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A REMARKABLE INTEGER SEQUENCE RELATED TO π AND $\sqrt{2}$

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Abstract

We prove that five ways to define entry **A086377** in the OEIS do lead to the same integer sequence.

– Dedicated to Jeff Shallit on the occasion of his 60th birthday

1. Introduction

In September of 2003 Benoit Cloitre contributed a sequence to the On-Line Encyclopedia of Integer Sequences [4], defined by him as $a_1 = 1$, and for $n \geq 2$ by

$$a_n = \begin{cases} a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\ a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\ a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence.} \end{cases} \quad (1)$$

The first 25 values of this sequence are

1, 4, 6, 8, 11, 13, 16, 18, 21, 23, 25, 28, 30, 33, 35, 37, 40, 42, 45, 47, 49, 52, 54, 57, 59.

The purpose of this paper is to prove the equivalence of five ways to define this integer sequence, most of them already conjecturally stated in the OEIS article on **A086377**. Besides a simplified recursion, the alternatives involve statements in terms of a morphic sequence, of a Beatty sequence, and of approximation properties linking a classical continued fraction of $\frac{4}{\pi}$ to that of $\sqrt{2}$.

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2. The Theorem

Theorem 1. *The following five definitions produce the same integer sequence:*

(a_n) defined by $a_1 = 1$ and for $n \geq 2$:

$$a_n = \begin{cases} a_{n-1} + 2 & \text{if } n \text{ is in the sequence,} \\ a_{n-1} + 2 & \text{if } n \text{ and } n-1 \text{ are not in the sequence,} \\ a_{n-1} + 3 & \text{if } n \text{ is not in the sequence, but } n-1 \text{ is in the sequence;} \end{cases}$$

(b_n) defined by $b_1 = 1$ and for $n \geq 2$:

$$b_n = \begin{cases} b_{n-1} + 2 & \text{if } n-1 \text{ is not in the sequence,} \\ b_{n-1} + 3 & \text{if } n-1 \text{ is in the sequence;} \end{cases}$$

(c_n) for $n \geq 1$ defined as the position of the n -th zero in the fixed point of the morphism

$$\phi : \begin{cases} 0 \mapsto 011 \\ 1 \mapsto 01 \end{cases} ;$$

(d_n) defined by $d_n = \lfloor (1 + \sqrt{2}) \cdot n - \frac{1}{2}\sqrt{2} \rfloor$ for $n \geq 1$;

(e_n) defined by $e_n = \lceil r_n \rceil = \lfloor r_n + \frac{1}{2} \rfloor$, with $r_1 = \frac{4}{\pi}$ and $r_{n+1} = \frac{n^2}{r_n - (2n - 1)}$, for $n \geq 1$.

At first we found it hard to believe the equivalence of these definitions, but a verification of the first 130000 terms ($a_{130000} = 313847$) convinced us to look for proofs.

3. Simplification and a Morphic Sequence

To show that (b_n) defines the same sequence as (a_n), simply note that $a_n - a_{n-1} \geq 2$ for all n : hence if n is in the sequence then $n - 1$ is not, and we can combine the first two cases in Equation (1).

In a comment to sequence **A086377**, Clark Kimberling asked if the integers in this sequence coincide with the positions of the zeroes in sequence **A189687**, which is the fixed point of the substitution

$$\phi : \begin{cases} 0 \mapsto 011 \\ 1 \mapsto 01 \end{cases} ,$$

defining the sequence (c_n) in the Theorem. It is not hard to see that this indeed produces the same as sequence (b_n) ; repeatedly applying the morphism ϕ to 0 produces after a few steps the initial segment

0110101011010110101101010110101011010110101011010110101101011...

The position c_n of the n -th zero is 2 ahead of c_{n-1} precisely when the latter is followed by a single 1, that is, when there is a 1 at position $n - 1$, and it is 3 ahead of c_{n-1} if that zero is followed by 11, which means that there was a 0 at position $n - 1$. Thus the rule is exactly that defining (b_n) .

4. Beatty Sequence

Every pair of real numbers α and β determines a Beatty sequence by

$$B(\alpha, \beta)_n := \lfloor n\alpha + \beta \rfloor, \quad n = 1, 2, \dots$$

The numbers α and β also determine sequences by

$$St(\alpha, \beta)_n := \lfloor (n + 1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor, \quad n = 1, 2, \dots,$$

which is a Sturmian sequence (of slope α), over the alphabet $\{0, 1\}$, provided that $0 \leq \alpha < 1$.

Thus Sturmian sequences are first differences of Beatty sequences (when $0 \leq \alpha < 1$), but Beatty sequences and Sturmian sequences are also linked in another way.

Lemma 1. *Let $\alpha > 1$ be irrational, and let $(s_n)_{n \geq 1}$ be given by $s_n = St(\frac{1}{\alpha}, -\frac{\beta}{\alpha})_n$, for some real number β with $\alpha + \beta > 1$ and such that $k\alpha + \beta \notin \mathbb{Z}$ for all positive integers k . Then $B(\alpha, \beta)$ is the sequence of positions of 1 in (s_n) .*

Proof. This is a generalization of Lemma 9.1.3 in [1], from homogeneous to inhomogeneous Sturmian sequences. The proof also generalizes:

there exists $k \geq 1 : n = \lfloor k\alpha + \beta \rfloor$ if and only if

there exists $k \geq 1 : n \leq k\alpha + \beta < n + 1$ if and only if

there exists $k \geq 1 : \frac{n - \beta}{\alpha} \leq k < \frac{n + 1 - \beta}{\alpha}$ if and only if

there exists $k \geq 1 : \lfloor \frac{n - \beta}{\alpha} \rfloor = k - 1$ and $\lfloor \frac{n - \beta}{\alpha} + \frac{1}{\alpha} \rfloor = k$ if and only if

$\lfloor \frac{n + 1}{\alpha} - \frac{\beta}{\alpha} \rfloor - \lfloor \frac{n}{\alpha} - \frac{\beta}{\alpha} \rfloor = 1$ if and only if

$St(\frac{1}{\alpha}, -\frac{\beta}{\alpha})_n = 1. \quad \square$

Our goal in this section is to prove that $(c_n) = (d_n)$. Let ψ be the morphism $\psi : \begin{cases} 0 \rightarrow 10 \\ 1 \rightarrow 100 \end{cases}$, and let w be the fixed point. Then

$$w = 10010101001010010100101001010010100101001010010100101001010010100 \dots,$$

which is obtained by exchanging 0s and 1s in the fixed point of ϕ , i.e., $\psi = E\phi E$, with E the exchange morphism given by $E(0) = 1, E(1) = 0$. So the positions of 0 in the fixed point of ϕ correspond to the positions of 1 in the fixed point w of ψ .

Let $\alpha_d = 1 + \sqrt{2}$ and $\beta_d = -\frac{1}{2}\sqrt{2}$; then $d_n = B(\alpha_d, \beta_d)_n$, for $n \geq 1$.

Applying Lemma 1, we deduce that d_n also equals the position of the n -th 1 in the Sturmian sequence $\text{St}(\alpha, \beta)$, generated by

$$\alpha = \frac{1}{\alpha_d} = \sqrt{2} - 1, \beta = \frac{-\beta_d}{\alpha_d} = 1 - \frac{1}{2}\sqrt{2}.$$

Lemma 2. $\text{St}(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2}) = w$.

Proof. This was already proved by Nico de Bruijn in 1981 ([2]), where it is the main example. See also Lemma 2 in [6]. Note, however, that our Sturmian sequences start at $n = 1$.

For a ‘modern’ proof as suggested by [3, Section 4], let ψ_1 and ψ_2 be the elementary morphisms given by $\psi_1(0) = 01, \psi_1(1) = 0$, and $\psi_2(0) = 10, \psi_2(1) = 0$. Then $\psi = \psi_2\psi_1E$. This implies that the fixed point w of ψ is a Sturmian word (see [5, Corollary 2.2.19]). To find its parameters (α, β) , use the 2D fractional linear maps that describe how the parameters of a Sturmian word change when one applies an elementary morphism. For Sturmian words starting at $n = 0$, the maps for E, ψ_1 and ψ_2 are² respectively (see [5, Lemma 2.2.17, Lemma 2.2.18, Exercise 2.2.6])

$$T_0(x, y) = (1-x, 1-y), T_1(x, y) = \left(\frac{1-x}{2-x}, \frac{1-y}{2-x}\right), T_2(x, y) = \left(\frac{1-x}{2-x}, \frac{2-x-y}{2-x}\right).$$

The change of parameters by applying ψ is therefore the composition

$$T_{210}(x, y) := T_2T_1T_0(x, y) = \left(\frac{1}{2+x}, \frac{2+x-y}{2+x}\right).$$

But the parameters α and β of w do *not* change when one applies ψ . This means that (α, β) is a fixed point of T_{210} , and one easily computes $\alpha = \sqrt{2} - 1$, and then $\beta = \frac{1}{2}\sqrt{2}$. Since our Sturmian words start at $n = 1$, we have to subtract α from β and obtain that $w = \text{St}(\sqrt{2} - 1, 1 - \frac{1}{2}\sqrt{2})$. □

²Actually there is a subtlety here involving the ceiling representation of a Sturmian sequence, but that does not apply in our case since $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$.

5. Converging Recurrence

In a comment to entry **A086377**, Joseph Biberstine conjectured a beautiful connection with the infinite continued fraction expansion

$$\frac{4}{\pi} = 1 + \frac{1^2}{3 + \frac{2^2}{5 + \frac{3^2}{7 + \frac{4^2}{9 + \frac{5^2}{11 + \dots}}}}}$$

derived from the arctangent function expansion. If we define R_n for $n \geq 1$ by

$$R_n = 2n - 1 + \frac{n^2}{2n + 1 + \frac{(n + 1)^2}{2n + 3 + \frac{(n + 2)^2}{2n + 5 + \dots}}}$$

then $R_1 = 4/\pi$ and $R_n = 2n - 1 + \frac{n^2}{R_{n+1}}$. We see that

$$\frac{R_n}{n} \frac{R_{n+1}}{n + 1} - \frac{2n - 1}{n} \frac{R_{n+1}}{n + 1} - \frac{n^2}{n(n + 1)} = 0.$$

This implies that if R_n/n converges, for $n \rightarrow \infty$, then it does so to a (positive) zero of $x^2 - 2x - 1$, that is, to $1 + \sqrt{2}$; cf. Lemma 3 below.

We consider now, conversely and slightly more generally, for any real $h \geq 1$, a sequence of positive numbers r_n satisfying

$$r_n = hn - 1 + \frac{n^2}{r_{n+1}} \tag{2}$$

for $n \geq 1$. We first show that this sequence is unique, i.e., there is a unique $r_1 > 0$ such that $r_n > 0$ for all $n \geq 1$, and give estimates for its terms.

Lemma 3. *For each $h \geq 1$, there is a unique sequence of positive real numbers $(r_n)_{n \geq 1}$ satisfying the recurrence (2). Moreover, we have for this sequence, for all $n \geq 1$,*

$$0 < r_n - \alpha n + c < \frac{(\alpha - c)(c - 1)}{\alpha n} \tag{3}$$

with $\alpha = \frac{h + \sqrt{h^2 + 4}}{2}$ and $c = \frac{1 + \alpha}{2\alpha - h} = \frac{1}{2} + \frac{h + 2}{2\sqrt{h^2 + 4}}$.

Proof. Let $f_n(x) = hn - 1 + n^2/x$. Suppose that a sequence of positive numbers r_n satisfies (2), i.e., that $f_n(r_{n+1}) = r_n$ for all $n \geq 1$. Then we have $r_n > hn - 1$ and thus $r_n < (h + 1/h)n$ for all $n \geq 1$. We deduce that there exists some $\delta > 0$ and $N \geq 1$ such that $r_n > (h + \delta)n$ for all $n \geq N$. Suppose that there is another sequence of positive numbers \tilde{r}_n satisfying (2). Since $|f'_n(x)| = |n/x|^2 < 1/(h + \delta)$ for all $x > (h + \delta)n$, we have

$$\begin{aligned} |r_N - \tilde{r}_N| &= |f_N f_{N+1} \cdots f_{n-1}(r_n) - f_N f_{N+1} \cdots f_{n-1}(\tilde{r}_n)| \\ &< \frac{|r_n - \tilde{r}_n|}{(h + \delta)^{n-N}} < \frac{n/h}{(h + \delta)^{n-N}} \end{aligned}$$

for all $n \geq N$, hence $r_N = \tilde{r}_N$, which implies that $r_n = \tilde{r}_n$ for all $n \geq 1$.

Next we show that

$$f_n(\alpha(n + 1) - c) < \alpha n - c + \frac{(\alpha - c)(c - 1)}{\alpha n}$$

and

$$f_n\left(\alpha(n + 1) - c + \frac{(\alpha - c)(c - 1)}{(n + 1)\alpha}\right) > \alpha n - c.$$

Indeed, using that $\alpha^2 = h\alpha + 1$ and $2\alpha c - hc = 1 + \alpha$, we have

$$\begin{aligned} (\alpha n + \alpha - c) f_n(\alpha(n + 1) - c) &= (hn - 1)(\alpha n + \alpha - c) + n^2 \\ &= (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c) \\ &< \alpha^2 n^2 + (\alpha^2 - 2\alpha c)n - (\alpha - c) + \frac{(\alpha - c)^2(c - 1)}{\alpha n} \\ &= (\alpha n + \alpha - c)\left(\alpha n - c + \frac{(\alpha - c)(c - 1)}{\alpha n}\right), \end{aligned}$$

and

$$\begin{aligned} &\left(\alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)}\right)(\alpha n - c) \\ &< \alpha^2 n^2 + (\alpha^2 - 2\alpha c)n - (\alpha - c) - \frac{c(\alpha - c)(c - 1)}{\alpha(n + 1)} \\ &< (h\alpha + 1)n^2 + (h\alpha - hc - \alpha)n - (\alpha - c) - \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)} \\ &< (hn - 1)\left(\alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)}\right) + n^2 \\ &= \left(\alpha n + \alpha - c + \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)}\right) f_n\left(\alpha(n + 1) - c + \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)}\right). \end{aligned}$$

As f_n is monotonically decreasing for $x > 0$, we deduce that

$$0 < f_n(x) - \alpha n + c < \frac{(\alpha - c)(c - 1)}{\alpha n}$$

for all x with $0 \leq x - \alpha(n + 1) + c \leq \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)}$. Then we also have

$$0 < f_n f_{n+1} \cdots f_{n+k-1}(\alpha(n + k) - c + x) - \alpha n + c < \frac{(\alpha - c)(c - 1)}{\alpha n}$$

for all $k, n \geq 1$, $0 \leq x - \alpha(n + k) + c \leq \frac{(\alpha - c)(c - 1)}{\alpha(n + k)}$. As f_n is contracting for $x \geq \alpha(n + 1) - c$, the intervals $[f_1 f_2 \cdots f_n(\alpha(n + 1) - c), f_1 f_2 \cdots f_n(\alpha(n + 1) - c + \frac{(\alpha - c)(c - 1)}{\alpha(n + 1)})]$ converge to a point r_1 . Then the numbers r_n given by (2) satisfy (3) for all $n \geq 1$. By the first paragraph of the proof, this is the unique sequence of positive numbers satisfying (2). \square

Now consider when $\alpha n - c + \frac{1}{2}$ is close to $\lceil \alpha n - c + \frac{1}{2} \rceil$. Let p_k/q_k be the convergents of the regular continued fraction $\alpha = [h; h, h, \dots]$, i.e., $q_{-1} = 0, q_0 = 1, q_{k+1} = hq_k + q_{k-1}$ for $k \geq 1, p_k = q_{k+1}$. Then we have

$$q_k = \frac{\alpha^{k+1} + (-1)^k / \alpha^{k+1}}{\alpha + 1/\alpha}$$

and thus

$$q_k \alpha - p_k = \frac{(-1)^k}{\alpha^{k+1}}. \tag{4}$$

Lemma 4. Let h be a positive integer and $\alpha = \frac{h + \sqrt{h^2 + 4}}{2}$. Then we have

$$\lceil \alpha n \rceil - \alpha n = \begin{cases} j/\alpha^{2k} & \text{if } n = jq_{2k-1}, k \geq 1, 1 \leq j < \alpha^{2k}, \\ (\alpha - 1)/\alpha^{2k+1} & \text{if } n = q_{2k-1} + q_{2k}, k \geq 0, \\ (\alpha + 1)/\alpha^{2k+2} & \text{if } n = q_{2k+1} - q_{2k}, k \geq 0, \end{cases}$$

and $n(\lceil \alpha n \rceil - \alpha n) \geq 1$ for all other $n \geq 1$.

Proof. The formulas for $n = jq_{2k-1}, n = q_{2k-1} + q_{2k}$ and $n = q_{2k+1} - q_{2k}$ are immediate from (4). By [7, Ch. 2, §5, Theorem 2], we have $n(\lceil \alpha n \rceil - \alpha n) \geq 1$ for all $n \geq 1$ that are not of the form $jq_k, 1 \leq j < \alpha/\sqrt{h}, q_k + q_{k-1}$ or $q_k - q_{k-1}$. Since $\alpha q_{2k} - \lfloor \alpha q_{2k} \rfloor = 1/\alpha^{2k+1}, \alpha(q_{2k} + q_{2k+1}) - \lfloor \alpha(q_{2k} + q_{2k+1}) \rfloor = (\alpha - 1)/\alpha^{2k+2}$ and $\alpha(q_{2k} - q_{2k-1}) - \lfloor \alpha(q_{2k} - q_{2k-1}) \rfloor = (\alpha + 1)/\alpha^{2k+1}$, we have $\lceil \alpha n \rceil - \alpha n > 1/2$ for $n = jq_{2k}, n = q_{2k} + q_{2k+1}$ and $n = q_{2k} - q_{2k-1}$. If moreover $n \geq 2$, then we have thus $n(\lceil \alpha n \rceil - \alpha n) \geq 1$ for these n as well. Since $q_0 + q_{-1} = 1$, the case $n = 1$ has already been treated. \square

We obtain that

$$n(\lceil \alpha n \rceil - \alpha n) = \begin{cases} \frac{j^2(1 - 1/\alpha^{4k})}{\sqrt{h^2 + 4}} & \text{if } n = jq_{2k-1}, k \geq 1, 1 \leq j < \alpha^{2k}, \\ \frac{h - (\alpha - 1)^2/\alpha^{4k+2}}{\sqrt{h^2 + 4}} & \text{if } n = q_{2k-1} + q_{2k}, k \geq 0, \\ \frac{h - (\alpha + 1)^2/\alpha^{4k+4}}{\sqrt{h^2 + 4}} & \text{if } n = q_{2k+1} - q_{2k}, k \geq 0. \end{cases}$$

The worst case for $n = q_{2k-1} + q_{2k}$ or $n = q_{2k+1} - q_{2k}$ is given by $n = q_{-1} + q_0 = 1$, hence

$$n(\lceil \alpha n \rceil - \alpha n) \geq h + 1 - \alpha = 1 - \frac{1}{\alpha}$$

for all $n \geq 1$ such that $n \neq q_{2k-1}$ for all $k \geq 1$.

Now we come back to the case $h = 2$ and consider the distance of $\alpha n - c + \frac{1}{2}$ to the nearest integer above $\alpha n - c + \frac{1}{2}$. Note that $c - \frac{1}{2} = \frac{1}{\sqrt{2}}$. We have

$$\begin{aligned} 2\left(\left\lceil \alpha n - \frac{1}{\sqrt{2}} \right\rceil - \alpha n + \frac{1}{\sqrt{2}}\right) &= 2\left\lceil \alpha n - \frac{1}{\sqrt{2}} \right\rceil - 1 - \alpha(2n - 1) \\ &\geq \lceil \alpha(2n - 1) \rceil - \alpha(2n - 1) > \frac{\alpha - 1}{2\alpha n}, \end{aligned}$$

where we have used that q_{2k-1} is even for all $k \geq 1$; thus

$$\left\lceil \alpha n - \frac{1}{\sqrt{2}} \right\rceil - \alpha n + \frac{1}{\sqrt{2}} > \frac{\alpha - 1}{4\alpha n}.$$

Since $(\alpha - c)(c - 1) = \frac{1}{4\alpha}$, we have

$$\alpha n - \frac{1}{\sqrt{2}} < r_n + \frac{1}{2} < \alpha n - \frac{1}{\sqrt{2}} + \frac{1}{4\alpha n} < \alpha n - \frac{1}{\sqrt{2}} + \frac{\alpha - 1}{4\alpha n} < \left\lceil \alpha n - \frac{1}{\sqrt{2}} \right\rceil$$

for all $n \geq 1$, thus $d_n = e_n$. This completes the proof of Theorem 1.

We remark that $h = 2$ cannot be replaced by an arbitrary positive integer in the previous paragraph. For example, for $h = 1$, we have $\alpha = \frac{1+\sqrt{5}}{2}$, $c = \frac{\alpha^2}{\sqrt{5}}$, $\lceil 137\alpha - c + \frac{1}{2} \rceil = 220$ and $\lceil r_{137} + \frac{1}{2} \rceil = 221$. However, computer simulations suggest that (for any h) we always have $\lceil \alpha n - c \rceil = \lceil r_n \rceil$.

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