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Capturing the Effect of Thickness on Size-Dependent Behavior of Plates with Nonlocal Theory

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Abstract

The effective elastic properties of nano-structures are shown to be strongly size-dependent. In this paper, using a three dimensional strong nonlocal elasticity, we have presented a formulation to capture the size-dependent behavior of plate structures as a function of their thickness. This paper discusses some new aspects of employing a three dimensional nonlocal formulation for analysis of plates, namely, the confining of the nonlocal kernel in the near-boundary regions at the two surfaces of the plate. To address this aspect, we have studied two different types of nonlocal kernels, one bounded in a finite domain of the structure and the other, non-bounded. This study shows that the influence of the plate's thickness on its bending stiffness can be captured within the nonlocal elasticity framework, and this influence highly depends on the bounding of the nonlocal kernel. Particularly, for a uniformly deformed plate with a homogeneous isotropic material, using the nonlocal formulation with the bounded domain reflects the physics of the problem better.

Keywords: Nonlocal theory, Finite-scale kernel, Nanoplate, thickness, size-dependency, continuum theory.

1. Introduction

Micro and nano electro-mechanical-systems (MEMS and NEMS) play key roles in a wide variety of modern applications, including nano-mechanical sensors, actuators, and many electronic devices. The performance of these devices is based on movements and deformations of their micro/nano mechanical components, such as cantilevers, double clamped beams or plates. Obviously, the further development of these devices requires a thorough understanding and modeling of their mechanical behavior. However, devices at nano-meter scale may exhibit mechanical properties not noticed at the macro-scale. Many theoretical methods such as molecular and atomistic simulations and size-dependent continuum theories are being developed to analyze this behavior. Molecular and atomistic simulations are generally time consuming and computationally expensive. Alternatively, continuum models offer superior computational efficiency.

Classical continuum mechanics is size independent and it cannot provide a good prediction for small scales. Therefore, size-dependent continuum theories have been introduced to account for these scaling effects [1, 2]. In an attempt to account for atomistic effects, these theories embed an internal material length scale. This makes it possible to qualify the size of a structure as "large" or "small" relative to its material length scale [1, 3, 4]. If "large", then these theories should converge to classical continuum theory, and, otherwise, they should reflect the size-dependence.

One of the best-known size-dependent continuum theories is non-local continuum theory, initiated in a general notation by Piola in 1846 [5, 6]. In nonlocal continuum theory, a material point is influenced by the state of all points of the body. The mathematical description of this theory relies on the introduction of additional contributions in terms of "gradients" or "integrals" of the strain field in the constitutive equations. This, respectively, leads to so-

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called “weak” or “strong” non-local models [7–9]. Although both models have been found to be largely equivalent [10], the weak (gradient) formulation requires stronger continuity on the displacements gradients. In addition, in cases that a well-defined spatial interaction exists in the material, the strong (integral) approach is preferred, because it models the nonlocality in a more transparent way [10].

In strong nonlocal theories, particularly formulated by Kröner in 1967 [2], and then by Eringen in 1977 [1, 4], the point-to-point relationship between stresses and strains does not hold anymore. Instead, the stress in each point is influenced by the strain of all points of the body. This influence is captured by a spacial integral over the body. The integral is weighted with a decaying kernel, which is designed to incorporate the long-range interaction between atoms in the continuum model. With the spacial integral, the dimensions of the body are brought into the constitutive equations, and thus, the constitutive equations will be size-dependent.

It is worth to mention here that closely related to strong nonlocal theory, the *peridynamics* theory has been developed by Silling [9]. In fact, in peridynamics, instead of spatial differential operators, integration over differences of the displacement field is used to describe the existing, possibly nonlinear, forces between particles of the solid body [9, 11]. However, in contrast to the peridynamic theory, the strong nonlocal theories rely on spatial integrations. The present study mainly focuses on the commonly used strong formulation given by Eringen.

The strong nonlocal theory has been used in many studies for modeling micro- or nanomechanical devices. In these studies, mechanical components such as thin-film elements and plate-like structures have been modeled with so-called *two-dimensional* non-local formulations, also known as “nonlocal plate theories” [12–14]. In these theories, the plate-like structures are generally modeled as a two-dimensional domain. In this way, the nonlocal contribution of the strain field in the transverse direction is ignored. Therefore, the size of a plate is only defined by its lateral dimensions, and thus, its thickness is not incorporated in its size-dependent behavior.

In plane stress problems, which are inherently two dimensional—such as the stress analysis near the crack tip in a thin plate [4]— ignoring the nonlocal effects in transverse direction is within reason. Also, for structures whose thickness is much smaller than the material length-scale, such as a monolayer graphene, the nonlocal effect in transverse direction is in fact meaningless [15]. However, modeling a plate as a two-dimensional domain and ignoring the nonlocal contribution in the transverse direction is not always valid. First of all, from a physical point of view, a nonlocal theory is supposed to incorporate the interaction between atoms in a continuum model and so its effect should exist in all directions [16]. Second, since the thickness of a plate is significantly smaller than its lateral dimensions, the length scale at which classical elasticity breaks down appears in the transverse direction first. Moreover, in problems in which there is a uniform strain field in the tangential directions, the nonlocal stress as a function of weighted average of strain in tangential directions is simply equal to the classical stress. This means the two-dimensional formulation fails to reflect any size-dependency. In such a case, it is likely that transverse non-locality would have a more pronounced size-dependence contribution.

In this paper, we particularly investigate how the strong three dimensional nonlocal formulation can incorporate the plate thickness. Moreover, we study the effect of thickness in the predicted size dependence of the overall flexural rigidity and elastic modulus of the plate.

It is worth to note that in nonlocal elasticity, as a consequence of including contributions of integrals of the strain field in the constitutive equations, the differential order of the governing equations changes. This results in additional boundary conditions which should physically reflect the surface properties of the material/structure. The latter, however, has not been discussed rigorously in literature so far and instead, the boundaries are often avoided in the respective analyses. When a three dimensional nonlocal formulation is employed in the analysis of plates, these extra boundary conditions should be defined on the upper and lower surface of the plate. In order to investigate the significance of these boundary conditions, two different treatments of the boundaries will be addressed in this paper.

This paper is structured as follows. In Section 2, the fundamentals of Eringen’s nonlocal elasticity theory, some important considerations and the basis of conventional nonlocal plate theory are reviewed. In Section 3, we will use a three dimensional nonlocal formulation to solve an example of uniformly deformed plate. For this purpose two types of boundary conditions will be applied for the nonlocal formulation. The results of this analysis will be discussed and compared to classical plate theory in Section 4. In the last section, the conclusions of this study will be presented.

2. Nonlocal elasticity theory

In linear nonlocal elasticity, the stress tensor ($\mathbf{t}$) for a homogenous and continuous domain is determined as [1, 17]:

$$\begin{aligned} t_{ij}(\mathbf{x}) &= \int\limits_{V_b} \alpha\left(|\mathbf{x}-\mathbf{x}'|, e_0 a\right) C_{ijkl}\varepsilon_{kl}(\mathbf{x}')dV(\mathbf{x}') \\ &= \int\limits_{V_b} \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \sigma_{ij}(\mathbf{x}')dV(\mathbf{x}') \end{aligned} \tag{1}$$

where $\varepsilon_{kl}(\mathbf{x}')$ are the classical Cauchy's strain components at the point $\mathbf{x}'$ and C_{ijkl} are the components of the elasticity tensor. Index k and l are the dummy index in Einstein's summation convention, and Cartesian coordinates have been assumed. The product of these two terms can be simply substituted with classical stress component $\sigma_{ij}(\mathbf{x}')$, as in the second line. Then, V_b is the volume of the body at hand. The function $\alpha(|\mathbf{x}-\mathbf{x}'|, e_0 a)$ is the non-local kernel representing the effect of long-range interactions [9] . This radial kernel reflects the nonlocal contribution of strain in all points $\mathbf{x}'$ of the body. The nonlocal kernel α is also a function of parameters a and e_0. The parameter a is the material characteristic length scale (e.g. atomic distance, lattice parameter, granular distance) [12], and e_0 is a constant for adjusting the model to match experiments or other models [4, 13, 16]. Other properties of the nonlocal kernel α will be discussed later in this section.

It should be stressed that the proof of existence of Cauchy's stress tensor is based on the equilibrium of contact forces with a force which is assumed to be continuous in space. We may use a similar assumption as well (as proposed in [5, 6]). Moreover, in strain gradient nonlocal theories, the constitutive equations are much more than one stress-strain relationship. Instead, so-called *double* or *hyper* stress components are defined associated to higher order strain gradients [18]. In the strong nonlocal theory, however, the basic equations for an isotropic solid can be expressed in its simplest form as in Equation 1 [1, 2, 7–10].

Accordingly, the nonlocal strain energy is expressible as [4]:

$$U_{nonlocal} = \frac{1}{2}\int\limits_{V_b} t_{ij}\varepsilon_{ij} dV. \tag{2}$$

Please note that this formulation of internal energy is a particular case of the formulation given by Kröner [2], provided that the kernel α reflects the local (short-range) as well the nonlocal (long-range) effects. The equilibrium equations in the nonlocal continuum theory are the same as for classical continuum theory, but represented in terms of the nonlocal stresses (t_{ij}) rather than the local stresses (σ_{ij}).

2.1. Nonlocal kernel

The function used as the nonlocal kernel ($\alpha(|\mathbf{x}-\mathbf{x}'|, e_0 a)$) needs to have the following characteristic properties;

1- To reflect the properties of atomistic long term interactions correctly, it acquires its maximum at $\mathbf{x} = \mathbf{x}'$ and monotonically decreases with $|\mathbf{x}-\mathbf{x}'|$.

2- To ensure that classical elasticity is included in the limit of a vanishing internal characteristic length, it must tend to Dirac's delta function when $e_0 a \to 0$. [17], i.e.

$$\lim_{e_0 a\to 0} \alpha\left(|\mathbf{x}-\mathbf{x}'|, e_0 a\right) = \delta\left(|\mathbf{x}-\mathbf{x}'|\right). \tag{3}$$

3- The stress at $\mathbf{x}$ should have the same contribution to the stress at $\mathbf{x}'$ as *vice versa*, thus, the nonlocal kernel is symmetric in its arguments $\mathbf{x}'$ and $\mathbf{x}$, i.e. $\alpha(\mathbf{x}, \mathbf{x}') = \alpha(\mathbf{x}', \mathbf{x})$.

4- According to Eringen's nonlocal continuum theory [1], the function α is normalized in the volume of the body (V_b):

$$\int\limits_{V_b} \alpha(|\mathbf{x}-\mathbf{x}'|)dV(\mathbf{x}') = 1. \tag{4}$$

This property assures that a uniform local strain field should also result in a uniform nonlocal stress field (See Equation 1), provided that the

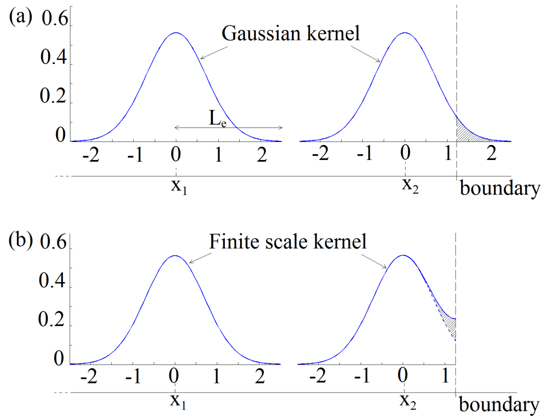

Figure 1: A schematic representation of one-dimensional kernels at two points within the body (x_1) and near the boundary region (x_2): (a) Gaussian density function, normalized on an infinite domain; (b) A bounded kernel which adapts its shape when the effective influence zone includes a boundary.

material is isotropic and homogeneous [19]. If the domain of the body at hand is large enough to be considered as an infinite domain, then this feature implies that α is always normalized on such an infinite domain. It should be noted that nonlocal kernels have an effective influence zone, V_e, centered around $\mathbf{x}$ and an effective cut-off length, L_e [16, 17]. Outside this influence zone, the function $\alpha(x-x')$ practically vanishes and, thus, it can be assumed that

$$\int_{V_e} \alpha(|\mathbf{x}-\mathbf{x}'|)dV(\mathbf{x}') \simeq \int_{V_\infty} \alpha(|\mathbf{x}-\mathbf{x}'|)dV(\mathbf{x}') = 1. \tag{5}$$

Many kernels have been suggested in literature with the above mentioned properties, and in general, all these kernels qualitatively lead to similar results [17, 20]. However, for a kernel with these properties, in points closer than its cut-off length (L_e) to the boundary, the influence zone of the kernel exceeds the boundary. Consequently, it only collects the nonlocal influence of the points $\mathbf{x}'$ inside the body (see Figure 1-a). Thus, the normalization condition would not be satisfied anymore.

To satisfy the normalization condition, either the analysis domain should be far from boundaries [1], and then we can use the common kernels; or, a modified bounded nonlocal kernel should be used. The shape of such a kernel is modified near boundaries such that it satisfies the normalization condition based on all material points available (Figure 1-b) [3, 10, 21, 22]. It is worth noting that the kernel function in such an expression cannot not be a function of $|\mathbf{x}-\mathbf{x}'|$ any more, but $\mathbf{x}$ and $\mathbf{x}'$.

It should be noted that bounding the nonlocal kernel in the finite domain of the body, results in bringing the dimensions and perhaps the shape of of the body into the definition of the nonlocal kernel. In order to introduce this modification to the nonlocal kernel mathematically, we first need to specify the final property of the kernel α.

5- The final feature of the function α, is an assumption, and physically speaking it is not really required [17]. It is assumed that the kernel α is the Green's function of an operator L. In other words, it is supposed that an operator L can be found for any kernel α where:

$$L\alpha(|\mathbf{x},\mathbf{x}'|) = \delta(|\mathbf{x},\mathbf{x}'|). \tag{6}$$

The later, as a matter of fact, is an assumption to convert integral equations to differential equations. Such an operator L can be used to relate the non-local stress to the local stress. Applying L to Equation 1 would yield:

$$Lt_{ij} = \sigma_{ij}. \tag{7}$$

Therefore, a choice of kernel implicitly defines a differential operator which transforms the nonlocal stress to the local one [19]. Equation 7 is commonly used instead of the definition given in Equation 1.

As a matter of fact, for solving Equations 7, one has to impose boundary conditions on t_{ij}. It should be reminded that Green's function of a boundary value problem should also satisfy the boundary conditions of the problem. Therefore, these boundary conditions are actually the tools to change the shape of the kernel in the near-boundary regions, as schematically shown in Figure 1-b.

In literature, most of the suggested operators L are second-order operators [17]. For the resulted second-order differential equation, the required boundary conditions involve either the zeroth- or first-order derivatives of the function. Thus, as an appropriate boundary condition for Equation 7, either the value of nonlocal stress components, their derivatives, or a linear combination of these two must be defined on the boundary. Fixing the stress components (t_{ij}) on the boundaries does not have any physical motivation. Hence, homogeneous or inhomogeneous boundary conditions on the first derivative of the stress components (e.i. Neumann B.C.) shall be employed.

Although the boundary conditions on the nonlocal kernel, which indicate the extra boundary conditions on the stress derivatives, are key elements in nonlocal elasticity, they are discussed in very few studies. In most publications, non-bounded kernels are used; and the boundary effects (like in Fig. 1-a,) are justified by the surface effects [3]. In a few studies, a homogeneous Neumann boundary condition is applied on the nonlocal kernel [3, 10, 22]. However, there is no discussion on the connection between the suggested mathematical boundary conditions and the underlying physics.
It should be mentioned that the added boundary conditions are a challenging problem in so-called weak nonlocal or higher order elasticity theories, as well. In strain gradient elasticity, for example, these boundary conditions automatically appear either for the second order gradient of the displacement components or associated *double stress* (i.e. the partial derivative of energy density to strain gradients) [18]. Although a clear interpretation of the mentioned boundary conditions is not provided in this theory either, it has been shown that the strain gradient and nonlocal formulation are largely equivalent *if* the appropriate boundary conditions are employed for the nonlocal kernel [10].

It is the authors' opinion that the boundary conditions on the nonlocal kernel should reflect the surface properties of the material/structure. From a physics point of view, the nonlocal kernel is supposed to incorporate the long range interactions of atoms into continuum mechanics. Hence, if its shape is varying near boundaries, it should result from the rearrangement of atoms near the surface of the pristine material. In addition, there are fewer possibilities for interaction between atoms near the surface, and this should also be reflected in the model.

We should keep in mind that the extra boundary conditions become very relevant in applying nonlocal elasticity to a plate. In such a case, the relevant boundaries are the lateral surfaces, for which the additional boundary conditions on the stress or the kernel should be defined. In this paper, we investigate the difference between the predicted mechanical response of the plate, when applying a non-bounded kernel (Fig. 1-a), or, a bounded one using a homogeneous Neumann boundary condition (Fig. 1-b). This is done for an example of uniform deformation of a plate.

2.2. Conventional nonlocal plate theories

In nonlocal elasticity, as described above, all formulations are three-dimensional. The points $\mathbf{x}$ and $\mathbf{x}'$ are arbitrary points in space and the integrals in Equations 1–2 involve the entire volume of the body. The kernel function α is a three-dimensional kernel, i.e. it reflects the nonlocal contribution of the strain field in all directions and its dimension is m^{-3}.

For thin plates, however, many studies have modeled the plate as a two-dimensional domain. In such models, the points $\mathbf{x}$ and $\mathbf{x}'$ are arbitrary points of the mid-plane of the plate and the integrals are taken over the surface area of the plate. The nonlocal kernel in such a model is a two-dimensional function, ignoring the components with respect to the transverse direction (z) [12–14, 23–25]. The dimension of a two-dimensional kernel is m^{-2} and it is normalized over the the area of the plate.

With above mentioned assumptions in the nonlocal plate theory, the nonlocal stress is defined as [14]:

$$\bar{t}_{ij}(\mathbf{x}) = \int_A \alpha\left(|\mathbf{x}-\mathbf{x}'|\right)\bar{\sigma}_{ij}(\mathbf{x}')dA(\mathbf{x}'), \quad (8)$$

where A is the surface area of the plate and i and j denote the in-plane coordinates. The over-line is employed specifically for vector form indication of the nonzero components of the second order tensors of stress and strain in the plate. The tangential resultants of stress ($\mathbf{N_{nl}}$) are then calculated by integrating the nonlocal stress components along the transverse direction (z) in the limits of plate thickness (h). Considering that $\alpha\left(|\mathbf{x}'-\mathbf{x}|\right)$ is not a function of z and, thus, it does not affect any integration in transverse direction, the tangential stress resultants and tangential stress couples can be written as:

$$
\begin{aligned}
\mathbf{N_{nl}} &= \int_h \bar{\mathbf{t}} dz \\
&= \int_h (\int_A \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \overline{\boldsymbol{\sigma}}(\mathbf{x}') dA(\mathbf{x}')) dz \\
&= \int_A (\int_h \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \overline{\boldsymbol{\sigma}}(\mathbf{x}') dz) dA(\mathbf{x}') \\
&= \int_A \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) (\int_h \overline{\boldsymbol{\sigma}}(\mathbf{x}') dz) dA(\mathbf{x}') \\
&= \int_A \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \mathbf{N^{cl}}(\mathbf{x}') dA(\mathbf{x}'),
\end{aligned}
\tag{9}
$$

and similarly the tangential stress couples ($\mathbf{M_{nl}}$) can be calculated as:

$$
\begin{aligned}
\mathbf{M_{nl}} &= \int_h \bar{\mathbf{t}} z dz \\
&= \int_A \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \int_h \overline{\boldsymbol{\sigma}}(\mathbf{x}') z dz dA(\mathbf{x}') \\
&= \int_A \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \mathbf{M^{cl}}(\mathbf{x}') dA(\mathbf{x}').
\end{aligned}
\tag{10}
$$

In these equations $\mathbf{N^{cl}}$ and $\mathbf{M^{cl}}$ are the tangential stress resultant and couples from the classical plate theory. The equations of motion are then identical as for the classical Kirchhoff plate theory, but based on the nonlocal tangential stress resultants and couples.

Notice that based on Equations 9 and 10, in nonlocal plate theories, the nonlocal tangential stress resultants and couples do not have any nonlocal contributions from the transverse direction. Therefore, the thickness of the plate does not have any influence on the size-dependency of the result. Considering that the kernel function is normalized over the area, if $\mathbf{N^{cl}}$ and $\mathbf{M^{cl}}$ are uniform in the plate, there would be no difference between the nonlocal and classical tangential stress resultants and couples, and therefore, the two solutions (classical and nonlocal) would predict similar mechanical responses.

This model is valid for inherently two dimensional plane-stress problems, where there is no variation of the classical stress field through the thickness. Otherwise, it cannot be motivated on the basis of the fundamentals of nonlocal theory or physics. However, it has been the basis of nonlocal plate theory which is commonly used for solving many other problems related to plates, where the stress varies through the thickness, as well.

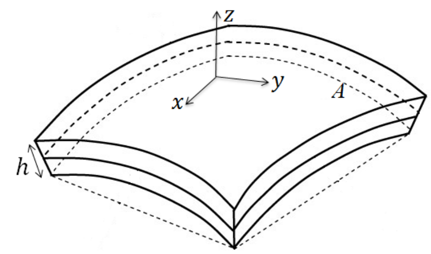

Figure 2: deflection of the plate with thickness h and surface area A.

3. Capturing effects of thickness in nonlocal plate theory

In this section, using an example of a plate with both a uniform stretch and a uniform curvature, the effect of thickness in nonlocal plate theory is studied. In fact, we use a three-dimensional nonlocal formulation in order to calculate the tangential resultant stresses and couples induced by such a deformation in the plate. With this solution we shall study the effect of thickness of the plate on its mechanical response as predicted by nonlocal elasticity theory. Moreover, we will compare the results with the classical plate theory. The solution discussed here can be extended to a plate with non-uniform deformation, as well.

Assume a plate with a uniform thickness h, and lateral area A (Figure 2). The lateral dimensions of the plate are much larger than its thickness. The mid-plane of the plate is subjected to both a uniform curvature $\boldsymbol{\kappa}$ and stretch $\boldsymbol{\gamma}$ (Figure 2). Using classical plate theories (based on plane-stress assumptions), the tangential strain $\boldsymbol{\varepsilon}$ at the interior of the plate, i.e. sufficiently far away from the edges, can be described as

$$
\overline{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix} = \begin{pmatrix} \gamma_{xx} \\ \gamma_{yy} \\ 2\gamma_{xy} \end{pmatrix} - z \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix}, \tag{11}
$$

where x and y denote the tangential coordinates. The transverse coordinate is z, and the midplane coincides with $z = 0$. Using classical linear constitutive equations, the classical tangential stresses follow as

$$\overline{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \mathbf{Q}\overline{\boldsymbol{\varepsilon}}, \tag{12}$$

where $\mathbf{Q}$ represents the elasticity tensor for a homogeneous and isotropic material and in a plane-stress problem:

$$\mathbf{Q} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \tag{13}$$

where E is the Young's modulus and ν is the Poisson ratio of the material. As a result, the nonlocal stress can be calculated:

$$\overline{\mathbf{t}} = \int_V \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \mathbf{Q}(\boldsymbol{\gamma} - z'\boldsymbol{\kappa}) dv(\mathbf{x}'). \tag{14}$$

It is worth noting that the classical plane-stress assumption ($\sigma_{zz} = 0$) as adopted in classical plate theory, now has directly lead us to a plane-stress condition for the nonlocal stresses as well ($t_{zz} = 0$). In particular, we can employ Equation 1 to explicitly prove this case:

$$\begin{aligned} \overline{t}_{zz}(\mathbf{x}) &= \int_{V_b} \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \overline{\sigma}_{zz}(\mathbf{x}') dV(\mathbf{x}') \\ &= \int_{V_b} \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) (0) dV(\mathbf{x}') = 0. \end{aligned} \tag{15}$$

In addition, we stress here that imposing the plane-stress assumption on the nonlocal stresses ($t_{zz} = 0$) will also result in plane-stress condition for the classical stress components ($\sigma_{zz} = 0$), provided that the kernel α is a positive definite function.

For deriving $\mathbf{t}$ in Equation 14, the appropriate nonlocal kernel $\alpha(|\mathbf{x}'-\mathbf{x}|)$ should be chosen. Here, a three-dimensional Gaussian function is chosen as the nonlocal kernel $\alpha(|\mathbf{x}'-\mathbf{x}|)$:

$$\alpha\left(|\mathbf{x}-\mathbf{x}'|\right) = \frac{1}{(\pi (e_0 a)^2)^{\frac{3}{2}}} \exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{(e_0 a)^2}\right). \tag{16}$$

This function has been reported to show an excellent agreement with the atomic dispersion curves of crystalline materials [1, 16]. The Gaussian density function is Green's function of the diffusion (or heat) equation in an infinite domain:

$$\nabla^2 t - \frac{\partial t}{\partial \tau} = 0, \tag{17}$$

where, $\tau = \frac{(e_0 a)^2}{4}$. We can also derive Green's function of the mentioned equation in a bounded domain by applying appropriate boundary conditions as discussed in Section 2.1. Here, both solutions with bounded and unbounded kernels are presented for the example at hand.

3.1. Gaussian kernel

To begin with, we neglect the nonlocal boundary effects in all boundaries of the plate and consider the unbounded kernel as in Equation 16. Without the boundary effects, the shape of the nonlocal kernel would be similar in all material points in different positions. A one-dimensional schematic of this kernel is shown in Figure 1-a. As this figure shows, the kernels are similar at the point $\mathbf{x} = \mathbf{x}_1$ within the body, and at $\mathbf{x} = \mathbf{x}_2$ where the influence length of the kernel exceeds the boundary. By using Equations 14 and 16, and by decoupling between in-plane and transverse directions, the nonlocal in-plane stress can be expressed as:

$$\begin{aligned} \overline{\mathbf{t}} &= \int_A \int_{-h/2}^{h/2} \alpha\left(|\mathbf{x}-\mathbf{x}'|\right) \mathbf{Q}(\boldsymbol{\gamma} - z'\boldsymbol{\kappa}) dz' dA' \\ &= \int_x \int_y \frac{\exp\left(-\frac{(x-x')^2+(y-y')^2}{(e_0 a)^2}\right)}{(\sqrt{\pi} e_0 a)^2} dx' dy' \times \\ &\quad \int_{-h/2}^{h/2} \mathbf{Q}(\boldsymbol{\gamma} - z'\boldsymbol{\kappa}) \frac{\exp\left(-\frac{(z-z')^2}{(e_0 a)^2}\right)}{(\sqrt{\pi} e_0 a)} dz' \end{aligned} \tag{18}$$

Notice that the function inside the first integral is a two-dimensional Gaussian kernel itself and is normalized on an infinite area. As mentioned before, the lateral geometries of the plate are considered to be much larger than its thickness and the plate dimensions can be assumed as infinite. In other words, near-boundary regions in these two directions can be ignored relative to the their dimensions. Therefore, the first integral equals to unity almost everywhere in the plate, and nonlocal stress can be simplified to the second integral only:

$$\overline{\mathbf{t}} \simeq \int_{-h/2}^{h/2} \mathbf{Q}(\boldsymbol{\gamma} - z'\boldsymbol{\kappa}) \frac{\exp\left(-\frac{(z-z')^2}{(e_0 a)^2}\right)}{(\sqrt{\pi} e_0 a)} dz'. \tag{19}$$

In fact, this result is valid in the points sufficiently far away from the edges. Note that due to the assumed uniformity of curvature and stretch, in-plane dimensions vanish from the equations and only the integration in transverse direction remains. Of course, for non-uniform deformation (i.e. $\boldsymbol{\kappa}(x,y)$ and $\boldsymbol{\gamma}(x,y)$), the in-plane non-locality would also remain in the formulations. The nonlocal stress can be solved and simplified to

$$\begin{aligned}\bar{\mathbf{t}} = \mathbf{Q}\boldsymbol{\gamma}&\left(\frac{1}{2}\left(\operatorname{erf}(\frac{z+h/2}{e_0a}) - \operatorname{erf}(\frac{z-h/2}{e_0a})\right)\right)\\ -\mathbf{Q}\boldsymbol{\kappa}&\left(\frac{1}{2}z\left(\operatorname{erf}(\frac{z+h/2}{e_0a}) - \operatorname{erf}(\frac{z-h/2}{e_0a})\right)\right.\\ &-\frac{1}{2}\frac{e_0a}{\sqrt{\pi}}\Bigg(\exp\left(-\frac{(z-h/2)^2}{(e_0a)^2}\right)\\ &\left.-\exp\left(-\frac{(z+h/2)^2}{(e_0a)^2}\right)\Bigg)\right),\end{aligned} \tag{20}$$

where erf is the error function. Notice that the choice of not normalizing the kernel function α results in a non uniform in-plane stress over the thickness even for the case of a uniform stretch (i.e., $\boldsymbol{\kappa} = 0$). Next, we can calculate the tangential stress resultants and couples in the plate:

$$\begin{aligned}\mathbf{N_{nl}} = \int_{-h/2}^{h/2} \bar{\mathbf{t}}dz = h\mathbf{Q}\boldsymbol{\gamma}\Big(&\operatorname{erf}(\eta)+\\ &\frac{1}{\eta\sqrt{\pi}}\left(\exp(-\eta^2)-1\right)\Big),\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{M_{nl}} = \int_{-h/2}^{h/2} \bar{\mathbf{t}}zdz = \frac{h^3}{12}\mathbf{Q}\boldsymbol{\kappa}\Bigg(&\operatorname{erf}(\eta) -\\ &\frac{1}{\sqrt{\pi}}\Big(\frac{2}{\eta}\exp(-\eta^2)\\ &+(3\eta^{-1}-2\eta^{-3})(1-\exp(-\eta^2))\Big)\Bigg).\end{aligned} \tag{22}$$

The parameter $\eta = \frac{h}{e_0a}$, which is the thickness normalized with the internal length scale of the material is introduced to simplify the formulations. Equations 21 and 22 can also be written as:

$$\mathbf{N_{nl}} = \lambda\mathbf{A}\boldsymbol{\gamma}, \tag{23}$$

$$\mathbf{M_{nl}} = \beta\mathbf{D}\boldsymbol{\kappa}. \tag{24}$$

Here, λ and β are introduced as the nonlocal modification factors on classical extensional (or membrane) stiffness matrix ($\mathbf{A} = h\mathbf{Q}$) and bending stiffness matrix $\mathbf{D} = \frac{h^3}{12}\mathbf{Q}$, respectively. These factors can be explicitly defined using the following equations:

$$\begin{aligned}\lambda &= \frac{1}{h}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \alpha(|z-z'|)dz'dz\\ &= \frac{1}{\eta\sqrt{\pi}}\left(\exp(-\eta^2)-1\right) + \operatorname{erf}(\eta),\end{aligned} \tag{25}$$

and,

$$\begin{aligned}\beta =& \frac{12}{h^3}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} -z'z\alpha(|z-z'|)\\ =&\operatorname{erf}(\eta) - \frac{1}{\sqrt{\pi}}\Big(\frac{2}{\eta}\exp(-\eta^2)\\ &+(3\eta^{-1}-2\eta^{-3})(1-\exp(-\eta^2))\Big).\end{aligned} \tag{26}$$

Note that a thickness of a plate that is large relative to the internal length scale (i.e., when $\eta \to \infty$,) results in nonlocal modification factors that converge to 1 and therefore, nonlocal theory converges to classical plate theory. This result will be discussed in the Section 4 (Results and Discussion).

The nonlocal modification factors λ and β were calculated assuming a uniform deformation of the plate. If a non-uniform stretch $\boldsymbol{\gamma}(x,y)$ and curvature $\boldsymbol{\kappa}(x,y)$ are assumed in Equation 11, the in-plane strain terms can also be expressed by $\boldsymbol{\varepsilon}(x,y)$. Using a similar formulation to Equations 12 to 22, to describe the nonlocal elasticity, results in nonlocal stress resultants and stress couples like:

$$\begin{aligned}\mathbf{N_{nl}} = &\int_A \mathbf{A}\boldsymbol{\gamma}(x,y)\frac{\exp\left(-\frac{(x-x')^2+(y-y')^2}{(e_0a)^2}\right)}{\left(\sqrt{\pi}e_0a\right)^2}dx'dy'\\ &\times\frac{1}{h}\int_Z\int_Z \frac{\exp\left(-\frac{(z-z')^2}{(e_0a)^2}\right)}{(\sqrt{\pi}e_0a)}dz'dz,\end{aligned} \tag{27}$$

$$\begin{aligned}\mathbf{M_{nl}} = &\int_A -\mathbf{D}\boldsymbol{\kappa}(x,y)\frac{\exp\left(-\frac{(x-x')^2+(y-y')^2}{(e_0a)^2}\right)}{\left(\sqrt{\pi}e_0a\right)^2}dx'dy'\\ &\times\frac{12}{h^3}\int_Z\int_Z \frac{\exp\left(-\frac{(z-z')^2}{(e_0a)^2}\right)}{(\sqrt{\pi}e_0a)}z'zdz'dz\end{aligned}$$

(28)

The stress resultants, as well as stress couples, consist of two terms; the first term is due to in-plane non-locality, which resembles Equations 9 and 10 and can be treated using existing nonlocal plate theories. The second terms in both equations are created by out of plane non-locality, and resemble the modification factor λ and β from Equations 25 and 26. This suggests that the same modification factors on the extensional and bending stiffness matrices can be introduced to conventional nonlocal plate theories, to argument them by a dependency on the plate thickness.

3.2. Bounded kernel

In Section 3.1, it was shown that only transverse non-locality has a dominant contribution in the nonlocal stress state in a uniformly deformed plate. For simplicity in this part, we use a one-dimensional nonlocal formulation in transverse direction only.

As mentioned before, the nonlocal kernel should be normalized in the volume of the structure at hand. Most kernels suggested in literature are derived for infinite domains. Therefore, these kernels are not normalized anymore in the near-boundary regions, where their influence zone exceeds the boundary. As an alternative solution, to ensure that Equation 4 holds everywhere in the body, one can benefit from the operator L mentioned in Equation 6. Particularly, the Green's function of such an operator in a boundary value problem is always normalized in the solution domain, and thus, it can still be employed as the nonlocal kernel [3, 10].

Similar to previous section, the Green's function of the diffusion equation (Equation 17) is chosen as the nonlocal kernel. As mentioned in Section 2.1, the homogeneous natural boundary condition (Neumann) is applied on Equation 17, and consequently, on its Green's function. This boundary condition is applied to the top and bottom surfaces of the plate ($z = \frac{h}{2}$ and $z = \frac{-h}{2}$):

$$\frac{\partial \alpha_h(\frac{h}{2}, z')}{\partial z'} = \frac{\partial \alpha_h(\frac{-h}{2}, z')}{\partial z'} = 0, \tag{29}$$

where α_h is the kernel bounded in $[-\frac{h}{2}, \frac{h}{2}]$. The solution of such a boundary value problem for any z and z' in $[-\frac{h}{2}, \frac{h}{2}]$ is [26]:

$$\alpha_h(z, z') = \sum_{n=-\infty}^{\infty} \big(\alpha(z - z' - 2nh) + \alpha(z + z' - (2n-1)h)\big), \tag{30}$$

where α is unbounded Green's function of the diffusion equation in an infinite domain, which is the Gaussian density function given by Equation 16.

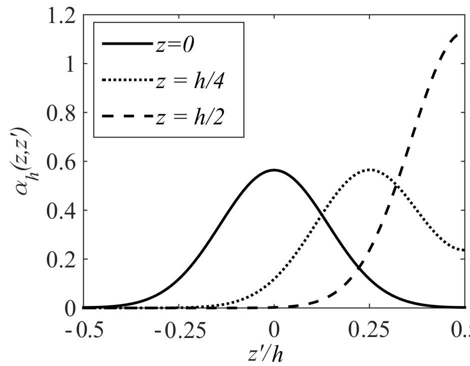

Figure 3: The Green's function of bounded diffusion equation for different locations and, for the case $h = 5e_0a$. The shape of the kernel varies so that it satisfies the boundary conditions and stays normalized.

The bounded kernel α_h is plotted in Figure 3 for three different locations with respect to the boundaries. As shown in Figure 3, the shape of the function α_h changes in regions near the boundaries such that the function remains normalized inside the volume of the structure. In other words: for any point z in $[-\frac{h}{2}, \frac{h}{2}]$, the following condition is satisfied:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_h(z, z')dz' = 1. \tag{31}$$

By substituting the kernel given in Equation 30, into Equation 14, the nonlocal stress terms are calculated. Consequently, the nonlocal tangential stress resultants and stress couples in the plate can be determined. As a result, a similar nonlocal modification factor λ_h and β_h on the extensional and bending stiffness matrices can be determined. Due to similarity of the procedure to the previous section, we avoid to repeat the whole calculations here. Accordingly, λ_h and β_h can be calculated using the following equations:

$$\lambda_h = \frac{1}{h} \int\limits_{Z} \int\limits_{Z'} -\alpha_h(z, z') dz' dz = 1, \tag{32}$$

$$\beta_h = \frac{12}{h^3} \int\limits_{Z} \int\limits_{Z'} -\alpha_h(z, z') z' z dz' dz. \tag{33}$$

Notice that normalizing the kernel function results in a modification factor for the stress resultant that is equal to 1, and the stress resultant $\mathbf{N_{nl}}$ is similar to that of the classical plate theory. This will be discussed and compared to the result of the calculations with a Gaussian kernel, in "Results and Discussion". It should be mentioned that the Green's function of the three dimensional diffusion equation in a finite domain is not separable, i.e. it cannot be decoupled mathematically in-plane and out-of-plane terms, as for the Gaussian kernel. Therefore, the effect of out-of-plane and in-plane non-locality cannot be separated anymore. Considering that for plates with semi-infinite geometry in tangential directions, the kernel needs to be bounded only in z direction. Thus, we introduce a new three-dimensional kernel:

$$\begin{aligned} \alpha'_h(\mathbf{x}, \mathbf{x}') =& \frac{\exp\left(-\frac{(x-x')^2+(y-y')^2}{(e_0 a)^2}\right)}{\left(\sqrt{\pi} e_0 a\right)^2} \times \\ & \sum_{n=-\infty}^{\infty} \Big(\alpha(z - z' - 2nh) + \\ & \alpha(z + z' - (2n-1)h)\Big). \end{aligned} \tag{34}$$

The new kernel is only bounded in transverse direction. Using the same process mentioned in the previous section, the new type of kernel can be employed to calculate the modification factor for the stiffness matrices (λ_h and β_h). These modification factor can be used to correct for the effects of thickness in the existing nonlocal plate theories, in problems with nonuniform deformations.

4. Results and Discussion

In Section 3, using the strong nonlocal formulation, also known as Eringen's nonlocal theory, the tangential stress resultants and stress couples were calculated in a plate with a uniform deformation. In order to consider the nonlocality in all directions, this calculation was performed with a three-dimensional kernel. It was shown that for a very thin plate, of which the lateral geometries are much larger than its thickness, the only remaining terms of the kernel are the ones expressing non-locality in transverse direction. It is emphasized that existing nonlocal beam or plate theories only account for in-plane non-locality in their formulation[13, 14, 23–25].

As mentioned in the Introduction, the reason for this common omission is that the nonlocal plate theories were initially introduced for inherently plane-stress problems, where the strain variation is the most significant in in-plane directions. As a result, the non-locality along the thickness could be ignored. In bending of plates, however, the strain gradient in the transverse direction is substantial. Consequently, this makes the effect of non-locality significant in that direction.

In this section, the effect of including the transverse non-locality on the stress and effective stiffness of the plate is discussed. First, consider a simple example of one-dimensionally stretched plate (i.e., $\boldsymbol{\gamma} = \begin{bmatrix}\gamma & 0 & 0\end{bmatrix}$, $\boldsymbol{\kappa} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$). Figure 4 shows the distribution of the tangential nonlocal stress in transverse direction for this example. Both solutions in Figure 4 (with bounded and non-bounded kernels) are normalized by the uniform stress as calculated by classical plate theory. As mentioned before, if the kernel is bounded and normalized in the domain, it ensures that a uniform strain field introduces a uniform stress field in the domain. Therefore, the nonlocal stress as calculated by a bounded kernel is uniformly distributed in the thickness and is similar to the classical stress. The nonlocal stress, as calculated with non-bounded kernel, is not uniform in the transverse direction. On the contrary, it has a sharp decrease near the surfaces of the plate. This sharp decrease of the lateral stress in the two surfaces of the structure is independent of the thickness and it always reduces to half of the classical stress. This is because for $z = h/2$ and $z = -h/2$, half of the nonlocal kernel exceeds the boundary and the other half is collecting the influence of the uniform strain inside the body. This behavior does not indeed describe physics or fundamentals of surface elasticity, and supports the reason behind the principle of normalization of the kernel in nonlocal theory.

Figure 5 shows the distribution of the tangen-

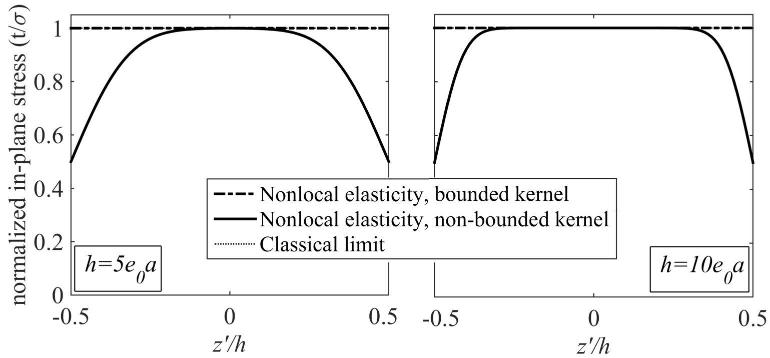

Figure 4: Normalized in-plane stress due to one dimensional stretch, corresponding to the nonlocal solutions for two different kernels and two different thicknesses $h = 10e_0a$ and $h = 5e_0a$. The transverse coordinate is normalized by the material length scale.

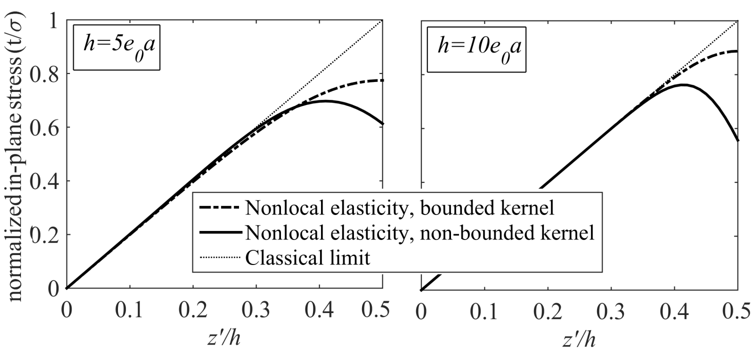

Figure 5: Normalized in-plane stress due to cylindrical bending, corresponding to the nonlocal solutions for two different kernels and two different thicknesses $h = 10e_0a$ and $h = 5e_0a$. The transverse coordinate is normalized by the material length scale.

tial nonlocal stress in transverse direction, in a simple example of cylindrical bending of a plate (i.e., $\boldsymbol{\kappa} = \begin{bmatrix}\kappa & 0 & 0\end{bmatrix}$, $\boldsymbol{\gamma} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$). Both solutions (with bounded and non-bounded kernels) are normalized by the maximum stress as calculated by classical plate theory.

As can be observed in Figure 5, the nonlocal in-plane stress does not vary linearly in the thickness of the plate as it does in the classical solution. The nonlocal stress in the near-boundary region is lower than the classical stress. This difference increases when the thickness of the plate gets smaller. However, inside the body, i.e. far enough from the boundaries, the difference between classical and nonlocal solutions vanishes. The length of the boundary region in which this difference is significant, is very close to the effective cut-off length of the kernel ($2e_0a$). Including the transverse nonlocality in the formulation, allows us to indicate some surface effects in elasticity of the structure.

There is a considerable difference between the nonlocal solutions with a non-bounded kernel and with a bounded kernel. When using a non-bounded kernel, the in-plane stress shows a sharp reduction near the surface which does not reflect a physical behavior. In contrast, the in-plane stress derived with a bounded kernel has a smooth increase near the surface.

The modification factors λ and β on the extensional and bending matrices stiffness were calculated in Section 3. These modification factors only include the effect of non-locality in the transverse direction. Therefore, they only depend on the plate's thickness h. Figure 6 shows the modification factor λ as a function of non-dimensional thickness η. If the chosen kernel is not bounded in transverse direction, the extensional stiffness of the plate is influenced by its thickness. Such a formulation predicts a softening when the thickness gets comparable to the material length scale. When the thickness of the

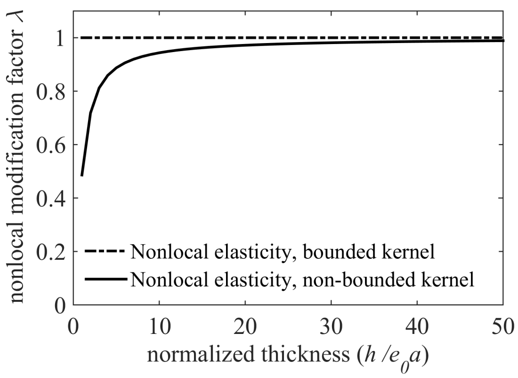

Figure 6: Nonlocal modification factor for extensional stiffness matrix as a function of the thickness of the plate normalized with internal length scale.

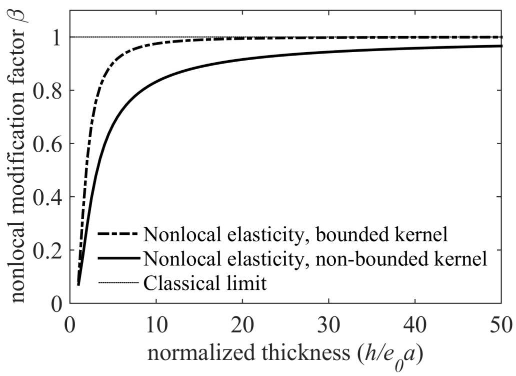

Figure 7: Nonlocal modification factor for bending stiffness matrix as a function of the thickness of the plate normalized with internal length scale.

plate is large, then the modification factor tends to unity which means stiffness will approach the classical limit $\mathbf{A} = h\mathbf{Q}$.

On the other hand, the bounded nonlocal kernel results in a constant extensional stiffness for the plate. As mentioned before, a uniform tensile strain in the plate, using a bounded kernel would result in a uniform nonlocal tensile stress equal to the classical stress. Therefore, the nonlocal solution with bounded kernel does not indicate the effective tensile modulus to change with the thickness.

Figure 7 shows the modification factor β for the nonlocal bending stiffness as a function of non-dimensional thickness η. When the thickness of the plate is relatively small, it can significantly influence the bending stiffness of a plate, as calculated by the nonlocal theory. When the thickness of the plate is relatively large, the modification factor tends to unity which means the bending stiffness will be equal to its classical limit $\mathbf{D} = \frac{h^3}{12}\mathbf{Q}$. According to these calculations, the thickness at which the difference between classical and nonlocal solutions gets more than 1% is $h = 16e_0a$.

In the calculation of the modifying factors λ and β, the initial calculation was based on a uniform deformation in a plate. However, it was shown that if the kernel can be decoupled in in-plane and out-of-plane directions, such modification factors can be used for a non-uniform deformation as well. In the latter case, the in-plane non-locality would have a contribution to the nonlocal stress resultants and stress couples. This contribution needs to be treated by existing nonlocal plate theories.

The results, as shown in Figures 6 and 7, suggest that the nonlocal solution with the bounded kernel reflects the size dependency of the elastic properties at a relatively smaller length scale. The difference between the two nonlocal solutions is relatively large, and is almost comparable to their differences with the classical elasticity. According to Eringen's nonlocal theory and several other publications [3, 10, 21], the solution with bounded kernel is considered to be the "correct" nonlocal solution.

Finally, we briefly discuss an example on how this theoretical approach can be applied in practice. For this purpose, a comparison is made to an experimental result provided by Sadeghian, *et al.* [27]. In that study, the size-dependence of the elastic behavior in Silicon nano-cantilevers has been experimentally investigated. The employed cantilevers in this study were 170 to 8 µm long, 20 to 8 µm wide and 1019 to 40 nm thick. Considering the large aspect ratio of the cantilevers they can be modeled as plates.

Figure 8 shows the experimentally obtained bending stiffness of the cantilevers when subjected to a non-uniform cylindrical bending. The bending stiffness in this graph is normalized with its classical amount $\frac{Et^3}{12(1-\nu^2)}$. The experimental results clearly show that the bending stiffness of the cantilevers is a function of their thickness.

For comparison the proposed nonlocal modification factors derived for three different length scales are also shown in this figure. The theoretical results are calculated using a bounded kernel. The very similar trend to the experiments can be modeled using the modification factor

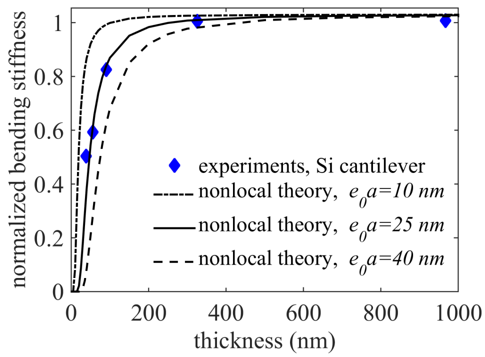

Figure 8: The normalized bending stiffness of a wide silicon cantilever in cylindrical bending based on pull-in measurements [27], and calculated with nonlocal theory with three different internal length scales.

proposed in this paper and the best match is achieved when $e_0 a = 25$ nm. We shall remind here that the internal length scale $e_0 a$ can be affected by the Silicon crystal properties as well as all the defects particularly on its surface. This excellent match between the model and the experimental results shows the potential for employing the proposed model in mechanical characterization of nano-structures. The observed scale effect can be captured with nonlocal elasticity theory effectively, and moreover, it is easy to implement.

5. Conclusions

In this paper, using a nonlocal elasticity theory, we have presented a formulation to capture the effect of thickness on size-dependent behavior of plates. We have discussed some new aspects and challenges of employing the strong three dimensional nonlocal formulation for analysis of plates. The presented formulation has been employed for a practical problem and is shown to be capable to describe the size effect observed experimentally.

Generally, in employing nonlocal elasticity formulation for plate problems, the non-locality in the constitutive equations is only considered in tangential directions of the plate. This, in turn, results in predicting a size dependent mechanical behavior which does not reflect any dependence to the plate thickness. This is while experimental results indicate otherwise [27]. Moreover, if the nonlocal elasticity aims to capture the long-range interactions between the atoms of the material in a continuous framework, their impact should be reflected in all directions, including the transverse direction.

The main problem in capturing non-locality in transverse direction for plates lies in confining the nonlocal kernel at the two surfaces of the structure. In this study, to investigate the effects of including the transverse non-locality in analysis of plates, we have employed two types of nonlocal kernels with bounded and non-bounded boundaries. In particular, the problem of uniform deformations of a plate has been studied with both types of kernels. The results show that using the nonlocal formulation with a bounded kernel can reflect the physics of the problem better. In fact, using a bounded kernel (i) for a given uniform local strain field, a nonlocal formulation predicts a uniform nonlocal stress field, (ii) stress components near the surface do not exhibit the sharp reduction, which occurs in case of employing a non-bounded kernel.

It should be mentioned here that although according to Eringen's theory of nonlocal elasticity, the solution given by a bounded kernel (finite-scale kernel) is suggested to provide the "correct" solution, there is no suggestion for a physical interpretation of the chosen boundary conditions on such a kernel. Thus, there is a definite need for a thorough study to define the reasoning behind the adaptation of the kernel in boundary regions. The authors suggest that calculating the suitable boundary condition for the nonlocal kernel in nonlocal elasticity theories (and other higher order elasticity theories) should be practicable using a molecular dynamics simulation or another atomistic model. These boundary conditions should not be problem dependent and instead they should reflect the physical properties in the surface of the structure. Otherwise, the nonlocal continuum theories will not be viable as they have to be adapted to every single problem.

Moreover, in practice, defects are in the nature of all materials. In a structure with a defected surface, the surface properties such as inhomogeneous elasticity should also be involved in the formulation. Furthermore, the effect of the surface defects should be reflected in the nonlocal kernel either via the boundary conditions or the internal length scale.

As a result of this study, two nonlocal modification factors on extensional and bending stiffness matrices have been presented to account for the effect of thickness in the nonlocal formulations. These modification factors are valid for any shape of the plates. Provided that the nonlocal kernel is separable in transverse and tangential coordinates, they can be used for solutions based on conventional nonlocal plate theories. The observed scaling effect and a good match to experimental results motivate future research into clear interpretation of the internal length scale and the boundary conditions.

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[1] A. C. Eringen, Nonlocal continuum field theories, Springer, 2002. doi:10.1007/b97697.

[2] E. Kröner, International Journal of Solids and Structures 3 (1967) 731–742. doi:10.1016/0020-7683(67)90049-2.

[3] A. Angela Pisano, P. Fuschi, International Journal of Solids and Structures 40 (2003) 13–23. doi:10.1016/S0020-7683(02)00547-4.

[4] A. C. Eringen, International Journal of Engineering Science 15 (1977) 177–183. doi:10.1016/0020-7225(77)90003-9.

[5] F. dell'Isola, U. Andreaus, L. Placidi, D. Scerrato, Intorno alle equazioni fondamentali del movimento di corpi qualsivogliono, considerati secondo la naturale loro forma e costituzione, Springer, 2014, pp. 1–370. doi:10.1007/978-3-319-00263-7.

[6] F. Dell'Isola, A. Della Corte, I. Giorgio, Mathematics and Mechanics of Solids (2016) 1081286515616034. doi:10.1177/1081286515616034.

[7] M. Di Paola, G. Failla, A. Pirrotta, A. Sofi, M. Zingales, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 371 (2013). doi:10.1098/rsta.2012.0433.

[8] J. Engelbrecht, M. Braun, Applied Mechanics Reviews 51 (1998) 475–488. doi:10.1115/1.3099016, 10.1115/1.3099016.

[9] S. Silling, R. Lehoucq, Advances in applied mechanics 44 (2010) 73–168. doi:10.1016/S0065-2156(10)44002-8.

[10] R. Peerlings, M. Geers, R. De Borst, W. Brekelmans, International Journal of solids and Structures 38 (2001) 7723–7746. doi:10.1016/S0020-7683(01)00087-7.

[11] O. Weckner, G. Brunk, M. A. Epton, S. A. Silling, E. Askari, Sandia National Laboratory Report J 1109 (2009) 2009.

[12] A. C. Eringen, International Journal of Engineering Science 22 (1984) 1113–1121. doi:10.1016/0020-7225(84)90112-5.

[13] P. Lu, H. Lee, C. Lu, P. Zhang, International Journal of Solids and Structures 44 (2007) 5289–5300. doi:10.1016/j.ijsolstr.2006.12.034.

[14] P. Lu, P. Zhang, H. Lee, C. Wang, J. Reddy, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 463 (2007) 3225–3240. doi:10.1098/rspa.2007.1903.

[15] W. Duan, C. M. Wang, Y. Zhang, Journal of Applied Physics 101 (2007) 024305–024305–7. doi:10.1063/1.2423140.

[16] R. Picu, Journal of the Mechanics and Physics of Solids 50 (2002) 1923–1939. doi:10.1016/S0022-5096(02)00004-2.

[17] A. C. Eringen, Journal of Applied Physics 54 (1983) 4703–4710. doi:10.1063/1.332803.

[18] D. C. Lam, F. Yang, A. Chong, J. Wang, P. Tong, Journal of the Mechanics and Physics of Solids 51 (2003) 1477–1508. doi:10.1016/S0022-5096(03)00053-X.

[19] S. Ghosh, V. Sundararaghavan, A. M. Waas, International Journal of Solids and Structures 51 (2014) 392–401. doi:10.1016/j.ijsolstr.2013.10.004.

[20] R. Maranganti, P. Sharma, Physical review letters 98 (2007) 195504. doi:10.1103/PhysRevLett.98.195504.

[21] R. Abdollahi, B. Boroomand, International Journal of Solids and Structures 50 (2013) 2758–2771. doi:10.1016/j.ijsolstr.2013.04.027.

[22] X. Ren, L. Truskinovsky, Journal of elasticity and the physical science of solids 59 (2000) 319–355. doi:10.1023/A:1011003321453.

[23] J. Peddieson, G. R. Buchanan, R. P. McNitt, International Journal of Engineering Science 41 (2003) 305–312. doi:10.1016/S0020-7225(02)00210-0.

[24] J. Reddy, International Journal of Engineering Science 48 (2010) 1507–1518. doi:10.1016/j.ijengsci.2010.09.020.

[25] Q. Wang, K. Liew, Physics Letters A 363 (2007) 236–242. doi:10.1016/j.physleta.2006.10.093.

[26] I. Stakgold, M. J. Holst, Green's functions and boundary value problems, volume 99, John Wiley & Sons, 2011. doi:10.1002/9780470906538.

[27] H. Sadeghian, C.-K. Yang, H. Goosen, P. J. F. Van Der Drift, A. Bossche, P. J. F. French, F. Van Keulen, Applied Physics Letters 94 (2009) 221903–221903–3. URL: http://dx.doi.org/10.1063/1.3148774. doi:10.1063/1.3148774.