Spatiotemporal Gaussian random fields using stochastic partial differential equations

by

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Preface

This thesis is the conclusion of my time as a master's student of Applied Mathematics at the faculty of Electrical Engineering, Mathematics & Computer Science at the Delft University of Technology, after a slightly unusual journey. It started at the Computer Science & Engineering program in the very same faculty, whose first-year courses taught me two valuable lessons:

1. proving mathematical statements is not as scary as those triangles in high school made me think;
2. I do not necessarily want to be a software engineer after all.

Seeing as how I quite enjoyed the mathematics courses, I did the natural thing and switched to the bachelor program Applied Physics. Interesting as it was at times, there were some minor indications that my heart was not fully in this choice either, such as the fact that for some courses the exam was my first encounter with the responsible professor. Despite this, I made it to the third year. I was required to do a minor, and I chose to try the minor Applied Mathematics, which happened to be a bridging minor giving access to the corresponding MSc programme.

I found myself more motivated to do the work for the minor courses, which was a sign that this may be a more suitable path for me. Indeed, the struggle of taking Real Analysis at the same time as Mathematical Structures was not enough to discourage me from diving into the deep end that was the MSc Applied Mathematics the following year. There, after a period of not being able to decide between different specializations, I contacted Kristin Kirchner who would go on to suggest this interesting topic on the interface of SPDE theory, numerics and statistics, leading ultimately to this thesis.

I would like to thank Kristin for offering this project and for her guidance and valuable feedback throughout, as well as Mark Veraar and Joris Bierkens for taking the time and effort to be part of my graduation committee. I also want to thank the 'DDD' crew for their significant part in keeping me sane throughout These Unprecedented Times™: Arie, David, Frank and whomever else incidentally tagged along. Lastly, I would like to thank some people from the Middelburg metropolitan area, in particular my parents for the occasional retreat and my friends Daan, Fabian, both Wouters and Pol. Although I saw less of them the past year or two, we still managed to fit in some good times.

Joshua Willems
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Introduction

In many areas of the environmental and social sciences, measurements spread across space and over time are collected in order to study uncertain phenomena evolving over time. A central goal of *spatiotemporal statistics* is to find statistical models which accurately describe such phenomena and whose parameters can be inferred from spatiotemporal data. Examples of application domains include neuroimaging [53], epidemiology [47], demography [27, 58] and finance [32].

The current state of data measurement and storage technology allows for the gathering of large and varied datasets. However, their applicability for spatiotemporal statistical inference is hindered by high computational costs, which necessitate a compromise in model accuracy in order to account for the available computational power. In this work, we consider an approach to spatiotemporal modeling which has the potential to strike a better balance between these metrics than existing methods. It is based on a class of *stochastic partial differential equation (SPDEs)* which we envision to possess desirable modeling properties, owing to a number of tunable parameters corresponding to interpretable properties of their solutions. The goal of this thesis is to study these SPDEs in terms of their analytical properties and covariance structure, and to take the first exploratory steps towards their efficient numerical simulation.

This introductory chapter is structured in the following way. In Section 1.1, we first provide a more detailed account of the present state of the art, the proposed method and the motivation behind this choice. This discussion is followed by Section 1.2, in which we sketch an outline of the chapters comprising this work and identify the contributions made to the body of knowledge regarding the class of SPDEs under consideration. This chapter is concluded with Section 1.3, which summarizes choices in the notation of certain mathematical concepts with which the reader is assumed to be familiar.

1.1. Background and motivation

The central objective in this work is to investigate *random fields* which possess useful properties for spatiotemporal statistical modeling. A spatiotemporal random field is defined to be a collection $(X(t,x))_{(t,x)\in\mathbb{T}\times\mathcal{D}}$ of random variables indexed by a given spatial domain $\mathcal{D}$ and time horizon $\mathbb{T}$; spatial random fields are of the form $(X(x))_{x\in\mathcal{D}}$. The most commonly used methods to define such random fields fall into two main categories: *second-order-based models* and *dynamical models*, see [22, 60].

A *second-order-based model* is defined by specifying the first and second moments of the random field; more precisely, this amounts to prescribing a mean function $\mu\colon \mathcal{I}\to\mathbb{R}$ and a covariance function $\varrho\colon \mathcal{I}\times\mathcal{I}\to\mathbb{R}$, where the index set $\mathcal{I}$ is $\mathbb{T}\times\mathcal{D}$ for a spatiotemporal field or $\mathcal{D}$ if the field is merely spatial. In this work we restrict ourselves to *Gaussian random fields (GRFs)*, which are random fields for which every finite subcollection of random variables is jointly Gaussian distributed; such fields are characterized completely by the first two moments.

In the special case of a spatial random field on a d-dimensional Euclidean domain $\mathcal{D}\subseteq\mathbb{R}^d$, the *Matérn covariance*, named after Swedish forestry statistician Bertil Matérn [51], is perhaps the most important and widely used covariance function [65]; it is given by

$$\varrho(x,y) := \varrho_0(x-y) := \frac{2^{1-\nu}\sigma^2}{\Gamma(\nu)}(\kappa\|x-y\|_{\mathbb{R}^d})^\nu K_\nu(\kappa\|x-y\|_{\mathbb{R}^d}), \quad x,y \in \mathcal{D}, \tag{1.1}$$

where Γ is the Gamma function, $\|\cdot\|_{\mathbb{R}^d}$ is the Euclidean norm on $\mathbb{R}^d$ and K_ν denotes the modified Bessel function of the second kind. The parameters $\sigma^2, \nu, \kappa > 0$ determine respectively the variance at each point, the spatial smoothness of the field and the practical correlation range between points. The utility of the Matérn model for inference from spatial data lies in the fact that each of these parameters is controllable and expresses a clearly interpretable property of the spatial field. Note that the covariance can be written in the form $\varrho(x,y) = \varrho_0(x-y)$; a field with such a covariance function is called *stationary*.

For the spatiotemporal case, it is difficult to find a practical second-order-based model analogous to the purely spatial Matérn model. There are two main reasons for this: firstly, the problem of finding suitable spatiotemporal covariance functions is challenging in general [21, 33, 37, 59, 61]. Secondly, simulating a random field at n points using a second-order-based method requires the factorization of an $n \times n$ covariance matrix which may be dense in general, in which case this operation incurs a computational cost of order $\mathcal{O}(n^3)$; this makes the computation impractically intensive for the high values of n typical in spatiotemporal applications. *Separable* covariance functions, which are simply the products of a purely spatial and a purely temporal covariance function, have been considered in an attempt to cope with these two limitations. Indeed, this is a straightforward way to devise spatiotemporal covariance functions and the resulting covariance matrices are easier to factorize since can be written as the Kronecker product of the spatial and temporal covariance matrices. Unfortunately, stationary and separable covariance functions generally do not result in realistic models for spatiotemporal processes, as noted for instance in [22, 52, 64].

The difficulty in defining and using covariance functions possessing the desirable properties of non-separability and/or non-stationarity have lead to the study of *dynamical models*, in which random fields are defined through their conditional probabilities or as solutions to SPDEs. In the remainder of this work, we focus on the SPDE approach.

As before, consider first the purely spatial situation. Using a power spectral density argument, Peter Whittle [67] showed that the Matérn model, which was introduced above as a second-order-based spatial model defined through the covariance function given by (1.1), admits a dynamical representation as the stationary solution to the SPDE

$$(\kappa^2 - \Delta)^\beta X(x) = \mathcal{W}(x) \quad \text{for all } x \in \mathcal{D} = \mathbb{R}^d. \tag{1.2}$$

Here Δ denotes the Laplacian, $\mathcal{W}$ denotes Gaussian spatial white noise and the fractional power is defined as $\beta := \frac{\nu}{2} + \frac{d}{4}$. This observation forms the basis for the technique proposed in the influential discussion paper by Lindgren, Rue and Lindström [48], who generated approximations to Matérn GRFs for $\beta \in \mathbb{N}$ by applying numerical methods to (1.2) on a bounded domain $\mathcal{D}$. The subsequent development of efficient numerical methods for fractional-order elliptic differential operators by Bonito and Pasciak [14] paved the way for extensions of the SPDE approach to more general fractional powers β, see [9, 20].

This approach to simulating approximations of spatial Matérn GRFs has a number of advantages as compared to the second-order-based method. The first of these is the ability to leverage a variety of computational techniques developed for the numerical approximation of (S)PDEs, such as finite element methods (FEM) [8, 9, 10, 20, 40, 48] and wavelets [11, 39], which often prove more efficient than factorizing covariance matrices. Secondly, this approach can be generalized in a natural way to allow for non-stationary fields; this is achieved by replacing the operator $\kappa^2 - \Delta$ in (1.2) with a uniformly elliptic differential operator L of the form

$$[Lu](x) := \kappa^2(x)u(x) - \nabla \cdot (\mathbf{A}(x)\nabla u(x)), \quad x \in \mathcal{D}.$$

Here, κ is allowed to be spatially varying and $\mathbf{A}$ is a function on $\mathcal{D}$ taking its values in the symmetric $d \times d$ matrices $\mathcal{D}$; this has been considered in e.g. [6, 8, 20, 34, 48]. A third advantage is that (1.2) can be formulated on more general surfaces and manifolds for all β [12, 39, 40]. This is not straightforward for the second-order-based model: for instance, it has been shown that replacing the Euclidean distance with the great circle distance in (1.1) does not generally yield a positive definite function on the sphere [38].

Turning back to the spatiotemporal case, we consider the matter of extending (1.2) to space-time in order to obtain a suitable dynamical spatiotemporal model. A natural choice of generalization would be the diffusion equation

$$(\partial_t + L^\beta)X(t,x) = \dot{\mathcal{W}}(t,x) \quad \text{for all } t \in \mathbb{T} \text{ and } x \in \mathcal{D}, \tag{1.3}$$

where $\dot{\mathcal{W}}$ denotes spatiotemporal Gaussian noise which is assumed to be white in time and white or possibly colored in space. In fact, we will take one step further in generality and consider the following stochastic initial boundary value problem, which is the central subject of this work:

$$\begin{cases} (\partial_t + L^\beta)^\gamma X(t,x) = \dot{\mathcal{W}}(t,x), & \text{for all } t \in (0,T) \text{ and } x \in \mathcal{D}; \\ X(0,x) = X_0(x) & \text{for all } x \in \mathcal{D}; \\ X(t,x) = 0 & \text{for all } x \in \partial\mathcal{D}; \end{cases} \tag{1.4}$$

where $\gamma > 0$ is an additional fractional exponent, X_0 is an initial random field on $\mathcal{D} \subsetneq \mathbb{R}^d$ and $T \in (0,\infty)$ is a finite time horizon.

The choice for (1.4) is motivated by the proposals recently made in [5], in which the authors analyze its infinite-space counterpart for $L = \kappa^2 - \Delta$ using Fourier techniques; similar models have been studied for instance in [2, 18, 42]. Firstly, it was found that the combination of β and γ governs the temporal smoothness of the solution; this property is important for the realistic modeling of spatiotemporal phenomena, and it extends the situation in the purely spatial model (1.2), where β controls spatial smoothness. A second finding was that the long-time behavior resembles that of the spatial model (1.2), which sets the SPDE from (1.4) apart from similar choices, such as $(\partial_t^\gamma + L^\beta)X = \dot{\mathcal{W}}$ which has been studied in [13, 28]. Note that the choice of homogeneous Dirichlet boundary conditions in (1.4) is mainly made for simplicity rather than realism of the model.

The aim of the present work is to take the first steps into the analytical and numerical investigation of (1.4). More precisely, we shall study a slightly more abstract formulation which will be introduced in Chapter 3, see (3.1).

1.2. Outline and contributions

The remainder of this thesis is structured as follows. Chapter 2 is a summary of preliminary notions from (functional) analysis, operator theory, probability theory and (S)PDE theory needed to understand the subsequent chapters. Although some of the results covered in Chapter 2 are non-trivial, they can usually be found in standard textbooks, to which we often refer for more details. The central topic of Chapter 3 is the analysis of (3.1), a more abstract counterpart of (1.4). The goals are to define solutions to the equation in a rigorous way and to investigate the effects of various fractional parameters on qualitative properties of solutions such as well-posedness, regularity and covariance structure. In Chapter 4, we propose a numerical method for computing approximations to a deterministic fractional ODE in time, which is a specialization of the SPDE considered throughout the rest of the work.

The main contributions of this work to the knowledge surrounding the proposed fractional SPDE approach are found throughout Chapter 3. We conclude with a short discussion and outlook in Chapter 5

Since, as stated above, it is not obvious *a priori* how to define solutions to (1.4) for fractional powers $\gamma \notin \mathbb{N}$, in Section 3.2 we study the unbounded operator $\partial_t + \mathcal{A}$ on the Bochner space $L^2(0,T;H)$, where H is some Hilbert space which can be thought of as $L^2(\mathcal{D})$. We show that its negative generates an exponentially stable product C_0-semigroup, expressed as the composition of a translation semigroup and the analytic C_0-semigroup corresponding to the negative of the unbounded, positive and self-adjoint operator A on H. This is the basis for a convenient representation of fractional powers $(\partial_t + \mathcal{A})^\gamma$, $\gamma \in \mathbb{R}$; in particular, we see that negative fractional powers can be defined in terms of an integral over the product semigroup, and the explicit representation of the semigroup allows us to derive a convolution formula for the action of $(\partial_t + \mathcal{A})^{-\gamma}$ on functions in $L^2(0,T;H)$.

Inspired by the convolution formula for $(\partial_t + \mathcal{A})^{-\gamma}$, in Section 3.3 we introduce a natural definition of the mild solution to (3.1), first in the case of a zero initial condition. This choice is then further motivated by noting its reduction to the familiar variation-of-constants formula in the case $\gamma = 1$, and by comparing it to a generalized weak variational solution concept based on taking Bochner inner products on both sides of (3.1); we show that the mild and weak solutions coincide under natural assumptions. Both formulations are subsequently extended to account for nonzero initial conditions.

In Section 3.4, we prove existence and spatiotemporal regularity of the mild solution by using direct estimates involving the stochastic convolution formula and the smoothing properties of the analytic C_0-semigroup generated by $-A$. These results are linked to conditions on the Hilbert–Schmidt norm of fractional powers of A and the covariance operator Q coloring the spatial part of the Gaussian noise. In particular, the results reflect that the temporal Hölder continuity and/or differentiability increases with γ, a temporal smoothing effect which is desired in applications. In the course of deriving the regularity results for nonzero initial conditions, we also show the continuous dependence of the mild solution on the initial datum.

In the last part of the analysis, we prove two results on the asymptotic covariance structure of solutions to (3.1), which are of interest from the viewpoint of statistical applications. First it is shown that for large times, the marginal spatial covariance of the mild solution can be expressed in terms of fractional powers of A and Q. In particular, this implies that if A and Q are chosen equal to fractional powers of the same operator, then the spatial covariance structure is eventually of the same form. Secondly, we consider the case $A = I$ and prove that the corresponding covariance is separable and that the temporal part of the covariance function is of Matérn type.

The contributions described above partially generalize well-known results about the *stochastic heat equation*, i.e. the non-fractional case $\gamma = \beta = 1$; these can be found in standard references such as [23]. To the best of our knowledge, the only prior results on similar equations with $\gamma \neq 1$ are those found in the recent work [66], which considers the regularity of a deterministic version of (1.4) with $\beta = 1$, $L = -\Delta$ and $\gamma \in (0,1)$. Besides treating a stochastic equation instead of a deterministic one, the present work furthermore differs from [66] in the approach taken to define fractional powers of the parabolic operator $\partial_t + A$ (where A is to be defined in Chapter 3), leading to the notion of a *(stochastic) fractional convolution*. This representation can be leveraged to directly derive well-posedness and regularity results for a larger class of operators A and for arbitrarily large values of γ, whereas the authors of [66] only consider $\gamma \in (0,1)$. The investigations into the covariance structure of solutions to (3.1) generalize certain results found in [5] to a larger class of spatiotemporal domains $\mathbb{T} \times \mathcal{D}$ and operators A.

1.3. Notation

This section will introduce some notation which is widely used throughout this work.

We take $\mathbb{N}$ to mean the natural numbers excluding zero and $\mathbb{N}^{\geqslant 0} := \mathbb{N} \cup \{0\}$. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are written as $\operatorname{Re} z$ and $\operatorname{Im} z$ respectively. Its argument is denoted $\arg z$ and is taken in $(-\pi, \pi]$. The indicator function of a set $A \subseteq X$ is denoted $\mathbf{1}_A$; recall that the indicator function $\mathbf{1}_A : X \to \{0,1\}$ is defined by $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$. We also use the related notation $\mathbf{1}\{\,\cdot\,\}$, which is an expression equal to 1 if the condition in between the brackets is true and 0 otherwise. For any function $f\colon A \to B$ and subsets $C \subseteq A, D \subseteq B$, define the image $f[C] := \{f(x)\colon x \in C\}$ and pre-image $f^{-1}[D] := \{x \in A\colon f(x) \in D\}$. If f is a linear mapping, then the square brackets may be omitted.

If $(X, \mathcal{T})$ is some topological space, then we write $\overline{A}$ for the closure of $A \subseteq X$. In a metric space, the open ball of radius $r > 0$ centered around $x \in X$ is written as $B(x,r)$. The Borel σ-algebra of the topological space $(X, \mathcal{T})$ is denoted by $\mathcal{B}(X)$; recall that this is the σ-algebra generated by all open sets $O \in \mathcal{T}$.

The space of continuous functions from an interval $J \subseteq \mathbb{R}$ to a Hilbert or Banach space X is denoted $C(J;X)$. We equip $C(J;X)$ with the supremum norm $\|f\|_{C(J;X)} := \sup_{t\in J}\big\|f(t)\big\|_X$, rendering it a Banach space. We write $C_{0,\{t\}}(J;X)$ for the space of continuous functions which vanish at the point $t \in J$. Recall the notion of the support of a function $f\colon J \to X$, namely $\operatorname{supp} f := \overline{\{t \in J\colon f(t) \neq 0\}}$; f is said to be compactly supported if $\operatorname{supp} f$ is compact as a subset of J, and the space consisting of such functions is denoted $C_c(J;X)$. For $n \in \mathbb{N}$, an n times continuously differentiable function is said to belong to $C^n(J;X)$. The Hölder-continuous functions with Hölder exponent $0 < \alpha \leqslant 1$ form the space

$C^{0,\alpha}(J;X)$. For a function $f \in C^{0,\alpha}(J;X)$, we define the Hölder seminorm by

$$|f|_{C^{0,\alpha}(J;X)} := \sup_{t\neq s\in J} \frac{\big\|f(t)-f(s)\big\|_X}{|t-s|},$$

which we recall is not a norm since it vanishes in particular for nonzero constant f. To mitigate this, one can for instance use the norm

$$\|f\|_{C^{0,\alpha}(J;X)} := |f|_{C^{0,\alpha}(J;X)} + \|f\|_{C(J;X)}$$

to render $C^{0,\alpha}(J;X)$ a Banach space. Functions which are n times continuously differentiable with α-Hölder continuous nth derivative are said to be members of $C^{n,\alpha}(J;X)$. For these spaces, we can combine the above norms to define the norm

$$\|f\|_{C^{n,\alpha}(J;X)} := \|f^{(n)}\|_{C^{0,\alpha}(J;X)} + \sum_{k=0}^{n-1} \|f^{(k)}\|_{C(J;X)},$$

where $f^{(k)}$ denotes the kth derivative of f. Defining $C^{n,0}(J;X) := C^n(J;X)$, the above definition also yields a norm on $C^n(J;X)$. Moreover, define $C^\infty(J;X) := \bigcap_{n\in\mathbb{N}} C^n(J;X)$.

For $p \in [1,\infty]$, the spaces of real-valued p-integrable functions defined on a measure space $(X,\mathcal{A},\mu)$, are denoted $L^p(X)$. They are Banach spaces when equipped with the norm $\|f\|_{L^p(X)} := (\int_X |f|^p \,\mathrm{d}\mu)^{1/p}$ if $p \in [1,\infty)$ and $\|f\|_{L^\infty(X)} := \operatorname{ess\,sup}_{x\in X} |f(x)|$, where $\operatorname{ess\,sup}$ denotes the essential supremum. For $p = 2$, the norm $\|\cdot\|_{L^2(X)}$ is induced by the inner product $(f,g)_{L^2(X)} := \int_X fg \,\mathrm{d}\mu$, so that $L^2(X)$ is a Hilbert space. As is common, we identify functions which are equal almost everywhere, i.e. functions which differ only on a μ-zero zet. We often abbreviate the phrase 'almost everywhere' to 'a.e.' and 'almost all' to 'a.a.'; similarly, we use the acronym 'a.s.' to mean 'almost surely', which is the same concept but for probability spaces $(\Omega,\mathcal{F},\mathbb{P})$. The corresponding Sobolev spaces with integrability $p \in [1,\infty]$ and weak differentiability $k \in \mathbb{N}$ are denoted $W^{k,p}(X)$; as a norm, we take the p-norm of the finite vector of $L^p(x)$ norms of all partial derivatives of f. We typically write $W^{k,2}(X) =: H^k(X)$ for the commonly used special case $p = 2$.

Hilbert spaces use either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ as their scalar fields. For statements which do not depend on the underlying field, we sometimes refer to the scalar field as $\mathbb{K}$, i.e. $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. The norm of a Hilbert space H will be written as $\|\cdot\|_H$ and its inner product as $(\,\cdot\,,\,\cdot\,)_H$. We write I for the identity operator on H. Given two Hilbert spaces U and H, the notation $U \hookrightarrow H$ expresses that U is continuously embedded in H, i.e. there exists a natural continuous injective map between U and H, typically the inclusion map, which we often denote by $\iota\colon U \to H$. The same notations are used for Banach spaces, with the obvious exception of the inner product.

The topological dual space of a Banach space E, consisting of the continuous linear functionals on E, is written as E'. Sometimes the dual space E' can be identified with a linear space $E^* \supset E$ in such a way that $x^* \in E^*$ if and only if $\langle x^*, \cdot\rangle \in E'$; here, for $x \in E$ and $x^* \in E^*$ the notation $\langle x, x^*\rangle := x^*(x)$ stands for the duality pairing of x and x^*. When the difference between E^* and E' is not worth emphasizing, we choose the notation E^* for the dual of E.

2 Preliminaries

The purpose of this chapter is to present the necessary theory to understand the rigorous definitions of the SPDEs introduced in the previous chapter and the (numerical) analysis in the following chapters. As such, we do not strive for generality: whenever a result is more easily understood by restricting ourselves to a specialized form which is sufficient for our purposes, we shall do so. Standard references will be often cited which treat the various subjects covered in the upcoming sections; these can be consulted for more details and for proofs of many of the statements collected in this chapter.

Throughout this chapter, we assume that U and H are separable Hilbert spaces over the complex scalar field unless otherwise specified. Sometimes we also use the Banach spaces E and F, for instance in Section 2.3.

2.1. Linear operators

This section introduces some theory of linear operators on Hilbert and/or Banach spaces. We sometimes omit the word 'linear' when it is clear from context that an operator is linear.

2.1.1. Bounded operators

Definition 2.1.1 (Bounded operator)**.** A linear operator $T \colon U \to H$ is called *bounded* if

$$\|T\|_{\mathscr{L}(U;H)} := \sup_{\|x\|_U = 1} \|Tx\|_H < \infty.$$

We denote by $\mathscr{L}(U;H)$ the space of bounded (equivalently, continuous) linear operators from U to H; equipping it with the operator norm yields a Banach space. If $U = H$, we simply write $\mathscr{L}(H)$.

Some bounded operators satisfy additional properties, thus giving rise to some interesting classes of operators in $\mathscr{L}(E;F)$. We summarize the additional properties which are of interest for the present work; the first of these properties is *compactness*.

Definition 2.1.2 (Compact operator)**.** A bounded linear operator $T \in \mathscr{L}(U;H)$ is called *compact* if it maps bounded subsets of U into relatively compact subsets of H, i.e. $\overline{TA} \subseteq H$ is compact for any given bounded $A \subseteq U$. We denote the subspace of compact operators by $\mathscr{K}(U;H)$.

Identifying H with its dual H^* by the Riesz representation theorem, we define the adjoint $T^* \in \mathscr{L}(H)$ of $T \in \mathscr{L}(H)$ to be the unique operator satisfying

$$(Tx, y)_H = (x, T^*y)_H \quad \forall x, y \in H.$$

Definition 2.1.3 (Self-adjoint operator)**.** An operator $T \in \mathscr{L}(H)$ is called *self-adjoint* if it satisfies $T^* = T$, i.e.

$$(Tx, y)_H = (x, Ty)_H \quad \forall x, y \in H.$$

Definition 2.1.4 (Positive (semi-)definite operator)**.** An operator $T \in \mathscr{L}(H)$ is called *positive semi-definite* or *non-negative* if it satisfies

$$(Tx, x)_H \geqslant 0 \quad \forall x \in H. \tag{2.1}$$

If the inequality is strict, then it is called *positive definite* or simply *positive*.

It can be checked using the polarization identity for inner products on complex Hilbert spaces that non-negative operators are necessarily self-adjoint. This result does not hold for real Hilbert spaces; intuitively, this is due to the fact that (2.1) is more restrictive on a complex Hilbert space than on a real Hilbert space.

For positive operators, we may introduce the *trace* as follows.

Definition 2.1.5 (Trace, of a non-negative operator)**.** For a non-negative operator $T \in \mathscr{L}(H)$, we define the *trace* as

$$\operatorname{tr}(T) := \sum_{j=1}^{\infty} (Te_j, e_j)_H, \tag{2.2}$$

where $(e_j)_{j\geqslant 1}$ is any orthonormal basis of H.

It can be shown that the above definition is indeed independent of the choice of orthonormal basis for H, thus showing the well-definedness of the trace.

In order to define the trace of general bounded operators $T \in \mathscr{L}(H)$, we introduce the *modulus* $|T| := (T^*T)^{1/2}$, which is a non-negative operator. The operator square root can be defined using fractional powers of operators, a concept which is introduced later in Section 2.5. It recovers some of the familiar properties of square roots of numbers, such as the fact that $T^{1/2}T^{1/2} = T$.

Definition 2.1.6 (Trace-class operator)**.** An operator $T \in \mathscr{L}(H)$ is said to be of trace class if $\operatorname{tr}|T| < \infty$. The space of trace-class operators will be denoted by $\mathscr{L}_1(H)$; it becomes a Banach space when equipped with the norm $\|T\|_{\mathscr{L}_1(H)} := \operatorname{tr}|T|$.

Definition 2.1.7 (Hilbert–Schmidt operator)**.** An operator $T \in \mathscr{L}(H)$ is called a *Hilbert–Schmidt operator* if it has finite *Hilbert–Schmidt norm*, i.e. if

$$\|T\|_{\mathscr{L}_2(H)} := \left(\sum_{j=1}^{\infty} \|Te_j\|_H^2 \right)^{1/2} < \infty \tag{2.3}$$

for any given orthonormal basis $(e_j)_{j\geqslant 1}$ of H. We denote by $\mathscr{L}_2(H)$ the set of Hilbert–Schmidt operators. Then, for $T, S \in \mathscr{L}_2(H)$, we may define the inner product

$$(T, S)_{\mathscr{L}_2(H)} := \sum_{j=1}^{\infty} (Te_j, Se_j)_H,$$

which renders $\mathscr{L}_2(H)$ a Hilbert space.

The relationship between bounded, compact, trace-class and Hilbert–Schmidt operators can be summarized by the chain of inclusions

$$\mathscr{L}_1(H) \subset \mathscr{L}_2(H) \subset \mathscr{K}(H) \subset \mathscr{L}(H).$$

In particular, we have the following inequalities:

$$\|T\|_{\mathscr{L}(H)} \leqslant \|T\|_{\mathscr{L}_2(H)} \leqslant \|T\|_{\mathscr{L}_1(H)}.$$

More details on trace-class and Hilbert–Schmidt operators, including proofs of the claims made about them in this section, may be found in [29, Chapter XI, Section 6].

2.1.2. Spectral properties of self-adjoint compact operators

First recall the definition of the *resolvent set* and *spectrum* of a bounded operator.

Definition 2.1.8. Given a $T \in \mathscr{L}(H)$, we define the *resolvent set* of T as

$$\rho(T) := \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ has a bounded two-sided inverse}\},$$

and its *spectrum* as $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Definition 2.1.9. Given $T \in \mathscr{L}(H)$ and $\lambda \in \rho(T)$, define the *resolvent operator*

$$R(\lambda, T) := (\lambda I - T)^{-1}.$$

For self-adjoint $T \in \mathscr{L}(H)$, the spectrum $\sigma(T)$ is real-valued. In fact, we have the following.

Proposition 2.1.10 (Spectrum of a self-adjoint bounded linear operator)**.** *Let $T \in \mathscr{L}(H)$ be self-adjoint. Setting*

$$m := \inf_{\|x\|=1} (Tx, x)_H, \quad M := \sup_{\|x\|=1} (Tx, x)_H,$$

it holds that $\{m, M\} \subseteq \sigma(T) \subseteq [m, M]$ and $\|T\|_{\mathscr{L}(H)} = \max\{|m|, |M|\}$.

Proof. See [16, Proposition 6.9]. □

The next theorem tells us that compact self-adjoint operators on H have spectra resembling those of diagonal matrices on finite-dimensional spaces.

Theorem 2.1.11. *Let $T \in \mathscr{K}(H)$ be self-adjoint. Then $\sigma(T)$ is at most countably infinite and has no accumulation point except possibly $\lambda = 0$, every nonzero $\lambda \in \sigma(T)$ is an eigenvalue of T with finite-dimensional eigenspace $E_\lambda := \{v \in H \colon Tv = \lambda v\}$ and there exists an orthonormal basis $(e_j)_{j\geqslant 1}$ of H consisting of eigenvectors of T.*

Proof. See [29, Corollary X.3.5]. □

We will call an orthonormal basis $(e_j)_{j\geqslant 1}$ consisting of eigenvectors of T with associated eigenvalues $(\lambda_j)_{j\geqslant 1}$, i.e. $Te_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, an *(orthonormal) eigenbasis* of T; an eigenbasis will sometimes be denoted more shortly as $(e_j, \lambda_j)_{j\geqslant 1}$.

Combining Theorem 2.1.11, with the compactness of trace-class and Hilbert–Schmidt operators yields different forms of formulas (2.2) and (2.3), namely

$$\|T\|_{\mathscr{L}_1(H)} = \sum_{j=1}^{\infty} \lambda_j, \tag{2.4}$$

$$\|T\|_{\mathscr{L}_2(H)} = \left(\sum_{j=1}^{\infty} \lambda_j^2 \right)^{1/2}, \tag{2.5}$$

by taking the eigenbasis $(e_j, \lambda_j)_{j\geqslant 1}$ of the compact positive operator $|T|$. The nonzero eigenvalues of $|T|$ are called *singular values*. In particular, if T is non-negative, then $T = |T|$ so that equations (2.4) and (2.5) are expressed in the nonzero eigenvalues of T itself.

2.1.3. Unbounded operators

In order to study linear operators on Hilbert/Banach spaces which are not necessarily bounded or even defined on the whole space, we introduce the notion of *unbounded* linear operators.

Definition 2.1.12. A linear operator A, which is defined on a linear subspace $\mathsf{D}(A)$ of H and takes its values in H, is called an *unbounded linear operator*. We denote by $\mathsf{D}(A)$ and $\mathsf{R}(A)$ the *domain* and *range* of A, respectively.

We often use the shorthand notation $(A, \mathsf{D}(A))$, which is taken to mean that A is an unbounded linear operator with domain $\mathsf{D}(A)$.

If $\mathsf{D}(A)$ is dense in H, then A is called *densely defined*. We define the *graph* of A by

$$\mathsf{G}(A) := \{(x, Ax) \colon x \in \mathsf{D}(A)\},$$

which can be equipped with the norm $\|(x, Ax)\|_{\mathsf{G}(A)} := \|x\|_H + \|Ax\|_H$. Similarly, we can define the norm $\|x\|_{\mathsf{D}(A)} := \|x\|_H + \|Ax\|_H$ for $\mathsf{D}(A)$; in this context, $\|\cdot\|_{\mathsf{D}(A)}$ is called the *graph norm* on $\mathsf{D}(A)$. An unbounded operator A is said to be *closed* if its graph is closed with respect to $\|\cdot\|_{\mathsf{G}(A)}$.

Definition 2.1.13 ((Maximal) accretive operator)**.** An unbounded linear operator $(A, \mathsf{D}(A))$ on a Hilbert space H is called *accretive* if for all $x \in \mathsf{D}(A)$ we have $\mathrm{Re}(Ax, x)_H \geqslant 0$. If moreover $\mathsf{R}(A + I) = H$, i.e. for all $f \in H$ there exists a $u \in \mathsf{D}(A)$ such that $Au + u = f$, then $(A, \mathsf{D}(A))$ is called *maximal accretive*.

2.2. Hilbert tensor product spaces

For the Hilbert spaces H and $\widetilde{H}$, we define the *algebraic tensor product space* $H \otimes \widetilde{H}$ as the tensor product of H and $\widetilde{H}$ in the sense of vector spaces. This means that $H \otimes \widetilde{H}$ consists of elements which can be written as

$$\sum_{n=1}^{N} x_n \otimes \widetilde{x}_n,$$

where $N \in \mathbb{N}$ and $x_n \in H, \widetilde{x}_n \in \widetilde{H}$ for all $n \in \{1, \dots, N\}$. Note that this representation is not unique; in fact, the mapping $(x, \widetilde{x}) \mapsto x \otimes \widetilde{x}$ is bilinear, so that for instance we have the identity

$$x \otimes \widetilde{x}_1 + x \otimes \widetilde{x}_2 = x \otimes (\widetilde{x}_1 + \widetilde{x}_2),$$

where both sides of the equation are different representations of the same element from $H \otimes \widetilde{H}$.

Various ways exist to extend the algebraic tensor space of two Hilbert spaces to a tensor product space which is itself a Banach or Hilbert space; each of these corresponds to the completion with respect to a different norm defined on the algebraic tensor product space. In this discussion, we limit ourselves to the *Hilbert product tensor space*. Given the elements

$$\hat{x} = \sum_{n=1}^{N} x_n \otimes \widetilde{x}_n \quad \text{and} \quad \hat{y} = \sum_{m=1}^{M} y_n \otimes \widetilde{y}_n$$

in $H \otimes \widetilde{H}$, we define the inner product

$$(\hat{x}, \hat{y})_{H \otimes \widetilde{H}} := \sum_{n=1}^{N} \sum_{m=1}^{M} (x_n, y_n)_H (\widetilde{x}_n, \widetilde{y}_n)_{\widetilde{H}}.$$

This is a well-defined inner product since can be shown to be independent of the choice of representation. The Hilbert product tensor space, also denoted $H \otimes \widetilde{H}$, is then obtained by taking the completion of the algebraic tensor product space with respect to the norm induced by $(\,\cdot\,,\,\cdot\,)_{H \otimes \widetilde{H}}$. If $H = \widetilde{H}$, then the abbreviation $H^{(2)} := H \otimes H$ is used.

2.3. Integration of vector-valued functions

2.3.1. Riemann integral

Throughout this work, we will encounter a variety of situations in which it is necessary to define the integral of a vector-valued function, for instance a function taking its values in the Banach space E. It turns out that for smooth enough integrands, say $f \in C([a, b]; E)$, the definition of the Riemann integral still makes sense, and the proofs of most familiar properties carry over upon replacing absolute values in the codomain by the norm $\|\cdot\|_E$. This can then be generalized to functions on $\mathbb{R}$ by defining improper Riemann integrals in the obvious way.

2.3.2. Bochner integral

For a function defined on a domain with less structure, such as a measurable space, the Riemann integral no longer makes sense and we would like to proceed as in the real-valued case by finding an analog to the Lebesgue integral. This is more difficult than for the Riemann case since the construction of the Lebesgue integral depends on the order structure of the real line. Nonetheless, we shall see that the *Bochner integral* is a suitable analog; its construction is the subject of this subsection.

Throughout this section, let $(A, \mathcal{A})$ denote a measurable space, i.e. A is a set and $\mathcal{A}$ is a σ-algebra on this set, and let the Banach space E be equipped with the Borel σ-algebra $\mathcal{B}(E)$ unless specified otherwise. Recall the notion of *measurability*: a function $f\colon A \to E$ is said to be measurable if $f^{-1}[B] \in \mathcal{A}$ for all $B \in \mathcal{B}(E)$. Moreover, recall that in the real-valued setting (i.e., $E = \mathbb{R}$), a function $f\colon A \to \mathbb{R}$ is measurable if and only if it is the pointwise limit of a sequence of *simple functions*. In the E-valued setting, a measurable function is not necessarily the pointwise limit of a sequence of simple functions; since the latter property is the more useful one for defining integrals, we introduce it as an alternative definition of measurability, called *strong measurability*.

Definition 2.3.1. A function $f\colon A \to E$ is said to be *simple* if there exist $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathcal{A}$ and $x_1, \dots, x_n \in E$ such that

$$f = \sum_{j=1}^{n} \mathbf{1}_{A_j} \otimes x_j,$$

where $(\mathbf{1}_{A_j} \otimes x_j)(\xi) := \mathbf{1}_{A_j}(\xi) \otimes x_j$. We call f *strongly measurable* if it is the pointwise limit of a function of simple functions.

The following theorem gives a characterization of strongly measurable functions.

Theorem 2.3.2 (Pettis measurability theorem)**.** *A function $f\colon A \to E$ is strongly measurable if and only if f takes its values in a separable closed subspace of E and $\langle f(\,\cdot\,), x^*\rangle\colon E \to \mathbb{R}$ is strongly measurable for all $x^* \in X^*$.*

Proof. See [41, Theorem 1.1.6]. □

An important corollary of this result is the fact that pointwise limits of sequences of strongly measurable functions are strongly measurable, see [41, Corollary 1.1.9]. The next result implies that the notions of measurability and strong measurability are equivalent if E is separable.

Proposition 2.3.3. *Let $f\colon A \to E$. Then these two statements are equivalent:*

(a) f is strongly measurable;

(b) f takes its values in a separable subspace of E and $f^{-1}[B] \in \mathcal{A}$ for all $B \in \mathcal{B}(E)$.

Proof. See [41, Corollary 1.1.10]. □

In the remainder of this section, suppose that we have a σ-finite measure space $(A, \mathcal{A}, \mu)$.

Definition 2.3.4 (Bochner integral of a simple function)**.** A simple function $f\colon A \to E$ is said to be *Bochner integrable* if the Lebesgue integral $\int_A \|f\|_E \,\mathrm{d}\mu$ is finite. In that case, writing $f = \sum_{j=1}^{n} \mathbf{1}_{A_j} \otimes x_j$ for some $n \in \mathbb{N}$, pairwise disjoint sets $A_1, \dots, A_n \in \mathcal{A}$ and $x_1, \dots, x_n \in E$, we define the *Bochner integral* of f by

$$\int_A f \,\mathrm{d}\mu := \sum_{j=1}^{n} \mu(A_j) x_j.$$

It can be shown that the value of the integral is well-defined in the sense that it does not depend on the choice of representation; i.e., if $f = \sum_{j=1}^{n} \mathbf{1}_{A_j} \otimes x_j = \sum_{i=1}^{m} \mathbf{1}_{B_i} \otimes y_i$ for $m \in \mathbb{N}$, $B_1, \dots, B_m \in \mathcal{A}$ and values $y_1, \dots, y_m \in E$, then $\sum_{j=1}^{n} \mu(A_j) x_j = \sum_{i=1}^{m} \mu(B_i) y_i$.

Definition 2.3.5 (Bochner integral)**.** A function $f\colon A \to E$ is said to be Bochner integrable if it is the pointwise limit of a sequence of integrable simple functions $(f_n)_{n\geqslant 1}$ and

$$\lim_{n\to\infty} \int_A \|f_n - f\|_E \,\mathrm{d}\mu = 0. \tag{2.6}$$

In that case, we define the Bochner integral of f by

$$\int_A f \,\mathrm{d}\mu := \lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu. \tag{2.7}$$

It is not immediately obvious that the Bochner integral is well-defined. First note that $\|f_n - f\|_E$ is measurable for all $n \geqslant 1$, since f_n and f are strongly measurable and $\|\cdot\|_E$ is continuous. The limit in equation (2.7) exists; this can be seen by checking that $(\int_A f_n \,\mathrm{d}\mu)_{n\geqslant 1}$ is Cauchy, which follows from (2.6) and an 'integral triangle inquality' which can be proved directly for simple functions. Lastly, it can be checked that the limit does not depend on the approximating sequence $(f_n)_{n\geqslant 1}$.

Now we collect some useful properties of the Bochner integral, some of which are analogous to familiar properties of the Lebesgue integral.

For a Bochner integrable function $f\colon A \to E$ and some $x^* \in E^*$, we have that

$$\left\langle \int_A f \,\mathrm{d}\mu, x^* \right\rangle = \int_A \langle f, x^* \rangle \,\mathrm{d}\mu. \tag{2.8}$$

A convenient characterization of Bochner integrability is the following.

Proposition 2.3.6. *A strongly measurable function $f\colon A \to E$ is Bochner integrable if and only if*

$$\int_A \|f\|_E \,\mathrm{d}\mu < \infty,$$

where the integral is a Lebesgue integral. In this case, we have the integral triangle inequality

$$\left\| \int_A f \,\mathrm{d}\mu \right\|_E \leqslant \int_A \|f\|_E \,\mathrm{d}\mu.$$

Proof. See [41, Proposition 1.2.2]. □

The following analog of the dominated convergence theorem (DCT) holds.

Theorem 2.3.7 (Dominated convergence theorem)**.** *Suppose that we have a sequence $(f_n)_{n\geqslant 1}$ of Bochner integrable functions from A to E which converges pointwise a.e. to some $f\colon A \to E$. If there exists a scalar-valued Bochner integrable function g on A such that $\|f_n\|_E \leqslant |g|$ for all $n \geqslant 1$, then f is Bochner integrable and we have*

$$\lim_{n\to\infty} \int_A \|f - f_n\|_E \,\mathrm{d}\mu = 0$$

and in particular

$$\lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu := \int_A f \,\mathrm{d}\mu.$$

Proof. See [41, Proposition 1.2.5]. □

Under certain conditions, we may interchange closed operators and Bochner integrals. Note that Definition 2.1.12, which introduces unbounded operators on Hilbert spaces and their domains, generalizes readily to Banach spaces.

Theorem 2.3.8. *Let $f\colon A \to E$ be Bochner integrable and let $(T, \mathsf{D}(T))$ be a closed linear operator from E to F. If f takes its values in $\mathsf{D}(T)$ a.e., and if the a.e.-defined function $Tf\colon A \to F$ is Bochner integrable, then $\int_A f \,\mathrm{d}\mu \in \mathsf{D}(T)$ and*

$$T \int_A f \,\mathrm{d}\mu = \int_A Tf \,\mathrm{d}\mu.$$

Proof. See [41, Theorem 1.2.4]. □

A last property of the Bochner integral which is worth noting is that any continuous function $f\colon [a, b] \to E$ is Bochner integrable and is value coincides with the Riemann integral.

2.3.3. Lebesgue–Bochner and vector-valued Sobolev spaces

By analogy with the Lebesgue spaces $L^p(A)$, $1 \leqslant p < \infty$, we define the *Lebesgue–Bochner space* (or simply *Bochner space*) $L^p(A;E)$ as the space consisting of strongly measurable functions $f\colon A \to E$ such that

$$\|f\|_{L^p(A;E)} := \left(\int_A \|f\|_E^p \,\mathrm{d}\mu\right)^{\frac{1}{p}} < \infty,$$

where we identify functions which are equal almost everywhere. In the same way as for the Lebesgue spaces, one shows that $(L^p(A;E), \|\cdot\|_{L^p(A;E)})$ is a Banach space. Also note that $L^1(A;E)$ is precisely the space of Bochner integrable functions by Proposition 2.3.6.

Now suppose that $A = (0,T)$ and $E = H$. Then for $p = 2$, the norm on $L^2(0,T;H)$ is induced by the inner product $(f,g)_{L^2(0,T;H)} := \int_0^T (f,g)_H \,\mathrm{d}\mu$ which renders it a Hilbert space. Since in this context we would like to define vector-valued Sobolev spaces, we first introduce *distributional derivatives* of Bochner integrable functions. Let $\mathscr{D}(0,T)$ denote the space of *test functions*, defined to be the set $C_c^\infty((0,T);\mathbb{R})$ of smooth and compactly supported real-valued functions equipped with the topology described in [24, Appendix "Distributions", §1.1.4]. An *H-valued distribution* is then a continuous linear mapping from $\mathscr{D}(0,T)$ to H. Given a distribution f, we define the *distributional derivative* f' by $f'(\phi) := -f(\phi')$ for $\phi \in \mathscr{D}(0,T)$; it can be shown that f' is indeed still linear and continuous, hence a well-defined H-valued distribution.

To a given $u \in L^p(0,T;H)$, where $1 \leqslant p < \infty$, we canonically associate the H-valued distribution f defined by

$$f(\phi) := \int_0^T u(t)\phi(t)\,\mathrm{d}t =: \langle u, \phi\rangle, \quad \phi \in \mathscr{D}(0,T).$$

If its distributional derivative f' admits the same type of representation, say $f'(\phi) = \langle v, \phi\rangle$ for some $v \in L^q(0,T;H), 1 \leqslant q < \infty$, then we say that u is *weakly differentiable* with *weak derivative* $\partial_t u := v$ and $\partial_t u \in L^q(0,T;H)$. This allows us to define *H-valued Sobolev spaces* as follows:

$$W^{1,p}(0,T;H) := \{u \in L^p(0,T;H)\colon \partial_t u \in L^p(0,T;H)\},$$

and again we set $H^1(0,T;H) := W^{1,2}(0,T;H)$. The latter can be endowed with the norm

$$\|u\|_{H^1(0,T;H)} := \left(\|u\|^2_{L^2(0,T;H)} + \|\partial_t u\|^2_{L^2(0,T;H)}\right)^{1/2},$$

which is associated to the obvious inner product and renders $H^1(0,T;H)$ a Hilbert space.

Vector-valued Sobolev spaces appear as trial and test spaces for weak variational formulations of spatiotemporal problems, see Section 2.7. Let $V \hookrightarrow H \simeq H^* \hookrightarrow V^*$ be a Gelfand triple — see for instance the 'dot spaces' at the end of Section 2.5 — and consider the spaces

$$\mathcal{X} := L^2(0,T;V) \quad \text{and} \quad \mathcal{Y} := L^2(0,T;V) \cap H^1(0,T;V^*),$$

where $\mathcal{X}$ is equipped with the standard Bochner 2-norm and $\mathcal{Y}$ with the norm

$$\|u\|_{\mathcal{Y}} := \left(\|u\|^2_{L^2(0,T;V)} + \|\partial_t u\|^2_{L^2(0,T;V^*)}\right)^{1/2},$$

which is again induced by an inner product which renders $\mathcal{Y}$ a Hilbert space. Since we have the continuous embedding $\mathcal{Y} \hookrightarrow \mathcal{C}([0,T];H)$, see [25, Chapter XVIII, §1, Theorem 1], pointwise evaluation makes sense so that we may define the closed subspaces

$$\mathcal{Y}_{0,\{0\}} := \{u \in \mathcal{Y}\colon u(0) = 0\} \quad \text{and} \quad \mathcal{Y}_{0,\{T\}} := \{u \in \mathcal{Y}\colon u(T) = 0\},$$

also equipped with the norm $\|\cdot\|_{\mathcal{Y}}$. Alternatively, we can endow $\mathcal{Y}_{0,\{T\}}$ with the equivalent norm

$$|||u|||_{\mathcal{Y}} := \left(\|u\|^2_{L^2(0,T;H)} + \|\partial_t u\|^2_{L^2(0,T;V^*)} + \|u(0)\|^2_H\right)^{1/2}.$$

These spaces and norms will be revisited in Section 2.7.

2.4. Semigroups of linear operators

For bounded operators $A \in \mathscr{L}(H)$, one defines the exponential operator e^{tA} for $t \in \mathbb{R}$ by the infinite sum

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \tag{2.9}$$

which converges absolutely in the $\mathscr{L}(H)$-norm by the convergence of the real exponential power series and the inequality $\|A^k\|_{\mathscr{L}(H)} \leqslant \|A\|_{\mathscr{L}(H)}^k$. The operator (or matrix) exponential naturally arises for instance in solutions to linear systems of ordinary differential equations.

2.4.1. Strongly continuous semigroups

If we could proceed similarly for unbounded operators $(A, \mathsf{D}(A))$, we would be able to solve partial differential equations such as the evolution equations which are discussed in Section 2.7. Unfortunately, definition (2.9) does not generalize to unbounded operators, since the defining series will generally not converge uniformly nor in the pointwise sense; however, to certain unbounded operators we can associate a family of bounded operators called a strongly continuous semigroup, which resembles the family $(e^{tA})_{t\geqslant 0}$ in the following sense.

Definition 2.4.1 (Strongly continuous semigroup)**.** A family $(S(t))_{t\geqslant 0}$ of bounded operators in $\mathscr{L}(H)$ is said to be a *strongly continuous semigroup* (or a *C_0-semigroup*) if it satisfies

(i) $S(0) = I$;

(ii) $S(t+s) = S(t)S(s)$ for all $t, s \geqslant 0$ (the *semigroup property*);

(iii) $\lim_{t\downarrow 0} \|S(t)x - x\|_H = 0$ for all $x \in H$ (*strong continuity*).

The aforementioned unbounded operator associated with a semigroup is defined as follows.

Definition 2.4.2 (Infinitesimal generator)**.** Given a C_0-semigroup $(S(t))_{t\geqslant 0}$, define the unbounded operator $A \colon \mathsf{D}(A) \subseteq H \to H$ by

$$\mathsf{D}(A) := \left\{ x \in H \colon \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \text{ exists} \right\},$$
$$Ax := \lim_{h\downarrow 0} \frac{S(h)x - x}{h}.$$

A is called the *(infinitesimal) generator* of $(S(t))_{t\geqslant 0}$.

In other words, Ax is defined to be the derivative at $t = 0$ of the *orbit* $t \mapsto S(t)x$, whenever $x \in H$ is such that this derivative exists. Note that a family of exponential operators $(e^{tA})_{t\geqslant 0}$ satisfies a similar property whenever it exists.

Next we collect some basic facts about C_0-semigroups, which will be used — sometimes without explicit mention — in the sequel.

Proposition 2.4.3 (Exponential boundedness)**.** *For any C_0-semigroup $(S(t))_{t\geqslant 0}$, there exist constants $M \geqslant 1$ and $w \in \mathbb{R}$ such that*

$$\|S(t)\|_{\mathscr{L}(H)} \leqslant Me^{wt} \quad \text{for all } t \geqslant 0.$$

Proof. See [57, Chapter 1, Theorem 2.2]. □

If $w = 0$ and $M = 1$ in the above proposition, we say that $(S(t))_{t\geqslant 0}$ is a *semigroup of contractions*.

Proposition 2.4.4. *Let $(S(t))_{t\geqslant 0}$ be a C_0-semigroup with generator A. Then it holds that*

(a) for all $x \in H$, the orbit $t \mapsto S(t)x$ is continuous on $[0, \infty)$;

(b) for all $x \in H$, we have $\int_0^t S(s)x \,\mathrm{d}s \in \mathsf{D}(A)$,

$$A \int_0^t S(s)x \,\mathrm{d}s = S(t)x - x$$

and both sides are equal to $\int_0^t S(s)Ax \,\mathrm{d}s$ whenever $x \in \mathsf{D}(A)$;

(c) *for all $x \in \mathsf{D}(A)$, the orbit $t \mapsto S(t)x$ is continuously differentiable on $[0, \infty)$, we have $S(t)x \in \mathsf{D}(A)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)x = AS(t)x = S(t)Ax;$$

(d) *A is closed and densely defined and determines the semigroup $(S(t))_{t \geqslant 0}$ uniquely.*

Proof. See Corollaries 2.3, 2.5 and Theorems 2.4, 2.7 in Chapter 1 of [57]. □

For the next result, which is sometimes useful in determining the domain of the generator $(A, \mathsf{D}(A))$ of a C_0-semigroup, we introduce the notion of a *core for A*, defined to be a subspace $D \subseteq \mathsf{D}(A)$ which is dense in $\mathsf{D}(A)$ with respect to its graph norm.

Proposition 2.4.5. *Let $(S(t))_{t \geqslant 0}$ be a C_0-semigroup with generator $(A, \mathsf{D}(A))$ on H. Any subspace $D \subseteq \mathsf{D}(A)$ which is dense in H with respect to $\| \cdot \|_H$ and invariant under $S(t)$, i.e. $S(t)D \subseteq D$ for all $t \geqslant 0$, is a core for A.*

Proof. See [30, Proposition II.1.7]. □

Theorem 2.4.6. *Let $(S(t))_{t \geqslant 0}$ be a C_0-semigroup on H and let $w \in \mathbb{R}, M \geqslant 1$ be as in Proposition 2.4.3. Then its generator $(A, \mathsf{D}(A))$ has the following properties.*

(a) *Whenever $\lambda \in \mathbb{C}$ is such that the integral $\int_0^\infty e^{-\lambda s} S(s) \,\mathrm{d}s$ exists as an improper Riemann integral converging in $\mathscr{L}(H)$, it holds that $\lambda \in \rho(A)$ and $R(\lambda, A)$ is given by this integral.*

(b) *For $\operatorname{Re} \lambda > w$, we have $\lambda \in \rho(A)$ with $R(\lambda, A)$ given by the integral from part (a) and $\| R(\lambda, A) \|_{\mathscr{L}(H)} \leqslant \frac{M}{\operatorname{Re}(\lambda) - w}$.*

Proof. See [30, Theorem II.1.10]. □

Some of the properties of generators which we encountered in the past few statements actually characterize the unbounded operators which generate strongly continuous semigroups. The first of these so-called *generation theorems* is the following. We state it separately for contraction semigroups before generalizing to arbitrary C_0-semigroups.

Theorem 2.4.7 (Hille–Yosida, contraction case)**.** *Let $(A, \mathsf{D}(A))$ be a closed and densely defined operator on H. Then the following statements are equivalent.*

(a) *$(A, \mathsf{D}(A))$ is the generator of a C_0-semigroup $(S(t))_{t \geqslant 0}$ of contractions;*

(b) *$(0, \infty) \subseteq \rho(A)$ and*

$$\| R(\lambda, A) \|_{\mathscr{L}(H)} \leqslant \frac{1}{\lambda} \quad \textit{for all } \lambda > 0;$$

(c) *$\{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda > 0\} \subseteq \rho(A)$ and*

$$\| R(\lambda, A) \|_{\mathscr{L}(H)} \leqslant \frac{1}{\operatorname{Re} \lambda} \quad \textit{for all } \operatorname{Re} \lambda > 0.$$

Proof. See [30, Theorem II.3.5]. □

From this, the general case can be deduced.

Theorem 2.4.8 (Hille–Yosida, general case)**.** *Let $(A, \mathsf{D}(A))$ be a closed and densely defined operator on H. Then the following statements are equivalent.*

(a) *$(A, \mathsf{D}(A))$ is the generator of a C_0-semigroup $(S(t))_{t \geqslant 0}$ such that $\|S(t)\|_{\mathscr{L}(H)} \leqslant M e^{wt}$ on $t \geqslant 0$ for some constants $w \in \mathbb{R}$ and $M \geqslant 1$;*

(b) *$(w, \infty) \subseteq \rho(A)$ and*

$$\| R(\lambda, A)^n \|_{\mathscr{L}(H)} \leqslant \frac{M}{(\lambda - w)^n} \quad \textit{for all } \lambda > w,\, n \in \mathbb{N};$$

(c) $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > w\} \subseteq \rho(A)$ *and*

$$\|R(\lambda, A)^n\|_{\mathscr{L}(H)} \leqslant \frac{M}{(\operatorname{Re}(\lambda) - w)^n} \quad \textit{for all } \operatorname{Re}\lambda > w,\ n \in \mathbb{N}.$$

Proof. See [30, Theorem II.3.8]. □

The following generation theorem characterizes generators of C_0-semigroups (of contractions) in terms of maximal accretiveness rather than resolvent bounds.

Theorem 2.4.9 (Lumer–Phillips)**.** *An unbounded operator $(A, \mathsf{D}(A))$ acting on H is maximal accretive if and only if $-A$ is the generator of a C_0-semigroup of contractions.*

Proof. A version for Banach spaces is stated and proved in [57, Chapter 1, Theorem 4.3]. To see that it reduces to the current formulation when the semigroup acts on a Hilbert space H, first note that by the Riesz representation theorem and Hahn–Banach theorem, the duality set of any given $h \in H$ is simply equal to $\{h\}$, i.e. $F(h) = \{h\}$ in the notation of [57]. Also note that it is formulated for (maximal) dissipative A, which is equivalent to $-A$ being (maximal) accretive. This shows that the definition of accretiveness used in [57] reduces to Definition 2.1.13 in the Hilbertian case.

To prove the 'only if' part of the theorem, we suppose that $(A, \mathsf{D}(A))$ is maximal accretive as defined in Definition 2.1.13 and first show that this implies that A (equivalently, $-A$) is densely defined. We do this by proving that $\mathsf{D}(A)$ has a trivial orthogonal complement; to this end, suppose that $f \in H$ is such that $(f, v)_H = 0$ for all $v \in \mathsf{D}(A)$. Then by maximal accretiveness there exists a $u \in \mathsf{D}(A)$ such that $Au + u = f$. But then

$$0 = \operatorname{Re}(f, u)_H = \operatorname{Re}(Au + u, u)_H = \|u\|_H^2 + \operatorname{Re}(Au, u)_H \geqslant \|u\|_H^2,$$

which implies $u = 0$ and thus $f = 0$. Applying part (a) of [57, Chapter 1, Theorem 4.3] to $-A$ with $\lambda_0 = 1$ then proves the forward implication.

For the 'if' part, let $-A$ be the generator of a C_0-semigroup and recall that generators of C_0-semigroups are always closed and densely defined, see Proposition 2.4.4(d). Now the implication follows from part (b) of [57, Chapter 1, Theorem 4.3] upon taking $\lambda = 1$. □

2.4.2. Analytic semigroups

The semigroups which we encounter in the problems studied in this work have better regularity properties than just strong continuity. The most relevant class of more regular semigroups is the class of *analytic semigroups*.

For the definition of this concept and the statement of some of its important properties, it is convenient to first define the open sector Σ_ω of angle $\omega \in (0, \pi]$ as

$$\Sigma_\omega := \{\lambda \in \mathbb{C} \setminus \{0\}\colon\ |\arg\lambda| < \omega\},$$

where we recall that the the argument is taken in $(-\pi, \pi]$.

Definition 2.4.10 (Analytic semigroup)**.** A C_0-semigroup $(S(t))_{t\geqslant 0}$ on H is called an *analytic semigroup on* Σ_ω (with $\omega \in (0, \frac{\pi}{2}]$) if the mapping $t \mapsto S(t)$ can be extended analytically to Σ_ω in such a way that the extension satisfies the semigroup property and

$$\lim_{\Sigma_{\omega'} \ni z \to 0} S(z)x = x \quad \text{for all } \omega' \in (0, \omega),\ x \in H.$$

If moreover

$$\sup_{z \in \Sigma_{\omega'}} \|S(z)\|_{\mathscr{L}(H)} < \infty \quad \text{for all } \omega' \in (0, \omega),$$

then the semigroup is called *bounded analytic*.

It turns out that the generators of bounded analytic semigroups can be characterized as negatives of the following class of operators.

Definition 2.4.11 (Sectorial operator)**.** A closed and densely defined operator $(A, \mathsf{D}(A))$ is called *sectorial of angle* $\omega \in (0, \pi)$ if $\sigma(A) \subseteq \overline{\Sigma}_\omega$ and for each $\omega' \in (0, \omega)$

$$\sup_{z \in \mathbb{C} \setminus \overline{\Sigma}_{\omega'}} \|zR(z, A)\|_{\mathscr{L}(H)} < \infty.$$

Sectorial operators then give rise to the following family of bounded operators.

Definition 2.4.12. Let $(A, \mathsf{D}(A))$ be a sectorial operator of angle $\omega \in (0, \pi/2)$. Define the bounded operators $S(0) := I$ and

$$S(z) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda z} (\lambda I + A)^{-1} \, \mathrm{d}\lambda, , \quad z \in \Sigma_{\frac{\pi}{2} - \omega}, \tag{2.10}$$

where Γ is the upwards oriented boundary of $\Sigma_{\pi - \omega'} \setminus B(0, 1)$ for some $\omega' \in (\omega, \pi)$.

Equation (2.10) is known as the inverse Laplace transform representation. As the notation suggests, the thus-defined family of bounded operators is in fact a bounded analytic semigroup with generator $-A$; conversely, every bounded analytic semigroup on Σ_ω has a generator which is ω-sectorial. In summary, we have the following.

Theorem 2.4.13. *Let* $(A, \mathsf{D}(A))$ *be an unbounded operator on* H *and let* $\omega \in (0, \pi/2)$*. Then the following statements are equivalent:*

(i) $-A$ *generates a bounded analytic semigroup on* Σ_ω*;*

(ii) A *is sectorial of angle* $\frac{\pi}{2} - \omega$*.*

These claims are proved in [30, Proposition II.4.3, Proposition II.4.4, Theorem II.4.6].

2.5. Fractional powers of operators

2.5.1. Definition and properties

In this section, we consider the question of how to define, given a closed *base operator* $(A, \mathsf{D}(A))$ on H and an exponent $\alpha \in \mathbb{R}$, the *fractional power* operator A^α in such a way that it behaves in some sense like a fractional power of a number. This turns out to be possible for a large class of base operators and spaces (e.g. closed operators on locally convex spaces) and exponents (e.g. complex powers $\alpha \in \mathbb{C}$); for more details on fractional powers on this level of generality, see [50].

We will limit ourselves to real powers and linear operators $(A, \mathsf{D}(A))$ on H such that $-A$ generates a C_0-semigroup $(S(t))_{t \geqslant 0}$ which is 'exponentially stable', i.e. there exist $M \geqslant 1$ and $w > 0$ such that

$$\|S(t)\|_{\mathscr{L}(H)} \leqslant M e^{-wt} \quad \text{for all } t \geqslant 0; \tag{2.11}$$

note the minus sign in the exponent. This assumption implies that $-A$ is closed and densely defined by part (d) of Proposition 2.4.4. Part (b) of Theorem 2.4.6 tells us that

$$\lambda \in \rho(-A) \quad \text{and} \quad \|(\lambda I + A)^{-1}\|_{\mathscr{L}(H)} \leqslant \frac{M}{w + \operatorname{Re} \lambda} \quad \text{for all } \operatorname{Re} \lambda > -w.$$

In particular, we have $0 \in \rho(A)$, which by definition of the resolvent means that A is boundedly invertible.

Note that this condition is satisfied when $-A$ is the generator of a bounded analytic semigroup with $0 \in \rho(A)$, since these two conditions imply that $A - wI$ is sectorial for some small enough $w > 0$, so that $-A + wI$ generates a bounded analytic semigroup by Theorem 2.4.13, which is given by $(e^{wt} S(t))_{t \geqslant 0}$ in this case. The uniform boundedness of $(e^{wt} S(t))_{t \geqslant 0}$, say with constant $M \geqslant 1$, then implies estimate (2.11).

This exponential stability of the semigroup guarantees the uniform convergence with respect to the operator norm topology of the following integral, which we use to define negative fractional powers of A:

$$A^{-\beta} := \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta - 1} S(t) \, \mathrm{d}t, \quad \beta > 0. \tag{2.12}$$

Note that the uniform convergence in $\mathscr{L}(H)$ of this integral implies that $A^{-\beta}$ is a bounded linear operator.

In addition to this semigroup representation of negative fractional powers, we also have the following equivalent representation which is valid for $0 < \beta < 1$:

$$A^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty t^{-\beta}(tI + A)^{-1}\,\mathrm{d}t, \quad \beta \in (0,1). \tag{2.13}$$

We remark that one cannot expect the above integral to converge for $\beta \geqslant 1$ in general, since even if $\|(tI + A)^{-1}\|_{\mathscr{L}(H)}$ is bounded near $t = 0$, the factor $t^{-\beta}$ grows too large as $t \to 0$, thus causing the integral to diverge.

Equation (2.13) is known as the *Balakrishnan formula*, named after A.V. Balakrishnan who studied fractional powers using similar representations, see [7]. Formula (2.13) can be derived from (2.12) using the representation of the resolvent given by Theorem 2.4.6,

$$(\lambda I + A)^{-1} = \int_0^\infty e^{-\lambda s} S(s)\,\mathrm{d}s \quad \text{for all } \operatorname{Re}\lambda > -w.$$

Substituting this representation into (2.13), using the Fubini theorem and changing variables then yields (2.12) for $\beta \in (0,1)$.

Now we wish to define arbitrary powers A^β for $\beta \in \mathbb{R}$. We set $A^0 := I$. It follows from $0 \in \rho(A)$ that A^{-1} is injective, and $A^{-\beta}$ inherits the injectivity so that we may define positive fractional powers of A as the unbounded linear operators $(A^\beta, \mathsf{D}(A^\beta))$ with $\mathsf{D}(A^\beta) := \mathsf{R}(A^{-\beta})$ and $A^\beta x := (A^{-\beta})^{-1}x$ for $x \in \mathsf{R}(A^{-\beta})$, where the latter denotes the left inverse of $A^{-\beta}$. We record this in the following definition.

Definition 2.5.1 (Fractional powers)**.** Let $(A, \mathsf{D}(A))$ be an unbounded operator on H whose negative is the generator of an exponentially stable semigroup $(S(t))_{t\geqslant 0}$. Let $\beta \in \mathbb{R}$ be given and

(i) for $\beta < 0$, define A^β through formula (2.12);

(i) for $\beta = 0$, set $A^\beta := I$;

(i) for $\beta > 0$ and $x \in \mathsf{R}(A^{-\beta}) =: \mathsf{D}(A^\beta)$, set $A^\beta x := (A^{-\beta})^{-1}x$.

The following theorem collects some important basic facts about the thus-defined fractional powers.

Theorem 2.5.2. *Let $(A, \mathsf{D}(A))$ be as in Definition 2.5.1. Then*

(a) for all $\beta > 0$, A^β is closed and densely defined;

(b) $\mathsf{D}(A^\alpha) \subseteq \mathsf{D}(A^\beta)$ whenever $\alpha \geqslant \beta > 0$;

(c) if $\alpha, \beta \in \mathbb{R}$, then $A^{\alpha+\beta}x = A^\alpha A^\beta x$ for every $x \in \mathsf{D}(A^{\max\{\alpha,\beta,\alpha+\beta\}})$;

(d) $(A^{-\beta})_{\beta\geqslant 0}$ is a C_0-semigroup of bounded linear operators.

Proof. See [57, Chapter 2, Corollary 6.5 and Theorem 6.8]. Note that Pazy assumes that $-A$ generates an analytic C_0-semigroup, but in fact only the resulting property (2.11) is used for the proofs of this theorem and the next. □

We have the following results regarding the interplay between the fractional powers A^β and the semigroup $(S(t))_{t\geqslant 0}$ generated by $-A$ in the case that $(S(t))_{t\geqslant 0}$ is bounded analytic.

Theorem 2.5.3. *Let $(A, \mathsf{D}(A))$ and $(S(t))_{t\geqslant 0}$ be as in Definition 2.5.1. If $(S(t))_{t\geqslant 0}$ is moreover a bounded analytic semigroup, then*

(a) $S(t)x \in \mathsf{D}(A^\beta)$ for all $t > 0$ and $\beta \geqslant 0$;

(b) $S(t)A^\beta x = A^\beta S(t)x$ for all $x \in \mathsf{D}(A^\beta)$, $t \geqslant 0$ and $\beta \geqslant 0$;

(c) if $\beta \geqslant 0$, the operator $A^\beta S(t)$ is bounded for all $t > 0$ and there exist constants $M_\beta \geqslant 1$, $w > 0$ such that

$$\|A^\beta S(t)\|_{\mathscr{L}(H)} \leqslant M_\beta t^{-\beta} e^{-wt} \quad \textit{for all } t > 0;$$

(d) for $0 < \beta \leqslant 1$ and $x \in \mathsf{D}(A^\beta)$ there exists a constant $C_\beta \geqslant 0$ such that

$$\|S(t)x - x\|_H \leqslant C_\beta t^\beta \|A^\beta x\|_H \quad \textit{for all } t \geqslant 0.$$

Proof. See [57, Chapter 2, Theorem 6.13] and the remark in the proof of Theorem 2.5.2. □

Now we consider the question of whether $-A^\beta$ generates a C_0-semigroup. To this end, we consider the sectoriality of fractional powers of sectorial operators.

Theorem 2.5.4. *Let $(A, \mathsf{D}(A))$ satisfy the assumptions in Definition 2.5.1 and suppose furthermore that A is sectorial of angle $\omega \in (0, \pi)$. If $\beta > 0$ is such that $\beta\omega < \pi$, then A^β is sectorial of angle less than or equal to $\beta\omega$.*

Proof. See [50, Theorem 5.4.1]. In order for this theorem to be applicable, we must first verify that the definition of fractional powers introduced in Chapter 5 of [50] is equivalent to Definition 2.5.1 whenever the base operator A satisfies the assumptions of this section.

If, as we assume, $0 \in \rho(A)$, then the authors similarly define positive fractional powers through the inverse of negative ones [50, Definition 5.1.2]; therefore, it suffices to check that their formulas for negative fractional powers coincide with (2.12).

Let $\tilde{A}^\beta$, $\beta \in \mathbb{R}$ denote the fractional powers defined as in [50]. The negative fractional powers $\tilde{A}^{-\beta}$ are obtained by applying a positive-power Balakrishnan formula to the bounded operator A^{-1}, which for $0 < \beta < 1$ yields

$$\tilde{A}^{-\beta}x = \frac{\sin \pi\beta}{\pi} \int_0^\infty t^{\beta-1}(tI + A^{-1})^{-1}A^{-1}x \,\mathrm{d}t, \quad x \in \mathsf{D}(A^{-1}) = H,$$

see [50, Definition 5.1.1 and Definition 3.1.1(i)]. Using properties of the inverse and the change of variables $s := t^{-1}$, we see that

$$\begin{aligned}
\tilde{A}^{-\beta} &= \frac{\sin \pi\beta}{\pi} \int_0^\infty t^{\beta-1}(tI + A^{-1})^{-1}A^{-1} \,\mathrm{d}t \\
&= \frac{\sin \pi\beta}{\pi} \int_0^\infty t^{\beta-1}(tA + I)^{-1} \,\mathrm{d}t \\
&= \frac{\sin \pi\beta}{\pi} \int_0^\infty s^{-\beta}(A + sI)^{-1} \,\mathrm{d}s,
\end{aligned}$$

i.e. we recover (2.13) and equivalently (2.12), thus $A^{-\beta} = \tilde{A}^{-\beta}$ for $0 < \beta < 1$. If $n < \beta < n+1$ for some $n \in \mathbb{N}$, then for all $x \in H$ we have

$$\tilde{A}^{-\beta}x := \tilde{A}^{n-\beta}A^{-n}x = A^{n-\beta}A^{-n}x = A^{-\beta}x,$$

where we used [50, Definition 3.1.1(iii)] followed by the case $0 < \beta < 1$ and Theorem 2.5.2(c). We have now shown that the mappings $\beta \mapsto \tilde{A}^{-\beta}$ and $\beta \mapsto A^{-\beta}$ agree on the dense subset $(0, \infty) \setminus \mathbb{N}$ of $[0, \infty)$. Since they are both strongly continuous, respectively by [50, Proposition 3.1.1] and Theorem 2.5.2(d), we obtain $A^{-\beta} = \tilde{A}^{-\beta}$ for all $\beta \geqslant 0$, which finishes the proof. □

In light of Theorem 2.4.13, we have as a corollary that $-A^\beta$ generates a bounded analytic semigroup if $\beta\omega < \pi/2$.

2.5.2. Dot spaces and Gelfand triples

Next we introduce for $\theta \geqslant 0$ the spaces $\dot{H}^\theta_A := \mathsf{D}(A^{\theta/2})$, which can be made a Hilbert space by equipping it with the inner product

$$(u, v)_{\dot{H}^\theta_A} := (A^{\theta/2}u, A^{\theta/2}v)_H.$$

Combining some of the previously mentioned properties of fractional power operators, it follows that for $0 \leqslant \eta < \theta$, the inclusion map from $\dot{H}^\theta_A$ to $\dot{H}^\eta_A$ is a dense continuous embedding. The negative-exponent spaces can be defined by $\dot{H}^{-\theta}_A := (\dot{H}^\theta_A)^*$ for $\theta \geqslant 0$. Then the map which restricts bounded linear functionals on $\dot{H}^\eta_A$ to $\dot{H}^\theta_A$ is again a continuous dense embedding. Finally, one can identify H with its dual via the Riesz representation theorem to obtain

$$\dot{H}^\theta_A \hookrightarrow \dot{H}^\eta_A \hookrightarrow H \simeq H^* \hookrightarrow \dot{H}^{-\eta}_A \hookrightarrow \dot{H}^{-\theta}_A.$$

More generally, a triplet

$$V \hookrightarrow H \hookrightarrow V^*,$$

where both embeddings are continuous and dense, is called a *Gelfand triple*. It does not make sense to identify V with its dual just as we did for H, but often it can be identified with a larger space containing H. More precisely, if we let $V' := \mathscr{L}(V;\mathbb{K})$ denote the actual dual of V, then we may often identify V' with some space $V^* \supset H$ in such a way that $g \in V^*$ if and only if $\langle g, \cdot\rangle \in V'$.

Example 2.5.5. Consider $V := \dot{H}^1_A$ and suppose that A is self-adjoint. To any given $v \in V$ we can naturally associate the functional

$$\langle Av, w\rangle := (A^{1/2}v, A^{1/2}w)_H = (v,w)_{\dot{H}^1_A}, \quad w \in V.$$

It turns out that the map $v \mapsto \langle Av, \cdot\rangle$ is an isometric isomorphism between V and V'. Equipping H with the norm $|||h|||_H := \|\langle AA^{-1}h, \cdot\rangle\|_{V'}$, it follows that we can extend the (unbounded) operator $A\colon \dot{H}^2_A \subset H \to H$ by density of $\dot{H}^2_A \hookrightarrow \dot{H}^1_A$ to an isomorphism $A\colon V \to V^*$, where V^* is defined as the completion of H with respect to the norm $|||\cdot|||_H$.

2.6. Elliptic second-order differential operators

The aim of this section is to define self-adjoint uniformly elliptic second-order differential operators and summarize some of their key properties. These operators are important because they arise in many situations, including the Whittle–Matérn SPDEs described in the introduction. Moreover, they are concrete nontrivial examples of operators which can be associated with analytic semigroups; in fact, we will see that the same holds for arbitrary positive fractional powers of such operators.

Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded, open and connected domain. Assume that

(i) $\mathbf{A} \in L^\infty(\mathcal{D};\mathbb{R}^{d\times d})$ is symmetric a.e. and satisfies the following *uniform ellipticity* condition:

$$\exists \theta > 0\colon \quad \xi^T\mathbf{A}(x)\xi \geqslant \theta\|\xi\|^2_{\mathbb{R}^d} \quad \text{for all } \xi \in \mathbb{R}^d \text{ and almost all } x \in \mathcal{D};$$

(ii) $\kappa \in L^\infty(\mathcal{D})$.

Define the bilinear form $\mathfrak{a}\colon H^1_0(\mathcal{D}) \times H^1_0(\mathcal{D}) \to \mathbb{R}$ by

$$\mathfrak{a}(u,v) := \int_{\mathcal{D}} [(\mathbf{A}(x)\nabla u(x), \nabla v(x))_{\mathbb{R}^d} + \kappa^2(x)u(x)v(x)]\,\mathrm{d}x, \quad u,v \in H^1_0(\mathcal{D}). \tag{2.14}$$

Proposition 2.6.1. *Let* $\mathfrak{a}$ *be as in (2.14) and define* $\|u\|_{\mathfrak{a}} := (\mathfrak{a}(u,u) + \|u\|^2_{L^2(\mathcal{D})})^{\frac{1}{2}}$ *for* $u \in H^1_0(\mathcal{D})$*. Then*

(a) $\mathfrak{a}$ *is* densely defined*, i.e.* $H^1_0(\mathcal{D}) \subset L^2(\mathcal{D})$ *is dense;*

(b) $\mathfrak{a}$ *is* symmetric*, i.e.* $\mathfrak{a}(u,v) = \mathfrak{a}(v,u)$ *for all* $u,v \in H^1_0(\mathcal{D})$*;*

(c) $\mathfrak{a}$ *is* accretive *(or* positive*), i.e.* $\mathfrak{a}(u,u) \geqslant 0$ *for all* $u \in H^1_0(\mathcal{D})$*;*

(d) $\mathfrak{a}$ *is* continuous*, i.e. there exists a constant* $C \geqslant 0$ *such that* $\mathfrak{a}(u,v) \leqslant C\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}}$ *for all* $u,v \in H^1_0(\mathcal{D})$*;*

(e) $\mathfrak{a}$ *is* coercive*, i.e. there exists a constant* $\delta > 0$ *such that* $\mathfrak{a}(u,u) \geqslant \delta\|u\|^2_{H^1(\mathcal{D})}$ *for all* $u \in H^1_0(\mathcal{D})$*.*

(f) $\mathfrak{a}$ *is* closed*, i.e. the space* $(H^1_0(\mathcal{D}), \|\cdot\|_{\mathfrak{a}})$ *is complete.*

Proof. The density of $H_0^1(\mathcal{D}) \subset L^2(\mathcal{D})$ with respect to the $L^2(\mathcal{D})$ norm is well-known, yielding (a). Statements (b)–(e) can be shown by estimates of the bilinear form; we will not show this in detail but rather only mention which assumptions are needed to derive each statement.

Part (b) follows from the assumption that $\mathbf{A}(x)$ is symmetric almost everywhere on $\mathcal{D}$. The proof of part (c) uses the uniform ellipticity assumption, which in particular implies that $\mathbf{A}(x)$ is positive definite a.e. Proving part (d) involves estimates using Cauchy–Schwarz and the boundedness of A and κ^2. For part (e), we need the Poincaré inequality, which holds since $\mathcal{D}$ is bounded, as well as the uniform ellipticity.

To show part (f), one proves that the norms $\|\cdot\|_{H^1(\mathcal{D})}$ and $\|\cdot\|_{\mathfrak{a}}$ are equivalent; this is sufficient since $H_0^1(\mathcal{D})$ is a closed subspace of $(H^1(\mathcal{D}), \|\cdot\|_{H^1(\mathcal{D})})$, so that $(H_0^1(\mathcal{D}), \|\cdot\|_{H^1(\mathcal{D})})$ is complete and the equivalence of $\|\cdot\|_{H^1(\mathcal{D})}$ and $\|\cdot\|_{\mathfrak{a}}$ would give the result. The equivalence is shown by using estimates similar to the continuity proof to prove $\|u\|_{\mathfrak{a}} \leqslant M\|u\|_{H^1(\mathcal{D})}$, while using uniform ellipticity to prove $\|u\|_{\mathfrak{a}} \geqslant m\|u\|_{H^1(\mathcal{D})}$, for some positive constants $m, M \geqslant 0$. $\square$

Next we introduce *the operator associated with* $\mathfrak{a}$, i.e. the unbounded linear operator $(L, \mathsf{D}(L))$ whose domain is defined by

$$\mathsf{D}(L) := \left\{u \in H_0^1(\mathcal{D})\colon \exists C \geqslant 0 \text{ such that } \left|\mathfrak{a}(u,v)\right| \leqslant C\|v\|_{L^2(\mathcal{D})} \ \ \forall v \in H_0^1(\mathcal{D})\right\}$$

and for all $u \in \mathsf{D}(L)$, the value Lu is defined as the unique element of $L^2(\mathcal{D})$ which satisfies

$$(Lu, v)_{L^2(\mathcal{D})} = \mathfrak{a}(u,v) \quad \forall v \in H_0^1(\mathcal{D});$$

such an element exists by the Riesz representation theorem and the density of $H_0^1(\mathcal{D})$ in $L^2(\mathcal{D})$.

The various properties listed in Proposition 2.6.1 correspond to analogous properties of the associated operator L. The symmetry of $\mathfrak{a}$ implies that L is self-adjoint [56, Proposition 1.24]. Moreover, $-L$ is the generator of a strongly continuous semigroup of contractions since $\mathfrak{a}$ is also densely defined, continuous and closed [56, Proposition 1.51]. In fact, we have that L is sectorial of arbitrarily small spectral angle $\omega > 0$ [56, Theorem 1.54], so that $-L$ generates a bounded analytic semigroup on any sector with angle strictly less than $\frac{\pi}{2}$. Hence, given an exponent $\beta > 0$, we can take $\omega > 0$ arbitrarily small while appealing to Theorem 2.5.4 and it follows that $-L^\beta$ generates an analytic semigroup as well.

By the Lax–Milgram lemma [56, Lemma 1.3], the coercivity of $\mathfrak{a}$ implies that L has an inverse which may be extended to a bounded operator $L^{-1}\colon H_0^1(\mathcal{D}) \to H_0^1(\mathcal{D})$. By the Rellich–Kondrachov compact embedding theorem, L^{-1} is compact [36, Theorem 6.3.1]. In view of the spectral theorem for self-adjoint compact operators (Theorem 2.1.11), we conclude that L has an orthonormal eigenbasis $(\lambda_j, e_j)_{j\geqslant 1}$ whose sequence of eigenvalues $(\lambda_j)_{j\geqslant 1}$ is positive and has no accumulation point so that it may be presumed to be in increasing order. In fact, we can be more precise about the eigenvalues of L in this particular situation: we have the following asymptotic relation, known as *Weyl's law*.

Theorem 2.6.2 (Weyl's law)**.** *Let L be the operator associated with the bilinear form $\mathfrak{a}$ defined in (2.14). Then there exist constants $c, C \geqslant 0$, depending only on the coefficient functions A, κ^2 and the domain $\mathcal{D} \subset \mathbb{R}^d$, such that the eigenvalues $(\lambda_j)_{j\geqslant 1}$ of L satisfy*

$$cj^{\frac{2}{d}} \leqslant \lambda_j \leqslant Cj^{\frac{2}{d}} \quad \forall j \in \mathbb{N}.$$

Proof. See [26, Theorem 6.3.1]. $\square$

2.7. Deterministic evolution equations

In this section, we discuss the so-called *inhomogeneous abstract Cauchy problem* (IACP):

$$\begin{aligned} u'(t) + Au(t) &= f(t), \quad t \in (0,T]; \\ u(0) &= u_0. \end{aligned} \tag{2.15}$$

Here we suppose that $(A, \mathsf{D}(A))$ is a linear operator on H whose negative generates an analytic C_0 semigroup $(S(t))_{t\geqslant 0}$. We assume that the initial value u_0 belongs to H. The right-hand side f belongs to the Bochner space $L^1(0,T;H)$. The interpretation of the derivative u' may differ throughout the section.

2.7.1. Solution concepts

In this section we introduce a variety of solution concepts for the IACP. The first subsection is concerned with three solution concepts which are most commonly considered in the context of evolution equations: classical, strong and mild solutions. After that, we cover the variational formulation of the IACP, which gives rise to the concept of weak solutions. Generally speaking, the first three solution concepts are more suited to the analysis of qualitative properties of the SPDE such as well-posedness, whereas the alternative variational viewpoint can be especially useful for numerical analysis, see [62, 63].

Classical, strong and mild solutions

Definition 2.7.1 (Classical solution)**.** We call u a *classical solution* to the IACP if

$$u \in C([0,T);H) \cap C((0,T);\mathsf{D}(A)),$$

its (classical) derivative satisfies $u' \in C((0,T);H)$ and (2.15) is satisfied pointwise.

For $f = 0$ it can be shown that $u(t) = S(t)u_0$ is a solution, which has the proper regularity to be a classical solution owing to the smoothing properties of analytic semigroups; in fact, it is the only such solution. For general $f \in L^1(0,T;H)$, a classical solution often does not exist. Indeed, f needs to be continuous in order to even make sense of pointwise evaluation of the evolution equation, but even for $f \in C((0,T);H)$ one can construct situations in which a classical solution does not exist. This suggests the need for a less restrictive solution concept in order to define solutions to the IACP.

We can formally integrate the evolution equation to obtain

$$u(t) = u_0 - \int_0^t Au(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s.$$

Based on the properties that a function u satisfying the above equation must have, we can formulate the following solution concept.

Definition 2.7.2 (Strong solution)**.** We call u a *strong solution* to the IACP if $u \in C([0,T);H)$, $u(0) = u_0$ and u is (classically) differentiable a.e. with $u' \in L(0,T;H)$, takes its values in $\mathsf{D}(A)$ a.e. and satisfies the evolution equation in (2.15) a.e.

This solution concept is less restrictive than that of a classical solution. Consequently, it turns out that the IACP has a unique solution whenever f is locally α-Hölder continuous with $\alpha > 0$, but not necessarily for general $f \in L^1(0,T;H)$. To mitigate this, we introduce the next solution concept.

First note that $t \mapsto S(t)u_0$ solves the following integrated form of the IACP for $f = 0$:

$$u(t) = u_0 - A\int_0^t u(s)\,\mathrm{d}s, \quad t \in [0,T);$$

this follows from the interplay between integration and semigroups noted in Proposition 2.4.4(b). Based on this observation, it makes sense to seek, given an $f \in L^1(0,T;H)$, a $u \in C([0,T);H)$ such that for all $t \in (0,T]$ we have $\int_0^t u(s)\,\mathrm{d}s \in \mathsf{D}(A)$ and

$$u(t) = u_0 - A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s. \tag{2.16}$$

It turns out that this is uniquely solved by the following variation of constants formula:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,\mathrm{d}s. \tag{2.17}$$

Definition 2.7.3 (Mild solution)**.** We call u a *mild solution* to the IACP if $u \in C([0,T);H)$, $u(0) = u_0$ and for all $t \in (0,T]$ we have $\int_0^t u(s)\,\mathrm{d}s \in \mathsf{D}(A)$ and u satisfies the integral equation (2.16). For problem (2.15) where $-A$ generates an analytic C_0-semigroup and $f \in L^1(0,T;H)$, the unique mild solution is given by (2.17).

More details on classical, strong and mild solutions can be found for instance in [57, Section 4.2].

Weak solutions

As mentioned at the beginning of this section, we can also take the variational approach to making sense of solutions to (2.15): roughly speaking, we associate to (2.15) a suitable bilinear form b and functional ℓ and we call u a solution to the variational problem if $b(u,v) = \ell(v)$ for all test functions v. The function space in which we seek the solution is called the *trial space*, whereas the space consisting of test functions is called the *test space*; they are both generally Bochner or (vector-valued) Sobolev spaces. Each choice of trial and test spaces, bilinear form b and functional ℓ then gives rise to a different notion of a weak variational solution.

We will now make these notions more precise. Assume for the remainder of this section that A satisfies some stronger assumptions than in the previous subsection: we assume that A is self-adjoint, positive and has a compact inverse. Recall from Section 2.5.2 the spaces $\dot{H}^\theta_A$ and set $V := \dot{H}^1_A$ so that $V^* = H^{-1}_A$ and we have the Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$. Note that A can now also be considered a bounded operator $A \in \mathscr{L}(V; V^*)$ instead of an unbounded operator $A \colon \mathsf{D}(A) \subset H \to H$. From Section 2.3.3, recall the spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Y}_{0,\{0\}}$ and $\mathcal{Y}_{0,\{T\}}$.

For the first variational formulation, we take $\mathcal{Y}$ as our trial space and $\mathcal{X}$ as the test space. Assuming for the moment that $u_0 = 0$, the trial space is more specifically $\mathcal{Y}_{0,\{0\}} \subset \mathcal{Y}$. Note that $\partial_t u + Au \in L^2(0,T;V^*)$, so that for a given $f \in L^2(0,T;V^*)$ and $v \in L^2(0,T;V) =: \mathcal{X}$ we can 'multiply' both sides of the equation by v, in the sense that we apply the duality pairing of $L^2(0,T;V)$ and $L^2(0,T;V^*)$. This gives rise to the following variational formulation:

$$\text{Find } u \in \mathcal{Y}_{0,\{0\}} \text{ such that } b_0(u,v) = \ell_0(v) \text{ for all } v \in \mathcal{X},$$

where the bilinear form $b_0 \colon \mathcal{Y}_{0,\{0\}} \times \mathcal{X} \to \mathbb{R}$ and the functional $\ell_0 \colon \mathcal{X} \to \mathbb{R}$ are respectively defined by

$$\begin{aligned} b_0(u,v) &:= \int_0^T \langle \partial_t u(t) + Au(t), v(t) \rangle \, \mathrm{d}t, && u \in \mathcal{Y}_{0,\{0\}},\ v \in \mathcal{X}; \\ \ell_0(v) &:= \int_0^T \langle f(t), v(t) \rangle \, \mathrm{d}t, && v \in \mathcal{X}. \end{aligned} \tag{2.18}$$

This approach can be extended to allow for nonzero u_0 — see [62] — but this will not be covered in this section.

To derive an alternative variational formulation, which works for arbitrary $u_0 \in H$, consider the case that u and v are smooth enough so that pointwise evaluation makes sense and the following integration by parts formula for V^*-valued distributional derivatives holds:

$$\int_0^T \langle \partial_t u(t), v(t) \rangle \, \mathrm{d}t = -\int_0^T \langle u(t), \partial_t v(t) \rangle \, \mathrm{d}t + (u(T), v(T))_H - (u(0), v(0))_H.$$

In the context of the IACP, we require that $u(0) = u_0$, which implies that $t \mapsto u(t) \in H$ should at least be continuous in a neighborhood of $t = 0$ in order for this requirement to make sense. If we furthermore assume $u \in \mathcal{X}$ and $v \in \mathcal{Y}_{0,\{T\}}$, where the latter also has a pointwise meaning as noted in Section 2.3.3, then we obtain

$$\int_0^T \langle \partial_t u(t), v(t) \rangle \, \mathrm{d}t = -\int_0^T \langle u(t), \partial_t v(t) \rangle \, \mathrm{d}t - (u_0, v(0))_H.$$

Using the self-adjointness of A, this implies that

$$\int_0^T \langle \partial_t u(t) + Au(t), v(t) \rangle \, \mathrm{d}t = \int_0^T \langle u(t), Av(t) - \partial_t v(t) \rangle \, \mathrm{d}t - (u_0, v(0))_H.$$

These observations motivate the definition of the following bilinear form $b \colon \mathcal{X} \times \mathcal{Y}_{0,\{T\}} \to \mathbb{R}$ and functional $\ell \colon \mathcal{Y}_{0,\{T\}} \to \mathbb{R}$:

$$\begin{aligned} b(u,v) &:= \int_0^T \langle u(t), Av(t) - \partial_t v(t) \rangle \, \mathrm{d}t, && u \in \mathcal{X},\ v \in \mathcal{Y}_{0,\{T\}}; \\ \ell(v) &:= \int_0^T \langle f(t), v(t) \rangle \, \mathrm{d}t + (u_0, v(0))_H, && v \in \mathcal{Y}_{0,\{T\}}. \end{aligned}$$

The associated variational formulation is

$$\text{Find } u \in \mathcal{X} \text{ such that } b(u,v) = \ell(v) \text{ for all } v \in \mathcal{Y}_{0,\{T\}}. \tag{2.19}$$

Note that (2.19) moves some regularity requirements from the trial functions to the test functions less as compared to (2.18). Therefore, solutions to (2.18) are sometimes called *strong variational solutions* in contrast to solutions to (2.19) which are called *weak variational solutions*. Note that the weak variational formulation does not place more stringent assumptions on f and u_0 than the mild solution concept; indeed, any $u_0 \in H$ is permitted and the functional ℓ associated to a given $f \in L^1(0,T;H)$ belongs to $\mathcal{Y}'$.

For the remainder of this section, we focus on the weak variational problem, noting that the other variational problems satisfy analogous properties to the ones which we cover for the weak variational problem.

An important property to establish is the well-posedness of the variational problem. To this end, we interpret the bilinear form as an operator from the trial space to the dual of the test space. I.e., we define $B \in \mathscr{L}(\mathcal{X};\mathcal{Y}'_{0,\{T\}})$ by $(Bu)(v) := b(u,v)$ for $u \in \mathcal{X}, v \in \mathcal{Y}_{0,\{T\}}$. Solving (2.19) then amounts to inverting B and computing $u = B^{-1}\ell$. Consequently, the bounded invertibility of B implies the well-posedness of the weak variational problem. This, in turn, is equivalent to the following there conditions on the bilinear form:

$$\|b\| := \sup_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}_{0,\{T\}}\setminus\{0\}} \frac{\left|b(u,v)\right|}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} < \infty \qquad (\textit{continuity}); \tag{2.20}$$

$$\beta := \inf_{u\in\mathcal{X}\setminus\{0\}} \sup_{v\in\mathcal{Y}_{0,\{T\}}\setminus\{0\}} \frac{\left|b(u,v)\right|}{\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}} > 0 \qquad (\textit{inf-sup condition}); \tag{2.21}$$

$$\forall v \in \mathcal{Y}_{0,\{T\}} \setminus \{0\}\colon \sup_{u\in\mathcal{X}\setminus\{0\}} \left|b(u,v)\right| > 0 \qquad (\textit{surjectivity}).$$

This result is known as the Banach–Nečas–Babuška theorem, see for instance [3]. For the weak variational problem and A satisfying some further conditions on its associated bilinear form $\mathfrak{a}(\,\cdot\,,\,\cdot\,)\colon V\times V \to \mathbb{R}$, these properties are shown to hold in [63, Theorem 2.2]. Hence in this situation, $B \in \mathscr{L}(\mathcal{X};\mathcal{Y}'_{0,\{T\}})$ is boundedly invertible and the weak variational problem is well-posed. If we equip $\mathcal{Y}$ with the equivalent norm $|||\cdot|||_{\mathcal{Y}}$ introduced in Section 2.3.3, then it can be derived from these results that B is in fact an isometric isomorphism.

2.7.2. Maximal regularity

Given a $p \in (1,\infty)$, we say that an operator $(A, \mathsf{D}(A))$ on H has *maximal L^p-regularity* if the abstract Cauchy problem has a unique solution $u \in L^p(0,T;\mathsf{D}(A)) \cap W^{1,p}_{0,\{0\}}(0,T;H)$, in the pointwise a.e. sense, for $u_0 = 0$ and any given $f \in L^p(0,T;H)$. We have the following result.

Theorem 2.7.4 (Maximal L^p-regularity)**.** *Let H be a Hilbert space and let $(A, \mathsf{D}(A))$ be an unbounded linear operator on H such that $-A$ generates an analytic C_0-semigroup. Then A has maximal L^p-regularity for $1 < p < \infty$.*

Proof. See [45, Corollary 1.7]. □

2.8. Hilbertian stochastic calculus

This section is devoted to setting up the stochastic theory needed to make sense of SPDE (1.3). In this section U, H and $\tilde{H}$ denote separable Hilbert spaces over the real scalar field. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space which is fixed throughout the section.

2.8.1. Hilbert-space-valued random variables

In this subsection, we collect some basic notions regarding H-valued random variables. A mapping $X\colon (\Omega,\mathcal{F},\mathbb{P}) \to (H, \mathcal{B}(H))$ is said to be a *random variable* if it is measurable. The measure $\mathbb{P}\circ X^{-1}$ defined by

$$(\mathbb{P}\circ X^{-1})(B) := \mathbb{P}(X^{-1}[B]) := \mathbb{P}(\{\omega\in\Omega\colon X(\omega)\in B\}), \quad B \in \mathcal{B}(H)$$

is called the *distribution* of X. Whenever a random variable X is $\mathbb{P}$-Bochner integrable, the *expected value* or *expectation* is defined as the Bochner integral

$$\mathbb{E}[X] := \int_\Omega X(\omega)\,\mathrm{d}\mathbb{P}(\omega).$$

The *characteristic function* φ_X of X is defined as

$$\varphi_X(h) := \mathbb{E}[e^{i(X,h)_H}], \quad h \in H,$$

and it completely characterizes the distribution of X.

If X is square integrable, i.e., if $X \in L^2(\Omega; H)$, then we can introduce the notion of its *covariance*. There are multiple ways to define the covariance of X. Using the Hilbert tensor product space introduced in Section 2.2, we define the covariance $\mathrm{Cov}(X) \in H^{(2)}$ by

$$\mathrm{Cov}(X) := \mathbb{E}[(X - \mathbb{E}[X]) \otimes (X - \mathbb{E}[X])].$$

Note the analogy of this definition with the scalar-valued case, where the tensor product reduces to multiplication of scalars.

A closely related concept is that of the *covariance operator*: a non-negative, self-adjoint, trace-class operator $Q_X \in \mathscr{L}_1(H)$ which satisfies

$$(\mathrm{Cov}(X), \varphi \otimes \psi)_{H^{(2)}} = (Q_X \varphi, \psi)_H \quad \text{for all } \varphi, \psi \in H. \tag{2.22}$$

It holds that a unique covariance operator can be associated to every $X \in L^2(\Omega; H)$. Note that by the definitions of $\mathrm{Cov}(X)$ and the inner product on $H^{(2)}$, we have

$$\begin{aligned}(\mathrm{Cov}(X), \varphi \otimes \psi)_{H^{(2)}} &= \mathbb{E}[((X - \mathbb{E}[X]) \otimes (X - \mathbb{E}[X]), \varphi \otimes \psi)_{H^{(2)}}] \\ &= \mathbb{E}[(X - \mathbb{E}[X], \varphi)_H (X - \mathbb{E}[X], \psi)_H],\end{aligned}$$

so that (2.22) becomes

$$\mathbb{E}[(X - \mathbb{E}[X], \varphi)_H (X - \mathbb{E}[X], \psi)_H] = (Q_X \varphi, \psi)_H. \tag{2.23}$$

Whenever it is obvious from context to which random variable we refer, the subscript X may be omitted from Q_X.

In applications, it is sometimes useful to consider another viewpoint of the covariance concept, namely the *covariance kernel*. It is defined for the frequently occurring situation when $H = L^2(\mathcal{D})$, where $\mathcal{D} \subseteq \mathbb{R}^d, d \in \mathbb{N}$. Since covariance operators are of trace class, they are in particular Hilbert–Schmidt, so that there exists a unique *kernel* $\varrho_X \in L^2(\mathcal{D} \times \mathcal{D})$ which satisfies

$$[Q_X f](y) = \int_{\mathcal{D}} \varrho_X(x, y) f(x)\,\mathrm{d}x \quad \text{for all } f \in L^2(\mathcal{D}) \text{ and almost all } y \in \mathcal{D};$$

see for instance [17, Section 3]. Combining this observation with (2.22) and (2.23), we see that

$$\begin{aligned}(Q_X f, g)_{L^2(\mathcal{D})} &= \int_{\mathcal{D}} \int_{\mathcal{D}} \varrho_X(x, y) f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \mathbb{E}[(X - \mathbb{E}[X], f)_{L^2(\mathcal{D})} (X - \mathbb{E}[X], g)_{L^2(\mathcal{D})}] \\ &= \mathbb{E}\Big[\int_{\mathcal{D}} (X(x) - \mathbb{E}[X(x)]) f(x)\,\mathrm{d}x \int_{\mathcal{D}} (X(y) - \mathbb{E}[X(y)]) g(y)\,\mathrm{d}y\Big].\end{aligned} \tag{2.24}$$

If ϱ_X were continuous and if we could validate the step

$$\begin{aligned}&\mathbb{E}\Big[\int_{\mathcal{D}} (X(x) - \mathbb{E}[X(x)]) f(x)\,\mathrm{d}x \int_{\mathcal{D}} (X(y) - \mathbb{E}[X(y)]) g(y)\,\mathrm{d}y\Big] \\ &\quad = \int_{\mathcal{D}} \int_{\mathcal{D}} \mathbb{E}[(X(x) - \mathbb{E}[X(x)])(X(y) - \mathbb{E}[X(y)])] f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

then a comparison with the first line of (2.24) would yield the identity

$$\varrho_X(x,y) = \mathbb{E}[(X(x) - \mathbb{E}[X(x)])(X(y) - \mathbb{E}[X(y)])] \quad \text{for all } x, y \in \mathcal{D},$$

which is the definition of the covariance function for spatial random fields $(X(x))_{x\in\mathcal{D}}$. Thus we see the connection between this concrete definition, which was also used in Chapter 1, and the more abstract definitions introduced in this chapter. It can also be seen from this identity that a covariance function is positive semi-definite, i.e. $\varrho_X(x,x) \geqslant 0$ for $x \in \mathcal{D}$.

2.8.2. Gaussian measures and random variables

Definition 2.8.1 (Gaussian measure on $\mathbb{R}$)**.** The *Gaussian measure* μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with mean $m \in \mathbb{R}$ and variance $\sigma^2 > 0$ is defined by

$$\mu(A) = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \mathrm{d}x, \quad A \in \mathcal{B}(\mathbb{R}).$$

For $\sigma^2 = 0$, we define μ by

$$\mu(A) = \begin{cases} 1 & \text{if } m \in A, \\ 0 & \text{if } m \notin A. \end{cases}$$

In this case, we say that the Gaussian measure is *degenerate*.

Definition 2.8.2. A real-valued random variable X is called a *Gaussian random variable* if its distribution is a Gaussian measure on $\mathbb{R}$. This situation is abbreviated to $X \sim \mathcal{N}(m, \sigma^2)$, where $m \in \mathbb{R}$ and $\sigma^2 \geqslant 0$ are respectively the mean and variance of the associated Gaussian measure.

Definition 2.8.3. We say that an H-valued random variable X is Gaussian if the real-valued random variable $(X, h)_H$ is Gaussian for all $h \in H$.

The following theorem collects the most important basic facts about H-valued Gaussian random variables.

Theorem 2.8.4. *Given an H-valued Gaussian random variable X, there exists a vector $m \in H$ and a non-negative, self-adjoint and trace-class operator $Q \in \mathscr{L}_1(H)$ such that*

$$\begin{aligned} \mathbb{E}[(X, h)_H] &= (m, h)_H && \forall h \in H, \\ \mathbb{E}[(X - m, h_1)_H (X - m, h_2)_H] &= (Qh_1, h_2)_H && \forall h_1, h_2 \in H. \end{aligned}$$

We call m the mean *and Q the* covariance operator *of X and we write $X \sim \mathcal{N}(m, Q)$. The characteristic function of X is*

$$\varphi_X(h) = e^{i(h,m)_H - \frac{1}{2}(Qh,h)_H} \quad \forall h \in H,$$

which implies that m and Q characterize the distribution of X.

Conversely, there exists a Gaussian random variable $X \sim \mathcal{N}(m, Q)$ given any such $m \in H$ and $Q \in \mathscr{L}_1(H)$.

Proof. The various claims made in this theorem are proved in Section 2.3.1 of [23]. □

2.8.3. Hilbert-space-valued random processes

This subsection is devoted to recalling some basic facts and definitions regarding stochastic processes and generalizing them to the H-valued situation. Recall that a *stochastic process* is an indexed family of random variables, usually indexed by a subset of the real numbers which carries the interpretation of a time horizon. Most commonly we consider $(X(t))_{t\geqslant 0}$ or $(X(t))_{t\in[0,T]}$ for some $T \in (0, \infty)$.

As was the case for random variables, we can define various notions of integrability for stochastic processes. A stochastic process is said to be *integrable* if

$$\|X(t)\|_{L^1(\Omega;H)} := \mathbb{E}\,\|X(t)\|_H < \infty \quad \forall t \in [0,T]$$

and *square integrable* if

$$\|X(t)\|_{L^2(\Omega;H)} := \big(\mathbb{E}\big[\|X(t)\|_H^2\big]\big)^{1/2} < \infty \quad \forall t \in [0,T].$$

For a fixed $\omega \in \Omega$, we call the mapping

$$t \mapsto X(t,\omega) := (X(t))(\omega), \quad t \in [0,T]$$

a *path* of the stochastic process $(X(t))_{t\in[0,T]}$.

Just as for single random variables, for an H-valued square-integrable stochastic process $(X(t))_{t\geqslant 0}$, we can define the concept of covariance in multiple ways. The covariance tensor $\operatorname{Cov}(X(t),X(s)) \in H^{(2)}$ is defined by

$$\operatorname{Cov}(X(t),X(s)) := \mathbb{E}[(X(t)-\mathbb{E}[X(t)]) \otimes (X(s)-\mathbb{E}[X(s)])], \quad t,s \geqslant 0.$$

For $t,s \geqslant 0$ we can always associate the covariance operator $Q_{X(t),X(s)} \in \mathscr{L}_1(H)$ associated to $\operatorname{Cov}(X(t),X(s))$. In the case of a spatiotemporal process, i.e. if moreover $H = L^2(\mathcal{D})$ for a given domain $\mathcal{D} \subseteq \mathbb{R}^d, d \in \mathbb{N}$, then we have a covariance kernel $\varrho_{X(t),X(s)} \in L^2(\mathcal{D}\times\mathcal{D})$.

If $t = s$, then trivially $\operatorname{Cov}(X(t),X(s)) = \operatorname{Cov}(X(t))$, which is simply the covariance of a single random variable as introduced in the previous section; in particular, the corresponding covariance operator is self-adjoint and non-negative and the kernel, if applicable, is positive semi-definite. If $t \neq s$, then it may occur that $Q_{X(t),X(s)}$ is not self-adjoint or non-negative and that $\varrho_{X(t),X(s)}$ is not positive semi-definite.

To understand why this is true, it is instructive to consider the case when $H = L^2(\mathcal{D})$ and the stochastic process $(X(t))_{t\geqslant 0}$ can be viewed as a single, square-integrable, Bochner-space-valued random variable $X \in L^2(\Omega; L^2(0,T;L^2(\mathcal{D}))) \cong L^2(\Omega; L^2((0,T)\times\mathcal{D}))$. Then by the theory of Section 2.8.1, there exists a kernel $\varrho_X \colon (0,T)\times\mathcal{D} \to \mathbb{R}$ such that

$$(\operatorname{Cov}(X), \varphi\otimes\psi)_{L^2((0,T)\times\mathcal{D})^{(2)}} = \int_{(0,T)\times\mathcal{D}}\int_{(0,T)\times\mathcal{D}} \varrho_X((t,x),(s,y))\varphi(t,x),\psi(s,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y,$$

see (2.24). On the other hand, assuming that all integrals can be interchanged and using the definition of the kernel $\varrho_{X(t),X(s)}$, we have

$$\begin{aligned}
(\operatorname{Cov}(X), \varphi\otimes\psi)_{L^2((0,T)\times\mathcal{D})^{(2)}} &= \mathbb{E}[(X-\mathbb{E}[X],\varphi)_{L^2((0,T)\times\mathcal{D})}(X-\mathbb{E}[X],\psi)_{L^2((0,T)\times\mathcal{D})}] \\
&= \mathbb{E}\Big[\int_0^T\int_0^T (X(t)-\mathbb{E}[X(t)],\varphi(t))_{L^2(\mathcal{D})}(X(s)-\mathbb{E}[X(s)],\psi(s))_{L^2(\mathcal{D})}\,\mathrm{d}t\,\mathrm{d}s\Big] \\
&= \int_0^T\int_0^T \mathbb{E}[(X(t)-\mathbb{E}[X(t)],\varphi(t))_{L^2(\mathcal{D})}(X(s)-\mathbb{E}[X(s)],\psi(s))_{L^2(\mathcal{D})}]\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^T\int_0^T\int_{\mathcal{D}}\int_{\mathcal{D}} \varrho_{X(t),X(s)}(x,y)\varphi(t,x)\psi(s,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

This informal argument shows that we can interpret $\varrho_{X(t),X(s)}(x,y)$ as $\varrho_X((t,x),(s,y))$ for (almost) all $t,s \in (0,T)$ and $x,y \in \mathcal{D}$. The former is positive semi-definite for $t = s$, so that $\varrho((t,x),(t,y)) \geqslant 0$ for all $x,y \in \mathcal{D}$; the latter is positive semi-definite in the sene that $\varrho_X((t,x),(t,x)) \geqslant 0$. However, there is no such guarantee for $\varrho_X((t,x),(s,y))$ if $t \neq s$.

2.8.4. Spatiotemporal noise

The noise term $\dot{W}$ occurring on the right-hand side of SPDE (1.3) represents spatiotemporal white or colored noise. Intuitively, spatiotemporal white noise should have the property that $\dot{W}(t,x)$ is a Gaussian random variable with mean zero and correlation

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y)$$

for all $t,s \geqslant 0$ and $x,y \in \mathcal{D}$, where we recall that $\mathcal{D} \subseteq \mathbb{R}^d$ is some spatial domain. In this context, δ is understood to be the Dirac delta. For colored noise, on the other hand, the spatial part of the noise may have a smoother correlation function; this can be expressed as

$$\mathbb{E}[\dot{W}^q(t,x)\dot{W}^q(s,y)] = \delta(t-s)q(x-y),$$

where q is a given real-valued function on the domain $\mathcal{D}$.

As a first step towards making this idea more mathematically rigorous, we adopt a different viewpoint: instead of considering $\dot{W}$ to be a (random) function of space and time, we view it as a random function of time which takes its values in a function space, say $L^2(\mathcal{D})$; this can subsequently be generalized to some Hilbert space H.

Next, we consider the following definition. In contrast to the usual situation where the index set of a stochastic process is a finite or infinite real interval thought of as time, in the following definition it is the Hilbert space H.

Definition 2.8.5 (H-isonormal Gaussian process)**.** Given a separable Hilbert space H, we call $(\mathcal{W}(h))_{h\in H} \subseteq L^2(\Omega)$ an *H-isonormal Gaussian process* if

(i) it is a Gaussian process, i.e. for any given $n \in \mathbb{N}$ and $h_1, \dots, h_n \in H$ we have that $(\mathcal{W}(h_1), \dots, \mathcal{W}(h_n))$ is an $\mathbb{R}^n$-valued Gaussian random variable with mean zero;

(ii) for all $h_1, h_2 \in H$ we have
$$\mathbb{E}[\mathcal{W}(h_1)\mathcal{W}(h_2)] = (h_1, h_2)_H.$$

The motivation for this definition is that an $L^2(\mathcal{D})$-isonormal Gaussian process should be interpreted as spatial white noise.

We note some properties of H-isonormal Gaussian processes. Firstly, we recall that Gaussian processes are characterized by their mean function
$$m(h) := \mathbb{E}[\mathcal{W}(h)], \quad h \in H,$$
and covariance function
$$c(h_1, h_2) := \mathbb{E}[\mathcal{W}(h_1)\mathcal{W}(h_2)] - \mathbb{E}[\mathcal{W}(h_1)]\mathbb{E}[\mathcal{W}(h_2)], \quad h_1, h_2 \in H.$$
Moreover, it follows from the definition that the mapping $h \mapsto \mathcal{W}(h)$ is in fact linear as a mapping from H to $L^2(\Omega)$.

An H-isonormal Gaussian process may be constructed in the following way.

Proposition 2.8.6. *Let $(\beta_j)_{j\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) real-valued random variables with distribution $\mathcal{N}(0,1)$ and let $(e_j)_{j\geqslant 1}$ be an orthonormal basis for H. Then*
$$\mathcal{W}(h) := \sum_{j=1}^{\infty} (h, e_j)_H \beta_j, \quad h \in H,$$
converges in $L^2(\Omega)$ and defines an H-isonormal Gaussian process.

Proof. See [15, Proposition 2.2]. □

2.8.5. Relation between real-valued Wiener and isonormal Gaussian processes

In this section we consider how isonormal Gaussian process relate to real-valued Wiener processes.

Given two stochastic processes $(X(t))_{t\geqslant 0}$ and $(Y(t))_{t\geqslant 0}$, we say that Y is a version of X if $\mathbb{P}(X(t) = Y(t)) = 1$ for all $t \geqslant 0$. If X and Y satisfy the stronger condition that $\mathbb{P}(X(t) = Y(t)\ \forall t \geqslant 0) = 1$, then they are said to be *indistinguishable*. Two versions of a stochastic process with almost surely continuous paths are in fact indistinguishable. Whenever a process has a version with almost surely continuous paths, we always consider that version unless specified otherwise.

Definition 2.8.7. A real-valued stochastic process $(W(t))_{t\geqslant 0}$ is called a *Wiener process* if

(i) $W(0) = 0$ almost surely;

(ii) the paths of $(W(t))_{t\geqslant 0}$ are continuous almost surely;

(iii) W has independent increments, meaning that the random variables
$$W(t_1) - W(t_0), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$$
are mutually independent for $0 \leqslant t_0 < t_1 < \dots < t_n, n \in \mathbb{N}$;

(iv) $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for all $t \geqslant s \geqslant 0$.

We can construct real-valued Wiener processes using $L^2(0,T)$-isonormal Gaussian processes in the following way.

Proposition 2.8.8. *Let $\mathcal{W}$ be an $L^2(0,T)$-isonormal Gaussian process and set $W(t) := \mathcal{W}(\mathbf{1}_{[0,t]})$ for $t \in [0,T]$. Then $(W(t))_{t\in[0,T]}$ has a continuous version which is a real-valued Wiener process.*

Proof. See [15, Proposition 2.4]. □

The proof of Proposition 2.8.8 uses the following well-known theorem, which gives a sufficient condition for the existence of versions with almost surely Hölder continuous paths, in order to establish property (ii) of Definition 2.8.7.

Theorem 2.8.9 (Kolmogorov–Chentsov)**.** *If there exist constants $p \in (1,\infty)$ and $\alpha \in (\frac{1}{p}, 1]$ such that $X \in C^{0,\alpha}([a,b]; L^p(\Omega; H))$, then there exists a version $Y \in \bigcap_{\beta\in(0,\alpha-\frac{1}{p})} L^p(\Omega; C^{0,\beta}([a,b]; H))$ of X.*

Proof. See for instance [54, Theorem 2.1]. □

Proposition 2.8.8 reflects the intuitive notion that 'white noise is the derivative of a Wiener process'. We keep this idea in mind in order to define spatiotemporal white noise: we wish to define a quantity $W(t)$ such that its time derivative would be spatiotemporal white noise. Such a process must behave as a Wiener process in time but as white noise in space. This gives rise to the following definition.

Definition 2.8.10. Given an $L^2(0, T\times\mathcal{D})$-isonormal Gaussian process, define

$$\mathcal{W}_t(\phi) := \mathcal{W}(\mathbf{1}_{[0,t]} \otimes \phi), \quad t \in [0,T],\, \phi \in L^2(\mathcal{D}),$$

where $(\mathbf{1}_{[0,t]} \otimes \phi)(s,x) := \mathbf{1}_{[0,t]}(s)\phi(x)$ for (almost) all $s \in (0,T]$ and $x \in \mathcal{D}$.

We then have that $\frac{1}{\sqrt{t}}\mathcal{W}_t(\,\cdot\,)$ is an $L^2(\mathcal{D})$-isonormal Gaussian process. As noted before, we can thus interpret $\mathcal{W}_t$ as spatial white noise for any given $t \in (0,T]$. We have the following analog to Proposition 2.8.6.

Proposition 2.8.11. *Let $(e_j)_{j\geqslant 1}$ be an orthonormal basis for $L^2(\mathcal{D})$ and define*

$$\beta_j(t) := \mathcal{W}(\mathbf{1}_{[0,t]} \otimes e_j) \quad \text{for all } j \in \mathbb{N},\, t \in [0,T]. \tag{2.25}$$

Then $(\beta_j)_{j\geqslant 1}$ are independent real-valued Wiener processes and we have almost surely

$$\mathcal{W}_t(\phi) = \sum_{j=1}^{\infty} \beta_j(t)(\phi, e_j)_{L^2(\mathcal{D})}. \tag{2.26}$$

Proof. See [15, Proposition 2.6]. □

2.8.6. Wiener processes on Hilbert spaces

Thie considerations from the previous subsection lead us to the following, more general definition.

Definition 2.8.12 (Cylindrical Wiener process)**.** Let $(e_j)_{j\geqslant 1}$ be an orthonormal basis for H and let $(\beta_j)_{j\geqslant 1}$ be the sequence of independent real-valued Wiener processes defined by (2.25). For any $t \in [0,T]$, we define the *cylindrical Wiener process* in H by

$$W(t) := \sum_{j=1}^{\infty} \beta_j(t) e_j,$$

where the convergence is almost surely in any Hilbert space $\widetilde{H} \supset H$ with Hilbert–Schmidt embedding.

The convergence is a special case of the following result.

Proposition 2.8.13. *Let $(e_j)_{j\geqslant 1}$ be an orthonormal basis for H and let $T \in \mathscr{L}_2(H;\tilde{H})$. If $(\beta_j)_{j\geqslant 1}$ is the sequence of independent real-valued Wiener processes defined by (2.25), then it holds for all $t \in [0,T]$ that $TW(t) := \sum_{j=1}^{\infty} \beta_j(t)Te_j$ is a well-defined element of $L^2(\Omega;\tilde{H})$, which is independent of the choice of orthonormal basis.*

Proof. Fix a $t \in [0,T]$ and assume without loss of generality that $n,m \in \mathbb{N}$ are such that $n \geqslant m$. Then, by the independence and zero mean of the real-valued Wiener processes $(\beta_j)_{j\geqslant 1}$, and the fact that $\beta_j(t) \sim \mathcal{N}(0,t)$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n}\beta_j(t)Te_j - \sum_{j=1}^{m}\beta_j(t)Te_j\right\|^2_{L^2(\Omega;\tilde{H})} &= \sum_{j=m+1}^{n}\|\beta_j(t)Te_j\|^2_{L^2(\Omega;\tilde{H})}\\ &= t\sum_{j=m+1}^{n}\|Te_j\|^2_{\tilde{H}} \leqslant t\,\|T\|^2_{\mathscr{L}_2(H;\tilde{H})} < \infty.\end{aligned}$$

The sum on the second line tends to zero as $n,m \to \infty$ as it is the difference of partial sums of a convergent series; by the Cauchy criterion it thus follows that the series $TW(t) := \sum_{j=1}^{\infty}\beta_j(t)Te_j$ converges in $L^2(\Omega;\tilde{H})$.

To show that $TW(t)$ is well-defined, i.e. the definition does not depend on the choice of orthonormal basis, let another orthonormal basis $(g_i)_{i\geqslant 1}$ for H be given. If $\widetilde{TW}(t)$ is the result of the definition using $(g_i)_{i\geqslant 1}$, then it holds almost surely that

$$\begin{aligned}TW(t) :&= \sum_{j=1}^{\infty}\beta_j(t)Te_j = \sum_{j=1}^{\infty}\mathcal{W}_t(e_j)Te_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(e_j,g_i)_H\mathcal{W}_t(g_i)Te_j =\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathcal{W}_t(g_i)(e_j,g_i)_HTe_j = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathcal{W}_t(g_i)Tg_i = \widetilde{TW}(t),\end{aligned}$$

where we used (2.26), the Fubini theorem, and the continuity of T. $\square$

We proceed with some examples showing how Proposition 2.8.13 may be applied in different contexts related to Wiener processes in Hilbert spaces.

Example 2.8.14. For all $h \in H$ it holds that the mapping $x \mapsto (h,x)_H$ belongs to $\mathscr{L}_2(H;\mathbb{R})$. Hence by Proposition 2.8.13 $(h,W(t))_H$ is a well-defined random process and

$$(h,W(t))_H := \sum_{j=1}^{\infty}\beta_j(t)(h,e_j)_H = \mathcal{W}_t(h).$$

Example 2.8.15. Let $H \subset \tilde{H}$ with Hilbert–Schmidt embedding, i.e. $\iota \in \mathscr{L}_2(H;\tilde{H})$, where ι denotes the inclusion map from H to $\tilde{H}$. Then we may identify $\iota W(t)$, which is a well-defined element of $L^2(\Omega;\tilde{H})$, with $W(t)$.

Note that a Hilbert space $\tilde{H} \supset H$ as in Example 2.8.15 can always be constructed: for any given orthonormal basis $(e_j)_{j\geqslant 1}$ of H, we let $\tilde{H}$ be such that $(\tilde{e}_j)_{j\geqslant 1} := (je_j)_{j\geqslant 1}$ is an orthonormal basis; i.e., $\tilde{h} \in \tilde{H}$ if and only if $\tilde{h} = \sum_{j=1}^{\infty} c_j\tilde{e}_j$ with $\sum_{j=1}^{\infty} c_j^2 < \infty$.

Example 2.8.16 (Spatiotemporal cylindrical Wiener process)**.** Suppose $H = L^2(\mathcal{D})$ and $\tilde{H} = H^{-s}(\mathcal{D})$ for $s > d/2$ and a bounded domain $\mathcal{D}$. For simplicity, we take s to be an integer. Also note that $H^{-s}(\mathcal{D})$ is not one of the dot spaces introduced in Section 2.5.2, but rather the dual of a conventional integer-order Sobolev space.

In [19, Theorem 2] it is stated that the embedding $H^s(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ is Hilbert–Schmidt. Since we can view the embedding $L^2(\mathcal{D}) \hookrightarrow H^{-s}(\mathcal{D})$ as its adjoint, it follows that the latter is also Hilbert–Schmidt. Thus, we have a concrete way of formulating the convergence of the series which defines the spatiotemporal cylindrical Wiener process.

Example 2.8.17 (Q-Wiener process)**.** Let a non-negative and self-adjoint operator $Q \in \mathscr{L}(U;H)$ be given. Define the space $\mathcal{H} := Q^{\frac{1}{2}}U$ with the inner product

$$(h_1, h_2)_{\mathcal{H}} := (Q^{-\frac{1}{2}}h_1, Q^{-\frac{1}{2}}h_2)_U, \quad h_1, h_2 \in \mathcal{H},$$

where $Q^{-\frac{1}{2}}$ denotes the operator pseudo-inverse of $Q^{\frac{1}{2}}$. This results in a Hilbert space.

Now consider the following definition:

$$W^Q(t) := Q^{\frac{1}{2}}W(t) := \sum_{j=1}^{\infty} \beta_j(t) Q^{\frac{1}{2}} e_j, \quad t \in [0,T],$$

where $(e_j)_{j\geqslant 1}$ and $(\beta_j)_{j\geqslant 1}$ are as in the previous propositions. If $\operatorname{tr} Q < \infty$, then we have that $Q^{\frac{1}{2}} \in \mathscr{L}_2(U;H)$ (equivalently, $\iota \in \mathscr{L}_2(\mathcal{H};H)$) so that W^Q takes its values in H by Proposition 2.8.13. In fact, one can show that W^Q satisfies a generalization of Definition 2.8.7 to H-valued processes, where condition (iv) is changed to $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$.

If $\operatorname{tr} Q = \infty$, then as before W^Q takes its values in a space which contains H with a Hilbert–Schmidt embedding, but which is possibly smaller than the space in which the cylindrical Wiener process W takes its values.

We call W^Q a *Q-Wiener process*. The space $\mathcal{H}$ is called the *reproducing kernel Hilbert space* (RKHS) of W^Q.

2.8.7. Stochastic integration

Having defined Wiener processes on Hilbert spaces, we can now make sense of stochastic integration with respect to a given H-valued Wiener processes W. For our purposes, it is sufficient to define the stochastic integral for deterministic integrands $\Phi\colon (0,T) \to \mathscr{L}(H;\tilde{H})$. We define the integral in terms of real-valued Itô integrals as follows. A more general approach, which allows for the integration of so-called *predictable* stochastic processes with certain integrability requirements, can be found for instance in Section 4.2 of [23].

Proposition 2.8.18. *Let W be a cylindrical Wiener process on H, let $\tilde{H}$ be a Hilbert space and let $\Phi\colon (0,T) \to \mathscr{L}(H;\tilde{H})$ satisfy*

$$\int_0^T \|\Phi(t)\|^2_{\mathscr{L}_2(H;\tilde{H})} \,\mathrm{d}t < \infty.$$

Given orthonormal bases $(e_j)_{j\geqslant 1}$ for H and $(g_i)_{i\geqslant 1}$ for $\tilde{H}$ and the sequence $(\beta_j)_{j\geqslant 1}$ of independent real-valued Wiener processes defined by (2.25), the stochastic integral defined by

$$\int_0^T \Phi(t)\,\mathrm{d}W(t) := \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}\int_0^T (\Phi(t)e_j, g_i)_{\tilde{H}} \,\mathrm{d}\beta_j(t)\Big) g_i$$

is a well-defined element of $L^2(\Omega;\tilde{H})$ which satisfies the Itô isometry*:*

$$\Big\| \int_0^T \Phi(t)\,\mathrm{d}W(t) \Big\|^2_{L^2(\Omega;\tilde{H})} = \int_0^T \|\Phi(t)\|^2_{\mathscr{L}_2(H;\tilde{H})}\,\mathrm{d}t.$$

Proof. The proof of the Itô isometry uses the isometry for real-valued stochastic integrals, see [15, Proposition 2.10]. To show that the definition does not depend on the choices of orthonormal bases for H and $\tilde{H}$, one argues as in the second part of the proof of Proposition 2.8.13. □

In light of Example 2.8.17, note that we can naturally define the stochastic integral against a Q-Wiener process by

$$\int_0^T \Phi(t)\,\mathrm{d}W^Q(t) := \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}\int_0^T (\Phi(t)Q^{\frac{1}{2}}e_j, g_i)_{\tilde{H}} \,\mathrm{d}\beta_j(t)\Big) g_i.$$

The Itô isometry can then be written using the RKHS $\mathcal{H}$:

$$\left\| \int_0^T \Phi(t) \,\mathrm{d}W^Q(t) \right\|^2_{L^2(\Omega;\tilde{H})} = \int_0^T \|\Phi(t) Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H;\tilde{H})} \,\mathrm{d}t = \int_0^T \|\Phi(t)\|^2_{\mathscr{L}_2(\mathcal{H};\tilde{H})} \,\mathrm{d}t.$$

The following special stochastic integral can also be useful.

Definition 2.8.19 (Weak stochastic integral)**.** Given $\Phi\colon (0,T) \to \mathscr{L}(H;\tilde{H})$ and $\Psi\colon (0,T) \to \tilde{H}$, we define the *weak stochastic integral* by

$$\int_0^t (\Psi(s), \Phi(s) \,\mathrm{d}W(s))_{\tilde{H}} := \int_0^t \tilde{\Phi}_\Psi(s) \,\mathrm{d}W(s), \quad t \in [0,T],$$

where $\tilde{\Phi}_\Psi\colon (0,T) \to \mathscr{L}(H;\mathbb{R})$ is defined by

$$\tilde{\Phi}_\Psi(t)h := (\Psi(t), \Phi(t)h)_{\tilde{H}}, \quad t \in (0,T),\ h \in H.$$

We conclude with a stochastic generalization of the Fubini theorem, which gives sufficient conditions to interchange deterministic and stochastic integrals.

Theorem 2.8.20 (Stochastic Fubini theorem)**.** *Let $\Phi\colon (0,T) \times (0,T) \to \mathscr{L}_2(H;\tilde{H})$ be strongly measurable with respect to the (product) Borel σ-algebras on both sides. If*

$$\int_0^T \left(\int_0^T \|\Phi(t,s)\|^2_{\mathscr{L}_2(\mathcal{H};H)} \,\mathrm{d}t \right)^{\frac{1}{2}} \mathrm{d}s < \infty,$$

then almost surely it holds that

$$\int_0^T \int_0^T \Phi(t,s) \,\mathrm{d}W(t) \,\mathrm{d}s = \int_0^T \int_0^T \Phi(t,s) \,\mathrm{d}s \,\mathrm{d}W(t).$$

Proof. This is the specialization to deterministic integrands of [23, Theorem 4.33]. □

3

Analysis of the SPDE

In this chapter, we study the following problem:

$$(\partial_t + A)^\gamma X = \dot{W}^Q, \quad X(0) = X_0 \in L^p(\Omega; \dot{H}^\eta_A). \tag{3.1}$$

Here we assume that A is a self-adjoint operator whose negative $-A$ generates an analytic C_0-semigroup $(S(t))_{t\geqslant 0}$ of contractions on the real and separable Hilbert space H, and $\dot{W}^Q$ is Q-Wiener spatiotemporal white noise and $\gamma, \eta \geqslant 0$ are real parameters. As before, we will often omit the Q from W^Q when no confusion arises as a result. Given a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ to which W^Q is adapted, we suppose that X_0 is $\mathcal{F}_0$-measurable. We call the SPDE (3.1) *non-fractional parabolic* if $\gamma = 1$ and *fractional parabolic* otherwise.

This chapter is structured as follows. Before addressing the matter of defining solution concepts for (3.1), in Section 3.1 we first consider the solution concepts known for the non-fractional parabolic problem and compare them to the deterministic solution concepts described in Section 2.7. Then in Section 3.2, we investigate the *parabolic operator* $\partial_t + A$ and the C_0-semigroup which it generates, upon which we conclude that fractional powers of this Bochner space operator can be defined using the theory described in Section 2.5. This will allow us to define the concept of a mild fractional solution in Section 3.3, where we also introduce a fractional variational weak solution concept and show that the two are equivalent whenever A and X_0 are such that a mild solution exists. Working with the mild solution, we derive well-posedness and spatiotemporal regularity results in Section 3.4, where the latter are linked to the Hilbert–Schmidt norm of certain fractional powers of the operators A and Q. Lastly, in Section 3.5 a connection is made with the statistical application by investigating briefly the marginal covariance structure and deriving an explicit expression for $\operatorname{Cov}(X(t))$ as $t \to \infty$.

3.1. Solution concepts for the non-fractional parabolic SPDE

The analysis of the non-fractional problem is more straightforward than that of the fractional one since the non-fractional SPDE can be written in the form of a stochastic evolution equation:

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = \mathrm{d}W^Q, \quad X(0) = X_0 \in L^p(\Omega; \dot{H}^\eta_A). \tag{3.2}$$

The most straightforward definition of a solution to (3.2) is obtained by formally integrating the equation, which suggests the formula

$$X(t) = X_0 - \int_0^t AX(s)\,\mathrm{d}s + W^Q(t), \quad \mathbb{P}\text{-a.s.} \tag{3.3}$$

Definition 3.1.1 (Strong solution)**.** An H-valued process $(X(t))_{t\in[0,T]}$ is called a *strong solution* to (3.2) if

(i) $X(t) \in \mathsf{D}(A)$ for almost all $t \in [0,T]$;

(ii) $\int_0^T \|AX(t)\|_H \,\mathrm{d}t < \infty$;

(iii) equation (3.3) is satisfied for all $t \in [0,T]$.

Keeping in mind the discussion in the previous chapter, it is clear that equation (3.3) only makes sense if $\operatorname{tr} Q < \infty$.

The next solution concept can be motivated by taking the inner product of (3.3) with a test vector and using the adjoint of A.

Definition 3.1.2 (H-weak solution)**.** An H-valued process $(X(t))_{t\in[0,T]}$ is called an H*-weak solution* to (3.2) if its paths are Bochner integrable $\mathbb{P}$-a.s. and for all $v \in \mathsf{D}(A)$ and almost all $t \in [0,T]$ we have

$$(X(t), v)_H = (X_0, v)_H - \int_0^t (X(s), Av)_H \,\mathrm{d}s + (W^Q(t), v)_H.$$

Compared to the strong solution, the H-weak solution concept places less stringent assumptions on both the solution $X(t)$, which is no longer required to belong to $\mathsf{D}(A)$, and on the covariance operator Q, which may have infinite trace since $((W^Q(t), v)_H)_{t\in[0,T]}$ is always a well-defined real-valued Wiener process. The terminology 'weak' and 'strong' is further justified by the fact that a strong solution is also a weak solution, which is an immediate consequence of the respective definitions.

Another possibility is to test (3.2) against functions from a Bochner space such as $L^2(0,T;H)$, by formally taking the corresponding inner product, integrating by parts and taking adjoints.

Definition 3.1.3 ($L^2(0,T;H)$-weak solution)**.** An H-valued process $(X(t))_{t\in[0,T]}$ is called an $L^2(0,T;H)$-*weak solution* to (3.2) if its paths are Bochner integrable $\mathbb{P}$-a.s. and for all $v \in C^1_{0,\{T\}}([0,T];\mathsf{D}(A))$ we have

$$(X, (-\partial_t + A)v)_{L^2(0,T;H)} = (X_0, v(0))_H + \int_0^T (v(t), \mathrm{d}W^Q(t))_H,$$

where the latter term is a weak stochastic integral. Note that ∂_t denotes a weak derivative, which coincides with the strong derivative for $v \in C^1_{0,\{T\}}([0,T];\mathsf{D}(A))$.

Since we moved the temporal derivative and the operator A from the trial function to the test function, this yields a solution concept analogous to the weak variational formulation introduced in Section 2.7 for the deterministic evolution equation. We continue the analogy with the deterministic situation by introducing the following stochastic mild solution concept.

Definition 3.1.4 (Mild solution)**.** Suppose that $-A$ generates a C_0-semigroup $(S(t))_{t\in[0,T]}$ in H satisfying

$$\int_0^T \|S(t)\|^2_{\mathscr{L}_2(\mathcal{H};H)} \,\mathrm{d}t < \infty, \tag{3.4}$$

where $\mathcal{H} := Q^{\frac{1}{2}} H$ is the RKHS of W^Q. The *mild solution* to (3.2) is given by

$$X(t) = S(t)X_0 + \int_0^t S(t-s)\,\mathrm{d}W^Q(s), \quad t \in [0,T].$$

Theorem 5.4 of [23] now tells us that (3.2) has a unique H-weak solution which coincides with the mild solution, given that (3.4) and the standing assumptions on A and X_0 hold. Under the same conditions, a mild solution is necessarily a $L^2(0,T;H)$-weak solution is necessarily a mild solution, as noted in [46, Lemma 3.2]; later we will see that it is equivalent to the mild solution, hence to the H-weak solution, see Propositions 3.3.2 and 3.3.4.

3.2. The parabolic operator

In order to define solution concepts in the fractional parabolic case, we need to find a suitable interpretation the fractional parabolic operator $(\partial_t + A)^\gamma$. The idea is to view $\partial_t + A$ as an unbounded linear operator defined on a subset of the Bochner space $L^2(0,T;H)$ and show that it generates a semigroup suitable for defining fractional powers, see Section 2.5.

First define

$$[\mathcal{A}v](\vartheta) := Av(\vartheta), \quad v \in \mathsf{D}(\mathcal{A}) := L^2(0,T;\mathsf{D}(A)), \text{ almost all } \vartheta \in (0,T);$$

this is the Bochner space counterpart of A. The parabolic operator is then defined as

$$(\partial_t + \mathcal{A})v := \partial_t v + \mathcal{A}v, \quad v \in \mathsf{D}(\partial_t + \mathcal{A}) := H^1_{0,\{0\}}(0,T;H) \cap L^2(0,T;\mathsf{D}(A)),$$

where $\partial_t v$ denotes the weak derivative of v in the Bochner space sense, see Section 2.3.3.

We now state and prove some properties of this operator, which will subsequently be combined into a proof that $-(\partial_t + \mathcal{A})$ generates a C_0-semigroup which we may use to define negative fractional powers. We first prove that the parabolic operator inherits the maximal accretivity from A.

Proposition 3.2.1. *If $-A$ generates an analytic semigroup of contractions on a Hilbert space H, then the parabolic operator $\partial_t + \mathcal{A} \colon \mathsf{D}(\partial_t + \mathcal{A}) \subset L^2(0,T;H) \to L^2(0,T;H)$ is maximally accretive.*

Proof. First note that the assumption that $-A$ generates an analytic semigroup of contractions implies by the Lumer-Phillips theorem that A is maximally accretive, hence in particular accretive. Thus $\mathcal{A}$ is also accretive, since for $v \in L^2(0,T;\mathsf{D}(A))$ we have

$$(\mathcal{A}v, v)_{L^2(0,T;H)} = \int_0^T \underbrace{(Av(t), v(t))_H}_{\geqslant 0} \,\mathrm{d}t \geqslant 0.$$

Moreover, it follows from integration by parts that the weak derivative operator is accretive: indeed, for $v \in H^1_{0,\{0\}}(0,T;H)$:

$$\begin{aligned}(\partial_t v, v)_{L^2(0,T;H)} = \int_0^T (\partial_t v(t), v(t))_H \,\mathrm{d}t &= -\int_0^T (v(t), \partial_t v(t))_H \,\mathrm{d}t + \|v(T)\|^2 - \|v(0)\|^2 \\ &= -(\partial_t v, v)_{L^2(0,T;H)} + \|v(T)\|^2,\end{aligned}$$

where we note that pointwise evaluation is meaningful since $H^1(0,T;H) \hookrightarrow C([0,T];H)$ [25, XVIII.1.2, Theorem 1].

Isolating $(\partial_t v, v)_{L^2(0,T;H)}$ yields

$$(\partial_t v, v)_{L^2(0,T;H)} = \frac{1}{2}\|v(T)\|^2 \geqslant 0.$$

It follows that $\partial_t + \mathcal{A}$ on $L^2(0,T;H)$ is accretive as the sum of accretive operators; it remains to show that it is in fact maximal accretive. This means that given an $f \in L^2(0,T;H)$, we need to find a $u \in \mathsf{D}(\partial_t + \mathcal{A}) = L^2(0,T;\mathsf{D}(A)) \cap H^1_{0,\{0\}}(0,T;H)$ such that

$$(\partial_t + \mathcal{A})u + u = f;$$

see Definition 2.1.13. But this amounts to the problem of maximal L^2-regularity of the abstract Cauchy problem

$$\partial_t u(t) + (A + I)u(t) = f(t), \quad \text{almost all } t \in (0,T).$$

Thus by Theorem 2.7.4 it suffices to check that $-(A+I)$ generates an analytic C_0-semigroup. Indeed, if we denote by $(S(t))_{t\geqslant 0}$ the semigroup generated by $-A$, then $-(I+A)$ generates $(e^{-t}S(t))_{t\geqslant 0}$, which clearly preserves all the properties needed for a semigroup to be analytic. □

Next we consider the operators $(\mathcal{S}(t))_{t\geqslant 0}$ on $L^2(0,T;H)$ defined by

$$[\mathcal{S}(t)v](\vartheta) := S(t)v(\vartheta), \quad t \geqslant 0,\ v \in L^2(0,T;H),\ \text{almost all } \vartheta \in (0,T). \tag{3.5}$$

This is the natural Bochner space version of the family $(S(t))_{t\geqslant 0}$. We expect this to be a C_0-semigroup generating $-\mathcal{A}$; the next proposition shows that this is indeed the case.

Proposition 3.2.2. *$(\mathcal{S}(t))_{t\geqslant 0}$ is a family of well-defined bounded operators on $L^2(0,T;H)$ forming a C_0-semigroup with infinitesimal generator $-\mathcal{A}$.*

Proof. We first verify that the operators defined in (3.5) are well-defined. Indeed, for $v \in L^2(0,T;\mathsf{D}(A))$ we have

$$\|\mathcal{A}v\|_{L^2(0,T;H)} = \left(\int_0^T \|Av(\vartheta)\|_H^2 \,\mathrm{d}\vartheta\right)^{1/2} \leqslant \|v\|_{L^2(0,T;\mathsf{D}(A))} . \tag{3.6}$$

Let $M \geqslant 1$ and $w \geqslant 0$ be as in Proposition 2.4.3, i.e. they satisfy $\|S(t)\|_{\mathscr{L}(H)} \leqslant Me^{wt}$ for all $t \geqslant 0$. Then for fixed $t \geqslant 0$ we have

$$\big\|\mathcal{S}(t)v\big\|_{L^2(0,T;H)} = \left(\int_0^T \|S(t)v(\vartheta)\|_H^2 \,\mathrm{d}\vartheta\right)^{1/2} \leqslant Me^{wt}\|v\|_{L^2(0,T;H)} . \tag{3.7}$$

This shows that $\mathcal{A}$ is well-defined and that $\mathcal{S}(t)$ is bounded for every $t \geqslant 0$.

We now check that $(\mathcal{S}(t))_{t\geqslant 0}$ is a strongly continuous semigroup. Clearly, $\mathcal{S}(0) = I$ and $\mathcal{S}(t+s) = \mathcal{S}(t)\mathcal{S}(s)$. To show that it is strongly continuous, let $v \in L^2(0,T;H)$, $0 < h < h'$ and note that

$$\begin{aligned}\big\|\mathcal{S}(h)v - v\big\|_{L^2(0,T;H)} &= \left(\int_0^T \big\|S(h)v(\vartheta) - v(\vartheta)\big\|_H^2 \,\mathrm{d}\vartheta\right)^{1/2} \\ &\leqslant \big\|\mathcal{S}(h)v\big\|_{L^2(0,T;H)} + \|v\|_{L^2(0,T;H)} \\ &\leqslant (Me^{wh'}+1)\|v\|_{L^2(0,T;H)} ,\end{aligned}$$

so that by dominated convergence we may pass to the limit $h \downarrow 0$ under the integral sign to deduce the strong continuity of $(\mathcal{S}(t))_{t\geqslant 0}$ from that of $(S(t))_{t\geqslant 0}$. This finishes the proof that $(\mathcal{S}(t))_{t\geqslant 0}$ is a C_0-semigroup.

We now consider its generator. Let $v \in L^2(0,T;\mathsf{D}(A))$ and consider for $0 < h < h'$:

$$\begin{aligned}\left\|\frac{1}{h}(\mathcal{S}(h)v - v) + \mathcal{A}v\right\|_{L^2(0,T;H)} &= \left(\int_0^T \left\|\frac{1}{h}(S(h)v(\vartheta) - v(\vartheta)) + Av(\vartheta)\right\|_H^2 \mathrm{d}\vartheta\right)^{1/2} \\ &\leqslant \left\|\frac{1}{h}(\mathcal{S}(h)v - v)\right\|_{L^2(0,T;H)} + \|\mathcal{A}v\|_{L^2(0,T;H)} .\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}\left\|\frac{1}{h}(\mathcal{S}(h)v - v)\right\|_{L^2(0,T;H)} &= \left(\int_0^T \left\|\frac{1}{h}(S(h)v(\vartheta) - v(\vartheta))\right\|_H^2 \mathrm{d}\vartheta\right)^{1/2} \\ &= \left(\int_0^T \left\|\frac{1}{h}\int_0^h S(s)Av(\vartheta)\,\mathrm{d}s\right\|_H^2 \mathrm{d}\vartheta\right)^{1/2} \\ &\leqslant \left(\int_0^T \frac{1}{h^2}\left[\int_0^h \big\|S(s)Av(\vartheta)\big\|_H \,\mathrm{d}s\right]^2 \mathrm{d}\vartheta\right)^{1/2} \\ &\leqslant \left(\int_0^T \frac{1}{h^2}\left[\int_0^h Me^{wh'}\big\|Av(\vartheta)\big\|_H \,\mathrm{d}s\right]^2 \mathrm{d}\vartheta\right)^{1/2} \\ &= Me^{wh'}\left(\int_0^T \big\|Av(\vartheta)\big\|_H^2 \,\mathrm{d}\vartheta\right)^{1/2} = Me^{wh'}\|\mathcal{A}v\|_{L^2(0,T;H)} .\end{aligned}$$

Applying this and (3.6) to (3.8) yields

$$\left\|\frac{1}{h}(\mathcal{S}(h)v - v) + \mathcal{A}v\right\|_{L^2(0,T;H)} \leqslant (Me^{wh'}+1)\|v\|_{L^2(0,T;\mathsf{D}(A))} .$$

This justifies the use of the dominated convergence theorem to conclude that

$$\lim_{h\downarrow 0}\frac{1}{h}(\mathcal{S}(h)v - v) = -\mathcal{A}v,$$

i.e., $L^2(0,T;\mathsf{D}(A))$ is contained in the domain of the generator of $(\mathcal{S}(t))_{t\geqslant 0}$, which is an extension of $\mathcal{A}$. Next, observe that $L^2(0,T;\mathsf{D}(A))$ is dense in $L^2(0,T;H)$ by the density of $\mathsf{D}(A)$ in H. Moreover, it follows from an argument analogous to (3.7) that $\mathcal{S}(t)$ maps $L^2(0,T;\mathsf{D}(A))$ to itself for each $t \geqslant 0$. Hence we have that $L^2(0,T;\mathsf{D}(A))$ is dense in the domain of the generator of $(\mathcal{S}(t))_{t\geqslant 0}$ with respect to the graph norm of the latter by Proposition 2.4.5. Since these observations together imply that the generator is the closure of $\mathcal{A}$, it suffices to prove that $\mathcal{A}$ is closed.

To this end, let the sequence $(v_n)_{n\geqslant 1} \subseteq \mathsf{D}(\mathcal{A}) = L^2(0,T;\mathsf{D}(A))$ be such that $v_n \to v$ and $\mathcal{A}v_n \to y$ in $L^2(0,T;H)$. We need to prove that $v \in L^2(0,T;\mathsf{D}(A))$ and $y = \mathcal{A}v$. Let $(v_{n_k})_{k\geqslant 1}$ be a subsequence such that $v_{n_k}(t) \to v(t)$ and $Av_{n_k}(t) \to y(t)$ in H for almost all $t \in (0,T)$. Then by the closedness of A it follows that $v(t) \in \mathsf{D}(A)$ and $y(t) = Av(t)$ for almost all $t \in (0,T)$. We use this to see

$$\begin{aligned}
\|v\|_{\mathsf{D}(\mathcal{A})} = \|v\|_{L^2(0,T;\mathsf{D}(A))} &= \left(\int_0^T \|v(\vartheta)\|^2_{\mathsf{D}(A)} \,\mathrm{d}\vartheta\right)^{1/2} \\
&= \left(\int_0^T \Big[\|v(\vartheta)\|_H + \|Av(\vartheta)\|_H\Big]^2 \,\mathrm{d}\vartheta\right)^{1/2} \\
&= \left(\int_0^T \Big[\|v(\vartheta)\|_H + \|y(\vartheta)\|_H\Big]^2 \,\mathrm{d}\vartheta\right)^{1/2} \\
&\leqslant C_2(\|v\|_{L^2(0,T;H)} + \|y\|_{L^2(0,T;H)}) < \infty,
\end{aligned}$$

so $v \in \mathsf{D}(\mathcal{A})$. Now the result follows from once again applying the dominated convergence theorem, this time to let $k \to \infty$ under the integral appearing in

$$\begin{aligned}
\big\|\mathcal{A}v - \mathcal{A}v_{n_k}\big\|_{L^2(0,T;H)} &= \left(\int_0^T \|y(\vartheta) - Av_{n_k}(\vartheta)\|^2_H \,\mathrm{d}\vartheta\right)^{1/2} \\
&\leqslant C_2(\|y\|_{L^2(0,T;H)} + \|\mathcal{A}v_{n_k}\|_{L^2(0,T;H)}) \\
&\leqslant C_2(\|y\|_{L^2(0,T;H)} + K),
\end{aligned}$$

where $K := \sup_{k\geqslant 1} \|\mathcal{A}v_{n_k}\|_{L^2(0,T;H)}$ is finite since $(\mathcal{A}v_{n_k})_{k\geqslant 1}$ converges in $L^2(0,T;H)$. This shows that $\mathcal{A}v = y$. $\square$

Next we consider the family of zero-padded right-translation operators $(\mathcal{T}(t))_{t\geqslant 0}$ on $L^2(0,T;H)$, defined by

$$[\mathcal{T}(t)v](\vartheta) := v(\vartheta - t)\mathbf{1}\{\vartheta \geqslant t\}, \quad t \geqslant 0,\ v \in L^2(0,T;H), \text{ almost all } \vartheta \in (0,T). \tag{3.9}$$

Proposition 3.2.3. *The family $(\mathcal{T}(t))_{t\geqslant 0}$ of bounded operators on $L^2(0,T;H)$ is a C_0-semigroup whose infinitesimal generator is given by the negative of the weak derivative $Rv := -\partial_t v$ for $v \in \mathsf{D}(R) = H^1_{0,\{0\}}(0,T;H)$.*

Proof. For each $t \geqslant 0$, it is clear that $\mathcal{T}(t)$ is well-defined as bounded linear map on $L^2(0,T;H)$; in fact, it is a contraction. It follows readily from the definition that $\mathcal{T}(0) = I$, and the semigroup property is satisfied since for $s,t \geqslant 0$, $\vartheta \in [0,T]$ in some suitable full-measure set and $v \in L^2(0,T;H)$, we have

$$\begin{aligned}
\mathcal{T}(t)\mathcal{T}(s)vt(\vartheta) &:= [\mathcal{T}(s)v](\vartheta - t)\mathbf{1}\{\vartheta \geqslant t\} \\
&= v(\vartheta - t - s)\mathbf{1}\{\vartheta - t \geqslant s\}\mathbf{1}\{\vartheta \geqslant t\} \\
&= v(\vartheta - (t+s))\mathbf{1}\{\vartheta \geqslant t+s\} = [\mathcal{T}(t+s)v](\vartheta).
\end{aligned}$$

To prove strong continuity, first let $v \in C_c((0,T];H)$ and let $\widetilde{v}$ denote its extension by zero to the interval $(-\infty,T]$. Then $\widetilde{v}$ is uniformly continuous by the compact support of v, so that given any $\varepsilon > 0$ we can choose $\delta > 0$ so small that $\|\widetilde{v}(\vartheta - t) - \widetilde{v}(\vartheta)\|_H < \varepsilon$ for all $\vartheta \in [0,T]$ and $t < \delta$. Then for some interval $[a,b]$ which contains the support of v, it follows that

$$\|\mathcal{T}(t)v - v\|^2_{L^2(0,T;H)} = \int_0^T \|\widetilde{v}(\vartheta - t) - \widetilde{v}(\vartheta)\|^2_H \,\mathrm{d}\vartheta \leqslant \varepsilon^2(b - a + \delta)$$

for all $t < \delta$; this proves $\lim_{t\downarrow 0} \|\mathcal{T}(t)v - v\|_{L^2(0,T;H)} = 0$. Since $C_c((0,T];H)$ is dense in $L^2(0,T;H)$, and $(\mathcal{T}(t))_{t\geqslant 0}$ is contractive, hence uniformly bounded, we get the strong continuity on the whole of $L^2(0,T;H)$; this proves that $(\mathcal{T}(t))_{t\geqslant 0}$ is a C_0-semigroup.

Now we turn to the generator of $(\mathcal{T}(t))_{t\geqslant 0}$. To this end, let an arbitrary $v \in C_c^1((0,T];H)$ be given and note that its zero extension $\widetilde{v}$ is continuously differentiable with classical (and hence, weak) derivative $\partial_\vartheta \widetilde{v} = \widetilde{\partial_\vartheta v}$ by the compact support of v in $(0,T]$. Given an arbitrary $\vartheta \in [0,T]$, the function $[\vartheta - T, \infty) \ni s \mapsto \widetilde{v}(\vartheta - s)$ is therefore continuously differentiable with derivative $s \mapsto -\widetilde{\partial_\vartheta v}(\vartheta - s)$ by the chain rule. Thus, by the fundamental theorem of calculus, we have

$$\begin{aligned}\mathcal{T}(t)v(\vartheta) - v(\vartheta) &= \widetilde{v}(\vartheta - t) - \widetilde{v}(\vartheta) \\ &= -\int_0^t \widetilde{\partial_s v}(\vartheta - s)\,\mathrm{d}s \\ &= -\int_0^t [\mathcal{T}(s)\partial_\vartheta v](\vartheta)\,\mathrm{d}s,\end{aligned}$$

It follows that

$$\mathcal{T}(t)v - v = -\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s$$

almost everywhere. On the other hand we know from Proposition 2.4.4(b) that if R denotes the generator of $(\mathcal{T}(t))_{t\geqslant 0}$, then we have

$$\mathcal{T}(t)v - v = R\int_0^t \mathcal{T}(s)v\,\mathrm{d}s,$$

hence combining the previous two displays yields

$$R\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = -\int_0^t \mathcal{T}(s)\partial_\theta v\,\mathrm{d}s. \tag{3.10}$$

Note that by continuity of $s \mapsto \mathcal{T}(s)v$, we have the following averaging result:

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = \mathcal{T}(0)v = v. \tag{3.11}$$

Dividing both sides of (3.10) by t, letting $t \downarrow 0$ and using averaging again, one obtains

$$\lim_{t\downarrow 0} R\frac{1}{t}\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = -\lim_{t\downarrow 0}\frac{1}{t}\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s = -\mathcal{T}(0)\partial_\vartheta v = -\partial_\vartheta v. \tag{3.12}$$

Since R is assumed to be the generator of a C_0-semigroup, it is in particular closed by Proposition 2.4.4(d); combining the closedness of R with (3.11) and (3.12) then yields $v \in \mathsf{D}(R)$ and $Rv = -\partial_\vartheta v$.

Since $C_c^1((0,T];H)$ is dense in $L^2(0,T;H)$ and $\mathcal{T}(t)C_c^1((0,T];H) \subseteq C_c^1((0,T];H)$ for all $t \geqslant 0$, we have that $C_c^1((0,T];H)$ is dense in $\mathsf{D}(R)$ with respect to the graph norm of R by Proposition 2.4.5. On the other hand, we have that $C_c^1((0,T];H)$ is dense in $H^1_{0,\{0\}}(0,T;H)$ and for all $v \in C_c^1((0,T];H)$ we have

$$\|v\|_{\mathsf{D}(R)} = \|v\|_{L^p(0,T;H)} + \|Rv\|_{L^p(0,T;H)} = \|\partial_\vartheta v\|_{L^p(0,T;H)} + \|v'\|_{L^p(0,T;H)} = \|v\|_{H^1_{0,\{0\}}(0,T;H)},$$

it follows that $\mathsf{D}(R) = H^1_{0,\{0\}}(0,T;H)$ and $Rv = -\partial_\vartheta v$ for all $v \in H^1_{0,\{0\}}(0,T;H)$. Now we may relabel $\partial_\vartheta v$ to $\partial_t v$ to obtain the result. $\square$

We can combine these results to obtain the following proposition, which says that the operator $\partial_t + \mathcal{A}$ generates a product semigroup which is exponentially stable.

Proposition 3.2.4. *The negative of the unbounded operator $\partial_t + \mathcal{A}$ on $L^2(0,T;H)$ with domain*

$$\mathsf{D}(\partial_t + \mathcal{A}) = H^1_{0,\{0\}}(0,T;H) \cap L^2(0,T;\mathsf{D}(A))$$

generates the C_0-semigroup $(\mathcal{S}(t)\mathcal{T}(t))_{t\geqslant 0}$ which satisfies

$$\big\|\mathcal{T}(t)\mathcal{S}(t)\big\|_{\mathscr{L}(L^2(0,T;H))} \leqslant Me^{-wt}, \quad t \geqslant 0 \tag{3.13}$$

for some $M \geqslant 0$ and $w > 0$.

Proof. It follows directly from definitions (3.5) and (3.9) that for $t \geqslant 0$, we have

$$(\mathcal{S}(t)\mathcal{T}(t)v)(\vartheta) = (\mathcal{T}(t)\mathcal{S}(t)v)(\vartheta) = S(t)v(\vartheta - t)\mathbf{1}\{\vartheta \geqslant t\}, \quad \vartheta \in (0,T),\ v \in L^2(0,T;H),$$

i.e., the semigroups $(\mathcal{T}(t))_{t\geqslant 0}$ and $(\mathcal{S}(t))_{t\geqslant 0}$ commute. From this we may conclude that $(\mathcal{T}(t)\mathcal{S}(t))_{t\geqslant 0}$ is a C_0-semigroup known as the *product semigroup*; its generator is an extension of $-\partial_t - \mathcal{A}$, and the domain of the generator contains $H^1_{0,\{0\}}(0,T;H) \cap L^2(0,T;\mathsf{D}(A))$ as a dense subset, see [30, p. 64]. This means that the generator is the closure of $-\partial_t - \mathcal{A}$, but since the parabolic operator is maximally accretive, its negative generates a semigroup and is thus closed, so we must have that $-\partial_t - \mathcal{A}$ is the generator of $(\mathcal{T}(t)\mathcal{S}(t))$. In order to check (3.13), we recall that $S(t)$ satisfies such an estimate by analyticity and see that it carries over to the product semigroup: given $v \in L^2(0,T;H)$ and $t \geqslant 0$, we have

$$\|\mathcal{T}(t)\mathcal{S}(t)v\|^2_{L^2(0,T;H)} = \int_t^T \|S(t)v(\vartheta - t)\|^2_H \,\mathrm{d}\vartheta \leqslant M^2 e^{-2\delta t}\|v\|^2_{L^2(0,T;H)},$$

and the desired operator norm estimate follows. □

Proposition 3.2.4 tells us that we are in a position to apply equation (2.12) to the parabolic operator, thus producing

$$(\partial_t + \mathcal{A})^{-\gamma} := \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1}\mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s, \tag{3.14}$$

which defines a bounded operator on $L^2(0,T;H)$ for all $\gamma > 0$. It follows that for a given $f \in L^2(0,T;H)$, we have

$$\begin{aligned}\big[(\partial_t + \mathcal{A})^{-\gamma} f\big](t) &:= \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1}[\mathcal{S}(s)\mathcal{T}(s)f](t)\,\mathrm{d}s \\ &= \frac{1}{\Gamma(\gamma)} \int_0^t s^{\gamma-1}S(s)f(t-s)\,\mathrm{d}s \\ &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}S(t-s)f(s)\,\mathrm{d}s\end{aligned} \tag{3.15}$$

for almost all $t \in (0,T)$. It follows from [23, Proposition 5.9] that $(\partial_t + \mathcal{A})^{-\gamma} f$ maps functions in $L^2(0,T;H)$ to functions in $C([0,T];H)$ for $\gamma > 1/2$. Thus, pointwise evaluation of $(\partial_t + \mathcal{A})^{-\gamma} f$ is meaningful, so that the previous display shows that $[(\partial_t + \mathcal{A})^{-\gamma} f](0) = 0$, i.e. $\mathsf{R}((\partial_t + \mathcal{A})^{-\gamma}) = \mathsf{D}((\partial_t + \mathcal{A})^{\gamma}) \subseteq C_{0,\{0\}}([0,T];H)$.

We can also consider the adjoint $(\partial_t + \mathcal{A})^{-\gamma *}$, which will be used in the next section to define solution concepts for the fractional parabolic SPDE and to show how they relate. The following lemma states two properties of the adjoint which are similar to the ones just discussed for the original operator.

Lemma 3.2.5. *For $\gamma > 1/2$, the adjoint negative fractional parabolic operator $(\partial_t + \mathcal{A})^{-\gamma *}$ maps from $L^2(0,T;H)$ to $C_{0,\{T\}}([0,T];H)$ and is given by*

$$[(\partial_t + \mathcal{A})^{-\gamma *} f](t) = \frac{1}{\Gamma(\gamma)} \int_t^T (t-s)^{\gamma-1}S(t-s)f(s)\,\mathrm{d}s \quad \textit{for all } f \in L^2(0,T) \textit{ and } t \in [0,T]. \tag{3.16}$$

Proof. Let $f, g \in L^2(0,T;H)$ be arbitrary, and consider

$$\begin{aligned}((\partial_t + \mathcal{A})^{-\gamma} f, g)_{L^2(0,T;H)} &= \int_0^T ([(\partial_t + \mathcal{A})^{-\gamma} f](t), g(t))_H \,\mathrm{d}t \\ &= \int_0^T \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}S(t-s)f(s)\,\mathrm{d}s, g(t)\right)_H \mathrm{d}t \\ &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T (\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1}S(t-s)f(s), g(t))_H \,\mathrm{d}s\,\mathrm{d}t;\end{aligned} \tag{3.17}$$

here, the interchange of inner product and integral on the third line is justified by (2.8). Next we would like to use Fubini's theorem to change the order of integration. To this end, we check that

$$
\begin{aligned}
&\int_0^T \int_0^T |(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1}S(t-s)f(s), g(t))_H| \,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant \int_0^T \int_0^T \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1}\,\|S(t-s)f(s)\|_H\,\|g(t)\|_H \,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leqslant M\int_0^T \int_0^t (t-s)^{\gamma-1}\,\|f(s)\|_H\,\mathrm{d}s\,\|g(t)\|_H\,\mathrm{d}t \\
&\quad\leqslant M\int_0^T \biggl(\int_0^t (t-s)^{2\gamma-2}\,\mathrm{d}s\biggr)^{\frac12} \bigl\|g(t)\bigr\|_H\,\mathrm{d}t\|f\|_{L^2(0,T;H)} \\
&\quad= C_\gamma M\int_0^T t^{\gamma-\frac12}\bigl\|g(t)\bigr\|_H\,\mathrm{d}t\|f\|_{L^2(0,T;H)} \\
&\quad\leqslant C_\gamma M\biggl(\int_0^T t^{2\gamma-1}\,\mathrm{d}t\biggr)^{\frac12}\|g\|_{L^2(0,T;H)}\|f\|_{L^2(0,T;H)} \\
&\quad= C_\gamma^{(2)} M T^\gamma \|g\|_{L^2(0,T;H)}\|f\|_{L^2(0,T;H)} < \infty.
\end{aligned}
$$

In this estimate, we used the Cauchy–Schwarz inequality for both H and $L^(0,T;\mathbb{R})$, the uniform boundedness of $(S(t))_{t\geqslant 0}$ with constant $M \geqslant 1$, and the fact that $\gamma > 1/2$ so that the integrals on the fourth and sixth lines admitted antiderivatives vanishing at zero. So we change the order of integration on the last line of (3.17) to obtain

$$
\begin{aligned}
((\partial_t+\mathcal{A})^{-\gamma}f, g)_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)}\int_0^T\int_0^T (\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1}S(t-s)f(s), g(t))_H\,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\gamma)}\int_0^T\int_0^T (f(s), \mathbf{1}_{(s,T)}(t)(t-s)^{\gamma-1}S(t-s)g(t))_H\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^T \biggl(f(s), \frac{1}{\Gamma(\gamma)}\int_s^T (t-s)^{\gamma-1}S(t-s)g(t)\,\mathrm{d}t\biggr)_H \mathrm{d}s,
\end{aligned}
$$

where we used the self-adjointness of A and hence that of $S(t-s)$ in the second line, and interchanged integrals and inner products as before on the last. Relabeling $s \leftrightarrow t$ and $g \leftrightarrow f$, this establishes identity (3.16).

The fact that $(\partial_t + \mathcal{A})^{-\gamma *}f$ belongs to $C_{0,\{T\}}([0,T];H)$ for all $f \in L^2(0,T)$ can be obtained by reasoning as for $(\partial_t + \mathcal{A})^{-\gamma}$. ☐

Lastly we note that $(\partial_t + \mathcal{A})^{-\gamma *} = ([\partial_t + \mathcal{A}]^*)^{-\gamma}$. To see that the fractional power on the right-hand side is indeed well-defined, we use [57, Corollary I.10.6] to see that $-[\partial_t + \mathcal{A}]^*$ is the generator of the C_0-semigroup $([\mathcal{S}(t)\mathcal{T}(t)]^*)_{t\geqslant 0}$, which clearly inherits the exponential boundedness from $(\mathcal{S}(t)\mathcal{T}(t))_{t\geqslant 0}$ since their norms are equal. The equality is then obtained as follows:

$$
(\partial_t+\mathcal{A})^{-\gamma *} = \biggl(\frac{1}{\Gamma(\gamma)}\int_0^\infty s^{\gamma-1}\mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s\biggr)^* = \frac{1}{\Gamma(\gamma)}\int_0^\infty s^{\gamma-1}[\mathcal{S}(s)\mathcal{T}(s)]^*\,\mathrm{d}s = ([\partial_t+\mathcal{A}]^*)^{-\gamma},
$$

where the first and last equalities are due to (3.14) and the second is a consequence of (2.8).

3.3. Solution concepts for the fractional parabolic SPDE

We now turn back to the matter of defining solutions to (3.1) for fractional powers γ. Having defined and investigated the parabolic operator $\partial_t + \mathcal{A}$, its domain and its fractional powers, we are now in particular able to invert the fractional parabolic operator $(\partial_t + \mathcal{A})^\gamma$. The observations from the end of the previous section suggest that the fractional parabolic equation (3.1) with initial datum $X_0 = 0$ has a mild solution which may be defined as the following 'fractional stochastic convolution':

$$
Z_\gamma(t) := \bigl[(\partial_t+\mathcal{A})^{-\gamma}\dot{W}\bigr](t) := \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}S(t-s)\,\mathrm{d}W(s), \quad \text{a.a. } t\in(0,T); \tag{3.18}
$$

which by the Itô isometry or the Burkholder–Davis–Gundy inequality is only well-defined if we assume the following analog to (3.4):

$$\int_0^T t^{2\gamma-2} \|S(t)\|^2_{\mathscr{L}_2(\mathcal{H};H)} \,\mathrm{d}t < \infty. \tag{3.19}$$

In order to provide a more rigorous justification for defining mild solutions to the fractional parabolic equation in this way, we would like to proceed as in the non-fractional case (Section 3.1), namely by finding a suitable weak solution concept which follows naturally from the fractional SPDE and show that it is equivalent to (3.18). For this, we use the following weak solution concept, which is based on $L^2(0,T;H)$ inner products and the weak stochastic integral; it is an analog to Definition 3.1.3 for the parabolic fractional problem with zero initial condition.

Definition 3.3.1 (Fractional weak solution, $X_0 = 0$)**.** A H-valued predictable process $(X(t))_{t\in[0,T]}$ is called a *weak solution* to (3.1) with $X_0 = 0$ if its paths are Bochner integrable $\mathbb{P}$-a.s. and for all $\varphi \in \mathsf{D}((\partial_t + \mathcal{A})^{\gamma*})$ it holds $\mathbb{P}$-a.s. that

$$(X, (\partial_t + \mathcal{A})^{\gamma*}\varphi)_{L^2(0,T;H)} = \int_0^T (\varphi(t), \mathrm{d}W(t))_H. \tag{3.20}$$

We now claim that the weak and mild solutions defined in this section are indeed equivalent.

Proposition 3.3.2. *If A satisfies the standing assumptions from this chapter and condition (3.19) holds, then Z_γ is the unique weak solution to (3.1) with $X_0 = 0$ in the sense of Definition 3.3.1.*

Proof. First we show that Z_γ is a weak solution. To this end, let $\varphi \in \mathsf{D}((\partial_t + \mathcal{A})^{\gamma*})$ be arbitrary and set $\psi := (\partial_t + \mathcal{A})^{\gamma*}\varphi$ for convenience. Then we use the definitions of the Bochner inner product and of Z_γ, pull an inner product inside the integral and use adjoints to obtain, $\mathbb{P}$-a.s.,

$$\begin{aligned}
(Z_\gamma, (\partial_t + \mathcal{A})^{\gamma*}\varphi)_{L^2(0,T;H)} &= (Z_\gamma, \psi)_{L^2(0,T;H)} \\
&= \frac{1}{\Gamma(\gamma)} \int_0^T \left(\int_0^t (t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W(s), \psi(t) \right)_H \mathrm{d}t \\
&= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T (\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W(s), \psi(t))_H \,\mathrm{d}t \\
&= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T (\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)\psi(t), \mathrm{d}W(s))_H \,\mathrm{d}t.
\end{aligned}$$

Note that the interchange of inner products and integrals is justified by a combination of (2.8) and the definition of the weak stochastic integral. Also by the definition of the weak stochastic integral, we have $\mathbb{P}$-a.s.

$$\frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T (\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)\,\mathrm{d}W(s), \psi(t))_H \,\mathrm{d}t = \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \Psi(s,t)\,\mathrm{d}W(s)\,\mathrm{d}t,$$

where the integrand $\Psi(s,t) \colon \mathcal{H} \to \mathbb{R}$ is defined for $s,t \in [0,T]$ by

$$\Psi(s,t)u := (\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)u, \psi(t))_H.$$

Note that, given an orthonormal basis $(g_j)_{j\geqslant 1}$ of $\mathcal{H}$,

$$\begin{aligned}
\|\Psi(s,t)\|^2_{\mathscr{L}_2(\mathcal{H};\mathbb{R})} &= \sum_{j=1}^\infty |\Psi(s,t)g_j|^2 \\
&= \sum_{j=1}^\infty |(\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)g_j, \psi(t))_H|^2 \\
&\leqslant \sum_{j=1}^\infty \|\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)g_j\|_H^2 \, \|\psi(t)\|_H^2 \\
&= \|\mathbf{1}\{s \leqslant t\}(t-s)^{\gamma-1} S(t-s)\|^2_{\mathscr{L}_2(\mathcal{H};H)} \, \|\psi(t)\|_H^2
\end{aligned}$$

From this, it follows that

$$\begin{aligned}\int_0^T \left(\int_0^T \|\Psi(s,t)\|^2_{\mathscr{L}_2(\mathcal{H};\mathbb{R})}\,\mathrm{d}s\right)^{\frac{1}{2}} \mathrm{d}t &\leqslant \int_0^T \left(\int_0^t \|(t-s)^{\gamma-1}S(t-s)\|^2_{\mathscr{L}_2(\mathcal{H};H)}\,\|\psi(t)\|_H^2\,\mathrm{d}s\right)^{\frac{1}{2}} \mathrm{d}t\\ &= \int_0^T \left(\int_0^t \|s^{\gamma-1}S(s)\|^2_{\mathscr{L}_2(\mathcal{H};H)}\,\mathrm{d}s\right)^{\frac{1}{2}} \|\psi(t)\|_H\,\mathrm{d}t\\ &\leqslant \int_0^T \left(\int_0^T \|s^{\gamma-1}S(s)\|^2_{\mathscr{L}_2(\mathcal{H};H)}\,\mathrm{d}s\right)^{\frac{1}{2}} \|\psi(t)\|_H\,\mathrm{d}t\\ &\leqslant T^{\frac{1}{2}} \int_0^T \|s^{\gamma-1}S(s)\|^2_{\mathscr{L}_2(\mathcal{H};H)}\,\mathrm{d}s\,\|\psi\|_{L^2(0,T;H)} < \infty,\end{aligned}$$

where we used the Cauchy–Schwarz inequality for $L^2(0,T;\mathbb{R})$ on the last line. Recall that the integral on the last line is finite owing to the assumption (3.19). Applying the stochastic Fubini theorem to change the order of integration and pulling the deterministic integral into the inner product yields

$$\begin{aligned}(Z_\gamma, (\partial_t+\mathcal{A})^{\gamma*}\varphi)_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)}\int_0^T \left(\int_s^T (t-s)^{\gamma-1}S^*(t-s)\psi(t)\,\mathrm{d}t, \mathrm{d}W(s)\right)_H\\ &= \int_0^T ([(\partial_t+\mathcal{A})^{-\gamma*}\psi](s), \mathrm{d}W(s))_H\\ &= \int_0^T (\varphi(s), \mathrm{d}W(s))_H,\end{aligned}$$

where we use (3.16) to derive the second line from the first, and subsequently recall that $\psi := (\partial_t + \mathcal{A})^{\gamma*}\varphi$. This proves that Z_γ is a weak solution.

Conversely, suppose that X is a weak solution, let an arbitrary $\psi \in L^2(0,T;H)$ be given and set $\varphi := (\partial_t+\mathcal{A})^{-\gamma*}\psi$. Substituting this into the definition of the weak solution gives

$$(X,\psi)_{L^2(0,T;H)} = \int_0^T ((\partial_t+\mathcal{A})^{-\gamma*}\psi(t), \mathrm{d}W(t))_H,$$

and reading the proof of the previous implication backwards, we see that

$$(X,\psi)_{L^2(0,T;H)} = (Z_\gamma,\psi)_{L^2(0,T;H)}$$

$\mathbb{P}$-a.s. for all $\psi \in L^2(0,T;H)$, hence $X = Z_\gamma$ in $L^2(0,T;H)$ $\mathbb{P}$-a.s. □

This covers the case $X_0 = 0$; from this we can deduce the nonzero initial value case using the linearity of SPDE (3.1). To this end, we ignore for the moment the domain of $(\partial_t+\mathcal{A})^\gamma$ and argue that X being a (weak or mild) solution to (3.1) should formally be equivalent to $Y(t) := X(t) - X_0$ solving

$$(\partial_t+\mathcal{A})^\gamma Y = \dot{\mathcal{W}} - A^\gamma X_0, \quad Y(0) = 0,$$

since the time derivative occurring in $(-\partial_t+\mathcal{A})^\gamma$ should not affect the time-independent function X_0. Then, taking $L^2(0,T;H)$ inner products and using adjoints, we can state an analog of (3.20): Y must satisfy

$$(Y, (\partial_t+\mathcal{A})^{\gamma*}\varphi)_{L^2(0,T;H)} = \int_0^T (\varphi(t), \mathrm{d}W(t))_H - (X_0, \mathcal{A}^{\gamma*}\varphi)_{L^2(0,T;H)}.$$

for all $\varphi \in \mathsf{D}((\partial_t+\mathcal{A})^{\gamma*})$. Turning back to $X(t) = Y(t) + X_0$ then gives rise to the following definition of a weak solution to the nonhomogeneous problem.

Definition 3.3.3 (Fractional weak solution)**.** A H-valued predictable process $(X(t))_{t\in[0,T]}$ is called a *weak solution* to (3.1) if its paths are Bochner integrable $\mathbb{P}$-a.s. and for all $\varphi \in \mathsf{D}((\partial_t+\mathcal{A})^{\gamma*})$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}(X, (\partial_t+\mathcal{A})^{\gamma*}\varphi)_{L^2(0,T;H)} = \int_0^T (\varphi(t), \mathrm{d}W(t))_H &+ (X_0, (\partial_t+\mathcal{A})^{\gamma*}\varphi)_{L^2(0,T;H)}\\ &- (X_0, \mathcal{A}^{\gamma*}\varphi)_{L^2(0,T;H)}.\end{aligned} \tag{3.21}$$

The next proposition states that we can associate a mild solution to the weak solution in the non-homogeneous case as well.

Proposition 3.3.4. *Define*

$$Z_0(\omega,t) := \frac{1}{\Gamma(\gamma)} A^\gamma \int_t^\infty s^{\gamma-1} S(s) X_0(\omega)\,\mathrm{d}s, \quad \textit{a.a. } t \in (0,T) \textit{ and } \omega \in \Omega.$$

If A satisfies the standing assumptions from this chapter and condition (3.19) holds, then $Z := Z_\gamma + Z_0$ is the unique weak solution to (3.1) in the sense of Definition 3.3.1.

Proof. First note that Z_0 is indeed well-defined. To see this, first fix a representative of X_0 and some $\omega \in \Omega$. The integrand appearing in the definition of Z_0 then belongs to $\mathsf{D}(A^\gamma)$ for all $s > t$ by analyticity of the semigroup $S(s)$, hence the same holds for the integral, which converges for the same reason that (2.12) does. Also note that

$$\begin{aligned} Z_0(\omega,t) &= \frac{1}{\Gamma(\gamma)} A^\gamma \int_0^\infty s^{\gamma-1} S(s) X_0(\omega)\,\mathrm{d}s - \frac{1}{\Gamma(\gamma)} A^\gamma \int_0^t s^{\gamma-1} S(s) X_0(\omega)\,\mathrm{d}s \\ &= X_0(\omega) - \frac{1}{\Gamma(\gamma)} A^\gamma \int_0^t s^{\gamma-1} S(s) X_0(\omega)\,\mathrm{d}s, \end{aligned}$$

since the integral in the first term of the first line equals $A^{-\gamma} X_0(\omega)$. Viewing $X_0(\omega)$ as a constant function in $L^2(0,T;H)$, this can be summarized as

$$Z_0(\omega) = X_0(\omega) - \mathcal{A}^\gamma (\partial_t + \mathcal{A})^{-\gamma} X_0(\omega). \tag{3.22}$$

Clearly, given another representative of X_0, this identity holds on a $\mathbb{P}$-a.s. set, hence Z_0 is well-defined. In what follows, we omit the argument ω from X_0 and Z_0; since all the identities in this proof involving random variables are meant in the almost sure sense, we assume that an ω is fixed throughout so that we may treat X_0 and Z_0 as if they are deterministic.

Now we wish to show that Z is a weak solution. Let $\varphi \in \mathsf{D}((\partial_t + \mathcal{A})^{\gamma *})$. In order to check (3.21), by Proposition 3.3.2 it suffices to establish

$$(Z_0, (\partial_t + \mathcal{A})^{\gamma *}\varphi)_{L^2(0,T;H)} = (X_0, (\partial_t + \mathcal{A})^{\gamma *}\varphi - \mathcal{A}^{\gamma *}\varphi)_{L^2(0,T;H)}, \tag{3.23}$$

which by (3.22) is equivalent to

$$(\mathcal{A}^\gamma (\partial_t + \mathcal{A})^{-\gamma} X_0, (\partial_t + \mathcal{A})^{\gamma *}\varphi)_{L^2(0,T;H)} = (X_0, \mathcal{A}^{\gamma *}\varphi)_{L^2(0,T;H)}. \tag{3.24}$$

First let $X_0 \in \mathsf{D}(A^\gamma)$. In this case, we have that

$$\frac{1}{\Gamma(\gamma)} A^\gamma \int_0^t s^{\gamma-1} S(s) X_0\,\mathrm{d}s = \frac{1}{\Gamma(\gamma)} \int_0^t s^{\gamma-1} S(s) A^\gamma X_0\,\mathrm{d}s$$

for almost all $t \in (0,T)$; this holds since A^γ is closed and commutes with $S(s)$ and since the Bochner integral on the right-hand side exists, which can be seen again by comparing with (2.12). It follows that

$$\mathcal{A}^\gamma (\partial_t + \mathcal{A})^{-\gamma} X_0 = (\partial_t + \mathcal{A})^{-\gamma} \mathcal{A}^\gamma X_0,$$

which in turn implies (3.24) after taking adjoints. Now we would like to use the density of $\mathsf{D}(A^\gamma)$ in H in order to obtain identity (3.23) for all $X_0 \in H$. To this end, it suffices to prove that the first arguments of the inner products on both sides of the equation depend continuously on X_0. For the right-hand side, this follows easily from $\|X_0\|_{L^2(0,T;H)} = T^{1/2}\|X_0\|_H$. For the left-hand side, observe that

$$\begin{aligned} \|Z_0\|^2_{L^2(0,T;H)} &= \int_0^T \left\| \frac{1}{\Gamma(\gamma)} A^\gamma \int_t^\infty s^{\gamma-1} S(s) X_0\,\mathrm{d}s \right\|_H^2 \mathrm{d}t \\ &\leqslant \frac{1}{\Gamma(\gamma)^2} \int_0^T \left(\int_t^\infty \|A^\gamma s^{\gamma-1} S(s) X_0\|_H\,\mathrm{d}s \right)^2 \mathrm{d}t \\ &\leqslant \frac{M^2}{\Gamma(\gamma)^2} \int_0^T \left(\int_t^\infty s^{-1} e^{-\delta s}\,\mathrm{d}s \right)^2 \mathrm{d}t\, \|X_0\|_H^2, \end{aligned}$$

where we use that $\|A^\gamma S(s)\|_{\mathscr{L}(H)} \leqslant M s^{-\gamma} e^{-\delta s}$ for some $M \geqslant 1, \delta > 0$ and all $s > 0$. The iterated integral is finite since

$$\begin{aligned}\int_0^T \left(\int_t^\infty s^{-1} e^{-\delta s} \,\mathrm{d}s \right)^2 \mathrm{d}t &\leqslant \int_0^\infty \left(\int_t^\infty s^{-1} e^{-\delta s} \,\mathrm{d}s \right)^2 \mathrm{d}t \\ &= \int_0^\infty \left(\int_{\delta t}^\infty u^{-1} e^{-u} \,\mathrm{d}u \right)^2 \mathrm{d}t = \delta^{-1} \log 4,\end{aligned}$$

where we used the change of variables $u = \delta s$ and an identity for the integral of a squared exponential integral [35, Identity 4.6.2]. This completes the proof that X is a weak solution.

Conversely, suppose that X is a weak solution. Let an arbitrary $\psi \in L^2(0,T;H)$ be given and define $\varphi := (\partial_t + \mathcal{A})^{-\gamma *}\psi$ so that $\varphi \in \mathsf{D}((\partial_t + \mathcal{A})^{\gamma *})$. Then (3.21) becomes

$$\begin{aligned}(X, \psi)_{L^2(0,T;H)} = \int_0^T ([(\partial_t + \mathcal{A})^{-\gamma *}\psi](t), \mathrm{d}W(t))_H + (X_0, \psi)_{L^2(0,T;H)} \\ -(X_0, \mathcal{A}^{\gamma *}(\partial_t + \mathcal{A})^{-\gamma *}\psi)_{L^2(0,T;H)}.\end{aligned}$$

Applying the corresponding implication of Proposition 3.3.2 yields

$$(X, \psi)_{L^2(0,T;H)} = (Z_\gamma + X_0, \psi)_{L^2(0,T;H)} - (X_0, \mathcal{A}^{\gamma *}(\partial_t + \mathcal{A})^{-\gamma *}\psi)_{L^2(0,T;H)},$$

but the same density argument as before now shows that

$$(X_0, \mathcal{A}^{\gamma *}(\partial_t + \mathcal{A})^{-\gamma *}\psi)_{L^2(0,T;H)} = (\mathcal{A}^{\gamma}(\partial_t + \mathcal{A})^{-\gamma} X_0, \psi)_{L^2(0,T;H)},$$

and thus

$$(X, \psi)_{L^2(0,T;H)} = (Z_\gamma + Z_0, \psi)_{L^2(0,T;H)}$$

for arbitrary $\psi \in L^2(0,T;H)$; the desired conclusion follows. □

3.4. Well-posedness and regularity of the mild solution

3.4.1. Regularity of the fractional stochastic convolution

This section is devoted to studying the spatial and temporal regularity of the fractional stochastic convolution Z_γ. The main results are Corollary 3.4.2 and Theorem 3.4.4, which are essentially extensions of [44, Lemma 4.1 and Theorem 4.2], respectively. They link the existence and regularity in the $L^p(\Omega), p \geqslant 2$ sense of mild solutions to the boundedness of the Hilbert–Schmidt norm of operators involving Q, A and their fractional powers. An application of the Kolmogorov–Chentsov continuity theorem allows us to partially transport these regularity result to the pathwise setting, see Theorem 3.4.8. This section is concluded with Example 3.4.9, which demonstrates the concrete conditions on the parameters $\alpha, \beta \geqslant 0$ when taking $A := L^\beta$ and $Q := \widetilde{L}^{-\alpha}$ for two given self-adjoint and uniformly elliptic operators L and $\widetilde{L}$; this is a very important example since it covers the Whittle–Matérn situation.

For given constants $a, b \in \mathbb{R}$ and $\sigma \geqslant 0$, define the function $\Phi_{a,b} \colon [0,T] \to \mathscr{L}(\mathcal{H}; \dot{H}^\sigma_A)$ by

$$\Phi_{a,b}(t) := t^a A^b S(t).$$

Moreover, we recall two estimates which hold for analytic semigroups such as $(S(t))_{t \geqslant 0}$ and which will be used for the estimates in what follows: (see [57, Theorem 6.13, p. 74])

$$\|A^\alpha S(t)\|_{\mathscr{L}(H)} \leqslant C_\alpha t^{-\alpha}, \qquad \alpha \geqslant 0,\ t > 0, \tag{3.25}$$

$$\|(S(t) - I)A^{-\alpha}\|_{\mathscr{L}(H)} \leqslant C_\alpha t^{\alpha}, \qquad 0 \leqslant \alpha \leqslant 1,\ t \geqslant 0. \tag{3.26}$$

Theorem 3.4.1 (Spatial regularity). *Let $a, b \in \mathbb{R}$ and $\sigma \geqslant 0$ be such that $b + \frac{\sigma}{2} \geqslant 0$ and $a - b - \frac{\sigma}{2} > -\frac{1}{2}$ and suppose that*

$$\|A^{b-a-\frac{1}{2}}\|_{\mathscr{L}_2(\mathcal{H}; \dot{H}^\sigma_A)} < \infty.$$

Then $t \mapsto \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s)$ belongs to $C([0,T]; L^p(\Omega; \dot{H}^\sigma_A))$ for $p \geqslant 2$.

Proof. We first check that $\int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s)$ is a well-defined element of $L^p(\Omega;\dot{H}^\sigma_A)$ for arbitrary $t \in [0,T]$. By the Burkholder–Davis–Gundy inequality, we know that this is equivalent to $\int_0^t \|s^a A^b S(s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s < \infty$. Let $t \in [0,T]$ and $x \in H$. Using the self-adjointness of A (and hence of $S(t)$) and changing variables from t to $2t$, we obtain

$$\begin{aligned}\int_0^T \left\|t^a A^b S(t)x\right\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}t &= \int_0^T (t^a A^b S(t)x, t^a A^b S(t)x)_{\dot{H}^\sigma_A}\,\mathrm{d}t \\ &= \int_0^T (t^{2a} A^{2b+\sigma} S(2t)x, x)_H\,\mathrm{d}t \\ &= 2^{-1-2a}\int_0^T (t^{2a} A^{2b+\sigma} S(t)x, x)_H\,\mathrm{d}t \\ &= 2^{-1-2a}\left(\int_0^T t^{2a} A^{2b+\sigma} S(t)x\,\mathrm{d}t, x\right)_H.\end{aligned}$$

Suppose for the moment that in fact $x \in \mathsf{D}(A^{2b+\sigma})$. Then we may interchange $S(t)$ with $A^{2b+\sigma}$ on the last line, and by equation (2.12), the integral on the last line converges in H as $T \to \infty$ if and only if $2a+1 > 0$, which holds since we assume $a > -1/2$. Thus, passing to this limit and using the continuity of inner products yields

$$\begin{aligned}\int_0^\infty \left\|t^a A^b S(t)x\right\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}t &= 2^{-1-2a}\Gamma(2a+1)(A^{-1-2a}A^{2b+\sigma}x, x)_H \\ &= 2^{-1-2a}\Gamma(2a+1)\,\|A^{b-a-\frac{1}{2}}x\|^2_{\dot{H}^\sigma_A}.\end{aligned}$$

It follows that

$$\int_0^T \left\|t^a A^b S(t)x\right\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}t \leqslant 2^{-1-2a}\Gamma(2a+1)\,\|A^{b-a-\frac{1}{2}}x\|^2_{\dot{H}^\sigma_A}. \tag{3.27}$$

Now we want to use the density of $\mathsf{D}(A^{2b+\sigma})$ in H to obtain (3.27) for arbitrary $x \in H$. To this end, we show that both sides of the inequality depend continuously on x. For the right-hand side, this follows from the boundedness of $A^{b-a-\frac{1}{2}+\frac{\sigma}{2}}$. For the left-hand side we make use of (3.25) to see that

$$\int_0^t \|s^a A^b S(s)x\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}s \leqslant C_{b,\sigma}\int_0^t s^{2a-2b-\sigma}\,\mathrm{d}s\,\|x\|^2_H = C_{a,b,\sigma,t}t^{2a-2b-\sigma+1}\,\|x\|^2_H,$$

which requires $b+\sigma/2 \geqslant 0$ for the estimate and $a-b-\sigma/2 > -1/2$ for the evaluation of the integral. If $x_n \to x$ is the approximating sequence in $\mathsf{D}(A^{2b+\sigma})$ converging in the H-norm, then we can estimate

$$\int_0^t \|s^a A^b S(s)x_n\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}s \leqslant C_{a,b,\sigma,t}t^{2a-2b-\sigma+1}\sup_{n\geqslant 1}\|x_n\|^2_H,$$

so that we may pull the limit as $n \to \infty$ inside the integral by dominated convergence, which gives establishes (3.27) for all $x \in H$. Finally, for the Hilbert–Schmidt norm this implies (using Fubini)

$$\int_0^t \|s^a A^b S(s)x\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s \leqslant 2^{-1-2a}\Gamma(2a+1)\,\|A^{b-a-\frac{1}{2}}x\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}.$$

In view of the assumption $\|A^{b-a-\frac{1}{2}}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} < \infty$, this shows that $Z_\gamma(t) \in L^p(\Omega;\dot{H}^\sigma_A)$ for all $t \in [0,T]$ and $p \geqslant 2$.

It remains to check the mean square continuity of Z_γ. For $t \geqslant 0$ and (without loss of generality) $h > 0$, we split the stochastic integral

$$\begin{aligned}&\int_0^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s) \\ &\quad= \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\,\mathrm{d}W(s).\end{aligned}$$

For $p \geqslant 2$, the Burkholder–Davis–Gundy inequality yields

$$\begin{aligned}
&\left\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\,\mathrm{d}W(s) \right\|_{L^p(\Omega;H)} \\
&\leqslant C_p \left[\int_t^{t+h} \|\Phi_{a,b}(t+h-s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s \right]^{\frac{1}{2}} + C_p \left[\int_0^t \left\|\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s \right]^{\frac{1}{2}} \\
&\leqslant 5C_p \left[\int_0^T \|\Phi_{a,b}(s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s \right]^{\frac{1}{2}} < \infty,
\end{aligned}$$

so that the continuity follows by taking the limit as $h \to 0$ inside the integrals on the second line, which is justified by dominated convergence owing to the bound on the last line. ☐

Taking $a = \gamma - 1$ and $b = 0$ yields the following statement for the parabolic stochastic convolution Z_γ.

Corollary 3.4.2 (Spatial regularity of Z_γ)**.** *Let $\gamma \in \mathbb{R}$ and $\sigma \geqslant 0$ be such that $\gamma - \frac{\sigma}{2} > \frac{1}{2}$ and suppose that*

$$\|A^{-\gamma+\frac{1}{2}}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} < \infty.$$

Then Z_γ belongs to $C([0,T];L^p(\Omega;\dot{H}^\sigma_A))$ for all $p \geqslant 2$.

Next we investigate the temporal regularity of Z_γ. We need some information about the derivatives of the integrands $\Phi_{a,b}(t)$.

Lemma 3.4.3. *For all $x \in H$, the orbit $t \mapsto \Phi_{a,b}(t)x$ belongs to $C^\infty((0,T];\dot{H}^\sigma_A)$ and its nth derivative is*

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Phi_{a,b}(t)x = \sum_{j=0}^n C_{a,j,n} t^{a-(n-j)} A^{b+j} S(t)x = \sum_{j=0}^n C_{a,j,n} \Phi_{a-(n-j),b+j}(t)x, \tag{3.28}$$

where

$$C_{a,j,n} = (-1)^j \binom{n}{j} \prod_{i=1}^{n-j} (a - (n-j) + i).$$

Proof. Fix an $x \in \mathcal{H}$. By analyticity, the orbit $t \mapsto S(t)x$ is in $C^\infty((0,T];H)$ with nth derivative $(-A)^n S(t)x \in \dot{H}^\sigma_A$ for all $t > 0$. Now fix $t \in (0,T]$ and let $\varepsilon := t/2$. Observe that

$$\begin{aligned}
{}^{(n)}(t) = [S(\cdot - \varepsilon)A^{b+\sigma/2}S(\varepsilon)x]^{(n)}(t) &= (-A)^n S(t-\varepsilon)A^{b+\sigma/2}S(\varepsilon)x \\
&= (-1)^n A^{n+b+\sigma/2} S(t)x,
\end{aligned}$$

where the limits for the derivatives are taken in the H norm. This is equivalent to saying that

$${}^{(n)}(t) = (-1)^n A^{n+b} S(t)x$$

with respect to the $\dot{H}^\sigma_A$ norm. The expression (3.28) for the nth derivative of $t \mapsto \Phi(t)x$ now follows from a general product rule. ☐

It is convenient to define for $t > 0$ the suggestively named operators $\Phi^{(n)}_{a,b}(t) \in \mathscr{L}(\mathcal{H};\dot{H}^\sigma_A)$ as

$$\Phi^{(n)}_{a,b}(t) := \sum_{j=0}^n C_{a,j,n} \Phi_{a-(n-j),b+j}(t) \tag{3.29}$$

by analogy with equation (3.28). In particular, denote $\Phi'_{a,b}(t) := \Phi^{(1)}_{a,b}(t)$.

Theorem 3.4.4. *Suppose the constants $\sigma \geqslant 0$, $n \in \mathbb{N}^{\geqslant 0}$, $0 < \tau \leqslant 1$ and $\gamma > n + \tau + \frac{1}{2} + \frac{\sigma}{2}$ are such that*

$$\|A^{n+\tau-\gamma+1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} < \infty.$$

Then for all $p \geqslant 2$, Z_γ belongs to $C^{n,\tau}([0,T];L^p(\Omega;\dot{H}^\sigma_A))$ with nth derivative

$$Z^{(n)}_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \Phi^{(n)}_{\gamma-1,0}(t-s)\,\mathrm{d}W(s). \tag{3.30}$$

The proof depends on the following intermediate results.

Lemma 3.4.5. *Let $a, b \in \mathbb{R}$, $\sigma \geqslant 0$ and $\delta > 0$ be such that*

$$a - \delta + \frac{1}{2} > \max\left\{0, b + \frac{\sigma}{2}\right\}$$

and

$$\|A^{\delta-a+b-1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} < \infty.$$

Then for $t \in [0,T]$ and $h > 0$ satisfying $t + h \in [0,T]$, it holds that

$$\left\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \leqslant C_{p,a,\delta} h^\delta \, \|A^{\delta-a+b-1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)},$$

where $p \geqslant 2$.

Proof. We apply the Burkholder–Davis–Gundy inequality followed by (3.25) to estimate the norm as follows:

$$\begin{aligned}
&\left\| \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\
&= \left\| \int_t^{t+h} (t+h-s)^a A^{a-\delta+1/2} S(t+h-s) A^{\delta-a+b-1/2}\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\
&\leqslant C_p \left[\int_t^{t+h} \left\| (t+h-s)^a A^{a-\delta+1/2} S(t+h-s) A^{\delta-a+b-1/2} \right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \mathrm{d}s \right]^{1/2} \\
&\leqslant C_{p,\delta,a} \left[\int_t^{t+h} (t+h-s)^{2\delta-1}\,\mathrm{d}s \right]^{1/2} \left\| A^{\delta-a+b-1/2} \right\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \\
&= C_{p,\delta,a} (2\delta)^{-1/2} h^\delta \left\| A^{\delta-a+b-1/2} \right\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)},
\end{aligned}$$

where we need $a - \delta + 1/2 \geqslant 0$ in order to apply (3.25) on the fourth line and $\delta > 0$ on the last line. □

Lemma 3.4.6. *Let $a, b \in \mathbb{R}$, $\sigma \geqslant 0$ and $0 < \delta \leqslant \min\{1, a + \frac{1}{2}\}$ be such that*

$$a - \delta + \frac{1}{2} > \max\left\{0, b + \frac{\sigma}{2}\right\}$$

and

$$\|A^{\delta-a+b-1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} < \infty.$$

Then for $t \in [0,T]$ and $h > 0$ satisfying $t + h \in [0,T]$, it holds that

$$\left\| \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \leqslant C_{p,a,\delta} h^\delta \, \|A^{\delta-a+b-1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}.$$

Proof. We use the Burkholder–Davis–Gundy inequality, then perform the change of variables $s \leftrightarrow t - s$ and subsequently split as follows:

$$\begin{aligned}
&\left\| \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\,\mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}^\sigma_A)} \\
&\leqslant C_p \left(\int_0^t \left\| (t+h-s)^a A^b S(t+h-s) - (t-s)^a A^b S(t-s) \right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \mathrm{d}s \right)^{1/2} \\
&= C_p \left(\int_0^t \left\| s^a A^b S(s) - (s+h)^a A^b S(s+h) \right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \mathrm{d}s \right)^{1/2} \\
&\leqslant C_p \left(\int_0^t \left\| s^a A^b [S(s) - S(s+h)] \right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \mathrm{d}s \right)^{1/2} \\
&\qquad + C_p \left(\int_0^t \left\| [s^a - (s+h)^a] A^b S(s+h) \right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \mathrm{d}s \right)^{1/2} \\
&=: C_p(T_1 + T_2).
\end{aligned}$$

Then, given an orthonormal basis $(g_j)_{j\geqslant 1}$ for $\mathcal{H}$, we have

$$\begin{aligned}
T_1 &= \left(\int_0^t \left\|s^a A^b[S(s)-S(h+s)]\right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s\right)^{1/2} \\
&= \left(\sum_{j=1}^\infty \int_0^t \left\|s^a(I-S(h))A^{-\delta}A^{\delta+b}S(s)g_j\right\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}s\right)^{1/2} \\
&\leqslant C_\delta h^\delta \left(\sum_{j=1}^\infty \int_0^t \left\|s^a A^{\delta+b}S(s)g_j\right\|^2_{\dot{H}^\sigma_A}\,\mathrm{d}s\right)^{1/2} \\
&\leqslant C_{\delta,a} h^\delta \,\|A^{\delta-a+b-1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,.
\end{aligned}$$

Here we used $0<\delta\leqslant 1$ to use estimate (3.26) on the third line; for the last line, we need to argue as in Theorem 3.4.1 and thus need $b+\delta+\frac{\sigma}{2}\geqslant 0$ and $a-b-\delta-\frac{\sigma}{2}>-\frac{1}{2}$.

For T_2 we introduce an arbitrary $\varepsilon>0$, use (3.25) and subsequently perform a change of variables $s\leftrightarrow s/h$ to see that

$$\begin{aligned}
T_2 &= \left(\int_0^t \left\|[s^a-(s+h)^a]A^b S(s+h)\right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}s\right)^{1/2} \\
&\leqslant C_{\delta,a}\left(\int_0^t \left\|[s^a-(s+h)^a](s+h)^{\delta-a-\frac{1}{2}-\varepsilon}A^{\delta-a+b-\frac{1}{2}-\varepsilon}\right\|^2_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,\mathrm{d}r\right)^{1/2} \\
&= C_{\delta,a}\left(\int_0^t [s^a-(s+h)^a]^2(s+h)^{2\delta-2a-1-2\varepsilon}\,\mathrm{d}s\right)^{1/2}\left\|A^{\delta-a+b-\frac{1}{2}-\varepsilon}\right\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \\
&\leqslant C_{\delta,a,\varepsilon}\left(\int_0^t [s^a-(s+h)^a]^2(s+h)^{2\delta-2a-1-2\varepsilon}\,\mathrm{d}s\right)^{1/2}\left\|A^{\delta-a+b-\frac{1}{2}}\right\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \\
&= C_{\delta,a,\varepsilon}h^c\left(\int_0^{t/h} [s^a-(s+1)^a]^2(s+1)^{2\delta-2a-1-2\varepsilon}\,\mathrm{d}s\right)^{1/2}\left\|A^{\delta-a+b-\frac{1}{2}}\right\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} \\
&\leqslant C_{\delta,a,\varepsilon}h^c\left(\int_0^{\infty} [s^a-(s+1)^a]^2(s+1)^{2\delta-2a-1-2\varepsilon}\,\mathrm{d}s\right)^{1/2}\left\|A^{\delta-a+b-\frac{1}{2}}\right\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}\,.
\end{aligned}$$

The first inequality in this display requires $a+\frac{1}{2}-\delta+\varepsilon\geqslant 0$, which is satisfied since $a-\delta+\frac{1}{2}>0$ by assumption. The introduction of the $\varepsilon>0$ causes the improper integral appearing on the last line to converge: as $s\downarrow 0$ the integrand is $\mathcal{O}(s^{2a})$ with $2a>-1$; as $s\to\infty$ it is $\mathcal{O}(s^{2\delta-3})$ with

$$2\delta-3-2\varepsilon\leqslant -1-2\varepsilon<-1.$$

This finishes the proof. $\square$

Combining Lemmas 3.4.5 and 3.4.6 yields the following corollary.

Corollary 3.4.7. *Let $a,b\in\mathbb{R}$, $\sigma\geqslant 0$ and $0<\delta\leqslant 1$ be such that*

$$a-\delta+\frac{1}{2}>\max\left\{0,b+\frac{\sigma}{2}\right\}$$

and

$$\|A^{\delta-a+b-1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)}<\infty.$$

Then

$$t\mapsto \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s)$$

belongs to $C^{0,\delta}([0,T];L^p(\Omega;\dot{H}^\sigma_A))$ for all $p\geqslant 2$.

Proof. For $t \geqslant 0$ and (without loss of generality) $h > 0$, we split the stochastic integral

$$\begin{aligned}\int_0^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s)\,\mathrm{d}W(s) \\ = \int_t^{t+h} \Phi_{a,b}(t+h-s)\,\mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)]\,\mathrm{d}W(s).\end{aligned}$$

The result then follows by applying Lemma 3.4.5 and Lemma 3.4.6 respectively. □

We can now prove Theorem 3.4.4.

Proof of Theorem 3.4.4. We first prove identity (3.30) by induction on n. The base case $n = 0$ is trivial, noting that $C_{a,0,0} = 1$ for any a. Now suppose that (3.30) holds for some $k \in \{0, \ldots, n-1\}$. Applying the induction hypothesis yields

$$\begin{aligned}\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_0^t \Phi_{\gamma-1,0}(t-s)\,\mathrm{d}W(s) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \Phi^{(k)}_{\gamma-1,0}(t-s)\,\mathrm{d}W(s) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \sum_{j=0}^{k} C_{\gamma-1,j,k} \Phi_{\gamma-1-(k-j),j}(t-s)\,\mathrm{d}W(s).\end{aligned}$$

Fixing an arbitrary $j \in \{0, \ldots, k\}$ and setting $\Psi := \Phi_{\gamma-1-(k-j),j}$, it suffices to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \Psi(t-s)\,\mathrm{d}W(s) = \int_0^t \Psi'(t-s)\,\mathrm{d}W(s), \tag{3.31}$$

where we reiterate that $\Psi'(t) := \Phi'_{\gamma-1-(k-j),j}(t)$ with the latter being defined as in (3.29). Indeed, having proved this for an arbitrary j, we have by linearity

$$\begin{aligned}\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_0^t \Phi_{\gamma-1,0}(t-s)\,\mathrm{d}W(s) &= \int_0^t \sum_{j=0}^{k} C_{\gamma-1,j,k} \Phi'_{\gamma-1-(k-j),j}(t-s)\,\mathrm{d}W(s) \\ &= \int_0^t \Phi^{(k+1)}_{\gamma-1,0}(t-s)\,\mathrm{d}W(s),\end{aligned}$$

where the latter identity holds in an operator sense; this can be seen by checking it through equation (3.28) for all $x \in H$.

By definition of the mean square derivative, proving (3.31) amounts to showing that

$$\lim_{h\to 0} \frac{1}{h}\Big(\int_0^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Psi(t-s)\,\mathrm{d}W(s) \Big) = \int_0^t \Psi'(t-s)\,\mathrm{d}W(s),$$

where the limit is taken with respect to the $L^2(\Omega; \dot{H}^\sigma_A)$ norm. To do so, let $t > 0$, let $h > 0$ without loss of generality and write

$$\begin{aligned}&\frac{1}{h}\Big(\int_0^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) - \int_0^t \Psi(t-s)\,\mathrm{d}W(s) \Big) - \int_0^t \Psi'(t-s)\,\mathrm{d}W(s) \\ &\quad = \frac{1}{h} \int_t^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) + \int_0^t \Big(\frac{\Psi(t+h-s) - \Psi(t-s)}{h} - \Psi'(t-s) \Big)\,\mathrm{d}W(s).\end{aligned}$$

We consider the $L^p(\Omega; \dot{H}^\sigma_A)$-norms of the last two integrals separately. For the first term, we apply Lemma 3.4.5 with $a = \gamma - 1 - (k-j)$, $b = j$ and $\delta = \tau + 1$, thus producing

$$\begin{aligned}\frac{1}{h} \Big\| \int_t^{t+h} \Psi(t+h-s)\,\mathrm{d}W(s) \Big\|_{L^p(\Omega; \dot{H}^\sigma_A)} &\leqslant C_{p,\gamma,j,k,\tau} h^\tau \, \| A^{\frac{1}{2}+\tau-\gamma+k+1} \|_{\mathscr{L}_2(\mathcal{H}; \dot{H}^\sigma_A)} \\ &\leqslant C_{p,\gamma,j,k,\tau} h^\tau \, \| A^{\frac{1}{2}+\tau-\gamma+n} \|_{\mathscr{L}_2(\mathcal{H}; \dot{H}^\sigma_A)},\end{aligned}$$

which tends to zero as $h \to 0$ since $\tau > 0$.

Now consider the $L^p(\Omega;\dot H^\sigma_A)$-norm of the other stochastic integral; the Burkholder–Davis–Gundy inequality, it suffices to consider

$$
\begin{aligned}
&C_p\left(\int_0^t \left\| \frac{\Psi(t+h-s)-\Psi(t-s)}{h} - \Psi'(t-s)\right\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)} \mathrm{d}s\right)^{1/2} \\
&\quad\leqslant \frac{1}{h}\left(\int_0^t \|\Psi(t+h-s)-\Psi(t-s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)}\,\mathrm{d}s\right)^{1/2} + \left(\int_0^t \|\Psi'(t-s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)}\,\mathrm{d}s\right)^{1/2}.
\end{aligned}
$$

To the first term, we can apply Lemma 3.4.6 with $a=\gamma-1-(k-j)$, $b=j$ and $\delta=1$:

$$
\begin{aligned}
\frac{1}{h}\left(\int_0^t \|\Psi(t+h-s)-\Psi(t-s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)}\,\mathrm{d}s\right)^{1/2} &\leqslant C_{\gamma,j,k}\,\|A^{\frac12-\gamma+k+1}\|_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)} \\
&\leqslant C_{\gamma,j,k}\,\|A^{\frac12+\tau-\gamma+n}\|_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)}\,.
\end{aligned}
$$

Note that

$$\Psi' = (\gamma-1-(k-j))\Phi_{\gamma-1-(k-j)-1,j} + \Phi_{\gamma-1-(k-j),j+1}.$$

Applying the estimate from Theorem 3.4.1 with $a=\gamma-1-(k-j)-1$, $b=j$ (respectively $a=\gamma-1-(k-j)$, $b=j+1$) yields

$$
\begin{aligned}
\int_0^t \|\Psi'(t-s)\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)}\,\mathrm{d}s &\leqslant C_{\gamma,k,j}\,\|A^{\frac12-\gamma+k+1}\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)} \\
&\leqslant C_{\gamma,k,j}\,\|A^{\frac12-\gamma+n+\tau}\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)}\,.
\end{aligned}
$$

Given an orthonormal basis $(g_j)_{j\geqslant 1}$ of $\mathcal{H}$, expanding the Hilbert-Schmidt norm and using Fubini's theorem, which allows us to interchange summation and integration for non-negative integrands/summands, gives

$$
\begin{aligned}
&\int_0^t \left\| \frac{\Psi(t-s+h)-\Psi(t-s)}{h} - \Psi'(t-s)\right\|^2_{\mathscr{L}_2(\mathcal{H};\dot H^\sigma_A)} \mathrm{d}s \\
&\quad= \int_0^t \sum_{j=1}^\infty \left\| \left[\frac{\Psi(t-s+h)-\Psi(t-s)}{h} - \Psi'(t-s)\right] g_j\right\|^2_{\dot H^\sigma_A} \mathrm{d}s \\
&\quad= \sum_{j=1}^\infty \int_0^t \left\| \left[\frac{\Psi(t-s+h)-\Psi(t-s)}{h} - \Psi'(t-s)\right] g_j\right\|^2_{\dot H^\sigma_A} \mathrm{d}s.
\end{aligned}
\tag{3.32}
$$

Passing to the limit $h\to 0$, we may take the limit under the sum by monotone convergence, and under the integrals by dominated convergence since we bounded them independently of h. Since the operator Ψ' was defined by analogy with equation (3.28), we have for the orbits

$$\lim_{h\to 0} \frac{\Psi(t-s+h)g_j - \Psi(t-s)g_j}{h} = \Psi'(t-s)g_j,$$

so (3.32) becomes zero in the limit $h\to 0$. This finishes the proof of the induction step, hence of identity (3.30).

It remains to be proven that the nth derivative, i.e. the right-hand side of equation (3.30), is τ-Hölder in mean square. Writing

$$\int_0^t \Phi^{(n)}_{\gamma-1,0}(t-s)\,\mathrm{d}W(s) = \sum_{j=0}^n C_{\gamma,j,n}\int_0^t \Phi_{\gamma-1-(n-j),j}(t-s)\,\mathrm{d}W(s),$$

we see that it follows by invoking Corollary 3.4.7 with $a=\gamma-1-(n-j)$, $b=j$ and $\delta=\tau$ for all $j\in\{0,\dots,n\}$. ☐

We can now use the Kolmogorov–Chentsov theorem in order to turn part of the temporal regularity results for Z_γ into pathwise continuity results.

Theorem 3.4.8. *Suppose the constants $\sigma \geqslant 0$, $0 < \tau \leqslant 1$ and $\gamma > \tau + \frac{1}{2} + \frac{\sigma}{2}$ are such that*

$$\|A^{\tau-\gamma+1/2}\|_{\mathscr{L}_2(\mathcal{H};\dot{H}^\sigma_A)} < \infty.$$

Then for all $p \geqslant 2$, there exists a version $\tilde{Z}_\gamma \in \bigcap_{\tau' \in (0,\tau)} L^p(\Omega; C^{0,\tau'}([0,T]; \dot{H}^\sigma_A))$ of Z_γ.

Proof. We first invoke Theorem 3.4.4 with $n = 0$, which implies that $Z_\gamma \in C^{0,\tau}([0,T]; L^q(\Omega; \dot{H}^\sigma_A))$ for all $q \geqslant 2$. Then the result follows by choosing $q \geqslant 2$ large enough and applying Theorem 2.8.9 with q instead of p. $\square$

The most important application of the preceding theory is the following example, which includes the Whittle–Matérn case.

Example 3.4.9. Let L and $\widetilde{L}$ be uniformly elliptic differential operators as described in Section 2.6. Suppose that L and $\widetilde{L}$ diagonalize with respect to the same orthonormal basis $(e_j)_{j\geqslant 1}$ of eigenvectors, i.e. there exist sequences of positive real numbers $(\lambda_j)_{j\geqslant 1}$ and $(\widetilde{\lambda}_j)_{j\geqslant 1}$ such that for all $j \in \mathbb{N}$ we have

$$Le_j = \lambda_j e_j \quad \text{and} \quad \widetilde{L}e_j = \widetilde{\lambda}_j e_j.$$

We set $A := L^\beta$ and $Q := \widetilde{L}^{-\alpha}$ for given exponents $\alpha, \beta \geqslant 0$. As remarked in Section 2.6, this is an admissible choice of A since $-A$ generates an analytic C_0-semigroup. Moreover, Q is indeed a non-negative, self-adjoint and bounded operator.

We first check when the conditions of Corollary 3.4.2 are satisfied. For simplicity, we consider $\sigma = 0$, and thus we suppose $\gamma > 1/2$. By the spectral mapping theorem for fractional powers of operators, see e.g. [50, Section 5.3], it holds that

$$Ae_j = L^\beta e_j = \lambda_j^\beta e_j \quad \text{and} \quad Qe_j = \widetilde{L}^{-\alpha} e_j = \widetilde{\lambda}_j^{-\alpha} e_j.$$

We use this to compute

$$\begin{aligned}\|A^{\frac{1}{2}-\gamma}\|^2_{\mathscr{L}_2(H;\mathcal{H})} = \|A^{\frac{1}{2}-\gamma}Q^{\frac{1}{2}}\|^2_{\mathscr{L}_2(H)} = \|L^{\beta(\frac{1}{2}-\gamma)}\widetilde{L}^{-\frac{\alpha}{2}}\|^2_{\mathscr{L}_2(H)} &= \sum_{j=1}^\infty \|L^{\beta(\frac{1}{2}-\gamma)}\widetilde{L}^{-\frac{\alpha}{2}}e_j\|^2_H \\ &= \sum_{j=1}^\infty \lambda_j^{\beta(1-2\gamma)}\widetilde{\lambda}_j^{-\alpha}\end{aligned}$$

Applying Weyl's law (Theorem 2.6.2) to both L and $\widetilde{L}$, it follows that there exist constants $c, C \geqslant 0$ such that

$$c\sum_{j=1}^\infty j^{\frac{2}{d}[\beta(1-2\gamma)-\alpha]} \leqslant \sum_{j=1}^\infty \lambda_j^{\beta(1-2\gamma)}\widetilde{\lambda}_j^{-\alpha} \leqslant C\sum_{j=1}^\infty j^{\frac{2}{d}[\beta(1-2\gamma)-\alpha]}.$$

Therefore, $\|A^{\frac{1}{2}-\gamma}\|_{\mathscr{L}_2(H;\mathcal{H})} < \infty$ if and only if

$$\beta(1-2\gamma) - \alpha < \frac{d}{2}.$$

In this case, Corollary 3.4.2 yields the existence of the stochastic convolution Z_γ belonging to $C([0,T]; L^p(\Omega; \dot{H}^\sigma_A))$ for all $p \geqslant 2$.

If we want to draw conclusion about the regularity of the solution, the stronger assumptions of Theorem 3.4.4 need to be satisfied. Namely, we need to assume that $\sigma \geqslant 0$, $n \in \mathbb{N}^{\geqslant 0}$ and $0 < \tau \leqslant 1$ are such that $\gamma > n + \tau + \frac{1}{2} + \frac{\sigma}{2}$; an analogous computation now shows that we furthermore need

$$\beta[\sigma + 1 - 2(\gamma - n - \tau)] - \alpha < \frac{d}{2}$$

in order to conclude that $Z_\gamma \in C^{n,\tau}([0,T]; L^p(\Omega; \dot{H}^\sigma_A))$ for all $p \geqslant 2$. If the above holds with $n = 0$, then by Theorem 3.4.8 we know that there exists a version $\tilde{Z}_\gamma \in \bigcap_{\tau' \in (0,\tau)} L^p(\Omega; C^{0,\tau'}([0,T]; \dot{H}^\sigma_A))$ of Z_γ.

3.4.2. Regularity of the initial value term

Now we turn to the 'initial value term' Z_0 in the mild solution to the fractional parabolic problem. We investigate its spatiotemporal regularity as we did before for the stochastic convolution, and additionally we prove the continuous dependence on the initial datum X_0.

Proposition 3.4.10. *Suppose that $X_0 \in L^p(\Omega; \dot{H}^\sigma_A)$ for some $2 \leqslant p < \infty$ and $\sigma \geqslant 0$. Then $Z_0 \in L^p(\Omega; C([0,T]; \dot{H}^\sigma_A))$, and the linear map $X_0 \mapsto Z_0$ is bounded from $L^p(\Omega; \dot{H}^\sigma_A)$ to $L^p(\Omega; C([0,T]; \dot{H}^\sigma_A))$.*

Proof. We first treat the deterministic case, i.e. we suppose that $X_0 = x \in \dot{H}^{\sigma+2\gamma}_A$. Then for all $t \geqslant 0$ we have

$$\frac{1}{\Gamma(\gamma)} \int_t^\infty \|s^{\gamma-1} S(s) A^\gamma x\|_{\dot{H}^\sigma_A} \,\mathrm{d}s \leqslant \frac{1}{\Gamma(\gamma)} \int_0^\infty \|s^{\gamma-1} S(s) A^\gamma x\|_{\dot{H}^\sigma_A} \,\mathrm{d}s = \|X_0\|_{\dot{H}^\sigma_A} < \infty,$$

hence $Z_0(t) = \frac{1}{\Gamma(\gamma)} \int_t^\infty s^{\gamma-1} S(s) A^\gamma x \,\mathrm{d}s$ and it follows that $Z_0(t)$ is continuous by dominated convergence. In particular, $\|Z_0(t)\|_{\dot{H}^\sigma_A}$ is continuous at $t = 0$ and $Z_0(0) = x$, so there exists some $t_0 > 0$ such that $\|Z_0(t)\|_{\dot{H}^\sigma_A} \leqslant 2\,\|x\|_{\dot{H}^\sigma_A}$ for all $0 \leqslant t < t_0$. On the other hand, for $t \geqslant t_0$ we have

$$\begin{aligned}
\|Z_0(t)\|_{\dot{H}^\sigma_A} &\leqslant \frac{1}{\Gamma(\gamma)} \int_t^\infty \|s^{\gamma-1} S(s) A^\gamma x\|_{\dot{H}^\sigma_A} \,\mathrm{d}s \\
&\leqslant \frac{1}{\Gamma(\gamma)} \int_{t_0}^\infty \|s^{\gamma-1} S(s) A^\gamma x\|_{\dot{H}^\sigma_A} \,\mathrm{d}s \\
&\leqslant C_\gamma \int_{t_0}^\infty s^{-1} e^{-ws} \,\mathrm{d}s\, \|x\|_{\dot{H}^\sigma_A}.
\end{aligned}$$

Here, $w > 0$ is a constant which exists by the exponential boundedness of $(S(t))_{t\geqslant 0}$. Noting that the integral on the last line is the exponential integral E_1 evaluated at $wt_0 > 0$, which is finite for positive arguments, we set $M := \max\{2, C_\gamma E_1(wt_0)\}$ and see that

$$\|Z_0\|_{C([0,T];\dot{H}^\sigma_A)} = \sup_{t\in[0,T]} \|Z_0(t)\|_{\dot{H}^\sigma_A} \leqslant M \|x\|_{\dot{H}^\sigma_A}. \tag{3.33}$$

By density of $\dot{H}^{\sigma+2\gamma}_A \subseteq \dot{H}^\sigma_A$, the above holds for all $x \in \dot{H}^\sigma_A$, and $x \mapsto Z_0$ is a bounded linear map from $\dot{H}^\sigma_A$ to $C([0,T]; \dot{H}^\sigma_A)$. Applying the above with $x := X_0(\omega)$ for $\omega \in \Omega$ in some suitable almost sure set and taking the $L^p(\Omega)$-norm on both sides of (3.33) now finishes the proof. ☐

Theorem 3.4.11. *Let $n \in \mathbb{N}^{\geqslant 0}$, $0 < \eta \leqslant 1$ and $2 \leqslant p < \infty$. Suppose that $X_0 \in L^p(\Omega; \dot{H}^{\sigma+2(\eta+n)}_A)$ and that γ satisfies either $\gamma \in \mathbb{N}$ or $n + \eta \leqslant \gamma$. Then $Z_0 \in L^p(\Omega; C^{n,\eta}([0,T]; \dot{H}^\sigma_A))$ and the linear map $X_0 \mapsto Z_0$ is bounded from $L^p(\Omega; \dot{H}^{\sigma+2(n+\eta)}_A)$ to $L^p(\Omega; C^{n,\eta}([0,T]; \dot{H}^\sigma_A))$.*

Proof. Again, we first consider the deterministic vector x instead of the random variable X_0. Note that the assumption $\eta > 0$ implies that the closed operator A^γ can be moved inside of the (Bochner) integral; this is valid since, for all $t \geqslant 0$,

$$\int_0^t \|A^\gamma s^{\gamma-1} S(s) x\|_{\dot{H}^\sigma_A} \,\mathrm{d}s \leqslant C_{\gamma,\eta} \int_0^t s^{-1+\eta} \,\mathrm{d}s\, \|A^\eta x\|_{\dot{H}^\sigma_A} = C_{\gamma,\eta} \frac{t^\eta}{\eta} \|A^\eta x\|_{\dot{H}^\sigma_A} < \infty,$$

where we use $\eta \leqslant \gamma$ in order for the first estimate to hold. It follows that we may write $Z_0(t) = x - \frac{1}{\Gamma(\gamma)} \int_0^t s^{\gamma-1} A^\gamma S(s) x \,\mathrm{d}s$.

Since the integrand $s \mapsto s^{\gamma-1} A^\gamma S(s) x$ is smooth as a map from $[0,T]$ to $\dot{H}^\sigma_A$ everywhere except possibly at $s = 0$, we expect its nth derivative to be of the following form, given by the fundamental theorem of calculus and the general Leibniz rule:

$$Z_0^{(n)}(t) = \sum_{k=0}^{n-1} C_{\gamma,k,(n-1)} t^{\gamma-(n-k)} A^{\gamma+k} S(t) x,$$

where k ranges from 0 to $n-1$ instead of n since the first differentiation merely removes the integral.

In order to check that $Z_0 \in C^n([0,T]; \dot{H}^\sigma_A)$, it suffices to prove that all the terms in the above sum are bounded as $t \downarrow 0$. Fix a $k \in \{0, \dots, n-1\}$ and observe that

$$
\begin{aligned}
\|t^{\gamma-(n-k)} A^{\gamma+k} S(t) x\|_{\dot{H}^\sigma_A} &= t^{\gamma-(n-k)} \, \|A^{\gamma-(n-k)-\eta} S(t) A^{n+\eta} x\|_{\dot{H}^\sigma_A} \\
&\leqslant \tilde{C}_{\gamma,n,k,\eta} t^\eta \, \|A^{n+\eta} x\|_{\dot{H}^\sigma_A} \,,
\end{aligned}
$$

which is finite at $t=0$ since $\eta \geqslant 0$. To apply the inequality in the above estimate, we need $n-k+\eta \leqslant \gamma$. If $\gamma \in \mathbb{N}$, then the coefficients $C_{\gamma,k,(n-1)}$ from the general Leibniz rule vanish whenever $\gamma-1-(n-k) < 0$, so that these terms can be ignored and we may assume $\gamma - 1 - (n-k) \geqslant 0$ in the above estimate. But this implies $n - k + \eta \leqslant \gamma - (1-\eta) \leqslant \gamma$ since $0 < \eta \leqslant 1$, so we see that the requirement $n-k+\eta \leqslant \gamma$ is always satisfied. If $\gamma \notin \mathbb{N}$, then we need to use that $n + \eta \leqslant \gamma$. This shows that $Z_0 \in C^n([0,T]; \dot{H}^\sigma_A)$.

We now consider the Hölder regularity of the nth derivative. Fix $t \in [0,T]$ and, without loss of generality, $h > 0$ small enough. Then, using the fundamental theorem of calculus, we can estimate the increment of Z_0 as follows:

$$
\begin{aligned}
\|Z_0^{(n)}(t+h) - Z_0^{(n)}(t)\|_{\dot{H}^\sigma_A} &= \left\| \int_t^{t+h} Z_0^{(n+1)}(s) \, \mathrm{d}s \right\|_{\dot{H}^\sigma_A} \\
&\leqslant \sum_{k=0}^{n} C_{\gamma,k,n} \int_t^{t+h} s^{\gamma-1-(n-k)} \, \|A^{\gamma+k} S(s) x\|_{\dot{H}^\sigma_A} \, \mathrm{d}s.
\end{aligned}
$$

For each $k \in \{0, \dots, n\}$, we estimate the corresponding term as before:

$$
\begin{aligned}
\int_t^{t+h} s^{\gamma-1-(n-k)} \, \|A^{\gamma+k} S(s) x\|_{\dot{H}^\sigma_A} \, \mathrm{d}s &\leqslant \tilde{C}^{(1)}_{\gamma,n,k,\eta} \int_t^{t+h} s^{\eta-1} \, \mathrm{d}s \; \|A^{n+\eta} x\|_{\dot{H}^\sigma_A} \\
&= \tilde{C}^{(2)}_{\gamma,n,k,\eta} [(t+h)^\eta - t^\eta] \; \|A^{n+\eta} x\|_{\dot{H}^\sigma_A} \\
&\leqslant \tilde{C}^{(3)}_{\gamma,n,k,\eta} h^\eta \; \|A^{n+\eta} x\|_{\dot{H}^\sigma_A} \,,
\end{aligned}
$$

where we need $n-k+\eta \leqslant \gamma$ on the fourth line, $\eta \neq 0$ for the fifth and $0 \leqslant \eta \leqslant 1$ for the last. As before, the first requirement is fulfilled for all k if $\gamma \in \mathbb{N}$ or $n+\eta \leqslant \gamma$. This proves that $Z_0 \in C^{n,\eta}([0,T]; \dot{H}^\sigma_A)$. The above estimates also show that the linear map $x \mapsto Z_0$ is bounded from $\dot{H}^{\sigma+2(n+\eta)}_A$ to $C^{n,\eta}([0,T]; \dot{H}^\sigma_A)$. As before, the proof is finished upon applying the above to $x := X_0(\omega)$ for almost all $\omega \in \Omega$ and taking the $L^p(\Omega)$-norm of the boundedness estimates. □

3.5. Covariance structure

The purpose of this section is to investigate the covariance of the solution to (1.4), where we limit ourselves to the case $X_0 = 0$ for the moment. In particular, we would like to obtain results analogous to those found in [5, Section 3]; in that work, Fourier techniques were used to study the covariance structure of the same type of SPDE on an infinite spatiotemporal domain $\mathcal{D} \times \mathbb{T} = \mathbb{R} \times \mathbb{R}^2$ using the operators $A := L^\beta$ and $Q := L^{-\alpha}$ where $L := \kappa^2 - \Delta$ is the Matérn operator and $\alpha, \beta \geqslant 0$.

Our formulation of the problem necessitates a different approach, since the boundaries of both the spatial domain $\mathcal{D} \subsetneq \mathbb{R}^d$ and the time horizon $\mathbb{T}$ limit the effectiveness of Fourier arguments. Instead, we proceed as in the previous sections of this chapter, namely by starting from (3.1) and working with the fractional stochastic convolution Z_γ.

The main results of this section are Propositions 3.5.2 and 3.5.3, which are analogs to Proposition 3.1 and Corollary 3.3 from [5], respectively. Indeed, we see that there is a price to be paid for the introduction of boundaries in the domain, since the results are less explicit and only hold asymptotically in time. The latter observation provides another reason, besides simplicity, to merely consider the case $X_0 = 0$: the desired covariance results only hold for large times, at which point one expects that the effects of the initial condition will have diminished, so that we may as well take $X_0 = 0$ in our model (1.4).

We start by considering the covariance operators corresponding to $\mathrm{Cov}(Z_\gamma(t), Z_\gamma(s)) \in H^{(2)}$ for given $t, s \geqslant 0$ and to $\mathrm{Cov}(Z_\gamma) \in L^2(0,T;H)^{(2)}$.

Proposition 3.5.1. *Suppose that A satisfies the standing assumptions for this chapter and suppose that $X_0 = 0$ and (3.19) hold. Then the following two statements hold.*

(a) *For arbitrary $t, s \geqslant 0$ and $x, y \in H$, we have*

$$(\mathrm{Cov}(Z_\gamma(t), Z_\gamma(s)), x \otimes y)_{H^{(2)}} = \left(\frac{1}{\Gamma(\gamma)^2} \int_0^{t \wedge s} [(t-\sigma)(s-\sigma)]^{\gamma-1} S(t-\sigma) Q S(s-\sigma) x \,\mathrm{d}\sigma, y \right)_H.$$

In other words, the covariance operator $Q_{Z_\gamma(t), Z_\gamma(s)}$ on H associated to $\mathrm{Cov}(Z_\gamma(t), Z_\gamma(s))$ is given by

$$Q_{Z_\gamma(t), Z_\gamma(s)} = \frac{1}{\Gamma(\gamma)^2} \int_0^{t \wedge s} [(t-\sigma)(s-\sigma)]^{\gamma-1} S(t-\sigma) Q S(s-\sigma) \,\mathrm{d}\sigma. \tag{3.34}$$

(b) *For all $\varphi, \psi \in L^2(0, T; H)$ we have*

$$(\mathrm{Cov}(Z_\gamma), \varphi \otimes \psi)_{L^2(0,T;H)^{(2)}} = \int_0^T \int_0^T (Q_{Z_\gamma(t), Z_\gamma(s)} \varphi(t), \psi(s))_H \,\mathrm{d}t \,\mathrm{d}s. \tag{3.35}$$

In other words, the covariance operator $\mathcal{Q}_{Z_\gamma}$ on $L^2(0, T; H)$ associated to $\mathrm{Cov}(Z_\gamma)$ is given by

$$[\mathcal{Q}_{Z_\gamma} \varphi](s) = \int_0^T Q_{Z_\gamma(t), Z_\gamma(s)} \varphi(t) \,\mathrm{d}t \quad \textit{for all } \varphi \in L^2(0,T;H) \textit{ and almost all } s \in (0,T).$$

Proof. First note that $\mathbb{E}[Z_\gamma(t)] = 0$ for all $t \geqslant 0$ by the corresponding property of the Wiener processes used to define the stochastic integral. Thus, we have for the covariance:

$$\begin{aligned}
&(\mathrm{Cov}(Z_\gamma(t), Z_\gamma(s)), x \otimes y)_{H^{(2)}} \\
&\quad = \mathbb{E}[(Z_\gamma(t), x)_H (Z_\gamma(s), y)_H] \\
&\quad = \mathbb{E}\left[\left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-\sigma)^{\gamma-1} S(t-\sigma) \,\mathrm{d}W(\sigma), x \right)_H \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\sigma)^{\gamma-1} S(s-\sigma) \,\mathrm{d}W(\sigma), y \right)_H \right] \\
&\quad = \frac{1}{\Gamma(\gamma)^2} \mathbb{E}\left[\int_0^t ((t-\sigma)^{\gamma-1} S(t-\sigma) \,\mathrm{d}W(\sigma), x)_H \int_0^s ((s-\sigma)^{\gamma-1} S(s-\sigma) \,\mathrm{d}W(\sigma), y)_H \right],
\end{aligned}$$

where the latter step holds by definition of the weak stochastic integral. Using a formula for the expectation of a product of stochastic integrals integrated for different lengths of time, see [23, Section 4.3], we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma)^2} \mathbb{E}\left[\int_0^t ((t-\sigma)^{\gamma-1} S(t-\sigma) \,\mathrm{d}W(\sigma), x)_H \int_0^s ((s-\sigma)^{\gamma-1} S(s-\sigma) \,\mathrm{d}W(\sigma), y)_H \right] \\
&\quad = \frac{1}{\Gamma(\gamma)^2} \int_0^{t \wedge s} (Q^{\frac{1}{2}} (t-\sigma)^{\gamma-1} S(t-\sigma) x, Q^{\frac{1}{2}} (s-\sigma)^{\gamma-1} S(s-\sigma) y)_H \,\mathrm{d}\sigma,
\end{aligned}$$

where we used the assumption that A, and hence $(S(t))_{t \geqslant 0}$, is self-adjoint. Using this fact once more, and subsequently applying the fact that inner products and integrals can be interchanged, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma)^2} \int_0^{t \wedge s} (Q^{\frac{1}{2}} (t-\sigma)^{\gamma-1} S(t-\sigma) x, Q^{\frac{1}{2}} (s-\sigma)^{\gamma-1} S(s-\sigma) y)_H \,\mathrm{d}\sigma \\
&\quad = \frac{1}{\Gamma(\gamma)^2} \int_0^{t \wedge s} [(t-\sigma)(s-\sigma)]^{\gamma-1} (S(s-\sigma) Q S(t-\sigma) x, y)_H \,\mathrm{d}\sigma \\
&\quad = \left(\frac{1}{\Gamma(\gamma)^2} \int_0^{t \wedge s} [(t-\sigma)(s-\sigma)]^{\gamma-1} S(s-\sigma) Q S(t-\sigma) x \,\mathrm{d}\sigma, y \right)_H.
\end{aligned}$$

This proves part (a). For part (b), let $\varphi, \psi \in L^2(0,T;H)$ be arbitrary and consider

$$
\begin{aligned}
(Z_\gamma, \varphi \otimes \psi)_{L^2(0,T;H)} &= \mathbb{E}[(Z_\gamma, \varphi)_{L^2(0,T;H)}(Z_\gamma, \psi)_{L^2(0,T;H)}] \\
&= \mathbb{E}\Big[\int_0^T (Z_\gamma(t), \varphi(t))_H \,\mathrm{d}t \int_0^T (Z_\gamma(s), \psi(s))_H \,\mathrm{d}s\Big] \\
&= \mathbb{E}\Big[\int_0^T \int_0^T (Z_\gamma(t), \varphi(t))_H (Z_\gamma(s), \psi(s))_H \,\mathrm{d}t \,\mathrm{d}s\Big] \\
&= \int_0^T \int_0^T \mathbb{E}[(Z_\gamma(t), \varphi(t))_H (Z_\gamma(s), \psi(s))_H] \,\mathrm{d}t \,\mathrm{d}s \\
&= \int_0^T \int_0^T (Q_{Z_\gamma(t), Z_\gamma(s)} \varphi(t), \psi(s))_H \,\mathrm{d}t \,\mathrm{d}s,
\end{aligned}
$$

where the Fubini theorem was used to interchange integrals and expectations and where part (a) was used in the last step. □

We can use this proposition to derive the next statement about the asymptotic spatial covariance of Z_γ. In the case that $H = L^2(\mathcal{D})$, the following proposition can be interpreted as saying that, for large t, the marginal spatial covariance of the random field $Z_\gamma(t,x)$ is determined by a spatial covariance operator consisting of fractional powers of Q and A.

Proposition 3.5.2. *Suppose that A satisfies the standing assumptions for this chapter and suppose that (3.19) holds. If $\gamma > 1/2$ and the spatial covariance operator Q of the Wiener process $\dot{\mathcal{W}}^Q$ commutes with $S(t)$ for all t, then the covariance operator of $Z_\gamma(t)$ asymptotically satisfies*

$$
\lim_{t\to\infty} Q_{Z_\gamma(t)} = \frac{\Gamma(\gamma - \frac{1}{2})}{2\sqrt{\pi}\Gamma(\gamma)} A^{1-2\gamma} Q,
$$

with convergence in the operator norm.

Proof. Letting $t = s$ causes equation (3.34) to reduce to

$$
Q_{Z_\gamma(t)} = \frac{1}{\Gamma(\gamma)^2} \int_0^t (t-s)^{2(\gamma-1)} S(t-\sigma) Q S(t-\sigma) \,\mathrm{d}\sigma.
$$

By the assumption that Q commutes with $S(t)$ for all $t \geqslant 0$, in conjunction with the semigroup property for $(S(t))_{t\geqslant 0}$, this can be written as

$$
Q_{Z_\gamma(t)} = \frac{1}{\Gamma(\gamma)^2} \int_0^t (t-s)^{2\gamma-2} Q S(2t - 2\sigma) \,\mathrm{d}\sigma.
$$

The change of variables $u := 2(t-\sigma)$ then yields

$$
Q_{Z_\gamma(t)} = \frac{1}{\Gamma(\gamma)^2} 2^{1-2\gamma} \int_0^{2t} u^{2\gamma-2} Q S(u) \,\mathrm{d}u.
$$

Since Q is bounded, it does not influence the convergence of the above integral in the operator norm topology as $t \to \infty$. By comparison with (2.12), we then see that the improper integral indeed exists as a bounded operator for $\gamma > 1/2$; we have

$$
\lim_{t\to\infty} Q_{Z_\gamma(t)} = \frac{1}{\Gamma(\gamma)^2} 2^{1-2\gamma} \int_0^\infty u^{2\gamma-2} Q S(u) \,\mathrm{d}u = \frac{\Gamma(2\gamma-1)}{\Gamma(\gamma)^2} 2^{1-2\gamma} Q A^{1-2\gamma}.
$$

Applying the Legendre duplication formula for the gamma function with $\gamma - 1/2 > 0$ then yields

$$
\Gamma(\gamma - \tfrac{1}{2})\Gamma(\gamma) = 2^{2-2\gamma}\sqrt{\pi}\Gamma(2\gamma - 1),
$$

which produces the prefactor given in the statement, thus completing the proof. □

The following result concerns a situation in which the covariance of Z_γ becomes separable into a distinct spatial and temporal part, see Section 1.1. Furthermore, the marginal temporal covariance turns out to behave like a Matérn function for large times. We see that these effects occur when taking $A = I$ in (3.1), which we note corresponds to setting the exponent $\beta = 0$ in (1.4).

Proposition 3.5.3. *Let $A = I$ and $\gamma > 1/2$. Then the covariance of Z_γ is separable and the temporal part of its covariance function is asymptotically of Matérn type. More precisely, there exists a function*

$$\varrho_{\textit{time}} \colon (0,\infty) \times (0,\infty) \to \mathbb{R}$$

such that

$$Q_{Z_\gamma(t), Z_\gamma(s)} = \varrho_{\textit{time}}(t,s)\, Q \quad \textit{for all } t,s \geqslant 0, \tag{3.36}$$

and for all $h \in \mathbb{R}$ we have

$$\lim_{t\to\infty} \varrho_{\textit{time}}(t+h,t) = \frac{2^{\frac{1}{2}-\gamma}}{\sqrt{\pi}\Gamma(\gamma)} h^{\gamma-\frac{1}{2}} K_{\gamma-\frac{1}{2}}(h). \tag{3.37}$$

Remark 3.5.4. Before proceeding with the proof of Proposition 3.5.3, we remark why equation (3.36) is related to the concept of separability as introduced for covariance functions. To this end, suppose that $H = L^2(\mathcal{D})$. Substituting (3.36) into (3.35) leads to

$$(\mathrm{Cov}(Z_\gamma), \varphi \otimes \psi)_{L^2(0,T;L^2(\mathcal{D}))^{(2)}} = \int_0^T \int_0^T \varrho_{\text{time}}(t,s) (Q\varphi(t), \psi(s))_{L^2(\mathcal{D})} \,\mathrm{d}t \,\mathrm{d}s$$

If Q is Hilbert–Schmidt, then we recall that it admits a representation as an integral operator for some covariance kernel $\varrho_{\text{space}} \in L^2(\mathcal{D} \times \mathcal{D})$, so that the above equation reads

$$\begin{aligned}
(\mathrm{Cov}(Z_\gamma), \varphi \otimes \psi)_{L^2(0,T;L^2(\mathcal{D}))^{(2)}} &= \int_0^T \int_0^T \varrho_{\text{time}}(t,s) \biggl(\int_{\mathcal{D}} \int_{\mathcal{D}} \varrho_{\text{space}}(x,y) \varphi(t,x) \psi(s,y) \,\mathrm{d}x \,\mathrm{d}y \biggr) \mathrm{d}t \,\mathrm{d}s \\
&= \int_0^T \int_0^T \int_{\mathcal{D}} \int_{\mathcal{D}} \varrho_{\text{time}}(t,s) \varrho_{\text{space}}(x,y) \varphi(t,x) \psi(s,y) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \,\mathrm{d}s,
\end{aligned}$$

supposing that the conditions of the Fubini theorem are met in order to interchange integrals in the last step. If, moreover, ϱ_{space} were known to be continuous, then this argument shows that the covariance kernel $\varrho \colon (\mathbb{T} \times \mathcal{D})^2 \to \mathbb{R}$ of Z_γ satisfies

$$\varrho((t,x),(s,y)) = \varrho_{\text{time}}(t,s) \varrho_{\text{space}}(x,y) \quad \text{for all } t,s \in \mathbb{T},\ x,y \in \mathcal{D},$$

which is precisely the definition of separability in the sense of covariance functions. We conclude that (3.36) is a generalization of separability to situations in which the spatial covariance operator Q is bounded but not necessarily trace-class or even Hilbert–Schmidt.

Proof of Proposition 3.5.3. The C_0-semigroup $(S(t))_{t\geqslant 0}$ generated by the bounded operator $A = I$ reduces to the family of exponential operators $(e^{-It})_{t \geqslant 0}$. Using the definition (2.9) of the latter, we obtain a simple representation:

$$e^{-It} := \sum_{k=0}^\infty \frac{(-t)^k}{k!} I^k = \sum_{k=0}^\infty \frac{(-t)^k}{k!} I = e^{-t} I \quad \text{for all } t \geqslant 0.$$

Let $s,t \geqslant 0$ be arbitrary. Substituting $(S(t))_{t\geqslant 0} = (e^{-t}I)_{t\geqslant 0}$ into (3.34) produces

$$\begin{aligned}
Q_{Z_\gamma(t), Z_\gamma(s)} &= \frac{1}{\Gamma(\gamma)^2} \int_0^{t\wedge s} [(t-\sigma)(s-\sigma)]^{\gamma-1} e^{-(t-\sigma)} Q e^{-(s-\sigma)} \,\mathrm{d}\sigma \\
&= \frac{1}{\Gamma(\gamma)^2} \int_0^{t\wedge s} [(t-\sigma)(s-\sigma)]^{\gamma-1} e^{-(t+s-2\sigma)} \,\mathrm{d}\sigma\, Q \\
&=: \varrho_{\text{time}}(t,s)\, Q,
\end{aligned}$$

where we pulled the bounded and σ-independent operator Q out of the integral. This proves (3.36).

Now $t \geqslant 0$ once again and, without loss of generality, let $h > 0$. We have

$$\varrho_{\text{time}}(t+h,t) = \frac{1}{\Gamma(\gamma)^2}\int_0^t [(t+h-\sigma)(t-\sigma)]^{\gamma-1} e^{-(2t+h-2\sigma)}\,\mathrm{d}\sigma.$$

The change of variables $u := 2t + h - 2\sigma$ yields

$$\begin{aligned}\varrho_{\text{time}}(t+h,t) &= \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\int_h^{2t+h} [(u+h)(u-h)]^{\gamma-1} e^{-u}\,\mathrm{d}u \\ &= \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\int_h^{2t+h} (u^2-h^2)^{\gamma-1} e^{-u}\,\mathrm{d}u,\end{aligned}$$

and thus,

$$\lim_{t\to\infty} \varrho_{\text{time}}(t+h,t) = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2}\int_h^{\infty} (u^2-h^2)^{\gamma-1} e^{-u}\,\mathrm{d}u, \tag{3.38}$$

Define the function $f\colon (0,\infty) \to \mathbb{R}$ by

$$f(u) := \begin{cases} 0, & 0 < u \leqslant h; \\ (u^2-h^2)^{\gamma-1}, & u > h. \end{cases}$$

Now note that the integral on the right-hand side of equation (3.38) is equal to $\mathcal{L}[f](1)$, i.e. the Laplace transform of f evaluated at $s = 1$. It follows from [55, Chapter II, Equation (13.21)] that

$$\mathcal{L}[f](s) = \frac{(2h)^{\gamma-\frac{1}{2}}\Gamma(\gamma)}{\sqrt{\pi}} s^{-(\gamma-\frac{1}{2})} K_{\gamma-\frac{1}{2}}(hs), \quad s > 0;$$

in order for this equation to hold, we use the assumption that $\gamma > 1/2$. Evaluating this at $s = 1$ and substituting the result into (3.38) produces (3.37). $\square$

4

Numerical approximation

As a first step towards the numerical analysis of the fractional parabolic problem, in this chapter we reduce (3.1) to a fractional ODE in time which can be solved directly, yet retains some of the novelty of the original problem due to the presence of the fractional exponent γ. We describe a numerical scheme which can be implemented for the approximation of solutions to this type of equation, and we explain the ideas behind this scheme. The reduction of the problem eases the exposition and implementation of the numerical schemes and the computation of the errors in the respective approximations. Gaining insight into the convergence behavior of various numerical schemes applied to the simpler problem then gives an idea of what to expect of analogous numerical approximation methods for the original problem.

We start this chapter by explaining the model problem in Section 4.1, namely a deterministic fractional initial value problem with zero initial condition. Subsequently, we explain in Section 4.2 the numerical schemes which we propose to study for future numerical experiments; these are combinations of weak variational time discretization methods with a *sinc quadrature* to deal with the fractional power.

4.1. Model problem

We consider the following reduction of (1.4). As our Hilbert space, we take $H = \mathbb{R}$ so that we may naturally identify the positive, self-adjoint operator A with some real constant $\lambda > 0$. Moreover, we take a deterministic right-hand side f instead of a noise term. In fact, we consider $f \equiv 1$ on $[0, T]$. Lastly, we limit ourselves to the case $u_0 = 0$. Together, this yields the following *deterministic fractional ODE*:

$$(\partial_t + \lambda)^\gamma u = 1, \quad u_0 = 0, \tag{4.1}$$

where $\gamma > 0$. Note that we retain the ∂_t notation for consistency with the previous chapters, keeping in mind that this actually refers to an ordinary derivative. This can be considered a minimal nontrivial fractional parabolic problem, which still poses a numerical challenge while requiring less effort in the implementation and analysis of numerical schemes compared to the infinite-dimensional problem.

Noting that the C_0-semigroup generated by $-A = -\lambda$ can be identified with $S(t) = e^{-\lambda t}$, applying the result of (3.15) and subsequently performing a change of variables $r(s) := \lambda(t - s)$ produces the exact solution

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} e^{-\lambda(t-s)} \, \mathrm{d}s \\ &= \frac{1}{\Gamma(\gamma)\lambda^\gamma} \int_0^{\lambda t} r^{\gamma-1} e^{-r} \, \mathrm{d}r \\ &= \frac{P(\lambda t, \gamma)}{\lambda^\gamma}, \end{aligned}$$

where $P(x, a)$ denotes the *lower incomplete gamma function*, defined by

$$P(x, a) := \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} \, \mathrm{d}t, \quad x, a > 0.$$

Since the (lower) incomplete gamma function can be computed to high precision in scientific computing environments such as MATLAB® and NumPy, the error of a numerical scheme can easily be computed for this model problem. This will be useful when studying the convergence behavior through numerical experiments.

4.2. Numerical schemes

In this section, we explain our proposed approach to computing numerical approximations to problem (4.1). First we describe finite element discretizations of the weak variational formulation of (4.1) for $\gamma = 1$, and subsequently we explain how the use of a *sinc quadrature* method allows us to deal with fractional exponents.

4.2.1. Nonfractional finite element time discretization

In this subsection, we limit ourselves to the non-fractional counterpart of (4.1), i.e. the case $\gamma = 1$, and describe an approach for the finite element time discretization of this problem based on weak variational formulations. Such techniques are described in more detail in [4] and [31, Section 65.1.5]; see also [43, Section 3.2].

The weak variational formulation (see Section 2.7) corresponding to the non-fractional model problem is

$$\text{Find } u \in \mathcal{X} \text{ such that } b(u,v) = \ell(v) \text{ for all } v \in \mathcal{Y}_{0,\{T\}}. \tag{4.2}$$

For the problem under consideration, the trial and test spaces are chosen to be respectively $\mathcal{X} = L^2(0,T)$ and $\mathcal{Y}_{0,\{T\}}$, the latter being the space of functions belonging to $\mathcal{Y} = H^1(0,T) \hookrightarrow C([0,T])$ which vanish at $t = T$. The bilinear form $b\colon \mathcal{X} \times \mathcal{Y}_{0,\{T\}} \to \mathbb{R}$ and functional $\ell\colon \mathcal{Y}_{0,\{T\}} \to \mathbb{R}$ are defined by

$$\begin{aligned} b(u,v) &:= \lambda \int_0^T u(t)v(t)\,\mathrm{d}t - \int_0^T u(t)v'(t)\,\mathrm{d}t, && u \in \mathcal{X},\ v \in \mathcal{Y}_{0,\{T\}}; \\ \ell(v) &:= \int_0^T v(t)\,\mathrm{d}t, && v \in \mathcal{Y}_{0,\{T\}}. \end{aligned} \tag{4.3}$$

We re-emphasize at this point that we have chosen to move the time derivative from the trial function to the test function using integration by parts.

In order to discretize (4.2), we introduce discrete trial and test subspaces $E^k \subset \mathcal{X}$ and $F^k \subset \mathcal{Y}_{0,\{T\}}$ respectively, both having the same finite and non-trivial dimension. Here the superscript $k > 0$ is a number indicating the temporal mesh size, where lower values of k indicate a finer mesh, as will be made precise later on. We look for a solution to the *discrete variational problem*

$$\text{Find } u^k \in E^k \text{ such that } b(u^k, v^k) = \ell(v^k) \text{ for all } v^k \in F^k. \tag{4.4}$$

By analogy with (2.21), we define the *discrete inf-sup constant* β_k by

$$\beta_k := \inf_{u \in E^k \setminus \{0\}} \sup_{v \in F^k \setminus \{0\}} \frac{|b(u,v)|}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}}.$$

Recall that $\|b\|$ is the norm of the bilinear form b, which was defined by (2.20).

The discrete variational problem is well-posed if and only if $\beta_k > 0$, see for instance [68, Section 2]. In that case, we also have the following estimate, see [68, Theorem 2]:

$$\|u - u^k\|_{\mathcal{X}} \leqslant \frac{\|b\|}{\beta_k} \inf_{w \in E^k} \|u - w\|_{\mathcal{X}}. \tag{4.5}$$

This estimate, which expresses that the error of the approximate solution u^k to u can be bounded by a positive constant multiplied by the least possible error given the discrete trial space E^k, is known as the *quasi-optimality bound* of the discrete variational problem.

Let a temporal mesh $\mathcal{T}$ be given, consisting of nodes $0 =: t_0 < t_1 < \cdots < t_N =: T$ for some $N \in \mathbb{N}$, so that the interval $[0, T]$ is split up into the N corresponding intervals in between the nodes. This mesh is used to construct a *discretization pair* $E^k \times F^k$. We now describe two particular choices of discretization pairs, namely the CN* and iE* discretizations. Both of these schemes use the discrete test space F^k consisting of continuous piecewise affine functions which correspond in the obvious way to the mesh $\mathcal{T}$ and which vanish at $t = T$, but they differ in the choice of discrete trial space E^k.

The CN* discretization takes the discrete trial space E^k to be the space of piecewise constant functions corresponding to $\mathcal{T}$. Its name is motivated by the observation that if the roles of the trial and test space are reversed — i.e., if E^k is chosen to be the space of piecewise affine continuous functions vanishing at $t = 0$ and F^k is chosen to be piecewise constant — then one obtains the Crank–Nicolson (CN) time stepping scheme. The name CN* should be read as 'adjoint Crank–Nicolson', referring moreover to the fact that the variational problems discretized by CN and CN* differ by an application of the adjoint parabolic operator in order to move the derivative from the trial to the test space.

A family of discretization pairs $(E^k \times F^k)_{k>0}$ is said to be *uniformly stable* if $\inf_{k>0} \beta_k > 0$, i.e. if $(\beta_k)_{k>0}$ is uniformly bounded away from zero. Uniform stability is a desirable property since, by the quasi-optimality estimate (4.5), it implies that the convergence of the method is governed by the approximation properties of E^k as $k \downarrow 0$. The CN* discretization is not uniformly stable: for a fixed mesh, the discrete inf-sup constant β_k corresponding to the CN* discretization tends to zero as $\lambda \to \infty$, as observed in [4, Equation (2.3.10)]. More precisely, as noted in [43, Section 3.2.1], β_k depends in the same way on the *parabolic CFL number*. This leads to a stability condition which states that the scheme is only stable if the time step size k decreases fast enough relative to the spatial mesh Δx as the latter tends to zero. Therefore, the CN* scheme may require a large number of time steps to ensure convergence when applied to a parabolic PDE on a fine spatial mesh.

The other discretization method we describe is the iE* discretization, in which the discrete trial space E^k is taken to be the space of functions $w \in L^2(0, T)$ such that $w|_{[t_{n-1}, t_n]}$ can be transformed into the function $(0, 1) \ni s \mapsto (4 - 6s)$ through translation and dilation for each $n \in \{1, \ldots, N\}$. Its name is read as 'adjoint implicit Euler', since it relates to the implicit Euler time stepping scheme applied to the adjoint variational problem. It is part of the larger class of *Gauss–Radau* discretizations, for which it has been established in [1, Section 3.4] that stability can be obtained under mild conditions on the temporal mesh. To be precise, we have the following lower bound for the discrete inf-sup constant:

$$\beta_k \geqslant \frac{1}{\sqrt{2(1 + \max\{1, r\})}},$$

where

$$r := \max_{n \in \{1, \ldots, N\}} \frac{|t_n - t_{n-1}|}{|t_{n+1} - t_n|},$$

see [43, Proposition 3.1]. Since this estimate does not depend on the parameter $\lambda > 0$, the iE* discretization is uniformly stable with respect to λ for each k, in contrast to the situation for the CN* scheme. We also see that uniform stability with respect to $k > 0$ is obtained if we take a family of meshes such that r remains bounded. Another difference with the CN* scheme is the fact that the family $(E^k)_{k>0}$ of trial spaces chosen for iE* is not nested, i.e. we do not have $E^k \subset E^{k+1}$, and that piecewise constant functions cannot be approximated by functions in E^k. As a consequence, the approximation properties of $(E_k)_{k>0}$ are such that we cannot use the quasi-optimality (4.5) to draw conclusions regarding the convergence of the iE* scheme.

4.2.2. Sinc quadrature and the numerical approximation of fractional problems

This section describes how the application of the *sinc quadrature* leads us to define a numerical method which allows for fractional exponents γ in (4.1). This quadrature approximation reduces the numerical computation of solutions to (4.1) to the repeated solution of nonfractional problems, which in turn can be handled using a temporal discretization method such as either of the discretizations described in the previous section. We first give a general overview of sinc quadrature techniques; after that, we will see where it can be applied in the context of the fractional parabolic problem (4.1).

Sinc quadrature

Sinc numerical methods are a class of computational techniques which can be applied to a variety of numerical approximation problems; for instance, in the standard reference [49] by Lund and Bowers, the goal is to develop numerical schemes for solving ordinary differential equations. In the context of the present work, we are most interested in the theory underlying sinc methods for numerical interpolation and integration, the main ideas of which are summarized in this subsection.

As the name suggests, a central rule in this theory is played by the *sinc function*, defined here by

$$\operatorname{sinc} x := \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \in \mathbb{C} \setminus \{0\}, \\ 1, & x = 0. \end{cases}$$

It is used to define, for a function f on $\mathbb{R}$ and a constant $h > 0$, the *cardinal function expansion*

$$C(f,h)(x) := \sum_{\ell=-\infty}^{\infty} f(\ell h) \operatorname{sinc}\left(\frac{x - \ell h}{h}\right),$$

whenever the series on the right-hand side converges. It follows from the definition of the sinc function that $C(f,h)$ coincides with f on points $kh, k \in \mathbb{Z}$; in other words, $C(f,h)$ interpolates f between the nodes kh. One is then lead to wonder when it holds that $C(f,h) = f$ on the whole real line. It turns out that such functions are characterized as members of the *Paley–Wiener class* of functions, and that such functions in fact coincide with their cardinal function expansion on the whole complex plane.

Definition 4.2.1. Let $h > 0$. A function $f\colon \mathbb{C} \to \mathbb{C}$ is said to belong to the *Paley–Wiener class*, denoted $f \in B(h)$, if

(i) f is entire, i.e. it is analytic on the whole complex plane $\mathbb{C}$;

(ii) $f|_{\mathbb{R}} \in L^2(\mathbb{R})$, i.e. the restriction of f to the real line is square-integrable;

(iii) f is *of exponential type* π/h on the complex plane, meaning that there exists some $K > 0$ such that
$$\left|f(z)\right| \leqslant K \exp(\tfrac{\pi}{h}|z|) \quad \text{for all } z \in \mathbb{C}.$$

The exact interpolation result then reads as follows; its proof is based on the formulation of the Paley–Wiener theorem which states that functions in the Paley–Wiener class as defined here have compactly supported Fourier transforms.

Theorem 4.2.2. *Given $h > 0$, a function $f \in B(h)$ can be represented as*

$$f(z) = \sum_{\ell=-\infty}^{\infty} f(\ell h) \operatorname{sinc}\left(\frac{z - \ell h}{h}\right) \quad \textit{for all } z \in \mathbb{C}.$$

Proof. See [49, Theorem 2.5]. ☐

This leads to the following corollary, which asserts that the *trapezoidal rule* of integration over the real line is exact for integrable functions in the Paley–Wiener class.

Corollary 4.2.3. *For $h > 0$, $f \in B(h)$ and $f \in L^1(\mathbb{R})$, we have that*

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = h \sum_{\ell=-\infty}^{\infty} f(\ell h). \tag{4.6}$$

Proof. See [49, Corollary 2.6]. ☐

This exactness result is not yet feasible to use as the basis for a practical numerical method. Firstly, the conditions for a function to belong to the Paley–Wiener class are too restrictive for many integrands, in particular the requirement that f be entire. Secondly, the infinite sum on the right-hand side of (4.6) must be truncated to a finite sum. We expect that both taking $f \notin B(h)$ and truncating the series will introduce errors, leading to the question of what can be said about the rates of these respective errors as functions of h.

Since we are interested in the approximation of integrals over the real line, it is natural to study integrands f which may not necessarily be entire but which are analytic on a region containing the real line. For such functions, we may then hope that the truncated trapezoidal rule of integration still exhibits fast convergence to the exact solution as $h \downarrow 0$. This leads to the following definition.

Definition 4.2.4. Let $d > 0$ be given and let f be a function defined on the *infinite strip* of width $2d$:

$$S_d := \{z \in \mathbb{C} \colon z = x + iy, \, |y| < d\}.$$

If f is analytic in S_d, and moreover satisfies

$$\int_{-d}^{d} \big|f(t+iy)\big| \,\mathrm{d}y = \mathcal{O}(|t|^a) \quad \text{as}\, |t| \to \infty, \text{ for } a \in [0,1),$$

and for some $p \in [1, \infty)$,

$$N^p(f,d) := \lim_{y \uparrow d} \Big(\big\|f(\cdot + iy)\big\|_{L^p(\mathbb{R})} + \big\|f(\cdot - iy)\big\|_{L^p(\mathbb{R})} \Big) < \infty,$$

then f is said to belong to $B^p(d)$.

The use of the class $B^p(d)$ is illustrated by the next theorem, which states that the error of the trapezoidal rule applied to functions $f \in B^1(d)$ decreases exponentially.

Theorem 4.2.5. *Let $d, h > 0$ and $f \in B^1(d)$ be given. Then it holds that*

$$\left| \int_{-\infty}^{\infty} f(t) \,\mathrm{d}t - h \sum_{\ell=-\infty}^{\infty} f(\ell h) \right| \leqslant \frac{N^1(f,d)}{2\sinh(\pi d/h)} e^{-\pi d/h}.$$

It follows that the quadrature error *of the trapezoidal rule in this situation is of order $\mathcal{O}(e^{-2\pi d/h})$.*

Proof. See [49, Theorem 2.20]. □

The above result shows that the trapezoidal rule is a highly efficient numerical integration technique whenever the integrand belongs to the class $B^1(d)$, in the sense that an amount of quadrature nodes of order $\mathcal{O}(\log(\varepsilon^{-1}))$ is sufficient to approximate the integral up to an error of $\varepsilon > 0$. However, as mentioned before, we still need to consider the error introduced by truncating the infinite series arising from the trapezoidal rule to a finite sum. It is desirable to have conditions on the integrand which ensure that the truncation error is balanced with the quadrature error, i.e. that the truncation error also has an exponential convergence rate. As one may naturally expect, this requirement leads to growth conditions of the integrand as $t \to \pm\infty$; the precise formulation is given in the following result.

Theorem 4.2.6. *Let $d > 0$ be given and suppose that $f \in B^1(d)$. Let $\alpha, \beta, C > 0$ be constants such that*

$$\big|f(x)\big| \leqslant C \begin{cases} \exp(-\alpha|x|), & x < 0, \\ \exp(-\beta|x|), & x \geqslant 0. \end{cases}$$

Choosing $K^+, K^- \in \mathbb{N}$ such that

$$K^+ = \left\lceil \frac{\alpha}{\beta} K^- + 1 \right\rceil,$$

where $\lceil \cdot \rceil$ denotes the function which rounds numbers up to the next integer, and setting

$$h := \sqrt{\frac{2\pi d}{\alpha K^-}},$$

we have the following estimate for the error of the truncated trapezoidal rule:

$$\left| \int_{-\infty}^{\infty} f(t) \,\mathrm{d}t - h \sum_{\ell=-K^-}^{K^+} f(\ell h) \right| \leqslant C_f \exp(-(2\pi d \alpha K^-)^{1/2}).$$

Proof. See [49, Theorem 2.21]. □

Note that the summation bounds K^- and K^+ are generally not chosen equal to each other, i.e. the truncated summation is non-symmetric in general.

Application to fractional problems

We now describe how the sinc numerical methods outlined in the previous subsection inspire a numerical method which can be applied to the problem of approximating solutions to problems involving fractional powers of operators. The idea is due to Bonito and Pasciak, who first proposed and studied this approach in the context of fractional powers of elliptic operators, which was published as [14]. A major finding in this work was the fact that exponential convergence of the quadrature error was possible for a suitable integral [14, Section 3.3], thus giving rise to a highly efficient numerical method for fractional elliptic problems. This approach has been widely used since its inception, owing mainly to its ease of implementation using standard FEM techniques.

In order to give a rough outline of the idea, let L be some linear operator on a Hilbert space H which admits fractional powers, let $\gamma > 0$ be a fractional power, and consider the problem of finding a solution u to $L^\gamma u = f$ where u and f belong to some appropriate function spaces. Thus, we are interested in approximating $u = L^{-\gamma} f$. We restrict ourselves to $0 < \gamma < 1$ for ease of presentation, but the method can be generalized to arbitrary $\gamma > 0$ by splitting up the exponent into fractional and integer parts, which reduces the problem to a combination of the techniques we will describe.

The main idea now is to represent the fractional solution operator $L^{-\gamma}$ as an operator-valued integral which is amenable to approximation using the obvious generalization of the trapezoidal rule described in the previous subsection. Moreover, for practical purposes we seek an integrand which can be evaluated efficiently. To this end, we start by applying the Balakrishnan formula (2.13) to L^γ, yielding

$$L^{-\gamma} = \frac{\sin \pi\gamma}{\pi} \int_0^\infty t^{-\gamma}(tI + L)^{-1}\,\mathrm{d}t.$$

Indeed, this is a suitable starting point: we represent the fractional solution map by an integral over a real domain of an expression involving shifted non-fractional solution operators, for which we assume that there exists an efficient approximation technique. In order to obtain an integral over the whole real line, we employ the change of variables $t = e^{-2y}$ which produces

$$L^{-\gamma} = \frac{2\sin \pi\gamma}{\pi} \int_{-\infty}^\infty e^{2\gamma y}(tI + e^{2y}L)^{-1}\,\mathrm{d}y.$$

We approximate this integral by the truncated trapezoidal rule, finally resulting in the sinc quadrature approximation Q_q^γ of the solution operator $L^{-\gamma}$:

$$Q_q^\gamma = \frac{2q\sin \pi\gamma}{\pi} \sum_{\ell=-K^-}^{K^+} e^{2\gamma\ell q}(I + e^{2\ell q}L)^{-1}.$$

Here the $q > 0$ denotes the quadrature step size $K^-, K^+ \in \mathbb{N}$ are the summation bounds as in the previous subsection.

We would like to establish some suitable operator-valued analog of Theorem 4.2.6 for the approximation of the Balakrishnan integral for $L^{-\gamma}$, which would yield exponential convergence similar to what is observed in the scalar-valued case. In [14, Sections 3.3–3.4], this analysis was performed for the case that L is a positive, self-adjoint and uniformly elliptic differential operator. To this end, the authors first checked that the integral obtained when taking $L := \lambda \geqslant \lambda_0 > 0$ for some positive real constant λ_0 satisfies the conditions for Theorem 4.2.6. The application of the theorem then results in an error bound for $|\lambda^{-\gamma} - Q_q^\gamma|$ which is independent of $\lambda \geqslant \lambda_0$. This scalar-valued result can subsequently be extended to the operator-valued case by means of the orthonormal eigenbasis $(\lambda_j, e_j)_{j\geqslant 1}$ on H corresponding to L, using the fact that $\lambda_j \geqslant \lambda_1 > 0$ for all $j \geqslant 1$ given that the eigenvalues are chosen in increasing order, see [14, Theorem 3.5].

Now we propose to solve the fractional parabolic equation using the natural analog to the one described in [14]. I.e., we approximate the fractional parabolic solution operator as follows:

$$(\partial_t + \lambda)^{-\gamma} = \frac{2\sin \pi\gamma}{\pi} \int_{-\infty}^\infty e^{2\gamma y}(1 + e^{2y}[\partial_t + \lambda])^{-1}\,\mathrm{d}y. \tag{4.7}$$

Let $B_k \colon E^k \to (F^k)'$ be the linear operator corresponding to the discrete weak variational problem (4.4) using one of the time discretizations introduced in the previous section. Approximating the integral on the right-hand side of (4.7) by the truncated trapezoidal integration rule now yields

$$Q_{q,k}^{\gamma} = \frac{2q \sin \pi\gamma}{\pi} \sum_{\ell=-K^-}^{K^+} e^{2\gamma\ell q} (I_{E^k \to (F^k)'} + e^{2\ell q} B_k)^{-1}.$$

Note that computing an approximation $Q_{q,k}^{\gamma} f$ for some $f \in (F^k)'$ then amounts so solving a nonfractional problem for each term on the right-hand side of the previous display. More explicitly, given a source term $\mathbf{f}_k$ with respect to a basis for E^k, the solution $\mathbf{u}_{q,k}$ can be expressed in the following way:

$$\mathbf{u}_{q,k} = \frac{2q \sin \pi\gamma}{\pi} \sum_{\ell=-K^-}^{K^+} e^{2\gamma\ell q} ((1 + \lambda e^{2\ell q}) \mathbf{M}_k - e^{2\ell q} \mathbf{G}_k)^{-1} \mathbf{f}_k,$$

where $\mathbf{M}_k$ denotes the mass matrix and $\mathbf{G}_k$ denotes the Gram matrix corresponding to the term $\int_0^T u(t) v'(t) \, \mathrm{d}t$ in the definition of the bilinear form (4.3). Hence, we have finally arrived at the explicit form of the numerical method proposed to solve the fractional problem (4.1) using a combination of variational time discretizations and sinc quadrature methods.

5 Discussion and outlook

We have studied an SPDE-based method for the generation of Gaussian random fields suitable for the purposes of spatiotemporal statistics, which has the potential to be an efficient generalization of existing spatial methods. The main focus of this work was on the analysis of the SPDE (3.1) itself rather than that of the possible numerical approximations, the ideas for which were briefly introduced in Chapter 4.

We have introduced mild and weak solution concepts for (3.1) and showed under which assumptions on A and Q they are well-posed. Moreover, the spatiotemporal regularity of the SPDE was studied, in such a way that they can be linked explicitly to fractional parameters of the SPDE in practical situations. The assumptions made on for instance the operator A in the SPDE were motivated mainly by the situation known from applications, which furthermore facilitated direct arguments to derive regularity and well-posedness results. We believe that some results in this area can be extended to situations where, for instance, A is not self-adjoint or the C_0-semigroup generated by $-A$ is not necessarily analytic.

We have also considered the covariance structure of solutions to the SPDE, which was seen to asymptotically behave in a predictable way with regard to both the spatial and temporal marginal covariance. This confirms the potential for this SPDE model to be useful for spatiotemporal statistical inference applications.

In Chapter 4 we have described one possible way of dealing with the numerical approximation of a fractional initial value problem, which can be generalized to the PDE situation. It was based on the combination of variational time discretizations with the sinc quadrature techniques which have proven powerful in the solution of elliptic problems.

It is clear that many questions on this subject are still open for future study, particularly on the numerical side. The first challenge in that respect will be to prove error estimates with the aim of showing that the fractional parabolic (deterministic) PDEs are equally amenable to the sinc quadrature methods which have thus far only been considered for the elliptic case. It is expected that the regularity results will play a central role in the estimates which need to be derived here. The next question would be whether such a method can carry over to an efficient simulation technique for the corresponding stochastic PDE.

From the point of view of both (numerical) analysis and applications, it is also interesting to consider extending the problem considered in this work to more general (S)PDEs on manifolds such as the sphere. Obviously, the specific case of the sphere is a highly relevant example due to the prevalence of data collected across the globe. For this problem, one can attempt to answer the same questions as considered on the Euclidean domain in the present work.

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