Optimal Shrinkage-Based Portfolio Selection in High Dimensions

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Supplement to "Optimal shrinkage-based portfolio selection in high dimensions"[1]

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1 Appendix: Proofs

Here the proofs of the theorems are given. Recall that the sample mean vector and the sample covariance matrix are given by

$$\bar{\mathbf{y}}_n = \frac{1}{n}\mathbf{Y}_n\mathbf{1}_n = \boldsymbol{\mu}_n + \boldsymbol{\Sigma}_n^{\frac{1}{2}}\bar{\mathbf{x}}_n \quad \text{with} \quad \bar{\mathbf{x}}_n = \frac{1}{n}\mathbf{X}_n\mathbf{1}_n \tag{1.1}$$

and

$$\mathbf{S}_n = \frac{1}{n}\mathbf{Y}_n(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')\mathbf{Y}_n' = \boldsymbol{\Sigma}_n^{\frac{1}{2}}\mathbf{V}_n\boldsymbol{\Sigma}_n^{\frac{1}{2}} \quad \text{with} \quad \mathbf{V}_n = \frac{1}{n}\mathbf{X}_n(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')\mathbf{X}_n', \tag{1.2}$$

respectively. Later on, we also make use of $\tilde{\mathbf{V}}_n$ defined by

$$\tilde{\mathbf{V}}_n = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n' \tag{1.3}$$

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and the formula for the 1-rank update of usual inverse given by (c.f., Horn and Johnsohn (1985))

$$\mathbf{V}_n^{-1} = (\tilde{\mathbf{V}}_n - \bar{\mathbf{x}}_n\bar{\mathbf{x}}_n')^{-1} = \tilde{\mathbf{V}}_n^{-1} + \frac{\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n} \tag{1.4}$$

as well as the formula for the 1-rank update of Moore-Penrose inverse (see, Meyer (1973)) expressed as

$$\begin{aligned} \mathbf{V}_n^{+} &= \left(\tilde{\mathbf{V}}' - \bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\right)^{+} \\ &= \tilde{\mathbf{V}}_n^{+} - \frac{\tilde{\mathbf{V}}_n^{+}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^{+})^2 + (\tilde{\mathbf{V}}_n^{+})^2\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^{+})}{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^{+})^2\bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^{+})^3\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^{+})^2\bar{\mathbf{x}}_n)^2}\tilde{\mathbf{V}}_n^{+}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{+}\,. \end{aligned} \tag{1.5}$$

First, we present an important lemma which is a special case of Theorem 1 in Rubio and Mestre (2011).

Lemma 1.1. *Assume (A2). Let a nonrandom $p \times p$-dimensional matrix $\mathbf{\Theta}_p$ and a nonrandom $n \times n$-dimensional matrix $\mathbf{\Theta}_n$ possess a uniformly bounded trace norms (sum of singular values). Then it holds that*

$$\left| tr\left(\mathbf{\Theta}_p(\tilde{\mathbf{V}}_n - z\mathbf{I}_p)^{-1}\right) - m(z)tr\left(\mathbf{\Theta}_p\right)\right| \xrightarrow{a.s.} 0 \tag{1.6}$$

$$\left| tr\left(\mathbf{\Theta}_n(1/n\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n)^{-1}\right) - \underline{m}(z)tr\left(\mathbf{\Theta}_n\right)\right| \xrightarrow{a.s.} 0 \tag{1.7}$$

for $p/n \longrightarrow c \in (0, +\infty)$ as $n \to \infty$, where

$$m(z) = (x(z)-z)^{-1} \quad and \quad \underline{m}(z) = -\frac{1-c}{z} + cm(z) \tag{1.8}$$

with

$$x(z) = \frac{1}{2}\left(1 - c + z + \sqrt{(1-c+z)^2 - 4z}\right)\,. \tag{1.9}$$

Proof of Lemma 1.1: The application of Theorem 1 in Rubio and Mestre (2011) leads to (1.6) where $x(z)$ is a unique solution in $\mathbb{C}^+$ of the following equation

$$\frac{1-x(z)}{x(z)} = \frac{c}{x(z)-z}\,. \tag{1.10}$$

The two solutions of (1.10) are given by

$$x_{1,2}(z) = \frac{1}{2}\left(1 - c + z \pm \sqrt{(1-c+z)^2 - 4z}\right) . \tag{1.11}$$

In order to decide which of two solutions is feasible, we note that $x_{1,2}(z)$ is the Stieltjes transform with a positive imaginary part. Thus, without loss of generality, we can take $z = 1 + c + i2\sqrt{c}$ and get

$$\mathbf{Im}\{x_{1,2}(z)\} = \mathbf{Im}\left\{\frac{1}{2}\left(2 + i2\sqrt{c} \pm i2\sqrt{2c}\right)\right\} = \mathbf{Im}\left\{1 + i\sqrt{c}(1 \pm \sqrt{2})\right\} = \sqrt{c}\left(1 \pm \sqrt{2}\right) , \tag{1.12}$$

which is positive only if the sign " + " is chosen. Hence, the solution is given by

$$x(z) = \frac{1}{2}\left(1 - c + z + \sqrt{(1-c+z)^2 - 4z}\right) . \tag{1.13}$$

The second assertion of the lemma follows directly from Bai and Silverstein (2010). □

We note here that Lemma 1.1 is a special case of Theorem 1 in Rubio and Mestre (2011), where one has uniform convergence in the statement of the theorem. Although it is not precisely written in the statement of Theorem 1 in Rubio and Mestre (2011), this observation follows from its proof on page 600 where after showing pointwise convergence Rubio and Mestre additionally proved the uniform convergence by applying Montel's theorem. In short, they first show that the random sequence of analytic functions of interest forms a normal family and, thus, by Montel's theorem there exists a subsequence of it, which converges uniformly on each compact subset of $\mathbb{C} \setminus \mathbb{R}^+$ to an analytic function and this one vanishes almost surely on $\mathbb{C} \setminus \mathbb{R}^+$. And so, the entire sequence converges uniformly to zero on every compact subset of $\mathbb{C} \setminus \mathbb{R}^+$. Furthermore, it is mentioned on page 348 of Rubio et al. (2012) that the convergence in Theorem 1 of Rubio and Mestre (2011) is in fact uniform.

Moreover, the following result (see, e.g., Theorem 1 on page 176 in Ahlfors (1953)), known as the Weierstrass theorem on the uniform convergence, will be used in a sequel together with Lemma 1.1 in the proofs of the technical lemmas.

Theorem 1.1 (Weierstrass). *Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover, $f'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.*

Because the convergence in Lemma 1.1 is uniform over z on every compact subset of $\mathbb{C}\setminus\mathbb{R}^+$, the Weierstrass theorem allows us to interchange any derivative with respect to z and the limit $n \to \infty$. We will consider compact subsets, which are the small neighbourhoods of zero with $\Re(z) = 0$ (without loss of generality) because all of the times we will let $z \to 0$ in order to get specific limiting expressions of interest. For example, one may take Ω as a unit disk $|z| < 1$ and Ω_n as a disk $|z| < \varepsilon_n$ for some $\varepsilon_n \to 0$ as $n \to \infty$. The analyticity of the function $\text{tr}\left(\mathbf{\Theta}_p(\tilde{\mathbf{V}}_n - z\mathbf{I}_p)^{-1}\right)$ follows immediately from the properties of the Stieltjes transform.

Lemma 1.2. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Euclidean norms. Then it holds that*

$$\left|\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta} - (1-c)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}\right| \stackrel{a.s.}{\longrightarrow} 0\,, \tag{1.14}$$

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n \stackrel{a.s.}{\longrightarrow} c\,, \tag{1.15}$$

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} 0\,, \tag{1.16}$$

$$\left|\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta} - (1-c)^{-3}\boldsymbol{\xi}'\boldsymbol{\theta}\right| \stackrel{a.s.}{\longrightarrow} 0\,, \tag{1.17}$$

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n \stackrel{a.s.}{\longrightarrow} \frac{c}{(1-c)}\,, \tag{1.18}$$

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} 0 \tag{1.19}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

Proof of Lemma 1.2: Since the trace norm of $\boldsymbol{\theta}\boldsymbol{\xi}'$ is uniformly bounded, i.e.

$$||\boldsymbol{\theta}\boldsymbol{\xi}'||_{tr} \le \sqrt{\boldsymbol{\theta}'\boldsymbol{\theta}}\sqrt{\boldsymbol{\xi}'\boldsymbol{\xi}} < \infty,$$

we get from Lemma 1.1 that

$$|tr((\tilde{\mathbf{V}}_n - z\mathbf{I}_p)^{-1}\boldsymbol{\theta}\boldsymbol{\xi}') - m(z)tr(\boldsymbol{\theta}\boldsymbol{\xi}')| \stackrel{a.s.}{\longrightarrow} 0 \ \text{ for } p/n \to c < 1 \text{ as } n \to \infty$$

Furthermore, the application of $m(z) \to (1-c)^{-1}$ as $z \to 0$ leads to

$$|\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta} - (1-c)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}| \xrightarrow{a.s.} 0 \quad \text{for } p/n \to c < 1 \text{ as } n \to \infty\,,$$

which proves (1.14).

For deriving (1.15) we consider

$$\begin{aligned}
\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n &= \lim_{z\to 0} \operatorname{tr}\left[\frac{1}{\sqrt{n}}\mathbf{X}_n'\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)\right] \\
&= \lim_{z\to 0} \operatorname{tr}\left[\left(\frac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)\right] + z\operatorname{tr}\left[\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n\right)^{-1}\left(\frac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)\right],
\end{aligned}$$

where the last equality follows from the Woodbury formula (e.g., Horn and Johnsohn (1985)). The application of Lemma 1.1 and Theorem 1.1 lead to

$$\operatorname{tr}\left[\left(\frac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)\right] + z\operatorname{tr}\left[\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n\right)^{-1}\left(\frac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)\right] \xrightarrow{a.s.} [1 + (c-1) + czm(z)]\operatorname{tr}\left[\left(\frac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)\right]$$

for $p/n \longrightarrow c < 1$ as $n \to \infty$ where $m(z)$ is given by (1.8). Setting $z \to 0$ and taking into account $\lim_{z\to 0} m(z) = \dfrac{1}{1-c}$ we get

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n \xrightarrow{a.s.} 1 + c - 1 = c \text{ for } \frac{p}{n} \longrightarrow c \in (0,1) \text{ as } n \to \infty\,.$$

The result (1.16) was derived in Pan (2014) (see, p. 673 of this reference).

Next, we prove (1.17). It holds that

$$\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta} = \frac{\partial}{\partial z}\operatorname{tr}\left[\left(\tilde{\mathbf{V}}_n - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}\boldsymbol{\xi}'\right]\bigg|_{z=0} = \frac{\partial}{\partial z}\zeta_n(z)\bigg|_{z=0}$$

where $\zeta_n(z) = \operatorname{tr}\left[\left(\tilde{\mathbf{V}}_n - z\mathbf{I}\right)^{-1}\boldsymbol{\theta}\boldsymbol{\xi}'\right]$. From Lemma 1.1 $\zeta_n(z)$ tends a.s. to $m(z)\boldsymbol{\xi}'\boldsymbol{\theta}$ as $n \to \infty$. Furthermore,

$$\frac{\partial}{\partial z}m(z)\bigg|_{z=0} = \frac{\partial}{\partial z}\frac{1}{x(z)-z}\bigg|_{z=0} = -\frac{x'(z)-1}{(x(z)-z)^2}\bigg|_{z=0} = -\left.\frac{\frac{1}{2}\left(1 - \frac{1+c-z}{\sqrt{(1-c+z)^2-4z}}\right) - 1}{(x(z)-z)^2}\right|_{z=0} = \frac{1}{(1-c)^3}\,. \tag{1.20}$$

Consequently, using Lemma 1.1 and Theorem 1.1 we conclude

$$|\boldsymbol{\xi}'\mathbf{S}_n^{-2}\boldsymbol{\theta} - (1-c)^{-3}\boldsymbol{\xi}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\theta}| \xrightarrow{a.s.} 0 \quad \text{for } p/n \to c < 1 \text{ as } n \to \infty\,.$$

Let $\eta_n(z) = \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n - z\mathbf{I})^{-1}\bar{\mathbf{x}}_n$ and $\boldsymbol{\Theta}_n = \left(\dfrac{\mathbf{1}_n\mathbf{1}_n'}{n}\right)$. Then

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n = \frac{\partial}{\partial z}\eta_n(z)\bigg|_{z=0},$$

where

$$\begin{aligned}
\eta_n(z) &= \operatorname{tr}\left[\frac{1}{\sqrt{n}}\mathbf{X}_n'\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\frac{1}{\sqrt{n}}\mathbf{X}_n\boldsymbol{\Theta}_n\right] \\
&= \operatorname{tr}(\boldsymbol{\Theta}_n) + z\operatorname{tr}\left[(1/n\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n)^{-1}\boldsymbol{\Theta}_n\right] \xrightarrow{a.s.} 1 + z\underline{m}(z) = c + czm(z)
\end{aligned}$$

for $\dfrac{p}{n} \to c \in (0,1)$ as $n \to \infty$. Hence, application of Lemma 1.1 and Theorem 1.1 reveals

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n \xrightarrow{a.s.} cm(0) + cz\,\frac{\partial}{\partial z}m(z)\bigg|_{z=0} = \frac{c}{1-c}$$

for $\dfrac{p}{n} \to c \in (0,1)$ as $n \to \infty$.

Finally, we get

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta} = \frac{\partial}{\partial z}\operatorname{tr}\left[\bar{\mathbf{x}}_n'\left(\tilde{\mathbf{V}}_n - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}\right]\bigg|_{z=0} \xrightarrow{a.s.} 0$$

for $\dfrac{p}{n} \to c \in (0,1)$ as $n \to \infty$. □

Lemma 1.3. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Eu-*

clidean norms. Then it holds that

$$\left|\boldsymbol{\xi}'\mathbf{V}_n^{-1}\boldsymbol{\theta} - (1-c)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}\right| \xrightarrow{a.s.} 0\,, \tag{1.21}$$

$$\bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{1-c}\,, \tag{1.22}$$

$$\bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\theta} \xrightarrow{a.s.} 0\,, \tag{1.23}$$

$$\left|\boldsymbol{\xi}'\mathbf{V}_n^{-2}\boldsymbol{\theta} - (1-c)^{-3}\boldsymbol{\xi}'\boldsymbol{\theta}\right| \xrightarrow{a.s.} 0\,, \tag{1.24}$$

$$\bar{\mathbf{x}}_n'\mathbf{V}_n^{-2}\bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{c}{(1-c)^3}\,, \tag{1.25}$$

$$\bar{\mathbf{x}}_n'\mathbf{V}_n^{-2}\boldsymbol{\theta} \xrightarrow{a.s.} 0 \tag{1.26}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

Proof of Lemma 1.3: From (1.4) we obtain

$$\boldsymbol{\xi}'\mathbf{V}_n^{-1}\boldsymbol{\theta} = \boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta} + \frac{\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta}}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n} \xrightarrow{a.s.} (1-c)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$ following (1.14)-(1.16). Similarly, we get (1.22) and (1.23).

In case of (1.23), we get

$$\begin{aligned}\boldsymbol{\xi}'\mathbf{V}_n^{-2}\boldsymbol{\theta} &= \boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta} + \frac{\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta}}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n} + \frac{\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta}}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n}\\ &+ \bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n\frac{\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta}}{(1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n)^2} \xrightarrow{a.s.} (1-c)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$. Similarly,

$$\begin{aligned}\bar{\mathbf{x}}_n'\mathbf{V}_n^{-2}\bar{\mathbf{x}}_n &= \bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n + \frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n}\\ &+ \bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n\frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n}{(1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n)^2} = \frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n}{(1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n)^2} \xrightarrow{a.s.} \frac{c}{(1-c)^3}\end{aligned}$$

and

$$\begin{aligned}\bar{\mathbf{x}}_n'\mathbf{V}_n^{-2}\boldsymbol{\theta} &= \bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta} + \frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta}}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\boldsymbol{\theta}}{1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n}\\ &+ \bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-2}\bar{\mathbf{x}}_n\frac{\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\boldsymbol{\theta}}{(1-\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^{-1}\bar{\mathbf{x}}_n)^2} \xrightarrow{a.s.} 0\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$. □

Lemma 1.4. *Assume (A2). Let* $\boldsymbol{\theta}$ *and* $\boldsymbol{\xi}$ *be universal nonrandom vectors with bounded Euclidean norms and let* $\mathbf{P}_n = \mathbf{V}_n^{-1} - \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}}$ *where* $\boldsymbol{\eta}$ *is a universal nonrandom vectors with bounded Euclidean norm. Then it holds that*

$$\begin{aligned}
\boldsymbol{\xi}'\mathbf{P}_n\boldsymbol{\theta} &\xrightarrow{a.s.} (1-c)^{-1}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right), && (1.27)\\
\bar{\mathbf{x}}_n'\mathbf{P}_n\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{c}{1-c}, && (1.28)\\
\bar{\mathbf{x}}_n'\mathbf{P}_n\boldsymbol{\theta} &\xrightarrow{a.s.} 0, && (1.29)\\
\boldsymbol{\xi}'\mathbf{P}_n^2\boldsymbol{\theta} &\xrightarrow{a.s.} (1-c)^{-3}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right), && (1.30)\\
\bar{\mathbf{x}}_n'\mathbf{P}_n^2\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{c}{(1-c)^3}, && (1.31)\\
\bar{\mathbf{x}}_n'\mathbf{P}_n^2\boldsymbol{\theta} &\xrightarrow{a.s.} 0 && (1.32)
\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ *as* $n \to \infty$.

Proof of Lemma 1.4: It holds that

$$\boldsymbol{\xi}'\mathbf{P}_n\boldsymbol{\theta} = \boldsymbol{\xi}'\mathbf{V}_n^{-1}\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\theta}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} \xrightarrow{a.s.} (1-c)^{-1}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right)$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$ following (1.21). Similarly, we get

$$\bar{\mathbf{x}}_n'\mathbf{P}_n\bar{\mathbf{x}}_n = \bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\bar{\mathbf{x}}_n - \frac{\bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}\bar{\mathbf{x}}_n}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} \xrightarrow{a.s.} \frac{c}{1-c}$$

and

$$\bar{\mathbf{x}}_n'\mathbf{P}_n\boldsymbol{\theta} = \bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\theta} - \frac{\bar{\mathbf{x}}_n'\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\theta}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} \xrightarrow{a.s.} 0$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

The rest of the proof follows from the equality

$$\mathbf{P}_n^2 = \mathbf{V}_n^{-2} - \frac{\mathbf{V}_n^{-2}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} - \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-2}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}} + \boldsymbol{\eta}'\mathbf{V}_n^{-2}\boldsymbol{\eta}\frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-1}}{(\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta})^2}$$

and Lemma 1.3. □

Proof of Theorem 2.1: Let $q_n = \max\{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n, \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\}$. We get that $q_n > 0$ uniformly in p, since $\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n \geq s$ and $s > 0$ uniformly in p by Assumption (A3).

The optimal shrinkage intensity can be rewritten in the following way

$$\begin{aligned}
\alpha_n^* &= \beta^{-1}\frac{\hat{\mathbf{w}}_S'(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})-\mathbf{b}'(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})}{\hat{\mathbf{w}}_S'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S-2\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S+\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}} \qquad (1.33)\\
&= \beta^{-1}\frac{\frac{\mathbf{1}'\mathbf{S}_n^{-1}(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}+\gamma^{-1}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})-\mathbf{b}'(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})}{\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1})^2}+2\gamma^{-1}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}+\gamma^{-2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n-2\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}-2\gamma^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n+\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}\\
&= \beta^{-1}\frac{A_n^*}{B_n^*},
\end{aligned}$$

where

$$\begin{aligned}
A_n^* &= \frac{1}{q_n}\left(\frac{\mathbf{1}'\mathbf{S}_n^{-1}(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}+\gamma^{-1}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})-\mathbf{b}'(\boldsymbol{\mu}_n-\beta\boldsymbol{\Sigma}_n\mathbf{b})\right)\\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\mu}_n}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}\\
&- \beta\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{b}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}\\
&+ \gamma^{-1}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}-\beta\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}\\
&- \frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}+\beta\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}{q_n}
\end{aligned}$$

and

$$\begin{aligned}
B_n^* &= \frac{1}{q_n}\left(\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1})^2}+2\gamma^{-1}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}+\gamma^{-2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n\right.\\
&- \left.2\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}-2\gamma^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n+\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\right)\\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-2}(\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1})^2}\\
&+ 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}\\
&+ \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\\
&- 2\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}\\
&- 2\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}+\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}{q_n},
\end{aligned}$$

In the formulas for A_n^* and B_n^* the factors $q_n^{-1}\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n$ and $q_n^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}$ are bounded by one. Moreover, since $(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\le\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}$ for any $\mathbf{b}$ with $\mathbf{b}'\mathbf{1}=1$, we also get $q_n^{-1}(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\le 1$.

The Euclidean norms of the following vectors

$$\frac{\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}}, \quad \frac{\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}} \quad \text{and} \quad \frac{\boldsymbol{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}$$

are all equal to one. As a result, using $\bar{\mathbf{y}}_n = \boldsymbol{\mu}_n + \boldsymbol{\Sigma}_n^{1/2}\bar{\mathbf{x}}_n$ and $\mathbf{S}_n = \boldsymbol{\Sigma}_n^{1/2}\mathbf{V}_n\boldsymbol{\Sigma}_n^{1/2}$ and applying Lemma 1.3 we get

$$\begin{aligned}
\frac{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^{-1}\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} \xrightarrow{a.s.} (1-c)^{-1}, \\
\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^{-1}\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}} \xrightarrow{a.s.} (1-c)^{-1}\frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}, \\
\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{b}}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^{-1}\boldsymbol{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} \xrightarrow{a.s.} (1-c)^{-1}\frac{1}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}, \\
\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^{-2}\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} \xrightarrow{a.s.} (1-c)^{-3}
\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

Furthermore, from Lemma 1.3 and 1.4 using the equalities

$$\hat{\mathbf{Q}}_n = \boldsymbol{\Sigma}_n^{-1/2}\mathbf{P}_n\boldsymbol{\Sigma}_n^{-1/2} \quad \text{and} \quad \mathbf{P}_n\mathbf{V}_n^{-1} = \mathbf{V}_n^{-2} - \frac{\mathbf{V}_n^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^{-2}}{\boldsymbol{\eta}'\mathbf{V}_n^{-1}\boldsymbol{\eta}}$$

with $\boldsymbol{\eta} = \boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}/\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}$ we obtain with $\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n > 0$

$$\begin{aligned}
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} &= \frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{P}_n\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&\xrightarrow{a.s.} (1-c)^{-1}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}, \\
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} &= \frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{P}_n\boldsymbol{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n\boldsymbol{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} \\
&\xrightarrow{a.s.} (1-c)^{-1}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}, \\
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}} &= \frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{P}_n\mathbf{V}_n^{-1}\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n\mathbf{V}_n^{-1}\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}} \\
&\xrightarrow{a.s.} (1-c)^{-3}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}} = 0, \\
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} &= \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n^2\bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} + 2\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{P}_n^2\bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} + \frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{P}_n^2\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&\xrightarrow{a.s.} \frac{(1-c)^{-3}c + (1-c)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}
\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

Substituting the above results into the expressions of A_n^* and B_n^*, we get that

$$|A_n^* - A^*| \xrightarrow{a.s.} 0 \quad \text{and} \quad |B_n^* - B^*| \xrightarrow{a.s.} 0$$

with

$$\begin{aligned}
A^* &= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}(1-c)^{-1}\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{(1-c)^{-1}}\\
&- \beta\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}(1-c)^{-1}}{(1-c)^{-1}}\\
&+ \gamma^{-1}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{(1-c)^{-1}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} - \beta\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(1-c)^{-1}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}\\
&- \frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} + \beta\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}{q_n}\\
&= \frac{1}{q_n}\left(\frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} - \beta\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} + \frac{\gamma^{-1}}{1-c}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n - \frac{\gamma^{-1}\beta}{1-c}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\Sigma}_n\mathbf{b} - \mathbf{b}'\boldsymbol{\mu}_n + \beta\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\right)
\end{aligned}$$

and

$$\begin{aligned}
B^* &= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(1-c)^{-3}}{(1-c)^{-2}} + 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{0}{(1-c)^{-1}}\\
&+ \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{(1-c)^{-3}c + (1-c)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\\
&- 2\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}(1-c)^{-1}}{(1-c)^{-1}}\\
&- 2\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(1-c)^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{Q}_n\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} + \frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}{q_n}\\
&= \frac{1}{q_n}\left(\frac{1}{1-c}\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} + \gamma^{-2}\left(\frac{1}{(1-c)^3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n + \frac{c}{(1-c)^3}\right) - 2\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\right.\\
&- \left. 2\frac{\gamma^{-1}}{1-c}\left(\mathbf{b}'\boldsymbol{\mu}_n - \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\right) + \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\right).
\end{aligned}$$

Let $\alpha^* = A^*/B^*$. Then,

$$|\alpha_n^* - \alpha^*| \le \left|\frac{1}{B_n^*}(A_n^* - A^*)\right| + \left|\frac{A^*}{B^*B_n^*}(B^* - B_n^*)\right| \xrightarrow{a.s.} 0$$

Using the notations $V_{GMV} = \dfrac{1}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}$, $R_{GMV} = \dfrac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}$, $s = \boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n$, $R_b = \mathbf{b}'\boldsymbol{\mu}_n$ and making some technical manipulations we get the statement of Theorem 2.1.

$\square$

Proof of Corollary 2.1: (a) We first compute U_S/q_n with $q_n = \max\{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n, \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\}$ given by

$$\frac{1}{q_n}U_S = \frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\mu}_n - \frac{\gamma}{2}\frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S,$$

where

$$\begin{aligned}
\frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\mu}_n &= \frac{1}{q_n}\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}} + \gamma^{-1}\frac{1}{q_n}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\mu}_n \\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\mu}_n}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}} \\
&+ \gamma^{-1}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}(1-c)^{-1}\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{(1-c)^{-1}} \\
&+ \gamma^{-1}(1-c)^{-1}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} = \frac{1}{q_n}\left(R_{GMV} + \gamma^{-1}(1-c)^{-1}s\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S &= \frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1})^2} + 2\gamma^{-1}\frac{\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}} + \gamma^{-2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n \\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-2}(\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1})^2} + \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&+ 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(1-c)^{-3}}{(1-c)^{-2}} + \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{(1-c)^{-3}c + (1-c)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&+ 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{0}{(1-c)^{-1}} \\
&= \frac{1}{q_n}\left((1-c)^{-1}V_{GMV} + \gamma^{-2}\frac{c}{(1-c)^3} + \gamma^{-2}(1-c)^{-3}s\right)
\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

Finally, the equality

$$\frac{1}{q_n}U_{EU} = \frac{1}{q_n}R_{GMV} + \frac{1}{2}\gamma^{-1}\frac{1}{q_n}s - \frac{\gamma}{2}\frac{1}{q_n}V_{GMV},$$

implies the statement of the first part of the corollary.

(b) It holds that

$$\begin{aligned}\frac{1}{q_n}U_{GSE} &= \alpha^*\frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\mu}_n+(1-\alpha^*)\frac{1}{q_n}\mathbf{b}'\boldsymbol{\mu}_n\\ &- \frac{\gamma}{2}\left((\alpha^*)^2\frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S+2\alpha^*(1-\alpha^*)\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S+(1-\alpha^*)^2\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\right),\end{aligned}$$

where the asymptotic values of $\hat{\mathbf{w}}_S'\boldsymbol{\mu}_n/q_n$ and $\hat{\mathbf{w}}_S'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S/q_n$ are fund in part (a) and

$$\begin{aligned}\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S &= \frac{1}{q_n}\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}+\gamma^{-1}\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n\\ &= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}\mathbf{1}'\mathbf{S}_n^{-1}\boldsymbol{\Sigma}_n\mathbf{b}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^{-1}\mathbf{1}}\\ &+ \gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n\bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}\\ &\xrightarrow{a.s.} \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}(1-c)^{-1}}{(1-c)^{-1}}\\ &+ \gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(1-c)^{-1}\left(\mathbf{b}'\boldsymbol{\mu}_n-\frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\right)}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}\\ &= \frac{1}{q_n}\left(V_{GMV}+\gamma^{-1}(1-c)^{-1}(R_b-R_{GMV})\right).\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$.

Hence,

$$\begin{aligned}&\frac{1}{q_n}(U_{EU}-U_{GSE}) = (\alpha^*)^2\frac{1}{q_n}U_{EU}+(1-\alpha^*)^2\frac{1}{q_n}U_{EU}+2\alpha^*(1-\alpha^*)\frac{1}{q_n}U_{EU}\\ -\ &(\alpha^*)^2\frac{1}{q_n}U_S-\alpha^*(1-\alpha^*)\frac{1}{q_n}\hat{\mathbf{w}}_S'\boldsymbol{\mu}_n-(1-\alpha^*)^2\frac{1}{q_n}U_{\mathbf{b}}-\alpha^*(1-\alpha^*)\frac{1}{q_n}\mathbf{b}'\boldsymbol{\mu}_n+\gamma\alpha^*(1-\alpha^*)\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_S\\ \xrightarrow{a.s.}\ &(\alpha^*)^2\frac{1}{q_n}(U_{EU}-U_S)+(1-\alpha^*)^2\frac{1}{q_n}(U_{EU}-U_{\mathbf{b}})\\ +\ &\alpha^*(1-\alpha^*)\frac{1}{q_n}\left(2R_{GMV}+\gamma^{-1}s-\gamma V_{GMV}-R_{GMV}-\frac{\gamma^{-1}}{1-c}s-R_b+\gamma V_{GMV}+\frac{R_b-R_{GMV}}{1-c}\right)\\ =\ &(\alpha^*)^2\frac{1}{q_n}(U_{EU}-U_S)+(1-\alpha^*)^2\frac{1}{q_n}(U_{EU}-U_{\mathbf{b}})+\alpha^*(1-\alpha^*)\frac{c}{1-c}\frac{1}{q_n}(R_b-R_{GMV}-\gamma^{-1}s).\end{aligned}$$

for $p/n \longrightarrow c \in (0,1)$ as $n \to \infty$. □

For the proof of Theorem 2.2 we need several results about the properties of Moore-Penrose inverse which are summarized in the following three lemmas. Similarly as the proof of Lemma 1.2, we will use Lemma 1.1 and Theorem 1.1 in a sequel every time a derivative must be interchanged with the limit $n \to \infty$.

Lemma 1.5. *Assume (A2). Let* $\boldsymbol{\theta}$ *and* $\boldsymbol{\xi}$ *be universal nonrandom vectors with bounded Eu-*

clidean norms. Then it holds that

$$
\begin{aligned}
\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{+}\boldsymbol{\theta} &\xrightarrow{a.s.} c^{-1}(c-1)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}\,, & (1.34)\\
\boldsymbol{\xi}'(\tilde{\mathbf{V}}_n^{+})^2\boldsymbol{\theta} &\xrightarrow{a.s.} (c-1)^{-3}\boldsymbol{\xi}'\boldsymbol{\theta}\,, & (1.35)\\
\bar{\mathbf{x}}'\tilde{\mathbf{V}}_n^{+}\bar{\mathbf{x}}_n &= 1\,, & (1.36)\\
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^{+})^2\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{c}{c-1}\,, & (1.37)\\
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^{+})^3\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{c^3}{(c-1)^3}\,, & (1.38)\\
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^{+})^4\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{c^4+c^5}{(c-1)^5}\,, & (1.39)\\
\bar{\mathbf{x}}'\tilde{\mathbf{V}}_n^{+}\boldsymbol{\theta} &\xrightarrow{a.s.} 0\,, & (1.40)\\
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^{+})^2\boldsymbol{\theta} &\xrightarrow{a.s.} 0\,, & (1.41)\\
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^{+})^3\boldsymbol{\theta} &\xrightarrow{a.s.} 0 & (1.42)
\end{aligned}
$$

for $p/n \longrightarrow c \in (1,+\infty)$ as $n \to \infty$.

Proof of Lemma 1.5: It holds that

$$\tilde{\mathbf{V}}^{+} = \left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n'\right)^{+} = \frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n\right)^{-2}\frac{1}{\sqrt{n}}\mathbf{X}_n'$$

and, similarly,

$$(\tilde{\mathbf{V}}^{+})^i = \frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n\right)^{-(i+1)}\frac{1}{\sqrt{n}}\mathbf{X}_n' \quad \text{for} \quad i=2,3,4.$$

Let $\boldsymbol{\Theta} = \boldsymbol{\theta}\boldsymbol{\xi}'$. It holds that

$$
\begin{aligned}
\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^{+}\boldsymbol{\theta} &= \operatorname{tr}\left[\frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n\right)^{-2}\frac{1}{\sqrt{n}}\mathbf{X}_n'\boldsymbol{\Theta}\right] = \frac{\partial}{\partial z}\operatorname{tr}\left[\frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n\right)^{-1}\frac{1}{\sqrt{n}}\mathbf{X}_n'\boldsymbol{\Theta}\right]\Bigg|_{z=0}\,,\\
\boldsymbol{\xi}'(\tilde{\mathbf{V}}_n^{+})^2\boldsymbol{\theta} &= \operatorname{tr}\left[\frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n\right)^{-3}\frac{1}{\sqrt{n}}\mathbf{X}_n'\boldsymbol{\Theta}\right] = \frac{1}{2}\frac{\partial^2}{\partial z^2}\operatorname{tr}\left[\frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n\right)^{-1}\frac{1}{\sqrt{n}}\mathbf{X}_n'\boldsymbol{\Theta}\right]\Bigg|_{z=0}
\end{aligned}
$$

The application of Woodbury formula (matrix inversion lemma, see, e.g., Horn and Johnsohn (1985)),

$$\frac{1}{\sqrt{n}}\mathbf{X}_n\left(\frac{1}{n}\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I}_n\right)^{-1}\frac{1}{\sqrt{n}}\mathbf{X}_n' = \mathbf{I}_p + z\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1} \tag{1.43}$$

leads to

$$\begin{aligned}\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} &= \frac{\partial}{\partial z} z\mathrm{tr}\left[\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\boldsymbol{\Theta}\right]\Bigg|_{z=0}, \\ \boldsymbol{\xi}'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} &= \frac{1}{2}\frac{\partial^2}{\partial z^2} z\mathrm{tr}\left[\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\boldsymbol{\Theta}\right]\Bigg|_{z=0}.\end{aligned}$$

From the proof of Lemma 1.2 we know that the matrix $\boldsymbol{\Theta}$ possesses the bounded trace norm. Then the application of Lemma 1.1 leads to

$$\begin{aligned}\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} &\xrightarrow{a.s.} \frac{\partial}{\partial z}\frac{z}{x(z)-z}\Bigg|_{z=0}\boldsymbol{\xi}'\boldsymbol{\theta}, \\ \boldsymbol{\xi}'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} &\xrightarrow{a.s.} \frac{1}{2}\frac{\partial^2}{\partial z^2}\frac{z}{x(z)-z}\Bigg|_{z=0}\boldsymbol{\xi}'\boldsymbol{\theta}\end{aligned}$$

for $p/n \to c > 1$ as $n \to \infty$, where $x(z)$ is given in (1.9).

Let us make the following notations

$$\theta(z) = \frac{z}{x(z)-z} \quad \text{and} \quad \phi(z) = \frac{x(z) - zx'(z)}{z^2}.$$

Then the first and the second derivatives of $\theta(z)$ are given by

$$\theta'(z) = \theta^2(z)\phi(z) \quad \text{and} \quad \theta^{''}(z) = 2\theta(z)\theta'(z)\phi(z) + \theta^2(z)\phi'(z). \tag{1.44}$$

Using L'Hopital's rule, we get

$$\theta(0) = \lim_{z\to 0}\theta(z) = \lim_{z\to 0}\frac{z}{x(z)-z} = \lim_{z\to 0}\frac{1}{(x'(z)-1)} = \frac{1}{\frac{1}{2}\left(1 - \frac{1+c}{|1-c|}\right) - 1} = -\frac{c-1}{c}, \tag{1.45}$$

$$\phi(0) = \lim_{z\to 0}\phi(z) = \lim_{z\to 0}\frac{x(z)-zx'(z)}{z^2} = -\frac{1}{2}\lim_{z\to 0}x^{''}(z) = -\frac{1}{2}\lim_{z\to 0}\frac{-2c}{((1-c+z)^2-4z)^{3/2}} = \frac{c}{(c-1)^3}, \tag{1.46}$$

and

$$\begin{aligned}\lim_{z\to 0}\phi'(z) &= -\lim_{z\to 0}\frac{2(x(z)-zx'(z)) + z^2x^{''}(z)}{z^2} \\ &= -\lim_{z\to 0}\frac{2\phi(z) + x^{''}(z)}{z} = -\lim_{z\to 0}(2\phi'(z) + x^{'''}(z)),\end{aligned} \tag{1.47}$$

which implies

$$\phi'(0) = \lim_{z\to 0}\phi'(z) = -\frac{1}{3}\lim_{z\to 0} x'''(z) = -\frac{1}{3}\lim_{z\to 0}\frac{6c(z-c-1)}{((1-c+z)^2-4z)^{5/2}} = \frac{2c(c+1)}{(c-1)^5}\,. \tag{1.48}$$

Combining (1.44), (1.45), (1.46), and (1.48), we get

$$\theta'(0) = \lim_{z\to 0}\theta'(z) = \theta^2(0)\phi(0) = \frac{1}{c(c-1)}$$

and

$$\theta''(0) = \lim_{z\to 0}\theta''(z) = 2\theta^3(0)\phi^2(0) + \theta^2(0)\phi'(0) = \frac{2}{(c-1)^3}\,.$$

Hence,

$$\begin{aligned}
\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} &\xrightarrow{a.s.} \frac{1}{c(c-1)}\boldsymbol{\xi}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\theta} \;\text{ for } p/n\to c>1 \;\text{ as } n\to\infty,\\
\boldsymbol{\xi}'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} &\xrightarrow{a.s.} \frac{1}{(c-1)^3}\boldsymbol{\xi}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\theta} \;\text{ for } p/n\to c>1 \;\text{ as } n\to\infty\,.
\end{aligned}$$

Taking into account that

$$\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\bar{\mathbf{x}}_n = \frac{1}{n}\mathbf{1}_n'\mathbf{X}_n'\mathbf{X}_n(\mathbf{X}_n'\mathbf{X}_n)^{-2}\mathbf{X}_n'\mathbf{X}_n\mathbf{1}_n = \frac{1}{n}\mathbf{1}_n'\mathbf{1}_n = 1\,.$$

we get (1.36). Similarly, using

$$\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^i\bar{\mathbf{x}}_n = 1/n\mathbf{1}_n'(1/n\mathbf{X}_n'\mathbf{X}_n)^{-(i-1)}\mathbf{1}_n \;\text{ for }\; i=2,3,4$$

we get

$$\begin{aligned}
1/n\mathbf{1}_n'(1/n\mathbf{X}_n'\mathbf{X}_n)^{-1}\mathbf{1}_n &= \lim_{z\to 0}\mathrm{tr}[(1/n\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I})^{-1}\boldsymbol{\Theta}_n] \xrightarrow{a.s.} \underline{m}(0)\,,\\
1/n\mathbf{1}_n'(1/n\mathbf{X}_n'\mathbf{X}_n)^{-2}\mathbf{1}_n &= \lim_{z\to 0}\frac{\partial}{\partial z}\mathrm{tr}[(1/n\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I})^{-1}\boldsymbol{\Theta}_n] \xrightarrow{a.s.} \underline{m}'(0)\,,\\
1/n\mathbf{1}_n'(1/n\mathbf{X}_n'\mathbf{X}_n)^{-3}\mathbf{1}_n &= \frac{1}{2}\lim_{z\to 0}\frac{\partial^2}{\partial z^2}\mathrm{tr}[(1/n\mathbf{X}_n'\mathbf{X}_n - z\mathbf{I})^{-2}\boldsymbol{\Theta}_n] \xrightarrow{a.s.} \frac{1}{2}\underline{m}''(0)
\end{aligned}$$

for $p/n\to c>1$ as $n\to\infty$, where $\boldsymbol{\Theta}_n = 1/n\mathbf{1}_n\mathbf{1}_n'$.

Using that the elements of $\mathbf{X}_n$ are independent and identically distributed and the fact that

$n < p$ from Lemma 1.1 and Theorem 1.1 we get that

$$\underline{m}'(z) = \frac{1}{\underline{x}(z) - z} \quad \text{with} \quad \underline{x}(z) = \frac{1}{2}\left(1 - c^{-1} + z + \sqrt{(1 - c^{-1} + z)^2 - 4z}\right).$$

Thus,

$$\underline{m}(0) = \frac{1}{\underline{x}(0)} = \frac{1}{1 - c^{-1}} = \frac{c}{c-1},$$

which proves (1.37). Furthermore, we get

$$\underline{m}'(z) = -\frac{\underline{x}'(z) - 1}{(\underline{x}(z) - z)^2}$$

with

$$\underline{x}'(z) = \frac{1}{2}\left(1 + \frac{1}{2}\frac{2(1 - c^{-1} + z) - 4}{\sqrt{(1 - c^{-1} + z)^2 - 4z}}\right) = \frac{1}{2}\left(1 + \frac{-1 - c^{-1} + z}{\sqrt{(1 - c^{-1} + z)^2 - 4z}}\right),$$

and, consequently,

$$\underline{m}'(0) = -\frac{\underline{x}'(0) - 1}{\underline{x}(0)^2} = \frac{1}{(1 - c^{-1})^3} = \frac{c^3}{(c-1)^3}.$$

In order to prove (1.39), we compute

$$\underline{m}''(z) = -\frac{\underline{x}''(z)}{(\underline{x}(z) - z)^2} + 2\frac{(\underline{x}'(z) - 1)^2}{(\underline{x}(z) - z)^3}$$

where

$$\underline{x}''(z) = \frac{1}{2}\left(\frac{1}{\sqrt{(1 - c^{-1} + z)^2 - 4z}} - \frac{(-1 - c^{-1} + z)^2}{((1 - c^{-1} + z)^2 - 4z)^{3/2}}\right),$$

Hence,

$$\underline{m}''(0) = -\frac{\underline{x}''(0)}{\underline{x}(0)^2} + 2\frac{(\underline{x}'(0) - 1)^2}{\underline{x}(0)^3} = 2\frac{c^{-1} + 1}{(1 - c^{-1})^5} = 2\frac{c^4 + c^5}{(c-1)^5}.$$

For (1.40) we consider

$$\bar{\mathbf{x}}'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} = \operatorname{tr}\left[\mathbf{I}_p + z\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}\bar{\mathbf{x}}'\right] = \bar{\mathbf{x}}'\boldsymbol{\theta} + z\bar{\mathbf{x}}'\left(\tilde{\mathbf{V}}_n - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}\,.$$

Because of (1.16), it holds that $\bar{\mathbf{x}}'\left(\tilde{\mathbf{V}}_n - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}$ is uniformly bounded as $z \longrightarrow 0$. Moreover, $\bar{\mathbf{x}}'\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} 0$ as $p \longrightarrow \infty$ following Kolmogorov's strong law of large numbers (c.f., Sen and Singer (1993, Theorem 2.3.10), since $\boldsymbol{\theta}$ has a bounded Euclidean norm. Hence, $\bar{\mathbf{x}}'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} 0$ for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Finally, in the case of (1.41) and (1.42), we get

$$\begin{aligned}
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} &= \frac{\partial}{\partial z} z\operatorname{tr}\left[\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}\bar{\mathbf{x}}'\right]\Bigg|_{z=0} \stackrel{a.s.}{\longrightarrow} 0\,,\\
\bar{\mathbf{x}}'(\tilde{\mathbf{V}}_n^+)^3\boldsymbol{\theta} &= \frac{1}{2}\frac{\partial^2}{\partial z^2} z\operatorname{tr}\left[\left(\frac{1}{n}\mathbf{X}_n\mathbf{X}_n' - z\mathbf{I}_p\right)^{-1}\boldsymbol{\theta}\bar{\mathbf{x}}'\right]\Bigg|_{z=0} \stackrel{a.s.}{\longrightarrow} 0.
\end{aligned}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$. □

Lemma 1.6. *Assume (A2). Let $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ be universal nonrandom vectors with bounded Euclidean norms. Then it holds that*

$$\boldsymbol{\xi}'\mathbf{V}_n^+\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}\,, \tag{1.49}$$

$$\bar{\mathbf{x}}_n'\mathbf{V}_n^+\bar{\mathbf{x}}_n \stackrel{a.s.}{\longrightarrow} \frac{1}{c-1}\,, \tag{1.50}$$

$$\bar{\mathbf{x}}_n'\mathbf{V}_n^+\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} 0\,, \tag{1.51}$$

$$\boldsymbol{\xi}'(\mathbf{V}_n^+)^2\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} (c-1)^{-3}\boldsymbol{\xi}'\boldsymbol{\theta}\,, \tag{1.52}$$

$$\bar{\mathbf{x}}_n'(\mathbf{V}_n^+)^2\bar{\mathbf{x}}_n \stackrel{a.s.}{\longrightarrow} \frac{c^2}{(c-1)^3}\,, \tag{1.53}$$

$$\bar{\mathbf{x}}_n'(\mathbf{V}_n^+)^2\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} 0 \tag{1.54}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Proof of Lemma 1.6: From (1.5) we get

$$\begin{aligned}
\boldsymbol{\xi}'\mathbf{V}_n^+\boldsymbol{\theta} &= \boldsymbol{\xi}'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^+\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} + \boldsymbol{\xi}'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta}}{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n}\\
&+ \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2}\boldsymbol{\xi}'\tilde{\mathbf{V}}_n^+\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} \stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1}\boldsymbol{\xi}'\boldsymbol{\theta}
\end{aligned}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$ following (1.34)-(1.36). Similarly, we get

$$\begin{aligned}
\bar{\mathbf{x}}_n' \mathbf{V}_n^+ \bar{\mathbf{x}}_n &= \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n - \frac{\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n + \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\
&+ \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \xrightarrow{a.s.} \frac{1}{c-1}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathbf{x}}_n' \mathbf{V}_n^+ \boldsymbol{\theta} &= \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} - \frac{\bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} + \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta}}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\
&+ \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \xrightarrow{a.s.} 0
\end{aligned}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Now, we consider the equality

$$\begin{aligned}
\left[(\tilde{\mathbf{V}}_n - \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n')^+\right]^2 &= \left(\tilde{\mathbf{V}}_n^+ - \frac{\tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \right)^2 \\
&= (\tilde{\mathbf{V}}_n^+)^2 + \left[\frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ - \frac{(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n}\right]^2 \\
&- \frac{2(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 + (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \\
&+ \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} (\tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+) \\
&= (\tilde{\mathbf{V}}_n^+)^2 + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^4 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ - \frac{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^3} \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \\
&+ \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \left[(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2\right] \\
&- \frac{(\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 + \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 + (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\boldsymbol{\xi}' (\mathbf{V}_n^+)^2 \boldsymbol{\theta} &= \boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} + \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^4 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} - \frac{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^3} \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} \\
&+ \frac{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n)^2} \left[\boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta} + \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta}\right] \\
&- \frac{\boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \boldsymbol{\theta} + \boldsymbol{\xi}' \tilde{\mathbf{V}}_n^+ \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^3 \boldsymbol{\theta} + \boldsymbol{\xi}' (\tilde{\mathbf{V}}_n^+)^3 \bar{\mathbf{x}}_n \bar{\mathbf{x}}_n' \tilde{\mathbf{V}}_n^+ \boldsymbol{\theta}}{\bar{\mathbf{x}}_n' (\tilde{\mathbf{V}}_n^+)^2 \bar{\mathbf{x}}_n} \xrightarrow{a.s.} (c-1)^{-3} \boldsymbol{\xi}' \boldsymbol{\theta}\,,
\end{aligned}$$

$$
\begin{aligned}
\bar{\mathbf{x}}_n'(\mathbf{V}_n^+)^2\bar{\mathbf{x}}_n &= \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n + \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^4\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2} - \frac{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^3} + 2\frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n}\\
&- \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n - 2\frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n}\\
&= \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^4\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2} - \frac{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^3} \xrightarrow{a.s.} \frac{c^2}{(c-1)^3}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\mathbf{x}}_n'(\mathbf{V}_n^+)^2\boldsymbol{\theta} &= \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} + \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^4\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2}\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} - \frac{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n)^2}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^3}\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta}\\
&+ \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n}{(\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n)^2}\left[\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta} + \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta}\right]\\
&- \frac{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\boldsymbol{\theta} + \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\boldsymbol{\theta} + \bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^3\bar{\mathbf{x}}_n\bar{\mathbf{x}}_n'\tilde{\mathbf{V}}_n^+\boldsymbol{\theta}}{\bar{\mathbf{x}}_n'(\tilde{\mathbf{V}}_n^+)^2\bar{\mathbf{x}}_n} \xrightarrow{a.s.} 0
\end{aligned}
$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$. $\square$

Lemma 1.7. *Assume (A2). Let* $\boldsymbol{\theta}$ *and* $\boldsymbol{\xi}$ *be universal nonrandom vectors with bounded Euclidean norms and let* $\mathbf{P}_n^+ = \mathbf{V}_n^+ - \frac{\mathbf{V}_n^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^+}{\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\eta}}$ *where* $\boldsymbol{\eta}$ *is a universal nonrandom vectors with bounded Euclidean norm. Then it holds that*

$$
\begin{aligned}
\boldsymbol{\xi}'\mathbf{P}_n^+\boldsymbol{\theta} &\xrightarrow{a.s.} c^{-1}(c-1)^{-1}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right), \qquad (1.55)\\
\bar{\mathbf{x}}_n'\mathbf{P}_n^+\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{1}{c-1}\,, \qquad (1.56)\\
\bar{\mathbf{x}}_n'\mathbf{P}_n^+\boldsymbol{\theta} &\xrightarrow{a.s.} 0\,, \qquad (1.57)\\
\boldsymbol{\xi}'(\mathbf{P}_n^+)^2\boldsymbol{\theta} &\xrightarrow{a.s.} (c-1)^{-3}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right), \qquad (1.58)\\
\bar{\mathbf{x}}_n'(\mathbf{P}_n^+)^2\bar{\mathbf{x}}_n &\xrightarrow{a.s.} \frac{c^2}{(c-1)^3}\,, \qquad (1.59)\\
\bar{\mathbf{x}}_n'(\mathbf{P}_n^+)^2\boldsymbol{\theta} &\xrightarrow{a.s.} 0 \qquad (1.60)
\end{aligned}
$$

for $p/n \longrightarrow c \in (1, +\infty)$ *as* $n \to \infty$.

Proof of Lemma 1.7: It holds that

$$
\boldsymbol{\xi}'\mathbf{P}_n^+\boldsymbol{\theta} = \boldsymbol{\xi}'\mathbf{V}_n^+\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\mathbf{V}_n^+\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\theta}}{\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\eta}} \xrightarrow{a.s.} c^{-1}(c-1)^{-1}\left(\boldsymbol{\xi}'\boldsymbol{\theta} - \frac{\boldsymbol{\xi}'\boldsymbol{\eta}\boldsymbol{\eta}'\boldsymbol{\theta}}{\boldsymbol{\eta}'\boldsymbol{\eta}}\right)
$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$ following (1.34). Similarly, we get

$$\bar{\mathbf{x}}_n' \mathbf{P}_n^+ \bar{\mathbf{x}}_n = \bar{\mathbf{x}}_n' \mathbf{V}_n^+ \bar{\mathbf{x}}_n - \frac{\bar{\mathbf{x}}_n' \mathbf{V}_n^+ \boldsymbol{\eta}\boldsymbol{\eta}' \mathbf{V}_n^+ \bar{\mathbf{x}}_n}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} \xrightarrow{a.s.} \frac{1}{c-1}$$

and

$$\bar{\mathbf{x}}_n' \mathbf{P}_n^+ \boldsymbol{\theta} = \bar{\mathbf{x}}_n' \mathbf{V}_n^+ \boldsymbol{\theta} - \frac{\bar{\mathbf{x}}_n' \mathbf{V}_n^+ \boldsymbol{\eta}\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\theta}}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} \xrightarrow{a.s.} 0$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

The rest of the proof follows from the equality

$$(\mathbf{P}_n^+)^2 = (\mathbf{V}_n^+)^2 - \frac{(\mathbf{V}_n^+)^2 \boldsymbol{\eta}\boldsymbol{\eta}' \mathbf{V}_n^+}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} - \frac{\mathbf{V}_n^+ \boldsymbol{\eta}\boldsymbol{\eta}' (\mathbf{V}_n^+)^2}{\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta}} + \boldsymbol{\eta}' (\mathbf{V}_n^+)^2 \boldsymbol{\eta} \frac{\mathbf{V}_n^+ \boldsymbol{\eta}\boldsymbol{\eta}' \mathbf{V}_n^+}{(\boldsymbol{\eta}' \mathbf{V}_n^+ \boldsymbol{\eta})^2}$$

and Lemma 1.6. □

Proof of Theorem 2.2: Let $q_n = \max\{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n, \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}\}$. From Assumption (A3) get that $q_n > 0$ uniformly in p, since $\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n \geq s$ and $s > 0$ uniformly in p.

In case of $c > 1$, the optimal shrinkage intensity is given by

$$\begin{aligned}
\alpha_n^+ &= \beta^{-1} \frac{\hat{\mathbf{w}}_{S^*}' (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\hat{\mathbf{w}}_{S^*}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_{S^*} - 2\mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_{S^*} + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} \\
&= \beta^{-1} \frac{\frac{\mathbf{1}' \mathbf{S}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-1} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\frac{\mathbf{1}' \mathbf{S}_n^* \boldsymbol{\Sigma}_n \mathbf{S}_n^* \mathbf{1}}{(\mathbf{1}' \mathbf{S}_n^* \mathbf{1})^2} + 2\gamma^{-1} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \boldsymbol{\Sigma}_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-2} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n - 2 \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{S}_n^* \mathbf{1}}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} - 2\gamma^{-1} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{Q}}_n^* \bar{\mathbf{y}}_n + \mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}} \\
&= \beta^{-1} \frac{A_n^+}{B_n^+},
\end{aligned}$$

where

$$\begin{aligned}
A_n^+ &= \frac{1}{q_n} \left(\frac{\mathbf{1}' \mathbf{S}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b})}{\mathbf{1}' \mathbf{S}_n^* \mathbf{1}} + \gamma^{-1} \bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* (\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) - \mathbf{b}'(\boldsymbol{\mu}_n - \beta \boldsymbol{\Sigma}_n \mathbf{b}) \right) \\
&= \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}}{q_n} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n)^{-1/2} \mathbf{1}' \mathbf{S}_n^* \boldsymbol{\mu}_n}{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&- \beta \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1/2} (\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b})^{-1/2} \mathbf{1}' \mathbf{S}_n^* \boldsymbol{\Sigma}_n \mathbf{b}}{(\mathbf{1}' \boldsymbol{\Sigma}_n^{-1} \mathbf{1})^{-1} \mathbf{1}' \mathbf{S}_n^* \mathbf{1}} \\
&+ \gamma^{-1} \frac{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \boldsymbol{\mu}_n}{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} - \beta \gamma^{-1} \frac{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\bar{\mathbf{y}}_n' \hat{\mathbf{Q}}_n^* \boldsymbol{\Sigma}_n \mathbf{b}}{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} \\
&- \frac{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}}{q_n} \frac{\mathbf{b}' \boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n' \boldsymbol{\Sigma}_n^{-1} \boldsymbol{\mu}_n} \sqrt{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}} + \beta \frac{\mathbf{b}' \boldsymbol{\Sigma}_n \mathbf{b}}{q_n}
\end{aligned}$$

and

$$
\begin{aligned}
B_n^+ &= \frac{1}{q_n}\left(\frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{(\mathbf{1}'\mathbf{S}_n^*\mathbf{1})^2} + 2\gamma^{-1}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} + \gamma^{-2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n\right.\\
&- \left. 2\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} - 2\gamma^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n + \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\right)\\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-2}(\mathbf{1}'\mathbf{S}_n^*\mathbf{1})^2}\\
&+ 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\mathbf{1}}\\
&+ \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\\
&- 2\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\mathbf{1}}\\
&- 2\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} + \frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}{q_n}.
\end{aligned}
$$

Using the equalities

$$
\bar{\mathbf{y}}_n = \boldsymbol{\mu}_n + \boldsymbol{\Sigma}_n^{1/2}\bar{\mathbf{x}}_n \text{ and } \mathbf{S}_n^* = \boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^+\boldsymbol{\Sigma}_n^{-1/2},
$$

the facts that $q_n^{-1}\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n \le 1$, $q_n^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b} \le 1$, $q_n^{-1}(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1} \le 1$ and that the Euclidean norms of the following vectors

$$
\frac{\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}},\quad \frac{\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}} \quad\text{and}\quad \frac{\boldsymbol{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}
$$

are equal to one, the application of Lemma 1.6 yields

$$
\begin{aligned}
\frac{\mathbf{1}'\mathbf{S}_n^*\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^+\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} \stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1},\\
\frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^+\boldsymbol{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}} \stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1}\frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}},\\
\frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{b}}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}\mathbf{V}_n^+\boldsymbol{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} \stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1}\frac{1}{\sqrt{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}},\\
\frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} &= \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1/2}(\mathbf{V}_n^+)^2\boldsymbol{\Sigma}_n^{-1/2}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} \stackrel{a.s.}{\longrightarrow} (c-1)^{-3}
\end{aligned}
$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Finally, from Lemma 1.6 and 1.7 as well as by using the equalities

$$\hat{\mathbf{Q}}_n^* = \mathbf{\Sigma}_n^{-1/2}\mathbf{P}_n^+\mathbf{\Sigma}_n^{-1/2} \quad \text{and} \quad \mathbf{P}_n^+\mathbf{V}_n^+ = (\mathbf{V}_n^+)^2 - \frac{\mathbf{V}_n^+\boldsymbol{\eta}\boldsymbol{\eta}'(\mathbf{V}_n^+)^2}{\boldsymbol{\eta}'\mathbf{V}_n^+\boldsymbol{\eta}}$$

with $\boldsymbol{\eta} = \mathbf{\Sigma}_n^{-1/2}\mathbf{1}/\sqrt{\mathbf{1}'\mathbf{\Sigma}_n^{-1/2}\mathbf{1}}$ we obtain

$$\begin{aligned}
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} &= \frac{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1/2}\mathbf{P}_n^+\mathbf{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n^+\mathbf{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&\stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}, \\
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\mathbf{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}} &= \frac{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1/2}\mathbf{P}_n^+\mathbf{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n^+\mathbf{\Sigma}_n^{1/2}\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}} \\
&\stackrel{a.s.}{\longrightarrow} c^{-1}(c-1)^{-1}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\mathbf{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}}, \\
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\mathbf{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}} &= \frac{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1/2}\mathbf{P}_n^+\mathbf{V}_n^+\mathbf{\Sigma}_n^{-1/2}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}} + \frac{\bar{\mathbf{x}}_n'\mathbf{P}_n^+\mathbf{V}_n^+\mathbf{\Sigma}_n^{-1/2}\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}} \\
&\stackrel{a.s.}{\longrightarrow} (c-1)^{-3}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\mathbf{1}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}}} = 0, \\
\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\mathbf{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} &= \frac{\bar{\mathbf{x}}_n'(\mathbf{P}_n^+)^2\bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} + 2\frac{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1/2}(\mathbf{P}_n^+)^2\bar{\mathbf{x}}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} + \frac{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1/2}(\mathbf{P}_n^+)^2\mathbf{\Sigma}_n^{-1/2}\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&\stackrel{a.s.}{\longrightarrow} \frac{(c-1)^{-3}c^2 + (c-1)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}
\end{aligned}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Substituting the above results into the expressions of A_n^* and B_n^*, we get that

$$|A_n^* - A^*| \stackrel{a.s.}{\longrightarrow} 0 \quad \text{and} \quad |B_n^* - B^*| \stackrel{a.s.}{\longrightarrow} 0$$

with

$$\begin{aligned}
A^+ &= \frac{(\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}c^{-1}(c-1)^{-1}\mathbf{1}'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{c^{-1}(c-1)^{-1}} \\
&- \beta\frac{(\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b})^{-1/2}c^{-1}(c-1)^{-1}}{c^{-1}(c-1)^{-1}} \\
&+ \gamma^{-1}\frac{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{c^{-1}(c-1)^{-1}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n} - \beta\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}}{q_n}\frac{c^{-1}(c-1)^{-1}\boldsymbol{\mu}_n'\mathbf{Q}_n\mathbf{\Sigma}_n\mathbf{b}}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}} \\
&- \frac{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}} + \beta\frac{\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}}{q_n} \\
&= \frac{1}{q_n}\left(\frac{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}} - \beta\frac{1}{\mathbf{1}'\mathbf{\Sigma}_n^{-1}\mathbf{1}} + \frac{\gamma^{-1}}{c(c-1)}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n - \frac{\gamma^{-1}\beta}{c(c-1)}\boldsymbol{\mu}_n'\mathbf{Q}_n\mathbf{\Sigma}_n\mathbf{b} - \mathbf{b}'\boldsymbol{\mu}_n + \beta\mathbf{b}'\mathbf{\Sigma}_n\mathbf{b}\right)
\end{aligned}$$

and

$$
\begin{aligned}
B^{+} &= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(c-1)^{-3}}{c^{-2}(c-1)^{-2}} + 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{0}{c^{-1}(c-1)^{-1}}\\
&+ \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{(c-1)^{-3}c^2+(c-1)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\\
&- 2\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}c^{-1}(c-1)^{-1}}{c^{-1}(c-1)^{-1}}\\
&- 2\gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{c^{-1}(c-1)^{-1}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{Q}_n\boldsymbol{\mu}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} + \frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}{q_n}\\
&= \frac{1}{q_n}\Bigg(\frac{c^2}{c-1}\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}} + \frac{\gamma^{-2}}{(c-1)^3}\left(c^2+\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n\right) - 2\frac{1}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\\
&- 2\frac{\gamma^{-1}}{c(c-1)}\left(\mathbf{b}'\boldsymbol{\mu}_n - \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\right) + \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\Bigg).
\end{aligned}
$$

Let $\alpha^+ = A^+/B^+$. Hence,

$$
|\alpha_n^+ - \alpha^+| \le \left|\frac{1}{B_n^+}(A_n^+ - A^+)\right| + \left|\frac{A^+}{B^+B_n^+}(B^+ - B_n^+)\right| \xrightarrow{a.s.} 0.
$$

This completes the proof of Theorem 2.2. □

Proof of Corollary 2.2: (a) With $q_n = \max\{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n, \mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\}$ we get

$$
\frac{1}{q_n}U_S = \frac{1}{q_n}\hat{\mathbf{w}}_{S^*}'\boldsymbol{\mu}_n - \frac{\gamma}{2}\frac{1}{q_n}\hat{\mathbf{w}}_{S^*}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_{S^*},
$$

where from the proof of Theorem 2.2 we have

$$
\begin{aligned}
\frac{1}{q_n}\hat{\mathbf{w}}_{S^*}'\boldsymbol{\mu}_n &= \frac{1}{q_n}\frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\mu}_n}{\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} + \gamma^{-1}\frac{1}{q_n}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\mu}_n\\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\mu}_n}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\mathbf{1}}\\
&+ \gamma^{-1}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\\
&\xrightarrow{a.s.} \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}c^{-1}(c-1)^{-1}\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{c^{-1}(c-1)^{-1}}\\
&+ \gamma^{-1}c^{-1}(c-1)^{-1}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} = \frac{1}{q_n}\left(R_{GMV} + \gamma^{-1}c^{-1}(c-1)^{-1}s\right)
\end{aligned}
$$

and

$$\begin{aligned}
\hat{\mathbf{w}}_{S^*}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_{S^*} &= \frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{(\mathbf{1}'\mathbf{S}_n^*\mathbf{1})^2} + 2\gamma^{-1}\frac{\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} + \gamma^{-2}\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n \\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-2}(\mathbf{1}'\mathbf{S}_n^*\mathbf{1})^2} + \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{\bar{\mathbf{y}}_n'\hat{\mathbf{Q}}_n^*\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&+ 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n)^{-1/2}\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}}{q_n}\frac{(c-1)^{-3}}{c^{-2}(c-1)^{-2}} + \gamma^{-2}\frac{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{q_n}\frac{(c-1)^{-3}c^2 + (c-1)^{-3}\boldsymbol{\mu}_n'\mathbf{Q}_n\boldsymbol{\mu}_n}{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n} \\
&+ 2\gamma^{-1}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}}{q_n}\frac{0}{c^{-1}(c-1)^{-1}} \\
&= \frac{1}{q_n}\left(c^2(c-1)^{-1}V_{GMV} + \gamma^{-2}\frac{c^2}{(c-1)^3} + \gamma^{-2}(c-1)^{-3}s\right)
\end{aligned}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Finally, using that

$$\frac{1}{q_n}U_{EU} = \frac{1}{q_n}R_{GMV} + \frac{1}{2}\gamma^{-1}\frac{1}{q_n}s - \frac{\gamma}{2}\frac{1}{q_n}V_{GMV},$$

we obtain the statement of the first part of the corollary.

(b) It holds that

$$\begin{aligned}
\frac{1}{q_n}U_{GSE} &= \alpha^+\frac{1}{q_n}\hat{\mathbf{w}}_{S^*}'\boldsymbol{\mu}_n + (1-\alpha^+)\frac{1}{q_n}\mathbf{b}'\boldsymbol{\mu}_n \\
&- \frac{\gamma}{2}\left((\alpha^+)^2\frac{1}{q_n}\hat{\mathbf{w}}_{S^*}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_{S^*} + 2\alpha^+(1-\alpha^+)\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_{S^*} + (1-\alpha^+)^2\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}\right),
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{w}}_{S^*} &= \frac{1}{q_n}\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{S}_n^*\mathbf{1}}{\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} + \gamma^{-1}\frac{1}{q_n}\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n \\
&= \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}\mathbf{1}'\mathbf{S}_n^*\boldsymbol{\Sigma}_n\mathbf{b}}{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1}\mathbf{1}'\mathbf{S}_n^*\mathbf{1}} \\
&+ \gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{\mathbf{b}'\boldsymbol{\Sigma}_n\hat{\mathbf{Q}}_n^*\bar{\mathbf{y}}_n}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} \\
&\xrightarrow{a.s.} \frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{(\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1})^{-1/2}(\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b})^{-1/2}c^{-1}(c-1)^{-1}}{c^{-1}(c-1)^{-1}} \\
&+ \gamma^{-1}\frac{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}}{q_n}\frac{c^{-1}(c-1)^{-1}\left(\mathbf{b}'\boldsymbol{\mu}_n - \frac{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}{\mathbf{1}'\boldsymbol{\Sigma}_n^{-1}\mathbf{1}}\right)}{\sqrt{\boldsymbol{\mu}_n'\boldsymbol{\Sigma}_n^{-1}\boldsymbol{\mu}_n}\sqrt{\mathbf{b}'\boldsymbol{\Sigma}_n\mathbf{b}}} \\
&= \frac{1}{q_n}\left(V_{GMV} + \gamma^{-1}c^{-1}(c-1)^{-1}(R_b - R_{GMV})\right)
\end{aligned}$$

for $p/n \longrightarrow c \in (1, +\infty)$ as $n \to \infty$.

Hence,

$$
\begin{aligned}
& \frac{1}{q_n}(U_{EU} - U_{GSE}) = (\alpha^+)^2 \frac{1}{q_n} U_{EU} + (1-\alpha^+)^2 \frac{1}{q_n} U_{EU} + 2\alpha^+(1-\alpha^+) \frac{1}{q_n} U_{EU} \\
- \quad & (\alpha^+)^2 \frac{1}{q_n} U_S - \alpha^+(1-\alpha^+) \frac{1}{q_n} \hat{\mathbf{w}}'_{S^*} \boldsymbol{\mu}_n - (1-\alpha^+)^2 \frac{1}{q_n} U_{\mathbf{b}} - \alpha^+(1-\alpha^+) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\mu}_n + \gamma \alpha^+ (1-\alpha^+) \frac{1}{q_n} \mathbf{b}' \boldsymbol{\Sigma}_n \hat{\mathbf{w}}_{S^*} \\
\stackrel{a.s.}{\longrightarrow} \quad & (\alpha^+)^2 \frac{1}{q_n}(U_{EU} - U_S) + (1-\alpha^+)^2 \frac{1}{q_n}(U_{EU} - U_{\mathbf{b}}) \\
+ \quad & \alpha^+(1-\alpha^+) \frac{1}{q_n} \left(2R_{GMV} + \gamma^{-1}s - \gamma V_{GMV} - R_{GMV} - \frac{\gamma^{-1}c^{-1}}{c-1} s - R_b + \gamma V_{GMV} + \frac{R_b - R_{GMV}}{c(c-1)} \right) \\
= \quad & (\alpha^+)^2 \frac{1}{q_n}(U_{EU} - U_S) + (1-\alpha^+)^2 \frac{1}{q_n}(U_{EU} - U_{\mathbf{b}}) + \alpha^+(1-\alpha^+) \frac{1+c-c^2}{c(c-1)} \frac{1}{q_n}(R_b - R_{GMV} - \gamma^{-1}s).
\end{aligned}
$$

for $p/n \longrightarrow c \in (0, 1)$ as $n \to \infty$. □

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