A first glimpse at phase transitions

Ferromagnetism

by

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to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be defended publicly on Monday June 27, 2022 at 13:00.

Student number:	5182417	
Project duration:	April 18, 2022 – June 30, 2022	
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An electronic version of this thesis is available at http://repository.tudelft.nl/.

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Abstract

The Curie-Weiss model is a simplification of the Ising model to show the existence of a phase transition for ferromagnetism. In this thesis, we study the behaviour of sums of these dependent variables. We prove in general that under the appropriate assumptions, we can still conclude a version of the Law of Large Numbers. We also find that if there exists a certain $m \in \mathbb{R}$, $\lambda > 0$ and integer $k \geq 1$, we have that $(S_n - nm)/n^{1/2k}$ converges to $\exp\left(-\lambda s^{2k}/(2k)!\right)$ in distribution.

For the Curie-Weiss model this means that for β, which is a constant proportion to inverse temperature, we find that if $\beta \in (0,1)$ we have $S_n/n \to \delta(s)$ and $S_n/\sqrt{n} \to N(0, \sigma^2)$ in distribution where $\sigma^2 = (1-\beta)^{-1} - 1$. At $\beta = 1$ there occurs a phase transition, we still have that $S_n/n \to \delta(s)$, but now $S_n/n^{3/4} \to \exp\left(-s^4/12\right)$. When $\beta > 1$ we can find an $m > 0$ such that $S_n/n \to \frac{1}{2}[\delta(s-m) + \delta(s+m)]$.

We also study the Curie-Weiss model where we assume that it is under the influence of a magnetic field. We prove that we do not find a phase transition, and we always have $S_n/n \to \delta(s-m)$ in distribution for some $m \in \mathbb{R}$. Next to this we find that $(S_n - nm)/\sqrt{n}$ always converges to a normal distribution.

Lay Summary

When a magnetized piece of metal is heated to a specific temperature, it will lose it's magnetic susceptibility. This is known as a phase transition and at which temperature this happens is determined by Curie's Law. In this thesis, we will look at the particles of the metal that behave like tiny magnets by themselves. If we assume that the particle can be magnetic into one of two directions, up or down, we can say that it takes the value 1 if it points up and -1 when it points down. If all the particles point up, then the average is 1 and the metal is very magnetic in this direction. But if there are as much particles pointing down as up, then the average is zero and the metal does not show signs of magnetism. Physics tells us that when a particle decides on the direction it wants to take, it is influenced by the direction of other particles around it, where the amount of influence is determined by the temperature. So all of the particles together follow a certain rule on how they want to be oriented. In this thesis we will research if we find the same results as Curie's Law, if we assume that the particles follow the rules given by what is know as the Curie-Weiss model.

Summary

We want to determine if we can show that there occurs a phase transition in a piece of metal, by modelling the magnetic properties on the basis of the distribution of magnetic moments (spins) of the particles. The distribution of the direction of these spins is influenced by neighboring particles. We will assume that they only take one of two possible directions which we will denote by $\{1, -1\}$. Another simplifying assumption is that all of the spins influence each other in the system, this means that we don't have the geometry which determines which particles are neighbours. This leads to the use of the Curie-Weiss model which gives an expression for joint distribution of the spins, with a parameter $\beta \geq 0$ which is proportional to inverse temperature.

Because the random variables are dependent for $\beta > 0$, we will prove two theorems in this thesis as an alternative to the Law of Large Numbers and the Central Limit Theorem, both of which are not applicable. We will define a function G which will be of great importance in the proofs of these theorems, and is determined by a measure ρ, which in turn is determined by the assumption about conditions the metal is in. This leads to the statement $S_n \to \sum_i b_i \delta(s - m_i)$ in distribution where the b_i's and m_i's are real and determined by ρ. Moreover, we find that under the right assumptions on ρ, there exists a unique $m \in \mathbb{R}$, $\lambda > 0$ and integer $k \geq 1$ such that $(S_n - nm)/n^{1/2k} \to \exp\left(-\lambda s^{2k}/(2k)!\right)$. With these results we can conclude that there exists a specific temperature such that there occurs a phase transition.

We also extend the model by assuming that the metal is under the influence of a magnetic field. This leads to the conclusion that there does not occur a phase transition, and we always have $S_n \to \delta(s - m)$ is distribution for some $m \in \mathbb{R}$.

Contents

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Introduction

When a magnetic piece of metal is heated, the ability to be magnetized weakens up to a specific temperature, where this ability disappears altogether. Once the metal is cooled down, it does not return to it's original state. Only if the metal is put under the influence of a strong enough magnetic field, the magnetization comes back.

The phenomenon of ferromagnetism is caused by the angular momentum of individual electrons all pointing in the same direction. Wilhelm Lenz invented the Ising model as an approximation of the actual physical phenomenon and gave his student Ernst Ising the assignment to study whether this model could explain the phase transition caused by the increasing temperature. Instead of orienting in any direction on a three-dimensional sphere this model assumed that the particles could only be in one of two different states given by $\{1, -1\}$. These states are often referred to as spin and because spins which are aligned have a lower energy than those who are opposed, nature prefers neighboring spins which have the same state. The edges of the graph on which the particles live determine whether they are neighbours and is usually considered to be $\mathbb{Z}^d$, with $d \in \mathbb{N}$.

It is, however, easier to assume that each spin is influenced by all the other spins in the system. In this case we talk about the Curie-Weiss model. Suppose the system consists of n distinct atoms. Given a vector $\sigma = (\sigma_i)_{i=1,\dots,n}$ where each σ_i represents the spin of an individual particle, we assume that the spins tend to align. Which is why we introduce the term

$$\frac{\beta}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j, \tag{1.1}$$

where $\beta \geq 0$ is a constant proportional to the inverse temperature. When two spins σ_i and σ_j of opposite sign are multiplied in the summation, they have a negative contribution, whereas two spins of equal sign have a positive contribution. Thus the more spins are oriented in the same direction, the larger the outcome of the term in (1.1). To put the parameter β to use we define the joint distribution of σ as

$$\mathbb{P}_\beta^n(\{\sigma\}) = \frac{1}{Z_n}\exp\left(\frac{\beta}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\right), \tag{1.2}$$

where Z_n is a normalization constant. We now see that the magnitude of β plays a big role in the distribution of the mass of the probability measure $\mathbb{P}_\beta^n$. When we increase β, then so does the probability of a configuration where the majority of spins has the same sign. For β close to zero, the contribution of the exponent 1.1 starts to decrease, and the amount of aligned spins loses it's importance.

When we considering the case $\beta = 0$ in (1.2) we find that this corresponds physically to infinite temperature and mathematically to independence. This implies that the individual σ_i's have a uniform distribution on $\{-1, 1\}$. If we set $S_n = \frac{1}{n}\sum_{i=1}^{n}\sigma_i$, then together with $\mathbb{E}[\sigma_i] = 0$ and $\mathbb{E}\left[\sigma_i^2\right] = 1$, both the

Law of Large Numbers and the Central Limit Theorem hold with

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad \text{a.s.,} \tag{1.3}$$

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}} = N(0,1) \quad \text{in distribution.} \tag{1.4}$$

Although this is a satisfying result, it does not say much about the case where $\beta > 0$ since we lose our assumption of independence. We already suggested that for small values of β we might observe behaviour which coincides with the independent case, where for large values of β it is more likely to observe a configuration σ where S_n moves away from zero. Since the value of the sum S_n/n determines the strength of the magnetization in the system, it is common to denote the magnetization by $m_n(\sigma) = S_n/n$, but throughout the thesis we will often mention S_n/n instead of m_n.

In Chapter 2 we will derive that for values of β close enough to zero we do in fact find that S_n/n converges to a discrete distribution with all it's mass centered at the point zero. We even find that $S_n/\sqrt{n}$ converges to a normal distribution. When $\beta = 1$ we will find the same convergence for S_n/n, but it will appear that we don't have normality in the limit anymore. But under the appropriate scaling, there still exists a limiting distribution. The convergence to the Dirac distribution is in a way comparable to (1.3). Intuitively, this means that when the temperature is high enough, the influence of the spins among each other is negligible compared to the energy in the system caused by the temperature.

When β becomes greater we see that the term in the exponent of (1.2) will give large portions of it's mass to configurations of σ where most of the spins are aligned. In this case it is unlikely that S_n/n converges the point zero in some way, since it favors a configuration where the absolute value of the sum S_n is large. In Chapter 3 we prove two theorems which will show us that for $\beta > 1$ the limit S_n/n will converge to a linear combination of Dirac measures. This is where we can make a distinction in the limiting behaviour by changing the parameter β and hence observe a phase transition when $\beta = 1$. These theorems will also generalize the obtained results from Chapter 2 such that we can study more complex variations of (1.2). Most of the theorems and lemmas found in Chapter 4 are based on results in [1] and is often an identical statement under a different hypothesis.

The model in (1.2) assumed that the spins where only influenced by the temperature in the system, it is however more interesting to study a system that is under the influence of a magnetic field. If one brings a piece of metal such as iron into a magnetic field, the magnetic properties of the metal change. In the model this can be explained by the spins which tend to align with the direction of the field, this translates to a joint distribution given by

$$\mathbb{P}_{\beta h}(\{\sigma\}) = \frac{1}{Z_n} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j + h\sqrt{\beta} \sum_{i=1}^{n} \sigma_i \right), \tag{1.5}$$

where $h \in \mathbb{R}$ is a constant proportional to the direction and magnitude of the field. From a physical point of view it makes more sense to replace the term replaces $h = \sqrt{\beta}\tilde{h}J^{-1}$ and $\beta = J\tilde{\beta}$, such that one can write the exponent as $\exp\big(-\tilde{\beta}\mathcal{H}(\sigma)\big)$, where

$$\mathcal{H}(\sigma) = -\frac{J}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j - \tilde{h} \sum_{i=1}^{n} \sigma_i,$$

denotes the Hamiltonian as described in [3]. However, it is helpful to use the expression as in (1.5) because it will be more convenient to work with in Chapter 4. In this chapter we find that the behaviour at $\beta = 1$ in the previous model vanishes due to the presence of the magnetic field, where S_n/n converges to a point of mass whenever $h \neq 0$. At the end of the chapter we will briefly consider an extreme case where β is several orders of magnitude larger than h. We know that for large values of β, in the model described by (1.2), S_n/n converges to two different points of mass. Regardless of the strength of the magnetic field, we will always find that S_n/n converges to one point if the distribution is given by (1.5). When h is extremely small, this convergence happens very slow. Since a piece of metal only has a finite amount of atoms, say n, one might not observe this result in an experiment.

2

Limit behaviour of the Curie-Weiss model

In this chapter we will analyse a specific case of the Curie-Weiss model where the joint distribution is given by $\mathbb{P}_\beta^n$ in (1.2). This will serve as an illustrative example for the more general results in later chapters. The ultimate goal is to determine under which circumstances we will observe a phase transition, which means that we have study the behaviour of S_n/n by looking at different values of β. Because (1.3) and (1.4) do not hold for $\beta > 0$, we are to find out how large sums of these dependent random variables are distributed. We will start by setting up some general notation which will help us identify similar objects in later chapters.

Given a sequence of random variables $X_1, \ldots, X_n$ with joint distribution $\mathbb{P}_\beta^n$, we will express this distribution in a more general form. Since the X_i take values in $\{-1, 1\}$ and in the independent case $\beta = 1$ we had a uniform distribution for an individual X_i, we will introduce the following measure

$$\rho(\{x\}) = \begin{cases} \frac{1}{2} \text{ if } x = \pm 1, \\ 0 \text{ otherwise.} \end{cases} \tag{2.1}$$

We can use this measure ρ to express the $\mathbb{P}_\beta^n$ in terms of an integral

$$\mathbb{P}_\beta^n((X_1, \ldots, X_n) \in A) = \int_A \frac{1}{Z_n} \exp\left[\frac{\beta}{2n}\left(\sum_{i=1}^n \sigma_i\right)^2\right] \prod_{j=1}^n \mathrm{d}\rho(\sigma_j). \tag{2.2}$$

Because ρ in (2.1) is a discrete and finite measure it is helpful to introduce the Dirac measure which is defined by

$$\delta(\{x\}) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

where we may also write $\delta(x) = \delta(\{x\})$. Furthermore, if we want to give all mass to a point $m \in \mathbb{R}$ not necessarily zero, we write $\delta_m(x) = \delta(x - m)$. Thus we can recover ρ in the form $\rho = \frac{1}{2}(\delta_1 + \delta_{-1})$. We can now factor out β in (2.2) by introducing

$$\rho_\beta = \frac{1}{2}\left(\delta_{\sqrt{\beta}} + \delta_{-\sqrt{\beta}}\right), \tag{2.4}$$

and substitute $x = \sqrt{\beta}\sigma$ for which we have $\mathrm{d}\rho_\beta(x) = \mathrm{d}\rho(x/\sqrt{\beta})$. This gives us the expression

$$\mathbb{P}_\beta^n((X_1, \ldots, X_n) \in A) = \int_A \frac{1}{Z_n} \exp\left[\frac{(x_1 + \cdots + x_n)^2}{2n}\right] \prod_{j=1}^n \mathrm{d}\rho_\beta(x_j), \tag{2.5}$$

which is the notation we will use in this and following chapters. Because we will study the same object in later chapters but for an arbitrary measure ρ, we write $\mathrm{d}\rho$ instead of $\mathrm{d}\rho_\beta$ from now on.

2.1. Distribution of sums

In this section we will prove claims about how $S_n = \sum_{i=1}^n X_i^{(n)}$ is distributed, where $X_1^{(n)}, \dots, X_n^{(n)}$ is a sequence of random variables with distribution (2.5). We will however assume that ρ is an arbitrary measure and thus write

$$\frac{1}{Z_n}\exp\left[\frac{(x_1+\cdots+x_n)^2}{2n}\right]\prod_{j=1}^n \mathrm{d}\rho(x_j). \tag{2.6}$$

In this expression we notice the summation of the variables x_i, which is useful to derive an expression for the probability measure of S_n. In the following lemma we will derive the distribution of S_n.

Lemma 2.1.1. *Let $X_1^{(n)}, \dots, X_n^{(n)}$ be a sequence of random variables with joint distribution* (2.6) *and let $S_n = \sum_{i=1}^n X_i^{(n)}$. Then the distribution of S_n is given by*

$$\frac{1}{Z_n}\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x), \tag{2.7}$$

where ρ^{*n} is the n-fold convolution of ρ with itself.

Proof. We will prove this claim by showing that the distribution of S_n can be expressed in terms of a Radon-Nikodym derivative with respect to the measure ρ^{*n}. First note that for a sequence of independent random variables $Y_1^{(n)}, \dots, Y_n^{(n)}$ with distribution ρ, the distribution measure of $Y = \sum_{i=1}^n Y_i^{(n)}$ is given by Theorem 15.1 from [2] as

$$\rho^{*n}(A) = \int_A \mathbb{1}_{\{x_1+\cdots+x_n=x\}}\prod_{j=1}^n \mathrm{d}\rho(x_j).$$

Inspired by (2.6) which contains the sum $\sum_i x_i$, we can define a new measure ν using the Radon-Nikodym derivative with respect to ρ^{*n}

$$\frac{\mathrm{d}\nu}{\mathrm{d}\rho^{*n}} = \frac{1}{Z_n}e^{Y^2/2n},$$

which will give the expression

$$\nu(A) = \int_A \frac{1}{Z_n}e^{x^2/2n}\mathrm{d}\rho^{*n}(x).$$

Observe now that we can obtain the distribution of S_n very easily in terms of ρ^{*n} because

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{S_n\in A}\right] &= \int \frac{1}{Z_n}\mathbb{1}_{\{x_1+\cdots+x_n\in A\}}\exp\left[\frac{(x_1+\ldots+x_n)^2}{2n}\right]\prod_{j=1}^n \mathrm{d}\rho(x_j)\\ &= \int_A \frac{1}{Z_n}\exp\left(\frac{x^2}{2n}\right)\mathbb{1}_{\{x_1+\cdots+x_n=x\}}\prod_{j=1}^n \mathrm{d}\rho(x_j)\\ &= \int_A \frac{1}{Z_n}\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x) = \nu(A).\end{aligned}$$

Hence S_n has distribution ν which was indeed what we meant to prove in (2.7). □

We now have an expression for the distribution of S_n. What we want to know is if the limit of S_n/n converges to a point of mass, and similar to the case of $\beta = 1$, whether we have normality in the limit of $S_n/\sqrt{n}$. When we observe that the latter is not possible, it might just be that we do have convergence of S_n/n^α, where $\alpha \neq \frac{1}{2}$. For this reason we want to study the behaviour of

$$\frac{S_n}{n^{1-\gamma}},$$

where γ is arbitrary, but only the cases where $\gamma \in \left[0, \frac{1}{2}\right]$ will appear to be relevant. Because ρ is assumed to be a discrete measure in this chapter, the distribution function is a summation which is different for each n. This is very inconvenient, which is why we slightly modify the object such that the cumulative distribution function is given by

$$\int_{-\infty}^{\theta} f(s)\mathrm{d}s,$$

for some density function f with respect to the Lebesgue measure. This can be done by adding an independent random variable which also is characterized by some density function. When $\gamma = \frac{1}{2}$, we might find a normal distribution in the limit. Thus if we add $W \sim N(0,1)$ to the object $S_n/\sqrt{n}$, and the resulting limiting distribution is a normal distribution, we can argue that $S_n/\sqrt{n}$ is also normal in the limit.

We have a different scaling of S_n when $\gamma < \frac{1}{2}$, thus we do not expect a normal distribution in the limit. Hence the added factor must lose it's contribution as $n \to \infty$. For this reason we will try to find the distribution of the object

$$\frac{W}{n^{\frac{1}{2}-\gamma}} + \frac{S_n}{n^{1-\gamma}}, \tag{2.8}$$

with $W \sim N(0,1)$ with W independent of S_n. The assumption that W is independent of S_n is necessary, in order to draw conclusions about the limiting distribution of $S_n/n^{1-\gamma}$. Why this is the case becomes clear when we state and prove the following lemma.

Lemma 2.1.2. *Suppose that for each n, W_n, and Y_n, are independent random variables such that $W_n \to \nu$, where*

$$\hat{\nu}(a) = \int e^{iax}\mathrm{d}\nu(x) \neq 0, \text{ all } a \in \mathbb{R}.$$

Then $Y_n \to \mu$ if and only if $W_n + Y_n \to \nu * \mu$.

Proof. First suppose that $Y_n \to \mu$ and $u_1, u_2 \in \mathbb{R}$, because of the independence we can apply Corollary 14.1 from [2] and we have

$$\varphi_{W_n,Y_n}(u_1,u_2) = \varphi_{W_n}(u_1)\varphi_{Y_n}(u_2) \tag{2.9}$$

where φ_{W_n} and φ_{Y_n} are the characteristic functions of W_n, and Y_n respectively. Now Lévy's Continuity Theorem tells us that

$$\varphi_{W_n}(u_1)\varphi_{Y_n}(u_2) = \hat{\nu}(u_1)\hat{\mu}(u_2) \quad \text{as } n \to \infty, \tag{2.10}$$

where $\hat{\mu}(a) = \int e^{iax}\mathrm{d}\mu(x)$. Thus if we denote the random variables W and Y such that $W_n \to W$ and $Y_n \to Y$ we obtain that W and Y are independent. Hence using (2.9) and (2.10) together with Theorem 15.2 from [2] we obtain

$$\varphi_{W_n+Y_n}(u) = \varphi_{W_n}(u)\varphi_{Y_n}(u) \to \varphi_W(u)\varphi_Y(u) = \varphi_{W+Y}(u) \quad \text{as } n \to \infty \tag{2.11}$$

where $W + Y$ has distribution $\nu * \mu$ which proves the implication.

Now assume that $W_n + Y_n \to \nu * \mu$ where $\nu * \mu$ is the distribution of $W + Y$ for some random variable Y with distribution μ, and we know that ν is the distribution of W to which W_n weakly converges. Note that these are independent by definition. This gives us a similar argument using (2.9)-(2.11) which leads to

$$\varphi_{W_n}(u)\varphi_{Y_n}(u) \to \varphi_W(u)\varphi_Y(u) \quad \text{as } n \to \infty,$$

for all $u \in \mathbb{R}$. And the assumption that φ_W is nonzero on $\mathbb{R}$ gives us pointwise convergence of $\phi_{Y_n} \to \phi_Y$ which implies $Y_n \to \mu$ and proves the converse. □

Now that we are at the point where we can immediately derive the limiting distribution of $S_n/n^{1-\gamma}$, we will derive the distribution of the object in (2.8) in the following claim.

Claim 2.1.3. *Given that S_n has distribution* (2.7) *with ρ as in* (2.4), *then for some $W \sim N(0,1)$ independent of S_n we have that*

$$\frac{W}{n^{\frac{1}{2}-\gamma}} + \frac{S_n}{n^{1-\gamma}} \sim \frac{\exp(-nG(s/n^{\gamma}))\mathrm{d}s}{\int \exp(-nG(s/n^{\gamma}))\mathrm{d}s}. \tag{2.12}$$

where the function $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$G(s) = \frac{s^2}{2} - \ln\cosh\left(s\sqrt{\beta}\right). \tag{2.13}$$

Proof. Define the interval $I := \left(-\infty, n^{1-\gamma}\theta\right]$ which will be helpful regarding notation. This gives the following equation

$$\mathbb{P}\left(\frac{W}{n^{\frac{1}{2}-\gamma}} + \frac{S_n}{n^{1-\gamma}} \leq \theta\right) = \mathbb{P}\left(\sqrt{n}W + S_n \leq n^{1-\gamma}\theta\right) = \mathbb{P}\left(\sqrt{n}W + S_n \in I\right). \tag{2.14}$$

We know that the distributions of $\sqrt{n}W$ and S_n are given by $N(0,n)$ and (2.7) respectively. It then follows by Theorem 15.1 from [2] that as they are independent, the distribution measure of $\sqrt{n}W + S_n$ is given by the convolution product of $\sqrt{n}W$ and S_n their measures. This is

$$\begin{aligned}&\left[N(0,n) * \exp\left(x^2/2n\right)\rho^{*n}\right](A)\\ &\quad = \int\int \mathbb{1}_I(u+x)\frac{1}{\sqrt{2\pi n}}\exp\left(-\frac{u^2}{2n}\right)\mathrm{d}u\frac{1}{Z_n}\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x)\end{aligned}$$

Continuing with (2.14) we obtain

$$\begin{aligned}\mathbb{P}\left(\sqrt{n}W + S_n \in I\right) &= \frac{1}{Z_n}\int_I \mathrm{d}\left[N(0,n) * \exp\left(x^2/2n\right)\rho^{*n}\right](x)\\ &= \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int\int \mathbb{1}_I(u+x)\exp\left(-\frac{u^2}{2n}\right)\mathrm{d}u\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x).\end{aligned} \tag{2.15}$$

Now if we use substitution for $s = u + x$, we obtain

$$\int \mathbb{1}_I(u+x)\exp\left(-\frac{u^2}{2n}\right)\mathrm{d}u = \int \mathbb{1}_I(s)\exp\left(-\frac{(s-x)^2}{2n}\right)\mathrm{d}s, \tag{2.16}$$

and expanding $(s-x)^2$ leads to

$$\frac{-(s-x)^2 + x^2}{2n} = -\frac{s^2}{2n} + \frac{sx}{n}. \tag{2.17}$$

Because we want an integral with respect to the Lebesgue measure, we use Tonelli-Fubini's Theorem to interchange $\mathrm{d}s$ and $\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x)$. We then obtain the following for (2.15)

$$\begin{aligned}&\frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int\int \mathbb{1}_I(u+x)\exp\left(-\frac{u^2}{2n}\right)\mathrm{d}u\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x)\\ &\quad\overset{(2.16)}{=} \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int\int \mathbb{1}_I(s)\exp\left(-\frac{(s-x)^2}{2n}\right)\mathrm{d}s\exp\left(\frac{x^2}{2n}\right)\mathrm{d}\rho^{*n}(x)\\ &\quad\overset{(2.17)}{=} \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\int\exp\left(\frac{sx}{n}\right)\mathrm{d}\rho^{*n}(x)\mathrm{d}s.\end{aligned} \tag{2.18}$$

The n-fold convolution product of ρ with itself is defined as the distribution measure for the sum of n independent random variables with distribution ρ, hence if we let $Y_1, Y_2, \ldots, Y_n$ be sequence of random variables which satisfy this condition it follows that

$$\begin{aligned}\int\exp\left(\frac{sx}{n}\right)\mathrm{d}\rho^{*n}(x) &= \mathbb{E}\left[\exp\left(\frac{s\sum_{i=1}^n Y_i}{n}\right)\right] = \mathbb{E}\left[\prod_{i=1}^n \exp\left(\frac{sY_i}{n}\right)\right]\\ &= \prod_{i=1}^n \mathbb{E}\left[\exp\left(\frac{sY_i}{n}\right)\right] = \mathbb{E}\left[\exp\left(\frac{sY_1}{n}\right)\right]^n.\end{aligned} \tag{2.19}$$

Where we used that the Y_i's are independent and identically distributed. Since our measure ρ is given by (2.4) we can calculate the expectation in (2.19)

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{sY_1}{n}\right)\right] = \int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho(x) &= \frac{1}{2}\left[\exp\left(\frac{s\sqrt{\beta}}{n}\right) + \exp\left(-\frac{s\sqrt{\beta}}{n}\right)\right] \\ &= \cosh\left(\frac{s\sqrt{\beta}}{n}\right).\end{aligned} \tag{2.20}$$

Returning to (2.18) we now have

$$\begin{aligned}&\frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho^{*n}(x)\mathrm{d}s \\ &\overset{(2.19)}{=} \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\mathbb{E}\left[\exp\left(\frac{sY_1}{n}\right)\right]^n \mathrm{d}s \\ &\overset{(2.20)}{=} \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\left[\cosh\left(\frac{s\sqrt{\beta}}{n}\right)\right]^n \mathrm{d}s \\ &= \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\exp\left(n\ln\cosh\left(\frac{s\sqrt{\beta}}{n}\right)\right)\mathrm{d}s,\end{aligned} \tag{2.21}$$

where we used that $a^n = \exp(n\ln a)$ for any $a > 0$ in the last equation. Observe that we can rewrite the term in the exponent as

$$-\frac{s^2}{2n} + n\ln\cosh\left(\frac{s\sqrt{\beta}}{n}\right) = -n\left(\frac{(s/n)^2}{2} - \ln\cosh\left(\frac{s}{n}\sqrt{\beta}\right)\right) = -nG(s/n), \tag{2.22}$$

where G is given by (2.13). Since we want an expression for the distribution we need to change the upperbound to θ by means of substitution. This will be obtained by substituting t for $sn^{\gamma-1}$ which will give an extra factor $n^{1-\gamma}$ in front of the integral. At last we can put it together and obtain for (2.21)

$$\begin{aligned}&\frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\exp\left(n\ln\cosh\left(\frac{s\sqrt{\beta}}{n}\right)\right)\mathrm{d}s \\ &\overset{(2.22)+(2.13)}{=} \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_{-\infty}^{n^{1-\gamma}\theta} \exp\left(-nG\left(\frac{s}{n}\right)\right)\mathrm{d}s \\ &\overset{t=sn^{\gamma-1}}{=} \frac{1}{Z_n}\frac{n^{1-\gamma}}{\sqrt{2\pi n}}\int_{-\infty}^{\theta} \exp\left(-nG\left(\frac{t}{n^{\gamma}}\right)\right)\mathrm{d}t,\end{aligned} \tag{2.23}$$

where the integrand has the desired form. To arrive at the normalization constant we have to let $\theta \to \infty$ which will give

$$\lim_{\theta\to\infty}\mathbb{P}\left(\frac{W}{n^{\frac{1}{2}-\gamma}} + \frac{S_n}{n^{1-\gamma}} \leq \theta\right) = 1.$$

And because $\theta \to \infty$ gives us the required integral and we obtain the expression

$$\frac{1}{Z_n}\frac{n^{1-\gamma}}{\sqrt{2\pi n}}\int \exp\left(-nG\left(\frac{t}{n^{\gamma}}\right)\right)\mathrm{d}t = 1 \;\Leftrightarrow\; \frac{1}{Z_n}\frac{n^{1-\gamma}}{\sqrt{2\pi n}} = \frac{1}{\int \exp\left(-nG\left(\frac{t}{n^{\gamma}}\right)\right)\mathrm{d}t}. \tag{2.24}$$

Hence it follows from (2.23) and (2.24) that (2.12) holds if we replace t by s. □

It will turn out that in a more general setting the function G in (2.13) will play a very important role as it characterizes the distribution of objects similar to that of (2.8). When we are discussing convergence of random variables in the next sections and chapters, we will often write $X_n \to f(s)$ when a sequence $(X_n)_{n\geq 1}$ converges weakly to a measure ν, which is determined by a density function $f(s)$ with respect to the Lebesgue measure $\mathrm{d}s$. Hence if $\mathrm{d}\nu(s) \equiv f(s)\mathrm{d}s$, the statements $X_n \to f(s)$ and $X_n \to \nu$ are assumed to be equivalent.

2.2. Law of Large Numbers

In the previous section we found the distribution in (2.12) which will help us to obtain a limiting distribution for $\frac{S_n}{n}$. Thus if we take $\gamma = 0$, we have to look at the density function $\exp(-nG(s))$. For $\beta \in (0, 1]$ the function G will visually resemble one of the graphs in Figure 2.1. Observe that G has a unique global minimum at zero and that $G(s) > 0$ for $s \neq 0$. We obtain that

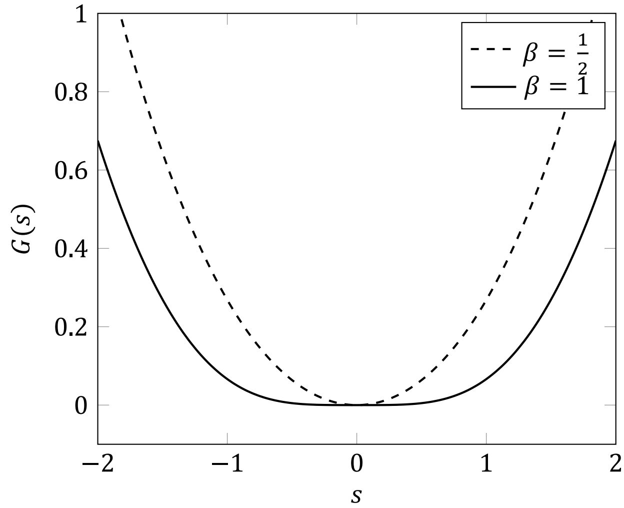

Figure 2.1: Plot of G for $\beta = \frac{1}{2}$ and $\beta = 1$.

$$\exp(-nG(s)) \to \begin{cases} 1, & \text{if } s = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.25}$$

as $n \to \infty$. Now that we know that all the mass is given at a single point we have to show that our distribution converges to the Dirac measure given by (2.3). Theorem 18.1 from [2] tells us that this is equivalent to showing that

$$\lim_{n\to\infty} \frac{\int e^{-nG(s)} h(s) \mathrm{d}s}{\int e^{-nG(s)} \mathrm{d}s} = \int h(x) \mathrm{d}\delta(x) = h(0), \tag{2.26}$$

for any continuous, bounded function h. We can use dominated convergence with

$$\exp(-nG(s))h(s) \to \begin{cases} h(0), & \text{if } s = 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{as } n \to \infty,$$

as in (2.25) for the pointwise convergence and we have the upperbound

$$\exp(-nG(s))h(s) \leq \exp(-G(s)) \max\{|h(s)| : s \in \mathbb{R}\},$$

for each $n \geq 1$ as $\exp(-G(s))$ is integrable by (2.24) for $n = 1$. Hence we obtain that in the left hand side of (2.26) the numerator converges to

$$\int e^{-nG(s)} h(s) \mathrm{d}s \to h(0) \quad \text{as } n \to \infty,$$

where the same holds for the denominator with $h \equiv 1$.

What we have shown now is that for some $W \sim N(0, 1)$ independent of S_n we have

$$\frac{W}{\sqrt{n}} + \frac{S_n}{n} \to \delta(s),$$

where we know that $W/\sqrt{n} \to \delta(s)$. It follows from Lemma 2.1.2 that this implies

$$\frac{S_n}{n} \to \delta(s), \tag{2.27}$$

for $0 < \beta \leq 1$ which is similar to (1.3). When $\beta > 1$ we see that G does not have an unique global minimum, but two global minima which is made visible in Figure 2.2. Because of this it is intuitively not possible to arrive at the same result as in (2.26), since all of the mass will be given to the two points where the global minima occur. We will develop additional tools in later chapters to show that there exists a limiting distribution for the case $\beta > 1$ as well.

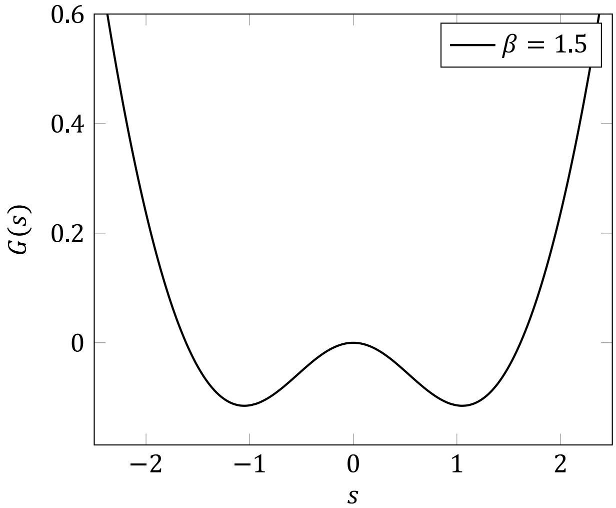

Figure 2.2: Plot of G for $\beta = 1.5$.

2.3. The Central Limit Theorem

Now that we have shown that a weaker version of the Law of Large Numbers exists under the condition that $0 < \beta \leq 1$, we will do the same with respect to the Central Limit Theorem. For a sequence of independent and identically distributed random variables we would have that

$$\lim_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1) \quad \text{in distribution,} \tag{2.28}$$

where μ and σ are the mean and standard deviation respectively, which corresponds to (1.4). But since we don't have independence in the Curie-Weiss model for $\beta > 0$, the argument is not applicable. However, we do have the expression (2.12) which can help us find a normal distribution in the limit. Note that in (2.28) we subtract $n\mu$ to center the limiting distribution around zero, in the following derivation we use (2.27) as a motivation to take $\mu = 0$. Hence we will try to show that

$$W + \frac{S_n}{\sqrt{n}} \to N(0,\sigma^2) \quad \text{as } n \to \infty,$$

for some $\sigma > 0$, using the known expression for the left-hand side of this equation, which is given by (2.12).

Similarly to the derivation of the limit (2.26) in Section 2.2, we will use point convergence of the density function in (2.12). To achieve this we use that G has the following Taylor expansion around $s = 0$

$$G(s) = \frac{s^2}{2} - \beta\frac{s^2}{2} + \beta^2\frac{s^4}{12} + \mathcal{O}(s^6). \tag{2.29}$$

So whenever $\beta \in (0, 1)$ we have

$$-nG(s/n^\gamma) = -n\left[(1-\beta)\frac{(s/n^\gamma)^2}{2} + \mathcal{O}\big((s/n^\gamma)^4\big)\right] \\ = -(1-\beta)\frac{s^2}{2n^{2\gamma-1}} + \mathcal{O}\left(\frac{s^4}{n^{4\gamma-1}}\right),$$

which implies that for $\gamma = \frac{1}{2}$ we obtain the pointwise convergence

$$\exp\big(-nG(s/n^{1/2})\big)h(s) \to \exp\left(-\frac{s^2}{2(1-\beta)^{-1}}\right)h(s) \quad \text{as } n \to \infty, \tag{2.30}$$

for any bounded, continuous function h. Observe that when $\beta = 1$ we lose the quadratic term in (2.29). This will lead to a limiting distribution which is not normal but gives

$$-nG(s/n^\gamma) = -n\left[\frac{(s/n^\gamma)^4}{12} + \mathcal{O}\big((s/n^\gamma)^6\big)\right] \\ = -\frac{s^4}{12n^{4\gamma-1}} + \mathcal{O}\left(\frac{s^6}{n^{6\gamma-1}}\right).$$

By the same reasoning as for $\beta \in (0, 1)$ we will choose $\gamma = \frac{1}{4}$ and obtain

$$\exp\big(-nG(s/n^{1/4})\big)h(s) \to \exp\left(-\frac{s^4}{12}\right)h(s) \quad \text{as } n \to \infty. \tag{2.31}$$

Now we can apply Theorem 18.1 from [2] again to obtain the limiting distribution. First observe that we can bound $\cosh\big(s\sqrt{\beta}\big) \le e^{|s|\sqrt{\beta}}$ and find that

$$G(s) = \frac{s^2}{2} - \ln\cosh\big(s\sqrt{\beta}\big) \ge \frac{s^2}{2} - \ln\big(e^{|s|\sqrt{\beta}}\big) \\ \ge \frac{s^2}{2} - |s|\sqrt{\beta}. \tag{2.32}$$

Then by (2.32) we have

$$\exp(-nG(s/n^\gamma))h(s) \overset{\gamma\in(0,1)}{\le} \exp\left(-\frac{s^2}{2} + |s|\sqrt{\beta}\right)\max\{|h(s)| : s \in \mathbb{R}\}, \tag{2.33}$$

thus we can apply dominated convergence with (2.30) and (2.33) to prove that for $\beta \in (0, 1)$ we have

$$\frac{\int \exp\big(-nG\big(s/n^{1/2}\big)\big)h(s)\mathrm{d}s}{\int \exp\big(-nG\big(s/n^{1/2}\big)\big)\mathrm{d}s} \to \int \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{s^2}{2\sigma^2}\right)h(s)\mathrm{d}s \quad \text{as } n \to \infty,$$

where $\sigma^2 = (1-\beta)^{-1}$. The same goes for $\beta = 1$ where (2.31) and (2.33) give

$$\frac{\int \exp\big(-nG\big(s/n^{1/4}\big)\big)h(s)\mathrm{d}s}{\int \exp\big(-nG\big(s/n^{1/4}\big)\big)\mathrm{d}s} \to \frac{\int \exp\left(-\frac{s^4}{12}\right)h(s)\mathrm{d}s}{\int \exp\left(-\frac{s^4}{12}\right)\mathrm{d}s} \quad \text{as } n \to \infty.$$

What we have shown is that for $\beta \in (0, 1)$ we obtain the limiting distribution

$$W + \frac{S_n}{\sqrt{n}} \to N(0, (1-\beta)^{-1}) \quad \text{as } n \to \infty, \tag{2.34}$$

where $W \sim N(0, 1)$. We can apply Lemma 2.1.2 together with Theorem 16.3 from [2] and conclude

$$\frac{S_n}{\sqrt{n}} \to N(0, (1-\beta)^{-1} - 1) \quad \text{as } n \to \infty, \tag{2.35}$$

for $\beta \in (0, 1)$. We could not arrive at a normal distribution in the limit for $\beta = 1$ using the same scaling as in (2.34), but we can divide S_n by a different order of n and obtain the following expression

$$\frac{W}{n^{1/4}} + \frac{S_n}{n^{3/4}} \to \exp\left(-\frac{s^4}{12}\right)\mathrm{d}s \quad \text{as } n \to \infty,$$

where the normalization constant is left out. Because the term $\frac{W}{n^{1/4}}$ converges to $\delta(s)$ we can apply Lemma 2.1.2 again to obtain

$$\frac{S_n}{n^{3/4}} \to \exp\left(-\frac{s^4}{12}\right)\mathrm{d}s \quad \text{as } n \to \infty. \tag{2.36}$$

Thus a certain change happens at $\beta = 1$. For smaller values of β we have a normal distribution in the limit where S_n is scaled by the same term $\sqrt{n}$ as in the Central Limit Theorem. But the limiting distribution takes another form when $\beta = 1$ and S_n requires a different scaling to obtain this.

What we observe is that by changing the physical properties of the model, in this case the parameter β which is proportional to inverse temperature, the behaviour of S_n changes drastically when β approaches 1 from below. In the next chapter we will generalize the methods that we used throughout this chapter and see why we could not find a limiting distribution for $\beta > 1$ as well. Using one of these methods it becomes possible to prove the claim at the end of section 2.2, where we stated that S_n/n converges to two different points with equal probability. In this case it is unreasonable to find an expression as in (2.28) without additional assumptions, as we do not have converge to the mean.

3

General

In Chapter 2 we have shown that for a specific case of dependent random variables, we can still apply the known limit theorems and obtain satisfactory results. We also saw that there exists a type of boundary for the Curie-Weiss model at $\beta = 1$. We know that the limit behaviour changes at this point but are unable to prove the actual existence of a phase change. In this chapter we will state and prove two theorems which will agree to the already obtained results and also lead to additional conclusions that we could not make in Section 2.2. In order to do this we must impose a condition on ρ, hence throughout the chapter we always assume that our measure ρ satisfies the condition

$$\int \exp\left(\frac{x^2}{2}\right) \mathrm{d}\rho(x) < \infty. \tag{3.1}$$

This excludes the case where ρ has a normal distribution for example, but is necessary to ensure we can apply the lemmas stated in this chapter.

3.1. Lemmas

The function G in (2.13) was not introduced without purpose, and the term $\cosh\left(s\sqrt{\beta}\right)$ appeared to be derived from (2.20). For this reason we will define the general form of G together with some properties in the following Lemma.

Lemma 3.1.1. *Suppose that ρ satisfies* (3.1) *and define $G : \mathbb{R} \to \mathbb{R}$ by*

$$G(s) = \frac{s^2}{2} - \ln \int \exp(sx) \mathrm{d}\rho(x). \tag{3.2}$$

Then G is real analytic and $G(s) \to \infty$ as $|s| \to \infty$. Thus, G has only a finite number of global minima. Also,

$$\int \exp(-nG(s)) \mathrm{d}s < \infty \textit{ for any } n \in \{1, 2, ...\}. \tag{3.3}$$

Proof. We will prove that $G(s) \to \infty$ as $|s| \to \infty$ by showing that the term $s^2/2$ goes much faster to infinity than $\ln \int \exp(sx) \mathrm{d}\rho(x)$. For the integral we have for some $L > 0$ and $s \in \mathbb{R}$

$$\begin{aligned} \int \exp(sx) \mathrm{d}\rho(x) &\leq \int \exp(|sx|) \mathrm{d}\rho(x) \\ &\leq \int_{|x| \leq L} \exp(|sx|) \mathrm{d}\rho(x) + \int_{|x| > L} \exp(|sx|) \mathrm{d}\rho(x) \\ &\leq \rho([-L, L]) \exp(L|s|) + \int_{|x| > L} \exp(|sx|) \mathrm{d}\rho(x), \end{aligned}$$

and if we use the identity

$$\frac{1}{2}(|s| - |x|)^2 = \frac{s^2}{2} - |sx| + \frac{x^2}{2} \geq 0, \tag{3.4}$$

and $\rho([-L,L]) \leq 1$ we obtain

$$\int \exp(sx)\mathrm{d}\rho(x) \overset{(3.4)}{\leq} \exp(L|s|) + \int_{|x|>L} \exp\left(\frac{s^2}{2} + \frac{x^2}{2}\right)\mathrm{d}\rho(x).$$

Since we can let $L = \sqrt{|s|}$ as we're letting $|s| \to \infty$, we find that

$$\begin{aligned}\int \exp(sx)\mathrm{d}\rho(x) &\leq \exp\left(|s|^{3/2}\right) + \exp(s^2/2)\int_{|x|>\sqrt{|s|}} \exp(x^2/2)\mathrm{d}\rho(x) \\ &= o\big(\exp(s^2/2)\big), \quad |s| \to \infty,\end{aligned} \tag{3.5}$$

because of the assumption (3.1) on ρ. Hence we obtain

$$\begin{aligned}\lim_{|s|\to\infty} G(s) &= \lim_{|s|\to\infty} \frac{s^2}{2} - \ln \int \exp(sx)\mathrm{d}\rho(x) \\ &= \lim_{|s|\to\infty} \ln \exp\left(\frac{s^2}{2}\right) - \ln \int \exp(sx)\mathrm{d}\rho(x) \\ &= \lim_{|s|\to\infty} -\ln\left(\frac{\int \exp(sx)\mathrm{d}\rho(x)}{\exp(s^2/2)}\right) \overset{(3.5)}{=} \infty.\end{aligned}$$

In order to prove (3.3) we start with $n = 1$, and use $a = \exp(\ln a)$ for $a > 0$ together the same expansion as in (2.17) (but we take $n = 1$). We also use Tonelli-Fubini's Theorem before we apply the expansion, this will help us to separate the terms only dependent on x. This gives

$$\begin{aligned}\int e^{-G(s)}\mathrm{d}s &= e^{-s^2/2+\ln\int \exp(sx)\mathrm{d}\rho(x)}\mathrm{d}s = \int e^{-s^2/2}\int e^{sx}\mathrm{d}\rho(x)\mathrm{d}s \\ &= \int\int e^{-s^2/2+sx}\mathrm{d}s\mathrm{d}\rho(x) \overset{(2.17)}{=} \int\int e^{-(s-x)^2/2}\mathrm{d}s e^{x^2/2}\mathrm{d}\rho(x).\end{aligned} \tag{3.6}$$

We now have the term $e^{-(s-x)^2/2}$ inside as the integrand with respect to $\mathrm{d}s$, which is the density of a normal distribution with mean x without the normalization constant. Since we assumed (3.1) we obtain from (3.6)

$$\int\int e^{-(s-x)^2/2}\mathrm{d}s e^{x^2/2}\mathrm{d}\rho(x) = \int \sqrt{2\pi}e^{x^2/2}\mathrm{d}\rho(x) \overset{(3.1)}{<} \infty. \tag{3.7}$$

Now for $n \geq 2$, we want to bound the integral in (3.3) by using that it converges for $n = 1$. We know that G tends to infinity and can therefore find an $M > 0$ such that $G(s) > 0$ for all $|s| > M$. This gives the following upperbound

$$\int_{|x|>M} e^{-nG(s)}\mathrm{d}s = \int_{|x|>M} \left(e^{-G(s)}\right)^n\mathrm{d}s \leq \int_{|x|>M} e^{-G(s)}\mathrm{d}s \overset{(3.7)}{<} \infty. \tag{3.8}$$

We also know that G is continuous because it is analytic, hence G attains a global minimum on the compact set $[-M,M]$. Thus for $|s| \leq M$ we can also find an upperbound on this domain

$$\begin{aligned}\int_{|x|\leq M} e^{-nG(s)}\mathrm{d}s &\leq \int_{|x|\leq M} \exp(-n\min\{G(s) : s \leq M\})\mathrm{d}s \\ &\leq 2M\exp(-n\min\{G(s) : s \leq M\}) < \infty.\end{aligned} \tag{3.9}$$

Now (3.8) and (3.9) give us

$$\int e^{-nG(s)}\mathrm{d}s = \int_{|x|\leq M} e^{-nG(s)}\mathrm{d}s + \int_{|x|>M} e^{-nG(s)}\mathrm{d}s < \infty$$

as desired. □

In Section 2.1 we proved several claims about the distribution of S_n, which will appear to be the foundation of the proofs in this section. Especially Lemma 2.1.1, which does not assume anything on ρ and makes it applicable in any case. Moreover, Claim 2.1.3 showed the importance of the function G which we defined in a more general way in Lemma 3.1.1. We will state and prove the following lemma which is a generalization of the claim. Because of this the proof will be analogous to that of Claim 2.1.3 and we will often refer to it.

Lemma 3.1.2. *Suppose $W \sim N(0,1)$ is independent of S_n for all $n \geq 1$. Then given γ and m real,*

$$\frac{W}{n^{\frac{1}{2}-\gamma}} + \frac{S_n - nm}{n^{1-\gamma}} \sim \frac{\exp(-nG(s/n^\gamma + m))\mathrm{d}s}{\int \exp(-nG(s/n^\gamma + m))\mathrm{d}s} \tag{3.10}$$

Proof. Because of the the term nm in (3.10) we will define the interval I such that we arrive at the same expression as (2.14), but now $I := \left(-\infty, n^{1-\gamma}\theta + nm\right]$ and we obtain

$$\mathbb{P}\left(\frac{W}{n^{\frac{1}{2}-\gamma}} + \frac{S_n - nm}{n^{1-\gamma}} \leq \theta\right) = \mathbb{P}\left(\sqrt{n}W + S_n \in I\right). \tag{3.11}$$

Because the distribution of S_n is known by Lemma 2.1.1, equations (2.15)-(2.18) follow by the same reasoning. Hence we can continue (3.11) to arrive at

$$\mathbb{P}\left(\sqrt{n}W + S_n \in I\right) = \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho^{*n}(x)\mathrm{d}s. \tag{3.12}$$

By (2.19) and $a^n = \exp(n \ln a)$ for any $a > 0$, we have for an random variable Y with distribution ρ

$$\int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho^{*n}(x) = \mathbb{E}\left[\exp\left(\frac{sY}{n}\right)\right]^n = \exp\left(n \ln \int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho(x)\right).$$

Hence similar to (2.21), we have that (3.12) becomes

$$\begin{aligned}&\frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho^{*n}(x)\mathrm{d}s \\ &= \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\exp\left(n \ln \int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho(x)\right)\mathrm{d}s.\end{aligned} \tag{3.13}$$

From here we can again follow the proof of Claim 2.1.3 and just as in (2.22) we obtain

$$\begin{aligned}-\frac{s^2}{2n} + n \ln \int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho(x) &= -n\left(\frac{(s/n)^2}{2} - \ln \int \exp\left(\frac{s}{n}x\right)\mathrm{d}\rho(x)\right) \\ &= -nG\left(\frac{s}{n}\right),\end{aligned} \tag{3.14}$$

where G is given by (3.2). Now (3.13) leads to

$$\begin{aligned}&\frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_I \exp\left(-\frac{s^2}{2n}\right)\exp\left(n \ln \int \exp\left(\frac{sx}{n}\right)\mathrm{d}\rho(x)\right)\mathrm{d}s \\ &\overset{(3.14)}{=} \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_{-\infty}^{n^{1-\gamma}\theta + nm} \exp\left(-nG\left(\frac{s}{n}\right)\right)\mathrm{d}s,\end{aligned} \tag{3.15}$$

where we want the upperbound to depend on θ only. This can be obtained by means substitution of $t = sn^{\gamma-1} - mn^\gamma$ and gives

$$\begin{aligned}&= \frac{1}{Z_n}\frac{1}{\sqrt{2\pi n}}\int_{-\infty}^{n^{1-\gamma}\theta + nm} \exp\left(-nG\left(\frac{s}{n}\right)\right)\mathrm{d}s \\ &\overset{t = sn^{\gamma-1} - mn^\gamma}{=} \frac{1}{Z_n}\frac{n^{1-\gamma}}{\sqrt{2\pi n}}\int_{-\infty}^{\theta} \exp\left(-nG\left(\frac{t}{n^\gamma} + m\right)\right)\mathrm{d}t.\end{aligned} \tag{3.16}$$

Now the normalization constant can be obtained using the same technique as in (2.24) and leads to (3.10) as required. □

With the lemmas stated in this section up to this point, we are able to prove claims about the limit behaviour of random variables with a distribution given by (2.6) for an arbitrary ρ which satisfies (3.1). Observe that in the previous chapter we obtained that the limit behaviour was determined by the value of $G^{(2k)}(0)$ for some positive integer k, and that we only had reasonable results whenever $G^{(2k)}(0) > 0$. We also need this to be true in the same sense for the rest of the chapter, which is why we will introduce the following definition for G in (3.2).

Definition 3.1.1. Given a positive integer α, we write

$$\rho \sim (m_1, k_1; \dots; m_\alpha, k_\alpha)$$

if the set of global minima of G is $\{m_1, \dots, m_\alpha\}$ where the m_i's are distinct and the k_i's are positive integers, and for each $i = 1, \dots, \alpha$

$$G(s) = G(m_i) + \lambda_i \frac{(s - m_i)^{2k_i}}{(2k_i)!} + \mathcal{O}\big((s - m_i)^{2k_i+1}\big) \quad \text{as } s \to m_i, \tag{3.17}$$

where λ_i is a positive number. We call $k(m_i) = k_i$ the type and $\lambda(m_i) = \lambda_i$ the strength and the maximal type is defined as the largest of the k_i's.

In the case of $\beta \in (0, 1)$ this led to the expression

$$G(s) = \frac{1 - \beta}{2} s^2 + \mathcal{O}\big(s^4\big),$$

thus we have that the minimum at zero is of type $k = 1$ and strength $\lambda = 1 - \beta$. For $\beta = 1$ we obtained

$$G(s) = \frac{1}{12} s^4 + \mathcal{O}\big(s^6\big),$$

which means that $k = 2$ and $\lambda = 2$.

Because we want to show weak converge of a sequence $(X_n)_{n\geq 1}$ to X in $\mathbb{R}$, for which we will use Theorem 18.1 from [2] again, we have to prove that

$$\lim_{n\to\infty} \mathbb{E}[h(X_n)] = \mathbb{E}[h(X)], \tag{3.18}$$

for any continuous, bounded function h. Since these random variables are characterized by the density function given in the right-hand side of (3.10), the Dominated Convergence Theorem is a convenient tool to prove this limit. We will apply dominated convergence in the following way, if we want to determine the left-hand side of (3.18) where the X_n's distribution is given by (3.10), then we can choose a set V which excludes the global minima. This can be done by taking small intervals of radius $\delta > 0$ around the m_i's out of $\mathbb{R}$. We will then apply Lemma 3.1.3 to show that the integral over V goes to 0. We have already applied Lemma 3.1.4 at this point since δ is determined by this lemma. Moreover, the lemma provides for the use of dominated convergence on $\mathbb{R} \setminus V$, as it gives expressions for pointwise convergence and an integrable upperbound.

Lemma 3.1.3. *Assume that* $\rho \sim (m_1, k_1; \dots; m_\alpha, k_\alpha)$ *satisfies* (3.1). *We define*

$$f = \min\{G(s) | s \in \mathbb{R}\}$$

and let V *be any closed subset of* $\mathbb{R}$ *which contains no global minima of* G. *Then there exists* $\varepsilon > 0$ *such that*

$$e^{nf} \int_V e^{-nG(s)} \mathrm{d}s = \mathcal{O}(e^{-n\varepsilon}), \quad n \to \infty. \tag{3.19}$$

Proof. We will prove (3.19) by showing that on a closed set V without global minima, there exists a distance $\varepsilon > 0$ such that for every $S \in V$ we have $G(s) - f > \varepsilon$. Suppose that for $m \in \mathbb{R}$ we have $G(m) = f$, then for every $s \in V$ we must have $G(s) - G(m) > 0$. If this was not true then either V has to be open or $G(s) = f$ for some $s \in V$ and V contains a global minimum. Hence as there are only finitely many global minima we can find an $\varepsilon > 0$ such that $G(s) - f > \varepsilon$. Now let $\varepsilon > 0$ such that $G(s) > f + \varepsilon$ on V and $n \geq 1$, we find for every $s \in V$ that

$$e^{n(f-G(s))} = e^{nf-(n-1)G(s)-G(s)} < e^{nf-(n-1)(f+\varepsilon)} e^{-G(s)} = e^{-n\varepsilon} e^{f+\varepsilon} e^{-G(s)}, \tag{3.20}$$

where $e^{-n\varepsilon}$ is now the only term dependent on n. Since the left-hand side of this equation is the integrand we want to bound and we know for the upperbound that

$$\int_V e^{-G(s)} \mathrm{d}s \leq \int e^{-G(s)} \mathrm{d}s \overset{(3.3)}{<} \infty. \tag{3.21}$$

Hence (3.19) follows from

$$e^{nf}\int_V e^{-nG(s)}\mathrm{d}s \overset{(3.20)}{<} e^{-n\varepsilon}e^{f+\varepsilon}\int_V e^{-G(s)}\mathrm{d}s$$
$$< e^{-n\varepsilon}\left[e^{f+\varepsilon}\int_V e^{-G(s)}\mathrm{d}s\right] \overset{(3.21)}{=} \mathcal{O}(e^{-n\varepsilon}), \quad n\to\infty.$$

□

Lemma 3.1.4. *Define*

$$B(s;m) = G(s+m) - G(m).$$

Then for each $i = 1, \dots, \alpha$ *if we say* $k = k_i$ *and* $\lambda = \lambda_i$, *there exists* $\delta > 0$ *sufficiently small so that as* $n \to \infty$

$$nB\left(\frac{s}{n^{\frac{1}{2k}}}; m_i\right) \to \frac{\lambda}{(2k)!}s^{2k}, \tag{3.22}$$

and

$$nB\left(\frac{s}{n^{\frac{1}{2k}}}; m_i\right) \geq \frac{1}{2}\frac{\lambda}{(2k)!}s^{2k} \quad \textit{for } |s| < \delta n^{\frac{1}{2k}}. \tag{3.23}$$

Proof. In order to prove (3.22) and (3.23) we first clarify the following; since G is analytic we have that if $a \in \mathbb{R}$ then

$$G(s) = G(a) + \sum_{j=1}^{\infty} G^{(j)}(a)\frac{(s-a)^j}{j!}.$$

Because of the assumption of ρ we have $G^{(1)}(m_i) = \cdots = G^{(2k_i-1)}(m_i) = 0$ for every global minimum m_i of type $k = k_i$, where the strength is given by $\lambda = \lambda_i = G^{(2k_i)}(m_i) > 0$. Hence if we write $G(s+m_i)$ using the expansion around m_i we obtain

$$B(s;m_i) = G(s+m_i) - G(m_i) = \frac{\lambda}{(2k)!}s^{2k} + \sum_{j=2k+1}^{\infty} G^{(j)}(m_i)\frac{s^j}{j!}. \tag{3.24}$$

Because every term in the summation has an exponent $j \geq 2k+1 \geq 2k$, it will go to zero when we substitute $\frac{s}{n^{\frac{1}{2k}}}$ for s, first we obtain from (3.24)

$$B\left(\frac{s}{n^{\frac{1}{2k}}}; m_i\right) = \lambda\frac{s^{2k}}{n(2k)!} + \sum_{j=2k+1}^{\infty} G^{(j)}(m_i)\frac{s^j}{n^{j/2k}j!}. \tag{3.25}$$

Hence for $j \geq 2k+1$ we then have

$$\frac{1}{n^{j/2k}} \geq \frac{1}{n^{(2k+1)/2k}} = \frac{1}{n}\frac{1}{n^{1/2k}},$$

which we can combine with (3.25) such that (3.22) follows from

$$nB\left(\frac{s}{n^{\frac{1}{2k}}}; m_i\right) = n\left(\lambda\frac{s^{2k}}{n(2k)!} + \mathcal{O}\left(\frac{s^{2k+1}}{n^{1+1/2k}}\right)\right) = \frac{\lambda}{(2k)!}s^{2k} + \mathcal{O}\left(\frac{s^{2k+1}}{n^{1/2k}}\right), \tag{3.26}$$

as $n \to \infty$. Now for (3.23) we need to bound $B(s, m_i)$ from below. This can be done using the inequality

$$\begin{aligned} B(s;m_i) &= \frac{\lambda}{(2k)!}s^{2k} - \left[\frac{\lambda}{(2k)!}s^{2k} - B(s;m_i)\right] \\ &\geq \frac{\lambda}{(2k)!}s^{2k} - \left|B(s;m_i) - \frac{\lambda}{(2k)!}s^{2k}\right|, \end{aligned} \tag{3.27}$$

where we know that

$$\left|B(s;m_i) - \frac{\lambda}{(2k)!}s^{2k}\right| = \left|\sum_{j=2k+1}^{\infty} G^{(j)}(m_i)\frac{s^j}{j!}\right| = \mathcal{O}\left(|s|^{2k+1}\right), \quad s \to 0.$$

Hence we can find an $\delta > 0$ such that

$$\left|B(s;m_i) - \frac{\lambda}{(2k)!}s^{2k}\right| \leq \frac{1}{2}\frac{\lambda}{(2k)!}s^{2k}, \quad |s| < \delta, \tag{3.28}$$

and with (3.27) arrive at

$$B(s;m_i) \overset{(3.28)}{\geq} \frac{1}{2}\frac{\lambda}{(2k)!}s^{2k}, \quad |s| < \delta.$$

At last we obtain (3.23) from substituting $\frac{s}{n^{\frac{1}{2k}}}$ for s which gives

$$nB\left(\frac{s}{n^{\frac{1}{2k}}};m_i\right) \geq n\frac{1}{2}\lambda\frac{s^{2k}}{n(2k)!} = \frac{1}{2}\frac{\lambda}{(2k)!}s^{2k}, \quad |s| < \delta n^{\frac{1}{2k}},$$

and concludes the proof of the lemma. □

3.2. Law of Large Numbers

In the case $\beta \in (0,1]$ of the Curie-Weiss model, we showed that (2.27) holds. But for $\beta > 1$ we could not prove such a claim, which was implied by the fact that G had two distinct global minima $\pm m$ around zero which is illustrated in Figure 2.2. We claimed that we will therefore have that S_n/n converges to those two points and as G is an even function, they should have the same probability mass in the limit. The following theorem states that for an arbitrary ρ which agrees with Definition 3.1.1, we will have a linear combination of Dirac measures in the limit, which is exactly what we described for the Curie-Weiss model when $\beta > 1$.

Theorem 3.2.1. *Assume that $\rho \sim (m_1, k_1; \dots; m_\alpha, k_\alpha)$, then*

$$\frac{S_n}{n} \to \frac{\sum_{i=1}^{\alpha} b_i \delta(s - m_i)}{\sum_{i=1}^{\alpha} b_i}, \tag{3.29}$$

where

$$b_i = \begin{cases} [\lambda(m_i)]^{-\frac{1}{2k_i}}, & \text{if } k_i \text{ is maximal.} \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

Proof. By Lemmas 2.1.2 and 3.1.2 with $\gamma = 0$ and by definition of the weak convergence of measures we have to show that

$$\frac{\int e^{-nG(s)}h(s)\mathrm{d}s}{\int e^{-nG(s)}\mathrm{d}s} \to \frac{\sum_{i=1}^{\alpha} h(m_i)b_i}{\sum_{i=1}^{\alpha} b_i}, \tag{3.31}$$

for each bounded continuous function h. Because we have $W/\sqrt{n} \to \delta(s) = \nu$ and for some measure μ that

$$\nu * \mu(A) = \int \mathbb{1}_A(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y) = \int \mathbb{1}_A(x)\mathrm{d}\mu(x) = \mu(A), \tag{3.32}$$

we can apply Lemma 2.1.2 in the following way; if $W/\sqrt{n} + S_n/n \to \mu$, then we must have $S_n/n \to \mu$. Now for the left-hand side of (3.31), we want to split the integrals into two integrals over different domains, where one integral will have no contribution in the limit. Choose $\delta > 0$ such that (3.23) holds and that we have $\delta < \min\{|m_i - m_j| : 1 \leq i \neq j \leq \alpha\}$. With this condition we can define the closed set V by

$$V = \mathbb{R} \setminus \bigcup_{i=1}^{\alpha}(m_i - \delta, m_i + \delta),$$

which will contain no global minima of G. For the first integral which we will consider, we will put the factor e^{nf} in front such that we can apply (3.19). Since we have that h is bounded we can indeed apply (3.19) to

$$\begin{aligned} e^{nf}\int_V e^{-nG(s)}h(s)\mathrm{d}s &\leq \sup\{|h(s)| : s\in\mathbb{R}\}e^{nf}\int_V e^{-nG(s)}\mathrm{d}s \\ &= \mathcal{O}(e^{-n\varepsilon}),\quad n\to\infty, \end{aligned} \tag{3.33}$$

for some $\varepsilon > 0$. The factor e^{nf} in front of the integral is also necessary for the integral over the rest of the domain $\mathbb{R}\setminus V$, which contains the global minima of G. We will also put a factor $n^{1/2k}$ in front of the following integral, this will have no influence on the convergence of (3.33) as it is of order $\mathcal{O}(e^{-n\varepsilon})$. For each $i = 1,\dots,\alpha$ we have that m_i is a global minimum of G thus $f = G(m_i)$. Then if $k = k(m_i)$ and $\lambda = \lambda(m_i)$ and we substitute $s + m_i$ for s we obtain

$$\begin{aligned} n^{\frac{1}{2k}}e^{nf}\int_{m_i-\delta}^{m_i+\delta} e^{-nG(s)}h(s)\mathrm{d}s &= n^{\frac{1}{2k}}\int_{-\delta}^{\delta} e^{-nG(s+m_i)}e^{nf}h(s+m_i)\mathrm{d}s \\ &\overset{f=G(m_i)}{=} n^{\frac{1}{2k}}\int_{-\delta}^{\delta}\exp(-n(G(s+m_i)-G(m_i)))h(s+m_i)\mathrm{d}s. \end{aligned} \tag{3.34}$$

We now recognize $nB(s;m_i)$ in the last equality and together with the substitution of $s/n^{1/2k}$ for s we obtain

$$\begin{aligned} &n^{\frac{1}{2k}}\int_{-\delta}^{\delta}\exp(-nB(s;m_i))h(s+m_i)\mathrm{d}s \\ &\quad= \int_{|s|<\delta n^{\frac{1}{2k}}}\exp\left(-nB\left(\frac{s}{n^{\frac{1}{2k}}};m_i\right)\right)h\left(\frac{s}{n^{\frac{1}{2k}}}+m_i\right)\mathrm{d}s, \end{aligned} \tag{3.35}$$

where $nB\left(\frac{s}{n^{\frac{1}{2k}}};m_i\right)$ is in the form of (3.22) and (3.23). Because we chose $\delta > 0$ as in Lemma 3.1.4 we obtain an integrable function as upperbound

$$\begin{aligned} &\exp\left(-nB\left(\frac{s}{n^{\frac{1}{2k}}};m_i\right)\right)h\left(\frac{s}{n^{\frac{1}{2k}}}+m_i\right) \\ &\quad\leq \exp\left(-\frac{1}{2}\frac{\lambda}{(2k)!}s^{2k}\right)\sup\{|h(s)| : s\in\mathbb{R}\},\quad \text{for } |s|<\delta n^{\frac{1}{2k}}, \end{aligned}$$

since $k \geq 1$. Now by (3.22) we have pointwise convergence for

$$\begin{aligned} &\exp\left(-nB\left(\frac{s}{n^{\frac{1}{2k}}};m_i\right)\right)h\left(\frac{s}{n^{\frac{1}{2k}}}+m_i\right)\mathbb{1}_{|s|<\delta n^{\frac{1}{2k}}} \\ &\quad\to \exp\left(-\frac{\lambda}{(2k)!}s^{2k}\right)h(m_i)\quad \text{as } n\to\infty. \end{aligned}$$

Hence by the Dominated Convergence Theorem we know that the limit of the integral in (3.34) is

$$\begin{aligned} n^{\frac{1}{2k}}e^{nf}\int_{m_i-\delta}^{m_i+\delta} e^{-nG(s)}h(s)\mathrm{d}s &\to \int \exp\left(-\frac{\lambda}{(2k)!}s^{2k}\right)h(m_i)\mathrm{d}s\quad \text{as } n\to\infty, \\ &\to h(m_i)\lambda^{-1}\int\exp\left(-\frac{s^{2k}}{(2k)!}\right)\mathrm{d}s\quad \text{as } n\to\infty. \end{aligned}$$

Now we can apply (3.33) and (3.35) to the numerator of (3.31) and obtain

$$\begin{aligned} &n^{\frac{1}{2k}} e^{nf} \int e^{-nG(s)} h(s) \mathrm{d}s \\ &\quad = n^{\frac{1}{2k}} e^{nf} \int_V e^{-nG(s)} h(s) \mathrm{d}s + \sum_{i=1}^{\alpha} n^{\frac{1}{2k}} e^{nf} \int_{m_i-\delta}^{m_i+\delta} e^{-nG(s)} h(s) \mathrm{d}s \\ &\quad \to \mathcal{O}(n^{\frac{1}{2k}} e^{-n\varepsilon}) + \sum_{i=1}^{\alpha} h(m_i) b_i \int \exp\left(-\frac{s^{2k}}{(2k)!}\right) \mathrm{d}s \quad \text{as } n \to \infty, \end{aligned} \tag{3.36}$$

where b_i is only nonzero if k_i is of maximal type. This is because if one substitutes a k larger than the type k_i of m_i in (3.26), $nB\left(\frac{s}{n^{\frac{1}{2k}}}; m_i\right) = \mathcal{O}(n^{\alpha})$ as $n \to \infty$ for some $\alpha > 0$, hence the integral will not contribute in that case. At last we have to choose $h \equiv 1$ in (3.36) such that the denominator gives

$$\begin{aligned} \frac{\int e^{-nG(s)} h(s) \mathrm{d}s}{\int e^{-nG(s)} \mathrm{d}s} &= \frac{n^{\frac{1}{2k}} e^{nf} \int e^{-nG(s)} h(s) \mathrm{d}s}{n^{\frac{1}{2k}} e^{nf} \int e^{-nG(s)} \mathrm{d}s} \\ \to \frac{\sum_{i=1}^{\alpha} h(m_i) b_i}{\sum_{i=1}^{\alpha} b_i} \frac{\int \exp\left(-\frac{s^{2k}}{(2k)!}\right) \mathrm{d}s}{\int \exp\left(-\frac{s^{2k}}{(2k)!}\right) \mathrm{d}s} &= \frac{\sum_{i=1}^{\alpha} h(m_i) b_i}{\sum_{i=1}^{\alpha} b_i} \quad \text{as } n \to \infty, \end{aligned}$$

with (3.29) and (3.30) as desired. □

3.3. Central Limit Theorem

When we have that $\rho \sim (m, k)$, G has an unique global minimum of type k where Theorem 3.2.1 implies that $S_n/n \to \delta(s - m)$. Because we now have convergence to a single point in distribution, we might have a result analogous with the Central Limit Theorem. The following theorem states whenever this is the case.

Theorem 3.3.1. *Assume that $\rho \sim (m, k)$ is nondegenerate and satisfies* (3.1) *and m has strength $\lambda = \lambda(m)$. Then*

$$\frac{S_n - nm}{n^{1-\frac{1}{2k}}} \to \begin{cases} N(0, \lambda^{-1} - 1), & \text{for } k = 1, \\ c \exp\left(-\frac{\lambda}{(2k)!} s^{2k}\right), & \text{for } k \geq 2, \end{cases} \tag{3.37}$$

where c is a normalization constant and $\lambda^{-1} - 1 > 0$ for $k = 1$.

Proof. We will prove this in a similar way as we proved Theorem 3.2.1, but now we need to show by Lemmas 2.1.2 and 3.1.2 with $\gamma = \frac{1}{2k}$ that

$$\frac{\int \exp\left(-nG(s/n^{1/2k} + m)\right) h(s) \mathrm{d}s}{\int \exp\left(-nG(s/n^{1/2k} + m)\right) \mathrm{d}s} \to \frac{\int \exp\left(-\frac{\lambda}{(2k)!} s^{2k}\right) h(s) \mathrm{d}s}{\int \exp\left(-\frac{\lambda}{(2k)!} s^{2k}\right) \mathrm{d}s}, \tag{3.38}$$

for any bounded, continuous function h. If $k \geq 2$, then using the same reasoning as in (3.32), we can immediately conclude (3.37). For $k = 1$ we will obtain that

$$W + \frac{S_n}{\sqrt{n}} \to N(0, \lambda^{-1}),$$

but since $W \sim N(0, 1)$ is chosen independently of S_n, Lemma 2.1.2 and Theorem 16.3 from [2] imply that (3.37) follows. Now that we have an unique global maximum we will define the closed set V as follows

$$V = \mathbb{R} \setminus (m - \delta, m + \delta),$$

with $\delta > 0$ from Lemma 3.1.4. Thus we obtain by substituting for $s/n^{1/2k} + m$ for s and using that h is bounded

$$\begin{aligned} &\int_{|s|\geq\delta n^{1/2k}} \exp\Big(-nG\Big(\frac{s}{n^{1/2k}} + m\Big)\Big)h(s)\mathrm{d}s \\ &\qquad = n^{1/2k}\int_V \exp(-nG(s))h\Big(\frac{s}{n^{1/2k}} + m\Big)\mathrm{d}s \\ &\qquad \leq \sup\{|h(s)| : s \in \mathbb{R}\}n^{1/2k}\int_V \exp(-nG(s))\mathrm{d}s. \end{aligned} \tag{3.39}$$

We can now apply (3.19) to (3.39) such that for some $\varepsilon > 0$ we have

$$e^{nf}\int_{|s|\geq\delta n^{1/2k}} \exp\Big(-nG\Big(\frac{s}{n^{1/2k}} + m\Big)\Big)h(s)\mathrm{d}s = \mathcal{O}(n^{1/2k}e^{-n\varepsilon}), \quad n \to \infty. \tag{3.40}$$

On the rest of the domain $|s| < \delta n^{1/2k}$ we see that since $f = G(m)$ recognize $B\Big(\frac{s}{n^{1/2k}}; m\Big)$

$$\begin{aligned} &e^{nf}\int_{|s|<\delta n^{1/2k}} \exp\Big(-nG\Big(\frac{s}{n^{1/2k}} + m\Big)\Big)h(s)\mathrm{d}s \\ &\quad = \int_{|s|<\delta n^{1/2k}} \exp\Big(-n\Big(G\Big(\frac{s}{n^{1/2k}} + m\Big) - G(m)\Big)\Big)h(s)\mathrm{d}s \\ &\quad = \int_{|s|<\delta n^{1/2k}} \exp\Big(-nB\Big(\frac{s}{n^{1/2k}}; m\Big)\Big)h(s)\mathrm{d}s, \end{aligned}$$

where we can use dominated convergence since

$$\begin{aligned} &\exp\left(-nB\left(\frac{s}{n^{\frac{1}{2k}}}; m\right)\right)h(s) \\ &\qquad \leq \exp\left(-\frac{1}{2}\frac{\lambda}{(2k)!}s^{2k}\right)\sup\{|h(s)| : s \in \mathbb{R}\}, \quad \text{for } |s| < \delta n^{\frac{1}{2k}}, \end{aligned}$$

gives us an integrable upperbound by (3.23). And (3.22) implies that

$$\exp\left(-nB\left(\frac{s}{n^{\frac{1}{2k}}}; m\right)\right)h(s)\mathbb{1}_{|s|<\delta n^{\frac{1}{2k}}} \to \exp\left(-\frac{\lambda}{(2k)!}s^{2k}\right)h(s) \quad \text{as } n \to \infty.$$

Hence for the numerator and the denominator in (3.38) with $h \equiv 1$ we now have

$$\begin{aligned} &e^{nf}\int \exp\big(-nG(s/n^{1/2k} + m)\big)h(s)\mathrm{d}s \\ &\quad = e^{nf}\int_{|s|\geq\delta n^{1/2k}} \exp\big(-nG(s/n^{1/2k} + m)\big)h(s)\mathrm{d}s \\ &\qquad + e^{nf}\int_{|s|<\delta n^{1/2k}} \exp\big(-nG(s/n^{1/2k} + m)\big)h(s)\mathrm{d}s \\ &\quad \to \mathcal{O}(n^{1/2k}e^{-n\varepsilon}) + \int \exp\left(-\frac{\lambda}{(2k)!}s^{2k}\right)h(s)\mathrm{d}s \quad \text{as } n \to \infty, \end{aligned}$$

which concludes the following

$$\frac{W}{n^{\frac{1}{2}-\frac{1}{2k}}} + \frac{S_n - nm}{n^{1-\frac{1}{2k}}} \to \begin{cases} N(0, \lambda^{-1}), & \text{for } k = 1, \\ c\exp\Big(-\frac{\lambda}{(2k)!}s^{2k}\Big), & \text{for } k \geq 2. \end{cases} \tag{3.41}$$

And since we have that $\frac{W}{n^{\frac{1}{2}-\frac{1}{2k}}} \to \delta(s)$ for $k \geq 2$ we can apply the same reasoning from the proof of Theorem 3.2.1 and particularly (3.32) to conclude that

$$\frac{S_n - nm}{n^{1-\frac{1}{2k}}} \to c\exp\left(-\frac{\lambda}{(2k)!}s^{2k}\right) \quad \text{for } k \geq 2.$$

In Section 2.3 we applied Lemma 2.1.2 together with Theorem 16.3 from [2] to show that (2.35) follows from (2.34). We can use the same argument for the case $k = 1$ in (3.41) provided that $\lambda^{-1} - 1 > 0$. In order to show that $\lambda^{-1} - 1 > 0$ whenever $k = 1$, we will write $\lambda^{-1} - 1 = \frac{1-\lambda}{\lambda}$ and conclude that both the numerator and denominator are strictly positive. Let $\phi(s) = \int \exp(sx)\mathrm{d}\rho(x)$ such that the first and second derivatives of G are

$$\frac{\mathrm{d}G(s)}{\mathrm{d}s} = s - \frac{\phi'(s)}{\phi(s)},$$
$$\frac{\mathrm{d}^2G(s)}{\mathrm{d}s^2} = 1 - \frac{\phi(s)\phi''(s) - (\phi'(s))^2}{\phi^2(s)} \equiv 1 - \theta(s),$$

where the derivatives of ϕ are given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(s) = \int x\exp(sx)\mathrm{d}\rho(x),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\phi(s) = \int x^2\exp(sx)\mathrm{d}\rho(x).$$

We will rewrite θ such that we can gather the sum of three integrals into one and recognize the quadratic form $a^2 - 2ab + b^2 = (a-b)^2$, which will contribute to the argument that $\theta > 0$. We first write

$$\begin{aligned}\theta(s) &= \frac{\phi(s)\phi''(s) - (\phi'(s))^2}{\phi^2(s)} = \frac{\phi''(s)}{\phi(s)} - 2\frac{(\phi'(s))^2}{\phi^2(s)} + \frac{(\phi'(s))^2}{\phi^2(s)} \\ &= \frac{1}{\phi(s)}\left[\phi''(s) - 2\left(\frac{\phi'(s)}{\phi(s)}\right)\phi'(s) + \left(\frac{\phi'(s)}{\phi(s)}\right)^2\phi(s)\right],\end{aligned} \tag{3.42}$$

where the quadratic form becomes clear when we expand the following integrals

$$\begin{aligned}&\phi''(s) - 2\left(\frac{\phi'(s)}{\phi(s)}\right)\phi'(s) + \left(\frac{\phi'(s)}{\phi(s)}\right)^2\phi(s) \\ &= \int x^2e^{sx}\mathrm{d}\rho(x) - 2\left(\frac{\phi'(s)}{\phi(s)}\right)\int xe^{sx}\mathrm{d}\rho(x) + \left(\frac{\phi'(s)}{\phi(s)}\right)^2\int e^{sx}\mathrm{d}\rho(x) \\ &= \int\left[x^2 - 2x\left(\frac{\phi'(s)}{\phi(s)}\right) + \left(\frac{\phi'(s)}{\phi(s)}\right)^2\right]e^{sx}\mathrm{d}\rho(x) \\ &= \int\left[x - \left(\frac{\phi'(s)}{\phi(s)}\right)\right]^2e^{sx}\mathrm{d}\rho(x).\end{aligned}$$

Thus (3.42) becomes

$$\theta(s) = \frac{1}{\phi(s)}\int\left[x - \left(\frac{\phi'(s)}{\phi(s)}\right)\right]^2e^{sx}\mathrm{d}\rho(x), \tag{3.43}$$

which is strictly positive for all s. To see that we can't have $x - \left(\frac{\phi'(s)}{\phi(s)}\right) = 0$ for all $x \in \mathrm{supp}(\rho)$, we will derive a contradiction. Assume that we have $x - \left(\frac{\phi'(s)}{\phi(s)}\right) = 0$ for all $x \in \mathrm{supp}(\rho)$. Then for $x_1, x_2 \in \mathrm{supp}(\rho)$ we obtain

$$x_1 = \left(\frac{\phi'(s)}{\phi(s)}\right) = x_2,$$

which implies that $\mathrm{supp}(\rho) = \{x_1\}$ and is in contradiction with the assumption that ρ is nondegenerate. Hence $\theta(s) > 0$, and with the assumption that $G''(m) = \lambda > 0$ is the type $k = 1$, this leads to

$$\lambda^{-1} - 1 = \frac{1-\lambda}{\lambda} = \frac{1 - G''(m)}{G''(m)} = \frac{\theta(m)}{G''(m)} > 0.$$

Thus we can conclude (3.37) for all $k \geq 1$. □

3.4. Agreement of the Curie-Weiss model with the theorems

Let us briefly return to the Curie-Weiss model that was treated in Chapter 2 where ρ_β was given by (2.4). We observe that in the case of $\beta \in (0,1)$ we can use the the expansion given by (2.29) to express G_β as

$$G_\beta(s) = (1-\beta)\frac{s^2}{2} + \mathcal{O}(s^4) \quad \text{as } s \to 0,$$

with an unique global minimum at $s = 0$ and $G_\beta(0) = 0$. Now by Definition 3.1.1 we may write $\rho_\beta \sim (m = 0, k = 1)$ with strength $\lambda = 1 - \beta$. Theorem 3.2.1 applies here and gives

$$\frac{S_n}{n} \to \delta(s), \tag{3.44}$$

with $b = \lambda^{-1/2}$ being the only weight as there is one distinct global minimum. We can also apply Theorem 3.3.1 here and immediately come to the result

$$\frac{S_n}{\sqrt{n}} \to N(0, \lambda^{-1} - 1),$$

which is identical to (2.35). For $\beta = 1$ we can apply the same expansion of $G_{\beta=1}$ for s close to zero, but now we obtain

$$G_\beta(s) = \frac{s^4}{12} + \mathcal{O}(s^6).$$

By definition we have therefore $\rho_\beta \sim (m = 0, k = 2)$ with strength $\lambda = 2$, Theorem 3.2.1 now implies that we again have (3.44) as there still is a unique global minimum located at zero. Now that we have a minimum of type $k = 2$, we obtain a limiting distribution which is given by Theorem 3.3.1 as

$$\frac{S_n}{n^{3/4}} \to c\exp\left(-\frac{s^4}{12}\right),$$

and one easily verifies that this is the same result as in (2.36). For $\beta > 1$ we can only derive an implicit expression where the global minima of G_β occur. The first derivative of G_β is given by

$$\begin{aligned}\frac{\mathrm{d}G_\beta(s)}{\mathrm{d}s} &= s - \sqrt{\beta}\frac{\sinh\left(s\sqrt{\beta}\right)}{\cosh\left(s\sqrt{\beta}\right)} \\ &= s - \sqrt{\beta}\tanh\left(s\sqrt{\beta}\right),\end{aligned}$$

which implies that the global minima satisfy the condition

$$m = \sqrt{\beta}\tanh\left(m\sqrt{\beta}\right). \tag{3.45}$$

If $\beta > 1$ we indeed find that (3.45) has three solutions which is made visible in Figure 3.1. Note that apart from zero, the global minima occur at m and $-m$ for some $m \in \mathbb{R}$. This can be explained by the fact that G_β is an even function for all $\beta \geq 0$. Now as we have that $G''_\beta(s) = 1 - \theta(s)$ where θ is given by (3.42) we find

$$\frac{\mathrm{d}^2 G_\beta(s)}{\mathrm{d}s^2} = 1 - \beta\frac{\cosh^2(s\sqrt{\beta}) - \sinh^2(s\sqrt{\beta})}{\cosh^2(s\sqrt{\beta})} = 1 - \frac{\beta}{\cosh^2(s\sqrt{\beta})}. \tag{3.46}$$

If we want to use the limit theorems from this chapter we have to show that G_β has an expansion around m and $-m$ of the form in (3.17). Because G_β is even, the following argument is symmetric for m and $-m$, which is why we only focus on $m > 0$. We'll show that $G''_\beta(m) > 0$ for $\beta > 1$ using the following claim.

Claim 3.4.1. *Suppose that G_β is defined as in* (2.13) *and assume $\beta > 1$. Then for a global minimum $m > 0$ of G_β we have*

$$\frac{\mathrm{d}^2 G_\beta(m)}{\mathrm{d}s^2} > 0.$$

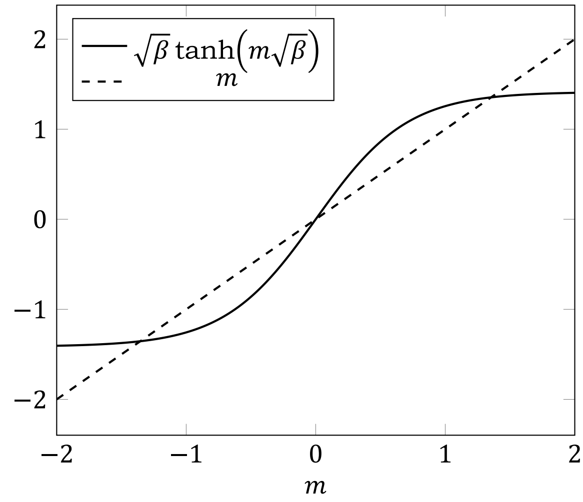

Figure 3.1: Minima of G when $\beta = 2$.

Proof. Because the second derivative of G_β is given by (3.46), we have to show that

$$\frac{\sqrt{\beta}}{\cosh\left(m\sqrt{\beta}\right)} < 1 \iff \sqrt{\beta}\tanh\left(m\sqrt{\beta}\right) < \sinh\left(m\sqrt{\beta}\right), \tag{3.47}$$

in order to prove the claim. We know that if m is a global minimum of G_β, it satisfies (3.45). And since the series of sinh is given by $\sinh(x) = \sum_{k\geq 0} \frac{x^{2k+1}}{(2k+1)!}$, we have an upperbound for $x > 0$

$$x < x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sinh(x). \tag{3.48}$$

This gives the following sequence of inequalities

$$\sqrt{\beta}\tanh\left(m\sqrt{\beta}\right) \overset{(3.45)}{=} m \overset{\beta>1}{<} m\sqrt{\beta} \overset{(3.48)}{<} \sinh\left(m\sqrt{\beta}\right),$$

proving (3.47) and hence the claim. □

We can now apply Claim 3.4.1 for the global minima satisfying (3.45) in the case of $\beta > 1$ to obtain the two expansions around $\pm m$

$$G(s) = G(m) + \lambda\frac{(s-m)^{2k}}{(2k)!} + \mathcal{O}\left((s-m)^{2k+1}\right) \quad \text{as } s \to m,$$
$$G(s) = G(-m) + \lambda\frac{(s+m)^{2k}}{(2k)!} + \mathcal{O}\left((s+m)^{2k+1}\right) \quad \text{as } s \to -m,$$

where $\lambda = G(m) = G(-m)$ and $\rho_\beta \sim (m, k = 1; -m, k = 1)$. Since we do not have a unique global minimum, we can not apply Theorem 3.3.1. But by Theorem 3.2.1 we get an interesting result, that is

$$\frac{S_n}{n} \to \frac{\lambda\delta(s-m) + \lambda\delta(s+m)}{2\lambda} = \frac{1}{2}[\delta(s-m) + \delta(s+m)].$$

Hence as $n \to \infty$, S_n/n takes one of the two values m or $-m$, both with probability $\frac{1}{2}$.

We can now conclude that there occurs a phase transition at $\beta = 1$, this is because for $\beta > 1$ we have that S_n/n converges to two different points with equal probability but as soon as $\beta \leq 1$ this convergence changes to a single point. For large values of β, the points m and $-m$ of the intersection in Figure 3.1 will tend to $\sqrt{\beta}$. And since we made the transformation $x_i = \sqrt{\beta}\sigma_i$ at the beginning of Chapter 2, the magnetization will tend to

$$m_n = \frac{1}{n}\sum_{i=1}^{n}\sigma_i = \frac{1}{\sqrt{\beta}}\frac{S_n}{n} \to \frac{1}{\sqrt{\beta}}\frac{1}{2}[\delta(s-m)+\delta(s+m)],$$

hence $\lim_{n\to\infty} m_n$ will be very close to either 1 or -1.

4

Extension of the Curie-Weiss model

4.1. Limit theorems

In this chapter we will consider the model described in (1.5), which had a probability measure given by

$$\mathbb{P}_{\beta h}(\{\sigma\}) = \frac{1}{Z_n}\exp\left(\frac{\beta}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j + h\sqrt{\beta}\sum_{i=1}^{n}\sigma_i\right).$$

In Chapter 3 we generalized the results found in Chapter 2, thus given an ρ which satisfies the conditions of Definition 3.1.1, we can immediately conclude the limit theorems (3.30) and (3.37). In order to work with these statements we have to put (1.5) in the form of (2.6). This can be done by changing the weights of the Dirac measures in (2.4), since we still have the dependence on β. If we use the following measure

$$\rho_{\beta,h} = \frac{e^h}{2\cosh h}\delta_{\sqrt{\beta}} + \frac{e^{-h}}{2\cosh h}\delta_{-\sqrt{\beta}}, \tag{4.1}$$

then (1.5) reduces to (2.6) where ρ is given by (4.1). Thus if we compute G from (3.2) we may be able to conclude one of the limit theorems. Given $\beta > 0$ and $h \in \mathbb{R}$ we have

$$\int e^{sx}\mathrm{d}\rho(x) = \frac{e^{s\sqrt{\beta}}e^h + e^{-s\sqrt{\beta}}e^{-h}}{2\cosh(h)} = \frac{\cosh\left(s\sqrt{\beta}+h\right)}{\cosh(h)},$$

Hence we define $G_{\beta,h} : \mathbb{R} \to \mathbb{R}$ as

$$G_{\beta,h}(s) = \frac{s^2}{2} - \ln\left(\frac{\cosh\left(s\sqrt{\beta}+h\right)}{\cosh(h)}\right),$$

which we know is analytic by Lemma 3.1.1. Similarly to the derivation of the derivatives and global minima of G_β in Section 3.4, we obtain that the first derivative is given by

$$\frac{\mathrm{d}G_{\beta,h}(s)}{\mathrm{d}s} = s - \sqrt{\beta}\tanh\left(s\sqrt{\beta}+h\right),$$

where a global minimum m of $G_{\beta,h}$ satisfies the condition

$$m = \sqrt{\beta}\tanh\left(m\sqrt{\beta}+h\right). \tag{4.2}$$

By the same reasoning we find that the second derivative of $G_{\beta,h}$ is given by

$$\frac{\mathrm{d}^2G_{\beta,h}(s)}{\mathrm{d}s^2} = 1 - \frac{\beta}{\cosh^2\left(s\sqrt{\beta}+h\right)}. \tag{4.3}$$

In Figure 4.1 we see that we have three points of intersection, but opposed to the case $h = 0$ which is illustrated in Figure 3.1, we have that the location of these points on the m axis is different. This is a result from the horizontal shift caused by h. If we increase $h > 0$ (or decrease for $h < 0$), then the graph is shifted up to a point where we only have one intersection m, which has the same sign as h. We will not observe this behaviour whenever $\beta \in (0,1]$, since the tangent of $\sqrt{\beta}\tanh\left(m\sqrt{\beta}+h\right)$ will not be steep enough such that the graph intersects at more then one point with m.

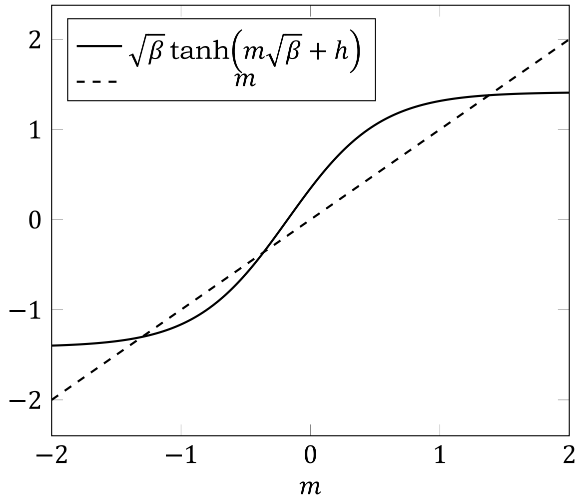

Figure 4.1: Minima of G when $\beta = 1.5$ and $h = \frac{1}{4}$.

We will make a proposition which states the limiting distributions of this model in the case where $h > 0$. One can derive the same results for $h < 0$ by changing signs. We already argued that we end up with one global minimum for a sufficiently large h. In the proposition we will show that this always is the case and are therefore able to apply Theorem 3.3.1.

Proposition 4.1.1. *Assume that $\rho_{\beta,h}$ is given by* (4.1) *and $\beta > 0$ and $h > 0$. If m is the largest solution to* (4.2)*, then*

$$\frac{S_n}{n} \to \delta(s-m), \tag{4.4}$$

$$\frac{S_n - nm}{\sqrt{n}} \to N(0,\sigma^2), \tag{4.5}$$

where $\sigma^2 = (G''_{\beta,h}(m))^{-1} - 1$.

Proof. If m is the largest solution to (4.2), then we have to show that it is a unique global minimum of $G_{\beta,h}$ before we can apply Theorem 3.3.1. Now let m_1 and m_2 be solutions to (4.2), where $m_2 \leq m_1$ and m_1 is the largest possible solution. So both are extreme points of $G_{\beta,h}$, but for m_1 to be a global minimum, we have to show that we indeed have $G_{\beta,h}(m_1) \leq G_{\beta,h}(m_2)$ and only have an equality if $m_1 = m_2$. To do this we define the function $g_\beta : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ as

$$g_\beta(s,h) = G_{\beta,h}(s).$$

The partial derivative with respect to h is given by

$$\begin{aligned}\frac{\partial g_\beta(s,h)}{\partial h} &= \frac{\partial}{\partial h}\left(\frac{s^2}{2} - \ln\left(\frac{\cosh\left(s\sqrt{\beta}+h\right)}{\cosh(h)}\right)\right)\\ &= \frac{\partial}{\partial h}\left(\ln(\cosh(h)) - \ln\left(\cosh\left(s\sqrt{\beta}+h\right)\right)\right)\\ &= \tanh(h) - \tanh\left(s\sqrt{\beta}+h\right),\end{aligned}$$

which implies that for any $h > 0$ we have for the solutions m_1 and m_2

$$\frac{\partial g_\beta(m_2,h)}{\partial h} = \tanh(h) - \tanh\left(m_2\sqrt{\beta} + h\right) = \tanh(h) - \frac{m_2}{\sqrt{\beta}}$$
$$\geq \tanh(h) - \frac{m_1}{\sqrt{\beta}} = \frac{\partial g_\beta(m_1,h)}{\partial h}.$$

Hence if m_1 is the largest solution to (4.2), then for any m_2 which is a solution too but not the largest, we have $G_{\beta,h}(m_1) < G_{\beta,h}(m_2)$ and we conclude that m_1 is an unique global minimum.

Because we claim that we have a normal distribution in the limit around m, we have to show that $G''_{\beta,h}(m) > 0$ such that we have $\rho_{\beta,h} \sim (m, k = 1)$ and can therefore apply Theorems 3.2.1 and 3.3.1. Similar to the proof of Claim 3.4.1, we are required to prove

$$\sqrt{\beta}\tanh\left(m\sqrt{\beta} + h\right) < \sinh\left(m\sqrt{\beta} + h\right),$$

where we refer to the expression for $G''_{\beta,h}$ in (4.3). For $\beta < 1$, this inequality holds since $\tanh(x) \leq \sinh(x)$ for all $x \geq 0$. For $\beta \geq 1$, we have

$$\sqrt{\beta}\tanh\left(m\sqrt{\beta} + h\right) \overset{(4.2)}{=} m \overset{\beta\geq 1}{<} m\sqrt{\beta} + h \overset{(3.48)}{<} \sinh\left(m\sqrt{\beta} + h\right),$$

hence whenever $h > 0$, we have that $G''_{\beta,h}(m) > 0$. To see that $\rho_{\beta,h} \sim (m, k = 1)$ with strength $\lambda(m) = G''_{\beta,h}(m)$, we give the expansion of $G_{\beta,h}$ around m

$$G_{\beta,h}(s) = G_{\beta,h}(m) + \frac{\mathrm{d}^2 G_{\beta,h}(m)}{\mathrm{d}s^2}\frac{(s-m)^2}{2!} + \mathcal{O}\left((s-m)^3\right) \quad \text{as } s \to m.$$

At last we can apply Theorems 3.2.1 and 3.3.1 to obtain (4.4) and (4.5). □

In the model that was treated in Chapter 2, there occurred a phase transition at $\beta = 1$ and we also observed that the limiting distribution was not normal at this point. When we try to find such a point in the current model, we will see that this is only possible in the case where $h = 0$, which is exactly the model of Chapter 2.

A limiting distribution which is not normal has to have a global minimum of type $k \geq 2$, which requires $G''_{\beta,h}(m) = 0$. Since we can find an expression for the second derivative in terms of the first

$$\begin{aligned} \frac{\mathrm{d}^2 G_{\beta,h}(s)}{\mathrm{d}s^2} &= 1 - \beta + \left(\sqrt{\beta}\tanh\left(s\sqrt{\beta} + h\right)\right)^2 \\ &= 1 - \beta + \left(s - \frac{\mathrm{d}G_{\beta,h}(s)}{\mathrm{d}s}\right)^2, \end{aligned} \tag{4.6}$$

it follows that the third derivative is given by

$$\frac{\mathrm{d}^3 G_{\beta,h}(s)}{\mathrm{d}s^3} = 2\left(s - \frac{\mathrm{d}G_{\beta,h}(s)}{\mathrm{d}s}\right)\left(1 - \frac{\mathrm{d}^2 G_{\beta,h}(s)}{\mathrm{d}s^2}\right).$$

We search for a minimum m of type $k \geq 2$, thus under the assumption $G'_{\beta,h}(m) = G''_{\beta,h}(m) = 0$ we obtain that

$$\frac{\mathrm{d}^3 G_{\beta,h}(m)}{\mathrm{d}s^3} = 2m.$$

Where $G^{(3)}_{\beta,h}(m)$ is only zero if $m = 0$, but from (4.6) we can conclude that as $m^2 = 1 - \beta$ we must have $\beta = 1$. Up to this point we have that if a global minimum m has type $k \geq 2$, then we must have $m = 0$. But if a global minimum occurs at $m = 0$, (4.2) implies that we must have $h = 0$. Hence this is the model we described in Chapter 2 for which we know that a phase transition happens at $\beta = 1$.

We conclude this section with the following results. Given a measure (4.1) for $\beta > 0$ and $h \in \mathbb{R} \setminus 0$, there exists a unique global minimum m which is of type $k(m) = 1$ and strength $\lambda(m) = 1 - \beta + m^2$. This results in the limiting distributions given by (4.4) and (4.5). Thus if a piece of metal is under the influence of a magnetic field, there appears to be no phase transition. But what we see is that when β approaches zero, G has a global minimum $m = 0$. When we increase β again, the minimum shifts to the right if $h > 0$ and vice-versa. This effect is illustrated in Figure 4.2, which can be compared to Figure 2.1 where we have the same values for β but $h = 0$. Note that for $\beta < 1$, the minimum forced to the right when $h > 0$. For $\beta > 1$ we see that the minimum with the same sign as h is favored over the other minimum we had in the case where $h = 0$. This is physically explained by the fact that the spins tend to align with the direction of the magnetic field.

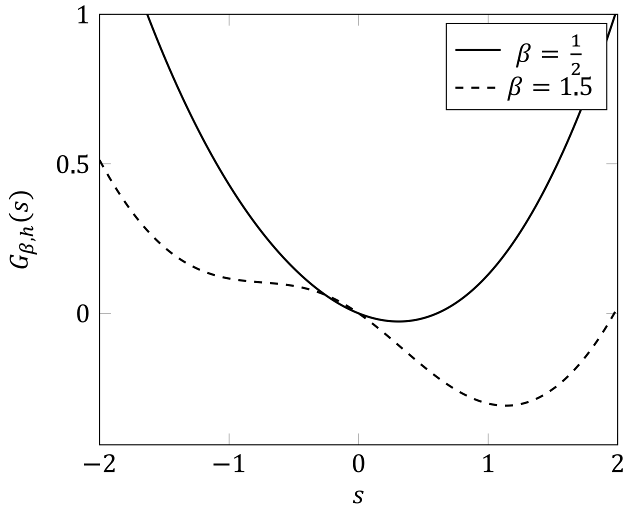

Figure 4.2: Plot of $G_{\beta,h}$ for $\beta = \frac{1}{2}$ and $\beta = 1.5$ and in both cases $h = \frac{1}{4}$.

4.2. Worked example of an extreme case

If a metal is under the influence of a very weak magnetic field and temperatures are very low, thus h is several orders of magnitude smaller than β, then we must have that (4.4) and (4.5) hold. We will consider a few cases where we take $\beta = 5 \cdot 10^2$ and h ranges from 10^{-5} to 10^{-10}. Since we have a global minimum m, and a local minimum at approximately $-m$, we might find that the values G attains at these local minima are also very close to each other. We have computed these values and they are given in Table 4.1. We see that $G_{\beta,h}(m_g)$ and $G_{\beta,h}(m_h)$ are equal up to a certain amount of decimal

h	m_g	$G_{\beta,h}(m_g)$	m_l	$G_{\beta,h}(m_l)$
10^{-5}	-22.36067977	-249.30686	22.36067978	-249.30684
10^{-7}	-22.36067977	-249.3068529	22.36067978	-249.3068527
10^{-10}	-22.36067977	-249.3068528195	22.36067978	-249.3068528193

Table 4.1: The global and local minima denoted by m_g and m_l respectively and the corresponding values $G_{\beta,h}$ attains for $\beta = 5 \cdot 10^2$ and different values of h.

points, and this amount increases as we let h becomes smaller. In the proof of Theorem 3.2.1 we used that there is a difference between $f = G(m_g)$ and $G(s)$ for any $s \in \mathbb{R}$ which is not a global minimum, such that we could conclude (3.33). If we can find a combination of β and h where this difference is of order 10^{-22}, then we have that (4.4) holds, but is not physically relevant as there are approximately $n = 10^{22}$ atoms in a gram of iron.

What we have shown is that under the influence of a magnetic field, we do not observe a phase transition since the atoms tend to align with the direction of the magnetic field. The limit in (4.4) agrees with this as the value of m increases monotonically with β. But whenever h is small enough, the convergence $S_n/n \to \delta(s - m)$ happens so slow in n that it is not physically relevant anymore.

5
Conclusion

We started this research by analyzing the limit behaviour of the Curie-Weiss model where a vector $\sigma = (\sigma_i)_{i=1,\dots,n}$ of spins has a joint distribution given by (1.2) and the distribution of the individual σ_i's is assumed to be given by the measure in (2.1). The case where $\beta = 0$ is ignored throughout the thesis up to Chapter 4 since we have independence and the know limit theorems in (1.3) and (1.4) hold. In Chapter 2 we concluded that for $\beta \in (0,1]$ we had the following limiting distributions

$$\frac{S_n}{n} \to \delta(s), \quad \text{if } \beta \leq 1, \tag{5.1}$$

$$\frac{S_n}{\sqrt{n}} \to N(0,(1-\beta)^{-1}-1), \quad \text{if } \beta \in (0,1), \tag{5.2}$$

$$\frac{S_n}{n^{1/4}} \to c\exp\left(-\frac{s^4}{12}\right), \quad \text{if } \beta = 1. \tag{5.3}$$

In Chapter 3 we generalized these results with respect to the measure ρ. After proving multiple lemmas about the properties of ρ and the function G defined in (3.2), we stated the main results in Theorems 3.2.1 and 3.3.1. We were able to extend the results (5.1)-(5.3) by the following statement

$$\frac{S_n}{n} \to \frac{1}{2}[\delta(s-m)+\delta(s+m)], \quad \text{if } \beta > 1, \tag{5.4}$$

where m is one of two global minima of G for $\beta > 1$. We concluded that there is a phase transition happening at $\beta = 1$, Since β is a constant proportional to inverse temperature, there must exists a temperature T_c which gives $\beta = 1$. Hence for temperatures $T < T_c$ there occurs spontaneous magnetization with a uniformly random direction, and where the strength of the magnetic force is determined by the value of m.

Chapter 4 extended the model that was treated in Chapter 3 by adding a magnetic field which using a constant h which is proportional to the strength of the magnetic field and it's direction. As expected we observed that the spins tend to align with the magnetic field and we did not observe a phase transition. This resulted in the limits given in (4.4) and (4.5). We briefly showed that these results may not be always reasonable from a physical point of view, as the influence of the magnetic field could be negligible when h is very small.

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