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ON SYSTEMS WITH QUASI-DISCRETE SPECTRUM

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Abstract. In this paper we re-examine the theory of systems with quasi-discrete spectrum initiated in the 1960's by Abramov, Hahn, and Parry. In the first part, we give a simpler proof of the Hahn–Parry theorem stating that each minimal topological system with quasi-discrete spectrum is isomorphic to a certain affine automorphism system on some compact Abelian group. Next, we show that a suitable application of Gelfand's theorem renders Abramov's theorem — the analogue of the Hahn-Parry theorem for measure-preserving systems — a straightforward corollary of the Hahn-Parry result.

In the second part, independent of the first, we present a shortened proof of the fact that each factor of a totally ergodic system with quasi-discrete spectrum (a "QDS-system") has again quasi-discrete spectrum and that such systems have zero entropy. Moreover, we obtain a complete algebraic classification of the factors of a QDS-system.

In the third part, we apply the results of the second to the (still open) question whether a Markov quasi-factor of a QDS-system is already a factor of it. We show that this is true when the system satisfies some algebraic constraint on the group of quasi-eigenvalues, which is satisfied, e.g., in the case of the skew shift.

1. Introduction

A classical problem in ergodic theory is to determine whether given (measure-preserving) dynamical systems are isomorphic or not, to determine complete sets of isomorphism invariants at least for some classes of dynamical systems and, possibly, to find canonical representatives for the corresponding isomorphism classes.

The oldest result of this type is the Halmos–von Neumann theorem from [HvN42], which says that the systems with *discrete spectrum* are isomorphic to compact Abelian group rotations, and the isomorphism class is completely determined by the point spectrum of the associated Koopman operator.

In [Abr62], the notion of a system with discrete spectrum was generalized by Abramov to (totally ergodic) systems with *quasi-discrete spectrum*. In analogy to the results of Halmos–von Neumann, Abramov could show that also this class has a complete isomorphism invariant (the "signature", in our terminology) and canonical representatives. Parallel to the original arguments of Halmos and von Neumann, Abramov first established a "theorem of uniqueness" telling that

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two systems with quasi-discrete spectrum with same signature are isomorphic, and then, in a "theorem of existence", showed that to each signature there is a special system — an affine automorphism on a compact Abelian group — with quasi-discrete spectrum and the given signature. As a combination of these two results, he then obtained the main "representation theorem" that each totally ergodic system with quasi-discrete spectrum is isomorphic to an affine automorphism on a compact Abelian group.

A couple of years later, Hahn and Parry [HP65] developed the corresponding theory for topological dynamical systems. Their approach was completely analogous: first to prove an isomorphism theorem, then a realization result; finally, as a corollary, the representation theorem. The results of Abramov and Hahn–Parry were incorporated by Brown into Chapter III of his classic book [Bro76]. Although Brown's presentation is more systematic, he essentially copied Abramov's proof of the isomorphism theorem.

One purpose of this article is to introduce a considerable simplification in the presentation of these results. In particular, we give direct proofs of the representation theorems to the effect that the isomorphism theorems become corollaries. We shall show, moreover, that the measure-preserving case is actually an immediate consequence of the topological case by virtue of a good choice of a topological model via Gelfand's theorem. (This underlines a general philosophy, already prominently demonstrated in the proof of the Halmos–von Neumann theorem in [EFHN, Chapter 17].)

Note that, in this approach, the realization results ("theorems of existence") are not needed any more neither for proving the representation nor the isomorphism theorem. Nevertheless, the realization theorems are completing the picture, and we include their proofs for the convenience of the reader.

In the second part (Section 5), we present a purely operator theoretic proof of a (generalization of a) result of Hahn and Parry from [HP68] which implies among other things that a factor of a totally ergodic system with quasi-discrete spectrum has again quasi-discrete spectrum. Using our notion of "signature" we also give a complete algebraic classification of the factors of such a system. These results are completely independent of the representation theorems of Sections 3 and 4.

In the last section we then discuss an application of these results to the problem of determining Markov quasi-factors of measure-preserving systems with quasi-discrete spectrum. We show that under certain algebraic assumptions on the signature of a system each Markov quasi-factor of the system is necessarily a factor.

Terminology and Notation. Throughout this article we generically write $\mathbf{K} = (K; \varphi)$ for topological and $\mathbf{X} = (\mathrm{X}; \varphi)$ for measure-preserving dynamical systems. This means that in the first case K is a compact Hausdorff space and $\varphi : K \to K$ is continuous, while in the second case $\mathrm{X} = (X, \Sigma_{\mathrm{X}}, \mu_{\mathrm{X}})$ is a probability space and $\varphi : X \to X$ is a measure-preserving measurable map. The topological system $\mathbf{K}$ is called **separable** if $\mathrm{C}(K)$ is separable as a Banach space, which is equivalent to K being metrizable. Analogously, the measure-preserving system $\mathbf{X}$ is **separable** if $\mathrm{L}^1(\mathrm{X})$ is separable as a Banach space. This is equivalent to Σ_{X} being countably generated (modulo null sets).

The corresponding **Koopman operators** on $\mathrm{C}(K)$ in the topological and on $\mathrm{L}^1(\mathrm{X})$ in the measure-preserving situation are generically denoted by T_φ or, if the dynamics is understood, simply by T.

In general, our terminology and notation is the same as in [EFHN]. In particular, if T is a bounded operator on a complex Banach space E, we write

$$\sigma_{\mathrm{p}}(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}$$

for the **point spectrum** of T. Given two measure-preserving systems $\mathbf{X} = (\mathrm{X};\varphi)$ and $\mathbf{Y} = (\mathrm{Y};\psi)$ we call each operator $S : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ a **Markov operator** if it is one-preserving, order-preserving and integral-preserving. A Markov operator S is a **Markov embedding** if it is a lattice homomorphism, a **Markov isomorphism** if it is a surjective Markov embedding, and **intertwining** if $ST_\varphi = T_\psi S$.

Two systems $\mathbf{X}$ and $\mathbf{Y}$ are **isomorphic** if there exists an intertwining Markov isomorphism between the respective L^1-spaces. (For the connection with the notion of *point isomorphism*, see [EFHN, Chapter 12].)

A **factor** of a measure-preserving system $\mathbf{X} = (\mathrm{X};\varphi)$ is a measure-preserving system $\mathbf{Y} = (\mathrm{Y};\psi)$ together with an intertwining Markov embedding $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$. Two factors

$$S_1 : \mathrm{L}^1(\mathrm{Y}_1) \to \mathrm{L}^1(\mathrm{X}) \quad \text{and} \quad S_2 : \mathrm{L}^1(\mathrm{Y}_2) \to \mathrm{L}^1(\mathrm{X})$$

are considered *the same* if $\operatorname{ran}(S_1) = \operatorname{ran}(S_2)$ or, equivalently, if there is an intertwining Markov isomorphism $S : \mathrm{L}^1(\mathrm{Y}_1) \to \mathrm{L}^1(\mathrm{Y}_2)$ such that $S_2S = S_1$. (See [EFHN, Section 13.4] for alternative descriptions of a factor.)

A **point factor map** of a system $\mathbf{X}$ to a system $\mathbf{Y}$ is a measurable and measure-preserving map $\pi : X \to Y$ such that $\pi \circ \varphi = \psi \circ \pi$ almost everywhere. The associated Koopman operator

$$S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X}), \qquad Sf := f \circ \pi$$

is then an intertwining Markov embedding, and hence constitutes a factor in our sense. By von Neumann's theorem, if the underlying probability spaces are standard, then every intertwining Markov embedding is induced by a point factor map, cf. [EFHN, Chapter 12] and, in particular, [EFHN, Appendix F].

2. Algebraic and Dynamic Preliminaries

Let us start with some purely algebraic preparations. The relevance of these will become clear afterwards when we turn to dynamical systems.

2.1. Signatures.

Suppose that G is a (multiplicative) Abelian group and $\Lambda : G \to G$ is a homomorphism. Consider the homomorphism

$$\Phi : G \to G, \quad \Phi(g) := g \cdot \Lambda g.$$

Then the binomial theorem yields

$$\Phi^n g = \prod_{j=0}^{n} (\Lambda^j g)^{\binom{n}{j}} \qquad (g \in G). \tag{2.1}$$

This is easy to see if one writes the group additively and notes that in this case $\Phi = (\mathrm{id} + \Lambda)$.

Let us define the increasing chain of subgroups

$$G_n := \ker(\Lambda^n) \qquad (n \geq 0),$$

so that $G_0 = \mathbf{1}$. Then $\Lambda : G_n \to G_{n-1}$ for $n \geq 1$. Moreover, Φ restricts to an automorphism on each G_n. (This is again easily seen by writing the group additively; the "Neumann series"

$$\Phi^{-1} = (\mathrm{id} + \Lambda)^{-1} = \sum_{j=0}^{\infty} (-1)^j \Lambda^j.$$

terminates when applied to elements of G_n, and yields the inverse Φ^{-1} of Φ.)

Recall that Λ is called **nilpotent** if $G = G_n$ for some $n \in \mathbb{N}$. We call Λ **quasi-nilpotent** if $G = \bigcup_{n \geq 0} G_n$. Note that if $\Lambda \in \mathrm{End}(G)$ is (quasi-)nilpotent, then so is its restriction to $\Lambda(G)$, as well as the induced homomorphism (by abuse of language)

$$\Lambda : G/G_n \to G/G_n, \quad \Lambda(gG_n) := (\Lambda g)\, G_n$$

for every $n \in \mathbb{N}$.

Recall that a group H is called **torsion-free** if H has no elements of finite order other than the neutral element.

Lemma 2.1. *Let H be an Abelian group and $\Lambda : H \to H$ a quasi-nilpotent homomorphism, with associated subgroups $H_n = \ker(\Lambda^n)$ as above. Then the induced homomorphism*

$$\Lambda : H_{n+1}/H_n \to H_n/H_{n-1}$$

is injective. Moreover, the following assertions are equivalent:

(i) *H_1 is torsion-free.*

(ii) *H is torsion-free.*

(iii) *H_{n+1}/H_n is torsion-free for every $n \geq 0$.*

Moreover, if (i)–(iii) *hold, then, with Φ defined as above, for each $m \geq 1$ and $h \in H_{m+1} \setminus H_m$ the elements*

$$\Phi^n h, \quad n \geq 0$$

are pairwise distinct modulo H_{m-1}.

Proof. The first assertion follows directly from the definition of H_n and renders straightforward the proof of the stated equivalence. For the remaining assertion, note that it follows from the binomial formula (2.1) that for all $m \geq 1$ and $n \geq 0$ and $h \in H_{m+1}$ one has

$$\Phi^n h = h(\Lambda h)^n \qquad (\mathrm{mod}\, H_{m-1}).$$

Hence, if $n \geq k \geq 0$ and $\Phi^n h = \Phi^k h \pmod{H_{m-1}}$, then $(\Lambda h)^{n-k} = 1 \pmod{H_{m-1}}$, which implies $n = k$, by (iii) and the fact that $\Lambda h \notin H_{m-1}$. □

A triple (G, Λ, ι) is called a **signature** if G is an Abelian group, $\Lambda : G \to G$ is a quasi-nilpotent homomorphism and

$$\iota : G_1 \to \mathbb{T}$$

is a monomorphism (= injective homomorphism), where $G_1 = \ker(\Lambda)$ as above. The **order** of the signature (G, Λ, ι) is

$$\mathrm{ord}(G, \Lambda, \iota) := \inf\{n \in \mathbb{N} : G = G_n\} \in \mathbb{N} \cup \{\infty\},$$

in the sense that the order is infinite if $G \neq G_n$ for all $n \in \mathbb{N}$, i.e., if Λ is not already nilpotent.

From the original signature (G, Λ, ι) one can canonically derive new signatures. First, one can pass to $(\Lambda(G), \Lambda, \iota)$ where we write, for simplicity, again Λ and ι for the respective restrictions of Λ to $\Lambda(G)$ and ι to $\Lambda(G_2) \leq G_1$.

Second, for each $n \in \mathbb{N}_0$ we obtain a derived signature $({}^{G}/_{G_n}, \Lambda, \widetilde{\iota})$ in the following way. The homomorphism Λ canonically induces a monomorphism (again denoted by Λ) at each step in the following chain:

$$G_{n+1}/G_n \xrightarrow{\Lambda} G_n/G_{n-1} \xrightarrow{\Lambda} \cdots \xrightarrow{\Lambda} G_1/G_0 = G_1/\{\mathbf{1}\} = G_1.$$

Hence $\widetilde{\iota} := \iota \circ \Lambda^n : G_{n+1}/G_n \to \mathbb{T}$ is a monomorphism. But G_{n+1}/G_n is precisely the kernel of Λ when considered as a quasi-nilpotent homomorphism on G/G_n.

A **morphism** $\alpha : (G, \Lambda, \iota) \to (\tilde{G}, \tilde{\Lambda}, \tilde{\iota})$ of signatures is every group homomorphism $\alpha : G \to \tilde{G}$ such that $\tilde{\Lambda} \circ \alpha = \alpha \circ \Lambda$ and $\tilde{\iota} \circ \alpha = \iota$ on G_1. It is then easily proved by induction that $\alpha|_{G_n}$ is injective for each $n \in \mathbb{N}$. Consequently, *every morphism α of signatures is injective.* If α is bijective then its inverse is also a morphism of signatures, and α is an **isomorphism**. For example, the derived signatures $(\Lambda(G), \Lambda, \iota)$ and $(G/G_1, \Lambda, \widetilde{\iota})$ are isomorphic via the induced isomorphism $\Lambda : G/G_1 \to \Lambda(G)$.

2.2. **Topological systems with quasi-discrete spectrum.** Signatures arise naturally in the context of dynamical systems. Let $\mathbf{K} = (K; \psi)$ be a topological dynamical system with Koopman operator T on $\mathrm{C}(K)$. Then the set

$$\mathrm{C}(K; \mathbb{T}) = \{f \in \mathrm{C}(K) : |f| = \mathbf{1}\}$$

is an Abelian group, and

$$\Lambda_{\mathbf{K}} : \mathrm{C}(K; \mathbb{T}) \to \mathrm{C}(K; \mathbb{T}), \qquad \Lambda_{\mathbf{K}} f := Tf \cdot \overline{f}$$

is a homomorphism, called the **derived homomorphism**. If $\mathbf{K}$ is understood, we simply write Λ in place of $\Lambda_{\mathbf{K}}$. Note that $Tf = f \cdot \Lambda f$, hence T takes the role of Φ from above. In particular, one has the formula

$$T^n f = \prod_{j=0}^{n} (\Lambda^j f)^{\binom{n}{j}} \tag{2.2}$$

for each $f \in \mathrm{C}(K; \mathbb{T})$ and $n \geq 0$.

Define $G_n(\mathbf{K}) := \ker(\Lambda^n)$ for $n \geq 0$ and

$$G(\mathbf{K}) := \bigcup_{n\geq 0} G_n(\mathbf{K}) = \bigcup_{n\geq 0} \ker(\Lambda^n).$$

(For simplicity, we write G_n and G if $\mathbf{K}$ is understood.) Then $G(\mathbf{K})$ is an Abelian group and $\Lambda = \Lambda_{\mathbf{K}}$ is a quasi-nilpotent homomorphism on it. Note that $G_1 = \ker(\Lambda) = \mathrm{fix}(T) \cap \mathrm{C}(K;\mathbb{T})$.

The elements of G_n are called (unimodular) **quasi-eigenvectors** of order $n-1$ (cf. Remark 2.3 below), and $G = \bigcup_{n=0}^{\infty} G_n$ is the group of all (unimodular) quasi-eigenvectors. Correspondingly, each element of

$$H_{n-1} = H_{n-1}(\mathbf{K}) := \Lambda(G_n)$$

is called a **quasi-eigenvalue** of order $n-1$, and

$$H := H(\mathbf{K}) := \bigcup_{n\geq 0} H_n = \bigcup_{n\geq 1} \Lambda(G_n) = \Lambda(G)$$

is the group of all quasi-eigenvalues. This terminology derives from the fact that the elements of G_n are precisely the unimodular solutions f of an equation $Tf = gf$, where $g \in G_{n-1}$ (in which case $g \in H_{n-1}$).

Let us now suppose that $\mathrm{fix}(T)$ *is one-dimensional*, i.e., consists of the constant functions only. (This is the case, e.g., if $\mathbf{K}$ is a minimal system.) Then the group G_2 consists of all the unimodular eigenfunctions of T corresponding to unimodular eigenvalues, and $H_1 = \Lambda(G_2)$ is the group of unimodular eigenvalues of T. (Indeed, since $f \in G_2$, the function Λf is constant and since $Tf = (\Lambda f)f$, Λf is a unimodular eigenvalue with eigenfunction f. Conversely, if $Tf = \lambda f$ for some nonzero function $f \in \mathrm{C}(K)$ and $\lambda \in \mathbb{T}$, then $|f| \in \mathrm{fix}(T)$, hence we can rescale and suppose without loss of generality that $|f| = 1$, i.e. f is unimodular. Then $\Lambda f = (Tf)\overline{f} = \lambda\mathbf{1} \in G_1$ and hence $f \in G_2$.)

Still under the hypothesis $\mathrm{fix}(T) = \mathbb{C}\mathbf{1}$, the mapping $\iota := \iota_{\mathbf{K}} : \mathrm{fix}(T) \to \mathbb{C}$, which maps a constant function to its value, is an isomorphism of vector spaces. (Note that $\iota(f) = f(x_0)$ for $f \in \mathrm{fix}(T)$ and arbitrary $x_0 \in K$.) Hence, its restriction $\iota : G_1 \to \mathbb{T}$ to G_1 is an isomorphism of groups, and (G, Λ, ι) is a signature.

The derived signature $\mathrm{sig}(\mathbf{K}) := (H, \Lambda, \iota)$ is called the **signature (of quasi-eigenvalues) of the system K**. Recall from above that this signature is, via Λ, isomorphic to the signature $({}^{G}\!/_{G_1}, \Lambda, \widetilde{\iota})$. The topological system $\mathbf{K}$ is said to have **quasi-discrete spectrum** if the linear hull of all quasi-eigenvectors is dense in $\mathrm{C}(K)$, i.e., if

$$\overline{\mathrm{span}}^{\mathrm{C}(K)}(G) = \mathrm{C}(K).$$

2.3. **Measure-preserving systems with quasi-discrete spectrum.** A similar construction and terminology applies for a measure-preserving system $\mathbf{X} = (\mathrm{X};\varphi)$ with Koopman operator T on $\mathrm{L}^1(\mathrm{X})$. Again one considers the **derived** group homomorphism $\Lambda = \Lambda_{\mathbf{X}}$ defined by

$$\Lambda f := \Lambda_{\mathbf{X}} f := Tf \cdot \overline{f}$$

on the Abelian group

$$\mathrm{L}^0(\mathrm{X};\mathbb{T}) := \{f \in \mathrm{L}^\infty(\mathrm{X}) : |f| = 1 \text{ (a.e.)}\},$$

which has kernel $G_1 := G_1(\mathbf{X}) := \mathrm{fix}(T) \cap \mathrm{L}^0(\mathrm{X};\mathbb{T})$, the group of unimodular fixed functions. As in the topological case, we let $G_n(\mathbf{X}) := \ker(\Lambda^n)$ for $n \geq 0$ be the group of (unimodular) **quasi-eigenvectors** of order $n-1$ and

$$G(\mathbf{X}) := \bigcup_{n\geq 0} G_n(\mathbf{X}) = \bigcup_{n\geq 0} \ker(\Lambda^n).$$

(Again, we write G_n and G if $\mathbf{X}$ is understood.) Analogously,

$$H_{n-1} = H_{n-1}(\mathbf{X}) = \Lambda(G_n)$$

is the group of (unimodular) **quasi-eigenvalues** of order $n-1$, and

$$H := H(\mathbf{X}) := \Lambda(G(\mathbf{X})) = \bigcup_{n=0}^{\infty} H_{n-1}$$

is the group of all quasi-eigenvalues.

Now suppose that the system $\mathbf{X}$ is ergodic. Then all fixed functions are essentially constant, so $G_1 = \ker(\Lambda) = \{c\mathbf{1} : c \in \mathbb{T}\}$. Again, we denote by

$$\iota = \iota_{\mathbf{X}} : G_1 \to \mathbb{T}, \qquad \iota(c\mathbf{1}) = c$$

the canonical monomorphism. Then (G, Λ, ι) is a signature. The derived signature

$$\mathrm{sig}(\mathbf{X}) := (H, \Lambda, \iota)$$

is called the **signature of the system X**.

The system $\mathbf{X}$ is said to have **quasi-discrete spectrum** if G is a total subset of $\mathrm{L}^2(\mathrm{X})$, i.e., if

$$\overline{\mathrm{span}}^{\mathrm{L}^2}(G) = \mathrm{L}^2(\mathrm{X}).$$

The simplest nontrivial system with quasi-discrete spectrum is the skew shift. We describe this system and compute its signature below.

Example 2.2 (Skew shift)**.** Let X be the two-dimensional torus $\mathbb{T}^2$ (written additively mod 1) with the Lebesgue measure, and φ be the transformation

$$\varphi : \mathbb{T}^2 \to \mathbb{T}^2, \qquad \varphi(x, y) := (x + \alpha, x + y)$$

for some irrational $\alpha \in (0, 1)$. The associated measure-preserving system $\mathbf{X} = (\mathrm{X}; \varphi)$ is called the skew shift. It is known that the skew shift is totally ergodic. (Cf. [EFHN, Prop. 10.17] for a proof of ergodicity; this proof can easily be adapted to yield even total ergodicity.)

Write $\mathrm{e}_k(t) := \mathrm{e}^{2\pi i k t}$ for $t \in (0, 1)$. Then some computation shows that

$$G_2(\mathbf{X}) = \{\lambda \mathrm{e}_k \otimes \mathbf{1} : k \in \mathbb{Z},\ \lambda \in \mathbb{T}\}$$

and

$$G_3(\mathbf{X}) = \{\lambda \mathrm{e}_k \otimes \mathrm{e}_l : k,\ l \in \mathbb{Z},\ \lambda \in \mathbb{T}\}.$$

It follows that $\mathbf{X}$ has quasi-discrete spectrum and $G(\mathbf{X}) = G_3(\mathbf{X})$ (see Corollary 5.5 below). Another little computation yields

$$\Lambda(\lambda \mathrm{e}_k \otimes \mathrm{e}_l) = \mathrm{e}_k(\alpha)(\mathrm{e}_l \otimes \mathbf{1}) \qquad (k, l \in \mathbb{Z},\ c \in \mathbb{T})$$

from which it follows that

$$H_0(\mathbf{X}) = \{\mathbf{1}\}, \qquad H_1(\mathbf{X}) = \{\mathrm{e}_k(\alpha)(\mathbf{1}\otimes\mathbf{1}) \,:\, k \in \mathbb{Z}\}$$

and

$$H(\mathbf{X}) = H_2(\mathbf{X}) = \{\mathrm{e}_k(\alpha)(\mathrm{e}_l\otimes\mathbf{1}) \,:\, k,l \in \mathbb{Z}\}.$$

This means that $\mathrm{sig}(\mathbf{X}) \simeq (\mathbb{Z}^2, \Lambda, \iota)$, where Λ is the two-step nilpotent homomorphism

$$\Lambda : \mathbb{Z}^2 \to \mathbb{Z}^2, \qquad \Lambda(k,l) = (l,0)$$

and $\iota : \mathbb{Z} \to \mathbb{T}$ is the embedding given by $\iota(k,0) := \mathrm{e}_k(\alpha)$.

Remark 2.3. Let us stress the fact that our terminology deviates slightly from the standard one (established first by Abramov and continued by successors). In Abramov's work, the labelling of the groups G_n is shifted to the effect that what we call G_n would be G_{n-1} in Abramov's terminology. We have chosen for this change in order to have a unified labelling for the significant subgroups associated with a quasi-nilpotent endomorphism of an Abelian group.

Other authors (e.g. Lesigne [Les93]) in the case of an ergodic system $\mathbf{X}$ write $E_0(T)$ for the set of eigenvalues and define recursively

$$E_k(T) = \{f \in \mathrm{L}^0(\mathrm{X};\mathbb{T}) \,:\, \Lambda f \in E_{k-1}(T)\}$$

for $k \geq 1$. This means that

$$E_0(T) = H_1(\mathbf{X}) \quad \text{and} \quad E_k(T) = G_{k+1}(\mathbf{X}) \quad \text{for } k \geq 1$$

if the system $\mathbf{X}$ is ergodic.

2.4. **Affine automorphisms.** Let Γ be a compact Abelian group, $\eta \in \Gamma$ and $\Psi : \Gamma \to \Gamma$ a continuous automorphism of Γ. Then the mapping

$$\psi : \Gamma \to \Gamma, \qquad \psi(\gamma) := \Psi(\gamma)\cdot\eta$$

is called an **affine automorphism**. The topological dynamical system $(\Gamma;\psi)$ is called an **affine automorphism system**, and denoted by $(\Gamma;\Psi,\eta)$. Clearly, the Haar measure is invariant under ψ, hence this gives rise also to a measure-preserving system $(\Gamma,\mathrm{m};\Psi,\eta)$.

Suppose that H is a (discrete) Abelian group, $\Lambda \in \mathrm{End}(H)$ is quasi-nilpotent and $\eta \in H^*$, the (compact) dual group. Then $\Phi : H \to H$, defined by $\Phi(h) = h\Lambda h$ is an automorphism of H. Passing to the dual group H^* we obtain the dual automorphism $\Phi^* \in \mathrm{Aut}(H^*)$, and $(H^*;\Phi^*,\eta)$ is an affine automorphism system.

The following result says that the conjugacy class of such an affine automorphism system is determined by Φ and the restriction of η to H_1.

Theorem 2.4. *Let H be a (discrete) Abelian group, $\Lambda : H \to H$ a quasi-nilpotent homomorphism, with induced automorphism $\Phi = \mathrm{I}\cdot\Lambda$ as above, and let $\eta \in H^*$. Then, for fixed $\gamma \in H^*$ the rotation map*

$$R_\gamma : H^* \to H^*, \qquad \chi \mapsto \chi\gamma$$

induces an isomorphism (conjugacy)

$$R_\gamma : (H^*;\Phi^*,\eta\Lambda^*\gamma) \xrightarrow{\cong} (H^*;\Phi^*,\eta)$$

of (topological) affine automorphism systems. Moreover, the set

$$\{\eta\,\Lambda^*\gamma \,:\, \gamma \in H^*\}$$

consists precisely of those $\chi \in H^*$ *which coincide with* η *on* H_1*.*

Proof. The first assertion is established by a straightforward computation. For the second, note first that $\Lambda^*\gamma = \gamma \circ \Lambda = 1$ on H_1. Hence $\eta\Lambda^*\gamma = \eta$ on H_1. Conversely suppose that $\chi \in H^*$ and $\chi = \eta$ on H_1. Then $\chi\eta^{-1} = 1$ on H_1, hence one can define $\gamma_1 : \Lambda(H) \to \mathbb{T}$ by

$$\gamma_1(\Lambda(h)) := (\chi\eta^{-1})(h) \qquad (h \in H).$$

Now, if $\gamma \in H^*$ extends γ_1, then $\chi = \eta\Lambda^*\gamma$ as claimed. Note that such an extension always exists since $\mathbb{T}$ is divisible, cf. also [EFHN, Prop.14.27] □

If (H, Λ, ι) is a signature and $\eta : H \to \mathbb{T}$ is any homomorphism that extends ι, then the affine automorphism system $(H^*; \Phi^*, \eta)$ is called **associated** with the signature (H, Λ, ι). By the result above, all affine automorphism systems associated with the same signature are topologically conjugate. Since the topological conjugacy is a rotation and hence preserves the Haar measure, it is also a conjugacy for the measure-preserving systems.

By the results of Hahn–Parry and Abramov (see Theorems 3.6 and 4.4 below), if (H, Λ, ι) is a signature such that H_1 is torsion-free, then any associated topological system $(H^*; \Phi^*, \eta)$ as well as the corresponding measure-preserving system $(H^*, \mathrm{m}; \Phi^*, \eta)$ has quasi-discrete spectrum with signature (H, Λ, ι).

3. Topological systems with quasi-discrete spectrum

From now on, we let $\mathbf{K} = (K; \psi)$ be a topological system such that $\mathrm{fix}(T)$ is one-dimensional, where T is, as always, the corresponding Koopman operator on $\mathrm{C}(K)$. Suppose that $\mathbf{K}$ has quasi-discrete spectrum with the additional property that the group of eigenvalues

$$H_1 = \Lambda(G_2) \cong {}^{G_2}\!/_{G_1} \lesssim \mathbb{T}$$

is torsion-free. Equivalently, by Lemma 2.1, the group H of all quasi-eigenvalues is torsion-free. Under these hypotheses we obtain the following result, obtained first by Hahn and Parry [HP65].

Lemma 3.1. *Let* $\mathbf{K}$ *be a topological system with quasi-discrete spectrum such that* $\dim \mathrm{fix}(T) = 1$ *and the group* H_1 *of unimodular eigenvalues is torsion free. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} (T^j f)(x) = 0 \quad \textit{for every } x \in K \textit{ and } f \in G \setminus G_1.$$

Proof. Let $f \in G_{k+1} \setminus G_k$ for some $k \geq 1$. For $n \geq k$ and $x \in K$,

$$T^n f(x) = \prod_{j=0}^{n} (\Lambda^j f)^{\binom{n}{j}}(x) = \prod_{j=0}^{k} [(\Lambda^j f)(x)]^{\binom{n}{j}} = f(x)\mathrm{e}^{2\pi \mathrm{i} p_x(n)},$$

where $p_x(n) = \sum_{j=1}^k \binom{n}{j}\theta_j(x)$ and $\theta_j(x) \in \mathbb{R}$ is such that $(\Lambda^j f)(x) = \mathrm{e}^{2\pi\mathrm{i}\theta_j(x)}$. The leading coefficient of the polynomial p_x is $\theta_k(x)/k!$, and this is irrational since $\Lambda^k f \in H_1$ and H_1 is torsion-free. By Weyl's equidistribution theorem,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (T^n f)(x) = f(x) \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}p_x(n)} = 0.$$

□

The lemma yields immediately the following theorem.

Theorem 3.2. *Let* $\mathbf{K}$ *be a topological system with quasi-discrete spectrum such that* $\dim \mathrm{fix}(T) = 1$ *and the group of quasi-eigenvalues is torsion-free. Then* $\mathbf{K}$ *is uniquely ergodic. Moreover, elements of* G *that are different modulo* G_1 *are orthogonal with respect to the unique invariant probability measure.*

Note that a uniquely ergodic system has a unique minimal subsystem (as every minimal subsystem is the support of an invariant measure, see [EFHN, Chapter 10]). Hence, we shall suppose in the following that $\mathbf{K}$ is *minimal*.

Lemma 3.3. *Let* $\mathbf{K}$ *be a minimal topological system with quasi-discrete spectrum. Then its group* H_1 *of unimodular eigenvalues is torsion-free if and only if* $\mathbf{K}$ *is totally minimal.*

Proof. If $\mathbf{K}$ is any totally minimal topological system, then every power T^m of its Koopman operator has one-dimensional fixed space. Hence, the group of unimodular eigenvalues is torsion-free.

Conversely, let $\mathbf{K} = (K; \psi)$ be a minimal system with quasi-discrete spectrum, such that H_1 is torsion-free. By Theorem 3.2, $\mathbf{K}$ has a unique invariant probability measure μ, say, which has full support. Now let, as above, be T the Koopman operator on $\mathrm{C}(K)$ of $\mathbf{K}$, and fix $m \in \mathbb{N}$. Any non-constant function in $\mathrm{fix}(T^m)$ would lead to T having an unimodular eigenvalue of order m, which is excluded. Hence, also $\mathrm{fix}(T^m)$ is one-dimensional. Moreover, it is easy to see from formula (2.2) that any quasi-eigenfunction for T is also a quasi-eigenfunction for T^m. It follows that also the system $(K; \psi^m)$ has quasi-discrete spectrum. The corresponding group of quasi-eigenvalues is a subgroup of H_1, hence torsion-free. By Theorem 3.2, (K, ψ^m) is uniquely ergodic, and since μ is ψ^m-invariant and has full support, it follows that $(K; \psi^m)$ is minimal. □

In the next step we show that a *totally minimal* topological system $\mathbf{K}$ of quasi-discrete spectrum is isomorphic to a specific affine automorphism system on a compact monothetic group.

Theorem 3.4 (Representation)**.** *Let* $\mathbf{K} = (K; \psi)$ *be a totally minimal topological system with quasi-discrete spectrum and signature* (H, Λ, ι)*. Then* $\mathbf{K}$ *is isomorphic to the affine automorphism system* $(H^*; \Phi^*, \eta)$*, where* $\Phi(h) = h\Lambda h$ *for* $h \in H$*, and* η *denotes any homomorphic extension of* $\iota : H_1 \to \mathbb{T}$ *to all of* H*.*

Proof. By Theorem 2.4 it suffices to find *one* extension $\eta \in H^*$ of ι such that $\mathbf{K}$ is isomorphic to $(H^*; \Phi^*, \eta)$. The proof will be given in several steps and employs the isomorphy (via Λ) of G/G_1 and H.

Fix $x_0 \in K$ and consider for each $x \in K$ the multiplicative functional

$$\delta_x : \mathrm{C}(K) \to \mathbb{C}, \qquad \delta_x(f) = \frac{f(x)}{f(x_0)}.$$

This restricts to a homomorphism $\delta_x : G \to \mathbb{T}$ that factors through G_1, hence induces a homomorphism $\delta_x : {}^{G}\!/_{G_1} \to \mathbb{T}$, i.e., $\delta_x \in ({}^{G}\!/_{G_1})^*$. We claim that the mapping

$$\delta : K \to ({}^{G}\!/_{G_1})^*, \qquad x \mapsto \delta_x$$

is a homeomorphism.

Since $\Gamma := ({}^{G}\!/_{G_1})^*$ carries the topology of pointwise convergence on ${}^{G}\!/_{G_1}$, δ is continuous. Since G separates the points of K, δ is injective. For the surjectivity it suffices to show that the induced Koopman operator

$$\Delta : \mathrm{C}(\Gamma) \to \mathrm{C}(K), \qquad (\Delta f)(x) := f(\delta_x).$$

is injective. To this end, note that $\{gG_1 : g \in G\} = {}^{G}\!/_{G_1} \cong \Gamma^*$ and, with this identification $\Delta(gG_1) = \frac{g}{g(x_0)}$ for $g \in G$. Moreover, by Theorem 3.2, if $gG_1 \neq hG_1$ then $g \perp h$ in $\mathrm{L}^2(K;\mu)$. Hence, $\Delta : \mathrm{span}(\Gamma^*) \to \mathrm{C}(K)$ is an isometry for the L^2-norms, i.e.,

$$\|f\|_{\mathrm{L}^2(\Gamma;\mathrm{m})} = \|\Delta f\|_{\mathrm{L}^2(K,\mu)} \leq \|\Delta f\|_{\mathrm{C}(K)} \qquad (f \in \mathrm{span}(\Gamma^*)).$$

Since the L^2-norm on $\mathrm{C}(\Gamma)$ is weaker than the uniform norm and $\mathrm{span}(\Gamma^*)$ is dense in $\mathrm{C}(\Gamma)$, it follows by approximation that

$$\|f\|_{\mathrm{L}^2(\Gamma;\mathrm{m})} \leq \|\Delta f\|_{\mathrm{C}(K)}$$

for *all* $f \in \mathrm{C}(\Gamma)$. And, since the L^2-norm is really a norm on $\mathrm{C}(\Gamma)$, i.e., the Haar measure has full support, Δ is injective.

Finally, we can — via the mapping δ — carry over the action ψ from K to Γ. For $x \in K$,

$$\begin{aligned}\delta_{\psi(x)}(f) &= \frac{f(\psi(x))}{f(x_0)} = \frac{(Tf)(x)}{f(x_0)} = \frac{f(x)}{f(x_0)}(\Lambda f)(x) = \frac{f(x)}{f(x_0)}\frac{(\Lambda f)(x)}{(\Lambda f)(x_0)}(\Lambda f)(x_0)\\ &= \delta_x(f)\cdot\delta_x(\Lambda f)\cdot(\Lambda f)(x_0) = (\delta_x \Lambda^* \delta_x)(f)\cdot(\Lambda f)(x_0).\end{aligned}$$

This means that

$$\delta_{\psi(x)} = \Phi^*(\delta_x)\eta$$

where $\eta(fG_1) = (\Lambda f)(x_0)$ for $f \in G$. Note that η restricts on ${}^{G_2}\!/_{G_1}$ to the canonical embedding of ${}^{G_2}\!/_{G_1} \to \mathbb{T}$. Hence, the theorem is completely proved. □

Corollary 3.5 (Isomorphism)**.** *Two minimal topological systems with quasi-discrete spectrum and torsion-free group of unimodular eigenvalues are conjugate if and only if their signatures are isomorphic.*

Proof. It is clear that two conjugate systems have isomorphic signatures. Conversely, any isomorphism of the associated signatures induces an isomorphism of associated affine automorphism systems, and by Theorem 3.4 this leads to an isomorphism of the original systems. □

In order to complete the picture, only the following result is missing.

Theorem 3.6 (Realization)**.** *Let (H, Λ, ι) be a signature and consider an associated affine automorphism system $\mathbf{K} := (H^*; \Phi^*, \eta)$. If H is torsion-free then $\mathbf{K}$ is totally minimal and has quasi-discrete spectrum with signature (isomorphic to) (H, Λ, ι).*

Proof. Denote by $K := H^*$ and $\varphi : K \to K$, $\varphi = \Phi^* \cdot \eta$. We denote, as usual, by T the Koopman operator on $\mathrm{C}(K)$, and define $\Lambda_T f := \overline{f} T f$ for $f \in \mathrm{C}(K; \mathbb{T})$. The associated subgroups of $\mathrm{C}(K; \mathbb{T})$ are

$$G_n = \ker(\Lambda_T^n) \quad (n \in \mathbb{N}_0) \quad \text{and} \quad G = \bigcup_{n \in \mathbb{N}} G_n.$$

We consider H as a subset of $\mathrm{C}(K; \mathbb{T})$. Define,

$$\Gamma_n := \mathbb{T} \cdot H_{n-1} \quad (n \in \mathbb{N}) \quad \text{and} \qquad \Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n.$$

A straightforward computation yields

$$Th = h \circ \varphi = \eta(h) \cdot h \cdot \Lambda h = \eta(h) \Phi h,$$

whence it follows that $\Lambda_T h = \eta(h) \Lambda h$ for $h \in H$. Consequently, $\Gamma_n \subseteq G_n$ for all $n \in \mathbb{N}$.

Let us compute the eigenspaces of T. Clearly, each $h \in H_1$ is an unimodular eigenvector of T with eigenvalue $\eta(h)$. Conversely, suppose that there is $f \in \mathrm{C}(K)$ with $Tf = \lambda f$. Since H is an orthonormal basis of $\mathrm{L}^2(K, \mathrm{m})$, the function f can be uniquely written as an $\mathrm{L}^2(K, \mathrm{m})$-convergent sum

$$f = \sum_{h \in H} \lambda_h h.$$

Applying T leads to

$$\lambda \sum_{h \in H} \lambda_h h = \lambda f = Tf = \sum_{h \in H} \lambda_h \eta(h) \Phi h$$

which, by comparison of coefficients, is equivalent to $\lambda \lambda_{\Phi h} = \lambda_h \eta(h)$ for all $h \in H$. In particular,

$$|\lambda_{\Phi h}| = |\lambda_h| \quad \text{for all } h \in H.$$

If $h \notin H_1$, by Lemma 2.1 the set $\{\Phi^n h : n \geq 0\}$ is infinite. Hence, $|\lambda_h| = 0$ if $h \notin H_1$. Moreover $\lambda \lambda_h = \eta(h) \lambda_h$ for $h \in H_1$, which is equivalent with $\lambda_h = 0$ or $\eta(h) = \lambda$. Since η is a monomorphism on H_1, we conclude that $\sigma_p(T) = \eta(H_1)$ and each eigenspace is one-dimensional and spanned by a function of H_1.

It follows that $\mathbf{K}$ has quasi-discrete spectrum and its group of eigenvalues is torsion-free. By Theorem 3.2, $\mathbf{K}$ is uniquely ergodic, and since the Haar measure is invariant and has full support, $\mathbf{K}$ is minimal. By Lemma 3.3, it is totally minimal.

Recall that $G_1 = \{c\mathbf{1} : c \in \mathbb{T}\}$ and consider the homomorphism of groups

$$\alpha : H \to G/G_1, \qquad \alpha(h) := hG_1.$$

Then α is injective, and it is easy to see that $\alpha : (H, \Lambda, \iota) \to (G/G_1, \Lambda_T, \tilde{\iota}))$ is a monomorphism of signatures.

It remains to be shown that α is an isomorphism, i.e., surjective. To this end, suppose that $g \in G$ is such that $gG_1 \notin \alpha(H)$. Then, by Theorem 3.2 again, $g \perp H$ in $\mathrm{L}^2(K, \mathrm{m})$. But H is an orthonormal basis, and hence $g = 0$, which is a contradiction to $|g| = \mathbf{1}$. □

Remark 3.7. Theorem 3.6 is due to Hahn and Parry [HP65, Theorem 4]. Our presentation is more detailed, for the sake of convenience.

4. Measure-preserving systems with quasi-discrete spectrum (Abramov's theorem)

We now turn to the measure-preserving case. Again, we start with the representation theorem.

Let $\mathbf{X} = (\mathrm{X}; \varphi)$ be a totally ergodic measure-preserving system with quasi-discrete spectrum. Its Koopman operator on $\mathrm{L}^1(\mathrm{X})$ is denoted by T, the homomorphism Λ on the group $\mathrm{L}^0(\mathrm{X}; \mathbb{T})$ is given by $\Lambda f := \overline{f} \cdot Tf$, and as before the subgroup G is given by

$$G = \bigcup_{n \geq 0} G_n, \qquad G_n := \ker(\Lambda^n) \qquad (n \geq 0).$$

Since the system is totally ergodic, $G_1 = \ker(\Lambda) = \mathrm{fix}(T) \cap \mathrm{L}^0(\mathrm{X}; \mathbb{T}) = \mathbb{T} \cdot \mathbf{1}$, and the group of eigenvalues $\Lambda(G_1) \cong {}^{G_2}\!/_{G_1}$ is torsion-free. That $\mathbf{X}$ has **quasi-discrete spectrum** means that the linear hull of G, $\mathrm{span}\, G$, is dense in $\mathrm{L}^2(\mathrm{X})$.

Consider now the closure

$$A := \mathrm{cl}_{\mathrm{L}^\infty} \mathrm{span}\, G$$

of $\mathrm{span}\, G$ in L^∞. Since G is multiplicative and T-invariant, A is a T-invariant, unital C^*-subalgebra of $\mathrm{L}^\infty(\mathrm{X})$. Hence, by an application of Gelfand's theorem, we can find a topological system $(K, \mu; \psi)$ together with a Markov isomorphism $\Psi : \mathrm{L}^1(K, \mu) \to \mathrm{L}^1(\mathrm{X})$ such that $T\Psi = \Psi T_\psi$ and $\Psi(\mathrm{C}(K)) = A$. (See [EFHN, Chapter 12] for details.) In the following we identify X with (K, μ) and A with $\mathrm{C}(K)$, drop explicit reference to the mapping Ψ, and write again T for the Koopman operator on $\mathrm{C}(K)$ of the mapping ψ. With these identifications being made, we now have $G \subseteq \mathrm{C}(K; \mathbb{T})$, and hence $\mathbf{K} := (K; \psi)$ is a topological system with quasi-discrete spectrum. The signature (H, Λ, ι) of this topological system is, by construction, the same as the signature of the original measure-preserving system. Moreover, since the measure μ on K has full support (also by construction), the system $\mathbf{K}$ is minimal (Theorem 3.2), hence even totally minimal by Lemma 3.3.

Now we can apply Theorem 3.4 to conclude that $\mathbf{K}$ is isomorphic to the affine automorphism system $(H^*; \Phi^*, \eta)$, where η is any homomorphic extension of ι to H. By virtue of this isomorphism, the measure μ on K is mapped to an invariant measure on H^*, which, by unique ergodicity of the systems, must therefore coincide with the Haar measure on H^*. The isomorphism of topological systems therefore extends to an isomorphism $\mathbf{X} \cong (H^*, \mathrm{m}; \Phi^*, \eta)$ of measure-preserving systems.

In effect, we have proved the following theorem, due to Abramov [Abr62, §4].

Theorem 4.1 (Representation)**.** *Let* $\mathbf{X} = (\mathrm{X}; \psi)$ *be a totally ergodic measure-preserving system with quasi-discrete spectrum and signature* (H, Λ, ι)*. Then* $\mathbf{X}$ *is isomorphic to the affine automorphism system* $(H^*, \mathrm{m}; \Phi^*, \eta)$*, where* $\Phi(h) = h\Lambda h$ *for* $h \in H$*, and* η *denotes any homomorphic extension of* $\iota : H_1 \to \mathbb{T}$ *to all of* H*.*

As in the topological case, the representation theorem implies readily the isomorphism theorem. The proof is completely analogous.

Corollary 4.2 (Isomorphism)**.** *Two totally ergodic measure-preserving systems with quasi-discrete spectrum are isomorphic if and only if their signatures are isomorphic.*

Remark 4.3. Recall that the notion of isomorphism used here is that of a *Markov isomorphism*, see Introduction. By a famous theorem of von Neumann, see [EFHN, Appendix E], if the underlying measure spaces are standard Lebesgue spaces, then Markov isomorphic systems are point isomorphic. Since a system $\mathbf{X}$ is Markov isomorphic to a standard Lebesgue system if and only if it is separable, restricting the results to standard Lebesgue spaces amounts to considering only signatures (H, Λ, ι) with a *countable* discrete group H.

Finally, as in the topological case, we complete the picture with the realization result. Its proof is—mutatis mutandis—the same as the proof of Theorem 3.6.

Theorem 4.4 (Realization)**.** *Let* (H, Λ, ι) *be a signature such that* H *is torsion-free. Then any associated (as above) measure-preserving affine automorphism system* $\mathbf{X} := (H^*, \mathrm{m}; \Phi^*, \eta)$ *is totally ergodic and has quasi-discrete spectrum with signature (isomorphic to)* (H, Λ, ι)*.*

Final Considerations. With the representation theorems at hand, one can confine to systems of the form $\mathbf{K} = (H^*; \Phi^*, \eta)$ (and their measure-theoretic analoga) when studying the fine structure of totally minimal/ergodic systems with quasi-discrete spectrum.

As H is the inductive limit of the Λ-invariant subgroups H_n, the system $\mathbf{K}$ is the inverse limit of the systems $(H_n^*; \Phi^*, \eta)$. We shall briefly indicate that each step in this chain is an abstract compact group extension by a continuous homomorphism.

The canonical embedding $H_n \subseteq H_{n+1}$ induces a canonical continuous epimorphism $H_{n+1}^* \to H_n^*$ with kernel

$$F := \{\gamma \in H_{n+1}^* \,:\, \gamma|_{H_n} = \mathbf{1}\}.$$

Since all groups are Abelian, the compact subgroup F of H_{n+1}^* acts by multiplication as automorphisms of the affine rotation system. Indeed, for $\gamma \in F$ one has $\Lambda^*\gamma = \gamma \circ \Lambda = 1$ on H_{n+1} and hence

$$\Phi^*(\chi\gamma) = (\chi\gamma)\Lambda^*(\chi\gamma)\eta = \big(\chi(\Lambda^*\chi)(\Lambda^*\gamma)\eta\big)\,\gamma = \big(\chi(\Lambda^*\chi)\eta\big)\,\gamma = \Phi^*(\chi)\,\gamma$$

for all $\chi \in H_{n+1}^*$. Hence $H_n^* \cong H_{n+1}^*/F$ not just as compact groups, but also as affine rotation systems.

It follows (e.g. by [Ell69, Prop. 6.6], but the proof can be simplified because of minimality) that each totally minimal system with quasi-discrete spectrum is distal.

From this one can eventually prove that *every totally ergodic system with quasi-discrete spectrum has zero entropy.* In fact, the proof is rather straightforward under the additional assumption that the system is separable, i.e., its L^1-space is separable. In that case, the group G/G_1 (notation from above) has to be countable by Theorem 3.2. Consequently, the L^∞-closed linear span of G is a separable C^*-algebra, hence its Gelfand space is metrizable. To sum up, the original system has a totally minimal and metrizable model which, as seen above, is distal. As Parry has shown in [Par68] (see also [Par81, Chap. 4, Thm. 17]) such systems have zero entropy. (Compare this proof with Abramov's from [Abr62, §5].)

In the general case, i.e., if $\mathrm{L}^1(\mathrm{X})$ is not separable, one may want to take advantage of the fact that the measure-theoretic entropy of a system is the supremum of the entropies of its separable factors. However, we do not see how to proceed from here directly without any further knowledge about the factors of a system with quasi-discrete spectrum.

It is our goal in the following section, built on [HP68], to provide such knowledge. We shall obtain a proof of the general statement — that *every* totally ergodic system with quasi-discrete spectrum, separable or not, has zero entropy — which does not use any of the results of the present and the preceding section.

5. Factors of systems with quasi-discrete spectrum

In this section, which is completely independent of Sections 3 and 4, we study factors of systems with quasi-discrete spectrum, recovering and extending results from [HP68].

A Technical Result.

Let $\mathbf{X}$ and $\mathbf{Y}$ be measure-preserving systems such that $\mathbf{Y}$ is a factor of $\mathbf{X}$. As is explained in [EFHN, Sec. 13.3] one can consider the space $\mathrm{L}^2(\mathrm{Y})$ as being a T-invariant subspace (in fact: a closed Banach sublattice containing the constants) of $\mathrm{L}^2(\mathrm{X})$. (Note that we do not require the dynamics to be invertible, and even if it was, our notion of a factor only requires T-invariance and not T-bi-invariance. A T-bi-invariant factor is called a **strict factor**, see [EFHN, Sec. 13.4].)

For simplicity, we shall abbreviate

$$X := \mathrm{L}^2(\mathrm{X}) \quad \text{and} \quad Y := \mathrm{L}^2(\mathrm{Y}) \subseteq \mathrm{L}^2(\mathrm{X}).$$

This is evidently an abuse of language, since usually X and Y denote the sets of the underlying probability spaces. However, base-space maps do not occur in this section, all arguments are purely operator theoretic, and it is better to have simple symbols for the function spaces rather than for the underlying sets.

Following this philosophy, we denote by

$$\mathbb{E}(\cdot \,|\, Y) : X \to Y$$

the conditional expectation (=Markov projection) onto the (L^2-space of the) factor. It is an easy exercise to establish the identity

$$T\mathbb{E}(f \,|\, Y) = \mathbb{E}(Tf \,|\, TY) \qquad (f \in X)$$

where $TY = T(Y) = \{Tf \,:\, f \in Y\}$, a factor as well.

Recall from above the abbreviation $\Lambda f := \overline{f}Tf$ and

$$G = \bigcup_{n\geq 0} G_n, \quad \text{where} \quad G_n = \{g \in X \,:\, |g| = 1 \text{ and } \Lambda^n g = \mathbf{1}\}.$$

The main technical result of Hahn and Parry from [HP68] is the following. We shall provide a new proof.

Theorem 5.1. *Let $\mathbf{X}$ be a measure-preserving system with Koopman operator T and let H be a subset of $G(\mathbf{X})$ such that*

$$h \in H \quad \Rightarrow \quad \overline{h}T^n h \in H \quad \text{for all } n \geq 1. \tag{5.1}$$

Let $\mathbf{Y}$ be a factor of $\mathbf{X}$ with the following property: whenever $h \in H$ and $f \in Y$ are such that $hf \in \bigcup_{n\in\mathbb{N}} \mathrm{fix}(T^n)$, then hf is a constant. Then, for each $h \in H$ either $h \in Y$ or $h \perp Y$.

Remarks 5.2. 1) In Hahn and Parry's original formulation, H was required to be a T-invariant subgroup of $G(\mathbf{X})$.

2) It follows from the representation

$$\overline{h}T^n h = \Lambda(h \cdot Th \cdot T^2 h \cdots T^{n-1}h)$$

that H satisfies the condition (5.1) if H is Λ-invariant and for $h \in H$ one has $h\,Th \cdots T^n h \in H$ for all $n \geq 1$.

3) If $H \neq \emptyset$ then $\mathbf{1} \in H$, so a factor $\mathbf{Y}$ satisfying the hypotheses of the theorem is necessarily totally ergodic. And if $\mathbf{Y}$ is totally ergodic, a function f as in the theorem has necessarily constant modulus.

For the proof we introduce the notation $H_n = H \cap G_n$ for $n \in \mathbb{N}_0$, so that $H = \bigcup_{n\geq 0} H_n$. Since H is Λ-invariant, Λ maps H_{n+1} into H_n. Note that T is not assumed to be invertible on Y. Therefore we introduce the factor

$$Y_\infty = \bigcap_{n\in\mathbb{N}} T^n Y,$$

which is the **invertible core** of Y, see [EFHN, Example 13.33]. As a consequence we have $TY_\infty = Y_\infty$, and hence $T\mathbb{E}(f \,|\, Y_\infty) = \mathbb{E}(Tf \,|\, Y_\infty)$ for each $f \in X$.

Lemma 5.3. *With the notation from above, $H \cap Y \subseteq Y_\infty$.*

Proof. We show $H_n \cap Y \subseteq Y_\infty$ by induction on $n \in \mathbb{N}_0$. Since H_0 consists of the function $\mathbf{1}$ only, the assertion is trivially true for $n = 0$. For the step from n to $n+1$, suppose that $h \in H_{n+1} \cap Y$. Then $\Lambda h = \overline{h}Th \in H_n \cap Y$, since $Y \cap \mathrm{L}^\infty$ is an algebra. By induction we conclude that $\Lambda h \in Y_\infty$. Now, use the identity $h = \overline{\Lambda h}Th$ together with the multiplicativity of T to prove inductively that $h \in T^m Y$ for each $m \in \mathbb{N}$. Hence, $h \in Y_\infty$ as claimed. □

We now turn to the proof of Theorem 5.1. Under the given hypothesis we shall prove by induction on $m \in \mathbb{N}_0$ the assertion

$$\forall\, h \in H_m : h \in Y \;\vee\; h \perp Y.$$

Note that for $m = 0$ this is trivially true. Let $m \in \mathbb{N}_0$, suppose that the assertion is true for m and let $h \in H_{m+1}$. We distinguish two cases.

First case. *There is* $n \in \mathbb{N}$ *such that* $\overline{h}T^n h \in Y$.
Then, by Lemma 5.3, $\overline{h}T^n h \in H \cap Y \subseteq Y_\infty$, and hence

$$T^n \mathbb{E}(h \,|\, Y) = \mathbb{E}(T^n h \,|\, T^n Y) = \mathbb{E}(h \,|\, T^n Y)(\overline{h}T^n h). \tag{5.2}$$

Since the function $\overline{h}T^n h$ has modulus equal to $\mathbf{1}$, it follows that

$$\|\mathbb{E}(h \,|\, Y)\|_2 = \|T^n \mathbb{E}(h \,|\, Y)\|_2 = \|\mathbb{E}(h \,|\, T^n Y)\|_2 .$$

But $T^n Y \subseteq Y$, and hence $\mathbb{E}(h \,|\, Y) = \mathbb{E}(h \,|\, T^n Y)$. It now follows from (5.2) that $\overline{h}\, \mathbb{E}(h \,|\, Y) \in \mathrm{fix}(T^n)$. Hence, by the assumption of the theorem, the function $h\, \overline{\mathbb{E}(h \,|\, Y)}$ is a constant. It follows that there is $c \in \mathbb{C}$ such that

$$\mathbb{E}(h \,|\, Y) = ch.$$

Since the eigenvalues of a projection can be only 0 and 1, it follows that $c \in \{0, 1\}$. If $c = 1$ then $h \in Y$; if $c = 0$ then $\mathbb{E}(h \,|\, Y) = 0$, i.e., $h \perp Y$. This settles the first case.

Second case. *For all* $n \in \mathbb{N}$, $\overline{h}T^n h \notin Y$.
Then, since $\overline{h}T^n h \in H_m$ and by the induction hypothesis, $\overline{h}T^n h \perp Y$ for all $n \in \mathbb{N}$. Applying T^k yields

$$0 \le k < n \quad \Rightarrow \quad T^k h \perp T^n h \pmod{T^k Y}$$

by which it is meant that $(T^k h)y \perp (T^n h)y'$ for all $y, y' \in T^k Y$. Now we define, for each $n \in \mathbb{N}_0$,

$$f_n := h\, \overline{T^n h}\, \mathbb{E}(T^n h \,|\, T^n Y) = h\, \overline{T^n h}\, T^n \mathbb{E}(h \,|\, Y).$$

By the preceding step we have $f_n \perp f_k$ whenever $n \neq k$. Moreover,

$$(\, f_n \,|\, h) = \int \overline{T^n h}\, \mathbb{E}(T^n h \,|\, T^n Y) = \int |\mathbb{E}(T^n h \,|\, T^n Y)|^2 = \|f_n\|_2^2$$

since $\mathbb{E}(\cdot \,|\, T^n Y)$ is an orthogonal projection. This shows that $\sum_n f_n$ is the orthogonal projection of h onto the subspace generated by the functions f_n. Hence, Bessel's inequality yields

$$1 = \|h\|_2^2 \ge \sum_n \|f_n\|_2^2 = \sum_n \|T^n \mathbb{E}(h \,|\, Y)\|_2^2 = \sum_n \|\mathbb{E}(h \,|\, Y)\|_2^2 .$$

Since the sum is infinite, we must have $\mathbb{E}(h \,|\, Y) = 0$, i.e., $h \perp Y$. This concludes the proof. □

Corollary 5.4. *Let* $\mathbf{X}$ *be a totally ergodic system. If* $f, g \in G(\mathbf{X})$ *are different modulo constant functions, then* $f \perp g$.

Proof. We let $\mathbf{Y}$ be the trivial factor and H be the smallest subset of $G = G(\mathbf{X})$ that contains $h := f\overline{g}$ and is invariant under all the mappings $f \mapsto \overline{f}T^n f$, $n \in \mathbb{N}$. Then the hypotheses of Theorem 5.1 are satisfied. It follows that either h is constant or $\int h = 0$. □

Corollary 5.5. *Let* $\mathbf{X}$ *be a totally ergodic system, let* $M \subseteq G(\mathbf{X})$ *be such that* $G_1 M \subseteq M$. *Then*

$$G(\mathbf{X}) \cap \overline{\mathrm{span}}(M) = M$$

where the closure is within $\mathrm{L}^2(\mathrm{X})$. *In particular, if* $n \in \mathbb{N}$ *is such that* $G_n(\mathbf{X})$ *is total in* $\mathrm{L}^2(\mathrm{X})$, *then* $G(\mathbf{X}) = G_n(\mathbf{X})$.

Proof. For the nontrivial inclusion, suppose that $f \in G(\mathbf{X}) \setminus M$. Then, since G_1 is the set of constant functions in $G(\mathbf{X})$ and $G_1 M \subseteq M$, f is different modulo constants from every element of M. By Corollary 5.4, $f \perp M$, and hence $f \notin \overline{\mathrm{span}}(M)$.

The second assertion follows from the first by letting $M = G_n(\mathrm{X})$. □

The lattice of factors.

With Theorem 5.1 at hand, we can turn to the main result of this section. Let $\mathbf{X}$ be a totally ergodic system with quasi-discrete spectrum, with its groups $G = G(\mathbf{X})$ and $H = H(\mathbf{X})$ of quasi-eigenvectors and quasi-eigenvalues, respectively, and its derived homomorphism $\Lambda = \Lambda_{\mathbf{X}}$. Recall that the factors of $\mathbf{X}$ can be identified with closed and T-invariant sublattices of $X = \mathrm{L}^2(\mathrm{X})$ containing the constants. As such, the factors form a (complete) lattice. To every factor $\mathbf{Y}$ of $\mathbf{X}$, we can form its group $H(\mathbf{Y})$ of quasi-eigenvalues, which is in a natural way a $\Lambda_{\mathbf{X}}$-invariant subgroup of $H(\mathbf{X})$. Indeed, with the notational conventions from above,

$$G(\mathbf{Y}) = Y \cap G$$

is the group of quasi-eigenvalues of $\mathbf{Y}$, and $\Lambda_{\mathbf{Y}} = \Lambda|_{Y \cap G}$. Hence, $H(\mathbf{Y}) = \Lambda(Y \cap G)$ is a Λ-invariant subgroup of H.

Conversely, let $K \leq H$ be any Λ-invariant subgroup of H. Then

$$\Lambda^{-1}(K) := \{f \in G : \Lambda f \in K\}$$

is a T-invariant subgroup of G containing G_1. Hence, $\mathrm{span}(\Lambda^{-1}(K))$ is a T-invariant subalgebra of $\mathrm{L}^\infty(\mathrm{X})$ containing the constants, and therefore its closure in L^2, $\overline{\mathrm{span}}(\Lambda^{-1}(K))$, is a factor.

The following theorem states that these mappings constitute a pair of mutually inverse order-preserving bijections between the lattice of factors on one side and the lattice of Λ-invariant subgroups on the other side.

Theorem 5.6. *Let $\mathbf{X}$ be a totally ergodic system with quasi-discrete spectrum, with group of quasi-eigenvectors $G = G(\mathbf{X})$ and derived homomorphism $\Lambda = \Lambda_{\mathbf{X}}$. Then the mappings*

$$Y \mapsto \Lambda(Y \cap G), \qquad K \mapsto \overline{\mathrm{span}}^{\mathrm{L}^2}(\Lambda^{-1}(K))$$

are mutually inverse isomorphisms between the lattice of factors $\mathbf{Y}$ of $\mathbf{X}$ and the lattice of Λ-invariant subgroups K of $H(\mathbf{X})$.

Proof. It remains to be shown that the two mappings are mutually inverse. Let $\mathbf{Y}$ be a factor and $K := \Lambda(Y \cap G)$. Then $\Lambda^{-1}(K) = Y \cap G$ since $G_1 \subseteq Y \cap G$. Denote $Y' := \overline{\mathrm{span}}(Y \cap G)$. Then $Y' \subseteq Y$, and we claim that $Y = Y'$.

Since by Corollary 5.4 the elements of G (modulo constants) form an orthonormal basis of X, the space $Y'^{\perp}$ is generated by those $f \in G$ such that $f \notin Y$. By Theorem 5.1, these functions also satisfy $f \perp Y$, so that $Y'^{\perp} \subseteq Y^{\perp}$. Hence, $Y \subseteq Y'$ as desired.

Conversely, let $K \leq H$ be any Λ-invariant subgroup and let $Y := \overline{\mathrm{span}}(\Lambda^{-1}(K))$. Corollary 5.5 applied with $M := \Lambda^{-1}(K)$ yields

$$M = G \cap \overline{\mathrm{span}}(M) = Y \cap G,$$

from which it follows that

$$K = \Lambda(\Lambda^{-1}(K)) = \Lambda(M) = \Lambda(Y \cap G)$$

as desired. □

Corollary 5.7. *Let* $\mathbf{X}$ *be a totally ergodic system with quasi-discrete spectrum, and let* $\mathbf{Y}$ *be a factor of* $\mathbf{X}$*. Then* $\mathbf{Y}$ *has quasi-discrete spectrum as well.*

Other Consequences.

In the remaining part of this section, we draw some other straightforward consequences of Theorems 5.1 and 5.6.

Corollary 5.8. *Let* $\mathbf{X}$ *be a totally ergodic measure-preserving system with quasi-discrete spectrum. Then* $\mathbf{X}$ *has zero entropy.*

Proof. As already noted, it suffices to show that every separable factor of $\mathbf{X}$ has zero entropy. By Corollary 5.7, such a factor has again quasi-discrete spectrum and by the observations from the end of the preceeding section, such systems have zero entropy.

However, one can proceed differently, without making use of the results of the previous sections. We denote as usual $\mathbf{X} = (\mathrm{X}; \varphi)$. Let $\mathcal{A}$ be a finite sub-σ-algebra of Σ_{X} and let $\mathbf{Y}$ be the factor with generating σ-algebra $\Sigma_{\mathrm{Y}} := \bigvee_{j=1}^{\infty} \varphi^{-j}\mathcal{A}$. By Corollary 5.7, $\mathbf{Y}$ has quasi-discrete spectrum and hence is invertible. It follows that $\mathcal{A} \subseteq \Sigma_{\mathrm{Y}}$, hence by [Wal82, Cor.4.14.1], $h(T, \mathcal{A}) = 0$. □

Theorem 5.9. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be totally ergodic factors of a system* $\mathbf{Z}$*, and suppose that* $\mathbf{X}$ *has quasi-discrete spectrum. Then the following assertions are equivalent:*

(i) *The factor system* $\mathbf{X} \wedge \mathbf{Y}$ *is trivial, i.e.,* $\mathbf{X} \wedge \mathbf{Y} = \{\mathbf{1}\}$.

(ii) *The factors* $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent, i.e.,* $\mathbf{X} \vee \mathbf{Y} \cong \mathbf{X} \times \mathbf{Y}$.

Here, $\mathbf{X} \times \mathbf{Y}$ denotes the usual direct product of the systems $\mathbf{X}$ and $\mathbf{Y}$.

Proof. The implication (ii)⇒(i) is clear. For the converse, suppose that $X \cap Y = \mathbb{C} \cdot \mathbf{1}$. We claim that $\mathbf{X}$ and $\mathbf{Y}$ satisfy the hypotheses of Theorem 5.1 with $H = G$ being the group of quasi-eigenvalues of $\mathbf{X}$. To this end, let $h \in G$, $0 \neq f \in Y$ and suppose that $T^n(hf) = hf$ for some $n \geq 1$. Taking the modulus yields $T^n |f| = |f|$, and since $\mathbf{Y}$ is totally ergodic, $|f|$ is constant. After rescaling we may suppose that $|f| = \mathbf{1}$. Then

$$\overline{h} T^n h = f \overline{T^n f} \in X \cap Y = \mathbb{C} \cdot \mathbf{1},$$

i.e., $T^n h = ch$ for some $|c| = 1$. Since ${}^{G}/_{G_1}$ is torsion-free (Lemma 2.1) it follows that $h \in G_1$, i.e., h is constant. But then $T^n f = f$ and hence also f is a constant. This establishes the claim.

Now fix again $h \in G$. Then Theorem 5.1 can be applied and yields either $h \perp Y$ or $h \in Y$, and in the latter case it follows by (i) that h is constant. In either case

$$\int hf = \int h \cdot \int f \quad \text{for all } f \in Y,$$

and since G generates X, (ii) is proved. □

The following consequence is [HP68, Cor.2.4].

Corollary 5.10. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be totally ergodic systems and suppose that* $\mathbf{X}$ *has quasi-discrete spectrum. Let* $T_{\mathbf{X}}$ *and* $T_{\mathbf{Y}}$ *be the respective Koopman operators. Then the following assertions are equivalent:*

(i) *The systems* $\mathbf{X}$ *and* $\mathbf{Y}$ *are disjoint;*

(ii) *The systems* $\mathbf{X}$ *and* $\mathbf{Y}$ *have no common factors except the trivial one;*

(iii) *The systems* $\mathbf{X}$ *and* $\mathbf{Y}$ *have no common factors with discrete spectrum except the trivial one;*

(iv) $\sigma_{\mathrm{p}}(T_{\mathbf{X}}) \cap \sigma_{\mathrm{p}}(T_{\mathbf{Y}}) = \{1\}$

Proof. The implications (i)⇒(ii)⇒(iii) are straightforward and the implication (ii)⇒(i) follows from Theorem 5.9.

To see that (iii) implies (iv), let $\Gamma := \sigma_{\mathrm{p}}(T_{\mathbf{X}}) \cap \sigma_{\mathrm{p}}(T_{\mathbf{Y}})$, a subgroup of $\mathbb{T}$. Let $G := \Gamma^*$ be the dual group, which is compact. Then for some $a \in G$ the rotation system $(G; a)$ is a factor of both $\mathbf{X}$ and $\mathbf{Y}$. Since this factor has discrete spectrum, by (iii) it follows that $G = \{\mathbf{1}\}$, i.e., $\Gamma = \{1\}$.

Finally, suppose that (iv) holds and that $\mathbf{X}$ and $\mathbf{Y}$ have the common factor $\mathbf{U}$. Then, by Corollary 5.7, $\mathbf{U}$ has quasi-discrete spectrum. The group of eigenvalues of $\mathbf{U}$ is a subgroup of $\sigma_{\mathrm{p}}(T_{\mathbf{X}}) \cap \sigma_{\mathrm{p}}(T_{\mathbf{Y}})$, which by (iv) is trivial. Hence $\mathbf{U}$ is trivial, so we have (ii). □

The following result appeared first in [ELD15, Lemma 2].

Corollary 5.11. *Let* $\mathbf{X}$ *be an ergodic system,* $m \in \mathbb{N}$ *and* $f \in G(\mathbf{X})$ *such that* $\Lambda^m_{\mathbf{X}} f \in \mathbb{T}$ *is not a root of unity. Then* $f \perp \mathbf{1}$.

Proof. Note that f cannot be a constant function. Let $c \in \mathbb{T}$ be not a root of unity, and let

$$H := \{h \in G(\mathbf{X}) \,:\, \exists m,\, n \in \mathbb{N} \text{ such that } \Lambda^m h = c^n \mathbf{1}\} \cup G_1(\mathbf{X}).$$

It is easy to see that H is Λ-invariant. Moreover, if $h \in H$, then $h \cdot Th \cdots T^k h \in H$ for each $k \geq 1$. We want to apply Theorem 5.1 (cf. Remark 5.2.2) to H and the trivial factor $\mathbf{Y}$. Take $h \in H$ such that $T^k h = h$ for some $k \geq 1$. If there are m, $n \in \mathbb{N}$ such that $\Lambda^m h = c^n \mathbf{1}$, then $g := \Lambda^{m-1} h$ is an eigenfunction of T with eigenvalue c^n. It follows that

$$c^{nk} g = T^k g = T^k \Lambda^{m-1} h = \Lambda^{m-1} T^k h = \Lambda^{m-1} h = g,$$

which implies that $nk = 0$, a contradiction. So $h \in G_1(\mathbf{X})$, i.e. h is constant. It follows that Theorem 5.1 can be applied, yielding that all non-constant functions in H are perpendicular to $Y = \mathbb{C}\mathbf{1}$. □

6. Markov Quasi-Factors

From now on we only consider separable and invertible measure-preserving systems. The Koopman operators are usually denoted by T, regardless of the system. Also, a totally ergodic system with quasi-discrete spectrum is called a **QDS-system** in the following.

A system $\mathbf{Y}$ is called a **Markov quasi-factor** of a system $\mathbf{X}$ if there is a Markov operator $M : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ such that

a) M is intertwining, i.e. $MT = TM$, and
b) the range of M is dense.

As always with Markov operators, these properties hold if and only if they hold for the restriction of M to the L^2-spaces. Moreover, there is a dual point of view by taking adjoints: $\mathbf{Y}$ is a Markov quasi-factor of $\mathbf{X}$ if there is an *injective* intertwining Markov operator $S : \mathrm{L}^1(\mathrm{Y}) \to \mathrm{L}^1(\mathrm{X})$.

Such an operator S must map eigenfunctions of $\mathbf{Y}$ to eigenfunctions of $\mathbf{X}$, resulting in $\sigma_p(T_{\mathbf{Y}}) \subseteq \sigma_p(T_{\mathbf{X}})$. Moreover, again since S is injective, the dimension of corresponding eigenspaces grows in passing from $\mathbf{Y}$ to $\mathbf{X}$. Hence, if $\mathbf{X}$ is (totally) ergodic, so is $\mathbf{Y}$.

Of course, if $\mathbf{Y}$ is a factor of $\mathbf{X}$, then it is also a Markov quasi-factor of $\mathbf{X}$. In general, the converse is wrong, see [Fra10, Proposition 4.4]. On the other hand, it is well known that a Markov quasi-factor of an ergodic system with discrete spectrum system is a factor. (The proof is easy: suppose that $\mathbf{Y}$ is a Markov quasi-factor of $\mathbf{X}$ where $M : \mathrm{L}^1(\mathrm{X}) \to \mathrm{L}^1(\mathrm{Y})$ is the corresponding intertwining Markov operator. Let $(\mathrm{e}_i)_i$ be the orthogonal basis of eigenfunctions of $\mathrm{L}^2(\mathrm{X})$. Then the linear span of $(M\mathrm{e}_i)_i$ is dense in $\mathrm{L}^2(\mathrm{Y})$. Furthermore, since M is intertwining, $M\mathrm{e}_i \neq 0$ implies that $M\mathrm{e}_i$ is an eigenfunction for all i. This shows that the system $\mathbf{Y}$ has discrete spectrum. Since $\sigma_p(T_{\mathbf{Y}}) \subseteq \sigma_p(T_{\mathbf{X}})$ as well, $\mathbf{Y}$ is a factor of $\mathbf{X}$.) It therefore has been an open question already for some time now whether the same is true for QDS-systems. In this section we give an affirmative answer in a class of QDS-systems with certain algebraic restrictions on the signature (Theorem 6.7). This class includes, for example, the skew-shift system from Example 2.2.

In what follows we shall employ the notion of the **derived factor** of a QDS-system. Suppose that $\mathbf{X}$ is a QDS-system with signature (H, Λ, η). Then $H' := \Lambda(H) \leq H$ is a Λ-invariant subgroup, hence Theorem 5.6 yields a unique factor $\partial\mathbf{X}$ of $\mathbf{X}$ with the signature $(H', \Lambda|_{H'}, \eta|_{H'\cap H_1})$. It is clear that if $\mathrm{ord}(H, \Lambda, \eta)$ is finite, then

$$\mathrm{ord}(H', \Lambda|_{H'}, \eta|_{H'\cap H_1}) = \mathrm{ord}(H, \Lambda, \eta) - 1.$$

It has been proved by Piekniewska that a Markov quasi-factor of a QDS-system is again a QDS-system [Pie13, Theorem 3.1.4]. We shall show that the argument there can be refined in order to obtain a bound on the order of the signature (Theorem 6.3 below). The proof, which is merely a closer inspection of the original one, rests on the following two powerful results from the literature.

Theorem 6.1. [Fra10, Proposition 5.1]. *If $\mathbf{Y}$ is a Markov quasi-factor of an ergodic system $\mathbf{X}$, then $\mathbf{Y}$ is a factor of some infinite ergodic self-joining of $\mathbf{X}$.*

Theorem 6.2. [Les93, Théorème 4]. *Let $\mathbf{X}$ be a totally ergodic system with group of quasi-eigenfunctions $G(\mathbf{X})$. Then for every $k \geq 0$ and every $f \in \mathrm{L}^2(\mathrm{X})$ the following assertions are equivalent:*

(i) $f \in G_{k+1}(\mathbf{X})^{\perp}$;

(ii) *For a.e. $x \in X$ and for each $P \in \mathbb{R}_k[t]$ and each continuous periodic function χ on $\mathbb{R}$ one has*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi(P(n))(T^n f)(x) = 0.$$

Here, $\mathbb{R}_k[t]$ denotes the space of all real polynomials of one variable of degree less or equal to k.

Proof. The case $k \geq 1$ is treated in Lesigne's paper [Les93]. (Recall from Remark 2.3 that $G_{k+1}(\mathbf{X}) = E_k(\mathbf{X})$ for $k \geq 1$ in Lesigne's terminology.) The case $k = 0$ holds by Birkhoff's ergodic theorem. □

It was observed in [Pie13] that the total ergodicity of $\mathbf{X}$ is not required for the proof of the implication (ii) ⇒ (i).

We can now state and prove the announced refinement of Piekniewska's result. Its proof is completely along the lines of her original argument.

Theorem 6.3. *Let $\mathbf{Y}$ be a Markov quasi-factor of a QDS-system $\mathbf{X}$. Then $\mathbf{Y}$ is again a QDS-system and*

$$\operatorname{ord}(H(\mathbf{Y}), \Lambda_{\mathbf{Y}}, \eta_{\mathbf{Y}}) \leq \operatorname{ord}(H(\mathbf{X}), \Lambda_{\mathbf{X}}, \eta_{\mathbf{X}}).$$

Proof. As $\mathbf{Y}$ is a Markov quasi-factor of $\mathbf{X}$ and $\mathbf{X}$ is totally ergodic, $\mathbf{Y}$ is totally ergodic and a factor of an ergodic (countably) infinite self-joining $\mathbf{Z}$, say, of $\mathbf{X}$ (Theorem 6.1). In this situation we may consider the different T-intertwining embeddings $J_n : \mathrm{L}^2(\mathbf{X}) \to \mathrm{L}^2(\mathbf{Z})$ which generate the joining $\mathbf{Z}$. For $f_1, \ldots, f_m \in \mathrm{L}^\infty(\mathrm{X})$ we abbreviate

$$f_1 \otimes \ldots \otimes f_m := (J_1 f_1) \cdot (J_2 f_2) \cdots (J_m f_m) \in \mathrm{L}^\infty(\mathrm{Z}).$$

It is then clear that if $k \in \mathbb{N}_0$ and each $f_j \in G_k(\mathbf{X})$, then $f_1 \otimes \ldots \otimes f_m \in G_k(\mathbf{Z})$.

Further, $\mathbf{X}$ is a QDS-system, $G(\mathbf{X})$ is a total subset of $\mathrm{L}^2(\mathbf{X})$. As $\mathbf{Z}$ is an infinite self-joining of $\mathbf{X}$, the elements of the form $f_1 \otimes \ldots \otimes f_m$ with each $f_m \in G(\mathbf{X})$ form a total subset of $\mathrm{L}^2(\mathbf{Z})$. In particular, $G(\mathbf{Z})$ is a total subset of $\mathrm{L}^2(\mathbf{Z})$.

Let now $k \in \mathbb{N}_0$ and suppose that $f \in \mathrm{L}^2(\mathrm{Y})$ is such that $f \perp G_{k+1}(\mathbf{Y})$. Then, by Theorem 6.2, for a.e. $y \in Y$, for each $P \in \mathbb{R}_k[t]$ and each continuous periodic function χ on $\mathbb{R}$ we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi(P(n)) T^n f(y) = 0. \tag{6.1}$$

Identifying f with an element in $\mathrm{L}^2(\mathrm{Z})$ we see that we may start the assertion with "for almost every $y \in Z$" here. Since the second implication of Theorem 6.2 does only require ergodicity, we conclude that $f \perp G_{k+1}(\mathbf{Z})$.

Consequently, if $f \perp G(\mathbf{Y})$, then $f \perp G(\mathbf{Z})$, which implies that $f = 0$. This shows that $\mathbf{Y}$ has quasi-discrete spectrum. Now suppose in addition that $k = \operatorname{ord}(H(\mathbf{X}), \Lambda_{\mathbf{X}}, \eta_{\mathbf{X}})$ is finite. Then $G_k(\mathbf{X})$ is total in $\mathrm{L}^2(\mathrm{X})$ and hence $G_k(\mathbf{Z})$ is total $\mathrm{L}^2(\mathrm{Z})$. As above, it follows that $G_k(\mathbf{Y})$ is total in $\mathrm{L}^2(\mathrm{Y})$.

In particular, if $f \in G_{k+1}(\mathbf{Y}) \setminus G_k(\mathbf{Y})$, then $f \perp G_k(\mathbf{Y})$, which implies that $f = 0$. This is impossible, so $G_{k+1}(\mathbf{Y}) \setminus G_k(\mathbf{Y}) = \emptyset$. And this was the claim. □

For the main theorem below it will be important to know that the derived factors are 'respected' under Markov quasi-factor maps of QDS-systems of order 2. This is our next step.

Theorem 6.4. *Let* $\mathbf{X}$ *be a QDS-system with signature* $(H(\mathbf{X}), \Lambda_{\mathbf{X}}, \eta_{\mathbf{X}})$ *of order 2. Let* $\mathbf{Y}$ *be a Markov quasi-factor of* $\mathbf{X}$*. Then* $\partial\mathbf{Y}$ *is a factor of* $\partial\mathbf{X}$*. In other words:*

$$\Lambda_{\mathbf{Y}}(H_2(\mathbf{Y})) \subseteq \Lambda_{\mathbf{X}}(H_2(\mathbf{X})) \tag{6.2}$$

when viewed as subgroups of $\mathbb{T}$*.*

Note that since $\mathbf{X}$ has order 2, $\mathbf{Y}$ is a QDS-system of order at most 2 (Theorem 6.3). Hence $\partial\mathbf{Y}$ is a QDS-system of order at most 1, i.e., a system with discrete spectrum. From the spectral considerations it follows that $H_1(\mathbf{Y}) \subseteq H_1(\mathbf{X})$ as subsets of $\mathbb{T}$.

Proof. Let, as in the proof of Theorem 6.3, be $\mathbf{Z}$ an infinite ergodic self-joining of $\mathbf{X}$ having $\mathbf{Y}$ as a factor. By definition of the groups $H_2(\mathbf{X})$ and $H_2(\mathbf{Y})$, (6.2) is the same as

$$\Lambda^2_{\mathbf{Y}}(G_3(\mathbf{Y})) \subseteq \Lambda^2_{\mathbf{X}}(G_3(\mathbf{X})). \tag{6.3}$$

In order to prove this, let $f \in G_3(\mathbf{Y})$ be such that

$$\Lambda^2_{\mathbf{Z}} f = c \in \Lambda^2_{\mathbf{Y}}(G_3(\mathbf{Y})) \setminus \Lambda^2_{\mathbf{X}}(G_3(\mathbf{X})).$$

Clearly, $c \in H_1(\mathbf{Y}) \subseteq H_1(\mathbf{X})$ is irrational. Let $f_1, \ldots, f_m \in G_3(\mathbf{X})$ be arbitrary. Then

$$\Lambda^2_{\mathbf{Z}}(\overline{f} \cdot (f_1 \otimes \cdots \otimes f_m)) = \overline{c} \cdot \Lambda^2_{\mathbf{X}}(f_1) \cdot \cdots \cdot \Lambda^2_{\mathbf{X}}(f_m) \in H_1(\mathbf{X}) \cap \overline{c}\Lambda^2_{\mathbf{X}}(G_3(\mathbf{X})).$$

As an element of $H_1(\mathbf{X})$, it is either irrational or is equal to 1. But, in fact, it cannot be equal to 1 because of the assumption $c \in \Lambda^2_{\mathbf{Y}}(G_3(\mathbf{Y})) \setminus \Lambda^2_{\mathbf{X}}(G_3(\mathbf{X}))$. We conclude by Corollary 5.11 that f is orthogonal to all such tensors. But then $f = 0$ (by the density of the span of the set of all tensors), a contradiction. □

Next, we recall some basic algebraic results. First of all, we state the following lemma. The proof can be found in [Lan02, Lemma 7.2].

Lemma 6.5. *Let* $f : A \to A'$ *be a surjective homomorphism of Abelian groups, and assume that* A' *is free. Let* B *be the kernel of* f*. Then there exists a subgroup* C *of* A *such that the restriction of* f *to* C *induces an isomorphism of* C *with* A'*, and such that* $A = B \oplus C$*.*

Using Lemma 6.5 and the fact that a subgroup of a free Abelian group is a free Abelian group as well (see [Lan02, Theorem 7.3]) one can easily prove the following lemma.

Lemma 6.6. *Let* H *be an Abelian group and let* $\pi : H \to H'$ *be a homomorphism such that the following assumptions hold:*

a) H' *is a free Abelian group;*

b) $\ker \pi \leq H$ *is a free Abelian group.*

Then there is a subgroup $K \leq H$ isomorphic via $\pi|_K$ to the subgroup $\operatorname{ran}\pi \leq H'$ *such that $H = K \oplus \ker \pi$. Any such K is a free Abelian group, and the group H is free Abelian as well.*

Finally, we arrive at the main theorem of this section.

Theorem 6.7. *Let $\mathbf{X}$ be a QDS-system with signature $(H(\mathbf{X}), \Lambda_{\mathbf{X}}, \eta_{\mathbf{X}})$ of order at most 2 such that the group $H_1(\mathbf{X})$ of eigenvalues is a free Abelian group. Then each Markov quasi-factor of $\mathbf{X}$ is a factor of $\mathbf{X}$.*

Proof. The system $\mathbf{X}$ is a QDS-system with signature $(H(\mathbf{X}), \Lambda_{\mathbf{X}}, \eta_{\mathbf{X}})$ of order at most 2. Since $\mathbf{Y}$ is a Markov quasi-factor of $\mathbf{X}$, $\mathbf{Y}$ is also a QDS-system with the signature $(H(\mathbf{Y}), \Lambda_{\mathbf{Y}}, \eta_{\mathbf{Y}})$ of order at most 2.

Our goal is to define the group homomorphisms α_1, α_2 such that $\alpha_2|_{H_1(\mathbf{Y})} = \alpha_1$, $\eta_{\mathbf{X}} \circ \alpha_1 = \eta_{\mathbf{Y}}$ and such that the diagram

$$\begin{array}{ccccc} \mathbf{1} & \longleftarrow & H_1(\mathbf{X}) & \overset{}{\underset{\Lambda_{\mathbf{X}}}{\longleftarrow}} & H_2(\mathbf{X}) \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_2} \\ \mathbf{1} & \longleftarrow & H_1(\mathbf{Y}) & \underset{\Lambda_{\mathbf{Y}}}{\longleftarrow} & H_2(\mathbf{Y}) \end{array}$$

is commutative. Then, by Theorem 5.6, the statement of the theorem follows.

As $\mathbf{Y}$ is a Markov quasi-factor of $\mathbf{X}$ one has a natural inclusion $H_1(\mathbf{Y}) \subseteq H_1(\mathbf{X})$, and we choose α_1 to be this inclusion map. Then clearly $\eta_{\mathbf{X}} \circ \alpha_1 = \eta_{\mathbf{Y}}$ as $\eta_{\mathbf{X}}$ and $\eta_{\mathbf{Y}}$ just map constant functions to their values.

In order to define the homomorphism α_2, observe that $\ker \Lambda_{\mathbf{Y}} = H_1(\mathbf{Y}) \subseteq H_2(\mathbf{Y})$. Fix a decomposition $H_2(\mathbf{Y}) = H_1(\mathbf{Y}) \oplus K$ for some free Abelian subgroup $K \leq H_2(\mathbf{Y})$, given by Lemma 6.6. We let $\alpha_2|_{H_1(\mathbf{Y})} := \alpha_1$.

Suppose that $\{\varepsilon_j\}_{j \in I}$ is a basis for K. Since $\Lambda_{\mathbf{Y}}(H_2(\mathbf{Y})) \subseteq \Lambda_{\mathbf{X}}(H_2(\mathbf{X}))$ by Theorem 6.4, for every basis element ε_j, there is $\delta_j \in H_2(\mathbf{X})$ such that

$$\alpha_1 \Lambda_{\mathbf{Y}}(\varepsilon_j) = \Lambda_{\mathbf{X}}(\delta_j).$$

Defining α_2 by $\alpha_2(\varepsilon_j) := \delta_j$ for every $j \in I$ completes the proof. □

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References

[ELD15] El Abdalaoui, E.H., Lemańczyk, M., and De La Rue, T. Automorphisms with quasi-discrete spectrum, multiplicative functions and average orthogonality along short intervals. *ArXiv e-Print* (2015), arXiv:1507.04132.

[Abr62] Abramov, L. M. Metric automorphisms with quasi-discrete spectrum. *Izv. Akad. Nauk SSSR Ser. Mat. 26* (1962), 513–530.

[Bro76] Brown, J. R. *Ergodic theory and topological dynamics*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. Pure and Applied Mathematics, No. 70.

[EFHN] Eisner, T., Farkas, B., Haase, M., and Nagel, R. *Operator Theoretic Aspects of Ergodic Theory.* Graduate Texts in Mathematics 272, Springer, 2015.

[Ell69] Ellis, R. *Lectures on topological dynamics.* W. A. Benjamin, Inc., New York, 1969.

[Fra10] Frączek, K. and Lemańczyk, M. A note on quasi-similarity of Koopman operators. *J. Lond. Math. Soc. (2)* 82 (2010), 361–375.

[HP65] Hahn, F., and Parry, W. Minimal dynamical systems with quasi-discrete spectrum. *J. London Math. Soc. 40* (1965), 309–323.

[HP68] Hahn, F., and Parry, W. Some characteristic properties of dynamical systems with quasi-discrete spectra. *Math. Systems Theory 2* (1968), 179–190.

[HvN42] Halmos, P. R., and von Neumann, J. Operator methods in classical mechanics. II. *Ann. of Math. (2) 43* (1942), 332–350.

[Lan02] Lang, S. *Algebra.* Graduate Texts in Mathematics 211, Springer-Verlag, New York, 2002.

[Les93] Lesigne, E. Spectre quasi-discret et théorème ergodique de Wiener-Wintner pour les polynômes. *Ergodic Theory Dynam. Systems* 13 (1993), 767–784.

[Par68] Parry, W. Zero entropy of distal and related transformations. In *Topological Dynamics (Symposium, Colorado State Univ., Ft. Collins, Colo., 1967).* Benjamin, New York, 1968, pp. 383–389.

[Par81] Parry, W. *Topics in ergodic theory*, vol. 75 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge-New York, 1981.

[Pie13] Piękniewska, A. Strong regularity of affine cocycles over irrational rotations. *PhD Thesis, Toruń* (2013).

[Wal82] Walters, P. *An Introduction to Ergodic Theory*, vol. 79 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1982.

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