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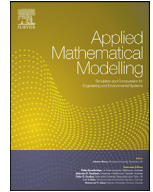
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Analysis of longitudinal oscillations in a vertically moving cable subject to nonclassical boundary conditions

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ABSTRACT

In this paper, we study a model of a flexible hoisting system, in which external disturbances exerted on the boundary can induce large vibrations, and so damage to the performance of the system. The dynamics is described by a wave equation on a slow time-varying spatial domain with a small harmonic boundary excitation at one end of the cable, and a moving mass at the other end. Due to the slow variation of the cable length, a singular perturbation problem arises. By using an averaging method, and an interior layer analysis, many resonance manifolds are detected. Further, a three time-scales perturbation method is used to construct formal asymptotic approximations of the solutions. It turns out that for a given boundary disturbance frequency, many oscillation modes jump up from order ε amplitudes to order $\sqrt{\varepsilon}$ amplitudes, where ε is a small parameter with $0 < \varepsilon \ll 1$. Finally, numerical simulations are presented to verify the obtained analytical results.

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1. Introduction

Varying-length cable systems are widely applied in a vast class of engineering problems which arise in industrial, civil, aerospace, mechanical, and automotive applications. Due to external excitations, large oscillations can occur when cables are lifted up or down. An example of these oscillations can be founded in mining cables, which are used to transport cargos in a cage between a working platform and the ground. External disturbances exerted on the cage or the cable can induce large vibrations and damage to the performance of the system. This phenomenon is caused by resonance. Resonance refers to the phenomenon that the amplitude of a mechanical system increases significantly when the excitation frequency of the mechanical system is close to a certain natural frequency of the system. In general, resonance is harmful, and can cause significant deformations and dynamic stresses in machinery and structures, and even can lead to accidents. In order to avoid resonances, it is necessary for us to study how external excitations influence the behavior of vertically translating cables for various boundary conditions. Most analytical solutions for axial or for transversal displacements of a moving cable focus on classical boundary conditions. Tan and Ying in [1] analyzed the axially moving cable based on wave propagation subject to a classical boundary condition. Zhu and Ni in [2] considered a class of translating media with moving Dirichlet boundary conditions. Sandilo and van Horssen in [3] studied auto-resonance phenomena in a space time-varying mechanical system

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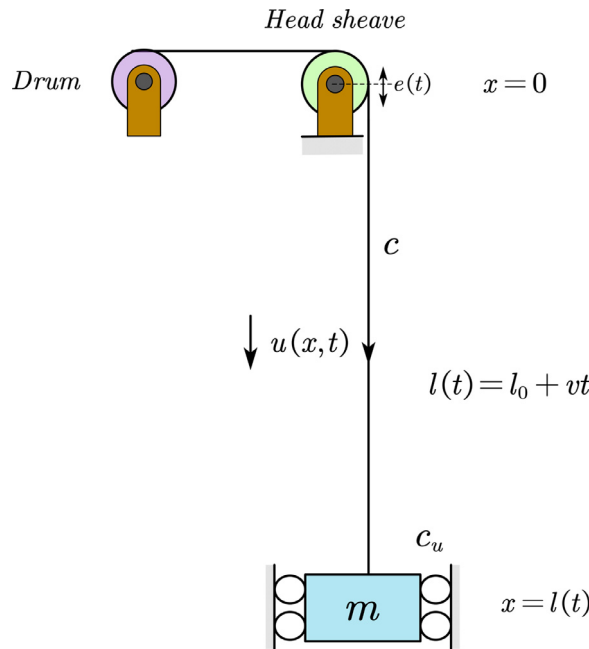


Fig. 1. The longitudinal vibrating cable with time-varying cable length $l(t)$.

with a moving Dirichlet boundary condition. Gaiko and van Horssen in [4] considered transverse vibrations of a traveling cable subject to a moving Dirichlet boundary condition with boundary damping, and in [5] the authors further discussed resonances and vibrations in an elevator cable system due to boundary sway. Chen et al. in [6] analyzed vibration responses for an axially translating cable of fixed length for classical mixed boundary conditions, and in [7] these authors studied for a traveling cable the energy dissipation and the energy exchange for fixed boundaries. Recently, researchers started to study longitudinal and transverse vibrations of moving cables or beams with moving nonclassical boundary conditions. Wang et al. in [8] investigated a coupled dynamic model for a flexible guiding hoisting system and presented the response of the system by numerical simulations. Crespo et al. in [9] introduced a model, and a numerical simulation of a stationary high-rise elevator system. Wang et al. in [10] studied the axial vibration suppression in a partial differential equation model for an ascending mining elevator cable system. These studies mainly focus on numerical simulations, and not on an analytical, mathematical analysis. For more information on numerical results for axially moving continua, the reader is referred to [11–16]. Compared to the analysis of systems subject to classical boundary conditions, the analytical study of axially moving systems with moving nonclassical boundary conditions is a challenging subject for research. Actually, for the problem with moving nonclassical boundary conditions, traditional, analytical methods, such as the method of separation of variables (SOV), and the (equivalent) Laplace transform method, can usually not be applied. Thus, it is necessary to develop analytical methods or to adapt existing methods to solve these types of problems from a mathematical view-point.

In [17], we developed and applied such methods to study a simple mathematical model, in which a one-dimensional and forced cable equation on a bounded, fixed interval was considered subject to a Dirichlet boundary condition at one end of the cable and a Robin boundary condition with a slowly varying time-dependent coefficient at the other end of the cable. In this paper we will study a real physical varying-length hoisting system model, such as a mining cable, in which the longitudinal vibrations in an axially moving cable with time-varying length are considered subject to a small harmonic boundary excitation at one end of the cable and a moving nonclassical boundary condition at the other end. This hoisting system consists of a drum, a head sheave, a driving motor, a hoisting moving conveyance, and a hoisting cable with time-varying length $l(t)$. The upper end of the vertical hoisting cable is located at $x = e(t)$, where the small displacement $e(t)$ of this upper end is supposed to be generated by the catenary system (consisting of drum, head sheave) in vertical direction. A flexible hoisting cable lets the hoisting conveyance run up and down (see Fig. 1). Compared to Wang et al. [17], the model in this paper is physically relevant, including a fundamental excitation in a boundary condition, a time-varying interval $(0, l(t))$, second order derivatives in a boundary condition, viscous damping, spatiotemporally varying tension, longitudinal stiffness and so on. An adapted version of the method of separation of variables, an averaging method, singular perturbation techniques, and a three time-scales perturbation method are applied to construct accurate, analytical approximations of the solutions of the problem. For the aforementioned reasons, averaging, determining the resonance zones, and constructing accurate approximations of solutions are much harder than for the problem as studied in [17]. In [17] we concluded that when the external force frequency satisfies a certain condition, then the resonance will occur in one oscillation mode only and no resonance will occur in the other modes, i.e., resonance emerges for only one time interval. However, in this paper, based

on a perturbation analysis of the formulated, mathematical problem for the cable equation, we come to the conclusion that for a given arbitrary excitation frequency, many oscillation modes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ amplitudes, i.e., resonance emerges for many times and the size of the resonance zone is of $O(\frac{1}{\sqrt{\varepsilon}})$. This analytical result is accurate and valuable for real applications.

The paper is organised as follows. In Section 2, the problem is formulated and some transformations are introduced to simplify the originally formulated problem. In Section 3, an interior layer analysis is presented. By introducing an adapted version of the method of separation of variables, by using averaging and singular perturbation techniques, the resonance zones are detected and the scalings are determined in the problem. By using these scalings, in Section 4 a three time-scales perturbation method is used to construct accurate, analytical approximations of the solutions of the problem. In Section 5 numerical approximations are presented by using a central finite difference scheme, which are in full agreement with the obtained, analytical approximations. In Section 6 we draw some conclusions based on the analytical and numerical results and also we discuss future research.

2. Formulation of the problem

Nomenclature:	
$u(x, t)$	the longitudinal displacement of the cable
$l(t)$	the length of the cable
$v = \dot{l}(t)$	the longitudinal velocity of the cable, v is assumed to be a constant.
ρ	the mass density of the cable
m	the mass of the hoisting conveyance
EA	the longitudinal stiffness, E Young's elasticity modulus, A the cross-sectional area of the cable
$T(x, t)$	the spatiotemporally varying tension in the cable
c	viscous damping coefficient in the cable
g	gravity
E_{gs}	initial gravitational potential energy
c_u	viscous damping coefficient
$e(t)$	the generated longitudinal displacement at the top of the vertical cable

By using the Hamilton's variational principle [18], the longitudinal vibrations of the axially moving hoisting rope in Fig. 1 are described by the following initial boundary value problem (see Appendix A):

$$\begin{cases} \rho(u_{tt} + 2vu_{xt} + v^2u_{xx}) - EAu_{xx} + c(u_t + vu_x) = 0, 0 \leq x \leq l(t), t > 0, \\ [m(u_{tt} + 2vu_{xt} + v^2u_{xx}) + EAu_x + c_u(u_t + vu_x)]|_{x=l(t)} = 0, t > 0, \\ u(0, t) = e(t), t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), 0 \leq x \leq l_0. \end{cases} \tag{1}$$

For the parameters v , c , c_u and the function $e(t)$, we make the following reasonable assumptions: the longitudinal velocity v is small compared to nominal wave velocity $\sqrt{\frac{EA}{\rho}}$; the viscous damping coefficients c and c_u are small; and the oscillation amplitudes $e(t)$ at $x = 0$ are small. Then, we can rewrite $v = \varepsilon v_0$, $c = \varepsilon c_0$, $c_u = \varepsilon c_{u0}$, $e(t) = \beta \sin(\alpha t)$ with $\beta = \varepsilon \beta_0$, where ε is a small parameter with $0 < \varepsilon \ll 1$. And $l(t) = l_0 + \varepsilon v_0 t$, where l_0 is the initial cable length. It is also assumed that both initial conditions are $O(\varepsilon)$, that is, $u_0(x) = O(\varepsilon)$, and $u_1(x) = O(\varepsilon)$.

To put problem (1) in a non-dimensional form, the following dimensionless parameters will be used: $u^* = \frac{u}{L}$, $x^* = \frac{x}{L}$, $t^* = \frac{t}{L} \sqrt{\frac{EA}{\rho}}$, $l^* = \frac{l}{L}$, $v^* = v \sqrt{\frac{\rho}{EA}}$, $c^* = \frac{cL}{\sqrt{EA\rho}}$, $c_u^* = \frac{Lc_u}{m} \sqrt{\frac{EA}{\rho}}$, $\beta^* = \frac{\beta}{L}$, $\alpha^* = L\alpha \sqrt{\frac{\rho}{EA}}$, $u_0^* = \frac{u_0}{L}$, $u_1^* = \sqrt{\frac{\rho}{EA}} u_1$, where L is the maximum length of the cable. The equations of motion in non-dimensional form then become:

$$\begin{cases} u_{tt} - u_{xx} = -2vu_{xt} - v^2u_{xx} - c(u_t + vu_x), 0 \leq x \leq l(t), t > 0, \\ u_{tt}(l(t), t) + \frac{\rho L}{m} u_x(l(t), t) = [-2vu_{xt} - v^2u_{xx} - c_u(u_t + vu_x)]|_{x=l(t)}, t > 0, \\ u(0, t) = e(t) = \beta \sin(\alpha t), t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), 0 \leq x \leq l_0, \end{cases} \tag{2}$$

where m , ρ , α , β , L and l_0 are positive constants, and where the asterisks (indicating the dimensionless variables and parameters) are omitted in problem (2) for convenience.

In order to simplify the integration of (2), it is convenient to transform the time-varying spatial domain $[0, l(t)]$ to a fixed domain $[0, 1]$ by introducing a new independent spatial coordinate $\xi = \frac{x}{l(t)}$. Since the function $u(x, t)$ becomes a new

function $\tilde{u}(\xi, t)$, all the partial derivatives have to be transformed in accordance with this new variable ξ as follows: $u_x = \frac{1}{l} \tilde{u}_\xi$, $u_{xx} = \frac{1}{l^2} \tilde{u}_{\xi\xi}$, $u_t = -\xi \frac{v}{l} \tilde{u}_\xi + \tilde{u}_t$, $u_{xt} = -\frac{v}{l^2} \tilde{u}_\xi + \frac{1}{l} \tilde{u}_{\xi t} - \xi \frac{v}{l^2} \tilde{u}_{\xi\xi}$, $u_{tt} = \tilde{u}_{tt} - 2 \frac{v}{l} \xi \tilde{u}_{\xi t} + \frac{v^2}{l^2} \xi^2 \tilde{u}_{\xi\xi} + \frac{2v^2}{l^2} \xi \tilde{u}_\xi$. Substituting these derivatives into (2), we obtain the following problem for $\tilde{u}(\xi, t)$:

$$\begin{cases} \tilde{u}_{tt} - \frac{1}{l^2} \tilde{u}_{\xi\xi} = \frac{2v}{l} \xi \tilde{u}_{\xi t} - \frac{2v}{l} \tilde{u}_{\xi t} - c\tilde{u}_t + O(\varepsilon^2), & 0 \leq \xi \leq 1, t > 0, \\ \tilde{u}_{t\xi}(1, t) + \frac{\rho l}{m} \tilde{u}_\xi(1, t) = [\frac{2v}{l} \xi \tilde{u}_{\xi t} - \frac{2v}{l} \tilde{u}_{\xi t} - c_u \tilde{u}_t]_{\xi=1} + O(\varepsilon^2), & t > 0, \\ \tilde{u}(0, t) = \tilde{e}(t) = \beta \sin(\alpha t), & t > 0, \\ \tilde{u}(\xi, 0) = \tilde{u}_0(\xi), \quad \tilde{u}_t(\xi, 0) = \tilde{u}_1(\xi), & 0 \leq \xi \leq 1, \end{cases} \tag{3}$$

where $l = l(t)$, $\tilde{u}_0(\xi) = u_0(\xi l_0)$, and $\tilde{u}_1(\xi) = u_1(\xi l_0) - \varepsilon \frac{\xi v_0}{l_0} u_\xi(\xi, 0)$.

In the following sections, we will construct analytical approximations of the solution of problem (3) on a time-scale of order $\frac{1}{\varepsilon}$ by an internal layer analysis and a three time-scales perturbation method. Moreover, to verify the analytical results, in Section 6 we will compare these analytical approximations with numerically obtained approximations.

3. Internal layer analysis

In this section, we determine resonance manifolds and their corresponding timescales by an adapted version of the method of separation of variables, by using averaging and singular perturbation techniques.

3.1. Transformations to homogeneous boundary conditions on a fixed domain

The partial differential equation in (3) has in the left-hand side of the equation a variable coefficient $\frac{1}{l^2}$. To remove this variable coefficient the Liouville-Green transformation (or equivalently the WKBJ method [19,20]) is used by introducing a new time-like variable $s(t)$ with

$$\frac{ds}{dt} = \frac{1}{l(t)}. \tag{4}$$

Substituting the derivative into (3), we obtain the problem for $\tilde{u}(\xi, s) = \tilde{u}(\xi, t)$ (see Appendix B). Further, in order to eliminate the non-homogeneous terms up to order ε^2 in the boundary condition at $\xi = 0$ and $\xi = 1$ in (3), the following transformation is used:

$$\tilde{u}(\xi, s) = W(\xi, s) + \varepsilon \frac{m\xi}{\rho l} (c_0 - c_{u0}) W_s(1, s) + \tilde{e}(s) + O(\varepsilon^2). \tag{5}$$

Thus, in order to obtain an order ε accurate approximation of the solution of $\tilde{u}(\xi, s)$, it is necessary and sufficient to construct an order ε accurate approximation of the solution of $W(\xi, s)$. From (5) and Appendix B, it follows that $W(\xi, s)$ has to satisfy:

$$\begin{cases} W_{ss} - W_{\xi\xi} = \varepsilon [(v_0 - c_0 \hat{l}) W_s + 2v_0(\xi - 1) W_{\xi s} + \beta_0 \alpha^2 \hat{l}^2 \sin(\frac{\alpha \hat{l}_0}{\varepsilon v_0} (e^{\varepsilon v_0 s} - 1)) \\ \quad - \frac{m\xi(c_0 - c_{u0})}{\rho l} W_{\xi\xi s}(1, s)] + O(\varepsilon^2), & 0 \leq \xi \leq 1, s > 0, \\ W_{\xi\xi}(1, s) + \frac{\rho \hat{l}}{m} W_\xi(1, s) = O(\varepsilon^2), \quad W(0, s) = O(\varepsilon^2), & s > 0, \\ W(\xi, 0) = W_0(\xi), \quad W_s(\xi, 0) = W_1(\xi), & 0 \leq \xi \leq 1, \end{cases} \tag{6}$$

where $W_0(\xi) = f(\xi) - \tilde{e}(0) + O(\varepsilon^2)$, $W_1(\xi) = g(\xi) - \tilde{e}_s(0) + O(\varepsilon^2)$. So the problem (3) is transformed into a simplified problem (6). In the following sections, accurate, analytical approximations of the solution $W(\xi, s)$ of problem (6) are constructed, and by using (4) and (5), accurate approximations of \tilde{u} of problem (3) can be obtained.

3.2. An adapted version of the method of separation of variables

First of all, in order to make the method of separation of variables applicable to problem (6), we consider problem (6) by neglecting the $O(\varepsilon)$ terms, that is,

$$\begin{cases} W_{ss} - W_{\xi\xi} = 0, & 0 \leq \xi \leq 1, s > 0, \\ W(0, s) = 0, \quad W_{\xi\xi}(1, s) + \frac{\rho \hat{l}(s)}{m} W_\xi(1, s) = 0, & s > 0, \\ W(\xi, 0) = W_0(\xi), \quad W_s(\xi, 0) = W_1(\xi), & 0 \leq \xi \leq 1, \end{cases} \tag{7}$$

where it should be noted that $\hat{l}(s) = l_0 e^{\varepsilon v_0 s}$. By defining a slow time variable $\tau = \varepsilon s$, which will be treated independently from the variable s , and so by defining $\tilde{l}(\tau) = l_0 e^{v_0 \tau}$, function $W(\xi, s)$ becomes a new function $W^*(\xi, s, \tau)$, and problem (7) becomes:

$$\begin{cases} W_{ss}^*(\xi, s, \tau) + 2\varepsilon W_{s\tau}^*(\xi, s, \tau) + \varepsilon^2 W_{\tau\tau}^*(\xi, s, \tau) - W_{\xi\xi}^*(\xi, s, \tau) = 0, \\ W^*(0, s, \tau) = 0, \quad W_{\xi\xi}^*(1, s, \tau) + \frac{\rho \tilde{l}(\tau)}{m} W_\xi^*(1, s, \tau) = 0, \quad \tilde{l}(\tau) = l_0 e^{v_0 \tau}, \\ W^*(\xi, 0, 0) = W_0(\xi), \quad W_s^*(\xi, 0, 0) + \varepsilon W_\tau^*(\xi, 0, 0) = W_1(\xi), \end{cases} \tag{8}$$

where $0 \leq \xi \leq 1$ and $s, \tau > 0$. Now let $T(s, \tau)X(\xi, \tau)$ be a nontrivial solution of (8). The general solution of (6) can be expanded in the following form (see Appendix C):

$$W(\xi, s) = \sum_{n=1}^{\infty} \bar{T}_n(s, \tau) \sin(\lambda_n(\tau)\xi), \tag{9}$$

where $\lambda_n(\tau)$ is the n th positive root of

$$\tan(\lambda_n(\tau)) = \frac{\rho L \bar{l}(\tau)}{m} \frac{1}{\lambda_n(\tau)}, \quad \bar{l}(\tau) = l_0 e^{\nu_0 \tau}, \tag{10}$$

and $\bar{T}_k(s, \tau)$ for $k = 1, 2, 3, \dots, s > 0, \tau > 0$ have to satisfy:

$$\left\{ \begin{aligned} \bar{T}_{k,ss} + \lambda_k^2(\tau) \bar{T}_k &= -2\varepsilon \bar{T}_{k,s\tau} + \varepsilon(\nu_0 - c_0 \bar{l}(\tau)) \bar{T}_{k,s} - 2 \sum_{n=1}^{\infty} \varepsilon c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} \bar{T}_{n,s} \\ &\quad + 2 \sum_{n=1}^{\infty} \varepsilon \nu_0 c_{n,k}^2(\tau) \bar{T}_{n,s} + \sum_{n=1}^{\infty} \varepsilon \frac{m(c_0 - c_{u0})}{\rho L} c_{n,k}^3(\tau) \bar{T}_{n,s} \\ &\quad + \varepsilon \beta_0 \alpha^2 \bar{l}^2(\tau) d_k(\tau) \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s} - 1)\right), \quad t, \tau \geq 0, \\ \bar{T}_k(0, 0) &= \frac{\int_0^1 \sigma(0, \xi) W_0(\xi) \sin(\lambda_k(0)\xi) d\xi}{\int_0^1 \sigma(0, \xi) \sin^2(\lambda_k(0)\xi) d\xi} = F_k, \\ \bar{T}_{k,s}(0, 0) + \varepsilon \bar{T}_{k,\tau}(0, 0) &= -\varepsilon \sum_{n=1}^{\infty} T_n(0, 0) \frac{d\lambda_n(\tau)}{d\tau} \Big|_{\tau=0} \frac{\int_0^1 \sigma(0, \xi) \xi \sin(\lambda_n(0)\xi) \cos(\lambda_k(0)\xi) d\xi}{\int_0^1 \sigma(0, \xi) \sin^2(\lambda_k(0)\xi) d\xi} \\ &\quad + \frac{\int_0^1 \sigma(0, \xi) W_1(\xi) \sin(\lambda_k(0)\xi) d\xi}{\int_0^1 \sigma(0, \xi) \sin^2(\lambda_k(0)\xi) d\xi} \\ &= G_k, \end{aligned} \right. \tag{11}$$

where $c_{n,k}^1(\tau), c_{n,k}^2(\tau), c_{n,k}^3(\tau)$ and $d_k(\tau)$ are functions of τ , and are given by:

$$\begin{aligned} c_{n,k}^1(\tau) &= \frac{\int_0^1 \sigma(\tau, \xi) \xi \cos(\lambda_n(\tau)\xi) \sin(\lambda_k(\tau)\xi) d\xi}{\int_0^1 \sigma(\tau, \xi) \sin^2(\lambda_k(\tau)\xi) d\xi}, \\ c_{n,k}^2(\tau) &= \frac{\lambda_n(\tau) \int_0^1 \sigma(\tau, \xi) (\xi - 1) \cos(\lambda_n(\tau)\xi) \sin(\lambda_k(\tau)\xi) d\xi}{\int_0^1 \sigma(\tau, \xi) \sin^2(\lambda_k(\tau)\xi) d\xi}, \\ c_{n,k}^3(\tau) &= \frac{\lambda_n^2(\tau) \sin(\lambda_n(\tau)) \int_0^1 \sigma(\tau, \xi) \xi \sin(\lambda_k(\tau)\xi) d\xi}{\int_0^1 \sigma(\tau, \xi) \sin^2(\lambda_k(\tau)\xi) d\xi}, \\ d_k(\tau) &= \frac{\int_0^1 \sigma(\tau, \xi) \sin(\lambda_k(\tau)\xi) d\xi}{\int_0^1 \sigma(\tau, \xi) \sin^2(\lambda_k(\tau)\xi) d\xi}. \end{aligned} \tag{12}$$

To simplify the formulas, we define a new dependent variable $\tilde{T}_k(s) = \bar{T}_k(s, \tau)$, for $k = 1, 2, 3, \dots$, yielding:

$$\left\{ \begin{aligned} \tilde{T}_{k,ss} + \lambda_k^2(\tau) \tilde{T}_k &= \varepsilon(\nu_0 - c_0 \bar{l}(\tau)) \tilde{T}_{k,s} - 2 \sum_{n=1}^{\infty} \varepsilon (c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} - \nu_0 c_{n,k}^2(\tau)) \tilde{T}_{n,s} \\ &\quad + \sum_{n=1}^{\infty} \varepsilon \frac{m(c_0 - c_{u0})}{\rho L} c_{n,k}^3(\tau) \tilde{T}_{n,s} + \varepsilon \alpha^2 \beta_0 \bar{l}^2(\tau) d_k(\tau) \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\nu_0 \tau} - 1)\right) + O(\varepsilon^2), \\ \tilde{T}_k(0) &= F_k, \quad \tilde{T}_{k,s}(0) = G_k, \end{aligned} \right. \tag{13}$$

where $F_k = O(\varepsilon), G_k = O(\varepsilon), s \geq 0$ and $\tau = \varepsilon s$. In the next subsection we will use the averaging method to detect resonance zones in problem (13), and to determine time-scales which describe the solutions of (13) accurately.

3.3. Averaging and resonance zones

The solution of the linear ordinary differential Eq. (13) with the slowly varying frequencies $\lambda_k(\tau)$ as given by (10), can be approximated by using the averaging method. In this section, by an interior layer analysis (including a rescaling and balancing procedure), the slowly varying frequencies $\lambda_k(\tau)$ lead to a description of many resonance manifolds and lead to time-scales which describe the solution of (13) sufficiently accurate. For the sake of convenience let us introduce the following standard transformations:

$$\phi_k(s) = \int_0^s \lambda_k(\varepsilon \bar{s}) d\bar{s} \quad \text{and} \quad \Phi = \frac{\alpha l_0}{\varepsilon \nu_0} (e^{\nu_0 \tau} - 1), \tag{14}$$

and according to an adapted version of the Lagrange variation of constants method, we assume that $\tilde{T}_k(s), \tilde{T}_{k,s}(s)$ are described by $A_k(s), B_k(s)$ in the following way:

$$\begin{aligned} \tilde{T}_k(s) &= A_k(s) \sin(\phi_k(s)) + B_k(s) \cos(\phi_k(s)), \\ \tilde{T}_{k,s}(s) &= \lambda_k(\tau)A_k(s) \cos(\phi_k(s)) - \lambda_k(\tau)B_k(s) \sin(\phi_k(s)). \end{aligned} \tag{15}$$

Then, by substituting (15) into problem (13), we obtain the following problem (where the dot \cdot represents differentiation with respect to s):

$$\begin{cases} \dot{A}_k = \tilde{A}_k(s) + \varepsilon \frac{\alpha^2 \beta_0^2 d_k(\tau)}{2\lambda_k(\tau)} (\sin(\Phi + \phi_k) + \sin(\Phi - \phi_k)), \\ \dot{B}_k = \tilde{B}_k(s) + \varepsilon \frac{\alpha^2 \beta_0^2 d_k(\tau)}{2\lambda_k(\tau)} (\cos(\Phi + \phi_k) - \cos(\Phi - \phi_k)), \\ \dot{t} = \varepsilon, \\ \dot{\Phi} = \alpha l_0 e^{v_0 \tau}, \\ \dot{\phi}_k = \lambda_k(\tau), \end{cases} \tag{16}$$

where

$$\begin{aligned} \tilde{A}_k(s) &= \frac{1}{2} \varepsilon (v_0 - c_0 \bar{l}(\tau)) [A_k(s) (\cos(2\phi_k(s)) + 1) - B_k(s) \sin(2\phi_k(s))] \\ &\quad + \varepsilon \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{2\lambda_k} [B_k(s) \sin(2\phi_k(s)) - A_k(s) (\cos(2\phi_k(s)) + 1)] \\ &\quad - \varepsilon \eta_{k,k}(\tau) [A_k(s) (\cos(2\phi_n(s)) + 1) - B_k(s) \sin(2\phi_k(s))] \\ &\quad - 2\varepsilon \sum_{n \neq k} \frac{\lambda_n(\tau)}{\lambda_k(\tau)} \eta_{n,k}(\tau) [A_n(s) \cos(\phi_n(s)) \cos(\phi_k(s)) - B_n(s) \sin(\phi_n(s)) \cos(\phi_k(s))], \\ \tilde{B}_k(s) &= -\frac{1}{2} \varepsilon (v_0 - c_0 \bar{l}(\tau)) [A_k(s) \sin(2\phi_k(s)) - B_k(s) (1 - \cos(2\phi_k(s)))] \\ &\quad + \varepsilon \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{2\lambda_k} [A_k(s) \sin(2\phi_k(s)) - B_k(s) (1 - \cos(2\phi_k(s)))] \\ &\quad + \varepsilon \eta_{k,k}(\tau) [A_k(s) \sin(2\phi_k(s)) - B_k(s) (1 - \cos(2\phi_n(s)))] \\ &\quad + 2\varepsilon \sum_{n \neq k} \frac{\lambda_n(\tau)}{\lambda_k(\tau)} \eta_{n,k}(\tau) [A_n(s) \cos(\phi_n(s)) \sin(\phi_k(s)) - B_n(s) \sin(\phi_n(s)) \sin(\phi_k(s))], \end{aligned} \tag{17}$$

and $\eta_{n,k}(\tau) = c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} - v_0 c_{n,k}^2(\tau) - \frac{m(c_0 - c_{u0})}{2\rho L} c_{n,k}^3(\tau)$. Resonance in (16), can be expected when $\dot{\Phi} - \dot{\phi}_k \approx 0$, or when $\dot{\Phi} + \dot{\phi}_k \approx 0$. But since $\alpha l_0 e^{v_0 \tau}$ and $\lambda_k(\tau) > 0$, resonance only will occur when

$$\alpha l_0 e^{v_0 \tau} \approx \lambda_k(\tau). \tag{18}$$

Since $\lambda_k(\tau)$ satisfies (10), that is, $\tan(\lambda_k(\tau)) = \frac{\rho L l_0 e^{v_0 \tau}}{m} \frac{1}{\lambda_k(\tau)}$, it follows that resonance occurs when

$$\lambda_k \approx \arctan\left(\frac{\rho L}{\alpha m}\right) + (k - 1)\pi, \quad k = 1, 2, \dots, \tag{19}$$

corresponding to the manifold τ around τ_k with

$$\tau_k = \frac{1}{v_0} \ln\left(\frac{1}{\alpha l_0} \lambda_k\right) = \frac{1}{v_0} \ln\left(\frac{\arctan\left(\frac{\rho L}{\alpha m}\right) + (k - 1)\pi}{\alpha l_0}\right), \quad \lambda_k \geq \alpha l_0, \quad k = 1, 2, \dots \tag{20}$$

From (20), we can conclude that no matter what the frequency is, there will be many resonance manifolds.

Outside the resonance manifold, we can average the right-hand side of the equations in (16) over ϕ_k and Φ while keeping A_k and B_k fixed [21]. Note that $\tilde{A}_k(s)$ and $\tilde{B}_k(s)$ are slowly varying, therefore they will not average out. The last terms of the first and second equations in (16) is the fast varying terms outside the resonance manifolds, therefore they will average out. Thus, the averaged equation for A_k and B_k now become

$$\begin{cases} \dot{A}_k^a = \left[\frac{1}{2} \varepsilon (v_0 - c_0 \bar{l}(\tau)) - \varepsilon c_{k,k}^1(\tau) \frac{d\lambda_k(\tau)}{d\tau} + \varepsilon v_0 c_{k,k}^2(\tau) + \varepsilon \frac{m(c_0 - c_{u0})}{2\rho L} c_{k,k}^3(\tau) - \varepsilon \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{2\lambda_k} \right] A_k^a, \\ \dot{B}_k^a = \left[\frac{1}{2} \varepsilon (v_0 - c_0 \bar{l}(\tau)) - \varepsilon c_{k,k}^1(\tau) \frac{d\lambda_k(\tau)}{d\tau} + \varepsilon v_0 c_{k,k}^2(\tau) + \varepsilon \frac{m(c_0 - c_{u0})}{2\rho L} c_{k,k}^3(\tau) - \varepsilon \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{2\lambda_k} \right] B_k^a, \end{cases} \tag{21}$$

where the upper index a indicates that this is the averaged function. From the expression for $c_{k,k}^1 > 0$, $c_{k,k}^2 = -\frac{1}{2}$ and $c_{k,k}^3 = 0$ in (12), we then obtain

$$A_k^a(s) = \frac{G_k}{\lambda_k(0)} e^{-\int_0^s \zeta(\varrho) d\varrho}, \quad B_k^a(s) = F_k e^{-\int_0^s \zeta(\varrho) d\varrho}, \tag{22}$$

with

$$\zeta(\tau) = \frac{1}{2}c_0\bar{l}(\tau) + c_{k,k}^1(\tau)\frac{d\lambda_k(\tau)}{d\tau} + \frac{d\lambda_k(\tau)}{d\tau}\frac{1}{2\lambda_k}, \tag{23}$$

and $G_k = O(\varepsilon)$, $F_k = O(\varepsilon)$ are given in (13). Hence, outside the resonance manifold the solution of system (13) is given by

$$\tilde{T}_k(s) = \frac{G_k}{\lambda_k(0)}e^{-\int_0^{s\varepsilon}\zeta(\varrho)d\varrho}\sin(\phi_k(s)) + F_k e^{-\int_0^{s\varepsilon}\zeta(\varrho)d\varrho}\cos(\phi_k(s)), \tag{24}$$

where $s = O(\frac{1}{\varepsilon})$. Observe that outside the resonance zone $\tilde{T}_k(s)$ remains order ε .

To study the behavior of the solution in the resonance zone we introduce $\psi = \Phi(t) - \phi_k(t)$ and rescale $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$ with $\bar{\tau} = O(1)$ and τ_k is given by (20). System (13) then becomes:

$$\begin{cases} \dot{A}_k = \tilde{A}_k(s) + \varepsilon\frac{\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}(\sin(\Phi + \phi_k) + \sin(\psi)), \\ \dot{B}_k = \tilde{B}_k(s) + \varepsilon\frac{\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}(\cos(\Phi + \phi_k) - \cos(\psi)), \end{cases} \tag{25}$$

combined with the slow/fast variables

$$\begin{cases} \dot{\tau} = \varepsilon, & \dot{\bar{\tau}} = \frac{\varepsilon}{\delta(\varepsilon)}, \\ \dot{\Phi} = \lambda_k(\tau_k)e^{\nu_0\delta(\varepsilon)\bar{\tau}}, \\ \dot{\phi}_k = \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}), \\ \dot{\psi} = \lambda_k(\tau_k)e^{\nu_0\delta(\varepsilon)\bar{\tau}} - \lambda_k(\tau_k + \delta(\varepsilon)\bar{\tau}) = (\nu_0\lambda_k(\tau_k) - \frac{d\lambda_k}{d\tau}|_{\tau=\tau_k})\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)), \end{cases} \tag{26}$$

where $\tilde{A}_k(s)$ and $\tilde{B}_k(s)$ are given by (17). By differentiating (10) with respect to τ , we obtain

$$\frac{1}{\cos^2(\lambda_k(\tau))}\frac{d\lambda_k(\tau)}{d\tau} = \frac{\rho Lv_0\bar{l}(\tau)}{m\lambda_k(\tau)} - \frac{\rho L\bar{l}(\tau)}{m\lambda_k^2(\tau)}\frac{d\lambda_k(\tau)}{d\tau} \Rightarrow \frac{d\lambda_k(\tau)}{d\tau} = \frac{\rho Lv_0\bar{l}(\tau)\lambda_k(\tau)\cos^2(\lambda_k(\tau))}{m\lambda_k^2(\tau) + \rho L\bar{l}(\tau)\cos^2(\lambda_k(\tau))}. \tag{27}$$

This implies for $\dot{\psi}$ (see (26)) that

$$\dot{\psi} = \gamma\delta(\varepsilon)\bar{\tau} + O(\delta^2(\varepsilon)), \quad \gamma = \frac{m\nu_0\lambda_k^3(\tau_k)}{m\lambda_k^2(\tau_k) + \rho L\bar{l}(\tau_k)\cos^2(\lambda_k(\tau_k))} \neq 0. \tag{28}$$

It now follows from (26) and (28) that a balance in system (26) occurs by choosing $\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon)$, that is, $\delta(\varepsilon) = \sqrt{\varepsilon}$. This is the size of the resonance zone. So, together with $\tau - \tau_k = \delta(\varepsilon)\bar{\tau}$, it follows from (26) that

$$\bar{\tau} = \sqrt{\varepsilon}(s - s_k), \quad s_k = \frac{\tau_k}{\varepsilon}. \tag{29}$$

Further, from (28), we obtain $\psi(s) = \psi(s_k) + \frac{1}{2}\gamma\varepsilon(s - s_k)^2$. Hence, in the resonance zone, we can write

$$\sin(\psi(s)) = \sin(\frac{1}{2}\gamma\varepsilon(s - s_k)^2 + \frac{\alpha l_0}{\varepsilon\nu_0}(e^{\varepsilon\nu_0 s_k} - 1) - \phi_k(s_k)), \quad s_k = \frac{\tau_k}{\varepsilon}, \tag{30}$$

where τ_k is given by (20). So, let us average system (25) over the fast variables. Then, the averaged equations for A_k and B_k become

$$\dot{A}_k^a = -\varepsilon\zeta(\tau)A_k^a + \frac{\varepsilon\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}\sin(\psi(s)), \quad \dot{B}_k^a = -\varepsilon\zeta(\tau)B_k^a - \frac{\varepsilon\alpha^2\beta_0\bar{l}^2(\tau)d_k(\tau)}{2\lambda_k(\tau)}\cos(\psi(s)), \tag{31}$$

where the upper index a indicates that this is the averaged function. It follows from (30) and (31) that A_k^a can be written as

$$\begin{aligned} A_k^a(s) &= \frac{G_k}{\lambda_k(0)}e^{-\int_0^{s\varepsilon}\zeta(\varrho)d\varrho} \\ &+ \varepsilon\alpha^2\beta_0e^{-\int_0^{s\varepsilon}\zeta(\varrho)d\varrho}\int_0^s\frac{\bar{l}^2(\varepsilon\bar{s})d_k(\varepsilon\bar{s})}{2\lambda_k(\varepsilon\bar{s})}e^{-\int_0^{-\varepsilon\bar{s}}\zeta(\varrho)d\varrho}\sin[\frac{1}{2}\gamma\varepsilon(\bar{s} - s_k)^2 \\ &+ \frac{\alpha l_0}{\varepsilon\nu_0}(e^{\varepsilon\nu_0 s_k} - 1) - \phi_k(s_k)]d\bar{s}, \end{aligned}$$

where $\zeta(\tau)$ is given by (23). For $\bar{s} = s_k + O(\frac{1}{\sqrt{\varepsilon}})$, $\tau_k = \varepsilon s_k$, we can observe that $\frac{\bar{l}^2(\varepsilon\bar{s})d_k(\varepsilon\bar{s})}{2\lambda_k(\varepsilon\bar{s})}e^{-\int_0^{-\varepsilon\bar{s}}\zeta(\varrho)d\varrho} = \frac{\bar{l}^2(\varepsilon s_k)d_k(\varepsilon s_k)}{2\lambda_k(\varepsilon s_k)}e^{-\int_0^{-\varepsilon s_k}\zeta(\varrho)d\varrho} + O(\sqrt{\varepsilon})$. Then, it follows from (12) that

$$A_k^a(s) = \frac{G_k}{\lambda_k(0)}e^{-\int_0^{s\varepsilon}\zeta(\varrho)d\varrho}$$

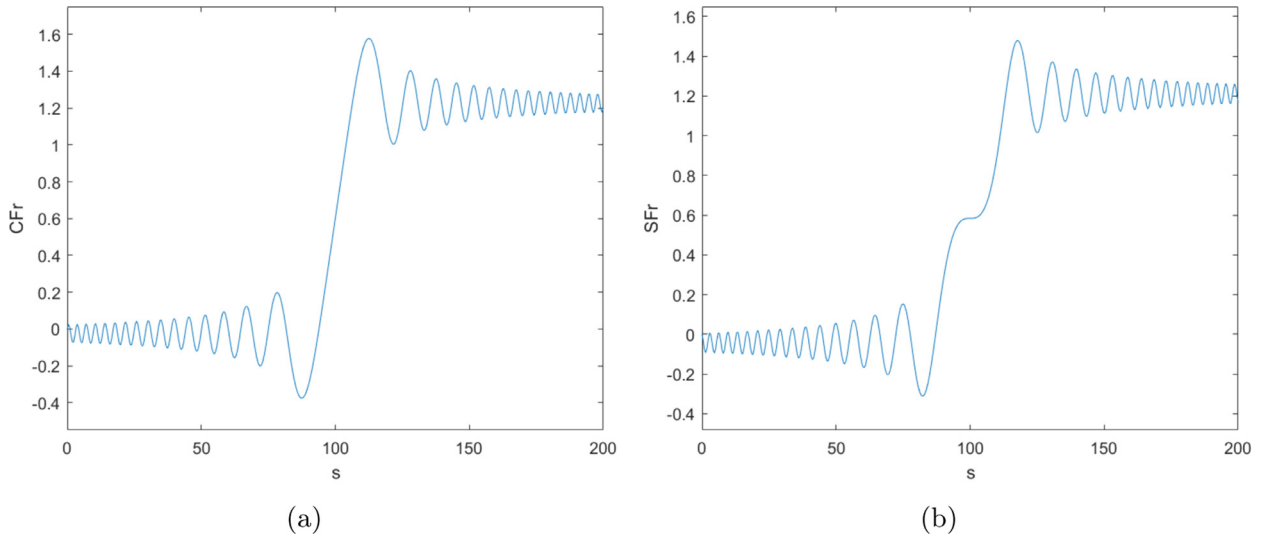


Fig. 2. (a) $C_{Fr}(s, s_k)$ has a resonance jump from $O(\sqrt{\varepsilon})$ to $O(1)$ around $s=100$. (b) $S_{Fr}(s, s_k)$ has a resonance jump from $O(\sqrt{\varepsilon})$ to $O(1)$ around $s=100$.

$$\begin{aligned}
 & + \frac{\varepsilon \alpha^2 \beta_0 \bar{l}^2(\tau_k) d_k(\tau_k)}{2\lambda_k(\tau_k)} \int_0^s \sin\left[\frac{1}{2} \gamma \varepsilon (\bar{s} - s_k)^2 + \frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right] d\bar{s} \\
 & + \text{high order terms in } \varepsilon.
 \end{aligned}$$

By setting $u = \sqrt{\frac{1}{2} \gamma \varepsilon} (\bar{s} - s_k)$, we obtain

$$\begin{aligned}
 & \varepsilon \int_0^s \sin\left[\frac{1}{2} \gamma \varepsilon (\bar{s} - s_k)^2 + \frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right] d\bar{s} \\
 & = \sqrt{\varepsilon} \bar{\alpha} \int_{-\sqrt{\varepsilon} \bar{\beta} s_k}^{\sqrt{\varepsilon} \bar{\beta} (s-s_k)} \sin(u^2 + \frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)) du \\
 & = \sqrt{\varepsilon} \bar{\alpha} \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right) C_{Fr}(s, s_k) + \sqrt{\varepsilon} \bar{\alpha} \cos\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right) S_{Fr}(s, s_k),
 \end{aligned}$$

where γ is given by (28), and where $\bar{\alpha} = \sqrt{\frac{2}{\gamma}}$, $\bar{\beta} = \sqrt{\frac{\gamma}{2}}$ and

$$C_{Fr}(s, s_k) = \int_{-\sqrt{\varepsilon} \bar{\beta} s_k}^{\sqrt{\varepsilon} \bar{\beta} (s-s_k)} \cos(u^2) du, \quad S_{Fr}(s, s_k) = \int_{-\sqrt{\varepsilon} \bar{\beta} s_k}^{\sqrt{\varepsilon} \bar{\beta} (s-s_k)} \sin(u^2) du, \quad s_k = \frac{\tau_k}{\varepsilon}. \tag{32}$$

Actually the presence of the Fresnel functions $C_{Fr}(s)$ and $S_{Fr}(s)$ cause resonance jumps in the system. The integrals $C_{Fr}(s)$ and $S_{Fr}(s)$ are plotted in Fig. 2 with $\varepsilon = 0.01$, $\bar{\beta} = 1$, and $s_k = 100$, respectively.

B_k^a can also be approximated in a similar expression as for A_k^a . So, in the resonance zone, the solution of $\tilde{T}_k(s)$ for problem (13) is given by

$$\begin{aligned}
 \tilde{T}_k(s) = & \sqrt{\varepsilon} M_k \left[\sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right) C_{Fr}(s, s_k) \right. \\
 & + \cos\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right) S_{Fr}(s, s_k) \sin(\phi_k(s)) - \left. \cos\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) \right. \right. \\
 & \left. \left. - \phi_k(s_k)\right) C_{Fr}(s, s_k) - \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k)\right) S_{Fr}(s, s_k) \cos(\phi_k(s)) \right] + O(\varepsilon),
 \end{aligned}$$

where $C_{Fr}(s, s_k)$ and $S_{Fr}(s, s_k)$ are given in (32), and

$$M_k = \frac{\bar{\alpha} \alpha^2 \beta_0 \bar{l}^2(\tau_k) d_k(\tau_k)}{2\lambda_k(\tau_k)}, \tag{33}$$

where $\bar{\alpha}$ is given in (32). Thus, the resonance always occurs for s near s_k and the size of the resonance zone in s is of $O(\frac{1}{\sqrt{\varepsilon}})$. For $O(\varepsilon)$ initial conditions and for an $O(\varepsilon)$ external, harmonic excitation, an $O(\sqrt{\varepsilon})$ amplitude modal response will occur. And for a fixed fundamental excitation frequency α , many resonance manifolds arise. The solution $\tilde{u}(\xi, s)$ in (5) (see also Appendix B (B.2)) is given by

$$\tilde{u}(\xi, s) = \sum_{k=1}^{\infty} \sqrt{\varepsilon} M_k \left[\sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1)\right) C_{Fr}(s, s_k) + \cos\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1)\right) \right]$$

$$\begin{aligned}
 & S_{Fr}(s, s_k) \sin\left(\int_{s_k}^s \lambda_k(\varepsilon \bar{s}) d\bar{s}\right) + \left(-\cos\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1)\right) C_{Fr}(s, s_k)\right. \\
 & \left. + \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1)\right) S_{Fr}(s, s_k)\right) \cos\left(\int_{s_k}^s \lambda_k(\varepsilon \bar{s}) d\bar{s}\right) \sin(\lambda_k(\varepsilon s) \xi) + O(\varepsilon).
 \end{aligned} \tag{34}$$

In the next section, the timescales as found by using the averaging method in this section will be used again to construct accurate approximations of the solutions for problem (13) by using a three-timescales perturbation method.

4. Formal approximation

4.1. Analysis results by using a three-timescales perturbation method

In this section the solution of problem (13) will be approximated by using a three-timescales perturbation method. This method can be applied to construct more accurate approximations of the solutions for problem (13) and can be applied to test the accuracy of the analytical results as obtained in the previous sections. It will turn out that the approximation as constructed in this section coincides up to order $\sqrt{\varepsilon}$ with the approximation as constructed in the previous section by using the averaging method. The Liouville-Green transformation and the following standard transformations are introduced (for fixed k) to study problem (13):

$$\tilde{T}_{k,s} = \lambda_k(\tau) \hat{T}_{k,\phi_k}, \quad \tilde{T}_{k,ss} = \lambda_k^2(\tau) \hat{T}_{k,\phi_k\phi_k} + \varepsilon \frac{d\lambda_k(\tau)}{d\tau} \hat{T}_{k,\phi_k}, \tag{35}$$

where $\tilde{T}_k(s) = \hat{T}_k(\phi_k(s))$ and $\phi_k(s)$ is given by (14). Substituting the transformations (35) into (13), we obtain the following problem for $\hat{T}_k(\phi_k)$:

$$\left\{ \begin{aligned}
 & \hat{T}_{k,\phi_k\phi_k} + \hat{T}_k = -\varepsilon \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{\lambda_k^2(\tau)} \hat{T}_{k,\phi_k} + \varepsilon(\nu_0 - c_0 \bar{l}(\tau)) \frac{1}{\lambda_k(\tau)} \hat{T}_{k,\phi_k} + 2 \sum_{n=1}^{\infty} \varepsilon \frac{\nu_0 c_{n,k}^2(\tau)}{\lambda_n(\tau)} \hat{T}_{n,\phi_n} \\
 & \quad - 2 \sum_{n=1}^{\infty} \varepsilon c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\phi_n} + \sum_{n=1}^{\infty} \varepsilon \frac{m(c_0 - c_{u0})}{\rho L} c_{n,k}^3(\tau) \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\phi_n} \\
 & \quad + \varepsilon \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\nu_0 \tau} - 1)\right) + O(\varepsilon^2), \quad t \geq 0, \\
 & \hat{T}_k(0) = F_k, \quad \hat{T}_{k,\phi_k}(0) = \frac{G_k}{\lambda_k(0)},
 \end{aligned} \right. \tag{36}$$

where $\tau = \varepsilon s$ is a function of ϕ_k . In the previous section, it was shown that (under certain conditions on the fundamental excitation frequency α) resonance can occur around times s_k , for $k = 1, 2, \dots$. In order to construct accurate approximations in the neighborhood of s_k , we rescale s with $s = \bar{s} + s_k$, $\tau = \varepsilon \bar{s} + \tau_k$, and $\phi_k(s) = \phi_k(\bar{s} + s_k) = \tilde{\phi}_k(\bar{s}) = \int_{-s_k}^{\bar{s}} \lambda_k(\tau_k + \varepsilon \bar{s}) d\bar{s}$. So, problem (36) can be rewritten for the function $\hat{T}_k(\tilde{\phi}_k)$ in:

$$\left\{ \begin{aligned}
 & \hat{T}_{k,\tilde{\phi}_k\tilde{\phi}_k} + \hat{T}_k = -\varepsilon \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{\lambda_k^2(\tau)} \hat{T}_{k,\tilde{\phi}_k} + \varepsilon(\nu_0 - c_0 \bar{l}(\tau)) \frac{1}{\lambda_k(\tau)} \hat{T}_{k,\tilde{\phi}_k} + 2 \sum_{n=1}^{\infty} \varepsilon \frac{\nu_0 c_{n,k}^2(\tau)}{\lambda_n(\tau)} \hat{T}_{n,\tilde{\phi}_n} \\
 & \quad - 2 \sum_{n=1}^{\infty} \varepsilon c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\tilde{\phi}_n} + \sum_{n=1}^{\infty} \varepsilon \frac{m(c_0 - c_{u0})}{\rho L} c_{n,k}^3(\tau) \frac{1}{\lambda_n(\tau)} \hat{T}_{n,\tilde{\phi}_n} \\
 & \quad + \varepsilon \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\nu_0 \tau} - 1)\right) + O(\varepsilon^2), \quad t \geq 0, \\
 & \hat{T}_k(0) = F_k, \quad \hat{T}_{k,\tilde{\phi}_k}(0) = \frac{G_k}{\lambda_k(0)},
 \end{aligned} \right. \tag{37}$$

where τ is a function of $\tilde{\phi}_k$. Next we study problem (37) in detail. The application of the straightforward expansion method to solve (37) will result in the occurrence of so-called secular terms which cause the approximations of the solutions to become unbounded on long timescales. And it has been shown in the previous section that the $O(\varepsilon)$ excitation can produce timescale of $O(\sqrt{\varepsilon})$. Therefore, to avoid these secular terms, we introduce three timescales $\tilde{s}_0 = \bar{s}$, $\tilde{s}_1 = \sqrt{\varepsilon} \bar{s}$, $\tilde{s}_2 = \varepsilon \bar{s}$, $\tau = \tilde{s}_2 + \tau_k$, and so $\tilde{\phi}_{k,0}$, $\tilde{\phi}_{k,1}$, $\tilde{\phi}_{k,2}$ are introduced as follows:

$$\tilde{\phi}_{k,0} = \int_a^{\tilde{s}_0} \lambda_k(\tau_k + \varepsilon \bar{s}) d\bar{s}, \quad \tilde{\phi}_{k,1} = \int_b^{\tilde{s}_1} \lambda_k(\tau_k + \sqrt{\varepsilon} \bar{s}) d\bar{s}, \quad \tilde{\phi}_{k,2} = \int_c^{\tilde{s}_2} \lambda_k(\tau_k + \bar{s}) d\bar{s},$$

where $a = -s_k$, $b = -\sqrt{\varepsilon} s_k$, $c = -\varepsilon s_k$. These scalings are based on the size of the resonance zone (which has been found in the previous section), and on the natural scalings for weakly nonlinear equations such as (37). By using the three timescales perturbation method, the function $\hat{T}_k(\tilde{\phi}_k; \sqrt{\varepsilon})$ is supposed to be a function of $\tilde{\phi}_{k,0}$, $\tilde{\phi}_{k,1}$, $\tilde{\phi}_{k,2}$, that is,

$$\hat{T}_k(\tilde{\phi}_k; \sqrt{\varepsilon}) = w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}). \tag{38}$$

By substituting (38) into (37), we obtain the following equations up to $O(\varepsilon\sqrt{\varepsilon})$:

$$\left\{ \begin{aligned} & \frac{\partial^2 w_k}{\partial \tilde{\phi}_{k,0}^2} + w_k + 2\sqrt{\varepsilon} \frac{\partial^2 w_k}{\partial \tilde{\phi}_{k,0} \partial \tilde{\phi}_{k,1}} + \varepsilon \left(2 \frac{\partial^2 w_k}{\partial \tilde{\phi}_{k,0} \partial \tilde{\phi}_{k,2}} + \frac{\partial^2 w_k}{\partial \tilde{\phi}_{k,1}^2} \right) + 2\varepsilon\sqrt{\varepsilon} \frac{\partial^2 w_k}{\partial \tilde{\phi}_{k,1} \partial \tilde{\phi}_{k,2}} \\ & = \varepsilon \left[- \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{\lambda_k^2(\tau)} \frac{\partial w_k}{\partial \tilde{\phi}_{k,0}} + (v_0 - c_0 \bar{l}(\tau)) \frac{1}{\lambda_k(\tau)} \frac{\partial w_k}{\partial \tilde{\phi}_{k,0}} + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{\alpha l_0}{\varepsilon v_0} (e^{v_0 \tau} - 1)\right) \right. \\ & \quad \left. - 2 \sum_{n=1}^{\infty} (c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} - v_0 c_{n,k}^2(\tau) - \frac{m(c_0 - c_{u0})}{2\rho L} c_{n,k}^3(\tau)) \frac{1}{\lambda_n(\tau)} \frac{\partial w_n}{\partial \tilde{\phi}_{n,0}} \right] \\ & \quad + \varepsilon\sqrt{\varepsilon} \left[- \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{\lambda_k^2(\tau)} \frac{\partial w_k}{\partial \tilde{\phi}_{k,1}} + (v_0 - c_0 \bar{l}(\tau)) \frac{1}{\lambda_k(\tau)} \frac{\partial w_k}{\partial \tilde{\phi}_{k,1}} \right. \\ & \quad \left. - 2 \sum_{n=1}^{\infty} (c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} - v_0 c_{n,k}^2(\tau) - \frac{m(c_0 - c_{u0})}{2\rho L} c_{n,k}^3(\tau)) \frac{1}{\lambda_n(\tau)} \frac{\partial w_n}{\partial \tilde{\phi}_{n,1}} \right], \\ & w_k(0, 0, 0; \sqrt{\varepsilon}) = \bar{F}_k, \quad \frac{\partial w_k}{\partial \tilde{\phi}_{k,0}}(0, 0, 0; \sqrt{\varepsilon}) + \sqrt{\varepsilon} \frac{\partial w_k}{\partial \tilde{\phi}_{k,1}}(0, 0, 0; \sqrt{\varepsilon}) + \varepsilon \frac{\partial w_k}{\partial \tilde{\phi}_{k,2}}(0, 0, 0; \sqrt{\varepsilon}) = \frac{G_k}{\lambda_k(0)}, \end{aligned} \right. \tag{39}$$

where $F_k = \varepsilon \bar{F}_k$ and $G_k = \varepsilon \bar{G}_k$ are $O(\varepsilon)$, and τ is a function of $\tilde{\phi}_{k,2}$. By using a three-timescales perturbation method, the function $w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon})$ is approximated by the formal asymptotic expansion

$$w_k(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) = \sqrt{\varepsilon} w_{k,0}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) + \varepsilon w_{k,1}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) + \varepsilon\sqrt{\varepsilon} w_{k,2}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) + O(\varepsilon^2). \tag{40}$$

By substituting (40) into problem (39), and after equating the coefficients of like powers in $\sqrt{\varepsilon}$, we obtain as: the $O(\sqrt{\varepsilon})$ -problem:

$$\frac{\partial^2 w_{k,0}}{\partial \tilde{\phi}_{k,0}^2} + w_{k,0} = 0, \quad w_{k,0}(0, 0, 0) = 0, \quad \frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,0}}(0, 0, 0) = 0, \tag{41}$$

the $O(\varepsilon)$ -problem:

$$\begin{aligned} \frac{\partial^2 w_{k,1}}{\partial \tilde{\phi}_{k,0}^2} + w_{k,1} &= -2 \frac{\partial^2 w_{k,0}}{\partial \tilde{\phi}_{k,0} \partial \tilde{\phi}_{k,1}} + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{\alpha l_0}{\varepsilon v_0} (e^{v_0 \tau} - 1)\right), \\ w_{k,1}(0, 0, 0) &= \bar{F}_k, \quad \frac{\partial w_{k,1}}{\partial \tilde{\phi}_{k,0}}(0, 0, 0) = - \frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,1}}(0, 0, 0) + \frac{\bar{G}_k}{\lambda_k(0)}, \end{aligned} \tag{42}$$

and the $O(\varepsilon\sqrt{\varepsilon})$ -problem:

$$\begin{aligned} \frac{\partial^2 w_{k,2}}{\partial \tilde{\phi}_{k,0}^2} + w_{k,2} &= -2 \frac{\partial^2 w_{k,1}}{\partial \tilde{\phi}_{k,0} \partial \tilde{\phi}_{k,1}} - 2 \frac{\partial^2 w_{k,0}}{\partial \tilde{\phi}_{k,0} \partial \tilde{\phi}_{k,2}} - \frac{\partial^2 w_{k,0}}{\partial \tilde{\phi}_{k,1}^2} + [(v_0 - c_0 \bar{l}(\tau)) \lambda_k(\tau) \\ & \quad - \frac{d\lambda_k(\tau)}{d\tau}] \frac{1}{\lambda_k^2(\tau)} \frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,0}} - 2 \sum_{n=1}^{\infty} [c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} - v_0 c_{n,k}^2(\tau) \\ & \quad - \frac{m(c_0 - c_{u0})}{2\rho L} c_{n,k}^3(\tau)] \frac{1}{\lambda_n(\tau)} \frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,0}}, \\ w_{k,2}(0, 0, 0) &= 0, \quad \frac{\partial w_{k,2}}{\partial \tilde{\phi}_{k,0}}(0, 0, 0) = - \frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,2}}(0, 0, 0) - \frac{\partial w_{k,1}}{\partial \tilde{\phi}_{k,1}}(0, 0, 0). \end{aligned} \tag{43}$$

The $O(\sqrt{\varepsilon})$ - problem has as solution

$$w_{k,0}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) = C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) \sin(\tilde{\phi}_{k,0}) + C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) \cos(\tilde{\phi}_{k,0}), \tag{44}$$

where $C_{k,1}$ and $C_{k,2}$ are still unknown functions of the slow variables $\tilde{\phi}_{k,1}$ and $\tilde{\phi}_{k,2}$, and they can be determined by avoiding secular terms in the solutions of the $O(\varepsilon)$ - and the $O(\varepsilon\sqrt{\varepsilon})$ - problems (see Appendix A). Before entering the resonance zone, the following result is found:

$$C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = 0, \quad C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = 0, \tag{45}$$

and inside of the resonance zone, to avoid secular terms in the solution $w_{k,1}$ and $w_{k,2}$, it turns out that

$$C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \bar{\alpha} \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{2\lambda_k(\tau)} [\sin(\vartheta(s_k)) \bar{C}_{Fr}(\tilde{s}_1) + \cos(\vartheta(s_k)) \bar{S}_{Fr}(\tilde{s}_1)],$$

$$C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \tilde{\alpha}\alpha^2\beta_0\bar{l}^2(\tau) \frac{d_k(\tau)}{2\lambda_k(\tau)} [\cos(\vartheta(s_k))\bar{C}_{Fr}(\tilde{s}_1) - \sin(\vartheta(s_k))\bar{S}_{Fr}(\tilde{s}_1)], \tag{46}$$

where γ is given by (28), $\tilde{\alpha}$ and $\tilde{\beta}$ are given by (32),

$$\vartheta(s_k) = \frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1) - \phi_k(s_k), \quad \bar{C}_{Fr}(\tilde{s}_1) = \int_{\tilde{\beta} b}^{\tilde{\beta} \tilde{s}_1} \cos(u^2) du, \quad \bar{S}_{Fr}(\tilde{s}_1) = \int_{\tilde{\beta} b}^{\tilde{\beta} \tilde{s}_1} \sin(u^2) du. \tag{47}$$

Further, to obtain more accurate approximations of problem (39), the $O(\varepsilon)$ - problem and the $O(\varepsilon\sqrt{\varepsilon})$ - problem can also be solved by using a similar analysis as for the $O(\sqrt{\varepsilon})$ - problem in Appendix D. At this moment, only the first term in the expansion of the solution for the cable problem is important from the physical point of view. So, to shorten the paper, we are not interested in high-order approximations.

Thus, from (40), an approximation of the solution of Eq. (39) is given by $w(\xi, s) = \sum_{n=1}^{\infty} \sqrt{\varepsilon} w_{n,0} + O(\varepsilon)$, where $w_{n,0}$ is given by (44). It follows from (20), for a given value of α , that around $\tau = \tau_n = \frac{1}{\nu_0} \ln(\frac{\arctan(\frac{\rho L}{\alpha m}) + (n-1)\pi}{\alpha l_0})$ the n th oscillation mode jumps up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. For such a jump the inequality $\arctan(\frac{\rho L}{\alpha m}) + (n-1)\pi \geq \alpha l_0$ needs to be satisfied. This implies that it might occur that the first few modes do not show this jump, but all that the higher order modes do. Before entering the resonance zone for the n th oscillation mode $w_{n,0} \equiv 0$ and in the resonance zone the n th oscillation mode $w_{n,0}$ is given by

$$\begin{aligned} & w_{n,0}(\tilde{\phi}_{n,0}, \tilde{\phi}_{n,1}, \tilde{\phi}_{n,2}; \sqrt{\varepsilon}) \\ &= \frac{1}{2} \alpha^2 \beta_0 \bar{l}^2(\tau_n) \frac{d_n(\tau_n)}{\lambda_n(\tau_n)} \tilde{\alpha} [\sin(\vartheta(s_n))\bar{C}_{Fr}(\tilde{s}_1) + \cos(\vartheta(s_n))\bar{S}_{Fr}(\tilde{s}_1)] \sin(\tilde{\phi}_{n,0}) \\ &\quad - \frac{1}{2} \alpha^2 \beta_0 \bar{l}^2(\tau_n) \frac{d_n(\tau_n)}{\lambda_n(\tau_n)} \tilde{\alpha} [\cos(\vartheta(s_n))\bar{C}_{Fr}(\tilde{s}_1) - \sin(\vartheta(s_n))\bar{S}_{Fr}(\tilde{s}_1)] \cos(\tilde{\phi}_{n,0}) \\ &= M_n [(\sin(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_n} - 1))\bar{C}_{Fr}(\tilde{s}_1) + \cos(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_n} - 1))\bar{S}_{Fr}(\tilde{s}_1)) \sin(\tilde{\phi}_{n,0} - \phi_n(s_n)) \\ &\quad + (-\cos(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_n} - 1))\bar{C}_{Fr}(\tilde{s}_1) + \sin(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_n} - 1))\bar{S}_{Fr}(\tilde{s}_1)) \cos(\tilde{\phi}_{n,0} - \phi_n(s_n))], \end{aligned} \tag{48}$$

where $\tilde{\phi}_{k,0} - \phi_k(s_k) = \int_{s_k}^s \lambda_k(\varepsilon \bar{s}) d\bar{s}$, γ , $\tilde{\alpha}$, $\vartheta(s_n)$, $\bar{C}_{Fr}(\tilde{s}_1)$ and $\bar{S}_{Fr}(\tilde{s}_1)$ are given by (28), (32), (47) and M_n is given by (33). The solution $w_{n,0}$ in (48) implies a resonance jump from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$ around τ_n in the n th oscillation mode. Thus, the solution $\tilde{u}(\xi, s)$ in (5) (see also Appendix B (B.2)) is given by

$$\tilde{u}(\xi, s) = \sum_{k=1}^{\infty} \sqrt{\varepsilon} w_{k,0}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) \sin(\lambda_k(\varepsilon s)\xi) + O(\varepsilon). \tag{49}$$

The solution in (49) is in agreement with (34), that is, the one obtained by the averaging method. All in all, introducing the following notation $\chi(t) = \frac{1}{\varepsilon \nu_0} \ln(\frac{l(t)}{l_0})$, according to the solution of $\tilde{u}(\xi, s)$ in (49), we obtain as approximation for the solution $\tilde{u}(\xi, t)$ of problem (3)

$$\begin{aligned} \tilde{u}(\xi, t) &= \sum_{k=1}^{\infty} \sqrt{\varepsilon} M_k [(\sin(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1))\bar{C}_{Fr}(\sqrt{\varepsilon}(\chi(t) - s_k)) + \cos(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1))\bar{S}_{Fr}(\sqrt{\varepsilon}(\chi(t) \\ &\quad - s_k))) \sin(\int_{s_k}^{\chi(t)} \lambda_k(\varepsilon \bar{s}) d\bar{s}) + (-\cos(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} - 1))\bar{C}_{Fr}(\sqrt{\varepsilon}(\chi(t) - s_k)) + \sin(\frac{\alpha l_0}{\varepsilon \nu_0} (e^{\varepsilon \nu_0 s_k} \\ &\quad - 1))\bar{S}_{Fr}(\sqrt{\varepsilon}(\chi(t) - s_k))) \cos(\int_{s_k}^{\chi(t)} \lambda_k(\varepsilon \bar{s}) d\bar{s})] \sin(\lambda_k(\varepsilon \chi(t))\xi) + O(\varepsilon), \end{aligned} \tag{50}$$

where $s_k = \frac{\tau_k}{\varepsilon}$ and τ_k is given by (20), \bar{C}_{Fr} , $\bar{S}_{Fr}(\tilde{s}_1)$ are given by (47), M_k is given by (33) and $\xi = \frac{x}{l(t)}$.

4.2. Numerical results

In this section we will present numerical simulations of the vibration response as computed and based on the analytical expressions (50). The computations are performed by using the following parameters:

$$\varepsilon = 0.01, \quad l_0 = 3, \quad \nu_0 = 1, \quad c_0 = 2, \quad c_{u0} = 1, \quad \rho = 1, \quad m = 10, \quad L = 10, \quad \beta_0 = 1, \quad \alpha = 1. \tag{51}$$

For simplicity, let us assume that only the initial displacement is prescribed, so that

$$\tilde{u}_0(\xi) = \varepsilon \sin(1.5\xi), \quad \tilde{u}_1(\xi) = 0, \quad 0 \leq \xi \leq 1. \tag{52}$$

It is worth mentioning that the following numerical results are computed based on $O(\varepsilon)$ approximations. Higher-order approximations are neglected due to their insignificant and small contribution to the solution. By using (18), we see that the resonance occurs around time instants s_k satisfying

$$\alpha l_0 e^{\nu_0 \tau} = \lambda_k, \quad \tau = \varepsilon s. \tag{53}$$

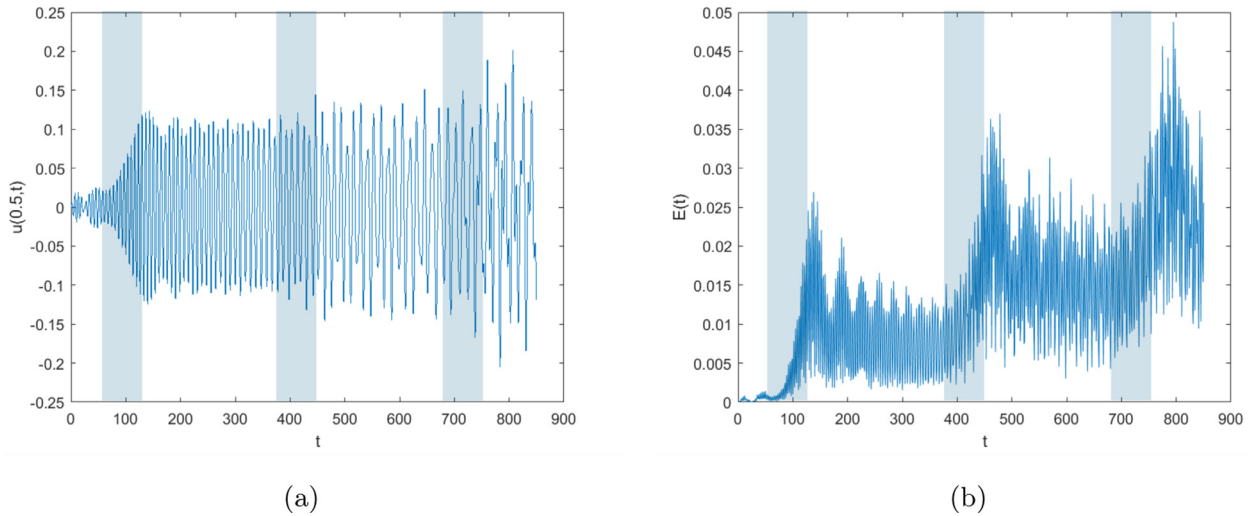


Fig. 3. Perturbation method: (a) Displacements of the mid-point of the cable. (b) The energy of the cable. The shadowed bands represent the resonance zones.

By using the Liouville-Green transformation with $\frac{ds}{dt} = \frac{1}{l(t)}$, we obtain $\alpha l(t) = \lambda_k$, $l(t) = l_0 + \varepsilon \nu_0 t$, which implies that

$$t_k = \frac{\lambda_k - \alpha l_0}{\varepsilon \alpha \nu_0}, \quad k \in \mathbb{N}, \tag{54}$$

where λ_k is given by (19). From the analysis in Section 4.1, we observe that the resonance times depend on the mode numbers k . Resonance for the first oscillation mode does not occur in this numerical example. For the second, third and fourth oscillation modes, resonance emerges for times $t_2 \approx 92.7, t_3 \approx 406.8, t_4 \approx 721.0$, respectively. The solution $\tilde{u}(\xi, t)$ in (50), and its corresponding energy are illustrated in Fig. 3, respectively.

5. Numerical approximation

In this section we will directly integrate problem (3) with a numerical method. To solve (3) numerically, we first rewrite (3) as

$$\begin{cases} \tilde{u}_{tt} - \frac{1}{l^2} \tilde{u}_{\xi\xi} = \frac{2\nu}{l} \xi \tilde{u}_{\xi t} - \frac{2\nu}{l} \tilde{u}_{\xi t} - c \tilde{u}_t + O(\varepsilon^2), & 0 \leq \xi \leq 1, t > 0, \\ \tilde{u}_{\xi\xi}(1, t) + \frac{\rho l l'}{m} \tilde{u}_\xi(1, t) = [(c - c_u) l^2 \tilde{u}_t]_{\xi=1} + O(\varepsilon^2), & \tilde{u}(0, t) = \tilde{e}(t) = \beta \sin(\alpha t), \quad t > 0, \\ \tilde{u}(\xi, 0) = \tilde{u}_0(\xi), \quad \tilde{u}_t(\xi, 0) = \tilde{u}_1(\xi), & 0 \leq \xi \leq 1. \end{cases} \tag{55}$$

By using the transformation $\tilde{u}(\xi, t) = \check{u}(\xi, t) + \beta \sin(\alpha t) + \xi \frac{m l'}{\rho l} (c - c_u) \check{u}_t(1, t)$, problem (55) can be written as

$$\begin{cases} \check{u}_{tt} - \frac{1}{l^2} \check{u}_{\xi\xi} = \frac{2\nu}{l} (\xi - 1) \check{u}_{\xi t} - c \check{u}_t + \alpha^2 \beta \sin(\alpha t) - \frac{m \xi (c - c_u)}{\rho l l'} \check{u}_{\xi\xi t}(1, t) + O(\varepsilon^2), & 0 \leq \xi \leq 1, t > 0, \\ \check{u}_{\xi\xi}(1, t) + \frac{\rho l l'}{m} \check{u}_\xi(1, t) = O(\varepsilon^2), & \check{u}(0, t) = 0, \quad t > 0, \\ \check{u}(\xi, 0) = \check{u}_0(\xi), \quad \check{u}_t(\xi, 0) = \check{u}_1(\xi), & 0 \leq \xi \leq 1, \end{cases} \tag{56}$$

where $0 \leq \xi \leq 1$ and $t > 0$. For problem (56), we first discretize the partial differential equation in (56) in the ξ -coordinate by using a central finite difference scheme. Then, we rewrite the so-obtained discretized equation in a matrix form and use the numerical time integration method of Crank-Nicolson (see Appendix E). We will use the same parameter values (51) and initial conditions (52) as for the analytic approximation, which is presented in the previous section (see also Fig. 3). Fig. 4 show the displacements at $\xi = 0.5$ and the vibratory energy of the cable, respectively, for times up to $t = 850$.

Comparison with Fig. 3: both Figs. 3 and 4 illustrate that resonances emerge at times $t_1 \approx 92.7, t_2 \approx 406.8, t_3 \approx 721.0$. In the resonance zones the displacements and the energy increase, and between these zones, stay constant (approximately). Around the first resonance time t_1 , the displacement amplitudes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. Around the second resonance time t_2 and the third resonance time t_3 , the amplitudes change again at the $O(\sqrt{\varepsilon})$ level, where ε is a small parameter with $\varepsilon = 0.01$. Moreover, we can observe that, between the resonance times, the frequency ranges are similar in Figs. 3 and 4, and the sizes of the resonance zones are of $O(\frac{1}{\sqrt{\varepsilon}})$. Thus, the numerical simulations in Fig. 4 agree very well with the analytical results as presented in Fig. 3, respectively.

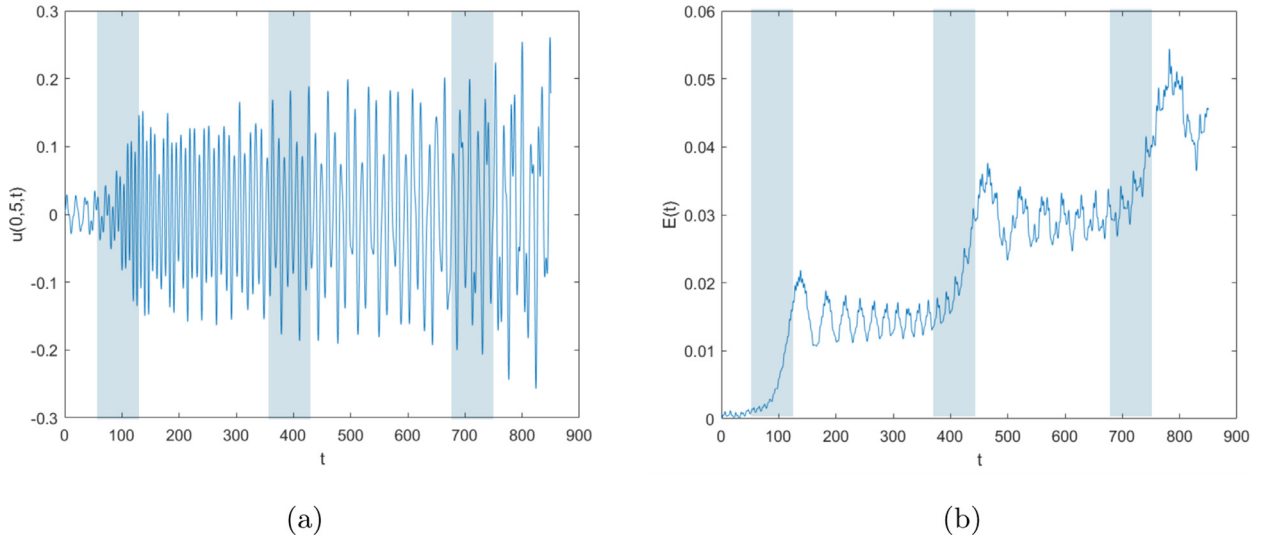


Fig. 4. Numerical method: (a) Displacements of the mid-point of the cable. (b) The energy of the cable. The shadowed bands represent the resonance zones.

6. Concluding remarks

In this paper, the longitudinal vibrations and associated resonances in an hoisting system due to a harmonic excitation at one of its boundaries have been studied. The problem is described by a partial differential equation (PDE) on a time-varying spatial interval with a small harmonic disturbance at one end and a moving nonclassical boundary condition at the other end. By assuming that the small harmonic boundary disturbance is of order ε and by assuming that the initial values are also small and of order ε , it is shown in this paper that for a given arbitrary boundary disturbance frequency, many oscillation modes jump up from $O(\varepsilon)$ to $O(\sqrt{\varepsilon})$. To obtain these results an adapted version of the method of separation of variables is introduced and presented, and perturbation methods, (such as averaging methods, singular perturbation techniques, and multiple timescales perturbation methods) are used. Furthermore, explicit, and accurate approximations of the solution of the initial-boundary value problem are constructed. These approximations are valid on time-scales of order ε^{-1} . Also approximations of the solution of the initial-boundary value problem are computed by using a numerical method. These numerical approximations are in full agreement with the analytically obtained approximations. The presented methods clearly indicate how more complicated problems can now be treated analytically. Also more complicated boundary conditions and changes of cable length over time can be included in the analysis of these problems. Finally, it should be remarked that we intend to apply the presented analytical approach to nonlinearly coupled transverse and longitudinal vibrations of axially moving cables. For these problems the partial differential equations, the boundary conditions and the nonlinear terms are expected to give challenges, which might be solved by applying the approach as has been presented in this paper.

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Appendix A. The derivation of motion (1)

According to Fig. 1, the partial differential equation (PDE) can be derived by Hamilton’s variational principle:

$$\int_{t_1}^{t_2} (\delta E_k(t) - \delta E_p(t) + \delta W_c(t)) dt = 0. \tag{A.1}$$

The Kinetic energy $E_k(t)$ can be represented as $E_k(t) = \frac{1}{2} \rho \int_0^{l(t)} (\frac{Du}{Dt} + v)^2 dx + \frac{1}{2} m (\frac{Du}{Dt} + v)^2 |_{x=l(t)}$, the Potential energy $E_p(t)$ can be expressed as $E_p(t) = \frac{1}{2} EA \int_0^{l(t)} u_x^2 dx + \int_0^{l(t)} T u_x dx + E_{gs} - \int_0^{l(t)} \rho g u dx - m g u |_{x=l(t)}$, and

$$\begin{aligned} \delta E_k(t) - \delta E_p(t) = & \rho \int_0^{l(t)} (\frac{Du}{Dt} + v) \delta \frac{Du}{Dt} dx + m (\frac{Du}{Dt} + v) \delta \frac{Du}{Dt} |_{x=l(t)} \\ & - [EA \int_0^{l(t)} u_x \delta u_x dx + \int_0^{l(t)} T \delta u_x dx - \int_0^{l(t)} \rho g \delta u dx - m g \delta u |_{x=l(t)}], \end{aligned} \tag{A.2}$$

where the operator $\frac{Du}{Dt}$ is defined as $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = u_t + vu_x$. The virtual work δW_c done by the distributed and the lumped damping force is given by

$$\delta W_c(t) = - \int_0^{l(t)} c \frac{Du}{Dt} \delta u dx - c_u \frac{Du}{Dt} \delta u|_{x=l(t)}. \tag{A.3}$$

By substituting the Eqs. (A.2)-(A.3) into (A.1), we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^{l(t)} \rho \left(\frac{Du}{Dt} + v \right) \delta \frac{Du}{Dt} dx dt + \int_{t_1}^{t_2} m \left(\frac{Du}{Dt} + v \right) \delta \frac{Du}{Dt} |_{x=l(t)} dt \\ & - EA \int_{t_1}^{t_2} \int_0^{l(t)} u_x \delta u_x dx dt - \int_{t_1}^{t_2} \int_0^{l(t)} T \delta u_x dx dt + \int_{t_1}^{t_2} \int_0^{l(t)} \rho g \delta u dx dt \\ & + \int_{t_1}^{t_2} mg \delta u|_{x=l(t)} dt - \int_{t_1}^{t_2} \int_0^{l(t)} c \frac{Du}{Dt} \delta u dx dt - \int_{t_1}^{t_2} c_u \frac{Du}{Dt} \delta u|_{x=l(t)} dt = 0. \end{aligned} \tag{A.4}$$

By integrating by parts the integrals in (A.4) it then follows that (A.4) can be rewritten in:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^{l(t)} [-\rho(u_{tt} + 2vu_{xt} + v^2u_{xx}) + EAu_{xx} + T_x + \rho g - c(u_t + vu_x)] \delta u dx dt \\ & + \int_{t_1}^{t_2} [-m(u_{tt} + 2vu_{xt} + v^2u_{xx} + au_x) - EAu_x - T + mg - c_u(u_t + vu_x)] \delta u|_{x=l(t)} dt \\ & + \int_{t_1}^{t_2} [\rho v(u_t + vu_x + v) + EAu_x + T] \delta u|_{x=0} dt = 0. \end{aligned} \tag{A.5}$$

So, the initial boundary value problem of the system can be obtained from (A.5) as

$$\rho(u_{tt} + 2vu_{xt} + v^2u_{xx}) - EAu_{xx} - T_x - \rho g + c(u_t + vu_x) = 0, \quad 0 \leq x \leq l(t), \quad t > 0, \tag{A.6}$$

$$[m(u_{tt} + 2vu_{xt} + v^2u_{xx}) + EAu_x + T - mg + c_u(u_t + vu_x)]|_{x=l(t)} = 0, \quad t > 0, \tag{A.7}$$

$$EAu_x + T + \rho v(u_t + vu_x + v)|_{x=0} = 0, \quad t > 0. \tag{A.8}$$

Note that (A.7) and (A.8) are the natural boundary conditions. However, the natural boundary condition (A.8) is not appropriate for our problem, since the cable at the top has an assumed and prescribed displacement $e(t)$, which is supposed to be generated by the catenary system (consisting of drum, head sheave) in vertical direction. Thus, the boundary condition is given by $u(e(t), t) = e(t)$, $t \geq 0$. By using the Taylor expansion for $u(x, t)$ in x for $x = 0$, and by assuming that $e(t)$ and $u(x, t)$ are small, the boundary condition

$$e(t) = u(e(t), t) = u(0, t) + e(t) \frac{\partial u}{\partial x}(0, t) + O(e^2(t))$$

can be approximated by $u(0, t) = e(t)$. Since the tension $T(x, t)$ is given by $T(x, t) = [m + \rho(l(t) - x)]g$, $0 \leq x \leq l(t)$, it then follows that the initial boundary value problem for the axially moving hoisting rope is given by (1).

Appendix B. Transformation to a fixed domain

By introducing a new time-like variable $s(t)$ with $\frac{ds}{dt} = \frac{1}{l(t)}$, $l(t) = \hat{l}(s) = l_0 e^{\varepsilon v_0 s}$. All partial derivatives then become $s = \frac{1}{\varepsilon v_0} \ln(\frac{l(t)}{l_0})$, $\tilde{u}_t = \frac{1}{l} \tilde{u}_s$, $\tilde{u}_{\xi t} = \frac{1}{l} \tilde{u}_{\xi s}$, $\tilde{u}_{tt} = \frac{1}{l^2} \tilde{u}_{ss} - \frac{v}{l} \tilde{u}_s$, $\tilde{e}(t) = \beta \sin(\frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s} - 1))$, where $\tilde{u}(\xi, t) = \tilde{u}(\xi, s)$. Substituting these derivatives into (3), we obtain the following problem for $\tilde{u}(\xi, s)$:

$$\begin{cases} \tilde{u}_{ss} - \tilde{u}_{\xi\xi} = v\tilde{u}_s + 2v\xi\tilde{u}_{\xi s} - 2v\tilde{u}_{\xi s} - c\hat{l}\tilde{u}_s + O(\varepsilon^2), & 0 \leq \xi \leq 1, \quad s > 0, \\ \tilde{u}_{ss}(1, s) + \frac{\rho\hat{l}}{m}\tilde{u}_{\xi}(1, s) = [v\hat{l}\tilde{u}_s + 2v\xi\tilde{u}_{\xi s} - 2v\tilde{u}_{\xi s} - c_u\hat{l}\tilde{u}_s]|_{\xi=1} + O(\varepsilon^2), & s > 0, \\ \tilde{u}(0, s) = \beta \sin(\frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s} - 1)), & s > 0, \\ \tilde{u}(\xi, 0) = f(\xi), \quad \tilde{u}_s(\xi, 0) = g(\xi), & 0 \leq \xi \leq 1, \end{cases} \tag{B.1}$$

where $\hat{l} = \hat{l}(s)$, $f(\xi) = \tilde{u}_0(\xi)$ and $g(\xi) = l_0 \tilde{u}_1(\xi)$. By using the PDE, the boundary condition at $\xi = 1$ can be rewritten and we obtain from (B.1) the following problem:

$$\begin{cases} \tilde{u}_{ss} - \tilde{u}_{\xi\xi} = v\tilde{u}_s + 2v\xi\tilde{u}_{\xi s} - 2v\tilde{u}_{\xi s} - c\hat{l}\tilde{u}_s + O(\varepsilon^2), & 0 \leq \xi \leq 1, \quad s > 0, \\ \tilde{u}_{\xi\xi}(1, s) + \frac{\rho\hat{l}}{m}\tilde{u}_{\xi}(1, s) = (c - c_u)\hat{l}\tilde{u}_s(1, s) + O(\varepsilon^2), & s > 0, \\ \tilde{u}(0, s) = \tilde{e}(s) = \beta \sin(\frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s} - 1)), & s > 0, \\ \tilde{u}(\xi, 0) = f(\xi), \quad \tilde{u}_s(\xi, 0) = g(\xi), & 0 \leq \xi \leq 1. \end{cases} \tag{B.2}$$

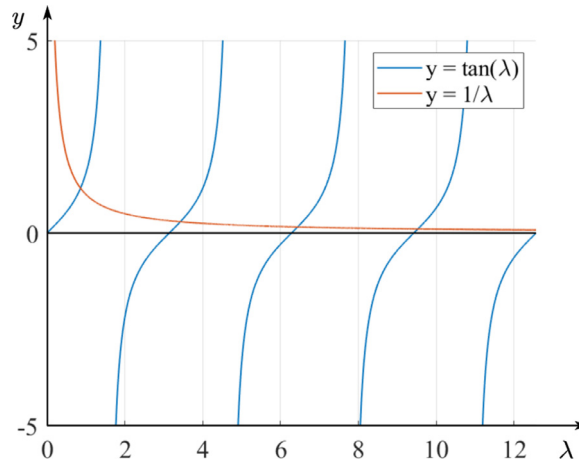


Fig. C.1. For $s=0$, intersection points of $y = \tan \lambda$ and $y = \frac{1}{\lambda}$ are giving $\lambda_n(0)$.

Appendix C. An adapted version of the method of separation of variables

By substituting $T(s, \tau)X(\xi, \tau)$ into the partial differential equation in (8), we obtain

$$\frac{T_{ss}(s, \tau)}{T(s, \tau)} + O(\varepsilon) = \frac{X_{\xi\xi}(\xi, \tau)}{X(\xi, \tau)}, \quad 0 \leq \xi \leq 1, \quad s > 0, \quad \tau > 0. \tag{C.1}$$

The $O(1)$ part of the left-hand side of Eq. (C.1) is a function of s and τ , and the right-hand side is a function of ξ and τ . To be equal, both sides need to be equal to a function of τ . Let this function be $-\lambda^2(\tau)$ (which will be defined later), so we obtain from (C.1) by neglecting terms of order ε : $\frac{T_{ss}(s, \tau)}{T(s, \tau)} = \frac{X_{\xi\xi}(\xi, \tau)}{X(\xi, \tau)} = -\lambda^2(\tau)$, implying:

$$X_{\xi\xi}(\xi, \tau) + \lambda^2(\tau)X(\xi, \tau) = 0, \quad T_{ss}(s, \tau) + \lambda^2(\tau)T(s, \tau) = 0, \quad 0 \leq \xi \leq 1, \quad s > 0, \quad \tau > 0. \tag{C.2}$$

In accordance with the first equation for $X(\xi, \tau)$ in (C.2) and boundary conditions in (E.3), a nontrivial solution $X_n(\xi, \tau)$ is

$$X_n(\xi, \tau) = B_n(\tau) \sin(\lambda_n(\tau)\xi), \tag{C.3}$$

where $B_n(\tau)$ is an arbitrary function of τ only, and $\lambda_n(\tau)$ is given by (10). Assuming that $\frac{\rho l_0}{m} = 1$, the values of $\lambda_n(0)$ can be obtained in Fig. C.1. It should be observed that the eigenfunctions $X_n(\xi, \tau)$ are orthogonal on $0 < \xi < 1$. And so,

The general solution of (7)-(8) can be expanded in the form in (9). By substituting Eq. (9) into the nonhomogeneous governing equation and initial conditions in (6), we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} [(\bar{T}_{n,ss} + 2\varepsilon\bar{T}_{n,s\tau} + \lambda_n^2(\tau)\bar{T}_n) \sin(\lambda_n(\tau)\xi) + 2\varepsilon\xi \frac{d\lambda_n(\tau)}{d\tau} \bar{T}_{n,s} \cos(\lambda_n(\tau)\xi)] \\ = & \sum_{n=1}^{\infty} \varepsilon[(v_0 - c_0\bar{l}(\tau))\bar{T}_{n,s} \sin(\lambda_n(\tau)\xi) + 2(\xi - 1)v_0\lambda_n(\tau)\bar{T}_{n,s} \cos(\lambda_n(\tau)\xi) \\ & + \frac{m(c_0 - c_{u0})\lambda_n^2(\tau)\xi}{\rho L} \bar{T}_{n,s} \sin(\lambda_n(\tau)\xi)] + \varepsilon\beta_0\alpha^2\bar{l}^2(\tau) \sin(\frac{\alpha l_0}{\varepsilon v_0}(e^{\varepsilon v_0 s} - 1)) + O(\varepsilon^2), \\ & \sum_{n=1}^{\infty} [\bar{T}_n(0, 0) \sin(\lambda_n(0)\xi)] = W_0(\xi), \end{aligned} \tag{C.4}$$

$$\sum_{n=1}^{\infty} [(\bar{T}_{n,s}(0, 0) + \varepsilon\bar{T}_{n,\tau}(0, 0)) \sin(\lambda_n(0)\xi) + \varepsilon\bar{T}_n(0, 0) \frac{d\lambda_n(0)}{d\tau} \xi \cos(\lambda_n(0)\xi)] = W_1(\xi). \tag{C.4}$$

Now, let $\sigma(\tau, \xi) = 1 + \frac{2m}{\rho l l(\tau)}\delta(\xi - 1)$ be a weight function, where $\delta(\xi - 1)$ is the Dirac delta function (with $\delta(\xi - 1) = 0$ for $\xi \neq 1$, and $\int_0^1 \delta(\xi - 1)d\xi = \frac{1}{2}$). By multiplying the first equation in (C.4) by $\sigma(\tau, \xi) \sin(\lambda_k(\tau)\xi)$, and the second and third equations in (C.4) with $\sigma(0, \xi) \sin(\lambda_k(0)\xi)$, by integrating the so-obtained equation from $\xi = 0$ to $\xi = 1$, and by using the fact that the $\sin(\lambda_k(\tau)\xi)$ functions subject to the inner product with weight function $\sigma(\tau, \xi)$ are orthogonal on $0 \leq \xi \leq 1$, it follows that $\bar{T}_k(s, \tau)$ for $k = 1, 2, 3, \dots$, and $s > 0, \tau > 0$ have to satisfy (11).

Appendix D. The construction of the functions $C_{k,1}$ and $C_{k,2}$

First of all, by using the initial conditions in Eq. (41), it follows that $C_{k,1}(0, 0) = C_{k,2}(0, 0) = 0$. Then, we shall solve the $O(\varepsilon)$ – problem (42). This problem (outside as well as inside the resonance manifold) can be written as

$$\frac{\partial^2 w_{k,1}}{\partial \tilde{\phi}_{k,0}^2} + w_{k,1} = -2\left[\frac{\partial C_{k,1}}{\partial \tilde{\phi}_{k,1}} \cos(\tilde{\phi}_{k,0}) - \frac{\partial C_{k,2}}{\partial \tilde{\phi}_{k,1}} \sin(\tilde{\phi}_{k,0})\right] + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{\alpha l_0}{\varepsilon v_0} (e^{v_0 \tau} - 1)\right), \tag{D.1}$$

$$w_{k,1}(0, 0, 0) = \bar{F}_k, \quad \frac{\partial w_{k,1}}{\partial \tilde{\phi}_{k,0}}(0, 0, 0) = -\frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,1}}(0, 0, 0) + \bar{G}_k. \tag{D.2}$$

Outside of the resonance zone, it should be observed that the last term in Eq. (D.1) does not give rise to secular terms in $w_{k,1}$. To avoid secular terms outside the resonance zone, it follows from (D.1) that $C_{k,1}$ and $C_{k,2}$ have to satisfy the following conditions $\frac{\partial C_{k,1}}{\partial \tilde{\phi}_{k,1}} = 0$, $\frac{\partial C_{k,2}}{\partial \tilde{\phi}_{k,1}} = 0$, which has as solutions

$$C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \bar{C}_{k,1}(\tilde{\phi}_{k,2}), \quad C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \bar{C}_{k,2}(\tilde{\phi}_{k,2}), \tag{D.3}$$

where $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are still unknown functions of the slow variable $\tilde{\phi}_{k,2}$, and can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})$ – problem (43). Since $C_{k,1}(0, 0) = C_{k,2}(0, 0) = 0$, this implies that $\bar{C}_{k,1}(0) = \bar{C}_{k,2}(0) = 0$. Now we consider the $O(\varepsilon)$ equation inside the resonance zone and observe that inside the resonance zone, the last term in Eq. (D.1) gives rise to secular terms in $w_{k,1}$. According to (30), we can write $\sin\left(\frac{\alpha l_0}{\varepsilon v_0} (e^{v_0 \tau} - 1)\right) = \sin\left(\frac{1}{2} \gamma \bar{s}_1^2 + \frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s_k} - 1) - \phi_k(s_k) + \tilde{\phi}_{k,0}\right)$, where γ is given by (28). So we can rewrite Eq. (D.1) inside the resonance zone as

$$\begin{aligned} \frac{\partial^2 w_{k,1}}{\partial \tilde{\phi}_{k,0}^2} + w_{k,1} = & [-2 \frac{\partial C_{k,1}}{\partial \tilde{\phi}_{k,1}} + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{1}{2} \gamma \bar{s}_1^2 + \frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s_k} - 1) - \phi_k(s_k)\right)] \cos(\tilde{\phi}_{k,0}) \\ & + [2 \frac{\partial C_{k,2}}{\partial \tilde{\phi}_{k,1}} + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \cos\left(\frac{1}{2} \gamma \bar{s}_1^2 + \frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s_k} - 1) - \phi_k(s_k)\right)] \sin(\tilde{\phi}_{k,0}). \end{aligned}$$

In order to remove secular terms, it follows that $C_{k,1}$ and $C_{k,2}$ have to satisfy

$$-2 \frac{\partial C_{k,1}}{\partial \tilde{\phi}_{k,1}} + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \sin\left(\frac{1}{2} \gamma \bar{s}_1^2 + \vartheta(s_k)\right) = 0, \quad 2 \frac{\partial C_{k,2}}{\partial \tilde{\phi}_{k,1}} + \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k^2(\tau)} \cos\left(\frac{1}{2} \gamma \bar{s}_1^2 + \vartheta(s_k)\right) = 0,$$

where $\vartheta(s_k) = \frac{\alpha l_0}{\varepsilon v_0} (e^{\varepsilon v_0 s_k} - 1) - \phi_k(s_k)$ and $\frac{\partial C_{k,i}}{\partial \tilde{\phi}_{k,1}} = \frac{1}{\lambda_k(\tau)} \frac{\partial C_{k,i}}{\partial \bar{s}_1}$, $i = 1, 2$. Thus,

$$\begin{aligned} C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) &= \bar{C}_{k,1}(\tilde{\phi}_{k,2}) + \frac{1}{2} \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{\lambda_k(\tau)} \int_b^{\bar{s}_1} \sin\left(\frac{1}{2} \gamma \bar{s}_1^2 + \vartheta(s_k)\right) d\bar{s}_1 \\ &= \bar{C}_{k,1}(\tilde{\phi}_{k,2}) + \bar{F}(\bar{s}_1, \tau), \\ C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) &= \bar{C}_{k,2}(\tilde{\phi}_{k,2}) - \bar{G}(\bar{s}_1, \tau), \end{aligned} \tag{D.4}$$

where

$$\begin{aligned} \bar{F}(\bar{s}_1, \tau) &= \bar{\alpha} \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{2 \lambda_k(\tau)} [\sin(\vartheta(s_k)) \bar{C}_{Fr}(\bar{s}_1) + \cos(\vartheta(s_k)) \bar{S}_{Fr}(\bar{s}_1)], \\ \bar{G}(\bar{s}_1, \tau) &= \bar{\alpha} \alpha^2 \beta_0 \bar{l}^2(\tau) \frac{d_k(\tau)}{2 \lambda_k(\tau)} [\cos(\vartheta(s_k)) \bar{C}_{Fr}(\bar{s}_1) - \sin(\vartheta(s_k)) \bar{S}_{Fr}(\bar{s}_1)], \end{aligned} \tag{D.5}$$

and where $\bar{C}_{k,1}$ and $\bar{C}_{k,2}$ are still unknown functions of the slow variables $\tilde{\phi}_{k,2}$. The undetermined behaviour with respect to $\tilde{\phi}_{k,2}$ can be used to avoid secular terms in the $O(\varepsilon\sqrt{\varepsilon})$ – problem (43). Taking into account the secularity conditions, the general solution of $w_{k,1}$ is given by

$$w_{k,1}(\tilde{\phi}_{k,0}, \tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}; \sqrt{\varepsilon}) = D_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) \sin(\tilde{\phi}_{k,0}) + D_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) \cos(\tilde{\phi}_{k,0}), \tag{D.6}$$

where $D_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})$ and $D_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})$ are unknown functions of $\tilde{\phi}_{k,1}$ and $\tilde{\phi}_{k,2}$. By using the initial conditions in Eq. (42), the values of $D_{k,1}(0, 0)$ and $D_{k,2}(0, 0)$ are given by the following equations $D_{k,1}(0, 0) = \frac{\bar{C}_k}{\lambda_k(0)} - \frac{\partial C_{k,2}}{\partial \tilde{\phi}_{k,1}}(0, 0)$, $D_{k,2}(0, 0) = \bar{F}_k$.

The $O(\varepsilon\sqrt{\varepsilon})$ – problem (43) outside and inside the resonance manifold can be written as

$$\begin{aligned} \frac{\partial^2 w_{k,2}}{\partial \tilde{\phi}_{k,0}^2} + w_{k,2} = & -2\left[\frac{\partial D_{k,1}}{\partial \tilde{\phi}_{k,1}} \cos(\tilde{\phi}_{k,0}) - \frac{\partial D_{k,2}}{\partial \tilde{\phi}_{k,1}} \sin(\tilde{\phi}_{k,0})\right] - 2\left[\frac{\partial C_{k,1}}{\partial \tilde{\phi}_{k,2}} \cos(\tilde{\phi}_{k,0}) - \frac{\partial C_{k,2}}{\partial \tilde{\phi}_{k,2}} \sin(\tilde{\phi}_{k,0})\right] \\ & - \left[\frac{\partial^2 C_{k,1}}{\partial \tilde{\phi}_{k,1}^2} \sin(\tilde{\phi}_{k,0}) + \frac{\partial^2 C_{k,2}}{\partial \tilde{\phi}_{k,1}^2} \cos(\tilde{\phi}_{k,0})\right] \end{aligned}$$

$$\begin{aligned}
 & +[(v_0 - c_0 \bar{l}(\tau))\lambda_k(\tau) - \frac{d\lambda_k(\tau)}{d\tau}] \frac{1}{\lambda_k^2(\tau)} [C_{k,1} \cos(\tilde{\phi}_{k,0}) - C_{k,2} \sin(\tilde{\phi}_{k,0})] \\
 & - 2[c_{k,k}^1(\tau) \frac{d\lambda_k(\tau)}{d\tau} - v_0 c_{k,k}^2(\tau)] \\
 & - \frac{m(c_0 - c_{u0})}{2\rho L} c_{k,k}^3(\tau) \frac{1}{\lambda_k(\tau)} [C_{k,1} \cos(\tilde{\phi}_{k,0}) - C_{k,2} \sin(\tilde{\phi}_{k,0})] \\
 & - 2 \sum_{n \neq k}^{\infty} [c_{n,k}^1(\tau) \frac{d\lambda_n(\tau)}{d\tau} - v_0 c_{n,k}^2(\tau)] \\
 & - \frac{m(c_0 - c_{u0})}{2\rho L} c_{n,k}^3(\tau) \frac{1}{\lambda_n(\tau)} [C_{n,1} \cos(\tilde{\phi}_{n,0}) - C_{n,2} \sin(\tilde{\phi}_{n,0})] \\
 w_{k,2}(0, 0, 0) = 0, \quad \frac{\partial w_{k,2}}{\partial \tilde{\phi}_{k,0}}(0, 0, 0) = -\frac{\partial w_{k,0}}{\partial \tilde{\phi}_{k,2}}(0, 0, 0) - \frac{\partial w_{k,1}}{\partial \tilde{\phi}_{k,1}}(0, 0, 0).
 \end{aligned} \tag{D.7}$$

To avoid secular terms in the solution $w_{k,2}$ of Eq. (D.7), outside the resonance zone, it follows from (D.7) that $D_{k,1}$, $D_{k,2}$, $\bar{C}_{k,1}$, and $\bar{C}_{k,2}$ have to satisfy:

$$-2 \frac{\partial D_{k,1}}{\partial \tilde{\phi}_{k,1}} - 2 \frac{\partial \bar{C}_{k,1}}{\partial \tilde{\phi}_{k,2}} + \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} \bar{C}_{k,1} = 0, \quad 2 \frac{\partial D_{k,2}}{\partial \tilde{\phi}_{k,1}} + 2 \frac{\partial \bar{C}_{k,2}}{\partial \tilde{\phi}_{k,2}} - \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} \bar{C}_{k,2} = 0, \tag{D.8}$$

where $\bar{\zeta}(\tau) = (v_0 - c_0 \bar{l}(\tau)) - \frac{d\lambda_k(\tau)}{d\tau} \frac{1}{\lambda_k(\tau)} - 2c_{k,k}^1(\tau) \frac{d\lambda_k(\tau)}{d\tau} + 2v_0 c_{k,k}^2(\tau) + 2 \frac{m(c_0 - c_{u0})}{2\rho L} c_{k,k}^3(\tau)$. If we solve Eq. (D.8) for $D_{k,1}$ and $D_{k,2}$ and integrate Eqs. (D.8) with respect to $\tilde{\phi}_{k,1}$, we observe that the solutions will be unbounded in $\tilde{\phi}_{k,1}$ due to terms which are only depending on $\tilde{\phi}_{k,2}$. Therefore, to have secular-free solutions, the following conditions have to be imposed independently

$$\frac{\partial \bar{C}_{k,1}}{\partial \tilde{\phi}_{k,2}} - \frac{1}{2} \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} \bar{C}_{k,1} = 0, \quad \frac{\partial \bar{C}_{k,2}}{\partial \tilde{\phi}_{k,2}} - \frac{1}{2} \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} \bar{C}_{k,2} = 0. \tag{D.9}$$

For $\frac{\partial \bar{C}_{k,i}}{\partial \tilde{\phi}_{k,2}} = \frac{1}{\lambda_k(\tau)} \frac{\partial \bar{C}_{k,i}}{\partial \tilde{s}_2^2}$, $i = 1, 2$, we obtain $\bar{C}_{k,1} = \bar{C}_{k,1}(0) e^{\int_0^\tau \frac{1}{2} \bar{\zeta}(\tau) d\tau}$, $\bar{C}_{k,2} = \bar{C}_{k,2}(0) e^{\int_0^\tau \frac{1}{2} \bar{\zeta}(\tau) d\tau}$. Since $\bar{C}_{k,1}(0) = \bar{C}_{k,2}(0) = 0$ and $\lambda_k(\tau)$ is bounded, it follows from Eq. (D.9) that outside the resonance zone

$$C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = 0, \quad C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = 0. \tag{D.10}$$

Inside the resonance zone, to avoid secular terms in the solution $w_{k,2}$ of Eq. (D.7), the following conditions have to be imposed

$$-2 \frac{\partial D_{k,1}}{\partial \tilde{\phi}_{k,1}} - 2 \frac{\partial \bar{C}_{k,1}}{\partial \tilde{\phi}_{k,2}} - 2 \frac{\partial \tilde{F}(\tilde{s}_1, \tau)}{\partial \tilde{\phi}_{k,2}} + \frac{1}{\lambda_k^2(\tau)} \frac{\partial^2 \tilde{G}(\tilde{s}_1)}{\partial \tilde{s}_1^2} + \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} (\bar{C}_{k,1} + \tilde{F}(\tilde{s}_1, \tau)) = 0 \tag{D.11}$$

$$2 \frac{\partial D_{k,2}}{\partial \tilde{\phi}_{k,1}} + 2 \frac{\partial \bar{C}_{k,2}}{\partial \tilde{\phi}_{k,2}} - 2 \frac{\partial \tilde{G}(\tilde{s}_1, \tau)}{\partial \tilde{\phi}_{k,2}} - \frac{1}{\lambda_k^2(\tau)} \frac{\partial^2 \tilde{F}(\tilde{s}_1)}{\partial \tilde{s}_1^2} - \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} (\bar{C}_{k,2} - \tilde{G}(\tilde{s}_1, \tau)) = 0. \tag{D.12}$$

If we solve Eqs. (D.11) and (D.12) for $D_{k,1}$ and $D_{k,2}$ and integrate respect to $\tilde{\phi}_{k,1}$, we observe that the solutions will be unbounded in $\tilde{\phi}_{k,1}$ due to terms which are only depending on $\tilde{\phi}_{k,2}$. Therefore, to have secular-free solutions, the following conditions have to be imposed independently

$$\frac{\partial \bar{C}_{k,1}}{\partial \tilde{\phi}_{k,2}} = \frac{1}{2} \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} \bar{C}_{k,1}, \quad \frac{\partial \bar{C}_{k,2}}{\partial \tilde{\phi}_{k,2}} = \frac{1}{2} \bar{\zeta}(\tau) \frac{1}{\lambda_k(\tau)} \bar{C}_{k,2}. \tag{D.13}$$

Since $\bar{C}_{k,1}(0) = \bar{C}_{k,2}(0) = 0$, it follows from Eq. (D.13) that inside the resonance zone

$$\bar{C}_{k,1}(\tilde{\phi}_{k,2}) = 0, \quad \bar{C}_{k,2}(\tilde{\phi}_{k,2}) = 0, \quad C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \tilde{F}(\tilde{s}_1, \tau), \quad C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2}) = \tilde{G}(\tilde{s}_1, \tau), \tag{D.14}$$

where $\tilde{F}(\tilde{s}_1, \tau)$ and $\tilde{G}(\tilde{s}_1, \tau)$ are given by (D.5). Thus, we obtain the functions of $C_{k,1}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})$ and $C_{k,2}(\tilde{\phi}_{k,1}, \tilde{\phi}_{k,2})$ in (D.10) and (D.14). Similarly, we also can obtain the solution $w_{k,1}$ of $O(\varepsilon)$ problem and the solution $w_{k,2}$ of $O(\varepsilon\sqrt{\varepsilon})$ problem by using the above analysis. In order to shorten the paper, this derivation is omitted.

Appendix E. Discretization and energy

To solve (56) numerically, it is convenient to rewrite the second order partial differential equation as a system of two coupled first-order partial differential equations:

$$\check{u}_t = \check{v}, \quad \check{v}_t = \frac{1}{l^2} \check{u}_{\xi\xi} + \varepsilon \left[\frac{2v_0}{l} (\xi - 1) \check{v}_\xi - c_0 \check{v} + \alpha^2 \beta \sin(\alpha t) - \frac{m\xi(c - c_u)}{\rho l l} \check{v}_{\xi\xi}(1, t) \right]. \tag{E.1}$$

Next, let us use mesh grids $\xi_j = (j - 1)\Delta\xi$ for $j = 1, 2, n, n + 1$ with $n\Delta\xi = 1$. By introducing the differences, $\check{u}_{\xi\xi}(\xi_j, t) = \frac{\check{u}_{j+1} - 2\check{u}_j + \check{u}_{j-1}}{(\Delta\xi)^2} + O((\Delta\xi)^2)$, $\check{v}_{\xi}(\xi_j, t) = \frac{\check{v}_{j+1} - \check{v}_{j-1}}{2\Delta\xi} + O((\Delta\xi)^2)$, $\check{v}_{\xi\xi}(\xi_j, t) = \frac{\check{v}_{j+1} - 2\check{v}_j + \check{v}_{j-1}}{(\Delta\xi)^2} + O((\Delta\xi)^2)$, it follows how system (E.1) can be discretized, yielding:

$$\begin{cases} \frac{d\check{u}}{dt}(\xi_j, t) = \check{v}_j, \\ \frac{d\check{v}}{dt}(\xi_j, t) = r(\check{u}_{j+1} - 2\check{u}_j + \check{u}_{j-1}) + q_j(\check{v}_{j+1} - \check{v}_{j-1}) - \varepsilon c_0 v_j + p_j \check{v}_n - p_j \check{v}_{n-1} + \varepsilon \alpha^2 \beta_0 \sin(\alpha t), \end{cases}$$

where $r = \frac{1}{l^2(\Delta\xi)^2}$, $q_j = \frac{\varepsilon v_0(\xi_j - 1)}{l\Delta\xi}$, $p_j = \frac{\varepsilon m(c_0 - c_{u0})\xi_j}{\rho l(2 + \Delta\xi)\Delta\xi}$ for $j=1,2,\dots,n$. Further,

$$R = \begin{pmatrix} -2r & r & 0 & \dots & \dots & 0 \\ r & -2r & r & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & r & -2r & r \\ 0 & \dots & \dots & 0 & c & -c \end{pmatrix} \in \mathbb{R}^{n \times n}, \text{ where } c = \frac{2\rho L}{l(2m + \rho l l \Delta\xi)\Delta\xi}, \text{ and}$$

$$P = \begin{pmatrix} -\varepsilon c_0 & q_1 & 0 & \dots & 0 & -p_1 & p_1 \\ -q_2 & -\varepsilon c_0 & q_2 & \dots & 0 & -p_2 & p_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -q_{n-2} & -\varepsilon c_0 & q_{n-2} - p_{n-2} & p_{n-2} \\ 0 & \dots & 0 & 0 & -q_{n-1} & -\varepsilon c_0 - p_{n-1} & q_{n-1} + p_{n-1} \\ 0 & \dots & \dots & \dots & 0 & d_n - q_n - p_n & e_n - \varepsilon c_0 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

where $d_n = \frac{\rho l l \Delta\xi - 2m}{2m + \rho l l \Delta\xi}$, $e_n = \frac{4m}{2m + \rho l l \Delta\xi} + p_n$. The four matrices \emptyset, I, R and P compose the system matrix M :

$$M = \begin{pmatrix} \emptyset & I \\ R & P \end{pmatrix} \in \mathbb{R}^{2n \times 2n},$$

where \emptyset is the zero matrix, and I is the identity matrix. In addition, let us introduce the following vector: $w = (u_1(\xi_1, t), u_2(\xi_2, t), \dots, u_n(\xi_n, t), v_1(v_1, t), v_2(\xi_2, t), \dots, v_n(\xi_n, t))^T$, $s = (\underbrace{0, 0, \dots, 0}_{n \text{ times}}, \underbrace{\bar{s}, \bar{s}, \dots, \bar{s}}_{n \text{ times}})^T$, where $\bar{s} = \varepsilon \alpha^2 \beta_0 \sin(\alpha t)$. So,

system (E.1) can be written in the following matrix form: $\frac{dw}{dt} = Mw + s$. In order to perform a time integration, we apply the Crank-Nicolson method. Introducing the mesh grid in time, $t_k = k\Delta t$ for $k=1,2,\dots,n$, we obtain

$$w^{k+1} = Dw^k + \frac{\Delta t}{2} (I - \frac{\Delta t}{2} M^{k+1})^{-1} (s^{k+1} + s^k), \tag{E.2}$$

where I is the identity matrix and $I \in \mathbb{R}^{2n \times 2n}$ and $D = (I - \frac{\Delta t}{2} M^{k+1})^{-1} (I + \frac{\Delta t}{2} M^k)$.

The total mechanical energy of the problem (8) is given by

$$E(t) = \frac{1}{2} \int_0^{l(t)} [\rho(u_t + v u_x)^2 + EAu_x^2] dx + \frac{m}{2} [u_t(l(t), t) + v u_x(l(t), t)]^2.$$

Using the dimensionless quantities, we rewrite the energy in a dimensionless form:

$$E(t) = \frac{1}{2} EAL \int_0^{l(t)} [(u_t + v u_x)^2 + u_x^2] dx + \frac{EAm}{2\rho} [u_t(l(t), t) + v u_x(l(t), t)]^2.$$

In order to define the energy on the interval (0,1), we obtain problem (10) by using the following transformation $\xi = \frac{x}{l(t)}$:

$$E(t) = \frac{EAL}{2l(t)} \int_0^1 [(l(t)\check{u}_t + (1 - \xi)v\check{u}_{\xi})^2 + \check{u}_{\xi}^2] d\xi + \frac{EAm}{2\rho l^2(t)} [l(t)\check{u}_t(1, t) + (1 - \xi)v\check{u}_{\xi}(1, t)]^2. \tag{E.3}$$

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