonged because this will have adverse effects on future operations and increase the working hours of bus drivers beyond the labor union contractual agreements. Avoiding schedule sliding yields the following inequality constraints:

$$\omega_{l,n} \leq \delta_l^{max}, \quad \forall n \in N_l, \ \forall l \in L \tag{9}$$

with $\omega_{l,n} = \omega_{l,n}(x, \xi, \zeta)$, which ensures that all trips $n \in N_l$ of line l have arrived at the last stop and have completed all passenger alightings before time δ_l^{max}.

III. Mathematical Program of the Robust, Network-Wide Synchronization Problem

The proposed network-wide synchronization problem that explicitly considers uncertain travel and dwell times is formulated as a *robust* optimization problem (see, e.g., [49]). The mathematical program can be succinctly written as:

$$\begin{aligned} Q: \ & \min_{x} \ \max_{\xi, \zeta} \ f(x, \xi, \zeta) \\ & \text{s.t.:} \ \ x \in \mathcal{F}(\xi, \zeta) \\ & \qquad = \{x \mid (x, h, a) \text{ satisfies Eqs.(1)-(2), (5)-(9)}\} \\ & \qquad x_{l,1} = \delta_l^{min} \quad l \in L \\ & \qquad \xi_{l,s}^{min} \leq \xi_{l,n,s} \leq \xi_{l,s}^{max} \quad n \in N_l, \ s \in \widehat{S}_l l \in L \\ & \qquad \zeta_{l,s}^{min} \leq \zeta_{l,n,s} \leq \zeta_{l,s}^{max} \quad n \in N_l, \ s \in \widehat{S}_l, \ l \in L \end{aligned} \tag{10}$$

Program Q is a min(i)max optimization problem and ranks the *designs* (in our case, the different dispatching time solutions) based on their worst-case outcomes. The robust dispatching times x (i.e., x that solves Q) perform best at worst-case travel time and dwell time noises, (ξ, ζ). We note that in Q the environmental variables (ξ, ζ) play the role of the *adversary* of a design x.

A. Solution Existence and Reformulation

The optimization problem Q is difficult to solve numerically. Intuitively, the feasible set $\mathcal{F}(\xi, \zeta)$ depends on the choice of the noise parameters, (ξ, ζ), while the choice of the noise depends on the choice of x. In this section, we formulate a *relaxed* problem of Q for ensuring feasibility. Thereby, we analyze program Q in more detail.

As shown in Theorem 1, for a given noise pair (ξ^0, ζ^0), the objective function $f(x, \xi^0, \zeta^0)$ is continuous, quadratic and convex (with respect to x). Therefore, the parametric optimization problem (with parameters (ξ^0, ζ^0))

$$P(\xi^0, \zeta^0): \quad \min_{x} f(x, \xi^0, \zeta^0) \quad \text{s.t.} \quad \begin{array}{l} x \in \mathcal{F}(\xi^0, \zeta^0) \\ x_{l,1} = \delta_l^{min}, l \in L \end{array} \tag{11}$$

can be easily solved to global optimality if the corresponding feasible set $\mathcal{F}(\xi^0, \zeta^0)$ is compact and non-empty.

Theorem 1: Given (ξ^0, ζ^0), $P(\xi^0, \zeta^0)$ is a convex optimization problem, which has a unique global minimizer (if any) with respect to x.

Proof: Note that the feasible set $\mathcal{F}(\xi^0, \zeta^0)$ is defined by linear (in)equalities. Hence, it is a closed polyhedron (and thus a convex set). We prove that $f(x, \xi^0, \zeta^0)$ is convex with respect to x. Note that $g_{l,n,s}(h) := (h_{l,n,s} - h_l^*)^2$

is a strictly convex function with respect to $h_{l,n,s}$. Indeed, $\frac{\partial^2 g_{l,n,s}}{\partial h_{l,n,s}^2} > 0$. We define matrix A and (noise-dependent) vector b so that for any x, $Ax + b = h$. We need to prove that $\tilde{g}_{l,n,s}(x) = g_{l,n,s}(Ax + b)$ is a convex function with respect to x. Now, let x^0, x^1 be arbitrary, and $\lambda \in [0, 1]$. Then, $\tilde{g}_{l,n,s}(\lambda x^0 + (1-\lambda)x^1) = g_{l,n,s}(A(\lambda x^0 + (1-\lambda x^1) + b) = g_{l,n,s}(\lambda h^0 + (1-\lambda)h^1) \leq \lambda g_{l,n,s}(h^0) + (1-\lambda) g_{l,n,s}(h^1) = \lambda \tilde{g}_{l,n,s}(x^0) + (1-\lambda)\tilde{g}_{l,n,s}(x^1)$. We note that $x^0 \neq x^1$ does not imply $Ax^0 + b \neq Ax^1 + b$. Since $f(x, \xi^0, \zeta^0) = \sum_{l,n,s} \frac{\omega_l}{4} \tilde{g}_{l,n,s}(x)$, we proved that $f(x, \xi^0, \zeta^0)$ is a convex function with respect to x. □

From the above theorem we establish that $P(\xi^0, \zeta^0)$ can be easily solved to global optimality if the corresponding feasible set is non-empty. Note though that we cannot expect that feasible set $\mathcal{F}(\xi^0, \zeta^0)$ is non-empty for any (ξ^0, ζ^0). We make the following observations:

- The equality constraints of Eqs.(1)-(2), (7) can be always satisfied because they just set the values of functions $a_{l,n,s}$, $h_{l,n,s}$ and $\omega_{l,n} = \omega_{l,n}(x, \zeta, \xi)$ which are unbounded in $\mathbb{R}_{\geq 0}$ and can receive any value dictated by x, ξ, ζ.
- The constraints of Eq.(8) are independent of the noise ξ^0, ζ^0 ensuring that $\exists\ x^*$ for which they are satisfied.
- $\exists\ x^*$ that satisfies the physical (hard) layover constraints of Eq.(6) because $x_{l,n'}$ is not bounded from above from Eqs.(1)-(2), (7)-(8).
- A solution that avoids missed synchronizations or schedule sliding (i.e., satisfies Eq.(5), (9)) might not exist for some travel time and dwell time noise instances.

To support our last observation, we provide a condition under which the schedule sliding constraints of Eq.(9) cannot be satisfied.

Lemma 1: For some noise (ξ^0, ζ^0) *so that* $\sum_{s \in \widehat{S}_l} (t_{l,n,z} + \xi^0_{l,n,z}) + \sum_{s \in \widehat{S}_l} (k_{l,n,z} + \zeta^0_{l,n,z}) > \delta_l^{max} - \delta_l^{min}$, *the inequalities of Eq.*(9) *cannot be satisfied.*

Proof: Trip n must have been completed before δ_l^{max} for ensuring that the daily operations do not result in schedule sliding. Let $\beta_0(\xi, \zeta) = \sum_{s \in \widehat{S}_l} (t_{l,n,z} + \xi_{l,n,z}) + \sum_{s \in \widehat{S}_l} (k_{l,n,z} + \zeta_{l,n,z})$. Hence, $x_{l,n} + \beta_0(\xi, \zeta) \leq \delta_l^{max}$ should hold for any (ξ, ζ) in order to satisfy Eq.(9). However, $x_{l,n}$ has a lower bound of δ_l^{min} and $x_{l,n} + \beta_0(\xi, \zeta) \leq \delta_l^{max}$ dictates that $\beta_0(\xi, \zeta)$ should always be less than $\delta_l^{max} - \delta_l^{min}$ in order to satisfy Eq.(9). Therefore, for some ξ^0, ζ^0 so that $\beta_0(\xi, \zeta) > \delta_l^{max} - \delta_l^{min}$, $\nexists\ x$ such that the constraints of Eq.(9) are satisfied. □

We therefore introduce a pragmatic approach to handle a (possible) empty feasible set. We relax the schedule sliding, synchronization, and layover constraints by introducing *penalty terms* to the objective function that add penalties when (at least one) of the respective constraints is violated.

First, we relax the schedule sliding constraints. We introduce the functions $\varphi_{l,n}$, $l \in L$, $n \in N_l$, defined as:

$$\varphi_{l,n}(x, \xi, \zeta) := c_\varphi \cdot \max\{0, (\omega_{l,n} - \delta_l^{\max})\}^2, \tag{12}$$

where $c_\varphi \gg 0$ is a non-negative constant with a *sufficiently high* value for ensuring that the satisfaction of schedule sliding constraints is prioritized. This sufficiently high value of c_φ is determined in practice by starting with a small value, minimizing the penalized objective function with this small value and then increasing this value incrementally until reaching solution stability.

For any fixed noise (ξ^0, ζ^0), a penalty function $\varphi_{l,n}(x, \xi^0, \zeta^0)$ penalizes any dispatching time $x_{l,n}$ for which Eq.(9) is violated. $\varphi_{l,n}(x, \xi^0, \zeta^0)$ is a convex function with respect to x. The squared value of $(\omega_{l,n} - \delta_l^{max})^2$ ensures that trips which are significantly prolonged beyond the time limit δ_l^{max} are penalized more severely than others which are close to δ_l^{max} (a widespread strategy in exterior point penalty methods [50]).

We propose to *relax* also the transfer synchronization constraints in Eq.(5). Similarly to Eq.(12), we introduce for any $(b, n, m) \in B_{lj}, l, j \in L$:

$$\mu_{lj}^{bnm}(x, \xi, \zeta) := c_\mu \big(a_{j,m,b} - a_{l,n,b}\big)^2 \tag{13}$$

to penalize violated synchronization constraints. Here, $c_\mu \gg 0$ is a *sufficiently* high value. $\mu_{lj}^{bnm}(x, \xi, \zeta)$ increases the value of the penalized objective function every time a synchronization is missed (i.e., the arrival times of trips that should be synchronized are not equal). In addition, for any given noise instance, (ξ^0, ζ^0), $\mu_{lj}^{bnm}(x, \xi^0, \zeta^0)$ is a convex function with respect to x.

Similar to previous penalty functions, we penalize violated layover times, i.e., for any $(n, n') \in C_l$, with $l \in L$,

$$\kappa_{l,n,n'}(x, \xi, \zeta) := c_\kappa (\max\{0, \omega_{l,n} + \psi_l - x_{l,n'}\})^2,$$

and $c_\kappa \gg 0$. Note that the layover constraints are "hard" constraints (i.e., if a bus has not completed its previous trip, it cannot start its next one). Therefore, they should be prioritized over the transfer synchronization and schedule sliding constraints which are "soft" constraints and can be violated (i.e., if necessary, a synchronization can be missed). To ensure this prioritization, weight factor c_κ is typically given a sufficiently higher value than weight factors c_ϕ and c_μ.

It is worth noting that $\phi_{l,n}$, $\mu_{l,j}^{bnm}$, and $\kappa_{l,n,n'}$ are all mappings from (x, ξ, ζ) onto $\mathbb{R}_{\geq 0}$. Consequently, the sum of all penalty functions is non-negative.

The penalized objective function now becomes:

$$\begin{aligned} \tilde{f}(x, \xi, \zeta) := f(x, \xi, \zeta) &+ \sum_{l \in L} \sum_{n \in N_l} \varphi_{l,n}(x, \xi, \zeta) \\ &+ \sum_{l \in L} \sum_{j \in L} \sum_{(b,n,m) \in B_{lj}} \mu_{lj}^{bnm}(x, \xi, \zeta) \\ &+ \sum_{l \in L} \sum_{(n,n') \in C_l} \kappa_{l,n,n'}(x, \xi, \zeta) \end{aligned} \tag{14}$$

which maintains to be a convex function (with respect to x) for any given noise instance (ξ^0, ζ^0), with $\tilde{f}(x, \xi^0, \zeta^0) \geq f(x, \xi^0, \zeta^0)$ for all x given that the sum of convex functions is a convex function. The robust optimization program Q is reformulated to the relaxed program $\tilde{Q}$ that includes the

penalized objective function $\tilde{f}(x,\xi,\zeta)$:

$$\begin{aligned}\tilde{Q}: \ &\min_{x} \ \max_{\xi,\zeta} \ \tilde{f}(x,\xi,\zeta)\\ &\text{s.t.: } x \in \tilde{\mathcal{F}}(\xi,\zeta)\\ &= \{x \mid (x,h,a) \text{ satisfies Eqs.(1)-(2), (7)-(8)}\}\\ &\quad x_{l,1} = \delta_l^{min}, \quad l \in L\\ &\quad \xi_{l,s}^{min} \le \xi_{l,n,s} \le \xi_{l,s}^{max}, \quad n \in N_l, \ s \in \widehat{S}_l, \ l \in L\\ &\quad \zeta_{l,s}^{min} \le \zeta_{l,n,s} \le \zeta_{l,s}^{max}, \quad n \in N_l, \ s \in \widehat{S}_l, \ l \in L \quad (15)\end{aligned}$$

Note that the corresponding feasible set $\tilde{\mathcal{F}}(\xi,\zeta)$ does not include the inequality constraints of Eqs.(5)-(6) and (9) and $\tilde{\mathcal{F}}(\xi,\zeta) \neq \emptyset$ for all (ξ,ζ). Note also that the feasible set that corresponds to $\tilde{Q}$ is compact.

From a mathematical perspective, we have relaxed program Q into an easier-to-study problem $\tilde{Q}$. For any given noise instance (ξ^0,ζ^0), we can find the optimal dispatching time x by solving

$$\tilde{P}(\xi^0,\zeta^0): \quad \min_{x} \tilde{f}(x,\xi^0,\zeta^0) \quad \begin{matrix} x \in \tilde{\mathcal{F}}(\xi^0,\zeta^0) \\ x_{l,1} = \delta_l^{min}, \ l \in L \end{matrix} \quad (16)$$

in which (ξ^0,ζ^0) are parameters. $\tilde{P}(\xi^0,\zeta^0)$ can be solved to global optimality since it is a convex optimization problem.

IV. Solution Method

In some problems, the worst values of (ξ,ζ) are easy to guess based on prior problem knowledge and the minimax problem is reduced to a classical minimization one. In our case though, the worst-case values of the environmental variables (ξ,ζ) depend on the settings of the design variables x in a way that is not intuitively obvious.

To solve our minimax problem, one can employ evolutionary algorithms [51], [52]. However, they do not guarantee convergence and do not exploit the convexity of $\tilde{P}(\xi^0,\zeta^0)$ because they treat the objective function as a black box. Other brute-force methods for solving the minimax problem can be employed when the design and environmental variables can take values in the *discrete* space resulting in general or zero-sum games [53] where the minimax solution is the same as the Nash equilibrium.

Notwithstanding, the fact that our minimax problem is solved in the continuous space and the worst-case values of the environmental variables (ξ,ζ) depend on the settings of the design variables x requires other strategies. One prominent strategy is the minimax approximation strategy that relaxes the original problem by introducing and updating a small discrete set of points in the continuous space of the environmental variables [54]. For a discussion with respect to the optimality conditions of the minimax problem, we refer to [54]–[56].

A. Relaxation for the Minimax Optimization

The minimax problem $\tilde{Q}$ searches for the dispatching time x that minimizes the worst-case performance $\max_{\xi,\zeta} \tilde{f}(x,\xi,\zeta)$. This problem is relaxed by performing the maximization over a finite set $\mathcal{R}_e$ instead of all possible $(\xi,\zeta) \in X_e = ([\xi_{l,n,s}^{min}, \xi_{l,n,s}^{max}] \times [\zeta_{l,n,s}^{min}, \zeta_{l,n,s}^{max}])_{l,n,s}$.

For any *discretization* $\mathcal{R}_e \subset X_e$, we introduce the optimization problem

$$\begin{aligned}\tilde{Q}(\mathcal{R}_e): \ &\min_{x} \ \max_{\xi,\zeta} \ \tilde{f}(x,\xi,\zeta)\\ &\text{s.t.: } x \in \tilde{\mathcal{F}}(\xi,\zeta)\\ &= \{x \mid x \text{ satisfies Eqs.(1)-(2), (7)-(8)}\}\\ &\quad x_{l,1} = \delta_l^{min}, \quad l \in L\\ &\quad (\xi_{l,n,s}, \quad \zeta_{l,n,s}) \in \mathcal{R}_e \quad (17)\end{aligned}$$

Given $\mathcal{R}_e$, program $\tilde{Q}(\mathcal{R}_e)$ has favorable mathematical properties compared to $\tilde{Q}$.

To solve this numerically, [57] proposed to start with an $\mathcal{R}_e$ of just one randomly chosen point $(\xi^0,\zeta^0) \in X_e$. Then, $x^0 \triangleq \{\text{Solves } \tilde{Q}(\mathcal{R}_e) \text{ for } \mathcal{R}_e = \{\xi^0,\zeta^0\}\}$ is the best set solution in the continuous space of design variables. Given x^0, the next step searches for $(\zeta^1,\xi^1) \in X_e$ that disturbs the overall performance as much as possible, i.e., we solve

$$T(x^0): \quad \max_{\xi,\zeta} \tilde{f}(x^0,\xi,\zeta) \quad \text{s.t.} \quad \begin{matrix} \xi_{l,s}^{min} \le \xi_{l,n,s} \le \xi_{l,s}^{max} \\ \zeta_{l,s}^{min} \le \zeta_{l,n,s} \le \zeta_{l,s}^{max} \\ \text{Eqs.(1)-(2), (7)} \end{matrix} \quad (18)$$

If the maximum possible disturbance (ξ^1,ζ^1) does not worsen the performance too much, that is, $\tilde{f}(x^0,\xi^1,\zeta^1) - \tilde{f}(x^0,\xi^0,\zeta^0) < \epsilon$ for some threshold $\epsilon \in \mathbb{R}_{\ge 0}$, then x^0 is an *acceptable approximation* of the minimax problem $\tilde{Q}$ and the search terminates. If not, the point (ξ^1,ζ^1) is added to the set $\mathcal{R}_e$ and the procedure is repeated (alg.1) [54].

Algorithm 1 Minimax Approximation via Relaxation of the Environmental Variables

0: Set $\epsilon \in \mathbb{R}_{\ge 0}$;
1: Choose randomly $\xi^0,\zeta^0 \in [\xi_{l,s}^{min}, \xi_{l,s}^{max}], [\zeta_{l,s}^{min}, \zeta_{l,s}^{max}]$ and set $\mathcal{R}_e \leftarrow (\xi^0,\zeta^0)$, set $k = 0$.
2: Solve $\tilde{Q}(\mathcal{R}_e)$ and obtain x^k;
3: Solve $T(x^k)$ and obtain (ξ^{k+1},ζ^{k+1});
4: If $\tilde{f}(x^k,\xi^{k+1},\zeta^{k+1}) - \tilde{f}(x^k,\xi^k,\zeta^k) < \epsilon$, STOP. Else, extend $\mathcal{R}_e \leftarrow \mathcal{R}_e \cup \{\xi^k,\zeta^k\}$, $k \leftarrow k+1$ and go to Step 2.

The proof that this algorithm satisfies the necessary optimality conditions of a locally optimal minimax solution is provided in the Appendix.

V. Numerical Experiments

A. Demonstration Using an Idealized Network

Fig.1 shows the idealized network under consideration. Even though the demonstration includes a small network with two bus lines, the analysis can be expanded to a full-scale city network without loss of generality.

The transfer stop of bus lines l, j in our idealized network is $B_{lj} = \{2\}$. Bus lines l, j involve two trips each, $N_l = \{1,2\}$ and $N_j = \{1,2\}$. The first trip of bus line l should be dispatched at $\delta_l^{min} =$ 8:00 am (or 28,800 sec from the beginning of the day) and the first trip of bus line j at $\delta_j^{min} =$

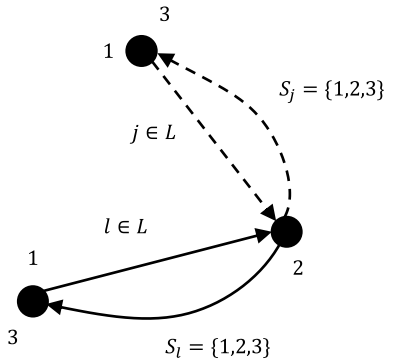

Fig. 1. Idealized bus network with two bus lines $l, j \in L$.

TABLE I
EXPECTED TRAVEL TIME AND DWELL TIME VALUES FOR THE IDEALIZED BUS NETWORK IN SECONDS

bus line l				bus line j			
Trip	Stop	$t_{l,n,s}$	$k_{l,n,s}$	Trip	Stop	$t_{j,m,z}$	$k_{j,m,z}$
n=1	s=2	450	20		z=2	610	38
	s=3	445	22		z=3	480	33
n=2	s=2	450	24		z=2	590	46
	s=3	460	25		z=3	490	32

8:02 am (or 28,920 sec). In the idealized scenario, each trip is operated by a different bus.

Each trip of bus line l needs to synchronize its arrival time at stops $b \in B_{lj}$ with the arrival time of the corresponding trip of line j (2 synchronizations in total). The ideal time headways between successive bus trips at bus stops are $h_l^* = 460$ sec for line l and $h_j^* = 600$ sec for line j. In addition, to prevent schedule sliding, all trips of bus lines l and j should have been completed before 10:00 am, thus $\delta_l^{max} = \delta_j^{max} = 36,000$ sec. The expected inter-station travel times and dwell times at stops are presented in Table I.

In this scenario, the disturbances of the environmental variables can take values from the sets $[\xi_{l,s}^{min}, \xi_{l,s}^{max}] = [-60 \text{ sec}, +60 \text{ sec}]\ \forall l \in L, \forall s \in \widehat{S}_l$ and $[\zeta_{l,s}^{min}, \zeta_{l,s}^{max}] = [-10 \text{ sec}, +20 \text{ sec}],\ \forall l \in L, \forall s \in \widehat{S}_l$. In addition, the minimum and maximum allowed dispatching headways for ensuring a minimum level of service are $h_l^{min} = 120$ sec and $h_l^{max} = 720$ sec, $\forall l \in L$.

To find a robust design, we apply Alg.1 with $\epsilon = 0.05$. We initialize our set $\mathcal{R}_e$ by selecting a random noise $(\xi^0, \zeta^0) \in X_e$ and setting $\mathcal{R}_e \leftarrow (\xi^0, \zeta^0)$. Let

$$\xi^0 = \begin{cases} (\xi_{l,1,2} = 60, \xi_{l,1,3} = 60, \xi_{l,2,2} = 60, \xi_{l,2,3} = 60) \\ (\xi_{j,1,2} = -60, \xi_{j,1,3} = -60, \xi_{j,2,2} \\ \qquad = -60, \xi_{j,2,3} = -60) \end{cases}$$

and

$$\zeta^0 = \begin{cases} (\zeta_{l,1,2} = 20, \zeta_{l,1,3} = 20, \zeta_{l,2,2} = 20, \zeta_{l,2,3} = 20) \\ (\zeta_{j,1,2} = -20, \zeta_{j,1,3} = -20, \zeta_{j,2,2} \\ \qquad = -20, \zeta_{j,2,3} = -20) \end{cases}$$

where all values are expressed in seconds.

The solution of $\tilde{Q}(\mathcal{R}_e)$ can be easily obtained by solving program $\tilde{P}(\xi^0, \zeta^0)$. That is, $x^0 \triangleq \{\text{Solves } \tilde{P}(\xi^0, \zeta^0)\}$. To solve the nonlinear $\tilde{P}(\xi^0, \zeta^0)$, we employ sequential quadratic programming (SQP) [58] in Python 3.6 using SciPy. SQP finds a local minimizer of the continuous nonlinear constrained optimization problem $\tilde{P}(\xi^0, \zeta^0)$ which is a globally optimal solution given the convexity of $\tilde{f}$ for any given noise (ξ^0, ζ^0). The resulting solution is:

$$x^0 = \begin{cases} (x_{l,1} = 28800, \quad x_{l,2} \simeq 29400) \text{ in sec} \\ (x_{j,1} = 28920, \quad x_{j,2} \simeq 29380) \text{ in sec} \end{cases}$$

with $\tilde{f}(x^0, \xi^0, \zeta^0) \simeq 2.56\text{E+}8$.

To obtain the worst-case noise (ξ^1, ζ^1) for x^0, we solve the maximization problem $T(x^0)$. This yields

$$\xi^1 = \begin{cases} (\xi_{l,1,2} = -60, \xi_{l,1,3} = -60, \xi_{l,2,2} = -60, \xi_{l,2,3} = 60) \\ (\xi_{j,1,2} = 60, \xi_{j,1,3} = 60, \xi_{j,2,2} = 60, \xi_{j,2,3} = -60) \end{cases}$$

and

$$\zeta^1 = \begin{cases} (\zeta_{l,1,2} = 20, \zeta_{l,1,3} = -10, \zeta_{l,2,2} = 20, \zeta_{l,2,3} = 20) \\ (\zeta_{j,1,2} = -10, \zeta_{j,1,3} = 20, \zeta_{j,2,2} \\ \qquad = -10, \zeta_{j,2,3} = -10) \end{cases}$$

with $\tilde{f}(x^0, \xi^1, \zeta^1) \simeq 2.17\text{E+}9$. Given that $\tilde{f}(x^0, \xi^1, \zeta^1) - \tilde{f}(x^0, \xi^0, \zeta^0) \not< \epsilon$, we add (ξ^1, ζ^1) to $\mathcal{R}_e$ and solve the updated $\tilde{Q}(\mathcal{R}_e)$. The updated $\tilde{Q}(\mathcal{R}_e)$ is solved by solving $\tilde{P}(\xi^k, \zeta^k)$ for all $(\xi^k, \zeta^k) \in \mathcal{R}_e$ and return x^k that minimizes the worst-case performance for the environmental variables in $\mathcal{R}_e$. For (ξ^1, ζ^1), the solution of $\tilde{P}(\xi^1, \zeta^1)$ is:

$$x^1 = \begin{cases} (x_{l,1} = 28800, \quad x_{l,2} \simeq 29519) \text{ in sec} \\ (x_{j,1} = 28920, \quad x_{j,2} \simeq 29259) \text{ in sec} \end{cases}$$

and the performance of designs x^0, x^1 for the environmental variables $[(\xi^0, \zeta^0), (\xi^1, \zeta^1)] \in \mathcal{R}_e$ is presented in Table II.

TABLE II
$\tilde{f}(x, \xi, \zeta)$

	(ξ^0, ζ^0)	(ξ^1, ζ^1)
x^0	2.56E+8	2.17E+9
x^1	8.22E+8	1.60E+9

From Table II, the solution of $\tilde{Q}(\mathcal{R}_e)$ with the lowest worst-case performance for the environmental variables in $\mathcal{R}_e$ is $x^* = x^1$. The corresponding performance is 1.60E+9.

In the next iteration, we obtain the worst-case noise (ξ^2, ζ^2) for x^1 by solving $T(x^1)$. This yields

$$\xi^2 = \begin{cases} (\xi_{l,1,2} = -60, \xi_{l,1,3} = -60, \xi_{l,2,2} = 60, \xi_{l,2,3} = 60) \\ (\xi_{j,1,2} = 60, \xi_{j,1,3} \\ \qquad = 60, \xi_{j,2,2} = -60, \xi_{j,2,3} = -60) \end{cases}$$

and

$$\zeta^2 = \begin{cases} (\zeta_{l,1,2} = 20, \zeta_{l,1,3} = -10, \zeta_{l,2,2} = 20, \zeta_{l,2,3} = 20) \\ (\zeta_{j,1,2} = -10, \zeta_{j,1,3} = 20, \zeta_{j,2,2} \\ \qquad = -10, \zeta_{j,2,3} = -10) \end{cases}$$

with $\tilde{f}(x^1, \xi^2, \zeta^2) \simeq 2.18\text{E+}9$. Given that $\tilde{f}(x^1, \xi^2, \zeta^2) - \tilde{f}(x^1, \xi^1, \zeta^1) \not< \epsilon$, we add (ξ^2, ζ^2) to $\mathcal{R}_e$. The updated $\tilde{Q}(\mathcal{R}_e)$

TABLE III
$\tilde{f}(x, \xi, \zeta)$

	(ξ^0, ζ^0)	(ξ^1, ζ^1)	(ξ^2, ζ^2)
x^0	2.56E+8	2.17E+9	1.60E+9
x^1	8.22E+8	1.60E+9	2.18E+9
x^2	4.00E+8	1.67E+9	1.74E+9

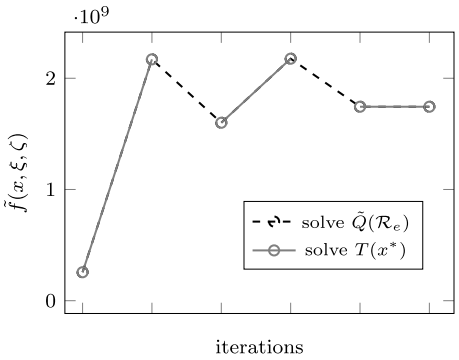

Fig. 2. Implementation of Alg.1. Solve $\tilde{Q}(\mathcal{R}_e)$ indicates the worst-case performance at $\mathcal{R}_e$ and solve $T(x^k)$ the worst-case performance at X_e.

is solved which returns solution:

$$x^2 = \begin{cases} (x_{l,2} = 28800, \quad x_{l,2} \simeq 29520) \text{ in sec} \\ (x_{j,2} = 28920, \quad x_{j,2} \simeq 29380) \text{ in sec} \end{cases}$$

We observe no change in the worst-case scenario after we solve $T(x^2)$, i.e,. $(\xi^3, \zeta^3) = (\xi^2, \zeta^2)$, and the algorithm is terminated. The performances of the designs x^0, x^1, x^2, for all $(\xi^k, \zeta^k) \in \mathcal{R}_e$ are presented in Table III.

Fig.2 summarizes the worst-case performance of the respective incumbent solution at $\mathcal{R}_e$ and X_e at each iteration.

B. Investigating the Performance of Robust Designs for the Bus Network of the Hague

In this application, we solve the robust synchronization problem for the bus network of The Hague, the Netherlands. To devise the bounds of our travel and dwell time adversary, we use Automated Vehicle Location (AVL) and Automated Passenger Count (APC) data from 1 month (March 2015). As illustrated in Fig.3, the network of The Hague consists of 8 bi-directional urban bus lines, yielding $|L| = 16$.

In this case study, we consider the planning period of this experiment from 7:00am to 8:00am, with each bus line operating with a frequency of 6 departures per hour. The stops are illustrated in Fig.3 including the two major interchange hubs, namely at The Hague Central Station, and The Hague HS Station.

The advantage of our approach compared to stochastic optimization is that we do not need any stochastic information about the travel and dwell times of all daily trips. Hence, our method can be applied even if the historical travel and dwell times do not follow a *specific probability distribution*. Consequently, we can directly use empirical data as input in our minimax problem without fitting probability distributions.

Defining realistic lower and upper limits for the travel time and dwell time noises, $(\xi_{l,s}^{min}, \xi_{l,s}^{max})$, $(\zeta_{l,s}^{min}, \zeta_{l,s}^{max})$, plays an important role in finding robust designs. By definition, a robust design has the best performance in the worst-case scenario.

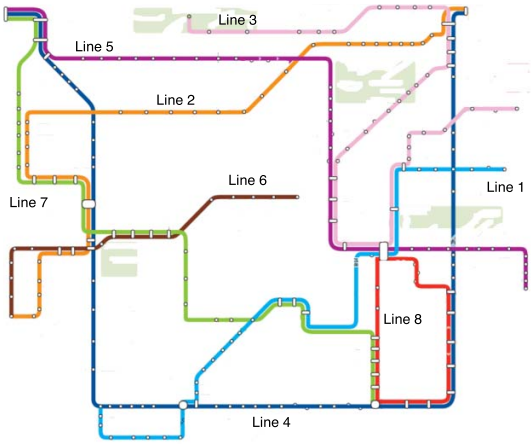

Fig. 3. Bus lines in the Hague.

The worst-case scenario depends on the adversary (in our case, the travel and dwell time noise). If we impose strict limitations on our adversary (i.e., consider that the travel and dwell times are always equal to their expected values), this will result in designs that perform well on average, but are not able to cope with changes. In contrary, if our adversary is not limited (i.e., the travel times are allowed to take unrealistically high values), our robust design will perform the best at scenarios that never occur in practice, whereas it might underperform in common-case scenarios.

To examine the importance of the limits of the adversary in robust designs, we generate the following designs using Alg.1:

- Design (i) - this design is optimal with respect to the expected travel and dwell times. When deriving design (i), the adversary is inactive;
- Robust Design (ii) - this design is robust to an adversary (ξ, ζ) that is allowed to take any value within the 45^{th} and 55^{th} percentile of the 1-month travel time data, and the 47.5^{th} and 52.5^{th} percentile of the dwell times;
- Robust Design (iii) - this design is robust to an adversary that takes values within the 40^{th} and 60^{th} percentile;
- Robust Design (iv) - this design is robust to an adversary that takes travel time values within the 30^{th} and 70^{th} percentile, and dwell time values within the 35^{th} and 65^{th} percentile, respectively.

To investigate the performance of implementing designs (i)-(iv) in realistic operations, we sample AVL and APC data from March 2015 and evaluate the performance of each design.

After applying designs (i)-(iv) at each day, the results in terms of network-wide regularity (Eq.(4)) and waiting times at transfer stops (Eq.(5)) are presented in Table IV. Table IV summarizes the results and reports the average (over the days) of the daily performances and the performance under the worst-case scenario of design (iv).

From the results in Table IV, one can note that the average performance of the robust designs (ii)-(iv) on the 30-day data is inferior to design (i). In reverse, robust designs (ii)-(iv) overperform in days with disruptions demonstrating that are capable of withstanding unexpected events.

The performance deterioration on the average case and the performance improvement on disrupted cases when using robust designs (ii)-(iv) instead of the deterministic design

TABLE IV
VALIDATION RESULTS AFTER APPLYING DESIGNS (I)-(IV) ON THE 30-DAY DATA

	Average Transfer Waiting Time (min)	
	Worst-case scenario (iv)	average performance
Design (i)	4.18	2.30
Robust Design (ii)	3.94	2.33
Robust Design (iii)	3.75	2.39
Robust Design (iv)	3.42	2.61
	Average Network-wide Regularity (min)	
	Worst-case scenario (iv)	average performance
Design (i)	5.96	2.77
Robust Design (ii)	5.86	2.81
Robust Design (iii)	5.81	2.83
Robust Design (iv)	5.76	2.88

(i) are summarized in Fig.4. Since the service regularity in Table IV is relatively stable regardless of the implemented design, Fig.4 presents only the results of the average transfer waiting times.

Fig.4 indicates that design (i) performed worse than the robust designs by 5.74%-18.18% when applied in a day with disruptions. This is in line with the results reported from the daily operations of schedules that are optimized for the average case without considering potential travel/dwell time fluctuations [12].

Designing robust schedules for more extreme scenarios (i.e., design (iv) where the adversary travel time was allowed to take values from the 30$^{\text{th}}$ to the 70$^{\text{th}}$ percentile) results in:

- improved performance in disrupted cases (performance improved by 18.18%);
- significant deterioration in common-case scenarios (average performance deterioration of 11.88%).

In contrast, designing robust designs to milder disruptions (i.e., design (iii)) strikes a better balance between the performance improvement in disrupted conditions and common-case conditions demonstrated by:

- a 10.29% performance improvement in disrupted days;
- a 3.77% performance deterioration on average.

VI. DISCUSSION

Unlike stochastic optimization, our approach does not require the laborious estimation of probability distributions for each inter-station travel time and dwell time.

It is clear from the analysis in Fig.4 that there is a trade-off between: (a) robust designs that impose stricter limits to the adversary (i.e., designs (ii)-(iii)) and result in solutions that perform better at common-case scenarios, and (b) robust designs that prepare for a wide range of disruptions (i.e., design (iv)) and overperform at extreme-case scenarios while under-performing in cases closer to the average.

This sensitivity of the generated robust designs to the limitations of the adversary can be exploited by bus operators. This can be instrumental in generating designs that fit their specific needs/preferences. For instance, in our case study in The Hague, designs that are robust to adversaries that can take values from the 40$^{\text{th}}$ to the 60$^{\text{th}}$ percentile of the observed

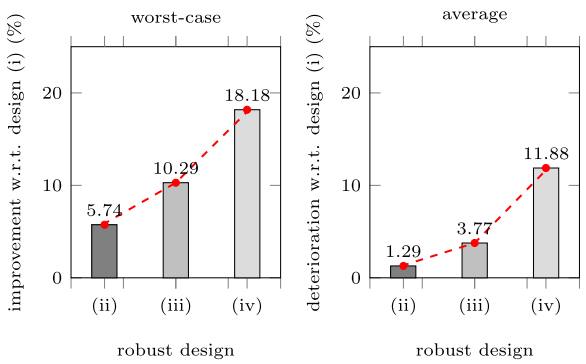

Fig. 4. Transfer waiting time improvement in disrupted cases and deterioration on the average performance when applying robust designs (ii)-(iv) with respect to (w.r.t.) the application of design (i).

data lead to favorable trade-offs between the performance improvement in disrupted cases and the deterioration on the average performance. Other bus networks might exhibit different behavior and the range of disruptions to which our design is robust should be studied meticulously on a case-by-case basis. This can be achieved by changing the bounds of the uncertainty sets from which the environmental variables (i.e., travel and dwell times) can receive their values when computing different robust designs with Alg.1.

VII. CONCLUSION

This study formulated the multi-line synchronization problem considering the potential variability in the travel and dwell times, the regularity of individual bus lines and the operational regulatory constraints such as schedule sliding prevention and layover time limits. After proving that for some travel and dwell time noise levels schedule sliding and missed synchronizations cannot be prevented, a flexible problem formulation was introduced that incorporates the constraint violations with the use of penalties.

In future studies, a broader set of robust timetables can be examined by solving the mathematical program ($\tilde{Q}$) for different percentages of travel and dwell time deviations from the average case. This will facilitate the selection of "dominant" solution(s) that yield the highest payoffs in terms of service regularity and synchronization improvements at both common-case scenarios and abnormal ones. In addition, the potential of our robust solution method can be examined in networks where there is a hierarchy (e.g. regional train and bus) in the services and the network can be synchronized considering a "feeder model" [31]–[34].

APPENDIX

We consider the minimization problem

$$\min_{x} f(x, y) \quad \text{s.t} \quad x \in \mathcal{F}.$$

which minimizes objective function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for a given (parameter) y over the (polyhedral convex) feasible set

$$\mathcal{F} = \{ x \,\big|\, \quad Ax \leq b \}, \quad \text{with} \quad A \in \mathbb{R}^{l \times n}, \; b \in \mathbb{R}^l$$

The value of $y \in \mathbb{R}^m$ is known to vary within in a compact subset $Y \subseteq \mathbb{R}^m$, which leads to the (robust) minimax problem

$$R: \qquad \min_{x \in \mathcal{F}} \max_{y \in Y} f(x, y).$$

Here, $f(x, y)$ is a continuous and continuously differentiable function with respect to (x, y), and convex in x. We assume in the remainder that $\mathcal{F}$ is bounded and non-empty ($\mathcal{F} \neq \emptyset$).

Following [55], for all x, the function

$$\phi(x) := \max_{y \in Y} f(x, y)$$

has directional derivatives at x in any direction $h \in \mathbb{R}^n$, given by

$$d\phi(x; h) := \lim_{t \to 0^+} \frac{\phi(x + th) - \phi(x)}{t} = \max_{y \in Y(x)} \nabla_x f(x, y)^T h \tag{19}$$

with $Y(x) := \{y \in Y : f(x, y) = \phi(x)\}$.

For any $x \in \mathcal{F}$, we define T_x to be the *cone of tangent directions* to $\mathcal{F}$ at x (see [59]), i.e.,

$$T_x := \{ \lambda(y - x) \big| \lambda \geq 0, y \in \mathcal{F}. \}$$

Theorem 2: Let $x \in \mathcal{F}$ be a minimizer of R, then the following condition holds:

$$s^x \in T_x \quad \Rightarrow \quad d\phi(x; s^x) \geq 0 \tag{20}$$

We consider $\bar{R}$, the discretized (with respect to $y^i \in Y$, $i = 1, 2, \ldots, k$) minimax problem of R:

$$\bar{R}: \quad \min_{x \in \mathcal{F}} \max_{y^i, i = 1,2,\ldots,k} f(x, y^i)$$

Given that $f(x, y)$ is a convex function with respect to x for a given $y \in Y$,

$$\chi(x) := \max_{y^i, i = 1,2,\ldots,k} f(x, y^i),$$

is also a convex function with respect to x. For all $x, h \in \mathbb{R}^n$, $\chi(x)$ has directional derivatives at x along h [55], given by

$$d\chi(x; h) := \lim_{t \to 0^+} \frac{\chi(x + th) - \chi(x)}{t} = \max_{i \in I(x)} \nabla_x f(x, y^i)^T h$$

with

$$I(x) := \{ i \in \{1, 2, \ldots, k\} \big| f(x, y^i) = \chi(x) \}$$

Theorem 3: $x \in \mathcal{F}$ is a global minimizer of $\bar{R}$ if and only if

$$s^x \in T_x \quad \Rightarrow \quad d\chi(x; s^x) \geq 0$$

holds.

Theorem 4: Assume that $x \in \mathcal{F}$ is a global minimizer of $\bar{R}$, and that

$$\max_{y \in Y} f(x, y) = \max_{y^i, i = 1,2,\ldots,k} f(x, y^i),$$

then x satisfies condition (20).

Proof: Suppose that $x \in \mathcal{F}$ solves $\bar{R}$, but that it does not satisfy condition (20) of Theorem A.1, i.e.,

$$d\phi(x; s^x) < 0 \quad \text{for some } s^x \in T_x. \tag{21}$$

Let $s^x \in T_x$ be so that the condition in (21) holds. By the definition in (19), it follows that

$$\nabla_x f(x, y)^T s^x \leq d\phi(x; s^x) < 0$$

for all $y \in Y(x)$. However, by assumption of the theorem we have that there exists a $y^i \in Y$, $i = 1, 2, \ldots, k$, for which $f(x, y^i) = \max_{y \in Y} f(x, y)$, i.e., $y^i \in Y(x)$, and (using Theorem A.2)

$$\nabla_x f(x, y^i)^T s^x \geq 0.$$

holds. Hence, we reached a contradiction. □

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