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AN ALTERNATIVE PROCESS-BASED APPROACH TO PREDICTING THE RESPONSE OF WATER SATURATED POROUS MEDIA TO HYDRODYNAMIC LOADS

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Methods have been developed to predict how hydrodynamic loads acting on nearly saturated porous media are transmitted to the subsoil. In line with the effective stress principle of Terzaghi, these methods apply the boundary conditions that the effective stresses at the surface of a porous medium are zero, and that the pore water pressures carry the full load. Here a new approach is presented which is based on defining a stress and a stress gradient as boundary conditions. The stress gradient follows from the momentum balance equation thereby assuring that the solution abides by D'Alemberts principle of minimization of virtual work. The corresponding solution is in full accordance with the volume and momentum balance equations of the linear elastic soil matrix and the volume and momentum balance equations of the pore water across the computational domain. The new method is thereby able to correctly reproduce measurements of pore pressure changes due to hydrodynamic loads under the assumption of a porous medium consisting of incompressible particles and pore water which could either be compressible or incompressible. The advantages of the proposed method is that it requires one less boundary condition at the surface of the porous medium. The method is therefore able to predict the magnitude of the effective stresses on a soil surface. Due to the ability to retain the assumption of incompressible water, the method has also become independent on a calibration parameter. The results of the method induce questions with respect to the validity of Terzaghi's principle of effective stress at the boundary when porous media are subjected to hydrodynamic loads.

KEY WORDS: *Elastic, harmonic, liquifaction, momentum, porous, saturated, sea bed, waves*

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1. INTRODUCTION

Despite the fact that the interaction between fluid flow and subsequent deformation of porous media are ubiquitous in nature, simulating these processes remains a challenge. The boundary value problem for porous media is given as main reason for this challenge. Models aim to predict the behaviour of porous media after defining the effective stresses, shear stresses and pore water pressures as loads on the boundary. This paper shows that by solely imposing the pressure and shear stresses at the boundary of a porous media subjected to hydraulic hydrodynamic loads, a unique solution of the volume and momentum balance can be obtained in accordance with the volume and momentum balance equations that can correctly reproduce the pressure profiles observed during experiments.

Experiments have shown that water infiltrates into porous media when subjected to waves (Davies, 1996; Ye and Jeng, 2011; Tong et al., 2020). Consequently either the porous medium needs to expand, or the water needs to compress. Several researchers have already attempted to capture the associated processes. Moshagen and Torum

(1975) and Massel (1976) both assumed that the bed was rigid and non-deformable. For incompressible water, the resulting Laplace Equation indicates that the pore water pressures must attenuate with depth without any phase shift, independent on the permeability and elastic properties of the porous medium. In order to explain the observed phase shifts in porous media, Moshagen and Torum (1975) assumed that water was compressible but the bed rigid. This gives a type of heat equation for the time dependent change in pore water pressures. Yamamoto et al. (1987); Jeng and Hsu (1996); Ye and Jeng (2011) extended this work with a poro-elastic description of the porous medium by solving the equations of Biot (1941). They noted that for incompressible water the resulting pore pressure response agrees with the results of the Laplace equation and is directly transmitted to deeper soil layers without any phase shift. Yamamoto et al. (1978) noted that an attenuation of pressures often observed during experiments, must be associated with deformation of the porous medium. Models that aim to reproduce these observations often solve Biot's equations whereby the pore water is assumed compressible via inclusion of a calibrated Skempton coefficient (Skempton, 1954).

In line with Terzaghi's effective stress concept, in all these studies the a-priori assumption made for the surface boundary was that the pore water pressures match the hydrodynamic pressures under the waves and that the effective stresses are zero (Ye and Yeng, 2011; Zhang et al. 2011). When looking at an example of overtopping waves running over a grass cover, the choice for such boundary conditions does not match observations of grass covers being pulled from the soil (Van Bergeijk, 2022). In order to address this, here we solve the constitutive relations for a linear elastic saturated porous media consisting of a mixture of a pore fluid and incompressible particles. The pore fluid could be taken either as compressible or incompressible. By enforcing that the momentum balance equations must be valid at the boundaries of the computational domain it was possible to solve the constitutive relations without having to make an a-priori assumption about the distribution of effective stresses and pore water fluid at the surface of the porous medium. The influence of acceleration terms have thereby been ignored, which is a common applied assumption in Soil Mechanics (Verruijt, 2012). The results have been validated against experimental data. The analysis shows that assuming that the effective stresses at the surface of a saturated linear elastic porous medium are zero and that the incompressible pore water carries the full hydrodynamic load, conflicts with the momentum balance equations and therefore conflicts with d'Alemberts principle of minimum energy.

Section 2 presents the constitutive relations that have been resolved and which boundary and initial conditions have been enforced. Section 3 presents the general analytical solutions of the constitutive relations for a 2D case, and how these compare to the analytical solutions of Madsen (1978). Section 4 discusses how the analytical solutions were used to derive a unique solution of the physical problem. This analytical solution is validated in Section 5, after which the results are discussed in Section 6 and conclusions are drawn and presented in Section 7. Additional information on the derivation of the momentum balance equations resolved is given in the appendix.

2. METHODOLOGY

Prior to discussing the method developed for simulating the behaviour of the porous medium, first the volume and momentum balance equations for the porous medium are revisited. The application of the principle of minimization of virtual work, outlined in Appendix A, resulted in two sets of mass and momentum balance equations. One set describing a porous medium consisting of individual particles, and one set describing the behaviour of the pore water. Together with the volume balance equations these have been used to describe the response of porous media to hydrodynamic loads.

2.1 Volume balance equation

The volume balance equation for compressible or incompressible pore water is given for the Cartesian (x_1, x_2, x_3) coordinates and time t by

$$p\beta\frac{\partial P}{\partial t} + \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}\left(p\frac{\partial v_i}{\partial t}\right) = 0, \tag{1}$$

where p denotes the porosity of the medium, β denotes the compressibility of the pore water, P denotes the pore water pressure and v_i for $i = 1, 2, 3$ denotes the Einstein summation of the 2D displacement vector of the pore water with $[x_1, x_2, x_3] = [x, y, z]$. The volume balance equation describing the change in porosity due to the movement of incompressible particles in a porous medium is given by

$$\frac{\partial(1-p)}{\partial t} + \frac{\partial}{\partial x_i}\left[(1-p)\frac{\partial u_i}{\partial t}\right] = 0, \tag{2}$$

where u_i denotes the Einstein's summation convention for the displacement vector of the porous medium. Adding the volume balance equation of the pore water to the volume balance equation of the particles gives the volume balance equation for the porous medium

$$\frac{\partial}{\partial x_i}\left\{p\left[\frac{\partial(v_i - u_i)}{\partial t}\right]\right\} + \frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial t}\right) + p\beta\frac{\partial P}{\partial t} = 0, \tag{3}$$

where $\frac{\partial u_i}{\partial x_i} = \epsilon_{\text{vol}}$, which is the volumetric strain of the porous medium.

2.2 Momentum balance equations

The derivation of the momentum balance equations for a 2D case is given in Appendix A. For small pore sizes the system of momentum balance equations for the porous medium and the pore water behaves like an overdamped elastic system. The impact of the accelerations of the porous medium and water is therefore expected to be negligible with respect to the impact of the particle-particle and particle-water interaction. Also the gravitational terms have been neglected to arrive at a solution which solely describes the dynamic stresses. The equations have therefore been reduced to

$$-\mu\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\lambda + 2\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{4}$$

$$\mu\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\lambda + 2\mu)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{5}$$

$$\frac{\partial P}{\partial x} + \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{6}$$

$$\frac{\partial P}{\partial z} + \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0, \tag{7}$$

where K_s [m/s] denotes the hydraulic conductivity, γ_w [N/m^2] denotes the specific weight, ρ_w denotes the density of the pore water, and ρ_p denotes the particle density. The Lamé constants λ and μ are related to the Elasticity Modulus E, and Poisson ratio ν of the porous medium as (Verruijt, 2012)

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{8}$$

$$\mu = \frac{E}{2(1+\nu)}. \tag{9}$$

The analytical solutions derived in this paper are solutions of these momentum balance equations in both the computational domain and at the boundaries. It is often assumed that solving these momentum balance equations requires the effective stresses, shear stresses and pressures to be defined a-priori at the boundary. This gives a problem as solely an external normal stress and a shear stress are often known a-priori. To address this, the divergence and curl of the momentum balance equations have been taken and combined with the volume balance equations to arrive at a new set of equations. In doing so the order of the equations is increased. Consequently both a stress and a stress gradient are required to be defined at the boundary. By taking the divergence, a transformation is performed on the momentum balance equations. In line with this, the transformation must also be performed on the stress boundary conditions giving a stress gradient. The divergence of the shear stress acting in the x-direction and the normal stress

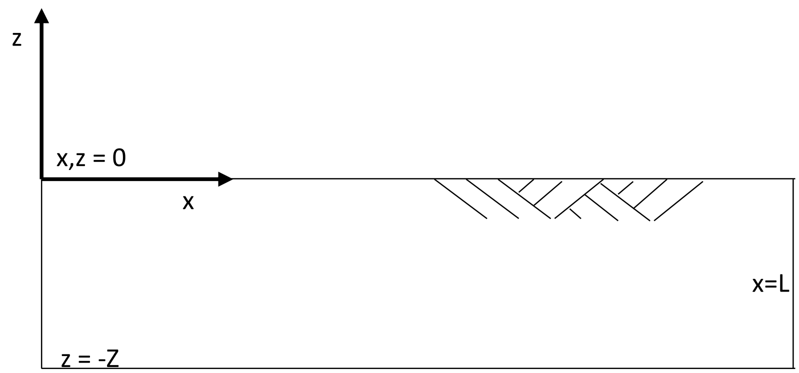

FIG. 1: Computational domain and coordinate directions.

acting in the z-direction results in the momentum balance equations normal to the boundary, given by Equation (5). Below is shown that by using this approach a unique solution of the volume and momentum balance equations is found.

The momentum balance equation are solved analytically in this paper in the $x-$domain $[0\ L]$ and the $z-$domain $[-Z\ 0]$ (see Figure 1), where $z = 0$ denotes the soil surface. The spatial distribution of the normal stresses acting on the surface of the porous medium is thereby expressed as a cosine transform. The spatial distribution of the shear stresses is expressed by a sine transform. In line with these assumptions, the displacements at the left and right boundaries of the computational domain are given by $\frac{du_z}{dx}|_{x=0} = \frac{du_z}{dx}|_{x=L} = 0$ and $u_x|_{x=0} = u_x|_{x=L} = 0$.

2.3 Vorticity equation

A constitutive equation for the vorticity ω is found when taking the curl of the momentum balance equations. The vorticity is thereby defined as the curl of the displacement field, which is given by

$$\omega = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}. \tag{10}$$

The Darcy's friction terms in Equations (4) and (5) have been replaced by the pressure gradients via substituting Equations (6) and (7). Taking the curl of the resulting momentum balance equations results in

$$-\mu\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\right] = 0. \tag{11}$$

which is independent on the volumetric strain and the pressure. The boundary conditions for the vorticity are yet unknown. Substituting the definition for the vorticity given by Equation (10) in Equation (11) gives

$$\mu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial z^2}\right) = 0. \tag{12}$$

This constitutive relation for the vorticity forms the first constitutive relation used for resolving the boundary value problem addressed in this paper. The solution for Equation (12) is given in Section 3.

2.4 Volumetric strain equation

Besides the curl also the divergence of Equations (4) and (5) has been taken, giving

$$
\begin{aligned}
&-\frac{2\mu}{2}\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)-\lambda\frac{\partial^2\epsilon_{\text{vol}}}{\partial x^2}-\mu\frac{\partial^3 u_x}{\partial x^3}-\frac{\gamma_w}{K_s}\frac{\partial}{\partial t}\frac{\partial p(v_x-u_x)}{\partial x}\\
&-\frac{2\mu}{2}\frac{\partial^2}{\partial x\partial z}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)-\lambda\frac{\partial^2\epsilon_{\text{vol}}}{\partial z^2}-\mu\frac{\partial^3 u_z}{\partial z^3}-\frac{\gamma_w}{K_s}\frac{\partial}{\partial t}\frac{\partial p(v_z-u_z)}{\partial z}=0.
\end{aligned}
\tag{13}
$$

Substituting Equation (3) now gives

$$
\frac{\gamma_w}{K_s}\frac{\partial\epsilon_{\text{vol}}}{\partial t}-(\lambda+2\mu)\left(\frac{\partial^2\epsilon_{\text{vol}}}{\partial x^2}+\frac{\partial^2\epsilon_{\text{vol}}}{\partial z^2}\right)=-\frac{\gamma_w}{K_s}p\beta\frac{\partial P}{\partial t},
\tag{14}
$$

which describes the quasi-steady redistribution of tensile and compressive stresses in the elastic porous medium while accounting for the effects of damping. In the case of no damping due to the absence of pore water, by accounting for the effects of the local acceleration, the divergence of the momentum balance equations would give a wave equation. This wave equation is commonly known to correctly describe the behaviour of elastic media without damping. Equation 14 is therefore likely to correctly predict the redistribution of stresses in an overly damped elastic porous medium. The solution to the constitutive relation for the volumetric strain is the second constitutive relation required to solve the boundary value problem addressed here.

2.5 Pressure relation

The constitutive relation for the pore water pressures follows from taking the divergence of the momentum balance equations for the pore water given by Equation (6) and (7). After substituting Equation (3) this gives

$$
\frac{\gamma_w}{K_s}p\beta\frac{\partial P}{\partial t}-\frac{\partial^2 P}{\partial x^2}-\frac{\partial^2 P}{\partial z^2}=-\frac{\gamma_w}{K_s}\frac{\partial\epsilon_{\text{vol}}}{\partial t},
\tag{15}
$$

which is the storage equation. When the solution of the momentum balance equations is also a solution of Equations (15) and (14), then volume is conserved. Based on the solution of the volumetric strain the pressure field P can be determined. Equation (15) forms the third constitutive relation used in resolving the boundary value problem addressed in this paper.

2.6 Relations for the displacements

Based on solutions for Equation (12) and (14) the horizontal and vertical displacements can be found. The volumetric strain and vorticity have been related to the horizontal and vertical displacement via the following equations

$$
-\frac{\partial^2 u_x}{\partial x^2}-\frac{\partial^2 u_x}{\partial z^2}=-\frac{\partial\omega}{\partial z}-\frac{\partial\epsilon_{\text{vol}}}{\partial x},
\tag{16}
$$

$$
-\frac{\partial^2 u_z}{\partial x^2}-\frac{\partial^2 u_z}{\partial z^2}=\frac{\partial\omega}{\partial x}-\frac{\partial\epsilon_{\text{vol}}}{\partial z}.
\tag{17}
$$

These relations respectively form the fourth and fifth ingredient used to resolve the boundary value problem addressed in this paper.

2.7 Conditions

The boundary value problem that this paper addresses is that at the boundary of a porous medium often the user needs to define both the effective stresses and pore water pressures. This user defined split in pressures and effective stresses solely determines the model results, thereby severely limiting the use of geotechnical models to those situations for which the boundary conditions can be determined well, and limiting the accuracy of predictions. The approach presented in this paper addresses this problem and thereby aims to significantly improve the understanding of the interaction between surface water and porous media.

2.7.1 Boundary conditions

The Cartesian coordinate system is given in Figure 1. At $z = 0$, the boundary conditions are given by the normal stress component acting on the porous medium given by $\sigma_{zz}|_{z=0} = F_{zz}$, and the shear stress component $\sigma_{xz}|_{z=0} = F_{xz}$. Appropriate boundary conditions are found by taking the curl or divergence of the surface stresses. The divergence of the stresses acting on the boundary gives the additional boundary condition that at the surface the choice in boundary conditions must be in line with the vertical momentum balance equation.

At $z = 0$, the boundary conditions for $\omega, u_z, \epsilon_{\text{vol}}$ at $z = 0$ depend on the boundary condition for the normal stress and shear stress, which will be addressed in Section 4. However at this stage it is already known that the solutions at the surface must adhere to the vertical momentum balance equation given by

$$\mu \frac{\partial \omega}{\partial x}|_{z=0} = (\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z}|_{z=0} - \frac{\partial P}{\partial z}|_{z=0}. \tag{18}$$

The normal stresses and shear stresses acting on the surface of the porous medium are given by the following Fourier Transform in time and respectively cosine and sine transforms in space.

$$F_{xz} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \text{Re}\left[s_{n,k} e^{\frac{2\pi i n t}{T}} \sin\left(\frac{\pi k x}{L}\right)\right], \tag{19}$$

$$F_{zz} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \text{Re}\left[n_{n,k} e^{\frac{2\pi i n t}{T}} \cos\left(\frac{\pi k x}{L}\right)\right]. \tag{20}$$

Here Re refers to the real part of the Fourier transform and $i^2 = -1$. The resulting model is therefore able to account for fully time dependent and spatially dependent harmonic boundary conditions. The boundary conditions at $z = -Z$ have also been defined such that the vertical momentum balance equation is abided by. Consequently these are given by

$$\omega|_{z=-Z} = u_z|_{z=-Z} = \frac{\partial P}{\partial z}|_{z=-Z} = \frac{\partial \epsilon_{\text{vol}}}{\partial z}|_{z=-Z} = \frac{\partial u_x}{\partial z}|_{z=-Z} = 0. \tag{21}$$

The boundary conditions at $x = 0$ and $x = L$ which are in line with the situation of a standing wave periodically increasing and decreasing the load on a horizontally confined soil body are given by

$$u_x|_{x=0,L} = \frac{\partial u_z}{\partial x}|_{x=0,L} = \omega|_{x=0,L} = \frac{\partial \epsilon_{\text{vol}}}{\partial x}|_{x=0,L} = \frac{\partial P}{\partial x}|_{x=0,L} = 0, \tag{22}$$

These boundary conditions are consistent with the horizontal momentum balance equations for the porous medium at $x = 0$ and $x = L$.

For a case of a traveling wave moving at a constant rate c over a porous medium, the boundary condition for the pressures has been decomposed in time and space using respectively the Fourier Transform and the discrete cosine transform. The shear stresses at the boundary have been decomposed in time and space using respectively the Fourier Transform and the discrete sine transform. The boundary conditions set at $x = 0$ and $x = L$ do not correspond well with the situation of a moving wave. However, Equations (14) and (15) indicate that the deeper soil layers will respond with a delay on what happens at the surface. due to the phase difference of the volumetric strain and pressures with depth, defining appropriate boundary conditions at $x = 0$ and $x = L$ becomes a significant challenge. It is expected that the influence of transforming a traveling wave in the sum of standing waves is limited to an area close to the vertical boundaries. As long as the area of interest is located far enough away from these areas, the impact is assumed negligible.

2.7.2 Initial conditions

The initial conditions follow from the assumption that all stresses and displacements must be 0 when no stresses act on the surface. The differential equation for the volumetric strain is first order in time. For that reason it is necessary to set an initial condition for the volumetric strain. However, it should be noted that when no stresses are acting on the surface initially also the pressure must be 0 at $t = 0$. It has been assumed that

$$u_z|_{t=0} = u_x|_{t=0} = \epsilon_{\text{vol}}|_{t=0} = \omega|_{t=0} = P|_{t=0} = 0. \tag{23}$$

3. SOLUTIONS OF THE CONSTITUTIVE RELATIONS

The differential equations have been solved analytically using separation of variables. The found analytical solutions have been compared against the analytical solution of Madsen (1978) whose analytical solution can be found in textbooks (Verruijt, 2006; Verruijt, 2014). First the solution of the pressure and volumetric strain is discussed.

3.1 Solution for the volumetric strain and pressure

The pressure and volumetric strain are given by the following system of equations

$$\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t} - (\lambda + 2\mu)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial x^2} - (\lambda + 2\mu)\frac{\partial^2 \epsilon_{\text{vol}}}{\partial z^2} = -\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t}, \tag{24}$$

$$\frac{\gamma_w}{K_s} p\beta \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} = -\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t}. \tag{25}$$

Rewriting this coupled set of equations in a matrix format gives.

$$\frac{\gamma_w}{K_s}\begin{pmatrix} \frac{1}{(\lambda+2\mu)} & \frac{p\beta}{(\lambda+2\mu)} \\ 1 & p\beta \end{pmatrix}\begin{pmatrix} \epsilon_{\text{vol}} \\ P \end{pmatrix}_t - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \epsilon_{\text{vol}} \\ P \end{pmatrix}_{x^2} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \epsilon_{\text{vol}} \\ P \end{pmatrix}_{z^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{26}$$

where the subscripts refer to the differentiation. The eigenvalues of the not-unit matrix are $p\beta + \frac{1}{(\lambda+2\mu)}$ and 0. The eigenvalue 0 refers to a steady state case and is therefore of less interest. Substituting these gives the eigenvectors

$$\begin{pmatrix} \frac{1}{\lambda+2\mu} \\ 1 \end{pmatrix}, \begin{pmatrix} -p\beta \\ 1 \end{pmatrix}. \tag{27}$$

The solution is now given by the differential equation

$$\frac{\gamma_w}{K_s}\left(p\beta + \frac{1}{(\lambda + 2\mu)}\right)\frac{\partial S}{\partial t} - \frac{\partial^2 S}{\partial x^2} - \frac{\partial^2 S}{\partial z^2} = 0, \tag{28}$$

where $S = [\frac{1}{\lambda+2\mu}\epsilon_{\text{vol}}, P]$. When initially no stresses are acting on the surface then the displacements, pressures and volumetric strain are 0, therefore S is initially 0. The boundary conditions at $x = 0$ and $x = L$ are given by $\frac{dS}{dx} = 0$. A generic solution of S is given by $S = G(t)G(x)G(z)$. This generic solution must abide by the periodically changing boundary conditions. Consequently, in line with the horizontal boundary conditions and the boundary conditions at $z = 0$ the solution for the volumetric strain is likely to be of the form

$$S = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left[e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right)G(z)\right]. \tag{29}$$

Substituting this in the constitutive relation for the volumetric strain gives

$$\left[\frac{\gamma_w}{K_s}\left(p\beta + \frac{1}{(\lambda + 2\mu)}\right)\frac{2\pi in}{T}G(z) + \frac{\pi^2 k^2}{L^2}G(z) - \frac{\partial^2 G(z)}{\partial z^2}\right]e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi ikx}{L}\right) = 0. \tag{30}$$

Equation (30) has been worked out in more detail. In order to find the analytical solution the following generic solution for $G(z)$ has been defined

$$G(z) = e^{(a+bi)z}. \tag{31}$$

Substituting this in Equation (30) gives

$$\frac{\gamma_w}{K_s}\left(p\beta + \frac{1}{(\lambda+2\mu)}\right)\frac{2\pi in}{T} + \frac{\pi^2k^2}{L^2} - \left(a^2 + 2abi - b^2\right) = 0. \tag{32}$$

A solution for the differential equation is now found for the conditions:

$$2ab = \frac{\gamma_w}{K_s}\left(p\beta + \frac{1}{(\lambda+2\mu)}\right)\frac{2\pi n}{T}, \tag{33}$$

$$\left(\frac{\pi^2k^2}{L^2} - a^2 + b^2\right) = 0. \tag{34}$$

Substituting $2abi = \frac{\gamma_w}{K_s}\frac{2\pi ni}{T}\left(p\beta + \frac{1}{(\lambda+2\mu)}\right)$ or $b = \frac{\gamma_w}{K_s}\frac{\pi n}{aT}\left(p\beta + \frac{1}{(\lambda+2\mu)}\right)$ gives

$$\left\{\frac{\pi^2k^2}{L^2} - a^2 + \left[\frac{\gamma_w}{K_s}\frac{\pi n}{aT}\left(p\beta + \frac{1}{(\lambda+2\mu)}\right)\right]^2\right\} = 0. \tag{35}$$

The solution for a now follows from

$$a_{\pm}^2 = \frac{-\frac{\pi^2k^2}{L^2} \pm \sqrt{\left[\frac{\pi^2k^2}{L^2}\right]^2 + 4\left[\frac{\gamma_w}{K_s}\frac{\pi n}{T}\left(p\beta + \frac{1}{(\lambda+2\mu)}\right)\right]^2}}{-2}. \tag{36}$$

The values for a_+^2 become negative leading to imaginary values of a which are in conflict with the chosen solution. Consequently the solution for a_-^2 has been used to solve a and b and to find the total solution. This leads to two different solutions for a and b. Any linear combination of these solutions abides by the differential equation. The unique solution is found by applying the boundary condition at $z = -Z$ given by $\frac{\partial \epsilon_{\text{vol}}}{\partial z}|_{z=-Z} = 0$. A part of the general solution of this differential equation is given by

$$G(z) = c_1 e^{(a_1+b_1i)z} + c_2 e^{(a_2+b_2i)z}. \tag{37}$$

Any condition at $z = 0$ for c_1 and c_2 simply gives a multiplication factor for the Fourier constants. Therefore the condition has been imposed that $c_1 + c_2 = 1$. The boundary conditions at $z = -Z$ for ϵ_{vol} and P give that $\frac{\partial G(z)}{\partial z}|_{z=-Z} = 0$. These conditions together give a unique solution for $G(z)$. The summation of the solution of S and its complex conjugate always results in real solutions. For that reason the full solution of S is given by

$$S = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}e^{\frac{2\pi in}{T}}\cos\left(\frac{\pi kx}{L}\right)\left[c_1e^{(a_1+b_1i)z} + c_2e^{-(a_2+b_2i)z}\right]\right\}. \tag{38}$$

The solutions for ϵ_{vol} and P now follow from the eigenvector belonging to the non-zero eigenvalue

$$\epsilon_{\text{vol}} = \frac{1}{(\lambda+2\mu)}S, \tag{39}$$

$$P = S. \tag{40}$$

When initially no stresses are acting on the surface, the volumetric strain must be 0. The value for $c_{k,n}$ therefore follows automatically from having a 0 stress acting on the boundaries of the domain at $t = 0$.

The analytical solution of Madsen (1978) is said to be a solution of Equation (25) and the momentum balance equations. According to the solution of Madsen, the pressure and volumetric strain are given by

$$\frac{P}{\mu} = -\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\left[2\frac{\pi k}{L}A_1 e^{-\frac{\pi k}{L}z} + (1+m)\left(r^2 - \frac{\pi^2 k^2}{L^2}\right)A_3 e^{-rz}\right]e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right), \tag{41}$$

$$\epsilon_{\text{vol}} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\left[2\frac{\pi k}{L}\theta A_1 e^{-\frac{\pi k}{L}z} - \left(r^2 - \frac{4\pi^2 k^2}{L^2}\right)e^{-rz}\right]e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right). \tag{42}$$

where $m = \frac{1}{1-2\nu}$, $\theta = p\beta\mu$ denotes the influence of the compressibility of water, and

$$r^2 = \frac{4\pi^2 k^2}{L^2} + \frac{2\pi in}{TC_c}\left(1 + \theta + m\theta\right) \tag{43}$$

where C_c is the coefficient of consolidation. Substituting Equations (42) and (41) in Equation (25) shows that the solution of Madsen abides by the Storage equation. However Equation (24) is not satisfied.

3.2 Homogeneous solutions pressure

In the case that the pore water is compressible, the homogeneous solution for the pressure is given by $P^H = 0$ when porous media are subjected to dynamic loads. Due to the presence of the time derivative of the pressure in Equation (24) it is not possible for the volumetric strain to be 0 or remain constant when the pressure is a function of time. Therefore the time derivative of the volumetric strain has to be non-zero leading to the particular solution of the pressure.

In case the pore water is considered incompressible $\beta = 0$, the pressure is given by the Poisson equation

$$-\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial z^2} = -\frac{\gamma_w}{K_s}\frac{\partial \epsilon_{\text{vol}}}{\partial t}. \tag{44}$$

The particular solution for the pressure then follows from $P = (\lambda + 2\mu)\epsilon_{\text{vol}}$. The total solution of this pressure equation in theory is given by a particular solution and a homogeneous solution.

The homogeneous solution of Equation (44) which also abides by the initial condition that $P|_{t=0} = 0$ is given by either $P^H = 0$ or

$$P^H = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{p_{n,k}e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right)\left[\cosh\left(\frac{\pi kz}{L}\right) + b_k\sinh\left(\frac{\pi kz}{L}\right)\right]\right\}. \tag{45}$$

This homogeneous solution abides by the initial conditions for the pressure. When the initial solution of the homogeneous solution of the pressure is 0, and the volumetric strain is initially 0, then also $P|_{t=0} = 0$. Since $\sum_{-\infty}^{\infty} c_n e^{\frac{2\pi int}{T}}|_{t=0} = 0$, the particular solution of the pressure abides by the initial conditions. This requires that the homogeneous solution must also adhere to the condition that $\sum_{-\infty}^{\infty} p_n e^{\frac{2\pi int}{T}}|_{t=0} = 0$. The unique total solution of the pressure has been determined in Section 4.

3.3 Solution for the vorticity equation

The differential equation for the vorticity is given by

$$-\mu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial z^2}\right) = 0. \tag{46}$$

Substituting $\omega = F(t)F(x)F(z)$ with $F(t)F(x) = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\lambda_{n,k} e^{\left(\frac{2\pi int}{T}\right)} \sin\left(\frac{\pi kx}{L}\right)$ gives a solution for the differential equation in line with the boundary conditions at $x = 0$ and $x = L$. This is found by solving the sum of the solution of the differential equation and its complex conjugate. The total solution of ω is given by either $\omega = 0$ or

$$\omega = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\left\{\lambda_{n,k} e^{\left(\frac{2\pi int}{T}\right)} \sin\left(\frac{\pi kx}{L}\right)\left[\cosh\left(\frac{\pi kz}{L}\right) + b_k \sinh\left(\frac{\pi kz}{L}\right)\right]\right\}. \tag{47}$$

This solution must conform the given definitions for the displacement.

According to the solution of Madsen (1978), the vorticity is given by

$$\omega = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\left\{\left[-A_1(2+2\theta+2m\theta)\frac{\pi k}{L}\right] e^{-\frac{\pi kz}{L}}\right\} e^{\frac{2\pi int}{T}} \sin\left(\frac{\pi kx}{L}\right). \tag{48}$$

which does conform Equation (46) and resembles Equation (47). However, the displacement relations for u_x and u_z from Madsen (1978) that give this expression for the vorticity give a non-zero volumetric strain that does not abide by Equation (24). It is unlikely that a function for u_x and u_z exists for which the curl and divergence are both non-zero and match Equations (46) and (24). For that reason ideally a solution for the displacement vector is needed for which either the curl is zero, or the divergence gives 0. The sum of these solutions will equal the total solution. However, there is no solution for u_x and u_z, for which $\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}$ is a solution of Equation (46) and for which $\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0$. For that reason it can be assumed that all displacements must be vorticity free, or $\omega = 0$ across the computational domain. This is in line with the potential flow theory often successfully applied to groundwater flows in non-deformable saturated media (Verruijt, 2012).

3.3.1 Vertical displacement relation

The vertical displacement is given by the relation

$$-\frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial z^2} = \frac{\partial \omega}{\partial x} - \frac{\partial \epsilon_{\mathrm{vol}}}{\partial z}. \tag{49}$$

The total solution of the displacement is given by the sum of the particular solution of the vertical displacement u_z^P and the homogeneous solution of the vertical displacement u_z^H. The vorticity contribution to the vertical displacement equals the homogeneous solutions of the vertical displacement, which has been derived further on in this paper. Here the volumetric strain contribution to the vertical displacement u_z^P has been worked out. Substituting the volumetric strain relation in Equation (49) gives

$$\begin{aligned} &-\frac{\partial^2 u_z^P}{\partial x^2} - \frac{\partial^2 u_z^P}{\partial z^2} = \\ &-\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\left\{c_{n,k}(a_1+b_1 i) e^{\frac{2\pi int}{T}} \cos\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z}\right]\right\} \\ &-\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\left\{c_{n,k}(a_2+b_2 i) e^{\frac{2\pi int}{T}} \cos\left(\frac{\pi kx}{L}\right)\left[c_2 e^{(a_2+b_2 i)z}\right]\right\}. \end{aligned} \tag{50}$$

Consequently the particular solution of the vertical displacement is given by

$$\begin{aligned} u_z^P = &\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\left\{c_{k,n}\left[\frac{(a_1+b_1 i)\,L^2}{\left(a_1^2+2a_1b_1 i-b_1^2\right)L^2-\pi^2k^2}\right] e^{\frac{2\pi int}{T}} \cos\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z}\right]\right\} \\ &+\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\left\{c_{k,n}\left[\frac{(a_2+b_2 i)\,L^2}{\left(a_2^2+2a_2b_2 i-b_2^2\right)L^2-\pi^2k^2}\right] e^{\frac{2\pi int}{T}} \cos\left(\frac{\pi kx}{L}\right)\left[c_2 e^{(a_2+b_2 i)z}\right]\right\}. \end{aligned} \tag{51}$$

This particular solution for the vertical displacement has been simplified by substituting the relations for a_i and b_i given by Equations (32), (33) and (34), which gives

$$u_z^P = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{k,n}\left[\frac{(a_1+b_1 i)}{(2a_1 b_1 i)}\right]e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z}\right]\right\}$$
$$+\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{k,n}\left[\frac{(a_2+b_2 i)}{(2a_2 b_2 i)}\right]e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right)\left[c_2 e^{(a_2+b_2 i)z}\right]\right\}. \tag{52}$$

The homogeneous solution for the vertical displacement is given by

$$u_z^H = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{\alpha_{n,k}e^{\frac{2\pi int}{T}}\cos\left(\frac{\pi kx}{L}\right)\left[\cosh\left(\frac{\pi kz}{L}\right)+b_k\sinh\left(\frac{\pi kz}{L}\right)\right]\right\} \tag{53}$$

The total solution of u_z is given by $u_z = u_z^P + u_z^H$. The term b_k in Equation (53) follows from the boundary condition at $z=-Z$ that $u_z|_{z=-Z}=0$. From the fact that $u_z^P|_{z=-Z}=0$ follows the condition $u_z^H|_{z=-Z}=0$.

3.3.2 Horizontal displacement relation

The constitutive relation for the horizontal displacement is given by the relation

$$-\frac{\partial^2 u_x}{\partial x^2}-\frac{\partial^2 u_x}{\partial z^2} = -\frac{\partial \omega}{\partial z}-\frac{\partial \epsilon_{\text{vol}}}{\partial x}. \tag{54}$$

Since $\omega = 0$, the contribution of ω to the total solution equals the homogeneous solution of the horizontal displacement. Substituting the relations for the volumetric strain gives the following particular solution for the horizontal displacement u_x^P

$$-\frac{\partial^2 u_x^P}{\partial x^2}-\frac{\partial^2 u_x^P}{\partial z^2} =$$
$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}\frac{\pi k}{L}e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z}\right]\right\}$$
$$+\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}\frac{\pi k}{L}e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\left[c_2 e^{(a_2+b_2 i)z}\right]\right\}. \tag{55}$$

Consequently the volumetric strain contribution to the horizontal displacement is given by

$$u_x^P = -\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}\left[\frac{\pi kL}{\left(a_1^2+2a_1 b_1 i-b_1^2\right)L^2-\pi^2k^2}\right]e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z}\right]\right\}$$
$$-\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}\left[\frac{\pi kL}{\left(a_2^2+2a_2 b_2 i-b_2^2\right)L^2-\pi^2k^2}\right]e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\left[c_2 e^{(a_2+b_2 i)z}\right]\right\}. \tag{56}$$

Substituting the relations from Equations (32), (33) and (34) now gives the horizontal displacement as

$$u_x^P = -\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}\left[\frac{\pi k}{(2a_1 b_1)L}\right]e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z}\right]\right\}$$
$$-\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left\{c_{n,k}\left[\frac{\pi k}{(2a_2 b_2)L}\right]e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\left[c_2 e^{(a_2+b_2 i)z}\right]\right\}. \tag{57}$$

The homogeneous solution for the horizontal displacement is given by

$$u_x^H = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \text{Re}\left\{ r_{n,k} e^{\frac{2\pi int}{T}} \sin\left(\frac{\pi kx}{L}\right)\left[\sinh\left(\frac{\pi kz}{L}\right) + b_k \cosh\left(\frac{\pi kz}{L}\right)\right]\right\}. \tag{58}$$

The total solution of the horizontal displacement is given by $u_x = u_x^P + u_x^H$.

3.4 Verification of the displacement relations

The total solution of the displacements must abide by the definitions for the volumetric strain and the vorticity. Substituting the definition of the volumetric strain gives

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \text{Re}\left\{ c_{n,k} e^{\frac{2\pi int}{T}} \cos\left(\frac{\pi kx}{L}\right)\left[c_1 e^{(a_1+b_1 i)z} + c_2 e^{(a_2+b_2 i)z}\right]\right\} = \epsilon_{\text{vol}}. \tag{59}$$

Substituting the expressions for the displacement in the definition for the vorticity gives

$$\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} = 0. \tag{60}$$

The divergence and curl of the homogeneous solutions of the horizontal and vertical displacements both give 0 for $r_{n,k} = -\alpha_{n,k}$ or

$$u_z^H = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \text{Re}\left\{ \alpha_{n,k} e^{\frac{2\pi int}{T}} \cos\left(\frac{\pi kx}{L}\right)\left[\cosh\left(\frac{\pi kz}{L}\right) + b_k \sinh\left(\frac{\pi kz}{L}\right)\right]\right\}, \tag{61}$$

$$u_x^H = -\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \text{Re}\left\{ \alpha_{n,k} e^{\frac{2\pi int}{T}} \sin\left(\frac{\pi kx}{L}\right)\left[\sinh\left(\frac{\pi kz}{L}\right) + b_k \cosh\left(\frac{\pi kz}{L}\right)\right]\right\}. \tag{62}$$

Using the expressions for the pressures and the displacements, and substituting these in the momentum balance equation for the pore water, an expression for the water displacement can easily be recovered. Since this expression is however not required for determining the stresses at the surface boundary, this derivation has not been worked out in this paper.

4. DEVELOPING THE UNIQUE SOLUTION

The unique solution must conform the vertical momentum balance equation at the surface given by

$$\mu\frac{\partial \omega}{\partial x} - (\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} + \frac{\partial P}{\partial z} = 0. \tag{63}$$

Substituting the expressions for the pressure and volumetric strain (see Equations (39) and (40)) in the momentum balance equation gives $(\lambda + 2\mu)\frac{\partial \epsilon_{\text{vol}}}{\partial z} = \frac{\partial P}{\partial z}$ everywhere. This shows that the pressure gradients balance the contribution of the volumetric strain to the effective stress.

For the case of compressible water, the homogeneous solution of the pressure is zero. Since there are no terms left to balance the vorticity gradient in the momentum balance equation. the vorticity gradient must be 0 to find a solution of the displacement field. Accordingly, the vorticity must be given by an arbitrary constant value, which in line with the initial conditions given $\omega = 0$.

In the case of incompressible water the homogeneous solution of the pressure is given by Equation (45). However, as still no solution for the displacement field exists for which the volumetric strain is zero and the vorticity matches a non-zero solution of Equation (10), the vorticity must be 0. In this case also $p_{n,k} = 0$.

Yamamoto et al. (1978); Ye and Jeng (2011); Moshagen and Torum (1975); Massel (1976); Davies (1996); Ye and Yeng (2011) all found that the pressure profile is given by the solution of the Laplace equation for boundary conditions given by a zero effective stress and a pressure which equals the total stress acting on the boundary. The analytical solution of this Laplace equation is given by Equation (45). This solution seems to violate the solution of the momentum balance equations of the particles and therefore does not adhere to d'Alembert principle of minimization of virtual work (see Appendix). The analytical solution of Madsen (1978) is a solution of the momentum balance equations for compressible water. However the solution of Madsen (1978) does not satisfy Equation 24.

After imposing the vertical momentum balance equation at the surface, the only remaining unknowns are the Fourier coefficients $c_{n,k}$ and $\alpha_{n,k}$. The boundary conditions at the surface are given by F_{xz} and F_{zz} (see Equations (19) and (20)). The two conditions with the two unknowns give a solvable system. When looking closely at the boundary conditions at the surface we see that the pressure is described by $P = (\lambda + 2\mu)\epsilon_{\text{vol}}$. On the other hand, the normal stress at the surface is given by

$$\sigma_{zz}|_{z=0} = -\lambda\epsilon_{\text{vol}} - 2\mu\frac{\partial u_z}{\partial z} + P = \sigma'_{zz} + P, \tag{64}$$

where σ'_{zz} denotes the effective stress. Equations (38) and (25) show that a positive value for the volumetric strain gives a positive contribution to the pressure but a negative contribution to the effective stress. Assuming that $\sigma_{zz}|_{z=0} = P|_{z=0} + \sigma'_{zz}|_{z=0} = F_{zz}$ gives a pressure at the surface which far exceeds the water pressure exerted on the surface by waves running over the porous medium. The assumption that $P|_{z=0} = F_{zz}$ gives a solution whereby the pore pressure at the surface matches the hydrodynamic load acting on the surface. In this case however the porous medium experiences a pulling force. The assumption that $\sigma'_{zz}|_{z=0} = F_{zz}$ also gives a solution whereby there is a discontinuity in water pressure over the surface. A solution whereby the effective stresses at the surface are 0 and that the pressure carries all the load is physically invalid as it does not adhere to the principle of minimization of virtual work. It is unlikely that a pressure discontinuity will be present at the surface of the porous medium. It has therefore been assumed that the boundary condition for the normal stress at $z = 0$ is given by $P|_{z=0} = F_{zz}$. With $c_1 + c_2 = 1$, the boundary conditions at $z = 0$ are now given by

$$F_{zz} = (\lambda + 2\mu)\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\text{Re}\left[c_{n,k}e^{\frac{2\pi int}{T}}e^{\frac{2\pi ikx}{L}}\right] = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\text{Re}\left[n_{nk}e^{\frac{2\pi int}{T}}e^{\frac{2\pi ikx}{L}}\right], \tag{65}$$

for the normal stress. The boundary condition for the shear stress is also needed to resolve the unknown Fourier components. Substituting $\tau = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$ gives the following condition for the shear stress at the surface

$$\begin{aligned} F_{xz} = &-\mu\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\text{Re}\left[\frac{2\pi ik}{L}\left(\frac{a_1 + b_1 i}{a_1 b_1}\right)c_{n,k}e^{\frac{2\pi int}{T}}e^{\frac{2\pi ikx}{L}}c_1\right] \\ &-\mu\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\text{Re}\left[\frac{2\pi ik}{L}\left(\frac{a_2 + b_2 i}{a_2 b_2}\right)c_{n,k}e^{\frac{2\pi int}{T}}e^{\frac{2\pi ikx}{L}}c_2\right] \\ &-2\mu\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left[\alpha_{n,k}\frac{2\pi ik}{L}e^{\frac{2\pi int}{T}}\sin\left(\frac{\pi kx}{L}\right)\right] \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\text{Re}\left[s_{nk}e^{\frac{2\pi int}{T}}e^{\frac{2\pi ikx}{L}}\right]. \end{aligned} \tag{66}$$

The unknowns in Equation (20) and (19) are the Fourier components $n_{n,k}$ and $\alpha_{n,k}$ which have been solved.

5. VALIDATION

The analytical solutions of the differential equations describing the porous medium have been solved in Matlab Version 2020a using the Fast Fourier Transform algorithm, the Discrete Cosine Transform and Discrete Sine Transform.

The number of summations were kept more than a factor of 2 higher than the number of wave lengths, or wave periods within the domain or time frame of the calculation. This way the aliasing effect whereby higher frequency harmonics would be mapped to lower frequency harmonics was prevented. For the studied cases harmonic waves with a known period were interacting with the seabed. The aliasing effect could easily be prevented. The solutions were validated against laboratory studies of Liu et al. (2015), and Li & Gao, (2022).

Liu et al. (2015) artificially induced periodic changes in water pressures on a soil column thereby simulating standing waves acting on a soil column. The wave parameters are given by a period of 9s a wave height of 3.5m at a water depth of 5.2m. The soil parameters provided by Liu et al. (2015) were determined experimentally and are a hydraulic conductivity of 1.8×10^{-4} m/sec, a shear modulus of 1.27×10^{7} N//m^2, a Poisson Ratio of 0.3, a porosity of 0.425, and a height h of the soil column of 1.8m. Due to the setup used by Liu et al. (2015), the shear stresses acting on the surface are 0. For validation of the theory against this test case the pore water was assumed incompressible. The Poisson Ratio and Shear Modulus have been substituted in Equations (8) and (9) to determine the values for 2μ and λ. These parameters have been entered into the analytical solution to obtain a blind prediction. The analytical solution has then been compared against the experimental outcomes. Figure 2 depicts the pressure distribution at

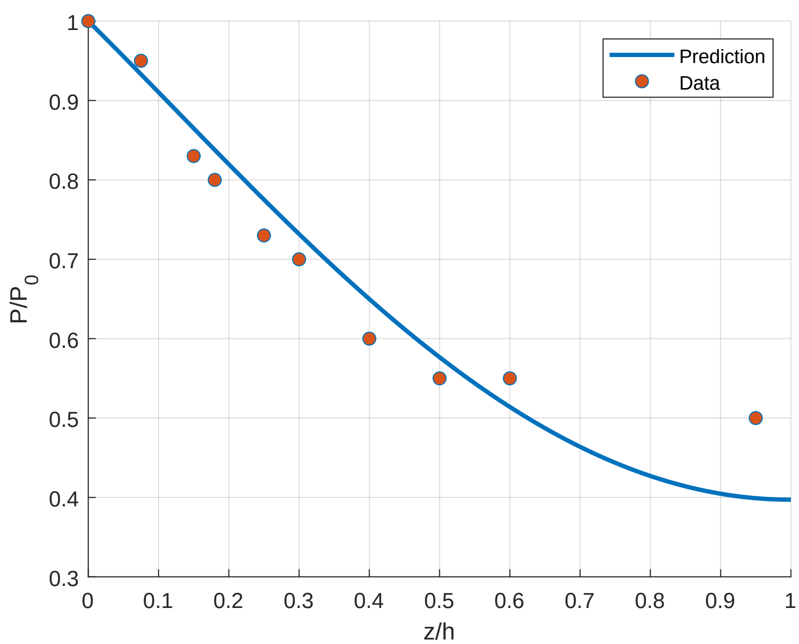

FIG. 2: The experimentally measured normalized pore water pressures against the normalized depth compared with the blind model prediction following from Equation (25).

the time of the maximum wave pressure and Figure 3 shows corresponding effective stress distribution. The figures clearly show that a hydraulic load acting on a porous medium causes a negative effective stress acting on the surface of the porous medium. This negative effective stress outs itself in terms of a pulling force. Figure 4 shows the phase difference predicted by the model for the given input parameters. Finally Figure 5 shows the influence of a change in hydraulic conductivity on the model results.

The advantage of the dataset provided by Liu et al. (2015) is that all parameters were determined experimentally. In other experiments, like e.g. (Li & Gao, 2022; Yamamoto et al., 1978) the soil parameters seem to follow from

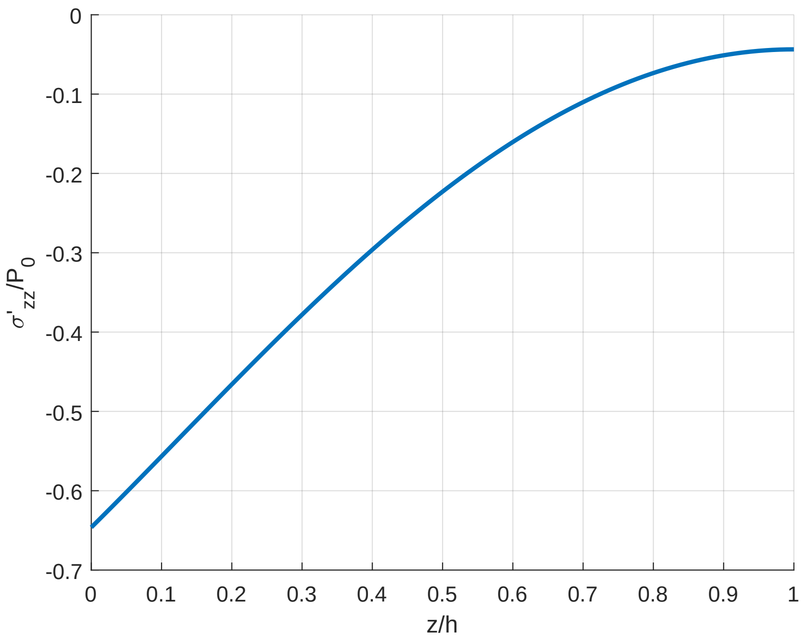

FIG. 3: Vertical normalized effective stress distribution versus the normalized depth for the pressure distribution given by Figure 2.

calibrating model results against experimental data. The predictions for many of the soil parameters are therefore a function on the model applied. Li & Gao (2022) presented values for the Shear Modulus and Poisson ration which result in values for the Young's Modulus ranging from 51 MPa for the medium sand to 62 Mpa for the fine sand bed. These high values are indicative of a dense packed sand bed (Geotechdata.info). This however does not correspond with the sand-raining technique used to prepare the bed. This raised questions to how realistic these values were. To evaluate how well the approach presented in this paper is able to reproduce the experimental results, an attempt was made to reproduce the findings of the experiment of (Li & Gao, 2022). The water was thereby assumed incompressible. The calculation domain was set at 41 m whereby the area of interest was located in the middle between 18 and 23 m. This way the impact of the boundary conditions at $x = 0$ and $x = L$ on the prediction on the response of the bed to a traveling wave was assumed to be small. For validation the Hydraulic conductivity was estimated based on the Kozeny-Carman Equation (Chapuis, R.P. &Aubertin, M.) given by

$$K_s = \frac{p^3 D_{50}^2}{180E - 6(1-p)^2}, \tag{67}$$

where D_{50} [m] denotes the mean grain size diameter. The values for the porosity and the D_{50} used in the experiment can be found in Table 1. The Shear moduli required were estimated such that these matched the situation of a loose sand bed. For the fine sand a Shear Modulus of 7.7 MPa was estimated. For the medium sand a value of 3.9MPa was estimated. This results a Young's Moduli of respectively 20 and 10MPa, which are more in line with the expected loose sand bed. The predicted change in pressure amplitude and phase shift for the fine sand are depicted in Figure 6 and those of the medium sand in Figure 7. Figure 8 depicts the predicted phase lag for the fine and course sand and how this aligns with the experimental data. As can be seen, the phase shift seems to decrease with depth, whereas

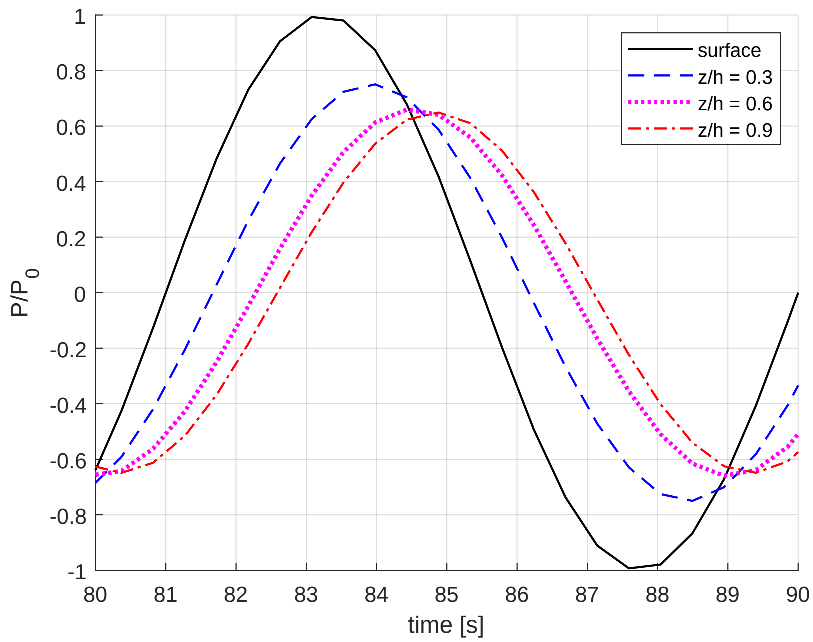

FIG. 4: Prediction of the phase difference of the normalized pore water pressures at different values of the normalized depth at 10 wave cycles.

the numerical results show a linear decrease. During the experiments also the observed change in pressure with depth was indicative of a non-constant value of the Shear Modulus. This could be attributed to the limited depth during the experiments, and the higher degree of consolidation of the deeper layers with respect to the surface layers.

No results have been shown for the case of a compressible bed. Equation (30) however shows that the same results as were obtained for an incompressible bed could be obtained for a compressible bed as long as the coefficient D from Equation (68) remains constant.

$$D = \frac{\gamma_w}{K_s}\left(p\beta + \frac{1}{(\lambda + 2\mu)}\right). \tag{68}$$

6. DISCUSSION

When modelling the effects of wave loads on porous media the assumption is often made that the effective stresses at the surface of a porous medium are 0 and that the pore water pressures carry the full load. These boundary conditions are in line with the effective stress concept of Terzaghi, who hypothesized that the total stress acting on a porous medium must equal the sum of the effective stress and the pore water pressures. In the case of statics and linear stress profiles the hypothesis of Terzaghi is in line with the momentum balance equations at the surface. The approach taken in this paper suggests that in the case of dynamic loads, it is not valid to impose these boundary conditions. Solving the 5 constitutive relations given in Section 2 requires solely 2 stress boundary conditions to find a unique solution in full accordance with the volume and momentum balance equations. Imposing the effective stress theory of Terzaghi gives a solution (Madsen, 1978) which does not conform all 5 of the constitutive relations outlined in Section 2.

Due to the strong coupling of the particular solutions of the constitutive equations for the pressure and volumetric

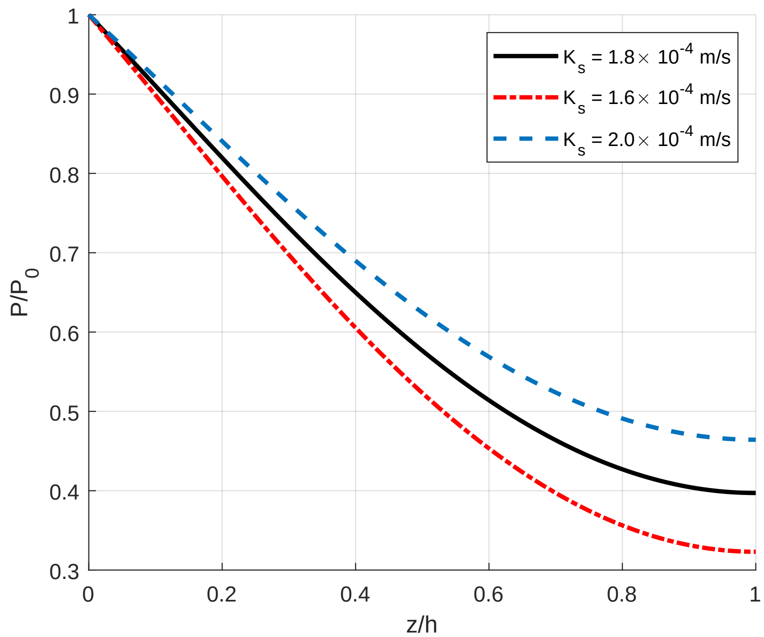

FIG. 5: Effect of changes in hydraulic conductivity on the prediction of the distribution of the normalized pore water pressures with the normalized depth.

strain given in Section 2, both terms balance each other in the momentum balance equations. No solution was found to exist for the vorticity induced displacement for which the behaviour of the vorticity could be described by the Laplace equation and for which the behaviour of the strain could be described by a heat equation. For that reason, it was deducted that the deformation of the soil is vorticity free under the assumption of a negligible influence of the acceleration terms in the momentum balance equations.

When the acceleration terms are included, Equations (12), (14) and (15) gain additional terms. As a consequence, displacements are no longer vorticity free. The homogeneous solutions of Equations (49) and (54) are however in this case zero to conform the momentum balance equations for compressible water. Including the vorticity via accounting for the influence of soil acceleration also has no effect on the prediction of the pore water pressure. The homogeneous solution of the pore water pressure for incompressible water is thereby not balanced by any other term in the momentum balance equations, indicating that the change in pressure remains only related to the change in volumetric strain. This coupling is a result of the Darcy friction terms in the momentum balance equations of both the pore water and the porous medium. Even after including the acceleration terms, only 2 boundary conditions are required to solve the dynamic behaviour of soil due to the strong coupling between the volumetric strain and the pore water pressure via the volume balance equation. The disadvantage of including the acceleration terms is that an additional initial condition is needed, which are difficult to define and complicate finding appropriate generic analytical solutions. The disadvantages therefore quickly outweigh the advantages of ignoring the acceleration terms.

When assuming that the pore water pressures at the surface must equal the external periodic hydraulic load, the analytical solutions presented in Section 3 are able to correctly predict the pressure distribution and phase shifts without having to revert to assumptions with regards to the compressibility of the pore water or particles (See Figures 2 and 8). Small changes between the measured and predicted pressure profile could be explained from spatial changes

TABLE 1: Parameters used for validation against the data of Li & Gao, (2022)

Wave parameters	Symbols [units]	Values	Values
Water depth	d [m]	0.5	0.5
Wave period	T [s]	1.2	1.2
Wave length	L [m]	2.05	2.05
Wave height	H [m]	0.13	0.095
Soil properties		Fine sand	Medium sand
Mean size of the grains	$D_{50}[mm]$	0.12	0.38
Effective size of the grains	D_{10} [mm]	0.03	0.23
Porosity	p	0.4	0.44
Poissons ratio	ν	0.3	0.27

in hydraulic conductivity, or shear modulus. As the porous medium expands the hydraulic conductivity increases, and the shear modulus tends to decrease. This interaction is not accounted for in the analytical solutions. Other aspects that could influence differences observed between the experimental data and the model results are the influence of the reflection of the pressure waves at the flume bed, or the impact of acceleration terms. However, considering the certainty to which soil parameters can be measured, the impact of the influence of for example assuming a constant value for the hydraulic conductivity is expected to be limited. This is also indicated in Figure 5. A 10% increase in hydraulic conductivity thereby has a similar effect as a 10% increase in Elasticity Modulus. Assuming incompressible water even results in more realistic predictions for the soil parameters after calibrating model results against experimental outcomes. At the surface of the porous medium the momentum balance equation remains satisfied. This results in negative effective stresses at the surface when the dynamic pore water pressures are positive (see Figure 3).

The approach outlined in this paper has some significant benefits over the current state-of-the-art approach. The first benefit is that one less boundary condition is required to solve the problem. Only a shear stress and either a normal stress or pressure are known. In the proposed approach therefore fewer assumptions are required to predict the behaviour of the soil. The second benefit is that models based on the assumption of compressible water suffer from the fact that the water content is often not known. The models therefore require extensive calibration. The method proposed here gives accurate predictions for the change in pore pressure under the assumption of incompressible water which is also commonplace in other fields of engineering like fluid mechanics. The predictive capacity of the models therefore increases after adopting the suggested solution method. The third benefit is that scalar equations like the equation for the volumetric strain and pressure are often easier to solve numerically than vector equations allowing for faster models. The fourth benefit is that the newly proposed method may offer insights into the physics behind erosion under high hydraulic loads as the effective stresses at the surface, induced by hydraulic loads become a model output. The hydraulic loads do need to be high enough to allow for the use of a continuum model. Further research is also required to facilitate this whereby the inclusion of acceleration terms also becomes important.

The analytical solutions which were derived in accordance with the momentum balance equations explain why grass covers have been witnessed to be pulled from the soil during wave overtopping events. It could also help explain the liquefaction of the soil which is often observed during wave soil interaction (Ye and Yeng, 2011). The analytical solutions given in this paper are valid for cases whereby soil is periodically loaded, the boundary conditions are smooth, and the final loading situation matches the initial loading situation and whereby initially no dynamic load is present. The degree of loading should be small enough to prevent significant compressibility of the soil particles. The soil should also be homogeneous, the domain should be small whereby also displacements at the boundary remain small and elastic.

For a widespread more practical use of the presented method, it is recommended to derive the set of five constitutive equations based on taking the divergence and curl of the momentum balance equations for non-homogeneous soils and to develop numerical approaches to solve these. Based on the strong coupling of the pore water pressure and volumetric strain, it may be possible to express one in terms of the other, simplifying the equations that need to be solved. Following the procedure outlined here only two stress boundary conditions will be required at the boundary.

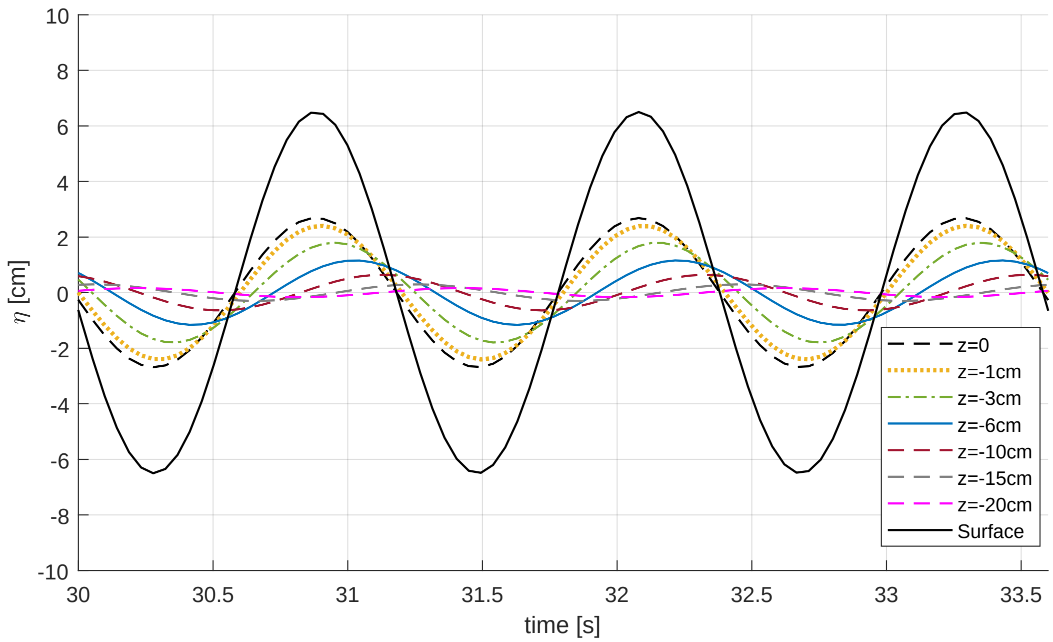

FIG. 6: Time series of the surface elevation and the delayed response of the hydraulic heads in a bed of fine sand.

7. CONCLUSIONS

The following conclusions have been drawn based on this study.

1. A set of constitutive relations has been derived for the displacements, vorticity, and volumetric strain which have been used to derive analytical solutions for the pressure, displacements and volumetric strain which are in full accordance with d'Alemberts principle of minimum energy.

2. By taking the divergence and curl of the momentum balance equations and by imposing the momentum balance equations acting normal to the boundary, on the boundary it becomes possible to reduce the number of required surface boundary conditions to solely one condition for the shear stress and one condition for the normal stress, or pressure.

3. A direct relation has been found between the pore water pressures and the volumetric strain.

4. Validation of the analytical solutions shows that the change in pressure profile could be predicted in accordance with measurements, indicating that the impact of the assumptions made is small.

5. The results indicate that imposing boundary conditions in line with Terzaghi's principle of effective stress leads to an incorrect description of the dynamic behaviour of porous media.

APPENDIX A.

The momentum balance equations have been derived by applying D'Alemberts principle of virtual work, which states that the total virtual work of the imposed forces plus the inertial forces vanish for reversible displacements. The

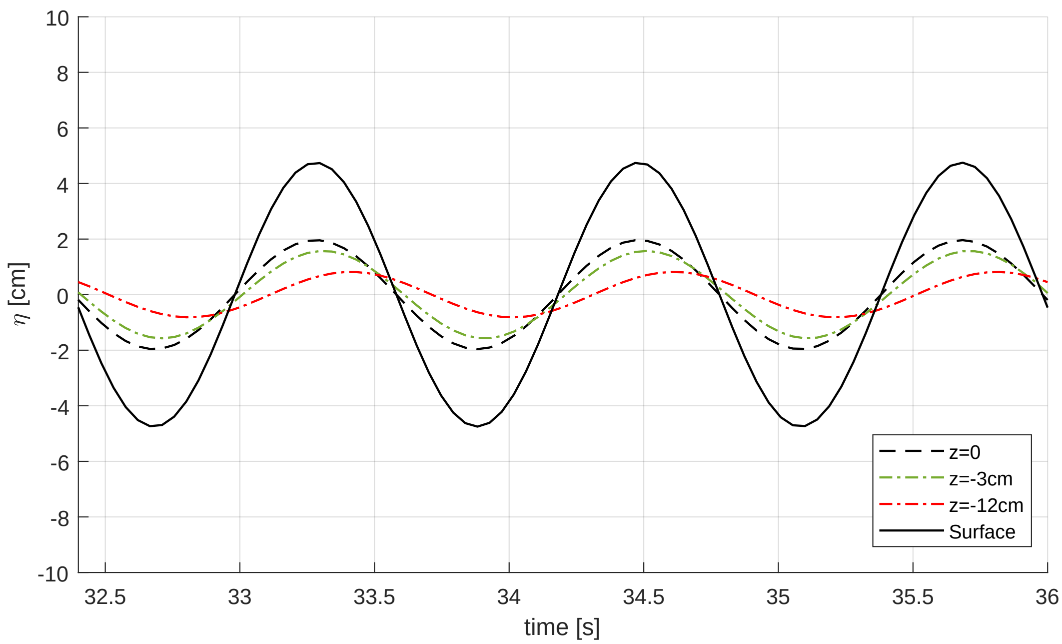

FIG. 7: Time series of the surface elevation and the delayed response of the hydraulic heads in a bed of medium sand.

system considered here is assumed to consist of incompressible water and incompressible particles. Although both are assumed incompressible, displacements of particles and flow of pore water between the porous medium consisting of particles, could induce a volume change of the porous medium, which is in line with Equation (3). It should however be noted that during this process no energy is lost, but only transferred between the particles and the pore water.

Body forces act on both the porous medium and pore water. Below these have been treated separately. The virtual work $\delta\hat{W}_g$ performed by body forces acting on both the porous medium and the pore water is given by

$$\delta\hat{W}_g = \int\limits_{\Omega_p} \rho_p g u_z^* d\Omega_p + \int\limits_{\Omega_w} \rho_w g v_z^* d\Omega_w. \tag{A.1}$$

Here Ω_p refers to the volume encompassed by the particles in the porous medium and Ω_w refers to the volume encompassed by the water. The asterix in the superscript of the displacement vectors indicates that it concerns virtual displacements. The sum of the virtual work performed by internal and external forcing, denoted by δW_σ is given by

$$\delta W_\sigma = \oint\limits_{S_p} \mathbf{u}_i^* \sigma_{ij}^p n_j dS_p - \int\limits_{\Omega_p} \epsilon_{ij}^{p*} \sigma_{ij}^p d\Omega_p + \oint\limits_{S_w} \mathbf{v}_i^* \sigma_{ij}^w n_j dS_w - \int\limits_{\Omega_w} \epsilon_{ij}^{w*} \sigma_{ij}^w d\Omega_w, \tag{A.2}$$

where σ_{ij}^p refers to the stress tensor for the porous medium and σ_{ij}^w refers to the stress tensor for the pore water. Here n_j denotes the normal vector.

The superscripts p and w respectively refer to the soil skeleton consisting of particles and the pore water. The surface integral domain S_p refers to that part of the surface consisting of particles. The surface integral domain S_w

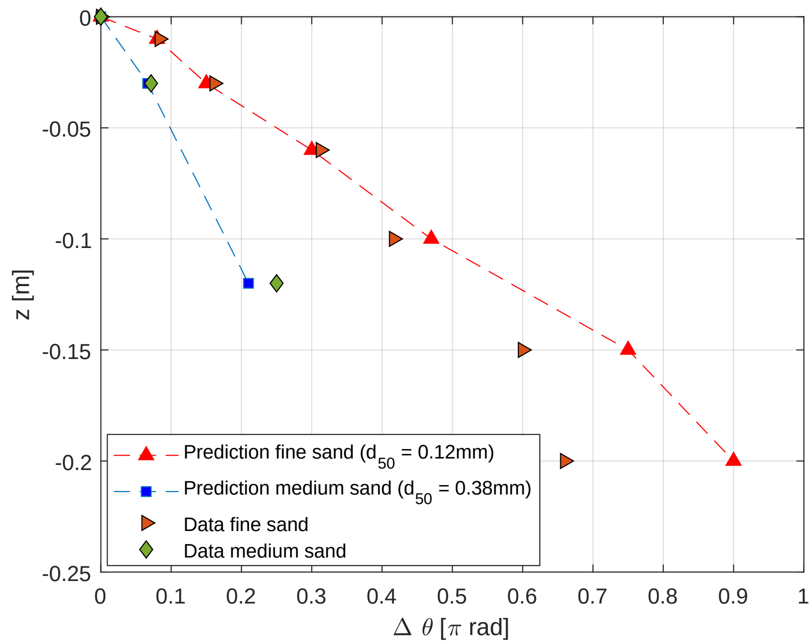

FIG. 8: Comparison of the predicted and measured phase lag for fine and course sand.

refers to that fraction of the surface consisting of water. The strain tensor denoted by ϵ is given by

$$\epsilon_{ij}^{*} = \frac{1}{2}\left[\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i}\right], \tag{A.3}$$

where ϵ_{ii} denotes the isotropic strain and ϵ_{ij} for $i \neq j$ denotes the deviatoric strain. Substituting Equation (A.3) into

Equation (A.2) and applying integration by parts to Equation (A.2) gives

$$\begin{aligned}
&\oint_{S_p} \mathbf{u}_i \sigma_{ij}^{p*} n_j dS_p - \int_{\Omega_p} \epsilon_{ij}^{p} \sigma_{ij}^{p*} d\Omega_p + \oint_{S_w} \mathbf{v}_i \sigma_{ij}^{w*} n_j dS_w - \int_{\Omega_w} \epsilon_{ij}^{w} \sigma_{ij}^{w*} d\Omega_w = \\
&\oint_{Sp} \left(\sigma_{xx}^{p} + \sigma_{xy}^{p} + \sigma_{xz}^{p}\right) u_x^* dS_p + \int_{\Omega_p} u_x^* \left(\frac{\partial \sigma_{xx}^{p}}{\partial x} + \frac{\partial \sigma_{xy}^{p}}{\partial y} + \frac{\partial \sigma_{xz}^{p}}{\partial z}\right) d\Omega_p \\
&+ \oint_{Sp} \left(\sigma_{yy}^{p} + \sigma_{xy}^{p} + \sigma_{zy}^{p}\right) u_y^* dS_p + \int_{\Omega_p} u_y^* \left(\frac{\partial \sigma_{yy}^{p}}{\partial y} + \frac{\partial \sigma_{yx}^{p}}{\partial x} + \frac{\partial \sigma_{zy}^{p}}{\partial z}\right) d\Omega_p \\
&+ \oint_{Sp} \left(\sigma_{zz}^{p} + \sigma_{xz}^{p} + \sigma_{yz}^{p}\right) u_z^* dS_p + \int_{\Omega_p} u_z^* \left(\frac{\partial \sigma_{zz}^{p}}{\partial z} + \frac{\partial \sigma_{xz}^{p}}{\partial x} + \frac{\partial \sigma_{yz}^{p}}{\partial y}\right) d\Omega_p \\
&+ \oint_{Sw} \left(\sigma_{xx}^{w} + \sigma_{xy}^{w} + \sigma_{xz}^{w}\right) v_x^* dS_w - \int_{\Omega_w} v_x^* \left(\frac{\partial \sigma_{xx}^{w}}{\partial x} + \frac{\partial \sigma_{xy}^{w}}{\partial y} + \frac{\partial \sigma_{xz}^{w}}{\partial z}\right) d\Omega_w \\
&+ \oint_{Sw} \left(\sigma_{yy}^{w} + \sigma_{xy}^{w} + \sigma_{zy}^{w}\right) v_y^* dS_w - \int_{\Omega_w} v_y^* \left(\frac{\partial \sigma_{yy}^{w}}{\partial y} + \frac{\partial \sigma_{yx}^{w}}{\partial x} + \frac{\partial \sigma_{zy}^{w}}{\partial z}\right) d\Omega_w \\
&+ \oint_{Sw} \left(\sigma_{zz}^{w} + \sigma_{xz}^{w} + \sigma_{yz}^{w}\right) v_z^* dS_w - \int_{\Omega_w} v_z^* \left(\frac{\partial \sigma_{zz}^{w}}{\partial z} + \frac{\partial \sigma_{xz}^{w}}{\partial x} + \frac{\partial \sigma_{yz}^{w}}{\partial y}\right) d\Omega_w = 0.
\end{aligned} \tag{A.4}$$

D'Alemberts principle now states that the total virtual work of the impressed forces plus the inertial forces equals 0. Adding the inertial forces to the virtual work expressed by the stresses and gravity leaves

$$\begin{aligned}
&\int_{\Omega_p} \left\{ u_i^* \left[\rho_p g_i + \frac{\partial \sigma_{ij}^{p}}{\partial x_j} - \frac{\partial^2 \rho_p u_i}{\partial t^2} - \frac{\partial}{\partial x_i} \left(\rho_p \frac{1}{2} \left(\frac{\partial u_i}{\partial t} \right)^2 \right) \right] \right\} d\Omega_p \\
&+ \int_{\Omega_w} \left\{ v_i^* \left[\rho_w g_i + \frac{\partial \sigma_{ij}^{w}}{\partial x_j} - \frac{\partial^2 \rho_w v_i}{\partial t^2} - \frac{\partial}{\partial x_i} \left(\rho_w \frac{1}{2} \left(\frac{\partial v_i}{\partial t} \right)^2 \right) \right] \right\} d\Omega_w = 0,
\end{aligned} \tag{A.5}$$

where $g_i = [0\ 0\ g]$. At the bottom boundary the virtual displacement of the water and the particles are prescribed as 0 because the virtual displacement must match the imposed displacement. For any arbitrary virtual work u_i^* and v_i^*, Equation (A.5) must result in 0. Consequently, this gives the following conditions

$$\begin{aligned}
&\rho_p g_i + \frac{\partial \sigma_{ij}^{p}}{\partial x_j} + \frac{\partial^2 \rho_p u_i}{\partial t^2} + \frac{\partial}{\partial x_j} \left[\rho_p \frac{1}{2} \left(\frac{\partial u_i}{\partial t} \right)^2 \right] = 0, \\
&\rho_w g_i + \frac{\partial \sigma_{ij}^{w}}{\partial x_j} + \frac{\partial^2 \rho_w v_i}{\partial t^2} + \frac{\partial}{\partial x_j} \left[\rho_w \frac{1}{2} \left(\frac{\partial v_i}{\partial t} \right)^2 \right] = 0.
\end{aligned} \tag{A.6}$$

The stresses are given by the effective stresses of the porous medium and the pore water stresses. The effective stresses of the porous medium and the water stresses have been defined as positive for compression. The relations of the stress gradients are now given by

$$\frac{\partial \sigma^p_{ii}}{\partial x_i} = -\frac{\partial}{\partial x_i}\left(\beta\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + \alpha \frac{\partial u_i}{\partial x_i}\right), \tag{A.7}$$

$$\frac{\partial \sigma^w_{ii}}{\partial x_i} = \mu \frac{\partial}{\partial x_i}\left(\frac{2}{3}\frac{\partial^2 v_j}{\partial x_j \partial t} - 2\frac{\partial^2 v_i}{\partial x_i \partial t}\right) + \frac{\partial P}{\partial x_i}, \tag{A.8}$$

$$\frac{\partial \sigma^p_{ij}|_{i\neq j}}{\partial x_j} = -\frac{a}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{A.9}$$

$$\frac{\partial \sigma^w_{ij}|_{i\neq j}}{\partial x_j} = -\mu \frac{\partial}{\partial x_j}\left(\frac{\partial^2 v_i}{\partial x_j \partial t} + \frac{\partial^2 v_j}{\partial x_i \partial t}\right). \tag{A.10}$$

The parameters β and α are related to the elasticity modules E and Poisson ratio ν as (Verruijt, 2001)

$$\beta = \frac{\nu E}{(1+\nu)(1-2\nu)}, \tag{8}$$

$$\alpha = \frac{E}{(1+\nu)}. \tag{9}$$

In Equation (A.8), P denotes the water pressure which is given by $P = \frac{1}{3}\left(\sigma^w_{xx} + \sigma^w_{yy} + \sigma^w_{zz}\right)$. These stresses are given on a continuum scale and hence relate to the total volume. A flow in a saturated porous medium can be represented as the collection of flows through individual tubes of different sizes. Under saturated conditions the pressure gradient over all tubes is deterministic and no water is exchanged between the tubes and the flow inside a tube is unidirectional. In the case of a laminar flow through a tube, the velocity profile of a flow in x-direction is parabolic in z-direction. The shape of the parabolic shape depends on the difference between the velocity of the particles and the velocity of the fluid. Consequently, substituting the parabolic flow profile in the expression of the shear stress of water gives

$$-\mu \frac{\partial}{\partial z}\left(\frac{\partial^2 v_x}{\partial z \partial t}\right) = \frac{\gamma_w}{K_s}\frac{\partial v_z - u_z}{\partial t}, \tag{A.11}$$

where K_s [m/s] denotes the expected value of the resistance experience by the flows in the distribution of pores of different sizes, which is represented by the hydraulic conductivity, and γ_w [N/m^3] is the specific weight. The particles in the porous medium experience shear stresses due to the motion of other particles and shear stresses due to the relative flow of the pore water. Hence, based on the principle of action equals minus reaction, also the Darcy friction term must be included in the momentum balance equation of the particles. The Darcy friction term thereby represents the exchange of energy between the porous medium and the pore water. The momentum balance equations that must be solved are for the porous medium for a 2D case

$$\begin{aligned}&\frac{\partial^2 \rho_p(1-p)u_x}{\partial t^2} + \rho_p(1-p)\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 u_x}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 u_x}{\partial t \partial z}\right)\right] + \rho_p(1-p)g_x \\ &-\frac{a}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0,\end{aligned} \tag{A.12}$$

$$\begin{aligned}&\frac{\partial^2 \rho_p(1-p)u_z}{\partial t^2} + \rho_p(1-p)\left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 u_z}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 u_z}{\partial t \partial z}\right)\right] + \rho_p(1-p)g_z \\ &+\frac{a}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - (\beta+\alpha)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) - \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0.\end{aligned} \tag{A.13}$$

In the case of water the friction dominates the normal stresses by a large amount. The friction in the pores is accounted for via a Darcy term. This leads to the following momentum balance equations for the pore water

$$\frac{\partial^2 \rho_w p v_x}{\partial t^2} + \rho_w p \left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 v_x}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 v_x}{\partial t \partial z}\right)\right] + \rho_w p g_x + \frac{\partial P}{\partial x} + \frac{\gamma_w}{K_s}\frac{\partial p(v_x - u_x)}{\partial t} = 0, \tag{A.14}$$

$$\frac{\partial^2 \rho_w p v_z}{\partial t^2} + \rho_w p \left[\frac{\partial x}{\partial t}\left(\frac{\partial^2 v_z}{\partial t \partial x}\right) + \frac{\partial z}{\partial t}\left(\frac{\partial^2 v_z}{\partial t \partial z}\right)\right] + \rho_w p g_z + \frac{\partial P}{\partial z} + \frac{\gamma_w}{K_s}\frac{\partial p(v_z - u_z)}{\partial t} = 0. \tag{A.15}$$

These momentum balance equations have been solved in this paper for a friction dominated case for which the acceleration terms have a negligible impact on the solution. The paper thereby focuses on determining the dynamic stresses. For that reason the gravitational terms have also been ignored.

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LIST OF FIGURES