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DOI
10.1109/ISIT.2016.7541716

Publication date
2016

Document Version
Accepted author manuscript

Published in
Proceedings ISIT 2016

On the Energy Benefit of Compute-and-forward for Multiple Unicasts

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***Abstract*—Compute-and-forward (CF) is a technique which exploits broadcast and superposition in wireless networks. In this paper, the CF energy benefit is studied for networks with unicast sessions and modeled by connected graphs. This benefit is defined as the ratio of the minimum energy consumption by traditional routing techniques, not using broadcast and superposition features, and the corresponding CF consumption. It is shown to be upper bounded by $\min(\bar{d}, K, 12\sqrt{K})$, where $\bar{d}$ and K are the average hop-count distance and the number of sessions, respectively. Also, it can be concluded that the energy benefit of network coding (NC) is also upper bounded by the same value, which is a new scaling law of the energy benefit for NC as a function of K.**

I. Introduction

Since network coding (NC) [1] was introduced, many studies considered the energy benefit of NC because it reduces the number of transmissions in wireless networks by letting the relay nodes transmit linear combinations of data packets. A broadcast scenario has been considered in [2], in which both centralized and decentralized schedules are considered and matching upper and lower bounds are given on the energy benefit. The problem of the energy benefit of NC in the multiple unicasts case, on the other hand, is more complicated. Upper bounds have been given for the energy benefit of NC for multiple unicasts in [3], [4]. In some specific networks, namely hexagonal lattice networks, a lower bound on the energy benefit for multiple unicasts of 2.4 has been derived in [5], which has been further improved to 3 in [6].

All of the above-mentioned studies considered the transmit energy only, which is not very practical in the scenarios that the energy consumed for receiving signals (for decoding, demodulation) is not negligible. Furthermore, it has also been shown in some studies, e.g., [7], that some NC based schemes decrease the number of transmissions at the cost of increasing the number of receptions. If the energy consumption for receiving is not negligible, some of the NC based schemes will have less improvement, or even no benefit at all. Hence, it is more general and practical to study the energy benefit in networks taking into account both the transmit and receive energy.

Compute-and-forward (CF), also known as reliable physical layer NC, is an advanced NC technique that allows the receivers to decode a linear combination of multiple messages after receiving the superposition of the physical layer signals of these messages [8]. It can benefit the network in energy consumption by reducing not only the number of transmissions, but also the number of receptions. It has been shown that in hexagonal lattice networks with a specific unicast session placement, the energy benefit of CF is in between 2 and 3, where the energy benefit is defined as the ratio of the minimum energy consumption by traditional routing techniques and the corresponding CF consumption [9].

The throughput benefit of CF over traditional routing, in a setting with K unicast sessions, was studied in [10]. In particular, this benefit was shown to be upper bounded by $3K$, while also a case was presented in which a benefit of $K/2$ is achieved. Hence, a CF gain in the order of K is possible with respect to throughput. In this paper, we show that gains in this order of magnitude are not to be expected from an energy perspective. Specifically, we show that the energy benefit is upper bounded by $\min(\bar{d}, K, 12\sqrt{K})$, where $\bar{d}$ is the average hop-count distance. Hence, CF energy saving gains beyond an order of $\sqrt{K}$ are not possible. This is a new scaling law for the energy benefit of CF and also NC in general networks with multiple unicast.

This paper is organized as follows. In Section II, we introduce our network model. In Section III, we give an upper bound of the energy improvement factor. This bound is obtained by combining 3 different upper bounds, which are presented in Subsections III-A, III-B, and III-C, respectively. At last, we conclude the results of this paper and give recommendations in Section IV.

II. Model

In this section, a network model similar to the one used in [10] is introduced. The network is represented by a connected graph, in which the vertices represent wireless nodes and the edges represent the wireless connectivity between two nodes. We focus on two features of wireless networks: the broadcast feature of wireless signals at the transmitters and the superposition feature of the wireless signals at the receivers. CF is capable of exploiting both of these two features, while in traditional routing schemes, neither of the two features is exploited. Thus, two transmission modes are proposed, namely TR (traditional routing) mode and CF mode. In these modes, these two features are allowed or disallowed. The formal definitions of the two modes will be given later in this section. First, we introduce our network model.

A. Network Model

The network is denoted by $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$. Here, $\mathcal{V}$ is a vertex set, $\mathcal{E}$ is an edge set, and $(\mathcal{V}, \mathcal{E})$ represents a connected, directed, and unweighted graph. Here, the elements in $\mathcal{V}$ are called *nodes*, and node $v \in \mathcal{V}$ is a neighbor of node $u \in \mathcal{V}$ if $(u, v) \in \mathcal{E}$. The wireless connectivity between two nodes is mutual. Hence, if $(u, v) \in \mathcal{E}$, $(v, u) \in \mathcal{E}$. $\mathcal{S} = \{S_1, S_2, \ldots, S_K\}$ is the session set and a session S_i is represented by its source a_i and destination b_i, i.e., $S_i = (a_i, b_i), a_i, b_i \in \mathcal{V}$. Here, we assume $a_i \neq a_j, b_i \neq b_j \forall i \neq j$. Further, the notation $\mathcal{A} = \{a_i | i \in \{1, 2, \ldots, K\}\}$ is used for the set of the sources and $\mathcal{B} = \{b_i | i \in \{1, 2, \ldots, K\}\}$ is used for the set of the destinations. We use the notation d_i for the (hop-count) distance of session S_i and $d(u, v)$ as the distance between node u and v. Hence, $d_i = d(a_i, b_i)$. We further define $d(\mathcal{V}', m), \mathcal{V}' \subseteq \mathcal{V}, m \in \mathcal{V}$ as $d(\mathcal{V}', m) = \min_{u \in \mathcal{V}'} d(u, m)$ and define $d(m, \mathcal{V}')$ in the same fashion. We let $\bar{d} = \frac{\sum_{i=1}^{K} d_i}{K}$.

For this network, assume that time is slotted and half-duplex is used. Messages are represented by symbols from finite field $\mathbb{F}_q$. The capacity of all edges is 1 message per time slot.

Note that this network model can be seen as a generalized version of the *protocol model* used in [3] with identical transmit and interference radii in the sense that it does not necessarily represent networks with certain geometric topologies.

B. Transmission Modes

We use the transmission modes proposed in [10], namely the TR and CF modes, to compare the energy consumption of traditional routing and CF. Here, we highlight the features of these transmission modes which are important w.r.t. the energy consumption. For details of these modes, we refer to [10].

1) TR Mode: TR mode represents traditional routing schemes in which broadcast and superposition are not exploited. Since network coding is not used, although the transmission of a node can be received by all neighbors, it can only be useful to one of them and is considered as interference by the other neighbors. Also, a transmission can be successfully received only if it is not interfered by other transmissions. Hence, at one time slot, a node can send a message to at most one neighbor and a node can receive a message only if just one of its neighbors is sending a message.

2) CF Mode: CF is a nested lattice codes [11] based technique which allows a node to decode linear combinations of the messages transmitted by its neighbors after the reception of the superposition of the physical layer signals of these messages [8].

In CF mode, as in NC, the broadcast feature is exploited by letting nodes transmit linear combinations of multiple messages. Thus, while transmitting, the message is broadcast to all of its neighbors. The superposition feature is exploited by the CF technique, i.e., a node is able to directly decode the sum of all messages transmitted by its neighbors in that time slot. [1]

C. Energy Consumption

In this paper, the energy consumption of any transmission scheme is always discussed in the context of a *round*. In each round, a transmission scheme should guarantee a new message from the corresponding source to be successfully decoded by each destination after the initial state of a long-term transmission. We then define E^X as the *minimum energy consumption* of any scheme for each round in mode $X \in \{\mathrm{TR}, \mathrm{CF}\}$. We use the notations e_t for the energy consumed (for broadcast, encoding, modulation, etc.) by a node to transmit (broadcast) a symbol from $\mathbb{F}_q$ to its neighbors and e_r for the energy consumed (for decoding, demodulation, etc.) by a node to receive a symbol (either one message or the sum of multiple messages) from $\mathbb{F}_q$.[2] The energy consumed for computing when CF is applied and all other energy consumption (supporting circuits, routing, signaling, etc.) are neglected. The *energy improvement factor* is defined as

$$J = E^{\mathrm{TR}} / E^{\mathrm{CF}}. \tag{1}$$

III. Upper Bound for the Energy Benefit

In this section, the upper bound of the energy improvement factor of CF for multiple unicasts is studied. The upper bound of J is derived by studying lower bounds for E^{CF} and establishing an explicit expression for E^{TR}. First, we give the main result of this section.

Theorem 1 (Upper bound of the energy benefit)**.** *For any network $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$, the energy benefit satisfies*

$$J \leq \min(\bar{d}, K, 12\sqrt{K}). \tag{2}$$

In the following three subsections, three upper bound of J which are $\bar{d}$, K, and $12\sqrt{K}$, respectively, will be derived.

[1]Note that as shown in [8], the rate for decoding the sum of n messages is not the same as the rate of transmitting one message via the same edge in TR mode. More precisely, the rate for decoding the sum of n symbol is $1/2 \log(1/n + \mathrm{SNR})$, while in TR mode the rate can be as high as $1/2 \log(1 + \mathrm{SNR})$. However, in high SNR scenarios, this difference is negligible. In this model, we neglect this rate difference and assume that the rate of decoding a individual message or a linear sum is 1 message per time slot in both cases.

[2]As introduced in [8] CF is based on lattice codes, which can be any linear code, e.g., low-density parity-check (LDPC) code, satisfying certain properties introduced in [11]. It is then clear that decoding a linear combination consumes no more energy than decoding a message encoded with the same linear channel codes in traditional routing. Hence, we assume that the decoding of both the individual message and the sum of multiple messages consumes energy e_r.

A. The $\bar{d}$ upper bound of J

First, the expression for E^{TR} is given.

Lemma 1 (Minimum energy consumption in TR mode)**.** *For any network* $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$*, the minimum energy consumption in TR mode is*

$$E^{\mathrm{TR}} = K\bar{d}(e_t + e_r). \tag{3}$$

Proof: An upper bound on E^{TR} can be given by any valid transmission scheme. Clearly, by letting all sessions send messages along their shortest paths, the energy consumption for each round is

$$E = \sum_{i=1}^{K} d_i(e_t + e_r) = K\bar{d}(e_t + e_r). \tag{4}$$

Hence we have the upper bound for E^{TR}. Then, it is clear that it is also the lower bound for E^{TR} since we assume that no network coding is allowed in TR mode. Thus the shortest-path routing strategy is optimal. ■

In the following lemma, a distance based upper bound of J is given.

Lemma 2. *For any network* $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$*, the energy benefit satisfies*

$$J \leq \bar{d}. \tag{5}$$

Proof: The proof of this lemma is straightforward since in CF mode, each source needs to transmit once and each destination needs to receive once in each round. Thus we have

$$E^{\mathrm{CF}} \geq K(e_t + e_r). \tag{6}$$

Combining this with (1) and (3) we finish our proof. ■

B. The K upper bound of J

In the next lemma, we show that the energy improvement factor is upper bounded by the number of sessions.

Lemma 3. *For any network* $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$*, the energy benefit satisfies*

$$J \leq K. \tag{7}$$

Proof: First, we introduce a notation N_r for the minimum number of non-source non-destination nodes needed to connect all sessions. More specifically, let $\mathcal{V}^* \subseteq \mathcal{V}\backslash(\mathcal{A} \cup \mathcal{B})$ and $\mathcal{E}^* = \{(u, v) \in \mathcal{E} | u, v \in \mathcal{V}^* \cup \mathcal{A} \cup \mathcal{B}\}$. For a network $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$, if the network $\mathbf{N}(\mathcal{V}^* \cup \mathcal{A} \cup \mathcal{B}, \mathcal{E}^*, \mathcal{S})$ is also a valid network model (all the sessions are connected), then $|\mathcal{V}^*| \geq N_r$. In other words, there does not exist a non-source non-destination node set $\mathcal{V}^*$ with $|\mathcal{V}^*| < N_r$ such that $\mathbf{N}(\mathcal{V}^* \cup \mathcal{A} \cup \mathcal{B}, \mathcal{E}^*, \mathcal{S})$ is a valid network.

Then we can prove

$$E^{\mathrm{CF}} \geq (N_r + K)(e_t + e_r) \tag{8}$$

by contradiction: If there exists a transmission scheme which consumes energy

$$E' < (N_r + K)(e_t + e_r) \tag{9}$$

in a round, since the sources have to transmit at least K times and the destinations have to receive at least K times, it is clear that in this scheme the energy consumption of all the other nodes is less than $N_r(e_t + e_r)$. Since in each round a node must transmit and receive at least once to function in the network if it is not a source or destination, we conclude that in this scheme there are less than N_r non-source non-terminal nodes involved, which contradicts the definition of N_r.

Combining (1), (3), and (8) we have

$$J \leq \frac{\sum_{i=1}^{K} d_i}{N_r + K}. \tag{10}$$

Now we prove $\sum_{i=1}^{K} d_i \leq K(N_r - 1) + K(K + 1)$ by considering the networks with the sum distance achieving this upper bound. More precisely, for given K and N_r, if a network $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$ has the maximum value of the sum distance $\sum_{i=1}^{K} d_i$ among all networks with the same K and N_r, it should have the following properties:

1) None of the sources is collocated with any destination. This can be proved by contradiction: Assume node u is both a_i and b_j, then we can find another network with node $u = a_i$, an additional node $v = b_j$, and edges $(u, v), (v, u)$ that has a larger sum distance.
2) It is a line network. First, it is straightforward that the network is acyclic since the sum distance of any network with cycles can be increased by removing edges to break the cycles. Then, we prove that any node can have at most two neighbors by contradiction. Assume node u has neighbors v_1, v_2, and v_3. Since the network is acyclic, without loss of generality (w.l.o.g.) we assume v_3 is closer to $\mathcal{V}^*$, i.e., $d(v_3, \mathcal{V}^*) \leq d(v_1, \mathcal{V}^*) = d(v_2, \mathcal{V}^*)$. We consider the network with edges $(u, v_1), (v_1, u)$ removed and $(v_1, v_2), (v_2, v_1)$ added. This network has a larger sum distance since the paths of all sessions involving v_1 are now 1 hop longer.
3) The paths of all sessions go through all the nodes in $\mathcal{V}^*$. This property is straightforward since any line network without this property can be easily modified to a network with this property and an increased sum distance.

With all the properties above, it can then be concluded that the maximum sum distance is achieved by a line network consisting of $N_r + 2K$ nodes. In this network, the sources and destinations are non-collocated and the paths of all sessions go through $\mathcal{V}^*$. It can be easily calculated the sum distance of this network is $K(N_r - 1) + K(K + 1)$. Then we have

$$J \leq \frac{\sum_{i=1}^{K} d_i}{N_r + K} \leq \frac{K(N_r - 1) + K(K + 1)}{N_r + K} = K. \tag{11}$$

■

C. The $12\sqrt{K}$ upper bound of J

In the following lemma, we give an upper bound of J which is in the order of $\sqrt{K}$.

Lemma 4. *For any network $\mathbf{N}(\mathcal{V},\mathcal{E},\mathcal{S})$, the energy benefit satisfies*

$$J < 12\sqrt{K}. \quad (12)$$

Before proving this lemma, we give the following lemmas.

Lemma 5. *For any network $\mathbf{N}(\mathcal{V},\mathcal{E},\mathcal{S})$, the minimum energy consumption in CF mode satisfies*

$$E^{\text{CF}} \geq \max_{\mathcal{S}^*\subseteq\mathcal{S}} \max(\sum_{a_i\in\mathcal{A}^*} d(a_i,\mathcal{B}^*), \sum_{b_i\in\mathcal{B}^*} d(\mathcal{A}^*,b_i))(e_t+e_r), \quad (13)$$

where $\mathcal{S}^*$ is a subset of $\mathcal{S}$ and $\mathcal{A}^*,\mathcal{B}^*$ are its source and destination sets, respectively.

The proof of this lemma is omitted here since it is simply a straightforward generalization of [4, Theorem 5.1].

Now we analyze the bound given in (13) by introducing the distance matrix.

Definition 1 (Distance matrix)**.** For a network $\mathbf{N}(\mathcal{V},\mathcal{E},\mathcal{S})$, a distance matrix denoted by D is a $K\times K$ matrix with $d(a_i,b_j)$ as the entry in the i-th row and the j-th column.

A distance matrix has the following properties.

Property 1. The entries on the diagonal are non-zero.

Property 2. If $d_{i,j}=0$, then $d_{i',j}, d_{i,j'} \neq 0$ for all $i'\neq i, j'\neq j$.

Property 3. $\forall i,j,k,l \in \{1,2,\ldots,K\}, k\neq i, l\neq j$, $d_{i,j} \leq d_{i,l}+d_{k,j}+d_{k,l}$.

The first property is trivial. The second property follows from our assumption in the model that a node cannot be the sources or destinations for multiple sessions. The third property follows from the fact that the route $a_i\to b_l\to a_k\to b_j$ is a valid path in the network and the length should not be smaller than $d(a_i,b_j)$.

For a subset $\mathcal{S}^*$ of $\mathcal{S}$, we denote the submatrix which contains only sessions in $\mathcal{S}^*$ by D^*, i.e., D^* is a submatrix with entries $d_{i,j}$ that $S_i,S_j\in\mathcal{S}^*$. We further use the notation $d^*_{i,j}$ to represent the (i,j)-th entry of D which is also an entry for D^*. Note that $d^*_{i,j}$ is not necessarily the (i,j)-th entry of D^*. Then, we can rewrite (13) as

$$E^{\text{CF}} \geq \max_{\mathcal{S}^*\subseteq\mathcal{S}} \max(\sum_i \min_j d^*_{i,j}, \sum_j \min_i d^*_{i,j})(e_t+e_r). \quad (14)$$

By analyzing (14) we derive the following lemma.

Lemma 6. *For a network $\mathbf{N}(\mathcal{V},\mathcal{E},\mathcal{S})$ we have*

$$E^{\text{CF}} \geq \frac{1}{6}\sqrt{K}d(e_t+e_r), \quad (15)$$

where $d=\min_{i=1}^K d_i$.

Proof: First, for $K<36$, (15) holds since (1), (3) and (7) hold.

Then, for $K\geq 36$, we discuss two cases.

- **Case 1**: There exists a row or column which contains at least $\sqrt{K}/2$ entries which are smaller than or equal to $d/3$, i.e., $\exists i$, $|\{j|d_{i,j}\leq d/3\}| \geq \sqrt{K}/2$ or $|\{j|d_{j,i}\leq d/3\}| \geq \sqrt{K}/2$.
  First we consider the case that $\exists i$, $|\{j|d_{i,j}\leq d/3\}| \geq \sqrt{K}/2$. Since the sessions can be arbitrarily indexed, w.l.o.g., we assume that the first $f\in\mathbb{Z}\cap[\frac{1}{2}\sqrt{K}+1,K]$ entries of the first row are smaller or equal than $d/3$ except the first one, i.e.,
  $$d_{1,j}\leq d/3, j\in\{2,3,\ldots,f\}. \quad (16)$$
  Now, for all $k\in\{2,3,\ldots,f\}, l\in\{2,3,\ldots,f\}$, we have
  $$\begin{aligned} d_{k,l} &\geq d_{k,k}-d_{1,l}-d_{1,k} \\ &\geq d_{k,k}-2d/3 \\ &\geq d/3, \end{aligned} \quad (17)$$
  where the first inequality follows from Property 3, the second inequality follows from (16), and the third inequality follows from the definition of d. Then we consider the submatrix D^* with the 2-fth rows and columns of D and have
  $$\sum_i \min_j d^*_{i,j} \geq (f-1)d/3 \geq \sqrt{K}d/6. \quad (18)$$
  For the case that $\exists i, |\{j|d_{j,i}\leq d/3\}| \geq \sqrt{K}/2$, with the same argument we have (18).
- **Case 2**: For any row or column, it contains fewer than $\sqrt{K}/2$ entries which are smaller or equal than $d/3$, i.e., $\forall i$, $|\{j|d_{i,j}\leq d/3\}| < \sqrt{K}/2$ and $|\{j|d_{j,i}\leq d/3\}| < \sqrt{K}/2$.
  In this case, w.l.o.g. we assume that the first $f\in\mathbb{Z}\cap[0,\frac{1}{2}\sqrt{K}+1)$ entries of the first row are smaller or equal than $d/3$ except the first one, i.e., $d_{1,j}\leq d/3, j\in\{2,3,\ldots,f\}$. Moreover, we assume that the entries in the rows $\mathbb{Z}\cap[g',g]$ of the first column are smaller or equal than $d/3$, i.e., $d_{i,1}\leq d/3, i\in\{g',g'+1,\ldots,g\}$. Here, $g',g\in\mathbb{Z}, g'\in(1,\frac{1}{2}\sqrt{K}+2], g\in[g',\sqrt{K}+1)$, and $g-g'<\sqrt{K}/2$.
  Then we consider the submatrix D_1 which is the matrix D with the 2nd to g-th rows and columns removed, i.e.,
  $$D_1 = \begin{bmatrix} d_{1,1} & d_{1,g+1} & \ldots & d_{1,K} \\ d_{g+1,1} & d_{g+1,g+1} & \ldots & d_{g+1,K} \\ \vdots & \vdots & \ddots & \vdots \\ d_{K,1} & d_{K,g+1} & \ldots & d_{K,K} \end{bmatrix}. \quad (19)$$
  Clearly, all entries in the first row and column (the blue parts) of D_1 are larger than $d/3$. Moreover, the property that any column or row contains less than $\sqrt{K}/2$ entries which are smaller or equal than $d/3$ still holds for this submatrix D_1. As a result, this puncturing process can be iteratively repeated for other rows and columns which still contain entries that are smaller or equal than $d/3$ (the red parts). During each iteration, less than $\sqrt{K}$ rows and columns are removed. After $h\in\mathbb{Z}\cap[\sqrt{K}/2,\sqrt{K}-1]$ iterations, we obtain a submatrix D_h in which all entries in at least h rows and columns are larger than $d/3$. Then,

if we consider $D^* = D_h$, we have $\sum_i \min_j d^*_{i,j} \geq hd/3$ and (18) holds.

Combining (18) with (14) we finish our proof. ■

Now, we prove Lemma 4 using Lemma 6.

Proof: Since the indexing of the sessions is arbitrary, w.l.o.g. we assume $d_1 \leq d_2 \leq \cdots \leq d_K$.

Now we consider a network $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S}^*)$ with $\mathcal{S}^* \subseteq \mathcal{S}$. Obviously, this network cannot consume more energy than $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$. Hence, by Lemma 6 we have a lower bound for E^{CF} for $\mathbf{N}(\mathcal{V}, \mathcal{E}, \mathcal{S})$

$$E^{\text{CF}} \geq \frac{1}{6}\sqrt{|\mathcal{S}^*|}\min_i d^*_{i,i}(e_t + e_r). \tag{20}$$

Combining all bounds obtained with all possible subsets, we have

$$E^{\text{CF}} \geq \frac{1}{6}\max_{\mathcal{S}^* \subseteq \mathcal{S}} \sqrt{|\mathcal{S}^*|}\min_i d^*_{i,i}(e_t + e_r). \tag{21}$$

Then, since $d_1 \leq d_2 \leq \ldots \leq d_K$, we do not need to consider all subsets of $\mathcal{S}$. More precisely, let us consider a subset $\mathcal{S}^* \subset \{S_k, S_{k+1}, \ldots, S_K\}$. It is clear that the lower bound of (20) for this subset is strictly smaller than the lower bound for the subset $\{S_k, S_{k+1}, \ldots, S_K\}$. Hence, we only consider the subsets $\mathcal{S}^* = \{S_k, \ldots, S_K\}, k \in \{1, 2, \ldots, K\}$ and (21) is simplified to

$$E^{\text{CF}} \geq \frac{1}{6}\max_{k=1}^{K} \sqrt{K-k+1}\, d_k(e_t + e_r). \tag{22}$$

Then we consider the energy improvement factor J. By (1), (3), and (22) we have

$$\begin{aligned} J &= \frac{E^{\text{TR}}}{E^{\text{CF}}} \\ &= \frac{\sum_{k=1}^{K} d_k(e_t + e_r)}{E^{\text{CF}}} \\ &\leq \frac{\sum_{k=1}^{K} d_k}{\frac{1}{6}\max_{k=1}^{K}\sqrt{K-k+1}\, d_k} \\ &\leq 6\left(\frac{d_1}{\sqrt{K}d_1} + \frac{d_2}{\sqrt{K-1}d_2} + \cdots + \frac{d_K}{\sqrt{1}d_K}\right) \\ &= 6\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}}\right) \\ &< 12\sqrt{K}. \end{aligned} \tag{23}$$

The last inequality can be proved by simple algebra, the details of the proof are omitted here. ■

Combining Lemmas 2-4, Theorem 1 is proved.

IV. Conclusion and Discussion

Unlike the throughput benefit for general networks given in [10], it is shown in Theorem 1 that the upper bound of the energy benefit of CF in (2) does not only depend on K but also the average distance of the sessions. Moreover, it is shown that the energy benefit is upper bounded by a factor of $\sqrt{K}$. This is a different phenomenon in comparison to the throughput benefit which can be as high as a factor of $K/2$ as shown in [10, Theorem 2].

Also, notice that CF is an advanced NC technique which includes the methods of plain NC. Hence we can also conclude that the energy benefit of NC over traditional routing is upper bounded by a factor of $\min(\bar{d}, K, 12\sqrt{K})$. In other words, our result can also be seen as an extension to [4, Theorem 5.1]. As far as we know, this is the best upper bound for the energy benefit of NC for multiple unicasts in general networks when K is large.

Another important remark is that for all the networks that have been considered in literature, e.g. [9], the energy improvement factors are no more than constants. Hence, it remains as an interesting problem to study whether there exists a network with an energy improvement factor at the order of $\sqrt{K}$ or that this upper bound can be further improved to a constant.

Acknowledgment

This work was supported by ERC Starting Grant 259530-ComCom.

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