Fluctuations for Interacting Particle Systems with Duality

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Chapter 1

Introduction

1.1 Scales in physics

Physics is about the understanding of fundamental forces and how the world that we observe can be explained with a minimal number of fundamental laws. This is in turn strongly related to the basic symmetries of nature and how these symmetries are manifested in conservation laws and Lagrangian actions. Physics has to deal with an enormous range of scales, from the "smallest" Planck scale $1.616255 \times 10^{-35} m$, towards the atomic scale $10^{-10} m$, towards the scale of large molecules, towards the scale of "daily objects" all the way until the scale of the universe $8.8 \times 10^{26} m$. The discovery of new phenomena in physics is often a consequence of the experimental accessibility of "new scales". E.g., in the 20th century, thanks to the discovery of radioactive phenomena, we became able to explore the atomic scale and its associated theoretical framework of quantum mechanics. The old laws (classical mechanics) can be understood as "emerging" from the new laws in a scaling limit (in this case the so-called classical limit where Planck's constant is scaled to zero). Similarly, precise measurements of the precession of the perihelion of Mercury, as well as of the bending of light by the gravitational field of the sun, pointed to the necessity of a new theory describing gravity at the scale of the universe. Once more, the old laws (classical Newtonian gravity) emerge as scaling limits (velocities much smaller than the speed of light and small mass densities). Finally, in the second half of the 20th century, thanks to heavy collider experiments, we discovered a wealth of new elementary particles and we understood that nucleons are constituted of quarks. This led to the "standard model", a quantum field theory by which we can describe weak, strong and electromagnetic interaction. The "old laws" of electromagnetism (Maxwell's equations) arise from this relativistic quantum field theory as a classical limit.

1.2 Statistical physics

1.2.1 Equilibrium

Statistical physics is the branch of physics where we want to understand macroscopic phenomena starting from the microscopic dynamics of individual agents or entities. These systems of agents not only represent systems of particles, but also spin systems or systems where energy is exchanged. Statistical physics is divided into two subareas: equilibrium statistical physics and non-equilibrium statistical physics. Let us first talk about the first.

In equilibrium statistical physics, we try to understand the laws of equilibrium thermodynamics -macroscopic laws- from the underlying micro-world. In particular, we aim to understand the phenomenon of phase transitions: how the same laws on the micro-scale can result in a variety of different behaviors on the macro-scale. Important examples are the liquid-gas, liquid-solid transitions, and phase transitions in magnetic systems, i.e., the phenomenon of ferromagnetism. In equilibrium, it is well understood how to describe the micro to macro transition, namely via the Boltzmann-Gibbs distribution, later in more mathematical terms formulated in the so-called DLR (Dobrushin-Lanford-Ruelle) formalism. In that sense, the study of equilibrium statistical mechanics reduces to the study of Gibbs measures as a function of parameters such as temperature and magnetic field. A milestone in our understanding of phase transitions in magnetic systems is the exact solution of the two-dimensional Ising model by Onsager in 1944. Further significant achievements in the mathematical theory of Gibbs measures are the universal properties of high-temperature Gibbs states, in the works of Dobrushin and Shlosman [31], the rigorous formulation of the cluster expansion by Minlos (and many others), building on earlier work of Mayer, and a general theory of contours and low-temperature states by Pirogov and Sinai [82], building on earlier work by Peierls. The mathematical theory of equilibrium statistical mechanics, i.e., of Gibbs measures is a well-established field, where the paradigms are well-defined. Even if in this field there are still important open problems (e.g., the liquid-solid transition), the mathematical paradigm is transparent, well-defined, and problems are defined within this paradigm. The field of Gibbs measures also has major interactions with and applications in other fields of mathematics such as ergodic theory and dynamical systems (works of Sinai, Ruelle, Bowen [11]), Markov process theory (works of Holley, Stroock, Zegarlinski [86]), and even in number theory (works of Knauf [59, 60], Newman).

1.2.2 Non-equilibirum

The situation of non-equilibrium statistical physics is very different. In non-equilibrium, we aim to understand macroscopic transport phenomena such as entropy production, heat conduction, particle transport, and phenomena in hydrodynamics or material science, starting from the motion of individual particles. The macroscopic laws for heat transport (Fourier law) or particle transport (Fick's law) were well-established long before statistical mechanics. However, the way they were derived was entirely phenomenological, based on intuitively plausible principles, such as the fact that the heat current is proportional to the temperature difference (Fourier law). In contrast with the setting of equilibrium statistical physics, there is no well-defined paradigm to define probability measures that are the analogs of Gibbs measures out of equilibrium. E.g., for a system in contact with two different temperatures, there is no "formula" which links the Hamiltonian of the microsystem to the probability measure (the so-called non-equilibrium steady-state) from which one can describe macro phenomena such as the emergence of heat current. The number of microscopic degrees of freedom (the position of individual particles) is so enormous that it is a hopeless and useless task to describe exactly the motion of particles. Indeed, even if one could do so, then there would still be the formidable task of deriving the macro-laws from this extremely complex high-dimensional motion.

In non-equilibrium we want to understand so-called "transient non-equilibrium", i.e., the phenomenon of relaxation to equilibrium, as well as "stationary non-equilibrium", i.e., the long-term behavior of systems in contact with non-equilibrium driving forces such as reservoirs at different temperatures (or chemical potentials), and/or bulk driving such an external field. The fundamental difference between "stationary equilibrium" and "stationary non-equilibrium" is that in the latter, we have breaking of time-reversal symmetry manifested by the presence of currents (which have a preferred direction). In that sense, we can roughly say that equilibrium is characterized by time-reversal symmetry, or detailed balance, whereas non-equilibrium implies the breaking of detailed balance.

1.3 Interacting particle systems

In the context of non-equlibrium statistical mechanics Interacting particle systems (IPS) are simple models describing basic rules of interaction among particles. In the seminal work [84] Frank Spitzer introduced several classes of these models on configuration space. The key property of this type of models is the fact that we assume that at the microscopic level particles follow some predescribed stochastic Markovian dynamics; we make a choice to simplify the microscopic motion of the particles. Instead of Hamiltonian mechanics (Newton's law), we describe particle motion by stochastic rules, i.e., the particles perform random walks and interact with each other, e.g., by exclusion (forbidden to be at the same place), or other repulsive or attractive interactions. The noise introduced in the microdynamics makes more accessible the task to derive the macro laws rigorously: the noise provides a natural source of relaxation to a "local equilibrium" state, and the macroscopic equations describe how the parameters governing the local equilibrium are evolving as a function of (macroscopic) space and time. In deterministic mechanical systems, we can argue that the source of noise is the ignorance of the initial conditions (we only know the initial macrostates of the system) combined with the chaotic motion of the microscopic degrees of freedom. This intuition can be made rigorous in “toy-deterministic systems” such as coupled chaotic maps ([72]). The advantage of working with IPS (which are “toy”-systems, or in the words of Dobrushin “caricatures of hydrodynamics”) is that we can rigorously define what it means to pass from the micro to the macro scale (the so-called hydrodynamic limit) and we can study much more than the emergence of the macro-equation.

Indeed, in IPS, the emergence of a macro equation (also called hydrodynamic equation) such as the heat equation can be understood as an infinite-dimensional law of large numbers. In analogy with the ordinary law of large numbers, we can ask for central limit behavior (equilibrium and non-equilibrium fluctuations around the hydrodynamic limit) and for large deviations (probabilities of deviations from the macro equation). Finally, we can study systems driven away from equilibrium by boundary reservoirs (boundary driven), bulk driven, or a combination of both. The field of hydrodynamic limits was developed in the 1980-1990’s. Probabilistic approaches based on duality, coupling were developed by de Masi, Presutti, and many others, see [28] and [27] for overviews. In the early 90’s Varadhan solved the problem of the large deviations from the hydrodynamic limit and developed a robust method for gradient systems (the so-called GPV or entropy method). Starting from these developments, many refinements and extensions of the entropy method were formulated, and a general theory of macroscopic non-equilibrium fluctuations was developed by Bertini, da Sole, Gabrielli, Landim [9]. In parallel, for a class of systems including the (symmetric as well as asymmetric) exclusion process coupled to boundaries, exact solutions with the so-called matrix product ansatz were developed by Derrida [30] and coworkers. This later developed into a research area, at present known under the name “integrable probability”.

1.4 Duality

Stochastic duality emerged as a fundamental tool in the study of IPS, from the very early stage of development of the field, see e.g. the foundational works of Spitzer [84] and Liggett [70]. This notion is analogous to that of integrable systems in the sense that those IPS that enjoy the property of duality are systems for which the BBGKY hierarchy closes, and as a consequence of this, the k-particle correlation functions obey closed-form equations (not involving higher correlations). Having correlation functions in a closed-form has proved itself useful in the derivation of many results [16], [18], [44], [45] to mention just a few examples. It is also precisely this property which is the common core of the IPS we study in this thesis. We will focus on three types of IPS; independent random walkers, exclusion processes (a model of exclusive interaction introduced by Spitzer in [84]), and inclusion processes (a process introduced in [44] and [45], which can be seen as the attractive counterpart of the exclusion process). In order to be able to use duality, we will work on the symmetric versions of them, and we will assume that the particles take positions in the infinite lattice $\mathbb{Z}^d$.

1.5 Macroscopic laws

1.5.1 Hydrodynamics

The first example of macroscopic behavior that we can deduce from the type of microscopic dynamics modeled by IPS is the so-called hydrodynamic limit. This is performed with the aim of rigorously deriving, starting from a microscopic IPS, a partial differential equation that describes the evolution of some macroscopic observables. We refer the reader to [28], [27] and [58] for a complete survey on the subject. For the concrete systems that we study in this thesis, the only conserved quantity is the total number of particles. Therefore, it is natural to expect that the desired PDE will describe the evolution of particles' density over time. This equation can be derived by suitably defining, at the microscopic level, a local density and looking at the way it changes as a rescaling parameter n (see Section 3.1.1 for more details) tends to infinity. This parameter intuitively represents the ratio between the macroscopic and the microscopic length scale (in some cases n is also related to the time scale and the total number of particles). Rigorously speaking, the hydrodynamic result is given in probabilistic terms; in particular, it corresponds to a type of law of large numbers.

1.5.2 Fluctuations from the hydrodynamic limit

Based on the idea that hydrodynamic limits correspond to a law of large number type of results, fluctuation limits are then the CLT counterpart to the hydrodynamic limit theorems. For this type of results, the quantity of interest at the microscopic level is a centered and suitably rescaled (by $\sqrt{n}$) version of the density field. The convergence is again given in probabilistic terms, but the limiting object is no longer the solution of a PDE but rather a generalized stochastic process, which is the solution to an SPDE. Of course, in the deterministic setting, this type of results are much harder to obtain [71]. Nevertheless, they should correspond to a qualitative picture of the chaotic behavior given by the sensitivity of Newton's equations to initial conditions. The first rigorous derivation of this type of limit theorems was given in [73] for a system of independent particles on $\mathbb{R}^d$. For results concerning IPS on the lattice $\mathbb{Z}^d$ we refer the reader to [27] and [85].

1.5.3 The role of conserved quantities: the Boltzmann-Gibbs principle

For the particle systems we study in this thesis, the only conserved quantity is the number of particles. As a consequence there exists a one-parameter family of homogeneous, reversible, and ergodic product measures; indexed by the particle density. This family of measures becomes a key ingredient in the study of equilibrium fluctuations from the hydrodynamic limit of such particle systems. A further consequence of the conservation of particles is the so-called Boltzmann-Gibbs principle. This principle states that the density fluctuation field is the slowest varying field, and other fluctuation fields can be replaced by their projections (with respect to the Hilbert space related to the one-parameter family of reversible measures) on the density field. Brox and Rost in [12] proved the validity of the principle for attractive zero-range processes. This principle was extended in our context, for a variant of the exclusion process, by De Masi, Presutti, Spohn and Wick in [26]. Later on, new proofs and generalizations came to light; see for example [68], [20], [81], among others. Most relevant to this thesis is the further generalization originally introduced in [46] in which a second-order version of this principle was established. This version allows us to replace local functionals of a conservative, one-dimensional stochastic process by a possibly nonlinear function of the conserved quantity.

1.6 Condensation

For IPS without restrictions on the number of particles per site, and for suitable attractive particle interactions, the systems can exhibit a condensation phenomenon. In simple words, condensation consists of a macroscopically significant portion of particles being concentrated at a single site or region. In more precise terms, this phenomenon consists of the existence of a critical density above which the system phase separates into a condensate and a homogeneous phase [24]. Of course, condensation phenomena have manifestations outside of physics; wealth condensation in macroeconomics [13], gelation in networks [64], traffic jamming [35], and coalescence in granular systems [32] are other examples just to mention a few.

Generally speaking, there are basically two tasks in the study of condensing particle systems:

Existence: The first step in this type of study is to show that indeed the systems condense, i.e., that a large proportion of particles is located at only one site with dominating probability.

Mestastability: The next step is to investigate the dynamical properties of the condensate.

In principle, this basic program can be applied to finite and infinite systems and for reversible and non-reversible versions of those systems. Nevertheless, the prototypical examples to study these questions are the ZRP and related models. In the ZRP, when initialized from a homogeneous distribution of particles, the condensate emerges from a coarsening process described in [34]. Despite its non-triviality, this coarsening process is understood heuristically. Despite its simplicity, the zero-range process is rich enough to exhibit a type of condensation analogous to the Bose-Einstein condensation of quantum physics. More precisely, when the rates of mass transfer in the ZRP depend on each site (heterogeneous systems), the condensation of particles is expected to occur in the site with the lowest rate [36]. In this case, condensation is analogous to Bose-Einstein condensation where the slowest site plays the role of the ground state.

Due to its attractive interaction, the SIP lives in the IPS realm that, under a particular regime, which we call the condensation regime, is expected to exhibit condensation phenomena. Nevertheless, for this system, the complete understanding of the coarsening process is still an open problem in the infinite-volume case with partial results in terms of Fourier-Laplace transforms given in [18]. It is precisely in [18] that the appearance of sticky Brownian motion as a relevant limiting object was first observed, hidden behind the Fourier-Laplace transform of some limiting variance. It was conjectured in the same paper that the emergence of sticky Brownian is a generic characteristic for systems with condensation and that it goes beyond the type of particle systems included in this thesis (i.e., beyond IPS with duality). In this thesis, with the help of self-duality and Dirichlet-form techniques, we obtain a precise scaling behaviour of the variance of the density field under the condensation regime giving a step forward in the understanding of the coarsening process.

1.7 Scope of this thesis

The work developed in this thesis lies in the intersection of fluctuation theory and applications of duality for IPS. It can be divided into two main parts. The first one consists of generalizations, in the context of duality, of two essential tools to derive fluctuation results, while the second part concerns the study of condensation phenomena from the point of view of fluctuation theory.

To be more precise, the first part contains two chapters. In Chapter 4, with the help of orthogonal duality polynomials, we obtain a quantitative generalization of the Boltzmann-Gibbs principle, both in equilibrium and local equilibrium, in the context of independent random walkers. In Chapter 5, with the help of orthogonal polynomial duality, we introduce a notion of higher-order fluctuation fields and characterize their scaling limits in terms of a recursive martingale problem, which formally corresponds to a notion of powers of generalized processes. This is done all at once for the three systems under consideration in this thesis.

In the second part, and for the particular case of the symmetric inclusion process, in Chapter 6, we obtain new relevant information about the dynamics of the coarsening process on the one-dimensional infinite lattice. Namely, employing Mosco convergence of Dirichlet forms and duality, we obtain an explicit scaling for the variance of the density field in the condensation regime

Outline

Chapter 4: Quantitative Boltzmann–Gibbs Principles via Orthogonal Polynomial Duality

In this chapter, we study fluctuation fields of orthogonal polynomials in the context of particle systems with duality. We thereby obtain a systematic orthogonal decomposition of the fluctuation fields of local functions, where the order of every term can be quantified. This implies a quantitative generalization of the Boltzmann–Gibbs principle. In the context of independent random walkers, we complete this program, including also fluctuation fields in a non-stationary context (local equilibrium). Similar results can be obtained for other interacting particle systems with duality, such as the symmetric exclusion process, under precise conditions on the n-particle dynamics.

Chapter 5: Higher-order fluctuation fields and orthogonal duality polynomials

In this chapter, inspired by the works in [5] and [47], we introduce what we call k-th-order fluctuation fields and study their scaling limits. This construction is done simultaneously for independent walkers, symmetric exclusion and inclusion processes in the d-dimensional Euclidean lattice. The explicit form of these higher-order fields resembles the one already introduced in [7] in the sense that both types of fields are based on an orthogonal decomposition of fluctuation fields of local functions that can be expressed in terms of orthogonal self-duality polynomials.

Thanks to the structure given by the orthogonal self-duality, we can mimic umbral calculus techniques and pretend that indices are exponents. This type of interpretation provides us with a setting in which we are able to understand these fields as some type of discrete analogues of powers of the well-known density fluctuation field. Later on, we make rigorous this idea by showing that indeed the weak limit of the k-th order fluctuation field satisfies a recursive martingale problem that formally corresponds to the SPDE associated with the k-th-power of a generalized Ornstein-Uhlenbeck process.

Chapter 6: Condensation of SIP particles and sticky Brownian motion

In this chapter, we study the symmetric inclusion process (SIP) in the condensation regime. We obtain an explicit scaling limit for the variance of the density field in this regime when initially started from a homogeneous product measure. This provides relevant new information on the coarsening dynamics of condensing interacting particle systems on the infinite lattice.

One of the novelties of this chapter is that our main result is obtained as an application of Mosco convergence of Dirichlet forms. Thanks to self-duality, the variance of the density field can be written in terms of the difference of the positions of two SIP particles. The process given by this difference is then showed to converge, in the sense of Mosco convergence of Dirichlet forms, to a two-sided sticky Brownian motion. This approach implies the convergence of the probabilities of the two SIP particles to be together at time t. This, combined with self-duality, allows us to obtain the explicit scaling for the variance of the fluctuation field.

The explicit scaling limit of the variance of the fluctuation field that we obtain can be expressed in terms of the two-sided sticky Brownian motion transition function. From this fact, we can clarify the qualitative picture of the coarsening process when started from a homogeneous product measure in the infinite lattice. Our results suggest the formation of large piles of particles that move together as ordinary Brownian motions and interact with each other as a consistent family of Brownian motions as introduced in [53].

Chapter 7: Perspectives

Chapter 7 is dedicated to a brief overview of some of the natural questions that may arise and that are still open problems. In this chapter, we present conjectures, ideas, and sketches of proofs of the results that we propose to explore as a follow up to the main results presented in this thesis.

In Section 7.1, related to the work Higher-order fluctuation fields and orthogonal duality polynomials, we generalize the context given in [5] and [47] to the higher-order fields case. By specifying this more general setting, we draw a future line of research in which discrete analogues of white-noise spaces are used to sketch the road towards the derivation of results concerning convergence to generalized wick renormalized powers of distributions.

Finally, in Section 7.2, as part of the perspectives on the work Condensation of SIP particles and sticky Brownian motion, we propose as a future line of research, the convergence of a system of k condensively rescaled SIP particles to a consistent family of Brownian motions. As an example of this conjecture, we show the convergence (in the Mosco sense) of the Dirichlet form associated with two SIP particles to a Dirichlet form expected to correspond to a pair of γ coupled Brownian motions.

Chapter 2

Mathematical preliminaries

2.1 The processes

In this section, we give a precise mathematical description of the interacting particle systems we consider in this thesis. As stated in the introduction, we will work with stochastic processes of the type of independent random walkers, inclusion, and exclusion processes:

Independent Random walkers: This is the simplest model for motion of particles. In this model, interaction of particles is neglected and particles jump according to mutually independent exponentially distributed clocks at rate $\alpha \in (0, \infty)$

Symmetric Exclusion Processes (SEP(α)): This particle system models the most elementary type of interaction. Namely, in this model, particles jump according to mutually independent Poisson jump processes, while the value of the parameter $\alpha \in \mathbb{N}$ determines the maximum number of particles allowed per site $x \in \mathbb{Z}^d$. In particular the case $\alpha = 1$ corresponds to the canonical exclusion process originally introduced by Spitzer in [84].

Symmetric Inclusion Processes (SIP(α)): This process also describes strong interaction but opposite to that of the exclusion process in the sense that particles attract each other. In this model, particles are equipped with two exponential clocks: one of rate α that represents the random walk jumps, and the other representing the inclusion dynamics has in principle rate 1 which is multiplied by the number of particles sitting in the arrival site. Contrary to the exclusion process, this time, the number of particles is unbounded,

Of the three types of processes we consider, without a doubt, the most studied is the exclusion process. The dynamics of this particle system is rich enough to describe strong interaction but of a nature simple enough to allow, in many cases, for uniform estimates that facilitate many proofs. Much less ubiquitous in the literature is the inclusion process. In this process the number of particles is unbounded, and therefore a priori several uniform estimates fail. Finally, the system of independent walkers, although this system neglects interaction, it serves well as a prototype to develop further theory.

In the rest of this chapter, we will introduce these systems of interacting particles in configuration space. In this perspective we treat particles as indistinguishable from each other, and we only keep track of the number of particles that are in each position $x \in \mathbb{Z}^d$. We will then introduce the notion of duality in Section 2.2.3 together with some applications.

2.1.1 The infinite configuration process

We now want to consider the dynamics of an infinite number of particles randomly hopping on the lattice $\mathbb{Z}^d$ according to any of the rules given by the Markov processes described above. Let us denote by $\{\eta(t) : t \geq 0\}$ the Markov process, in configuration notation, with state space Ω of the type $\Omega = \Lambda^{\mathbb{Z}^d}$ where $\Lambda = \mathbb{N}$ or $\Lambda = \{0, 1, \dots, \alpha\}$. I.e., for a configuration $\eta = (\eta_i : i \in \mathbb{Z}^d)$, η_i denotes the number of particles at site $i \in \mathbb{Z}^d$.

The parameters $(\sigma, \alpha) \in \{0, 1\} \times (0, \infty) \cup \{-1\} \times \mathbb{N}$ determine the type of interaction among particles as follows:

Exclusion Process

The choice $\sigma = -1$ results in exclusion interaction. For this process the parameter α takes values in the set of natural numbers, $\alpha \in [k] \subset \mathbb{N}$, as it determines the maximum number of particles allowed per site.

Independent Random Walkers

This particle system corresponds to the choice $\sigma = 0$ and the intensity parameter $\alpha \in \mathbb{R}$ regulates the rate at which the particles move independently from each other.

Inclusion Process

The choice $\sigma = 1$ gives rise to an interaction of inclusion-type consisting of particles attracting each other at rate $p(r)$. Moreover particles move independently from each other at rate $\alpha p(r)$ with $\alpha \in (0, \infty)$.

Remark 2.1.1. *The definition of the state space Ω is different in each case, depending on whether there are restrictions or not on the total number of particles allowed per site. This is finite for the exclusion process, thus, for SEP(α), we have $\Omega = \{0, 1, \ldots, \alpha\}^{\mathbb{Z}^d}$. The situation is different in the IRW and SIP cases, for which, in principle, there are no restrictions on the maximum number of particles. Nevertheless, one has to avoid explosions of the number of particles in a given site. For this reason, the characterization of Ω in these cases (i.e., for $\sigma \geq 0$) is a more subtle problem. For the moment, we will restrict to implicitly define Ω as the set of configurations in $\mathbb{N}^{\mathbb{Z}^d}$ whose evolution $\eta(t)$ is well-defined and belonging to Ω for all subsequent times $t \geq 0$. We refer the reader to [2] and [28] for examples on conditions sufficient to guarantee the well-definedness. A possible such subset is the set of tempered configurations. This is the set of configurations η such that there exist $C, \beta \in \mathbb{R}$ that satisfy $|\eta(x)| \leq C|x|^\beta$ for all $x \in \mathbb{R}$.*

In this notation, we can analytically describe the evolution of our systems via the following operator working on local functions $f : \Omega \to \mathbb{R}$ as

$$\mathscr{L} f(\eta) = \sum_{i \in \mathbb{Z}^d} \sum_{r \in \mathbb{Z}^d} p(r) \eta_i (\alpha + \sigma \eta_{i+r})(f(\eta^{i,i+r}) - f(\eta)) \tag{2.1}$$

where $\eta^{i,i+r}$ denotes the configuration obtained from η by removing a particle from position $i \in \mathbb{Z}^d$ and moving it to position $i + r$, i.e.,

$$\eta^{i,i+r} = \eta - \delta_i + \delta_{i+r}$$

In the rest of this thesis, unless stated otherwise, we will always assume that $p(r)$ is a symmetric, finite-range, irreducible Markov transition function on $\mathbb{Z}^d$:

1. Symmetry. The function $p : \mathbb{R}^d \to [0, \infty)$ is of the form:

   $$p(r_1, \ldots, r_d) = p(|r_1|, \ldots, |r_d|) \tag{2.2}$$

   and such that $p(r_{\sigma(1)}, \ldots, r_{\sigma(d)}) = p(r_1, \ldots, r_d)$ for all $\sigma \in \mathscr{P}(d)$, the set of permutations of $\{1, \ldots, d\}$.

2. Finite-range. There exists a finite subset of integer numbers $\mathscr{R} \subset \mathbb{Z}^d$ of the form $\mathscr{R} = [-R, R]^d \cap \mathbb{Z}^d$, for some $R \in \mathbb{N}$, $R > 1$, such that $p(r) = 0$ for all $r \notin \mathscr{R}$.

3. Irreducibility. For all pair of points $x, y \in \mathbb{Z}^d$ there exists a sequence of points $i_1 = x, \ldots, i_n = y$ such that

   $$\prod_{k=1}^{n-1} p(i_k - i_{k+1}) > 0.$$

We will also assume, without loss of generality, that $p(0) = 0$, and that

$$\sum_{r\in\mathscr{R}} p(r) = 1$$

and denote by χ the second moment:

$$\chi := \sum_{r\in\mathscr{R}} r_\ell^2 \cdot p(r), \qquad \text{for all} \quad \ell \in \{1,\dots,d\}. \tag{2.3}$$

The finite-range assumption is only there for technical reasons since the results in this thesis can be easily extended to the infinite-range case, as long as the transition function is such that the infinitesimal generator is well-defined.

Notice that when restricted to only one particle, all the above systems coincide with a random walker moving on $\mathbb{Z}^d$ at rate $\alpha \cdot p(r)$. i.e. with infinitesimal generator

$$L^{(1)} f(i) := \alpha \sum_{r\in\mathbb{Z}^d} p(r)(f(i+r) - f(i)) \tag{2.4}$$

for functions $f : \mathbb{Z}^d \to \mathbb{R}$.

This observation is simple but worthy to mention since it is commonly used in the context of interacting particle systems with duality for which, computations involving, in principle, an infinite number of particles can be reduced to computations involving only one independent walker.

2.1.1.1 Reversibility

A reversible measure for the generator (2.1) is a measure μ, which is non-identical zero, and such that the following detailed balance relation is satisfied:

$$\mu(\eta)c(\eta, \eta^{i,i+r}) = \mu(\eta^{i,i+r})c(\eta^{i,i+r}, \eta) \tag{2.5}$$

for every $r \in \mathscr{R}$ and where

$$c(\eta, \eta^{i,i+r}) = p(r)\, \eta_i \,(\alpha + \sigma\eta_{i+r}) \tag{2.6}$$

The particles systems considered in this thesis have a one-parameter family of homogeneous (w.r.t. translations) reversible and ergodic product measures $\nu_\rho, \rho > 0$, indexed by the particle density, i.e.,

$$\int \eta_0 \, d\nu_\rho = \rho. \tag{2.7}$$

The nature of the underlying dynamics and the type of reversible measure we obtain is regulated by the parameter $\sigma \in \mathbb{Z}$ as follows.

Exclusion Process

This system is well known to have reversible measures ν_ρ, $\rho \in (0, \alpha)$, that are products of Binomial distributions: $\nu_\rho = \otimes_{i\in\mathbb{Z}^d}\text{Binom}\left(\alpha, \frac{\rho}{\alpha}\right)$ whose marginals are given by

$$\mathbb{P}_{\nu_\rho}(\eta_i = n) = \frac{1}{Z_{\alpha,\rho}} \cdot \binom{\alpha}{n} \cdot \left(\frac{\rho}{\alpha-\rho}\right)^n, \quad \forall\, i \in \mathbb{Z}^d,$$

with normalizing constant

$$Z_{\alpha,\rho} = \left(\frac{\alpha}{\alpha-\rho}\right)^\alpha. \tag{2.8}$$

Independent Random Walkers

The reversible measures ν_ρ, $\rho > 0$ are products of Poisson distributions with parameter ρ, $\nu_\rho = \otimes_{i\in\mathbb{Z}^d}\text{Pois}(\rho)$, i.e. the marginals are given by

$$\mathbb{P}_{\nu_\rho}(\eta_i = n) = \frac{1}{Z_\rho} \cdot \frac{\rho^n}{n!}, \qquad Z_\rho = e^\rho, \qquad \forall\, i \in \mathbb{Z}^d.$$

Inclusion Process

The SIP is known to have products of Negative-Binomial distributions as reversible measures, i.e. ν_ρ, $\rho > 0$ with $\nu_\rho = \otimes_{i\in\mathbb{Z}^d}\text{Neg-Binom}\left(\alpha, \frac{\rho}{\rho+\alpha}\right)$ with marginals

$$\mathbb{P}_{\nu_\rho}(\eta_i = n) = \frac{1}{Z_{\alpha,\rho}} \cdot \binom{n+\alpha-1}{n} \cdot \left(\frac{\rho}{\alpha+\rho}\right)^n, \forall\, i \in \mathbb{Z}^d,$$

with normalizing constant

$$Z_{\alpha,\rho} = \left(\frac{\alpha+\rho}{\alpha}\right)^\alpha. \tag{2.9}$$

2.2 Duality

Generally speaking, we can consider duality as a tool that provides us with two different perspectives of the same object. The notion of duality, has in general many manifestations across mathematics. In particular, Interacting Particle Systems is among those areas that enjoy the applicability of this concept. In this area, many times, a nontrivial duality relation is used to prove properties of processes. The idea is that we have one Markov process $\{X_t\}_{t\geq 0}$ that we would like to analyze, and another process $\{\hat{X}_t\}_{t\geq 0}$ for which we already have sufficient information or that is easier to analyze. Then the duality relation allows us to transfer information from $\{\hat{X}_t\}_{t\geq 0}$ to $\{X_t\}_{t\geq 0}$, and vice-versa. More precisely, we have the following definition:

Definition 2.2.1. *Let $\{X_t\}_{t\geq 0}$ and $\{\hat{X}_t\}_{t\geq 0}$ be two Markov Processes with state spaces E and $\hat{E}$. Let also $D : \hat{E} \times E \to \mathbb{R}$ be a measurable function. The processes $\{X_t\}_{t\geq 0}$ and $\{\hat{X}_t\}_{t\geq 0}$ are said to be dual with respect to D if for all $x \in E$, $\hat{x} \in \hat{E}$ and $t \geq 0$ we have*

$$\mathbb{E}_x D(\hat{x}, X_t) = \hat{\mathbb{E}}_{\hat{x}} D(\hat{X}_t, x), \tag{2.10}$$

where $\mathbb{E}_x$ and $\mathbb{E}_{\hat{x}}$ denote expectation with respect to X_t and $\hat{X}_t$ when starting from x and $\hat{x}$ respectively, and both RHS and LHS are assumed to be finite. Moreover, the measurable structure on $\hat{E} \times E$ is given by the product of their Borel σ-algebras.

Remark 2.2.1. *Notice that in* (2.10) *we have implicitly assumed that the duality functions are integrable.*

Relation (2.10) can be also written in terms of semigroups:

$$(S_t D(\hat{x}, \cdot))(x) = (\hat{S}_t D(\cdot, x))(\hat{x}) \tag{2.11}$$

where S_t and $\hat{S}_t$ denote the semigroups associated to X_t and $\hat{X}_t$ respectively.

For the processes considered in this thesis, this relation is also equivalent to the corresponding relation at the level of generators. Namely,

$$\mathscr{L} D(\hat{x}, \cdot)(x) = \hat{\mathscr{L}} D(\cdot, x)(\hat{x}) \qquad \text{for all} \quad x \in E, \hat{x} \in \hat{E}. \tag{2.12}$$

Remark 2.2.2. *In order to have duality at the level of generators, we have to take care that the corresponding duality functions are in the domain of the corresponding generators. This is not always the case. For example, we have the duality between Brownian motion with reflection at zero (denoted by B_t^{ref}), and Brownian motion with absorption at zero (denoted by B_t^{abs}). If we denote by $\mathscr{L}^{ref}$ and $\mathscr{L}^{abs}$ the generators of B_t^{ref} and B_t^{abs}respectively, we have that their domains are given by:*

$$D(\mathscr{L}^{ref}) = \{f \in C_0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) : f', f'' \in C_0(\mathbb{R}_+), \quad f'(0^+) = 0\}$$

and

$$D(\mathscr{L}^{abs}) = \{f \in C_0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) : f', f'' \in C_0(\mathbb{R}_+), \quad f(0) = 0, \quad f''(0^+) = 0\}.$$

In this case, the duality function is

$$D(x, y) = I(x \leq y)$$

which, because of differentiability issues, is not in the domain of the generators of the two processes. We refer the reader to section A.1.4 in the appendix for the definition of those generators.

2.2.1 Self-Duality

The particular case in which the two processes involved in the duality relation are copies of each other is called self-duality. Let us make this type of self-duality more transparent by specializing it to the type of interacting particle systems relevant to this thesis and, instead of the lattice $\mathbb{Z}^d$, restricting to the case of a finite lattice V (only for this section). For a configuration $\xi \in \Omega$, let us denote by $\|\xi\|$ the number of particles it contains, i.e.,

$$\|\xi\| = \sum_x \xi_x \tag{2.13}$$

Let us denote by Ω_k the subset of Ω whose elements have exactly k particles. Additionally, we denote by Ω_f the set of configurations with a finite number of particles

$$\Omega_f = \bigcup_{k \in \mathbb{N}} \Omega_k \tag{2.14}$$

Remark 2.2.3. *Notice that in the case of a finite lattice V, the sets Ω and Ω_f are almost surely equal. This is because avoiding explosions (infinitely many particles coexisting on one site) implies that the number of particles at a given position are finite almost surely. Then, from the finiteness of the lattice, any valid configuration is almost surely finite. The notation Ω_f and Ω is used to be consistent with the rest of this thesis, where we extensively work with the infinite lattice case.*

A self-duality function will then be a function $D : \Omega_f \times \Omega \to \mathbb{R}$ such that:

$$\mathbb{E}_\eta\big[D(\xi, \eta_t)\big] = \mathbb{E}_\xi\big[D(\xi_t, \eta)\big] \tag{2.15}$$

for all $\xi \in \Omega_f, \eta \in \Omega$. Or, equivalently,

$$\mathscr{L} D(\xi, \cdot)(\eta) = \mathscr{L}^{(k)} D(\cdot, \eta)(\xi) \tag{2.16}$$

again for all $\xi \in \Omega_f, \eta \in \Omega$, and where $\mathscr{L}^{(k)}$ denotes the generator (2.1) restricted to configurations containing exactly k particles.

2.2.1.1 Triangular self-duality

For reversible particle systems there is a “cheap” duality function that is easy to find. Namely, let us denote by μ a reversible measure for the generator (2.1). I.e., a measure satisfying the detailed balance condition (2.5). Then the cheap duality function

$$D_{\text{cheap}}(\xi, \eta) = \prod_{i \in V} \mathbb{1}_{\{\eta_i = \xi_i\}} \frac{1}{\mu(\eta_i)} \tag{2.17}$$

indeed satisfies (2.16).

Despite its simplicity, finding a "cheap" self-duality function is a good first step to build more useful self-duality functions. More precisely, from the "cheap" one, we can construct new self-duality functions by acting with symmetries of the generator $\mathscr{L}$. A symmetry S of $\mathscr{L}$ is an operator satisfying the relation:

$$[\mathscr{L}, S] = 0$$

where $[\cdot,\cdot]$ denotes the Lie bracket or commutator, i.e.,

$$[\mathscr{L}, S] = \mathscr{L}S - S\mathscr{L}$$

The new duality function is then given by:

$$D_{\text{new}}(\xi,\eta) = SD_{\text{cheap}}(\xi,\eta)$$

In [17], using a Lie algebraic approach it is proven that the generator $\mathscr{L}$ defined in (2.1) admits a set of factorized symmetries that are constructed starting from suitable creation and annihilation operators. The existence of these symmetries, combined with the cheap duality function obtained by the reversible measures of Section 2.1.1.1, allow to compute a non-trivial factorized duality function that has a characteristic "triangular" form.

In all three cases, the self-duality functions are factorized polynomials. This particular form will be the case for all duality functions used in this thesis. More precisely,

$$D(\xi,\eta) = \prod_{i\in\mathbb{Z}^d} P(\xi_i,\eta_i) \tag{2.18}$$

where $P(0,n) = 1$, and $P(m,\cdot)$ is a polynomial of degree m.

Remark 2.2.4. *Notice that from the fact that the configuration ξ has a finite number of particles, we have that the product in the RHS of* (2.18) *only has a finite number of factors different from 1, and hence the product is well-defined.*

We then have the following triangular self-duality relations:

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In this case the self-duality function is a product of polynomials in the variable η. More precisely

$$D(\xi,\eta) = \prod_{i\in\mathbb{Z}^d} d(\xi_i,\eta_i). \tag{2.19}$$

The single-site duality polynomials are given by

$$d(m,n) = \mathbb{1}_{\{m\leq n\}}\,\frac{n!}{(m-n)!}. \tag{2.20}$$

Symmetric Exclusion Process

For SEP the single-site duality functions are

$$d(m,n) = \mathbb{1}_{\{m\leq n\}} \binom{n}{\alpha} (\alpha - m)!. \tag{2.21}$$

Symmetric Inclusion Process

For SIP we have

$$d(m,n) = \mathbb{1}_{\{m\leq n\}} \frac{n!}{(n-m)!} \frac{\Gamma(\alpha)}{\Gamma(\alpha+m)}. \tag{2.22}$$

2.2.1.2 k-point correlation functions

As we will see in the sections to come, the knowledge of k-point correlation functions is useful for many applications related to scaling limits of IPS. Immediate examples are Section 2.2.2 and Section 3.1 of this thesis, which make use of the one-point (expectations) and two-point correlations in the context of hydrodynamic limits.

As we mentioned earlier in Section 1.4, thanks to self-duality we can explicitly compute k-point correlation functions. This is due to the polynomial form of the triangular self-duality functions. For example, for the one-point and two-point correlations we have the following identities:

$$\eta_x = C_1(\alpha,\sigma)\cdot D(\delta_x,\eta) \tag{2.23}$$

$$\begin{aligned}\eta_x\cdot\eta_y &= \mathbb{1}_{x=y}\cdot\left[C_2(\alpha,\sigma)\cdot D(2\delta_x,\eta) + C_3(\alpha,\sigma)\cdot D(\delta_x,\eta)\right] \\ &+ \mathbb{1}_{x\neq y}\cdot C_4(\alpha,\sigma)\cdot D(\delta_x+\delta_y,\eta)\end{aligned} \tag{2.24}$$

From (2.23) and (2.24), together with self-duality we indeed obtained the one and two-point correlation functions. Namely, we have

$$\begin{aligned}\mathbb{E}_\eta\left[\eta_x(t)\right] &= C_1\cdot\mathbb{E}_\eta\left[D(\delta_x,\eta(t))\right] \\ &= C_1\cdot\mathbb{E}_x\left[D(\delta_{X(t)},\eta)\right]\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\mathbb{E}_\eta\left[\eta_x(t)\cdot\eta_y(t)\right] &= \mathbb{1}_{x=y}\cdot\left[C_2\cdot\mathbb{E}_\eta\left[D(2\delta_x,\eta(t))\right] + C_3\cdot\mathbb{E}_\eta\left[D(\delta_x,\eta(t))\right]\right] \\ &+ \mathbb{1}_{x\neq y}\cdot C_4\cdot\mathbb{E}_\eta\left[D(\delta_x+\delta_y,\eta(t))\right] \\ &= \mathbb{1}_{x=y}\cdot\left[C_2\cdot\mathbb{E}_{x,y}\left[D(\delta_{X(t)}+\delta_{Y(t)},\eta)\right] + C_3\cdot\mathbb{E}_x\left[D(\delta_{X(t)},\eta)\right]\right] \\ &+ \mathbb{1}_{x\neq y}\cdot C_4\cdot\mathbb{E}_{x,y}\left[D(\delta_{X(t)}+\delta_{Y(t)},\eta)\right]\end{aligned} \tag{2.26}$$

where for notational convenience we removed the dependence of the constants C_i, for $i \in \{1, \ldots, 4\}$, on the parameters α and σ.

Remark 2.2.5. *Notice that the RHS of* (2.25) *and* (2.26) *is much simpler. Thanks to self-duality we have simplified computations involving in principle an infinite number of particles to expectations involving only one and two particles respectively.*

The knowledge of k-point correlation functions has many applications and we mention just a few of them:

Scaling limits: Generally speaking, the scaling properties of a single dual particle determine the hydrodynamic equation. More precisely, the expectation of the density field converges to the solution of the hydrodynamic equation, which in our context is the linear heat equation. The variance of the density field is related to the behavior of two dual particles. From the scaling properties of their joint dynamics, one can understand both the stationary and non-stationary behavior of the variance of the density fluctuation field. In particular, quantities such as the effect of deviation from local equilibrium become accessible.

Correlation inequalities: Information about the k-point correlations can be obtained by controlling the dynamics of k dual particles. This has allowed for example to obtain correlation inequalities. An example of this is the work [45] that uses duality to find correlation inequalities for the SIP, the so-called Brownian momentum process, and the Brownian energy process.

Ergodic properties: By ergodic properties, we understand the characterization of the extreme points of the set of invariant measures and the characterization of which measures, over time, converge to a given extremal invariant measure. By duality, the characterization of invariant measures boils down to the understanding of bounded harmonic functions of the dual-process, which in our context is always a system of finitely many particles, i.e., simpler than the original system we started from (which in principle has infinitely many particles). In [70] Chapter 8, the ergodic properties of SEP (1) are completely studied using duality and a comparison inequality between exclusion particles and independent random walkers. In [65] using a similar approach this problem is solved for SIP.

Well-posedness of martingale problems: Duality has already been used, see e.g. [33], to show the uniqueness of solutions to martingale problems. More recently, in [29], duality has been used to show the existence of solutions as well.

Non-equilibrium systems Duality allows to analyze systems out of equilibrium. For example, in [16], a class of boundary driven systems is considered. These systems are placed in contact with proper reservoirs, working at different particle densities or different temperatures. Theese particle systems are showed to be dual to systems with absorbing boundaries (which are much simpler to analyse).

For the case of the type of duality we consider, the reader can find in [17] a method and worked examples to find dualities for certain types of interacting particle systems via operators commuting with the infinitesimal generator. Additionally, the work [79] exhausts the types of duality relations of factorized form possible for a class of particle systems that include the ones described by the infinitesimal generator (2.1). The approach is based on a relation between factorized duality functions and stationary product measures and an intertwining relation provided by generating functions. Additionally, from the perspective of population genetics, the works [75] and [74] have revealed strong connections of duality with the notions of symmetry and conserved quantities.

2.2.2 Applications of triangular self-duality: discrete heat equation

An application of self-duality related to Hydrodynamic limits is related to the so-called Kolmogorov equation. To make things transparent, consider the one-dimensional nearest neighbor symmetric inclusion process in $\mathbb{Z}$. In particular for $x \in \mathbb{Z}$ consider the function $f(\eta) = \eta_x$, we then have

$$\begin{aligned} \mathscr{L}\eta_x &= \mathscr{L}C_1 \cdot D(\delta_x, \eta) = C_1 \cdot \mathscr{L}D(\delta_x, \eta) \\ &= C_1 \cdot L^{(1)}D(\delta_x, \eta) = L^{(1)}C_1 \cdot D(\delta_x, \eta) \\ &= \frac{\alpha}{2}\big(\eta_{x+1} + \eta_{x-1} - 2\eta_x\big) \end{aligned} \tag{2.27}$$

where in the third line we used self-duality.

Then if we define the function

$$\Psi(x,t) = E_\eta[\eta_x(t)], \tag{2.28}$$

by the Hille-Yoshida Theorem A.1.1 we have

$$\frac{\partial}{\partial t}\Psi(x,t) = \frac{\alpha}{2}\big(\Psi(x+1,t) + \Psi(x-1,t) - 2\Psi(x,t)\big) \tag{2.29}$$

Using the notation Δ for the discrete Laplacian in 2.29, we obtain the Cauchy problem

$$\frac{\partial}{\partial t}\Psi(x,t) = \alpha\Delta\Psi(x,t) \tag{2.30}$$

with initial condition

$$\Psi(x,0) = E_\eta[\eta_x(0)] \tag{2.31}$$

Using Fourier analysis, and with single-particle dynamics we can express the solution as:

$$\begin{aligned}\Psi(x,t) &= E_\eta[\eta_x(t)] = E_x[\eta_{X(t)}] \\ &= \sum_y p_t(x,y)\eta_x(0) \\ &= \frac{1}{2\pi}\sum_y \eta_0(y)\int_{-\pi}^{\pi} e^{-t(2-2cosK)}e^{-ik(x-y)}dk \end{aligned} \tag{2.32}$$

where in the third equality, in order to compute $\eta_x(t)$, we also needed the initial configuration $\{\eta_0(x), x \in \mathbb{Z}\}$ and the transition kernel of a simple random walk.

For us, the fact that Ψ satisfies the Cauchy problem (2.30) motivates the idea that, given the right time-scaling (one that guarantees convergence of the discrete Laplacian), a function defined in the same way satisfies a Cauchy problem as well. In Section 3.1, we will deal with this idea again.

2.2.3 Orthogonal polynomial self-duality

We have a further special case of duality to be discussed. We are now interested in a type of self-duality that enjoys orthogonal properties with respect to the reversible measures ν_ρ. More precisely, the type of duality function will then be a function $\mathcal{D} : \Omega_f \times \Omega \to \mathbb{R}$ such that the following properties hold:

1. Self-duality.
$$\mathbb{E}_\eta\big[\mathcal{D}(\xi,\eta_t)\big] = \mathbb{E}_\xi\big[\mathcal{D}(\xi_t,\eta)\big] \tag{2.33}$$
for all $\xi \in \Omega_f, \eta \in \Omega$ (where we remind that $\eta \in \Omega$ is always chosen such that the process $\{\eta(t) : t \geq 0\}$ is well-defined when starting from η).

2. Factorized polynomials.
$$\mathcal{D}(\xi,\eta) = \prod_{i\in\mathbb{Z}^d} P(\xi_i,\eta_i) \tag{2.34}$$
where $P(0,n) = 1$, and $P(m,\cdot)$ is a polynomial of degree m.

3. Orthogonality: for $\xi, \xi' \in \Omega_f$
$$\int \mathcal{D}(\xi,\eta)\mathcal{D}(\xi',\eta)d\nu_\rho(\eta) = \delta_{\xi,\xi'}a_\rho(\xi) \tag{2.35}$$
where $a_\rho(\xi) = \|\mathcal{D}(\xi,\cdot)\|^2_{L^2(\nu_\rho)}$

Notice that this time these functions will depend on the parameter ρ, but from now and on we suppress this dependence in the notation in order to simplify it.

IRW: Charlier polynomials

The orthogonal self-duality functions, for Independent Random Walkers are products of Charlier polynomials. These polynomials can be expressed in terms of hypergeometric functions as follows:

$$C(m,n) = {}_2F_0\left[\begin{matrix} -m \quad -n \\ - \end{matrix}; -\frac{1}{\rho}\right]$$

Remark 2.2.6. *To avoid minor confusions please notice that in [41] a relation between "classical" and new orthogonal duality polynomials is given. By classical polynomials we mean*

$$d(m,n) = \frac{n!}{(n-m)!} \tag{2.36}$$

and the way in which they relate these types of duality polynomials is given by:

$$\mathcal{D}(\xi,\eta) = \prod_{x\in\mathbb{Z}^d} \sum_{j=0}^{\xi_x} \binom{\xi_x}{j} (-\rho)^{\xi_x - j} \frac{\eta_x!}{(\eta_x - j)!}. \tag{2.37}$$

Notice that expression (2.37) differs by a factor $-\rho^{|\xi|}$ from the traditional form of the Charlier polynomials found in the literature:

$$\tilde{\mathcal{D}}(\xi,\eta) = \prod_{x\in\mathbb{Z}^d} \sum_{j=0}^{\xi_x} \binom{\xi_x}{j} (-\rho)^{-j} \frac{\eta_x!}{(\eta_x - j)!}. \tag{2.38}$$

The factor $-\rho^{||\xi||}$ is however invariant under the dynamics of our process that conserves the total number of particles $||\xi(t)||$, and hence its addition preserves the duality property.

SEP(α): Krawtchouk polynomials.

Strictly speaking these polynomials do not satisfy a self-duality relation. However, under a proper normalization we can find a duality function in terms of them. The single-site duality polynomials are hence given by

$$d(m,n) = \frac{m!(\alpha-m)!}{\alpha!} K(m,n)$$

where $K(m,n)$ denotes the mth-order Krawtchouk polynomial.

These polynomials can be written in terms of hypergeometric functions as

$$K(m,n) = {}_2F_1\left[\begin{matrix} -m \quad -n \\ -\alpha \end{matrix}; \frac{1}{\rho}\right]$$

SIP(α): Meixner polynomials

As in the case of the SEP process, the polynomials that satisfy the single-site self-duality relation are given by the following normalization of the Meixner polynomials

$$d(m,n) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+m)} M(m,n)$$

where $M(m,n)$ has hypergeometric form

$$M(m,n) = {}_2F_1\left[\begin{matrix} -m & -n \\ & -\alpha \end{matrix} ; 1 - \frac{1}{\rho}\right]$$

For more details on orthogonal duality and a proof of self-duality with respect to this function we refer to [41] and [78]. In those papers a more complete study is provided, which includes the case of other processes such as exclusion and inclusion, among others.

Remark 2.2.7. *Notice that relations* (2.25) *and* (2.26) *can also be written in terms of orthogonal self-duality polynomials. With the advantage that in this case, thanks to orthogonality, the expressions become simpler.*

2.2.4 Application of orthogonal self-duality: time-covariances

Let $\xi, \xi' \in \Omega_f$, we denote by $p_t(\xi, \xi')$ the transition probability to go from the configuration ξ to ξ' in time t. The following is an elementary consequence of duality with orthogonal duality functions.

Lemma 2.2.1. *Let $\xi, \xi' \in \Omega_f$, then, for all processes considered we have*

$$\int \mathbb{E}_\eta \left[\mathcal{D}(\xi, \eta_t)\right] \mathcal{D}(\xi', \eta) d\nu_\rho(\eta) = p_t(\xi, \xi') a(\xi') \tag{2.39}$$

Proof. We use self-duality to compute

$$\begin{aligned}
\int \mathbb{E}_\eta \left[\mathcal{D}(\xi, \eta_t)\right] \mathcal{D}(\xi', \eta) d\nu_\rho(\eta) &= \int \mathbb{E}_\xi \left[\mathcal{D}(\xi_t, \eta)\right] \mathcal{D}(\xi', \eta) d\nu_\rho(\eta) \\
&= \sum_\zeta p_t(\xi, \zeta) \int \mathcal{D}(\zeta, \eta) \mathcal{D}(\xi', \eta) d\nu_\rho(\eta) \\
&= p_t(\xi, \xi') a(\xi')
\end{aligned}$$

which proves the result. □

Remark 2.2.8. *Notice that* (2.39) *in particular implies that if* η_0 *is initially distributed according to* ν_ρ*, we first have*

$$\mathbb{E}_{\nu_\rho}\left[\mathcal{D}(\xi,\eta_t)\right] = 0 \tag{2.40}$$

and then

$$Cov_{\nu_\rho}\left(\mathcal{D}(\xi,\eta_t)\mathcal{D}(\xi',\eta)\right) \geq 0 \tag{2.41}$$

i.e. duality orthogonal polynomials are positively correlated.

Lemma 2.2.1 provides a big simplification since it allows to transfer most of the uncertainty of our process $\{\xi(t), t \geq 0\}$ to the transition kernel $p_t(\xi,\xi')$ of two configurations in Ω_f. Recall that here $\{\xi(t), t \geq 0\}$ is a much simpler process, conserving only $\|\xi(t)\|$ over time, and thus easier to treat. In the Appendix, for the case of IRW, we provide an estimate of this kernel utilizing the local limit theorem.

Chapter 3

Equilibrium fluctuations in the context of duality

This thesis deals with the fluctuation theory for a class of interacting particle systems that enjoy the property of duality. In Section 2.2.3 we have already introduced the notion of duality and some of its basic applications. It is now time to present the precise mathematical context in which fluctuation theory is developed and in which our results are established.

Fluctuation theory concerns the study of scaling limits of the type of functional central limit theorems. As such, it is then convenient to spend one section talking about hydrodynamic limits. These types of scaling limits are the analogous results in the direction of the law of large numbers. Because of this analogy, we usually refer to fluctuation results just as fluctuations around the hydrodynamic limit.

3.1 Hydrodynamic limits

The raison d'être of the study of hydrodynamic limits is the rigorous derivation, starting from a microscopic particle system, of a partial differential equation that describes the evolution of some quantity. For the concrete systems that we study in this thesis, the only conserved quantity is the total number of particles. Therefore it is natural to expect that the desired PDE will describe the evolution of the density of particles over time. At the micro-level, i.e., at the particle systems level, the quantity corresponding to the particle density is the so-called empirical density field:

$$\pi_n(\eta) = \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d} \delta_{x/n} \eta_x \tag{3.1}$$

where δ_x denotes the Dirac mass at zero and n is a scaling parameter.

The empirical density field is a measure that assigns mass n^{-d} to each point $x/n \in \mathbb{R}^d$ for each particle that sits at position $x \in \mathbb{Z}^d$.

Remark 3.1.1. *Notice that, when we let η evolve in time, the empirical density field becomes a measure-valued random trajectory.*

Loosely speaking, a hydrodynamic result concerns the weak convergence in path space of this object (the empirical density field) to a deterministic trajectory that concentrates on the solution of a certain PDE. In particular, in the simplest of the versions of the systems considered in this thesis, the corresponding PDE is the heat equation.

3.1.1 From micro to macro: diffusive scaling

The rigorous derivation of the macroscopic equation requires rescaling in space and time. The idea is to go from the microscopic dynamics of the system in $\mathbb{Z}^d$ to the dynamics at the macroscopic scale in $\mathbb{R}^d$ in a way in which the lattice mesh goes to zero. The distance among points is controlled by a scaling factor of n (sometimes N). That is, a macroscopic point $x \in \mathbb{R}^d$ will correspond to the microscopic point $\lfloor nx \rfloor$. Given this shrinking of space, and to see a non-trivial evolution, we also need to rescale time. From the observation that, in a time t, a single particle typically moves a distance $\sqrt{t}/n$, we can deduce that, in order to see a non-trivial evolution, we should rescale time by a factor n^2, i.e., macro time t will correspond to micro time $n^2 t$. From now and on, we will call this type of rescaling diffusive scaling.

Remark 3.1.2. *In more generality, choosing a diffusive space-time scaling is not the only possibility. Nevertheless, anticipating the heat equation as the hydrodynamic equation (which has first-order time and second-order space derivatives), the diffusive rescaling becomes a sensible choice. There are many other possibilities to rescale in order to study scaling limits in general. As an example, we refer to Section 6.1.4.1 in which, in the context of the SIP, a different type of rescaling is introduced.*

3.1.2 Density Field

The standard approach to show the weak convergence result in hydrodynamic limits is to use martingale techniques. To optimally exploit these techniques, we usually consider the empirical measure (3.1) rescaled diffusively and integrated against an adequate set of test functions φ, i.e.,

$$\mathscr{Y}_t^n(\varphi, \eta) := \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d} \varphi(\tfrac{x}{n}) \eta_x(n^2 t) \tag{3.2}$$

The hydrodynamic result is then stated as follows

Theorem 3.1.1. *For each of the processes considered in this thesis, started from a product measure ν of slowing varying density profile $\rho_0 : \mathbb{R}^d \to \mathbb{R}$, for every $T > 0$, every $t \in [0,T]$, every $\varphi \in C_c^\infty(\mathbb{R}^d)$, and $\epsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left[\mid \mathscr{Y}_t^n(\varphi,\eta) - \int_{\mathbb{R}^d} \varphi(x)\rho(t,x)\,dx \mid > \epsilon \right] = 0 \tag{3.3}$$

where $\rho(t,x)$ is a weak solution of the heat equation:

$$\begin{cases} \partial_t \rho = \frac{\chi\alpha}{2}\Delta\rho \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases} \tag{3.4}$$

Remark 3.1.3. *To simplify the presentation of this type of results, we intentionally avoided giving details about the notions and necessary assumptions on the initial distribution of particles. We refer to [58], Chapter I and IV, for details on those matters.*

In order to prove results like Theorem 3.1.1, we have at our disposal martingale techniques that arise naturally in the context of Markov processes, see [28] for a complete survey on this and other approaches. Additionally, for the IPS that we consider in this thesis, we have the simplifying self-duality property. This property implies that already on the micro-level, we have the gradient condition and even more, a discrete heat equation for the mathematical expectation of the density (see Section 2.2.2). Moreover, self-duality allows us to control the relevant martingales to establish tightness and show vanishing variances. We refer to the author's master thesis [6] for a simple sketch on how to use self-duality to derive hydrodynamic results in the context of SIP.

3.2 Fluctuation theory

In this section, we consider the IPS with generator (2.1) started in equilibrium from one of the reversible measures ν_ρ defined in Section 2.1.1.1; for clearness of exposition, these systems are taken to be stationary in time, reversible and invariant under spatial shifts.

3.2.1 Density Fluctuation field

Recall that fluctuation theorems are the CLT counterparts of hydrodynamic limits. This means that the natural object to study fluctuations is the following:

$$\mathscr{X}_t^n(\varphi,\eta) := n^{d/2}\left(\mathscr{Y}_t^n(\varphi,\eta) - \mathbb{E}\mathscr{Y}_t^n(\varphi,\eta)\right) \tag{3.5}$$

We call this object the density fluctuation field. We can think of this field as a distribution-valued process acting on test functions $\varphi \in S(\mathbb{R}^d)$, where:

$$S(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \mid x^\beta D^\theta f \mid < \infty, \forall \beta, \theta \in \mathbb{N}\} \tag{3.6}$$

The path-space becomes then $D([0,T]; S'(\mathbb{R}^d))$, the set of paths that are right continuous with left limits. Fluctuation theory aims to show the convergence of the density fluctuation field (3.5) to a distribution-valued process that we will denote by $\mathscr{X}_t$.

3.2.2 Generalized Ornstein-Uhlenbeck process

Since the density fluctuation field inherits the Markov property from the process $\{\eta_t\}_{t\geq 0}$, we expect the conservation of the Markov property in the limit $n \to \infty$. Moreover, we are deriving an analogue to CLT type of results; hence, Gaussianity is also a desirable property of our limiting object $\mathscr{X}_t$. If on top of that, we restrict ourselves to the case of stationarity, the limiting field $\mathscr{X}_t$ should be then stationary as well. Taken together, the above restrictions limit the possibilities for $\mathscr{X}_t$. This is because the only distribution-valued stationary Gaussian Markov processes are generalized Ornstein-Uhlenbeck processes. To properly define this family of distribution-valued processes, we give Theorem 3.2.1 below. This theorem, which we state as in [58], deals with existence and uniqueness for the generalized Ornstein-Uhlenbeck process. Its proof can be found in [58] and in its original version in the work [52].

Let us consider the non-negative operator $\mathscr{U}$ acting on functions φ as follows:

$$\mathscr{U}\varphi = \frac{\chi\alpha}{2}\Delta\varphi \tag{3.7}$$

with domain $D(\mathscr{U}) \subseteq L^2(\mathbb{R}^d)$ and with χ, σ, ρ, and α as in Section 2.1.1. Let us also consider the operator

$$\mathscr{V}\varphi = \chi\rho(\alpha + \sigma\rho)\nabla\varphi \tag{3.8}$$

with corresponding domain $D(\mathscr{V}) \subseteq L^2(\mathbb{R}^d)$. The generalized Ornstein-Uhlenbeck process is then determined by the following martingale problem:

Theorem 3.2.1. *Let Q be a probability measure on $C([0,T]; S(\mathbb{R}^d))$. Assume that for every $\varphi \in S(\mathbb{R}^d)$, and every $t \geq 0$*

$$M_t(\varphi) = \mathscr{X}_t(\varphi) - \mathscr{X}_0(\varphi) - \int_0^t \mathscr{X}_s(\mathscr{U}(\varphi))ds \tag{3.9}$$

and

$$\big(M_t(\varphi)\big)^2 - \big\|\mathscr{V}(\varphi)\big\|^2 t \tag{3.10}$$

are $L^1(Q)$ $\mathscr{F}_t$-martingales. Then for every $0 \le s \le t$, and every subset A of $\mathbb{R}^d$

$$Q\left(\mathscr{X}_t(\varphi) \in A \mid \mathscr{F}_s\right) = \int_A \frac{1}{\sqrt{2\pi \int_0^{t-s} \|\mathscr{V} S_r\varphi\|_2^2 \, dr}} \exp\left(\frac{-(y-\mathscr{X}_s(S_{t-s}\varphi))^2}{\int_0^{t-s}\|\mathscr{V} S_r\varphi\|_2^2 dr}\right) dy \tag{3.11}$$

where S_t is the semigroup associated to $\mathscr{U}$. In particular the knowledge of Q restricted to $\mathscr{F}_0$ uniquely determines Q in the whole $C([0,T];S(\mathbb{R}^d))$.

Formally speaking, the solution of the above martingale problem is also a solution of the SPDE:

$$d\mathscr{X}_t = \tfrac{\chi\alpha}{2}\, \Delta\mathscr{X}_t \; dt + \sqrt{\chi\rho(\alpha+\sigma\rho)}\, \nabla d\mathscr{W}_t \tag{3.12}$$

where $\mathscr{W}_t(x)$ is a space-time white noise with covariance

$$cov[\mathscr{W}_t(x), \mathscr{W}_s(y)] = \min(t,s)\delta(x-y)$$

Remark 3.2.1. *Notice that the formal SPDE* (3.12) *is another option to justify the idea that fluctuation results concern small deviations from the typical behavior described by the hydrodynamic equation.*

3.2.3 Rigorous statement

The precise statement we can make is the following

Theorem 3.2.2. *The sequence of processes $\{\mathscr{X}_t^n : t \in [0,T]\}_{n\ge1}$, given by* (3.5)*, converges, as $n \to \infty$, in distribution with respect to the $J1$-topology of $D([0,T]; S'(\mathbb{R}^d))$ to the process $\{\mathscr{X}_t : t \in [0,T]\}$ being the unique solution of the martingale problem specified in Theorem 3.2.1.*

The usual strategy to show Theorem 3.2.2 consists in proving three things:

1. That the sequence of probability measures Q_n is tight.
2. The Gaussianity of all limiting points Q restricted to $\mathscr{F}_0$.
3. That the limit points solve the martingale problem given in Theorem 3.2.1.

We now sketch how we can prove the third point given above with the help of duality. For the Gaussianity of the limiting field, we refer to Chapter 11 of [58]. In [58], we can find a proof of the Gaussianity in the context of zero-range processes. This proof is adaptable to our case. For tightness, despite the fact of being approachable with duality techniques, we refer to Chapter 5 of this thesis for a proof in a more general context.

Let us start by observing that the density fluctuation field can be written in terms of orthogonal self-duality polynomials, defined in Section 2.2.3, as follows:

$$\mathscr{X}_t^n(\varphi,\eta) = \frac{C(\rho,\alpha,\sigma)}{n^{d/2}} \sum_{x\in\mathbb{Z}^d} \varphi(x/n)\mathcal{D}(\delta_x, \eta(n^2 t)) \tag{3.13}$$

where $C(\rho, \alpha, \sigma)$ is a constant that depends on each of the three particle systems we consider.

Remark 3.2.2. *The observation that it is possible to re-write the density field in terms of duality polynomials opens the possibility of defining more fields based on these polynomials and studying their scaling limits. We will indeed do this in Chapter 5, where we study the convergence of a bigger class of fields based on orthogonal self-dualities.*

3.2.4 The martingale problem

Let us move to the third point of the list. In order to show that the limiting Q indeed satisfies the martingale problem (3.9)-(3.10), we make use of the well known Dynking martingales. This means that, already at the micro-level, we have the following martingales:

$$M_t^n(\varphi) = \mathscr{X}_t^n(\varphi, \eta) - \mathscr{X}_0^n(\varphi, \eta) - n^2 \int_0^t \mathscr{L}\mathscr{X}_s^n(\varphi, \eta) ds \tag{3.14}$$

and

$$\left(M_t^n(\varphi, \eta)\right)^2 - n^2 \int_0^t \Gamma \mathscr{X}_s^n(\varphi, \eta) ds \tag{3.15}$$

where the operator Γ is the so-called carré -du-champ defined in Appendix A.2.1 of this thesis.

To show that indeed in the limit equations (3.9)-(3.10) are satisfied, the first step is to write the action of the generator and the carré -du-champ in terms of the density fluctuation field. More precisely, we need to show that

$$\mathscr{L}\mathscr{X}_s^n(\varphi, \eta) = \mathscr{X}_s^n(\mathscr{U}\varphi, \eta) + O(1/n) \tag{3.16}$$

and

$$\Gamma \mathscr{X}_s^n(\varphi, \eta) = \mathscr{V}\varphi + O(1/n), \tag{3.17}$$

where $O(1/n)$ converges to zero as $n \to \infty$ in the appropriate sense.

3.2.4.1 The drift

Let us start with (3.16). By linearity of the generator we have

$$\begin{aligned} \mathscr{L}\mathscr{X}_s^n(\varphi, \eta) &= n^2 \frac{C(\rho, \alpha, \sigma)}{n^{d/2}} \sum_{x \in \mathbb{Z}^d} \varphi(x/n) \mathscr{L}\mathcal{D}(\delta_x, \eta(s)) \\ &= n^2 \frac{C(\rho, \alpha, \sigma)}{n^{d/2}} \sum_{x \in \mathbb{Z}^d} \varphi(x/n) \mathscr{L}^{(1)}\mathcal{D}(\delta_x, \eta(s)) \end{aligned} \tag{3.18}$$

where in the second line we used the self-duality property with one particle.

Moreover, by compatibility and reversibility we have

$$\begin{aligned}\mathscr{L}\mathscr{X}_s^n(\varphi,\eta) &= n^2\frac{C(\rho,\alpha,\sigma)}{n^{d/2}}\sum_{x\in\mathbb{Z}^d}\varphi(x/n)L^{(1)}\mathcal{D}(g(x),\eta(s)) \\ &= n^2\frac{C(\rho,\alpha,\sigma)}{n^{d/2}}\sum_{x\in\mathbb{Z}^d}L^{(1)}(\varphi(x/n))\cdot\mathcal{D}(g(x),\eta(s))\end{aligned}$$

where g is a mapping that translates particle dynamics at the level of coordinates to dynamics at the level of configurations. We refer to [19] for more details on this mapping.

For simplicity let us restrict our exposition to the case $d=1$. By Taylor's Theorem we have

$$\begin{aligned}n^2L^{(1)}\varphi(x/n) &= \alpha\sum_{r\in\mathscr{R}}p(r)\big(\varphi(x+r/n)-\varphi(x/n)\big) \\ &= \alpha\sum_{r\in\mathscr{R}_+}p(r)\big(\varphi(x+r/n)+\varphi(x-r/n)-2\varphi(x/n)\big) \\ &= \frac{\chi\alpha}{2}\Delta\varphi(x/n)+\frac{\alpha}{3!}\sum_{r\in\mathscr{R}}p(r)\frac{r^3}{n}\varphi^{(3)}(z(x,r)/n) \qquad (3.19)\end{aligned}$$

where for each r we have that $z(x,r)$ is a point in the open interval

$$(\min\{x/n,x+r/n\},\max\{x/n,x+r/n\})$$

We then have

$$\mathscr{L}\mathscr{X}_s^n(\varphi,\eta)=\mathscr{X}_s^n(\mathscr{U}\varphi,\eta)+\frac{K}{n^{3/2}}\sum_{x\in\mathbb{Z}}\sum_{r\in\mathscr{R}}p(r)r^3\varphi(z(x,r)/n)\mathcal{D}(\delta_x,\eta(s)) \quad (3.20)$$

where K is a constant that incorporates any other constant in our computation.

Let us denote the second term in the RHS of (3.20) by $\mathscr{E}(\varphi,\eta_s)$, i.e.,

$$\mathscr{E}(\varphi,\eta_s):=\frac{K}{n^{3/2}}\sum_{x\in\mathbb{Z}}\sum_{r\in\mathscr{R}}p(r)r^3\varphi(z(x,r)/n)\mathcal{D}(\delta_x,\eta(s)) \qquad (3.21)$$

The following proposition makes explicit the sense in which $\mathscr{E}(\varphi,\eta_s)$ converges to zero.

Proposition 3.2.1. *For any $\varphi\in S(\mathbb{R})$ and any $t\in[0,T]$ we have*

$$\lim_{n\to\infty}\mathbb{E}_n\left[\left(\int_0^t\mathscr{E}(\varphi,\eta_s)ds\right)^2\right]=0, \qquad (3.22)$$

where $\mathbb{E}_n$ denotes the expectation on $D([0,T],\Omega)$ induced by the configuration process $\{\eta(n^2t) : t \geq 0\}$.

The proof of this result shows the convenience of orthogonal self-duality in particular.

Proof. By stationarity we have

$$\mathbb{E}_n\left[\left(\int_0^t \mathscr{E}(\varphi,\eta_s)ds\right)^2\right] = 2\int_0^t\int_0^s\int \mathbb{E}_\eta\left[\mathscr{E}(\varphi,\eta_{(s-u)})\right]\mathscr{E}(\varphi,\eta)\nu_\rho(d\eta)\,du\,ds \tag{3.23}$$

where we remind the reader that ν_ρ is one of the reversible measures from Section 2.1.1.1.

Let us denote by $V(\varphi,\eta)$ the integrand inside the two time integrals in the RHS of (3.23). The we can estimate it as follows

$$\begin{aligned} &V(\varphi,\eta(s-u)) \\ &= \int \mathbb{E}_\eta\left[\mathscr{E}(\varphi,\eta_{(s-u)})\right]\mathscr{E}(\varphi,\eta)\nu_\rho(d\eta) \\ &= \frac{K^2}{n^3}\sum_{x,y\in\mathbb{Z}}\sum_{r,r'\in\mathscr{R}} p(r)p(r')r^3r'^3\varphi(z(x,r)/n)\varphi(z(y,r')/n) \\ &\times \int \mathbb{E}_\eta\left[\mathcal{D}(\delta_x,\eta(s-u))\right]\mathcal{D}(\delta_y,\eta)\nu_\rho(d\eta) \\ &= \frac{K^2}{n^3}\sum_{x,y\in\mathbb{Z}}\sum_{r,r'\in\mathscr{R}} p(r)p(r')r^3r'^3\varphi(z(x,r)/n)\varphi(z(y,r')/n) \\ &\times p_{n^2(s-u)}(\delta_x;\delta_y)a(\delta_y) \end{aligned} \tag{3.24}$$

where in the last line we used Lemma 2.2.1. By the extra factor $\frac{1}{n}$ in front of the double summation, these last computations make clear that (3.23) vanishes as $n \to \infty$. $\square$

Remark 3.2.3. *For other particle systems like, for example, some zero-range processes, it is not possible to directly close the expression $\mathscr{L}\mathscr{X}_s^n(\varphi,\eta)$. i.e., to rewrite it in terms of the field itself eventually applied to a different test function. For such particle systems, H. Rost introduced in [80] a technical tool called the Boltzmann-Gibbs Principle (BGP). This tool allows to make a replacement in order to close the equation. As we have seen in the computations of this section,* *for particle systems that enjoy the property of self-duality, this result is not necessary. Nevertheless, it is interesting to explore the consequences of self-duality in the context of the Boltzmann-Gibbs principle. This is precisely the content of Chapter 4, where, via orthogonal self-duality, we provide quantitative versions of the BGP.*

3.2.4.2 Carré-du-champ

Now for the Carré-du-champ, i.e., for (3.17), we make use of the simpler formula (A.27) in Appendix A.2.1.

$$\Gamma \mathscr{X}_s^n(\varphi,\eta) = n^2 \sum_{i\in\mathbb{Z}^d}\sum_{r\in\mathbb{Z}^d} p(r)\eta_i(\alpha+\sigma\eta_{i+r})\Big(\mathscr{X}_s^n(\varphi,\eta^{i,i+r}) - \mathscr{X}_s^n(\varphi,\eta)\Big)^2 \tag{3.25}$$

Let us evaluate separately the term inside the square:

$$\begin{aligned}
&\mathscr{X}_s^n(\varphi,\eta^{i,i+r}) - \mathscr{X}_s^n(\varphi,\eta)\\
&= \frac{C(\rho,\alpha,\sigma)}{n^{d/2}} \sum_{x\in\mathbb{Z}^d} \varphi(x/n)\left[\mathcal{D}(\delta_x,\eta(s)) - \mathcal{D}(\delta_x,\eta(s)-\delta_i+\delta_{i+r})\right]\\
&= \frac{C(\rho,\alpha,\sigma)}{n^{d/2}}\big(\varphi(i+r/n)-\varphi(i/n)\big) \qquad (3.26)
\end{aligned}$$

This gives already a discrete gradient, which, in combination with the n^2 in (3.25) gives the following:

$$\Gamma \mathscr{X}_s^n(\varphi,\eta) \;=\; \frac{C^2}{n^d}\sum_{i\in\mathbb{Z}^d}\sum_{r\in\mathbb{Z}^d} p(r)\eta_i(\alpha+\sigma\eta_{i+r})\Big[n\big(\varphi(i+r/n)-\varphi(i/n)\big)\Big]^2 \tag{3.27}$$

To conclude, we simply argue by stationarity and the ergodic theorem applied to the initial measure ν_ρ, that indeed the RHS of (3.27) converges to:

$$\|\mathscr{V}\varphi\|_2^2$$

in the $L^2(\mathbb{P}_n)$ sense, where $\mathbb{P}_n$ is the probability on $D([0,T],\Omega)$ induced by the process $\{\eta(n^2t) : t \geq 0\}$.

Given that we have tightness and Gaussianity, at this point to conclude that indeed the martingale problem (3.9)-(3.10) is satisfied, it is enough to show uniform integrability of the sequence $\{M_t^n\}_{n\geq 1}$. For the proof of uniform integrability, we also refer to Chapter 5 of this thesis.

Part I

Fluctuation fields and Duality

Chapter 4

Quantitative Boltzmann–Gibbs Principles via Orthogonal Polynomial Duality

The Boltzmann-Gibbs principle is an important ingredient in the study of fluctuation fields of interacting particle systems [58]. It basically states that on the central limit scale, the fluctuation field of local functions can be replaced by a constant times the density fluctuation field, or in other words, it can be replaced by its projection on the one-dimensional space generated by the density fluctuation field (where projection has to be understood in an appropriate Hilbert space of macroscopic quantities [12]). The aim of the present chapter is to refine and quantify the Boltzmann-Gibbs principle in the context of particle systems with duality, using fluctuation fields of orthogonal polynomials. Indeed, it turns out that replacing the fluctuation field of a local function by its projection on the density field corresponds to the projection on the fluctuation fields of orthogonal polynomials of order one. Therefore, the Boltzmann Gibbs principle easily follows from an estimation of the covariance of fluctuation fields of orthogonal polynomials of order two and higher. In this chapter, for independent random walkers we quantify the precise order of these covariances of fluctuation fields of orthogonal (Charlier) polynomials of order k for all $k \in \mathbb{N}$, and therefore we are able to give an orthogonal decomposition of the fluctuation field of any local function, which is a generalization of the Boltzmann Gibbs principle. Next, still in the context of independent random walkers, we are able to extend this result in a non-equilibrium setting, using the fact that products of Poisson measures are preserved under this dynamics, i.e., a strong form of propagation of local equilibrium holds in that context.

One of the basic ingredients of our approach is stochastic duality, a property shared by a certain class of interacting particle systems such as independent random walkers [15], exclusion process, inclusion process, Brownian energy process, etc. (see [16] for a review on the subject). Thanks to duality the k-body correlation functions obey closed equations, not involving higher correlations. This has many implications, such as the possibility to study the decay properties of correlation functions [37] and to study small perturbations of the original process [25].

In this chapter we exploit a duality property with orthogonal polynomials (see e.g.[78]) combined with precise estimates (of local limit type) of the k-particle dynamics. Therefore, the results immediately apply in the context of the stationary symmetric exclusion process, and more generally for particle systems where these precise estimates (of local limit type) of the k-particle dynamics can be obtained (e.g. via the log-Sobolev inequality [67]).

The rest of this chapter is organized as follows: In section 4.1 we formally introduce our system of random walkers, and the basic concepts and properties needed for the development of this paper. In section 4.2, in the context of stationarity, we start by introducing our results for the simplest non-trivial example of second order and move to a generalization first to higher orders and in a next stage to more general functions. We present in section 4.3 an extension of these last results to a non-equilibrium setting. Finally in section 4.4, we show how under additional assumptions our results can be extended to other interacting particle systems.

4.1 Basic notions

4.1.1 Independent Random Walkers

In this chapter we consider the generator (2.1) for the special case $\sigma = 0$ and $\alpha = 1$. This means that we consider a system of Independent Random Walkers at rate 1. Recall that this is an interacting particle system where particles randomly hop on the lattice $\mathbb{Z}^d$ without interaction and with no restrictions on the number of particles per site. Configurations are denoted by η, ξ, ζ and are elements of $\Omega = \mathbb{N}^{\mathbb{Z}^d}$ (where $\mathbb{N}$ denotes the natural numbers including zero). We denote by η_i the number of particles at i in the configuration $\eta \in \Omega$. The generator (2.1), working on local functions $f : \Omega \to \mathbb{R}$, takes the form

$$\mathscr{L} f(\eta) = \sum_{i,j} p(i,j)\eta_i(f(\eta^{ij}) - f(\eta)) \tag{4.1}$$

where η^{ij} denotes the configuration obtained from η by removing a particle from i and putting it at j. Additionally, we assume that $p(i,j)$ is a translation invariant, symmetric, irreducible Markov transition function on $\mathbb{Z}^d$, i.e.,

1. $p(i,j) = p(j,i) = p(0, j-i)$.
2. $\sum_{j\in\mathbb{Z}^d} p(i,j) = 1$
3. There exists $R > 0$ such that $p(i,j) = 0$ for $|i-j| > R$.
4. For all $x, y \in \mathbb{Z}^d$ there exist $i_1 = x, \ldots, i_m = y$ such that

$$\prod_{l=1}^{m} p(i_l, i_{l+1}) > 0$$

Recall from Chapter 2 that for the associated Markov process on Ω, we use the notation $\{\eta(t) : t \geq 0\}$, i.e., $\eta_x(t)$ denotes the number of particles at time t at location $x \in \mathbb{Z}^d$. We also know from Chapter 2 that this particle system has a one-parameter family of homogeneous (w.r.t. translations), reversible and ergodic product measures $\nu_{\bar{\rho}}, \rho > 0$ with Poisson marginals

$$\nu_\rho(m) = \frac{\rho^m}{m!} e^{-\rho}$$

This family is indexed by the density of particles, i.e.,

$$\int \eta_0 \, d\nu_{\bar{\rho}} = \rho$$

Remark 4.1.1. *Recall that for IRW the initial configuration has to be chosen in a subset of configurations such that the process $\{\eta(t) : t \geq 0\}$ is well-defined. A possible such subset is the set of tempered configurations. This is the set of configurations η such that there exist $C, \beta \in \mathbb{R}$ that satisfy $|\eta(x)| \leq C(|x|^\beta + 1)$ for all $x \in \mathbb{R}^d$. We denote this set (with slight abuse of notation) still by Ω, because we will always start the process from such configurations, and this set has $\nu_{\bar{\rho}}$ measure 1 for all ρ. Since we will be working mostly in $L^2(\nu_{\bar{\rho}})$ spaces, this is not a restriction.*

4.1.2 Fluctuation fields

Let $S(\mathbb{R}^d)$ be the set of Schwartz functions on $\mathbb{R}^d$, and denote by $S'(\mathbb{R}^d)$ the corresponding distributions space (strong topological dual). Moreover we denote by τ_x the spatial shift, i.e., $\tau_x(\eta)_y = \eta_{y+x}$,. Fix $\varphi \in S(\mathbb{R}^d)$ and let $f : \Omega \to \mathbb{R}$ be a local function, we define its fluctuation field on scale n as

$$\mathsf{X_n}(f, \eta; \varphi) := a_n(f) \sum_{x\in\mathbb{Z}^d} \varphi(\tfrac{x}{n})(\tau_x f(\eta) - \psi_f(\rho)) \tag{4.2}$$

where

$$\psi_f(\rho) := \int f d\nu_{\bar{\rho}}, \qquad \tau_x f(\eta) := f(\tau_x \eta) \tag{4.3}$$

and $a_n(\cdot)$ is a suitable normalization constant depending on f. The field $\mathsf{X_n}(f,\eta;\cdot)$ is a Schwartz-distribution, i.e., an element of $S'(\mathbb{R}^d)$, associated to the configuration η. An important case is the density fluctuation field (3.5), where we have to make the choice $f(\eta) = \eta_0$, $a_n(f) = n^{-d/2}$.

The time-dependent fluctuation field at scale n is then defined as

$$\mathsf{X_n}(f,t;\varphi) = \mathsf{X_n}(f,\eta(n^2t);\varphi) \tag{4.4}$$

The diffusive rescaling anticipates the natural macroscopic time-scale in this symmetric process, which has the linear heat equation as hydrodynamic limit. $\{\mathsf{X_n}(f,t;\cdot),\ t \geq 0\}$ is then a Schwartz-distribution-valued stochastic process.

4.1.3 Boltzmann-Gibbs principle

The Boltzmann-Gibbs principle makes rigorous the idea that the density fluctuation field is the fundamental fluctuation field, because the density is the only (non-trivial) conserved quantity in the process under consideration. This means that one can replace, in first approximation, the fluctuation field of a function f by its "projection on the density field". For a local function f this projection is the fluctuation field of the function $P_1(f) := \psi_f'(\rho)(\eta_0 - \rho)$, where $\psi_f(\rho) = \int f d\nu_{\bar\rho}$.

The standard statement of the Boltzmann Gibbs principle is given in the following theorem.

Theorem 4.1.1. *For all f local, and $\varphi \in S(\mathbb{R}^d)$ and for all $T > 0$*

$$\lim_{n\to\infty} \mathbb{E}_{\nu_{\bar\rho}} \left[\left(\frac{1}{n^{d/2}} \int_0^T \left(\mathsf{X_n}(f,t;\varphi) - \mathsf{X_n}(P_1(f),t;\varphi) \right) dt \right)^2 \right] = 0 \tag{4.5}$$

We refer to [58] for the proof of Theorem 4.1.1 and for a comprehensive discussion of the result that is valid in a more general context and not only for the process considered in the present paper.

4.1.4 Fluctuation fields of orthogonal polynomials

For $k \in \mathbb{N}$ we denote by $\mathscr{H}_k$ the (real) Hilbert spaces generated by the polynomials $D(\xi,\cdot)$ with degree at most k, i.e. $||\xi|| \leq k$. We have of course the inclusion $\mathscr{H}_0 = \mathbb{R} \subset \mathscr{H}_1 \subset \mathscr{H}_2 \subset \ldots$ and the union of the spaces $\mathscr{H}_k$ is dense in $L^2(\nu_\rho)$. Moreover, for every $f \in L^2(\nu_\rho)$ its projection on $\mathscr{H}_k$ is given by

$$f_k = \sum_{\xi\in\Omega_f:||\xi||\leq k} \langle f, \mathcal{D}(\xi,\cdot)\rangle \frac{\mathcal{D}(\xi,\cdot)}{a(\xi)} \tag{4.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\nu_{\bar{\rho}})$ inner product, and $\mathcal{D}(\xi, \eta)$ the self-duality orthogonal polynomials of Section 2.2.3.

The aim of what follows is to show that the Boltzmann-Gibbs principle is an instance of a more general statement concerning the fluctuation behavior of functions which are orthogonal to $\mathscr{H}_k$ for some $k \in \mathbb{N}$. This is (in some sense to be explained below) the case for the function $f - P_1(f)$.

For $\xi \in \Omega_f$, with $||\xi|| = k$, $\varphi \in S(\mathbb{R}^d)$ we define the k-th order polynomial fluctuation field as

$$\begin{aligned} X_n(\xi, \eta, \varphi) \qquad &:= \sum_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right) \mathcal{D}(\xi, \tau_x \eta) \\ &= \sum_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right) \mathcal{D}(\tau_x \xi, \eta) \end{aligned} \tag{4.7}$$

4.2 Stationary case

4.2.1 Second-order polynomial field

We start with the simplest non-trivial example for independent random walkers started from a product measure with homogeneous Poisson marginals. To illustrate our point let us start with a simple computation, which contains all the important ingredients of the more general Theorem 4.2.1 below. Consider the field

$$X_n^{(2)}(\eta; \varphi) := X_n(2\delta_0, \eta, \varphi) = \sum_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right) \mathcal{D}(2\delta_x, \eta) \tag{4.8}$$

The notation $X_n^{(2)}$ suggests that this is in some sense the "second order" polynomial field. In the orthogonal polynomial language, this is the field of the second order Charlier polynomial

$$\mathcal{D}(2\delta_x, \eta) = \eta_x(\eta_x - 1) - 2\rho(\eta_x - \rho) - \rho^2 \tag{4.9}$$

Recall from Section 2.2.3 that

$$a(2\delta_0) = \int (\mathcal{D}(2\delta_x, \eta))^2 d\nu_\rho(\eta)$$

then we have the following.

Proposition 4.2.1. *The second-order polynomial field $X_n^{(2)}(\eta; \varphi)$ is such that*

1. *For $t > 0$ we have*

$$\mathbb{E}_{\nu_{\bar\rho}}\left[X_n^{(2)}(\eta(t);\varphi)\, X_n^{(2)}(\eta(0);\varphi)\right] = a(2\delta_0) \sum_{x,y\in\mathbb{Z}^d} \varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})(p_t(x,y))^2 \tag{4.10}$$

2. *As a consequence, for $t > 0$ we have*

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{E}_{\nu_{\bar\rho}}\left[X_n^{(2)}(\eta(n^2t);\varphi)X_n^{(2)}(\eta(0);\varphi)\right]\\ &= \frac{d\cdot a(2\delta_0)}{(2\pi t)^d}\int_{\mathbb{R}^{2d}} e^{-\frac{d|x-y|^2}{t}}\varphi(x)\varphi(y)dxdy\end{aligned} \tag{4.11}$$

Proof. The first statement follows from self-duality and Lemma 2.2.1. For the second statement we use that φ has compact support, call this support S, and define

$$M := \max\{d(x,y) : x,y\in S\} \tag{4.12}$$

it follows from Theorem C.1.2 that there exists $c = c(M)$ such that

$$\sup_{x:|x|\le Mn\sqrt{t}} p^{RW}_{n^2t}(x) \le \bar p_{n^2t}(x)\left(1+\frac{c}{n\sqrt{t}}\right)$$

with $\bar p_t(\cdot)$ as defined in (C.2). Then from (4.33) it follows that

$$\begin{aligned}&\mathbb{E}_{\nu_{\bar\rho}}\left[X_n^{(2)}(\eta(t);\varphi)X_N^{(2)}(\eta(0);\varphi)\right]\\ &= a(2\delta_0)\sum_{x,y\in S}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})\,\bar p_{n^2t}(x)\bar p_{n^2t}(y)\left(1+\frac{c}{n\sqrt{t}}\right)^2\\ &= a(2\delta_0)\cdot\frac{d}{(2\pi t)^d}\cdot\frac{1}{n^{2d}}\sum_{x,y\in S}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})e^{-\frac{d(z-y)^2}{tn^2}}\left(1+\frac{c}{n\sqrt{t}}\right)^2\end{aligned}$$

and letting $n\to\infty$ we obtain the r.h.s. of (4.11). □

In the current context the Boltzmann-Gibbs principle for the fluctuation field of the function $f = \eta_0(\eta_0 - 1)$ is a consequence of Proposition 4.2.1. We make this statement more transparent with the following corollary

Corollary 4.2.1. *The field $X_n^{(2)}(\eta(n^2t);\varphi)$ is such that, for all $T > 0$, there exist $C_1(T), C_2(T)$ and $C_3(T,d)$, such that for all n big enough*

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n^{(2)}(\eta(n^2t);\varphi)\,dt\right)^2\right] \le \begin{cases} C_1(T)\,n^{-1} & \text{if } d=1\\ C_2(T)\,\frac{\log(n)}{n^2} & \text{if } d=2\\ C_3(T,d)\,n^{-2} & \text{if } d\ge 3\end{cases} \tag{4.13}$$

More precisely, (4.11) *gives a better estimate, than the one of Theorem 4.1.1, for the order of the covariance of the fluctuation field on the diffusive time-scale as* $n \to \infty$.

Proof. Given the fact that the RHS of (4.11) has an indetermination at $t = 0$, we derive the following estimate for the integrand in (4.13)

$$\begin{aligned}
&\frac{1}{n^d}\mathbb{E}_{\nu_{\bar\rho}}\left[X_n^{(2)}(\eta(n^2t);\varphi)X_n^{(2)}(\eta(n^2s);\varphi)\right]\\
&= K_\rho \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\varphi(\tfrac{x}{n})p_{n^2(t-s)}(x,y)\sum_{y\in\mathbb{Z}^d}\varphi(\tfrac{y}{n})p_{n^2(t-s)}(x,y)\\
&\le K_\rho p_{n^2(t-s)}(0,0)\|\varphi\|_1\mathbb{E}_x\varphi(\tfrac{X_t}{n})\\
&\le K_\rho p_{n^2(t-s)}(0,0)\|\varphi\|_1\|\varphi\|_\infty
\end{aligned}$$

Plugging this into the LHS of (4.13) and after the substitution $r = t - s$ we obtain:

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n^{(2)}(\eta(n^2t);\varphi)\,dt\right)^2\right] \le C_{\rho,\varphi}\int_0^T\int_0^t p_{n^2r}(0,0)\,dr\,dt \tag{4.14}$$

where

$$C_{\rho,\varphi} = 2K_\rho\|\varphi\|_1\|\varphi\|_\infty$$

By the LCLT we have the following estimate

$$p_{n^2r}(0,0) \le \frac{d}{(2\pi n^2r)^{d/2}} \tag{4.15}$$

Let us now divide by cases according to the dimension d.

Case $d = 1$

In this case the LCLT gives the bound

$$p_{n^2r}(0,0) \le \frac{1}{n\sqrt{r}} \tag{4.16}$$

which then gives

$$\int_0^T\int_0^t p_{n^2r}(0,0)\,dr \le \frac{4}{3}T^{3/2}\,n^{-1} \tag{4.17}$$

substituting this bound in the RHS of (4.14) gives

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n^{(2)}(\eta(n^2t);\varphi)\,dt\right)^2\right] \le \frac{4}{3}T^{3/2}C_{\rho,\varphi}\,n^{-1} \tag{4.18}$$

Case $d = 2$

Now we have the bound

$$p_{n^2 r}(0,0) \leq \frac{2}{n^2 r} \tag{4.19}$$

This time, because of integrability issues, we cannot just substitute this bound in the RHS of (4.14). Instead, we distinguish the cases $|r| \geq \epsilon_n$ and $|r| < \epsilon_n$, where ϵ_n is to be optimized.

$$p_{n^2 r}(0,0) \leq \begin{cases} \frac{2}{n^2 r}, & \text{if } |r| \geq \epsilon_n \\ 1 & \text{if } |r| < \epsilon_n \end{cases} \tag{4.20}$$

then, for n large enough (i.e., at least $n \geq T$), we have

$$\begin{aligned} \int_0^t p_{n^2 r}(0,0)\, dr &\leq \epsilon_n + \frac{2}{n^2}\big(\log(t) - \log(\epsilon_n)\big) \\ &\leq \epsilon_n + \frac{2}{n^2}\big(\log(n) - \log(\epsilon_n)\big) \end{aligned} \tag{4.21}$$

Assuming ϵ_n is of the form $n^{-\alpha}$, we obtain

$$\int_0^t p_{n^2 r}(0,0)\, dr \leq n^{-\alpha} + \frac{2(\alpha+1)}{n^2}\log(n) \tag{4.22}$$

We conclude by taking for example $\alpha = 3$, in this case substituting this bound in the RHS of (4.14) gives

$$\mathbb{E}_{\nu_{\bar{\rho}}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n^{(2)}(\eta(n^2 t);\varphi)\, dt\right)^2\right] \leq 9\, T C_{\rho,\varphi}\, \frac{\log(n)}{n^2} \tag{4.23}$$

Case $d \geq 3$

In this case we move again from the approach of the first two cases. We will instead exploit the transience of the random walk in dimension $d \geq 3$.

Let us then start with a change of variables:

$$\int_0^t p_{n^2 r}(0,0)\, dr = \frac{1}{n^2}\int_0^{n^2 t} p_s(0,0)\, ds \tag{4.24}$$

We now claim that the time integral in the RHS of (4.24) is of order $O(1)$. In order to see that this is the case, let us introduce the local time at zero of a single walker, i.e.,

$$l_t^{(0)} = \int_0^t \mathbb{1}_{\{X_s = 0\}}\, ds \tag{4.25}$$

with this new notation we can express the time integral as follows

$$\int_0^{n^2 t} p_s(0,0) = \mathbb{E}_0^{1,d}\left[l_{n^2 t}^{(0)}\right] \tag{4.26}$$

where $\mathbb{E}_0^{1,d}$, denotes expectation with respect to a single d-dimensional random walker started at 0.

Furthermore, we have the bound

$$\int_0^{n^2 t} p_s(0,0) \le \mathbb{E}_0^{1,d}\left[l_{\infty}^{(0)}\right] \tag{4.27}$$

where

$$\mathbb{E}_0^{1,d}\left[l_{\infty}^{(0)}\right] := \lim_{t\to\infty} \mathbb{E}_0^{1,d}\left[l_{t}^{(0)}\right] \tag{4.28}$$

From the transience of the random walker in dimension $d \ge 3$, see for example [83], we have that indeed the expectation in the RHS of (4.27) is of order $O(1)$.

Hence we obtain:

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n^{(2)}(\eta(n^2 t);\varphi)\,dt\right)^2\right] \le C_3(T,d)\,n^{-2} \tag{4.29}$$

where

$$C_3(T,d) = T\,C_{\rho,\varphi}\,\mathbb{E}_0^{1,d}\left[l_{\infty}^{(0)}\right] \tag{4.30}$$

□

Back to the second-order polynomial fluctuation fields, and for the sake of transparency, we make explicit the dependence on the "coordinate points" x_1, x_2 and redefine the fields in terms of the orthogonal duality polynomials as follows:

$$X_n^{(2)}(x_1,x_2,\eta;\varphi) := \sum_{x\in\mathbb{Z}^d} \varphi(\tfrac{x}{n})\mathcal{D}(\delta_{x_1+x}+\delta_{x_2+x},\eta) \tag{4.31}$$

Notice then, that in Proposition 4.2.1 we treated for $x_1 = x_2 = 0$. It is necessary then to verify that Proposition 4.2.1 is not only the result of this particular choice we made, consider then for $x_1 \neq x_2$ the field

$$X_n^{(2),\neq}(x_1,x_2,\eta,\varphi) = \sum_{x\in\mathbb{Z}^d} \varphi(\tfrac{x}{n})(\eta_{x+x_1}-\rho)(\eta_{x+x_2}-\rho) \tag{4.32}$$

where the upper index $\neq$ refers to the fact that $x_1 \neq x_2$. We then have the following analogues of Proposition 4.2.1.

Proposition 4.2.2. *The second-order fluctuation field* $X_n^{(2),\neq}(x_1,x_2,\eta;\varphi)$ *is such that*

1. *For* $t>0$ *we have*

$$\begin{aligned}&\mathbb{E}_{\nu_{\bar\rho}}(X_n^{(2),\neq}(x_1,x_2,\eta(t);\varphi)X_n^{(2),\neq}(x_1,x_2,\eta(0);\varphi))\\ =\;& a(\delta_{x_1}+\delta_{x_2})\sum_{x,y\in\mathbb{Z}^d}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})p_t(x+x_1,x+x_2;y+x_1,y+x_2)\\ +\;& a(\delta_{x_1}+\delta_{x_2})\sum_{x,y\in\mathbb{Z}^d}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})p_t(x+x_1,x+x_2;y+x_2,y+x_1)\end{aligned} \tag{4.33}$$

2. *As a consequence, for* $t>0$ *we have*

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}_{\nu_{\bar\rho}}(X_n^{(2),\neq}(x_1,x_2,\eta(n^2t);\varphi)X_n^{(2),\neq}(x_1,x_2,\eta(0);\varphi))\\ &=\frac{2a(\delta_{x_1}+\delta_{x_2})d}{(2\pi t)^d}\int_{\mathbb{R}^{2d}}e^{-\frac{d|x-y|^2}{t}}\varphi(x)\varphi(y)dxdy\end{aligned} \tag{4.34}$$

Proof. The argument for the first statement is similar to the one in the proof of Proposition 4.2.1, the difference is that now

$$\mathcal{D}(\delta_{x+x_1}+\delta_{x+x_2},\eta)=(\eta_{x+x_1}-\rho)(\eta_{x+x_2}-\rho)$$

is the product of two first-order Charlier polynomials, which by the assumption of factorized polynomials allows us to proceed in the same way as before. Furthermore, in this case we have

$$\begin{aligned}&p_t(\delta_{x+x_1}+\delta_{x+x_2},\delta_{y+x_1}+\delta_{y+x_2})\\ &=p_t(x+x_1,x+x_2;y+x_1,y+x_2)+p_t(x+x_1,x+x_2;y+x_2,y+x_1)\end{aligned} \tag{4.35}$$

which is the source of the second term in (4.33). In the second statement is necessary to verify that x_1 and x_2 do not play a role in the leading order

$$\begin{aligned}&\mathbb{E}_{\nu_{\bar\rho}}(X_n^{(2),\neq}(x_1,x_2,\eta(n^2t);\varphi)X_n^{(2),\neq}(x_1,x_2,\eta(0);\varphi))\\ =\;& a(\delta_{x_1}+\delta_{x_2})\sum_{x,y\in\mathbb{Z}^d}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})p_{n^2t}(x+x_1,x+x_2;y+x_1,y+x_2)\\ +\;& a(\delta_{x_1}+\delta_{x_2})\sum_{x,y\in\mathbb{Z}^d}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})p_{n^2t}(x+x_1,x+x_2;y+x_2,y+x_1)\end{aligned} \tag{4.36}$$

The first term in the RHS of (4.36) can be treated in the same way as before. For the second term, we just have to notice

$$|x+x_1-y-x_2|^2+|x+x_2-y-x_1|^2=2|x-y|^2+2|x_1-x_2|^2$$

and proceed in the same way. □

Now we show how to generalize this result and discuss the case of higher-order fields.

4.2.2 Higher-order fields

Let $k \in \mathbb{N}$ and denote by $\mathbf{x} \in \mathbb{Z}^{kd}$ the coordinate vector $\mathbf{x} := (x_1, \dots, x_k)$, with $x_i \in \mathbb{Z}^d$, $i = 1, \dots, k$. We denote by $\xi(\mathbf{x})$ the configuration associated to $\mathbf{x}$, i.e. $\xi_x(\mathbf{x}) = \sum_{i=1}^k \mathbf{1}_{x=x_i}$. We define $||\mathbf{x}|| := ||\xi(\mathbf{x})|| = k$. Here x_i is the position of the i-th particle, where particles are labeled in such a way that the dynamics is symmetric. For a more extensive explanation of the labeled dynamics we refer the reader to [28]. We denote by $\hat{\tau}_z$, $z \in \mathbb{Z}^d$, the shift operator acting on the coordinate representation:

$$\hat{\tau}_z \mathbf{x} = (z + x_1, \dots, z + x_k), \qquad \text{and then} \qquad \tau_z \xi = \xi(\hat{\tau}_z \mathbf{x}) \tag{4.37}$$

Because of the translation invariance of the dynamics we have that

$$p_t(\xi(\hat{\tau}_y \mathbf{x}), \xi(\hat{\tau}_z \mathbf{x})) = p_t(\xi(\mathbf{x}), \xi(\hat{\tau}_{z-y} \mathbf{x})) \tag{4.38}$$

With an abuse of notation, we keep denoting by $p_t(\mathbf{x}, \mathbf{y})$ the transition probability of the labeled particles in the coordinate representation.

Remark 4.2.1. *The relation between the transition probabilities in the coordinate and in the configuration representations is given by*

$$p_t(\xi(\mathbf{x}), \xi(\mathbf{y})) = \sum_{\mathbf{x}' : \xi(\mathbf{x}') = \xi(\mathbf{y})} p_t(\mathbf{x}, \mathbf{x}') \tag{4.39}$$

Notice that it is precisely from relation (4.39) that a factor of 2 appears in Proposition 4.2.2 and not in Proposition 4.2.1. We can expect that in this general setting the difference among cases will become more cumbersome. To avoid any further notational difficulties we introduce the following:

Let $\mathscr{P}_k$ be the set of permutations of $\{1, \dots, k\}$, for $\sigma, \sigma' \in \mathscr{P}_k$ we define the following equivalence relation:

$$\sigma \sim \sigma' \quad \text{mod} \quad \mathbf{x} \qquad \text{iff} \quad x_{\sigma(i)} = x_{\sigma'(i)} \quad \forall i \in \{1, \dots, k\} \tag{4.40}$$

and define $\mathscr{P}_k(\mathbf{x}) := \mathscr{P}_k / \sim_{\mathbf{x}}$. Then we have

$$|\mathscr{P}_k(\mathbf{x})| = \frac{k!}{\prod_{i \in \mathbb{Z}^d} \xi_i(\mathbf{x})!} \tag{4.41}$$

For each $\sigma \in \mathscr{P}_k(\mathbf{x})$ we define the new coordinate vector $\mathbf{x}^{(\sigma)}$ such that

$$\mathbf{x}_i^{(\sigma)} = x_{\sigma(i)} \tag{4.42}$$

thus we can write

$$p_t(\xi(\mathbf{x}), \xi(\hat{\tau}_z\mathbf{x})) = \sum_{\mathbf{x}':\xi(\mathbf{x}')=\xi(\hat{\tau}_z\mathbf{x})} p_t(\mathbf{x}, \mathbf{x}') = \sum_{\sigma\in\mathscr{P}_k(\mathbf{x})} p_t(\mathbf{x}, \hat{\tau}_z\mathbf{x}^{(\sigma)}) \tag{4.43}$$

With a slight abuse of notation we denote by

$$X_n(\mathbf{x}, \eta, \varphi) := \sum_{z\in\mathbb{Z}^d} \varphi\left(\frac{z}{n}\right) \mathcal{D}(\hat{\tau}_z\mathbf{x}, \eta), \tag{4.44}$$

the k-th-order fluctuation field associated to the k-particles configuration $\mathbf{x}$. Then we have

Theorem 4.2.1. *Let $k := ||\mathbf{x}||$, then the k-th-order fluctuation field $X_n(\mathbf{x}, \eta, \varphi)$ is such that*

1. *For all $t > 0$*

$$\begin{aligned}&\mathbb{E}_{\nu_{\bar\rho}}\left[X_n(\mathbf{x}, \eta(t), \varphi) X_n(\mathbf{x}, \eta(0), \varphi)\right]\\ &= a(\xi(\mathbf{x})) \sum_{\sigma\in\mathscr{P}_k(\mathbf{x})} \sum_{y,z\in\mathbb{Z}^d} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n}) p_t(\mathbf{x}, \hat{\tau}_{z-y}\mathbf{x}^{(\sigma)})\end{aligned} \tag{4.45}$$

2. *As a consequence, for $t > 0$*

$$\begin{aligned}&\lim_{n\to\infty} n^{d(k-2)} \mathbb{E}_{\nu_{\bar\rho}}\left[X_n(\mathbf{x}, \eta(n^2t), \varphi) X_n(\mathbf{x}, \eta(0), \varphi)\right]\\ = \;& |\mathscr{P}_k(\mathbf{x})| a(\xi(\mathbf{x})) \frac{d^{k/2}}{(2\pi t)^{dk/2}} \int_{\mathbb{R}^{2d}} e^{-kd|z-y|^2/2t} \varphi(z)\varphi(y) dz dy\end{aligned} \tag{4.46}$$

Proof. The first statement of the theorem is a direct application of Lemma 2.2.1 and the fact that the function $a(\cdot)$ is translation invariant, i.e. $a(\xi(\hat{\tau}_z\mathbf{x})) = a(\xi(\mathbf{x}))$, for all $z \in \mathbb{Z}^d$.

$$\begin{aligned}&\mathbb{E}_{\nu_{\bar\rho}}\left[X_n(\mathbf{x}, \eta(t), \varphi) X_n(\mathbf{x}, \eta(0), \varphi)\right]\\ &= a(\xi(\mathbf{x})) \sum_{y,z\in\mathbb{Z}^d} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n})\, p_t(\xi(\hat{\tau}_y\mathbf{x}), \xi(\hat{\tau}_z\mathbf{x}))\end{aligned} \tag{4.47}$$

Then, from (4.38) and (4.47) it follows that

$$\begin{aligned}&\mathbb{E}_{\nu_{\bar\rho}}\left[X_n(\mathbf{x}, \eta(t), \varphi) X_n(\mathbf{x}, \eta(0), \varphi)\right]\\ &= a(\xi(\mathbf{x})) \sum_{\sigma\in\mathscr{P}_k(\mathbf{x})} \sum_{y,z\in\mathbb{Z}^d} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n})\, p_t(\mathbf{x}, \hat{\tau}_{z-y}\mathbf{x}^{(\sigma)})\end{aligned} \tag{4.48}$$

For the second statement observe that from translation invariance we have

$$p^{\mathrm{IRW}}_{n^2t}(\mathbf{x}, \hat{\tau}_{z-y}\mathbf{x}) = \left(p^{RW}_{n^2t}(z-y)\right)^k \tag{4.49}$$

Define $B_{M,n} := \{x \in \mathbb{Z}^d : |x| \leq nM\}$, then, since φ has a finite support, we have that there exists $M \geq 0$ such that

$$\begin{aligned}
&\sum_{y,z\in\mathbb{Z}^d} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n})\, p^{\mathrm{IRW}}_{n^2t}(\mathbf{x}, \hat{\tau}_{z-y}\mathbf{x}) \\
&= \sum_{y,z\in B_{M,n}} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n}) \left(p^{RW}_{n^2t}(z-y)\right)^k \\
&= \left(\frac{\sqrt{d}}{(2\pi t)^{d/2}}\right)^k \left(1+\frac{c}{n\sqrt{t}}\right)^k \frac{1}{n^{kd}} \sum_{y,z\in B_{M,n}} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n})\, e^{-\frac{kd|\frac{z}{n}-\frac{y}{n}|^2}{2t}}
\end{aligned}$$

for a suitable $c = c(M)$, the last equality coming from Theorem C.1.2. We have

$$\lim_{n\to\infty} \tfrac{1}{n^{2d}} \sum_{y,z\in\mathbb{Z}^d} \varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n})\, e^{-\frac{kd|\frac{z}{n}-\frac{y}{n}|^2}{2t}} = \int_{\mathbb{R}^{2d}} \varphi(y)\,\varphi(z)\; e^{-\frac{kd|z-y|^2}{2t}}\, dxdz$$

□

4.2.2.1 Quantitative Boltzmann-Gibbs principle

In the same spirit as Corollary 4.2.1, we can now state a refined quantitative version of the Boltzmann-Gibbs principle for higher-order fields.

Theorem 4.2.2. *Let $k \geq 3$, and $\mathbf{x}$ with $||\mathbf{x}|| = k$, then the k-th-order fluctuation field $X_n(\mathbf{x}, \eta, \varphi)$ is such that for all $T > 0$, there exist $C_1(T)$ and $C_2(T, k, d)$ such that, for all n big enough*

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n(\mathbf{x}, \eta(n^2t), \varphi)\, dt\right)^2\right] \leq \begin{cases} C_1(T)\, \frac{\log(n)}{n^2} & \text{if } k=3 \text{ and } d=1 \\ C_2(T,k,d)\, n^{-2} & \text{else} \end{cases} \tag{4.50}$$

Proof. Analogously to the case of two particles (see the proof of Corollary 4.2.1), and using observation (4.49) we first obtain the following estimate

$$\begin{aligned}
&\frac{1}{n^d}\mathbb{E}_{\nu_{\bar\rho}}\left[X_n(\mathbf{x}, \eta(n^2t), \varphi) X_n(\mathbf{x}, \eta(n^2s), \varphi)\right] \\
&\leq \left(p^{RW}_{n^2(t-s)}(0)\right)^{k-1} |\mathscr{P}_k(\mathbf{x})| a(\xi(\mathbf{x}))\|\varphi\|_1\|\varphi\|_\infty
\end{aligned}$$

Similar to the case of two particles, with a change of variables in the integration by time, we obtain

$$\mathbb{E}_{\nu_{\bar{\rho}}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n(\mathbf{x},\eta(n^2t),\varphi)\,dt\right)^2\right] \le C_{\rho,\varphi}\int_0^T\int_0^t (p^{RW}_{n^2r}(0,0))^{k-1}\,dr\,dt \tag{4.51}$$

where

$$C_{\rho,\varphi} = 2|\mathscr{P}_k(\mathbf{x})|a(\xi(\mathbf{x}))\|\varphi\|_1\|\varphi\|_\infty$$

By the LCLT

$$\left(p^{RW}_{n^2r}(0)\right)^{k-1} \le \begin{cases} \frac{d}{n^{(k-1)d}r^{(k-1)d/2}}, & \text{if } |r|\ge \epsilon_n \\ 1, & \text{otherwise}\end{cases}$$

where again the idea is to optimize ϵ_n.

We now have two cases:

Case I: $k=3$ **and** $d=1$

In this case we have the bound

$$\left(p^{RW}_{n^2r}(0)\right)^2 \le \begin{cases} \frac{1}{n^2r}, & \text{if } |r|\ge \epsilon_n \\ 1, & \text{otherwise}\end{cases}$$

Analogously to the case $k=2$ and $d=2$ we have that, for n large enough

$$\int_0^t (p^{RW}_{n^2r}(0,0))^{k-1}\,dr \le \epsilon_n + \frac{1}{n^2}\left(\log(n)-\log(\epsilon_n)\right)$$

Assume ϵ_n is of the form $n^{-\alpha}$ then we obtain

$$\int_0^t (p^{RW}_{n^2r}(0,0))^{k-1}\,dr \quad \le \quad n^{-\alpha} + \frac{(\alpha+1)}{n^2}\log(n) \tag{4.52}$$

We conclude by taking for example $\alpha=3$, in this case substituting this bound in the RHS of (4.51) gives

$$\mathbb{E}_{\nu_{\bar{\rho}}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n(\mathbf{x},\eta(n^2t),\varphi)\,dt\right)^2\right] \le 5\,TC_{\rho,\varphi}\,\frac{\log(n)}{n^2} \tag{4.53}$$

Case II: $(k-1)d \geq 2$

This case is completely analogous to the case of two particles and $d \geq 3$. Here we have

$$\begin{aligned}\int_0^t (p_{n^2r}^{RW}(0,0))^{k-1}\,dr &\leq \frac{1}{n^2}\,\mathbb{E}_0^{1,(k-1)d}\left[l_{n^2t}^{(0)}\right] \\ &\leq \frac{1}{n^2}\,\mathbb{E}_0^{1,(k-1)d}\left[l_{\infty}^{(0)}\right] \qquad (4.54)\end{aligned}$$

where the expectation in the RHS of (4.54) can, by independence of the $k-1$ walkers, be interpreted as the expected local time at the origin of a $(k-1)d$-dimensional random walker.

Recall that in this case the random walker is transient by the condition $(k-1)d \geq 3$ and hence the expectation of the local time is of order $O(1)$.

Finally, substituting (4.54) in the RHS of (4.51) gives

$$\mathbb{E}_{\nu_{\bar{\rho}}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n(\mathbf{x},\eta(n^2t),\varphi)\,dt\right)^2\right] \leq C_2(T,k,d)\,n^{-2} \qquad (4.55)$$

where

$$C_2(T,k,d) = T\,C_{\rho,\varphi}\,\mathbb{E}_0^{1,(k-1)d}\left[l_{\infty}^{(0)}\right] \qquad (4.56)$$

□

4.2.3 Fluctuation Fields of projections on $\mathscr{H}_k$

We can further generalize part (2) of Theorem 4.2.1 to a wider class of functions f. In this section we make such a generalization for a particular subset of $L^2(\nu_\rho)$. For $f \in L^2(\nu_\rho)$ we can use the fact that the union of the spaces $\mathscr{H}_l$ is dense in $L^2(\nu_\rho)$ to express f as follows

$$f(\eta) = \sum_{\substack{l\geq 0\\ \xi\in\Omega_f:\|\xi\|=l}} C_{l,\xi}\,\mathcal{D}(\xi,\eta) \qquad (4.57)$$

For the rest of this section we restrict ourselves to the set of functions $f \in L^2(\nu_\rho)$ satisfying the following condition

$$\sum_{\xi,\xi'\in\Omega_f:\|\xi\|=\|\xi'\|} |C_{l,\xi}C_{l,\xi'}|a(\xi') < \infty \qquad (4.58)$$

In particular all linear combinations of orthogonal duality polynomials satisfy (4.58).

Theorem 4.2.3. *Let f be a function such that the condition (4.58) is satisfied, and as before let f_{k-1} denote the projection of f on $\mathscr{H}_{k-1}$, then the field*

$$\mathsf{X_n}(f, k-1, \eta; \varphi) = \sum_{x \in \mathbb{Z}^d} (\tau_x f(\eta) - \tau_x f_{k-1}(\eta))\varphi(\tfrac{x}{n})$$

satisfies

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\mathsf{X_n}(f, k-1, \eta; \varphi)\mathsf{X_n}(f, k-1, \eta(n^2t); \varphi)\right] = O(n^{-d(k-2)})$$

Proof. After some simplifications due to orthogonality the field reads

$$\mathsf{X_n}(f, k-1, \eta; \varphi) = \sum_{x \in \mathbb{Z}^d} \varphi(\tfrac{x}{n}) \sum_{\substack{l \geq k \\ \xi \in \Omega_f : \|\xi\| = l}} C_{l,\xi} \tau_x \mathcal{D}(\xi, \eta)$$

We then compute

$$\begin{aligned} &\mathbb{E}_{\nu_{\bar\rho}}\left[\mathsf{X_n}(f, k-1, \eta; \varphi)\mathsf{X_n}(f, k-1, \eta(n^2t); \varphi)\right] \\ = \;& \sum_{x,y} \varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n}) \sum_{\substack{j,l \geq k \\ \xi \in \Omega_f : \|\xi\| = j \\ \xi' \in \Omega_f : \|\xi'\| = l}} C_{j,\xi} C_{l,\xi'}\, \mathscr{I}(x, y, \xi, \xi') \end{aligned}$$

where

$$\mathscr{I}(x, y, \xi, \xi') = \int \tau_x \mathcal{D}(\xi, \eta) E_\eta \left[\tau_y \mathcal{D}(\xi', \eta(n^2t))\right] d\nu_{\bar\rho}(\eta)$$

Then by Lemma 2.2.1 we have:

$$\begin{aligned} &\mathbb{E}_{\nu_{\bar\rho}}\left[\mathsf{X_n}(f, k-1, \eta; \varphi)\mathsf{X_n}(f, k-1, \eta(n^2t); \varphi)\right] \\ = \;& \sum_{x,y} \varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n}) \sum_{\substack{j \geq k \\ \xi \in \Omega_f : \|\xi\| = j \\ \xi' \in \Omega_f : \|\xi'\| = j}} C_{j,\xi}\, C_{j,\xi'}\, a(\xi')\, p_{n^2t}(\tau_y \xi', \tau_x \xi) \end{aligned}$$

From the LCLT we can also obtain that

$$p_{n^2t}(\tau_y \xi, \tau_x \xi') = \mathscr{O}(n^{-d\|\xi\|})$$

This, allows us to bound our expression of interest

$$\begin{aligned}
& n^{d(k-2)}\mathbb{E}_{\nu_{\bar\rho}}\left[\mathsf{X}_{\mathsf{n}}(f,k-1,\eta;\varphi)\mathsf{X}_{\mathsf{n}}(f,k-1,\eta(n^2t);\varphi)\right] \\
\leq \; & n^{d(k-2)}\sum_{x,y}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n}) \sum_{\substack{j\geq k \\ \xi\in\Omega_f:\|\xi\|=j \\ \xi'\in\Omega_f:\|\xi'\|=j}} \frac{M}{n^{dj}}|C_{j,\xi}C_{j,\xi'}|a(\xi') \\
= \; & \left(\frac{1}{n^{2d}}\sum_{x,y}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})\right) \sum_{\substack{j\geq k \\ \xi\in\Omega_f:\|\xi\|=j \\ \xi'\in\Omega_f:\|\xi'\|=j}} \frac{M}{n^{d(j-k)}}|C_{j,\xi}C_{j,\xi'}|a(\xi')
\end{aligned} \tag{4.59}$$

At this point we need to show that the last summation does not play a role in the leading order. But this comes from the fact that f satisfies condition (4.58). □

Analogously to Theorem 4.2.2 we provide a quantitative version of the Boltzmann-Gibbs principle for the current setting.

Theorem 4.2.4. *The field $\mathsf{X}_{\mathsf{n}}(f,k-1,\eta;\varphi)$ is such that, for all $T>0$, there exist $C_1(T), C_2(T), C_3(T)$ and $C_4(T,k,d)$ such that*

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T \mathsf{X}_{\mathsf{n}}(f,k-1,\eta(n^2t);\varphi)\,dt\right)^2\right] \leq \begin{cases} C_1(T)\,n^{-1} & \text{if } k=2 \text{ and } d=1 \\ C_2(T)\,\frac{\log(n)}{n^2} & \text{if } k=2 \text{ and } d=2 \\ C_3(T)\,\frac{\log(n)}{n^2} & \text{if } k=3 \text{ and } d=1 \\ C_4(T,k,d)\,n^{-2} & \text{else} \end{cases} \tag{4.60}$$

for all n big enough.

Proof. The proof is an exact replica of the arguments given in the proofs of Theorem 4.2.2 and Corollary 4.2.1. □

4.3 Non-stationary fluctuation fields

4.3.1 Second-order fields

Let us now start independent walkers from a product measure of non-homogeneous Poisson, with weakly varying density profile i.e., from the measure $\nu_{\bar\rho} = \otimes_{x\in\mathbb{Z}^d}\nu_{\rho(x)}$ where $\bar\rho\in\mathbb{R}^{\mathbb{Z}^d}$ and $\rho(x)$ is given by the relation $\bar\rho = (\rho(x))_{x\in\mathbb{Z}^d}$. We denote by $\mathcal{D}_{\bar\rho}$ the orthogonal polynomials, i.e.,

$$\mathcal{D}_{\bar\rho}(\xi,\eta) = \prod_{i\in\mathbb{Z}^d}\mathcal{D}_{\rho(i)}(\xi_i,\eta_i)$$

where $\mathcal{D}_{\rho(i)}$ denote the orthogonal polynomials w.r.t. Poisson with parameter $\rho(i)$.
We also denote by $\bar{\rho}_t = (\rho(x))_{x\in\mathbb{Z}^d}$, where $\rho_t(x) = \mathbb{E}_x\left[\rho(X_t)\right]$ and X_t denotes the continuous-time random walk. We now are interested in the fields

$$X_n(\xi,\bar{\rho},\varphi,t) := \sum_{x\in\mathbb{Z}^d} \varphi(\tfrac{x}{n})\mathcal{D}_{\bar{\rho}_{tn^2}}(\xi,\eta(n^2t)) \tag{4.61}$$

then the second-order field is

$$X_n^{(2)}(\bar{\rho},\varphi,t) := X_n(2\delta_0,\bar{\rho},\varphi,t) = \sum_x \varphi(\tfrac{x}{n})\mathcal{D}_{\bar{\rho}_{tn^2}}(2\delta_x,\eta(n^2t)) \tag{4.62}$$

Compared to previous notation, please notice the additional dependence on the parameter $\bar{\rho}$ and on time t.

We want to prove that the covariance of $X_n^{(2)}(\bar{\rho},\varphi,t)$ and $X_n^{(2)}(\bar{\rho},\varphi,s)$ is of order 1, as $N\to\infty$, exactly as in the stationary case. For this we start with the following result:

Lemma 4.3.1. *Let $\nu_{\bar{\rho}} := \otimes_{x\in\mathbb{Z}^d}\nu_{\rho(x)}$ be a product of non-homogeneous Poisson measures, then we have*

$$\int \mathbb{E}_\eta\left[\mathcal{D}_{\bar{\rho}_t(x)}(2\delta_x,\eta(t))\right]\mathcal{D}_{\rho(y)}(2\delta_y,\eta)\,d\nu_{\bar{\rho}}(\eta) = k_2(y)\,p_t(x,y)^2 \tag{4.63}$$

where

$$k_2(y) = \int \Big(\mathcal{D}_{\rho(y)}(2\delta_y,\eta)\Big)^2\,d\nu_{\bar{\rho}}(\eta)$$

Proof. Note that

$$\mathcal{D}_{\rho_t(x)}(2\delta_x,\eta_t) = \eta_x(t)(\eta_x(t)-1) - 2\rho_t(x)(\eta_x(t)-\rho_t(x)) - \rho_t(x)^2$$

hence

$$\begin{aligned} &\mathbb{E}_\eta\left[\mathcal{D}_{\bar{\rho}_t(x)}(2\delta_x,\eta_t)\right] \\ =\;& \mathbb{E}_\eta\left[\eta_x(t)(\eta_x(t)-1)\right] - 2\rho_t(x)\mathbb{E}_\eta\left[\eta_x(t)-\rho_t(x)\right] - \rho_t(x)^2 \end{aligned} \tag{4.64}$$

We now state the following:
Claim 1:

$$\int \mathbb{E}_\eta\left[\eta_x(t)-\rho_t(x)\right]\mathcal{D}_{\rho(y)}(2\delta_y,\eta)\,d\nu_{\bar{\rho}}(\eta) = 0$$

Indeed, by duality, $\mathbb{E}_\eta\left[\eta_x(t)-\rho_t(x)\right] = \sum_z p_t(x,z)(\eta_z-\rho(z))$ and the polynomial $(\eta_z-\rho(z))$ is in $L^2(\nu_{\bar{\rho}}(\eta))$ always orthogonal to $\mathcal{D}_{\rho(y)}(2\delta_y,\eta)$ because for $z\neq y$ both $(\eta_z-\rho(z))$ and $\mathcal{D}_{\rho(y)}(2\delta_y,\eta)$ have expectation zero, and when

$z = y$ because it is the inner product of the first order and second-order orthogonal polynomials, which is zero. So we only have to work out the expectation $\mathbb{E}_\eta\left[\eta_x(t)(\eta_x(t)-1)\right]$ which by duality equals

$$\sum_u p_t(x,u)^2\eta_u(\eta_u-1) + 2\sum_{u\neq v} p_t(x,u)p_t(x,v)\eta_u\eta_v$$

Claim 2:
For all u

$$\int \eta_u \mathcal{D}_{\rho(y)}(2\delta_y,\eta)d\nu_{\bar\rho}(\eta) = 0$$

Indeed, for $u \neq y$ this is true because of the product character of the measure and the fact that $\mathcal{D}_{\rho(y)}(2\delta_y,\eta)$ has zero expectation, and for $u = y$, $\eta_y = \eta_y - \rho(y) + \rho(y)$ which is the sum of the first orthogonal polynomial and a constant, which is in $L^2(\nu_{\bar\rho}(\eta))$ orthogonal to $\mathcal{D}_{\rho(y)}(2\delta_y,\eta)$.

Finally, we remark that for all $u \neq y$

$$\int \eta_u(\eta_u-1)\mathcal{D}_{\rho(y)}(2\delta_y,\eta)d\nu_{\bar\rho}(\eta) = 0$$

because of the product character of the measure and the fact that the polynomial $\mathcal{D}_{\rho(y)}(2\delta_y,\eta)$ has zero expectation. Finally,

$$\int \eta_y(\eta_y-1)\mathcal{D}_{\rho(y)}(2\delta_y,\eta)d\nu_{\bar\rho}(\eta) = \int (\mathcal{D}_{\rho(y)}(2\delta_y,\eta))^2 d\nu_{\bar\rho}(\eta)$$

because adding first order terms in η_y does not change the inner product with $\mathcal{D}_{\rho(y)}(2\delta_y,\eta)$. $\square$

As a consequence of Lemma 4.3.1 and using that a product of Poisson measures is reproduced at later times, we compute

$$\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}_{\nu_{\bar\rho}}\left[X_n^{(2)}(\bar\rho,\varphi,t)X_n^{(2)}(\bar\rho,\varphi,s)\right] \\
= & \lim_{n\to\infty} \mathbb{E}_{\nu_{\bar\rho_{sn^2}}}\left[X_n^{(2)}(\bar\rho,\varphi,t-s)X_n^{(2)}(\bar\rho,\varphi,0)\right] \\
= & \int \frac{e^{-\frac{(x-y)^2}{t-s}}}{2\pi(t-s)^{d/2}}\,\varphi(x)\varphi(y)\,\kappa_2(y)dxdy
\end{aligned} \tag{4.65}$$

where

$$\kappa_2(y) = \lim_{n\to\infty} k_2(ny)$$

which exists because the initial Poisson measure has a slowly varying density profile.

4.3.2 Higher-order fields: Non-stationary case

The aim of this section is to extend the results of the previous example to higher-order fields:

$$X_n(\mathbf{x}, \bar{\rho}, \varphi, t) = \sum_{x\in\mathbb{Z}^d} \varphi(\tfrac{x}{n}) \mathcal{D}_{\bar{\rho}_{tn^2}}(\hat{\tau}_{\mathbf{x}}\xi, \eta(n^2 t)) \tag{4.66}$$

We start then with a generalization of Lemma 4.3.1 to higher orders. As we already stated in Remark 2.2.6 in the case of independent random walkers, the orthogonal duality polynomials are related to the classical duality polynomials in the following way:

$$\mathcal{D}_{\bar{\rho}}(\xi, \eta) = \prod_{x\in\mathbb{Z}^d} \sum_{j=0}^{\xi_x} \binom{\xi_x}{j} (-\rho(x))^{\xi_x - j} d(j, \eta_x) \tag{4.67}$$

where $d(k, n)$ are the classical single-site duality polynomials.

Remark 4.3.1. *Notice that due to the non-homogeneity of the product measure, the duality property cannot be any longer guaranteed.*

Despite the previous remark, the special form of the Charlier polynomials allows us to reach the same conclusions as in the stationary case. Let us first make a simple observation:
Define $A(\xi, \eta, \bar{\rho})$ as the difference between the Charlier and classical polynomials of order $\|\xi\|$, i.e.

$$A(\xi, \eta, \bar{\rho}) := \mathcal{D}_{\bar{\rho}}(\xi, \eta) - \prod_{x\in\mathbb{Z}^d} d(\xi_x, \eta_x)$$

and notice that $A(\xi, \eta, \bar{\rho})$ is a polynomial of degree strictly less than $\|\xi\|$ and as a consequence it has an expansion, in terms of orthogonal polynomials, consisting only of polynomials of order strictly smaller than $\|\xi\|$. Therefore, by orthogonality we have

$$\int \mathbb{E}_\eta \left[A(\xi, \eta, \rho) \right] \mathcal{D}_{\rho_0}(\xi', \eta) d\nu_{\bar{\rho}_0}(\eta) = 0$$

for any configuration ξ' such that $\|\xi\| \le \|\xi'\|$. With this observation we are ready to state the following Lemma:

Lemma 4.3.2. *Let $\nu_{\bar{\rho}} := \otimes_{x\in\mathbb{Z}^d} \nu_{\rho(x)}$ be a product of non-homogeneous Poisson measures, and let $\rho_t(x) = \mathbb{E}_x \left[\rho(X_t) \right]$, where X_t denotes continuous-time random walk. Then we have*

$$\int \mathbb{E}_\eta \left[\mathcal{D}_{\bar{\rho}_t}(\xi, \eta(t)) \right] \mathcal{D}_{\bar{\rho}}(\xi', \eta) d\nu_{\bar{\rho}}(\eta) = p_t(\xi, \xi') a_0(\xi') \tag{4.68}$$

where $a_t(\xi) = \|\mathcal{D}_{\bar{\rho}_t}(\xi, \cdot)\|^2_{L^2(\nu_{\bar{\rho}})}$

Proof. We simply compute

$$
\begin{aligned}
& \int \mathbb{E}_\eta \left[\mathcal{D}_{\bar\rho_t}(\xi, \eta(t))\right] \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \int \mathbb{E}_\eta \left[\prod_x \sum_{j=0}^{\xi_x} \binom{\xi_x}{j} (-\rho_t)^{\xi_x - j} d(j, \eta(x,t))\right] \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \int \mathbb{E}_\eta \left[\prod_x d(\xi_x, \eta(x,t))\right] \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
+ & \int \mathbb{E}_\eta \left[A(\xi, \eta, \bar\rho)\right] \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \int \mathbb{E}_\xi \left[\prod_x d(\xi(x,t), \eta_x)\right] \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \int \sum_\zeta p_t(\xi, \zeta) \left(\prod_x d(\zeta_x, \eta_x) + A(\zeta, \eta, \bar\rho)\right) \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \int \sum_\zeta p_t(\xi, \zeta) \left(\prod_x \sum_{j=0}^{\zeta_x} \binom{\zeta_x}{j} (-\rho(x))^{\zeta_x - j} d(j, \eta_x)\right) \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \int \sum_\zeta p_t(\xi, \zeta) \mathcal{D}_{\bar\rho}(\zeta, \eta) \mathcal{D}_{\bar\rho}(\xi', \eta) d\nu_{\bar\rho}(\eta) \\
= & \; p_t(\xi, \xi') a_0(\xi') \qquad (4.69)
\end{aligned}
$$

where in the fourth and fifth line we subtracted and added zero respectively by using the orthogonality of $\mathcal{D}_{\bar\rho}(\xi', \eta)$ to lower-order polynomials in the expansion. □

We now state the non-stationary version of Theorem 4.2.1

Theorem 4.3.1. *Let $\nu_{\bar\rho} := \otimes_{x \in \mathbb{Z}^d} \nu_{\rho(x)}$ and $\rho_t(x)$ be as before, and let $k := ||\mathbf{x}||$, then*

1. *For all $t > 0$*

$$
\begin{aligned}
& \mathbb{E}_{\nu_{\bar\rho}} \left[X_n(\mathbf{x}, \bar\rho, \varphi, t) X_n(\mathbf{x}, \bar\rho, \varphi, 0)\right] \\
= & \; a_0\left(\sum_{i=1}^k \delta_{x_i}\right) \sum_{x,y} \varphi(\tfrac{x}{n}) \varphi(\tfrac{y}{n}) p_t \left(\sum_{i=1}^k \delta_{x+x_i}; \sum_{i=1}^k \delta_{y+x_i}\right)
\end{aligned}
\qquad (4.70)
$$

2. As a consequence, for $t > s > 0$

$$\begin{aligned}&\lim_{n\to\infty} N^{d(k-2)}\mathbb{E}_{\nu_{\bar\rho}}\left[X_n(\mathbf{x},\bar\rho,\varphi,t)X_n(\mathbf{x},\bar\rho,\varphi,s)\right]\\ =\;& \mathscr{K}(\mathbf{x};\rho)\frac{d^{k/2}}{(2\pi(t-s))^{dk/2}}\int_{\mathbb{R}^2} e^{-kd(x-y)^2/2(t-s)}\varphi(x)\varphi(y)dxdy\end{aligned}$$

with $\xi = \sum_{i=1}^k \delta_{x_i}$ and $\mathscr{K}(\mathbf{x};\rho)$ defined as in the stationary case.

Proof. Is a consequence of Lemma 4.3.2 together with the fact that a product of Poisson measure is reproduced at later times. We refer to Section 3 in Chapter I of [58] for a proof of the fact that local equilibrium is conserved by the time evolution in the case of IRW. □

With this last theorem, we have now the ingredients to obtain a quantitative Boltzmann-Gibbs principle

Corollary 4.3.1. *For all $T > 0$, there exist $C_1(T), C_2(T), C_3(T)$ and $C_4(T,k,d)$ such that for all n big enough*

$$\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_0^T X_n(\mathbf{x},\bar\rho,\varphi,t)\,dt\right)^2\right] \le \begin{cases} C_1(T)\,n^{-1} & \text{if } k=2 \text{ and } d=1\\ C_2(T)\,\frac{\log(n)}{n^2} & \text{if } k=2 \text{ and } d=2\\ C_3(T)\,\frac{\log(n)}{n^2} & \text{if } k=3 \text{ and } d=1\\ C_4(T,k,d)\,n^{-2} & \text{else}\end{cases} \tag{4.71}$$

Proof. The proof is essentialy the same than in all the previous cases. □

4.4 QBGP beyond independent random walkers

In the context of stationarity, the results of this Chapter are not exclusive for Independent Random Walkers. Hence in this section, we extend our results to a wider class of IPS. i.e., to those particle systems that enjoy the existence of orthogonal self-duality with respect to a stationary measure ν_ρ and that satisfy an additional condition in the transition kernel. Let then $\{\eta_t\}_{t\ge 0}$ be an IPS for which there exists an orthogonal self-duality function $\mathcal{D}: \Omega_f \times \Omega \to \mathbb{R}$ satisfying all the properties stated in section 2.2.3. As in the same section, we denote by $p_t(\xi,\xi')$ the transition probability to go from configuration ξ to ξ' in time t. Then this is enough to guarantee the validity of Lemma 2.2.1. Namely,

Lemma 4.4.1. *Let $\xi,\xi' \in \Omega_f$, then*

$$\int \mathbb{E}_\eta(\mathcal{D}(\xi,\eta_t))\mathcal{D}(\xi',\eta)d\nu_\rho(\eta) = p_t(\xi,\xi')a(\xi') \tag{4.72}$$

for all $\xi,\xi' \in \Omega_f$.

Furthermore, let us assume the following:

Assumption 1. *For all $\xi, \xi' \in \Omega_f$, the transition kernel satisfies the following estimate*

$$p_t(\xi, \xi') \le \frac{C}{(1+t)^{\|\xi\| d/2}} \tag{4.73}$$

This assumption is reasonable since in [67] estimates of this kind were already found for a wide class of interacting particle systems that, for example, includes generalized exclusion processes. The results of [67] are applicable as long as the process satisfies a logarithmic Sobolev inequality for the symmetric part of the generator. As before, for a fixed $\mathbf{x} \in \mathbb{Z}^{dk}$ we define the polynomial fluctuation field

$$X_n(\mathbf{x}, \eta, \varphi) := \sum_{z \in \mathbb{Z}^d} \varphi\left(\frac{z}{n}\right) \mathcal{D}(\hat{\tau}_z \mathbf{x}, \eta), \tag{4.74}$$

Then, from assumption (4.73) we can also conclude the following theorem

Theorem 4.4.1. *For all $T > 0$ there exist $C_1(T), C_2(T), C_3(T)$ and $C_4(T, x, d)$ such that, for all n big enough*

$$\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{n^{d/2}} \int_{[0,T]^2} X_n(\mathbf{x}, \eta(n^2 t), \varphi)\, dt\right)^2\right] \le \begin{cases} C_1(T)\, n^{-1} & \text{if } k = 2 \text{ and } d = 1 \\ C_2(T)\, \frac{\log(n)}{n^2} & \text{if } k = 2 \text{ and } d = 2 \\ C_3(T)\, \frac{\log(n)}{n^2} & \text{if } k = 3 \text{ and } d = 1 \\ C_4(T, k, d)\, n^{-2} & \text{else} \end{cases} \tag{4.75}$$

for all $x \in \mathbb{Z}^{dk}$.

4.4.1 BGP via local times and Green functions

If we want to show results of the BGP type for systems away from LCLT's or heat kernel estimates, we can still say something, but in this case, we lose the quantitative nature of the results.

Let again $\{\eta_t\}_{t \ge 0}$ be an IPS for which there exists an orthogonal self-duality function $\mathcal{D} : \Omega_f \times \Omega \to \mathbb{R}$ satisfying all the properties stated in Section 2.2.3. In the language of Section 4.2.2 we let

$$X_n(\mathbf{x}, \eta, \varphi) := \sum_{z \in \mathbb{Z}^d} \varphi\left(\frac{z}{n}\right) \mathcal{D}(\tau_z \mathbf{x}, \eta), \tag{4.76}$$

define the k-th-order fluctuation field associated to the k-particles configuration given by $\mathbf{x} \in \mathbb{Z}^{dk}$. We then have the following Boltzmann-Gibbs principle

Theorem 4.4.2. *For all $T > 0$ we have*

$$\lim_{n\to\infty} \mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{n^{d/2}}\int_{[0,T]} X_n(\mathbf{x},\eta(n^2t),\varphi)\,ds\right)^2\right] = 0 \tag{4.77}$$

for all $\mathbf{x} \in \mathbb{Z}^{dk}$.

Sketch of the proof

By an application of Lemma 4.4.1 and the fact that the function $a(\cdot)$ is translation invariant, i.e. $a(\xi(\hat{\tau}_z\mathbf{x})) = a(\xi(\mathbf{x}))$, for all $z \in \mathbb{Z}^d$, we have:

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{n^{d/2}}\int_{[0,T]} X_n(\mathbf{x},\eta(n^2t),\varphi)\,ds\right)^2\right] \\
&= \frac{2}{n^d}\int_0^T\int_0^t a(\xi(\mathbf{x}))\sum_{\sigma\in\mathscr{P}_k(\mathbf{x})}\sum_{y,z\in\mathbb{Z}^d}\varphi(\tfrac{y}{n})\varphi(\tfrac{z}{n})\,p_{n^2(t-s)}(\tau_y\mathbf{x},\tau_z\mathbf{x}^{(\sigma)})\,ds\,dt \\
&\le \frac{2}{n^d}\int_0^T\int_0^T a(\xi(\mathbf{x}))\sum_{\sigma\in\mathscr{P}_k(\mathbf{x})}\sum_{y,z\in\mathbb{Z}^d}\left|\varphi\left(\tfrac{y}{n}\right)\right|\left|\varphi\left(\tfrac{z}{n}\right)\right|\,p_{n^2r}(\tau_y\mathbf{x},\tau_z\mathbf{x}^{(\sigma)})\,dr\,dt \\
&\le \frac{2T}{n^d}\int_0^T a(\xi(\mathbf{x}))\sum_{\sigma\in\mathscr{P}_k(\mathbf{x})}\sum_{y,z\in\mathbb{Z}^d}\left|\varphi\left(\tfrac{y}{n}\right)\right|\left|\varphi\left(\tfrac{z}{n}\right)\right|\,p_{n^2r}(\tau_y\mathbf{x},\tau_z\mathbf{x}^{(\sigma)})\,dr
\end{aligned} \tag{4.78}$$

where $p_t(\mathbf{x},\tau_{z-y}\mathbf{x}^{(\sigma)})$ and $\mathscr{P}_k(\mathbf{x})$ are given in terms of the notation introduced at the beginning of Section 4.2.2.

Let us simplify the previous expression and introduce the following constant:

$$C = C(\mathbf{x},T,\rho) = 2\cdot T\cdot a_\rho(\xi(\mathbf{x})) \tag{4.79}$$

Taking the sup-norm of φ over $z \in \mathbb{Z}^d$ we obtain:

$$\begin{aligned}
&\mathbb{E}_{\nu_{\bar\rho}}\left[\left(\frac{1}{n^{d/2}}\int_{[0,T]} X_n(\mathbf{x},\eta(n^2t),\varphi)\,ds\right)^2\right] \\
&\le \frac{C\cdot\|\varphi\|_\infty}{n^d}\int_0^T\sum_{\sigma\in\mathscr{P}_k(\mathbf{x})}\sum_{y\in\mathbb{Z}^d}\left|\varphi\left(\tfrac{y}{n}\right)\right|\left(\sum_{z\in\mathbb{Z}^d}p_{n^2r}(\tau_y\mathbf{x}-\mathbf{x}^{(\sigma)},\tau_z\mathbf{0})\right)dr
\end{aligned} \tag{4.80}$$

We now rewrite the term inside brackets in the RHS of (4.80) as follows:

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} p_{n^2r}(\tau_y\mathbf{x}-\mathbf{x}^{(\sigma)},\tau_z\mathbf{0}) &= \sum_{\mathbf{z}\in\mathbb{Z}^{kd}} p_{n^2r}(\tau_y\mathbf{x}-\mathbf{x}^{(\sigma)},\mathbf{z})\,1\!\!1_{\{z_1=\ldots=z_k\}}\\ &= \mathbb{E}_{\tau_y\mathbf{x}}\left[1\!\!1_{\{X_1(n^2r)-x_{\sigma(1)}=\ldots=X_k(n^2r)-x_{\sigma(k)}\}}\right]\end{aligned} \tag{4.81}$$

where $\mathbb{E}_{\hat{\tau}_z\mathbf{x}}$ denotes the expectation of k particles in coordinate notation, each of them originally starting at position $(\tau_z\mathbf{x})_i\in\mathbb{Z}^d$.

Substitution of (4.81) in (4.80) gives:

$$\begin{aligned}&\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{n^{d/2}}\int_{[0,T]}X_n(\mathbf{x},\eta(n^2t),\varphi)\,ds\right)^2\right]\\ &\leq \frac{C\cdot\|\varphi\|_\infty}{n^d}\sum_{\sigma\in\mathscr{P}_k(\mathbf{x})}\int_0^T\sum_{y\in\mathbb{Z}^d}\left|\varphi\left(\tfrac{y}{n}\right)\right|\mathbb{E}_{\tau_y\mathbf{x}}\left[1\!\!1_{\{X_1(n^2r)-x_{\sigma(1)}=\ldots=X_k(n^2r)-x_{\sigma(k)}\}}\right]dr\end{aligned} \tag{4.82}$$

By the compact support of φ we can further estimate as follows:

$$\begin{aligned}&\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{n^{d/2}}\int_{[0,T]}X_n(\mathbf{x},\eta(n^2t),\varphi)\,ds\right)^2\right]\\ &\leq C_2\cdot\|\varphi\|_\infty^2\sum_{\sigma\in\mathscr{P}_k(\mathbf{x})}\int_0^T\mathbb{E}_{\tau_y\mathbf{x}}\left[1\!\!1_{\{X_1(n^2r)-x_{\sigma(1)}=\ldots=X_k(n^2r)-x_{\sigma(k)}\}}\right]dr\end{aligned} \tag{4.83}$$

where the constant C_2 depends on the previous constant C and the size of the box containing the support of φ.

At this point, we have two ways to show that the RHS of (4.83) indeed vanishes. We will briefly discuss them in the following un-numbered sections.

Local times

This possibility comes from the observation that, to conclude (4.77), it is enough to show that, for all y:

$$\lim_{n\to\infty}\frac{1}{n^2}\int_0^{n^2T}\mathbb{E}_{\tau_y\mathbf{x}}\left[1\!\!1_{\{X_i(s)-X_j(s)=X_{\sigma(i)}(0)-X_{\sigma(j)}(0)\quad\forall i,j\}}\right]ds=0 \tag{4.84}$$

By consistency the behaviour of the previous limit reduces to any random pair of particles. Therefore (4.84) in turn, can be achieved by showing that, for all y:

$$\lim_{n\to\infty} \frac{1}{n^2} \int_0^{n^2 t} \mathbb{E}_{\tau_y \mathbf{x}} \left[\mathbb{1}_{X_1(s)-X_2(s)=0} \right] ds = 0 \tag{4.85}$$

where the same expression is satisfied by substituting the point zero in the indicator function by any other point in space.

The experienced reader can identify the emergence of the local time at the origin of the difference of the positions of two particles in the LHS of (4.85). Therefore, condition (4.85) is easily accessible for the particle systems of interest in this thesis. This is due to the convergence of a system of two particles diffusively rescaled to two-dimensional Brownian motions.

Green functions

For the case $(k-1)d \geq 3$, there is a second possibility which comes from the observation that the RHS (without the limit) of (4.82) can be further estimated as follows:

$$\begin{aligned}
&\frac{1}{n^2} \int_0^{n^2 T} \mathbb{E}_{\tau_y \mathbf{x}} \left[\mathbb{1}_{\{X_1(n^2 r) - x_{\sigma(1)} = \ldots = X_k(n^2 r) - x_{\sigma(k)}\}} \right] ds \\
&\leq \frac{1}{n^2} \int_0^{\infty} \mathbb{E}_{\tau_y \mathbf{x}} \left[\mathbb{1}_{\{X_1(n^2 r) - x_{\sigma(1)} = \ldots = X_k(n^2 r) - x_{\sigma(k)}\}} \right] ds \\
&=: \frac{1}{n^2} G(d, k, \mathbf{x})
\end{aligned} \tag{4.86}$$

for any permutation $\sigma \in \mathscr{P}_k(\mathbf{x})$.

The function $G(d, k, \mathbf{x})$ that we just introduced should be thought of as a diagonal analogue of the Green's function for independent random walkers. Therefore, possessing information about this Green function for a given particle system (with orthogonal self-duality) should be enough to conclude or discard the limit (4.77).

As an example of what we claimed in the previous paragraph, let us consider the case of independent random walkers, for which the system is transient if $(k-1)d \geq 3$. This condition translates into $G(d, k, \mathbf{x})$ satisfying:

$$G(d, k, \mathbf{x}) < \infty \tag{4.87}$$

for every $\mathbf{x} \in \mathbb{Z}^{kd}$.

Chapter 5

Higher-order fluctuation fields and orthogonal duality polynomials

In the context of interacting particle systems with a conserved quantity (such as the number of particles) in [27, 58] one studies the time-dependent density fluctuation field

$$\mathscr{X}^{(n)}(\varphi, \eta(n^2 t)) = \frac{1}{n^{d/2}} \sum_{x \in \mathbb{Z}^d} \varphi(x/n)(\eta_x(n^2 t) - \rho).$$

Here φ denotes a test-function, and η_x the number of particles at site $x \in \mathbb{Z}^d$. The quantity $\mathscr{X}^{(n)}(\varphi, \eta(n^2 t))$ is then considered as a random time-dependent (Schwartz) distribution. In a variety of models with particle number conservation (such as zero-range processes, simple exclusion processes, etc.), this time-dependent field is proved to converge, at equilibrium, to a stationary infinite-dimensional Ornstein-Uhlenbeck process. This scaling limit behavior of the density fluctuation field can be thought of as a generalized central limit theorem, and, as such, as a correction or refinement of the hydrodynamic limit (or the ergodic theorem for the invariant measure, as it is the case of this paper), which can be thought of as a law of large numbers.

The usual strategy of proof (see e.g. Chapter 11 of [58]) is to start from the Dynkin martingale associated to the density field and prove convergence of the drift term via the Boltzmann-Gibbs principle (the drift term becomes in the scaling limit a function of the density field), and convergence of the noise term via a characterization of its quadratic variation (which becomes deterministic in the scaling limit). This then eventually leads to the informally written SPDE

$$d\mathscr{X}_t = D\Delta\mathscr{X}_t + \sigma(\rho)\nabla d\mathscr{W}_t$$

where ρ is the parameter of the invariant measure associated to the density, Δ denotes the Laplacian, and where $\sigma(\rho)\nabla d\mathscr{W}_t$ is an informal notation for Gaussian white noise with variance $\sigma^2(\rho)\int(\nabla\varphi)^2dx$.

In interacting particle systems with (self-)duality, the drift term in the equation for the density field is already microscopically (i.e., without rescaling) a (linear) function of the density field. As a consequence, closing the equation and proving convergence to the limiting Ornstein-Uhlenbeck process, is, for self-dual systems, particularly simple and does not require the use of a Boltzmann-Gibbs principle. This simplification suggests that, in that context, we can obtain more detailed results about fluctuation fields of more general observables. Orthogonal polynomial duality is a useful tool in the study of fluctuation fields, and associated Boltzmann-Gibbs principles, as we have seen in Chapter 4.

The density fluctuation field can be viewed as the lowest (i.e., first) order of a sequence of fields associated to orthogonal polynomials. Indeed, in all the models with orthogonal polynomial self-duality, the function $(\eta_x - \rho)$ is the first-order orthogonal polynomial up to a multiplicative constant. Orthogonal polynomials are indexed by finite-particle configurations, i.e., the dual configurations. If we denote by $\mathcal{D}(x_1, \dots x_k; \eta)$ the orthogonal polynomial associated to the dual configuration $\sum_{i=1}^{n}\delta_{x_i}$, then a natural field generalizing the density fluctuation field is

$$\mathscr{X}^{(n,k)}(\Phi,\eta) = n^{-kd/2}\sum_{x_i\in\mathbb{Z}^d}\mathcal{D}(x_1,\dots,x_k;\eta)\cdot\Phi\left(\tfrac{x_1}{n},\dots,\tfrac{x_k}{n}\right).$$

In the context of exclusion processes the case $k = 2$ (orthogonal polynomial of order 2) has been studied in [47], where this field, called the quadratic fluctuation field, is shown to converge, in the limit $n \to \infty$, to the solution of a martingale problem. The quadratic variation of this 2nd-order field is proven to be a function of the 1st-order field (the density field). From the result on the quadratic (k=2) field one can conjecture the existence of a more general structure where the kth-order orthogonal polynomials field satisfies, in the scaling limit, a martingale problem with quadratic variation depending on the $k-1$-order field.

In this chapter we show exactly the emergence of a scenario of this type: within a general class of models with orthogonal polynomial self-duality we consider the fluctuation fields associated to orthogonal polynomials and prove that they converge, in the scaling limit, to the solution of a recursive system of martingale problems. We believe that this can also be a first step in the direction of defining non-linear fields, such as the square of the density field, via approximation of the identity, i.e., via a singular linear observable (cf. [47]) of the field constructed in our paper.

The rest of this chapter is organized as follows. In Section 5.1 we define the basic models, and introduce orthogonal polynomial duality. In Section 5.2 we define the fluctuation fields, in Section 5.3 we introduce a coordinate version of the dual process, a technical tool that will prove to be useful later on. In Section 5.4 we state the main result, Theorem 5.4.1 below, and outline a strategy of its inductive proof. Finally, the rest of the sections are devoted to the proof of Theorem 5.4.1.

5.1 The models

5.1.1 The infinite configuration process

We consider an interacting particle system where an infinite number of particles randomly hop on the lattice $\mathbb{Z}^d$. Configurations are denoted by η, ξ, ζ and are elements of $\Omega \subseteq \mathbb{N}^{\mathbb{Z}^d}$ (where $\mathbb{N}$ denotes the natural numbers including zero). We denote by η_x the number of particles at x in the configuration $\eta \in \Omega$. We have in mind symmetric processes of the type independent random walkers, inclusion or exclusion. We fix two parameters $(\sigma, \alpha) \in \{0,1\} \times [0,\infty) \cup \{-1\} \times \mathbb{N}$ and we define the generator working on local functions $f : \Omega \to \mathbb{R}$ as

$$\mathscr{L} f(\eta) = \sum_{i \in \mathbb{Z}^d} \sum_{r \in \mathbb{Z}^d} p(r) \eta_i (\alpha + \sigma \eta_{i+r})(f(\eta^{i,i+r}) - f(\eta)) \tag{5.1}$$

where $\eta^{i,i+r}$ denotes the configuration obtained from η by removing a particle from i and putting it at $i+r$. The state space Ω has to be defined and its form depends on the choice of the parameters α and σ. We assume that $p(r)$ is a symmetric, finite-range, irreducible Markov transition function on $\mathbb{Z}^d$:

1. Symmetry. The function $p : \mathbb{R}^d \to [0, \infty)$ is of the form:

$$p(r_1, \ldots, r_d) = p(|r_1|, \ldots, |r_d|) \tag{5.2}$$

and such that $p(\mathbf{r}_\sigma) := p(r_{\sigma(1)}, \ldots, r_{\sigma(d)}) = p(r_1, \ldots, r_d)$ for all $\sigma \in \mathscr{P}(d)$, the set of permutations of $\{1, \ldots, d\}$.

2. Finite-range. There exists a finite subset $\mathscr{R} \subset \mathbb{Z}^d$ of the form $\mathscr{R} = [-R, R]^d \cap \mathbb{Z}^d$, for some $R \in \mathbb{N}$, $R > 1$, such that $p(r) = 0$ for all $r \notin \mathscr{R}$.

3. Irreducibility. For all $x, y \in \mathbb{Z}^d$ there exists a sequence $i_1 = x, \ldots, i_n = y$ such that

$$\prod_{k=1}^{n-1} p(i_k - i_{k+1}) > 0.$$

We will also assume, without loss of generality, that $p(0) = 0$, and denote by χ the second moment:

$$\chi := \sum_{r \in \mathscr{R}} r_\ell^2 \cdot p(r), \qquad \text{for all} \quad \ell \in \{1, \ldots, d\}. \tag{5.3}$$

Remark 5.1.1. *The symmetry assumption* (5.2) *is crucial in order to be able to have and apply orthogonal self-duality.*

For the associated Markov processes on Ω, we use the notation $\{\eta(t) : t \geq 0\}$, $\eta_x(t)$ denoting the number of particles at time t at location $x \in \mathbb{Z}^d$. These particle systems have a one-parameter family of homogeneous (w.r.t. translations) reversible and ergodic product measures $\nu_\rho, \rho > 0$, indexed by the density of particles, i.e.,

$$\int \eta_0 d\nu_\rho = \rho. \tag{5.4}$$

The nature of the underlying dynamics and the type of reversible measure we obtain is regulated by the parameter $\sigma \in \mathbb{Z}$ as follows.

Independent random walkers (IRW): This particle system corresponds to the choice $\sigma = 0$ and the intensity parameter $\alpha \in \mathbb{R}$ regulates the rate at which the walkers move. The reversible measures ν_ρ, $\rho > 0$ are products of Poisson distributions with parameter ρ, $\nu_\rho = \otimes_{i \in \mathbb{Z}^d} \mathrm{Pois}(\rho)$, i.e. the marginals are given by

$$\mathbb{P}_{\nu_\rho}(\eta_i = n) = \frac{1}{Z_\rho} \cdot \frac{\rho^n}{n!}, \qquad Z_\rho = e^{-\rho}, \qquad \forall\, i \in \mathbb{Z}^d.$$

Symmetric exclusion process (SEP(α)): The choice $\sigma = -1$ results in exclusion interaction. For this process the parameter α takes values in the set of natural numbers, $\alpha \in \mathbb{N}$, as it determines the maximum number of particles allowed per site. This system is well known to have reversible measures ν_ρ, $\rho \in (0, \alpha)$, that are products of Binomial distributions: $\nu_\rho = \otimes_{i \in \mathbb{Z}^d} \mathrm{Binom}\left(\alpha, \frac{\rho}{\alpha}\right)$ whose marginals are given by

$$\mathbb{P}_{\nu_\rho}(\eta_i = n) = \frac{1}{Z_{\alpha,\rho}} \cdot \binom{\alpha}{n} \cdot \left(\frac{\rho}{\alpha - \rho}\right)^n, \quad \forall\, i \in \mathbb{Z}^d.$$

with normalizing constant

$$Z_{\alpha,\rho} = \left(\frac{\alpha}{\alpha - \rho}\right)^\alpha \tag{5.5}$$

Symmetric inclusion process (SIP(α)): The choice $\sigma = 1$ gives rise to an interaction of inclusion-type consisting of particles attracting each other.

The SIP is known to have products of Negative-Binomial distributions as reversible measures, i.e. ν_ρ, $\rho > 0$ with $\nu_\rho = \otimes_{i\in\mathbb{Z}^d}\text{Neg-Binom}\left(\alpha, \frac{\rho}{\rho+\alpha}\right)$ with marginals

$$\mathbb{P}_{\nu_\rho}(\eta_i = n) = \frac{1}{Z_{\alpha,\rho}} \cdot \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\cdot n!}\left(\frac{\rho}{\alpha+\rho}\right)^n, \forall\, i \in \mathbb{Z}^d.$$

with normalizing constant

$$Z_{\alpha,\rho} = \left(\frac{\alpha+\rho}{\alpha}\right)^\alpha \tag{5.6}$$

Remark 5.1.2. *Notice that for the three processes we have that all moments are finite.*

The definition of the state space Ω is different in each case, depending on whether there are restrictions or not on the total number of particles allowed per site. This is finite for the exclusion process, thus, for SEP(α), we have $\Omega = \{0, 1, \ldots, \alpha\}^{\mathbb{Z}^d}$. The situation is different in the cases of IRW and SIP, for which, in principle, there are no restrictions. Nevertheless, one has to avoid explosions of the number of particles in a given site. For this reason the characterization of Ω in these cases (i.e. for $\sigma \geq 0$) is a more subtle problem whose treatment is beyond the scope of this thesis. Here we will restrict ourselves by implicitly defining Ω as the set of configurations in $\mathbb{N}^{\mathbb{Z}^d}$ whose evolution $\eta(t)$ is well-defined and belonging to Ω for all subsequent times $t \geq 0$. A possible such subset is the set of tempered configurations. This is the set of configurations η such that there exist $C, \beta \in \mathbb{R}$ that satisfy $|\eta(x)| \leq C(|x|^\beta + 1)$ for all $x \in \mathbb{R}^d$. We denote this set (with slight abuse of notation) still by Ω, because we will always start the process from such configurations, and this set has $\nu_{\bar{\rho}}$ measure 1 for all ρ.

5.1.2 The finite configuration processes

The process introduced in Section 5.1.1 can also be realized with a fixed finite number of particles. For a process with $k \in \mathbb{N}$ particles we denote by Ω_k its state space, more precisely:

$$\Omega_k = \Big\{\xi \in \Omega : \|\xi\| := \sum_{x\in\mathbb{Z}^d} \xi_x = k\Big\}. \tag{5.7}$$

We will then denote by $\{\xi(t) : t \geq 0\}$ the Ω_k-valued Markov process, with infinitesimal generator given by

$$\mathscr{L}^{(k)} f(\xi) = \sum_{i\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\xi_i(\alpha + \sigma\xi_{i+r})(f(\xi^{i,i+r}) - f(\xi)) \tag{5.8}$$

working on functions $f : \Omega_k \to \mathbb{R}$.

We now define the measure $\Lambda(\cdot)$, which is a product measure not depending on k:

$$\Lambda(\xi) = \prod_{i \in \mathbb{Z}^d} \lambda(\xi_i) \tag{5.9}$$

with

$$\lambda(m) = \begin{cases} \frac{1}{m!}, & m \in \mathbb{N} \quad \text{for } \sigma = 0 \quad \text{IRW} \\ \frac{\alpha!}{m!(\alpha-m)!}, & m \in \{0, \dots, \alpha\} \quad \text{for } \sigma = -1 \quad \text{SEP}(\alpha) \\ \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}, & m \in \mathbb{N} \quad \text{for } \sigma = 1 \quad \text{SIP}(\alpha) \end{cases} \tag{5.10}$$

Notice that by detailed balance we can verify that the measure $\Lambda(\cdot)$ is reversible. As a consequence of this reversibility we can infer that the k-particles generator $\mathscr{L}^{(k)}$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\Lambda$, i.e. for all $f, g \in L^2(\Omega_k, \Lambda)$ we have

$$\langle f, \mathscr{L}^{(k)} g \rangle_\Lambda = \langle \mathscr{L}^{(k)} f, g \rangle_\Lambda \tag{5.11}$$

where the inner product is given by

$$\langle f, g \rangle_\Lambda = \sum_{\xi \in \Omega_k} f(\xi) g(\xi) \Lambda(\xi). \tag{5.12}$$

5.1.3 Orthogonal polynomial self-duality

The processes defined in Section 5.1.1 share a self-duality property that will be crucial in our analysis. Define the set

$$\Omega_f = \bigcup_{k \in \mathbb{N}} \Omega_k \tag{5.13}$$

of configurations with a finite number of particles, the self-duality functions that we consider in this paper are functions $D_\rho : \Omega_f \times \Omega \to \mathbb{R}$ parametrized by the density $\rho > 0$ satisfying the following properties.

1. Self-duality:

$$\mathbb{E}_\eta \left[D_\rho(\xi, \eta(t)) \right] = \mathbb{E}_\xi \left[D_\rho(\xi(t), \eta) \right] \qquad \text{for all} \quad \xi \in \Omega_f, \eta \in \Omega \tag{5.14}$$

or, equivalently,

$$[\mathscr{L} D_\rho(\xi, \cdot)](\eta) = [\mathscr{L}^{(k)} D_\rho(\cdot, \eta)](\xi) \qquad \text{for all} \quad \xi \in \Omega_f, \eta \in \Omega. \tag{5.15}$$

2. Factorized polynomials:

$$D_\rho(\xi,\eta) = \prod_{i\in\mathbb{Z}^d} d_\rho(\xi_i,\eta_i)$$

where $d_\rho(0,n)=1$, and $d_\rho(k,\cdot)$ is a polynomial of degree k.

3. Orthogonality:

$$\int D_\rho(\xi,\eta) D_\rho(\xi',\eta)\, d\nu_\rho(\eta) = \delta_{\xi,\xi'} \cdot \frac{1}{\mu_\rho(\xi)} \tag{5.16}$$

where

$$\mu_\rho(\xi) := \left(\int D_\rho(\xi,\eta)^2\, d\nu_\rho(\eta) \right)^{-1}. \tag{5.17}$$

Remark 5.1.3. *Notice that, as a consequence of the orthogonality property* (5.16)*, we have that*

$$\int \mathbb{E}_\eta \left[D_\rho(\xi,\eta(t)) \right] \cdot D_\rho(\xi',\eta)\, d\nu_\rho(\eta) = p_t(\xi,\xi') \cdot \frac{1}{\mu_\rho(\xi')} \tag{5.18}$$

where $p_t(\cdot,\cdot)$ is the transition probability function of the dual process $\{\xi(t): t \geq 0\}$. Moreover, if we use the reversibility of the measure ν_ρ on the LHS of (5.18) *we obtain*

$$\begin{aligned} p_t(\xi,\xi') \cdot \frac{1}{\mu_\rho(\xi')} &= \int \mathbb{E}_\eta \left[D_\rho(\xi,\eta(t)) \right] \cdot D_\rho(\xi',\eta)\, d\nu_\rho(\eta) \\ &= \int D_\rho(\xi,\eta) \cdot \mathbb{E}_\eta \left[D_\rho(\xi',\eta(t)) \right]\, d\nu_\rho(\eta) \\ &= p_t(\xi',\xi) \cdot \frac{1}{\mu_\rho(\xi)} \end{aligned} \tag{5.19}$$

which, by detailed balance, implies the reversibility of the measure $\mu_\rho(\xi)$. This in turn implies that there exists a constant $c(k,\rho)$ such that

$$\Lambda(\xi) = c(k,\rho) \cdot \mu_\rho(\xi) \qquad \textit{for all} \quad \xi \in \Omega_k. \tag{5.20}$$

Remark 5.1.4. *Notice that by Remark 5.1.2 we have that $\mu_\rho(\xi) < \infty$ for every $\xi \in \Omega_f$. Moreover, the measure μ_ρ is not a probability measure.*

From now on we will often suppress the dependence on the parameter ρ, of the duality functions $D(\cdot,\cdot) = D_\rho(\cdot,\cdot)$, in order not to overload the notation. The same omission will be done for the single site duality-polynomials $d(\cdot,\cdot)$, and any other orthogonal polynomial introduced below.

For each of the processes we are considering, the orthogonal duality polynomials are given as follows.

IRW: Charlier polynomials. The duality polynomials are given by

$$d(m,n) = C(m,n)$$

where $C(m,\cdot)$ is the Charlier polynomial of degree m that we characterize by means of the following generating function:

$$\sum_{m=0}^{\infty} C(m,n)\cdot\frac{t^m}{m!} = e^{-t}\left(\frac{\rho+t}{\rho}\right)^n. \tag{5.21}$$

We can differentiate the RHS of (5.21) with respect to t, and evaluate at $t=0$, to obtain that the first three Charlier (and self-duality) polynomials are:

$$\begin{aligned} d(0,n) &= C(0,n) = 1, \\ d(1,n) &= C(1,n) = \frac{1}{\rho}(n-\rho), \\ d(2,n) &= C(2,n) = \frac{1}{\rho^2}\left(n(n-1) - 2\rho n + \rho^2\right). \end{aligned} \tag{5.22}$$

SEP(α): Krawtchouk polynomials. For the SEP the duality polynomials are given by

$$d(m,n) = \frac{m!(\alpha-m)!}{\alpha!}\cdot K(m,n)$$

where $K(m,\cdot)$ is the Krawtchouk polynomial of degree m whose generating function is

$$\sum_{m=0}^{\infty} K(m,n)\cdot t^m = (1-t)^\alpha\left(\frac{1+(\frac{\alpha-\rho}{\rho})t}{1-t}\right)^n. \tag{5.23}$$

With analogous computations to the IRW case, the first Krawtchouk polynomials are:

$$\begin{aligned} K(0,n) &= 1, \\ K(1,n) &= \frac{\alpha}{\rho}(n-\rho), \\ K(2,n) &= \left(\frac{\alpha}{\rho}\right)^2 n(n-1) - 2\left(\frac{\alpha}{\rho}\right)(\alpha-1)n + \alpha(\alpha-1), \end{aligned} \tag{5.24}$$

with corresponding single-site duality polynomials:

$$\begin{aligned} d(0,n) &= 1, \\ d(1,n) &= \frac{1}{\rho}(n-\rho), \\ d(2,n) &= \frac{2\alpha}{\rho^2(\alpha-1)}\left(n(n-1) - \frac{2\rho(\alpha-1)}{\alpha}n + \frac{\rho^2(\alpha-1)}{\alpha}\right). \end{aligned} \tag{5.25}$$

Remark 5.1.5. *Notice that these polynomials are only defined for the case $m, n \leq \alpha$.*

SIP(α): Meixner polynomials. In this case the polynomials satisfying the self-duality relation are given by the following normalization of the Meixner polynomials

$$d(m,n) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+m)} \cdot M(m,n) \tag{5.26}$$

where $M(m,\cdot)$ is the Meixner polynomial of degree m with generating function

$$\sum_{m=0}^{\infty} M(m,n) \cdot \frac{t^m}{m!} = (1-t)^{-\alpha} \left(\frac{1 - \frac{(\alpha+\rho)t}{\rho}}{1-t} \right)^n . \tag{5.27}$$

The first Meixner polynomials are:

$$\begin{aligned} M(0,n) &= 1, \\ M(1,n) &= -\frac{\alpha}{\rho}(n-\rho), \\ M(2,n) &= \left(\frac{\alpha}{\rho}\right)^2 n(n-1) - 2\left(\frac{\alpha}{\rho}\right)(\alpha+1)n + \alpha(\alpha+1). \end{aligned} \tag{5.28}$$

with corresponding single-site duality polynomials are:

$$\begin{aligned} d(0,n) &= 1, \\ d(1,n) &= -\frac{1}{\rho}(n-\rho), \\ d(2,n) &= \frac{\alpha}{\rho^2(\alpha+1)} \left(n(n-1) - \frac{2\rho(\alpha+1)}{\alpha} n + \frac{\rho^2(\alpha+1)}{\alpha} \right) \end{aligned} \tag{5.29}$$

We refer the reader to [61] and [23] for more details on these polynomials and their generating functions. For proofs of self-duality with these orthogonal polynomials we refer to [41] and [79].

5.2 Fluctuation fields

The density fluctuation field $\mathscr{X}$ is the stochastic object usually defined to study fluctuations of density around its expected limit. This field corresponds to a central limit type of rescaling of the density field, i.e.

$$\mathscr{X}_t^{(n)}(\varphi, \eta) := n^{-d/2} \sum_{x \in \mathbb{Z}^d} \varphi(x/n)(\eta_x(n^2 t) - \rho). \tag{5.30}$$

where φ is an element of the Schwartz space $S(\mathbb{R}^d)$, i.e., the space of all smooth functions whose derivatives are rapidly decreasing:

$$S(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} |x^\beta D^\theta \varphi| < \infty, \forall \beta, \theta \in \mathbb{N}\} \tag{5.31}$$

where $C^\infty(\mathbb{R}^d)$ is the space of smooth functions.

Fields of this type have been intensively studied in the literature. For different models, see for example [58] for the case of the ZRP, the sequence $\mathscr{X}_t^{(n)}$ is proven to converge to a limiting field $\mathscr{X}_t$ that is identified as the distribution-valued random variable satisfying the following martingale problem: for any $\varphi \in S(\mathbb{R}^d)$ the process

$$M_t(\varphi) = \mathscr{X}_t(\varphi) - \mathscr{X}_0(\varphi) - \frac{\chi\alpha}{2}\int_0^t \mathscr{X}_s(\Delta\varphi)ds \tag{5.32}$$

is a square integrable continuous martingale of quadratic variation given by the expression:

$$\chi\rho(\alpha+\sigma\rho)\big\|\nabla\varphi(x)\big\|^2 \cdot t. \tag{5.33}$$

Following a procedure analogous to the one given in Chapter 11, pages 290-291, of [58], the martingale problem (5.32)-(5.33) can be rewritten as:

$$\mathscr{X}_t(\varphi) = \mathscr{X}_0(\varphi) + \frac{\chi\alpha}{2}\int_0^t \mathscr{X}_s(\Delta\varphi)ds + \sqrt{\chi\rho(\alpha+\sigma\rho)}\big\|\nabla\varphi(x)\big\|\,\mathscr{W}_t(\varphi) \tag{5.34}$$

where $\mathscr{W}_t$ is a generalized Brownian motion with covariance

$$\mathrm{cov}\left[\mathscr{W}_t(\varphi), \mathscr{W}_s(\psi)\right] = \min(t,s)\int_{\mathbb{R}} \frac{\nabla\varphi(x)}{\big\|\nabla\varphi(x)\big\|}\frac{\nabla\psi(x)}{\big\|\nabla\psi(x)\big\|}\,dx. \tag{5.35}$$

Formally speaking, (5.34) is equivalent to say that the limiting field $\mathscr{X}_t$ satisfies the Ornstein-Uhlenbeck equation

$$d\mathscr{X}_t = \tfrac{\chi\alpha}{2}\,\Delta\mathscr{X}_t\,dt + \sqrt{\chi\rho(\alpha+\sigma\rho)}\,\nabla d\mathscr{W}_t, \tag{5.36}$$

where $\nabla d\mathscr{W}_t$ is to be interpreted as saying that the integral

$$\int_0^t \nabla d\mathscr{W}_s(\varphi) \tag{5.37}$$

is a continuous martingale of quadratic variation:

$$t\cdot\big\|\nabla\varphi(x)\big\|. \tag{5.38}$$

We refer the reader to [27] for a precise statement on the convergence for the case of the exclusion process, corresponding, in our setting, to the case $\alpha = 1$

and $\sigma = -1$.

The density field (5.30) can be written, in our context, in terms of our orthogonal polynomial dualities $D_\rho(\xi, \eta)$ by choosing $\xi \in \Omega_1$. Indeed, in all models considered we have that there exists a constant $c_{\sigma,\alpha,\rho}$ such that

$$D_\rho(\delta_x, \eta) = c_{\sigma,\alpha,\rho}\,(\eta_x - \rho) \tag{5.39}$$

where

$$c_{\sigma,\alpha,\rho} = \begin{cases} 1/\rho & \text{if } \sigma = 0 \\ -1/\rho & \text{if } \sigma = 1 \\ 1/\rho & \text{if } \sigma = -1 \end{cases} . \tag{5.40}$$

Later on, in order not to overload notation we will suppress the dependence on ρ and α and just write c_σ. From (5.39) we observe that the field (5.30) can be rewritten (modulo a multiplicative constant) as

$$\mathscr{X}_t^{(n,1)}(\varphi) = n^{-d/2} \sum_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right) D_\rho(\delta_x, \eta(n^2 t)) \tag{5.41}$$

where the superindex $(n, 1)$ suggests that, in some sense, this is the first-order density field. Using (5.39) and (5.36) the formal limiting SPDE for $\mathscr{X}_t$ is

$$d\mathscr{X}_t = \frac{\chi\alpha}{2}\,\Delta\mathscr{X}_t dt + c_\sigma \sqrt{\chi\rho(\alpha + \sigma\rho)}\,\nabla d\mathscr{W}_t \tag{5.42}$$

The observation that the field (5.30) can be expressed in terms of duality polynomials opens the possibility of defining higher-order fields and study their scaling limits. For $k \in \mathbb{N}$, $k \geq 1$ we define the k-th order field as

$$\begin{aligned} \mathscr{X}^{(n,k)}(\varphi^{(k)}, \eta) := \mathscr{Y}^{(n,k)}(\Phi, \eta) \quad &:= \quad n^{-kd/2} \sum_{\xi \in \Omega_k} \left(\prod_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right)^{\xi_x} \right) \Lambda(\xi) \cdot D_\rho(\xi, \eta) \\ &= \quad n^{-kd/2} \sum_{\xi \in \Omega_k} \left(\prod_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right)^{\xi_x} \cdot \lambda(\xi_x) \cdot d_\rho(\xi_x, \eta_x) \right) \end{aligned} \tag{5.43}$$

where $\varphi \in S(\mathbb{R}^d)$ is a test function, Λ is as in (5.9), and

$$\varphi^{(k)} := \bigotimes_{i=1}^{k} \varphi \tag{5.44}$$

$$\Phi(\xi) = \prod_{x \in \mathbb{Z}^d} \varphi(x)^{\xi_x}, \qquad \Phi_n(\xi) = \prod_{x \in \mathbb{Z}^d} \varphi\left(\tfrac{x}{n}\right)^{\xi_x} \tag{5.45}$$

In the rest of this work, we will refer to test functions of the type $\varphi^{(k)}$ as symmetric elements of the Schwartz space $S(\mathbb{R}^{kd})$. Likewise, the functions $\Phi : \Omega_k \to \mathbb{R}$, given by (5.45), will be considered as elements of the Schwartz space of test functions over configuration space.

Notice that there is no difference between $\mathscr{X}^{(n,k)}(\varphi^{(k)}, \eta)$ and $\mathscr{Y}^{(n,k)}(\Phi, \eta)$ besides that the latter works on test functions over configuration space, i.e., $\Phi \in S(\Omega_k)$, while the former works on test functions $\varphi^{(k)} \in S(\mathbb{R}^{kd})$. Then, using the notation

$$\mathfrak{D}_\rho(\xi, \eta) := \Lambda(\xi) \cdot D_\rho(\xi, \eta), \qquad \mathfrak{d}_\rho(m, n) = \lambda(m) \cdot d_\rho(m, n) \tag{5.46}$$

$$\mathfrak{D}_\rho(\xi, \eta) = \prod_{i \in \mathbb{Z}^d} \mathfrak{d}_\rho(\xi_i, \eta_i), \tag{5.47}$$

we can rewrite the k-th order field (5.43) as

$$\mathscr{Y}^{(n,k)}(\Phi, \eta) := n^{-kd/2} \sum_{\xi \in \Omega_k} \Phi_n(\xi) \cdot \mathfrak{D}_\rho(\xi, \eta) \tag{5.48}$$

and define:

$$\mathscr{Y}_t^{(n,k)}(\Phi) := \mathscr{Y}^{(n,k)}(\Phi, \eta(n^2 t)). \tag{5.49}$$

The choice of multiplying the duality function by the measure $\Lambda(\cdot)$ in (5.46) is dictated simply by computational convenience that, even if obscure at the moment, will be made clearer in the course of this work.

First example: second-order fluctuation fields for the SEP(1)

Let us specialize these fields to the case of the one-dimensional symmetric exclusion process for $k = 2$. This means that we are taking:

$$d = 1, \quad \alpha = 1, \text{ and } \quad \sigma = -1. \tag{5.50}$$

In this case we have:

$$\begin{aligned} \mathscr{X}^{(n,2)}(\varphi^{(2)}, \eta) &= \frac{1}{n} \sum_{\xi \in \Omega_2} \left(\prod_{x \in \mathbb{Z}} \varphi\left(\tfrac{x}{n}\right)^{\xi_x} \right) \Lambda(\xi) \cdot D_\rho(\xi, \eta) \\ &= \frac{1}{2n} \sum_{\substack{x,y \in \mathbb{Z} \\ x \neq y}} \varphi\left(\tfrac{x}{n}\right) \varphi\left(\tfrac{y}{n}\right) \Lambda(\delta_x + \delta_y)\, D_\rho(\delta_x + \delta_y, \eta) \\ &= \frac{1}{2\rho^2} \left[\frac{1}{n} \sum_{\substack{x,y \in \mathbb{Z} \\ x \neq y}} \varphi\left(\tfrac{x}{n}\right) \varphi\left(\tfrac{y}{n}\right) (\eta_x - \rho)\,(\eta_y - \rho) \right] \end{aligned} \tag{5.51}$$

where in the second line, in order to get rid of the sum at the diagonal, we used the fact that for SEP(1) we have $D_\rho(2\delta_x, \eta) = 0$. Notice that in the last line we used (5.9) and (5.25).

Remark 5.2.1. *Notice that the previous field corresponds, modulo a multiplicative factor, to the quadratic field introduced earlier in [47]. Also notice that the previous field is not the same as the quadratic field introduced in [5].*

Second example: second-order fluctuation fields for IRW(1)

Let us now look at the case of one-dimensional independent random walkers. This means that we are taking:

$$d = 1, \quad \alpha = 1, \text{ and } \quad \sigma = 0. \tag{5.52}$$

Analogous to the case of SEP(1), in this case we have:

$$\begin{aligned}
\mathscr{X}^{(n,2)}(\varphi^{(2)}, \eta) &= \frac{1}{n} \sum_{\xi \in \Omega_2} \left(\prod_{x \in \mathbb{Z}} \varphi\left(\tfrac{x}{n}\right)^{\xi_x} \right) \Lambda(\xi) \cdot D_\rho(\xi, \eta) \\
&= \frac{1}{2n} \sum_{\substack{x,y \in \mathbb{Z} \\ x \neq y}} \varphi\left(\tfrac{x}{n}\right) \varphi\left(\tfrac{y}{n}\right) \Lambda(\delta_x + \delta_y)\, D_\rho(\delta_x + \delta_y, \eta) \\
&+ \frac{1}{n} \sum_{x \in \mathbb{Z}} \varphi\left(\tfrac{x}{n}\right)^2 \Lambda(2\delta_x)\, D_\rho(2\delta_x, \eta) \\
&= \frac{1}{2n\rho^2} \sum_{\substack{x,y \in \mathbb{Z} \\ x \neq y}} \varphi\left(\tfrac{x}{n}\right) \varphi\left(\tfrac{y}{n}\right) (\eta_x - \rho) \left(\eta_y - \rho\right) \\
&+ \frac{1}{2n\rho^2} \sum_{x \in \mathbb{Z}} \varphi\left(\tfrac{x}{n}\right)^2 \left(\eta_x(\eta_x - 1) - 2\rho\, \eta_x + \rho^2\right)
\end{aligned} \tag{5.53}$$

Remark 5.2.2. *Notice that different to the case of* SEP(1)*, in this case we have that the second-order duality polynomials do not vanish and as a consequence we have a contribution coming from the diagonal (i.e., the second summation in the RHS of* (5.53)*).*

5.3 The coordinate process

Thinking of $k \in \mathbb{N}$ as the number of particles in our process, we want to introduce a family of permutation-invariant coordinate processes $\{X^{(k)}(t) : t \geq 0\}$ compatible with the finite configuration processes $\{\xi(t) : t \geq 0\}$ on Ω_k. Here the coordinate process is a Markov process on $\mathbb{Z}^{dk}$ with

$$X^{(k)}(t) = (X_1(t), \ldots, X_k(t)), \qquad X_i(t) \in \mathbb{Z}^d, \quad \forall i = 1, \ldots, k \tag{5.54}$$

$X_i(t)$ being the position of the i-th particle at time $t \geq 0$. For a further explanation of the notion of compatibility we refer the reader to [19].
Denote by $\mathbf{x} \in \mathbb{Z}^{kd}$ the coordinate vector $\mathbf{x} := (x_1, \ldots, x_k)$, with $x_i \in \mathbb{Z}^d$, for $i = 1, \ldots, k$. The coordinate process $\{X^{(k)}(t) : t \geq 0\}$ is defined by means of its infinitesimal generator:

$$L^{(k)} f(\mathbf{x}) = \sum_{i=1}^{k} \sum_{r \in \mathscr{R}} p(r) \left(\alpha + \sigma \sum_{\substack{j=1 \\ j \neq i}}^{k} \mathbf{1}_{x_j = x_i + r} \right) \left(f(\mathbf{x}^{i,i+r}) - f(\mathbf{x}) \right) \tag{5.55}$$

where $\mathbf{x}^{i,i+r}$ denotes $\mathbf{x}$ after moving the particle in position x_i to position $x_i + r \in \mathbb{Z}^d$. Notice that for $\mathbf{x} \in \mathbb{Z}^{kd}$ the compatible configuration $\xi(\mathbf{x}) \in \Omega_k$ is given by

$$\xi(\mathbf{x}) = \left(\xi_i(\mathbf{x}), i \in \mathbb{Z}^d \right) \qquad \text{with} \qquad \xi_i(\mathbf{x}) = \sum_{j=1}^{k} \mathbf{1}_{x_j = i}. \tag{5.56}$$

5.3.1 Product σ-finite reversible measures

It is possible to verify, by means of detailed balance, that the coordinate-process $\{X^{(k)}(t) : t \geq 0\}$ admits a reversible σ-finite measure that is given by

$$\Pi(\mathbf{x}) = \frac{\Lambda(\xi(\mathbf{x}))}{N(\xi(\mathbf{x}))} = \prod_{i \in \mathbb{Z}^d} \xi_i(\mathbf{x})! \cdot \lambda(\xi_i(\mathbf{x})) \qquad \text{for} \quad \mathbf{x} \in \mathbb{Z}^{kd} \tag{5.57}$$

where λ is given as in (5.9), and with

$$N(\xi) := |\{\mathbf{x} \in \mathbb{Z}^{kd} : \ \xi(\mathbf{x}) = \xi\}| = \frac{k!}{\prod_{i \in \mathbb{Z}^d} \xi_i!} \tag{5.58}$$

Then we can rewrite Π in the product form:

$$\Pi(\mathbf{x}) = \prod_{i \in \mathbb{Z}^d} \pi(\xi_i(\mathbf{x})), \qquad \mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^{kd} \tag{5.59}$$

with π given as follows:

$$\pi(m) = m! \cdot \lambda(m) = \begin{cases} 1, & m \in \mathbb{N} \quad \text{for } \sigma = 0 \quad \text{IRW} \\[2ex] \frac{\alpha!}{(\alpha - m)!}, & m \in \{0, \ldots, \alpha\} \quad \text{for } \sigma = -1 \quad \text{SEP}(\alpha) \\[2ex] \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)}, & m \in \mathbb{N} \quad \text{for } \sigma = 1 \quad \text{SIP}(\alpha) \end{cases} . \tag{5.60}$$

Given the measures Π, we now consider the spaces of permutation-invariant functions:

$$\hat{L}^2(\mathbb{Z}^{kd}, \Pi) := \left\{ f \in L^2(\mathbb{Z}^{kd}, \Pi) : f(\mathbf{x}) = f(\mathbf{x}_\sigma), \ \forall \sigma \in \mathscr{P}(k) \right\} \tag{5.61}$$

with $\mathscr{P}(k)$ denoting the set of all possible permutations of elements of the set $\{1, 2, 3, \dots k\}$. We endowed the space $\hat{L}^2(\mathbb{Z}^{kd}, \Pi)$ with the inner product given by:

$$\langle f, g \rangle_\Pi = \sum_{\mathbf{x} \in \mathbb{Z}^{kd}} f(\mathbf{x}) g(\mathbf{x}) \Pi(\mathbf{x}). \tag{5.62}$$

Remark 5.3.1. *Notice that any function $f \in \hat{L}^2(\mathbb{Z}^{kd}, \Pi)$ can be interpreted also as a function on the configuration space. In this work we will extensively use this fact by changing between interpretations sometimes from one line to another in the same derivation.*

Remark 5.3.2. *As a consequence of reversibility of the measures Π, we can infer that the k-particles generator $L^{(k)}$ is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\Pi$, i.e.*

$$\langle f, L^{(k)} g \rangle_\Pi = \langle L^{(k)} f, g \rangle_\Pi \tag{5.63}$$

for all $f, g \in \hat{L}^2(\mathbb{Z}^{kd}, \Pi)$.

5.3.2 The fluctuation fields in coordinate notation

It is possible to rewrite the fluctuation field (5.43) in the coordinate variables. Notice that in this context the test function Φ defined in (5.45) becomes a tensor function:

$$\Phi(\xi(\mathbf{x})) = \prod_{i=1}^{k} \varphi(x_i) \tag{5.64}$$

i.e. it is the homogeneous k-tensor test function $\varphi^{\otimes k} \in S(\mathbb{R}^{kd})$ of the form

$$\Phi \circ \xi = \varphi^{\otimes k} := \bigotimes_{i=1}^{k} \varphi \tag{5.65}$$

then, after a change of variable in the sum we can rewrite the k-th field as follows

$$\mathscr{X}^{(n,k)}(\varphi^{(k)}, \eta) = \mathscr{Y}^{(n,k)}(\Phi, \eta) = n^{-kd/2} \sum_{\mathbf{x} \in \mathbb{Z}^{kd}} \varphi^{(k)} \left(\tfrac{\mathbf{x}}{n}\right) \cdot \Pi(\mathbf{x}) \cdot D(\xi(\mathbf{x}), \eta). \tag{5.66}$$

Notice that we can also let the field $\mathscr{X}$ act on a general $f \in S(\mathbb{R}^{kd})$ as expected, i.e.,

$$\mathscr{X}^{(n,k)}(f, \eta) = n^{-kd/2} \sum_{\mathbf{x} \in \mathbb{Z}^{kd}} f\left(\tfrac{\mathbf{x}}{n}\right) \cdot \Pi(\mathbf{x}) \cdot D(\xi(\mathbf{x}), \eta). \tag{5.67}$$

Remark 5.3.3. *Because we deal with unlabeled particle systems it is natural to define the higher-order fluctuation fields acting on symmetric test functions Φ i.e. on elements of the Schwartz space $S(\mathbb{R}^{kd})$ that are permutation-invariant: $\Phi(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \Phi(x_1, \dots, x_k)$ for all $\sigma \in \mathscr{P}(k)$, the set of permutations of $\{1, \dots, k\}$.*

Remark 5.3.4. *The set of test functions of the form $\varphi^{\otimes k}$ is dense in the space of symmetric Schwartz test functions. This can be seen in two steps. First, linear combination of tensors are dense in $S(\mathbb{R}^{kd})$. Second, restricting to symmetric linear combinations of elements in $S(\mathbb{R}^{kd})$, we have that by polarization linear combinations of powers of the form $\varphi^{\otimes k}$ are dense in this restriction (see for example Remark 2.5 in [39]).*

5.4 Main result

5.4.1 Heuristics: macroscopic dynamics

The goal of this section is to provide some intuitions on the type of limiting field that we should expect for fields of order greater than one. We will start by considering the cases $k = 1, 2$ and, inspired by the results obtained in [47], we will propose a heuristic interpretation of the two SPDEs obtained as scaling limits and their relation. Based on this interpretation we will conjecture a possible generalization to the kth-order case. In Section 5.4.2 we will give the rigorous result confirming the validity of the conjecture.

Here we will informally use the notation $\mathscr{Y}_t^{(k)}$ and $\mathscr{X}_t^{(k)}$ for the distributional limits of $\mathscr{Y}^{(n,k)}$ and $\mathscr{X}^{(n,k)}$ respectively.

Recall that from (5.42) we know that formally the distribution valued first-order field $\mathscr{X}_t^{(1)}(x)$ is a solution to the Ornstein-Uhlenbeck equation

$$d\mathscr{X}_t^{(1)}(x) = \frac{\chi\alpha}{2}\,\Delta\mathscr{X}_t^{(1)}(x)\,dt + c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\nabla d\mathscr{W}_t(x) \tag{5.68}$$

where for $x \in \mathbb{R}^d$, $\mathscr{W}_t(x)$ is a space-time white noise and $\nabla d\mathscr{W}_t(x)$ should be interpreted as in (5.37)-(5.38). Additionally, from the martingale problem given in [47], we can deduce that the distribution-valued second-order field $\mathscr{X}_t^{(2)}(x, y)$ is a solution to the SPDE

$$\begin{aligned} d\mathscr{X}_t^{(2)}(x,y) &= \frac{\chi\alpha}{2}\,\Delta^{(2)}\mathscr{X}_t^{(2)}(x,y)dt + c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\mathscr{X}_t^{(1)}(x)\nabla d\mathscr{W}_t(y) \\ &+ c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\mathscr{X}_t^{(1)}(y)\nabla d\mathscr{W}_t(x) \end{aligned} \tag{5.69}$$

where $\mathscr{W}_t(x)$ is the same white noise as in (5.68) and $\Delta^{(2)}$ denotes the usual $2d$-dimensional Laplacian, which is the sum of the Laplacian in the x variable plus the Laplacian in the y variable.

The key idea to extrapolate these relations to higher orders is to interpret the non-linearity on the RHS of (5.69) as some product of fields, that we denote by $\diamond$, that satisfies the Leibniz rule of differentiation. This interpretation suggests that the second-order field $\mathscr{X}_t^{(2)}(x,y)$ is, in turn, a second power of the first-order field $\mathscr{X}_t^{(1)}(x)$. More precisely conjecturing

$$\mathscr{X}_t^{(2)}(x,y) = \mathscr{X}_t^{(1)}(x) \diamond \mathscr{X}_t^{(1)}(y),$$

since the product $\diamond$ follows the Leibniz rule we would have that

$$\begin{aligned}
d\mathscr{X}_t^{(2)}(x,y) &= d\left(\mathscr{X}_t^{(1)}(x) \diamond \mathscr{X}_t^{(1)}(y)\right) \\
&= d\mathscr{X}_t^{(1)}(x) \diamond \mathscr{X}_t^{(1)}(y) + \mathscr{X}_t^{(1)}(x) \diamond d\mathscr{X}_t^{(1)}(y) \\
&= \left(\frac{\chi\alpha}{2}\,\Delta\mathscr{X}_t^{(1)}(x)dt + c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\nabla d\mathscr{W}_t(x)\right) \diamond \mathscr{X}_t^{(1)}(y) \\
&+ \mathscr{X}_t^{(1)}(x) \diamond \left(\frac{\chi\alpha}{2}\,\Delta\mathscr{X}_t^{(1)}(y)dt + c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\nabla d\mathscr{W}_t(y)\right) \\
&= \frac{\chi\alpha}{2}\,\Delta^{(2)}\mathscr{X}_t^{(2)}(x,y)dt + c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\mathscr{X}_t^{(1)}(x) \diamond \nabla d\mathscr{W}_t(y) \\
&+ c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\,\mathscr{X}_t^{(1)}(y) \diamond \nabla d\mathscr{W}_t(x) \qquad (5.70)
\end{aligned}$$

which indeed agress with (5.69).

Remark 5.4.1. *This section is created with the intention to develop some intuition on the type of martingale problem we should expect for higher-order fields. The precise product to be used in this section is not relevant since after all our derivations are just made at a formal level. What is important is that the product should satisfy the Leibniz rule.*

After the discussion above, it seems natural to expect that the kth-order field is a kth $\diamond$-power of the first-order one. More precisely we conjecture that a relation of the type

$$\mathscr{X}_t^{(k)}(x_1,x_2,\dots,x_k) = \mathscr{X}_t^{(1)}(x_1) \diamond \mathscr{X}_t^{(1)}(x_2) \diamond \cdots \diamond \mathscr{X}_t^{(1)}(x_k).$$

is satisfied. If this holds true, computations analogous to (5.70) would imply the formal SPDE

$$d\mathscr{X}_t^{(k)}(\mathbf{x}) = \frac{\chi\alpha}{2}\,\Delta^{(k)}\mathscr{X}_t^{(k)}(\mathbf{x})dt + c_\sigma\sqrt{\chi\rho(\alpha+\sigma\rho)}\sum_{j=1}^{k}\mathscr{X}_t^{(k-1)}(\mathbf{x}^{-j}) \diamond \nabla d\mathscr{W}_t(x_j) \qquad (5.71)$$

where $\Delta^{(k)}$ is the kd-dimensional Laplacian, defined as the sum of the Laplacians at each coordinate and $\mathbf{x}^{-j}$ is the $(k-1)d$-dimensional vector obtained from $\mathbf{x}$ by removing its coordinate x_j.

In the following section we formulate rigorously the meaning of the heuristic equation, via a martingale problem.

5.4.2 Main theorem

Let us spend one paragraph to introduce the probability notions which are relevant for our main result. As we already mentioned, the kth-order fluctuation field can be considered as taking values in $S'(\mathbb{R}^k)$, the space of tempered distributions which is dual to $S(\mathbb{R}^k)$. Our original process η_{n^2t} has state space $\Omega^{(n)}$ corresponding to the rescaled lattice $\frac{1}{n}\mathbb{Z}$. We then denote by $\mathbb{P}_n$, respectively $\mathbb{E}_n$, the probability measure, respectively expectation, induced by the measure ν_ρ and the diffusively rescaled process η_{n^2t} on $D([0,T];\Omega^{(n)})$. We also denote by $Q_n^{(k)}$ the probability measure on $D([0,T];S'(\mathbb{R}^{kd}))$ induced by the density fluctuation field $\mathscr{X}_t^{(n,k)}$ over $\mathbb{P}_n$. Finally, for $t \geq 0$, we denote by $\mathscr{F}_t$ the σ-algebra in $D([0,T];S'(\mathbb{R}^{kd}))$ generated by the limiting first-order field $\mathscr{X}_s^{(1)}(\varphi)$ for $s \leq t$ and $\varphi \in S(\mathbb{R}^d)$.

Theorem 5.4.1. *The process $\{\mathscr{X}_t^{(n,k)} : t \in [0,T]\}$ converges in distribution, with respect to the J1-topology of $D([0,T];S'(\mathbb{R}^{kd}))$, as $n \to \infty$, to the process $\{\mathscr{X}_t^{(k)} : t \in [0,T]\}$ which is the unique solution of the following recursive martingale problem.*

Recursive martingale problem: *for any symmetric $\varphi^{(k)} \in S(\mathbb{R}^{kd})$ the process*

$$M_t^{(k)}(\varphi^{(k)}) = \mathscr{X}_t^{(k)}(\varphi^{(k)}) - \mathscr{X}_0^{(k)}(\varphi^{(k)}) - \frac{\chi\alpha}{2}\int_0^t \mathscr{X}_s^{(k)}(\Delta^{(k)}\varphi^{(k)})ds \tag{5.72}$$

is a continuous square integrable $\mathscr{F}_t$-martingale of quadratic variation

$$c_\sigma^2\chi\rho(\alpha+\sigma\rho)\int_0^t\int_{\mathbb{R}^d}\left\|\nabla\varphi(x)\right\|^2\left(\mathscr{X}_s^{(k-1)}(\varphi^{(k-1)})\right)^2 dx\, ds \tag{5.73}$$

with initial condition $\mathscr{X}_t^{(1)}$ given by the solution of (5.68).

Remark 5.4.2. *This recursive martingale problem is the rigorous counterpart of the formal SPDE* (5.71) *that we heuristically obtained.*

5.4.3 Strategy of the proof

We will prove Theorem 5.4.1 by using induction on k. In the proof we will take advantage of the fact that the base case, $k = 1$, is already proved in the literature. On the other hand, the inductive step will be proven by means of an approach based on the natural Dynkin martingales:

$$M_t^{(n,k)}(\Phi) = \mathscr{Y}_t^{(n,k)}(\Phi) - \mathscr{Y}_0^{(n,k)}(\Phi) - n^2\int_0^t \mathscr{L}\mathscr{Y}_s^{(n,k)}(\Phi)ds \tag{5.74}$$

and

$$N_t^{(n,k)}(\Phi) = (M_t^{(n,k)}(\Phi))^2 - n^2\int_0^t \Gamma\mathscr{Y}_s^{(n,k)}(\Phi)ds \tag{5.75}$$

where Γ is the so-called carré-du-champ operator given by:

$$\Gamma(f) = \mathscr{L}(f^2) - 2f\mathscr{L}(f). \tag{5.76}$$

Notice that the Dynkin martingales can also be expressed in terms of the fields $\mathscr{X}_t^{(n,k)}$.

Roughly our approach consists of the following steps:

1. we express the integrand term of equation (5.74) in terms of the kth-order fluctuation field $\mathscr{Y}^{(n,k)}$ using duality (Section 5.5.1);
2. we close the equation (5.75) by expressing the integrand in the RHS in terms of the $(k-1)$th-order fluctuation field $\mathscr{Y}^{(n,k-1)}$ (Section 5.5.2);
3. we show tightness for the sequence of probability measures $Q_n^{(k)}$ (Section 5.5.3);
4. finally we characterize the limiting field by showing uniqueness of the solution of the martingale problem (Sections 5.5.4-5.5.5).

5.4.4 Inductive argument

The proof is done by induction on the order of the field k. The base case $k = 1$, corresponding to the density fluctuation field (5.30), is assumed to be true. Indeed, as mentioned in Section 5.2, a proof of Theorem 5.4.1 for exclusion dynamics and zero-range processes (of which independent random walkers are a particular case) is given in [27] and [58] respectively. By similar arguments the result can be extended to the case of the inclusion process.

To implement the inductive argument we formalize the following inductive hypothesis that will be referred to several times in the course of the proof of Theorem 5.4.1.

Inductive hypothesis 5.4.1. *For any $k_0 \in \{1, 2, \ldots, k-1\}$ the sequence $\{\mathscr{X}_t^{(n,k_0)} : t \in [0,T]\}$ converges in distribution, with respect to the J1-topology of $D([0,T]; S'(\mathbb{R}^{k_0 d}))$, as $n \to \infty$ to the process $\{\mathscr{X}_t^{(k_0)} : t \in [0,T]\}$ being the unique solution of the following martingale problem.*

Martingale problem: *for any symmetric $\varphi^{(k_0)} \in S(\mathbb{R}^{k_0 d})$ the process*

$$M_t^{(k_0)}(\varphi^{(k_0)}) = \mathscr{X}_t^{(k_0)}(\varphi^{(k_0)}) - \mathscr{X}_0^{(k_0)}(\varphi^{(k_0)}) - \frac{\chi\alpha}{2}\int_0^t \mathscr{X}_s^{(k_0)}(\Delta^{(k_0)}\varphi^{(k_0)})ds \tag{5.77}$$

is a continuous square-integrable $\mathscr{F}_t$-martingale of quadratic variation

$$c_\sigma^2 \chi\rho(\alpha+\sigma\rho)\int_0^t \int_{\mathbb{R}^d} \left\|\nabla\varphi(x)\right\|^2 \left(\mathscr{X}_s^{(k_0-1)}(\varphi^{(k_0-1)})\right)^2 dx\, ds. \tag{5.78}$$

5.5 Proof of Theorem 5.4.1

5.5.1 Closing the equation for the drift term: $k \geq 2$

In order to close the equation (5.74) for the drift term (i.e., the integral term), thanks to Remark 5.3.2 we can just proceed as follows

$$\begin{aligned} n^2 \mathscr{L}\mathscr{Y}^{(n,k)}(\Phi,\eta) &= n^{-kd/2} \sum_{\xi\in\Omega_k} n^2 \Phi_n(\xi)\cdot[\mathscr{L}\mathfrak{D}(\xi,\cdot)](\eta) \\ &= n^{-kd/2} \sum_{\xi\in\Omega_k} n^2 \Phi_n(\xi)\cdot\Lambda(\xi)\cdot[\mathscr{L}D(\xi,\cdot)](\eta) \\ &= n^{-kd/2} \sum_{\xi\in\Omega_k} n^2 \Phi_n(\xi)\cdot\Lambda(\xi)\cdot[\mathscr{L}^{(k)}D(\cdot,\eta)](\xi) \\ &= n^{-kd/2} \sum_{\xi\in\Omega_k} n^2 [\mathscr{L}^{(k)}\Phi_n](\xi)\cdot\Lambda(\xi)\cdot D(\xi,\eta) \\ &= n^{-kd/2} \sum_{\xi\in\Omega_k} n^2 [\mathscr{L}^{(k)}\Phi_n](\xi)\cdot\mathfrak{D}(\xi,\eta). \end{aligned}$$

We proceed evaluating the action of the k-particles generator on Φ_n. We then have

$$\begin{aligned} n^2[\mathscr{L}^{(k)}\Phi_n](\xi) &= \sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\cdot\xi_x(\alpha+\sigma\xi_{x+r})\cdot n^2(\Phi_n(\xi^{x,x+r})-\Phi_n(\xi)) \\ &= \sum_{x\in\mathbb{Z}^d}\Phi_n(\xi-\delta_x)\sum_{r\in\mathscr{R}} p(r)\cdot\xi_x(\alpha+\sigma\xi_{x+r})\cdot\Delta_n^r\varphi(x) \\ &= \alpha\sum_{x\in\mathbb{Z}^d}\Phi_n(\xi-\delta_x)\cdot\xi_x\sum_{r\in\mathscr{R}} p(r)\cdot\Delta_n^r\varphi(x) \\ &+ \sigma\sum_{x\in\mathbb{Z}^d}\Phi_n(\xi-\delta_x)\sum_{r\in\mathscr{R}} p(r)\cdot\xi_x\xi_{x+r}\cdot\Delta_n^r\varphi(x). \end{aligned} \tag{5.79}$$

where

$$\Delta_n^r\varphi(x) = n^2\left(\varphi\left(\tfrac{x+r}{n}\right)-\varphi\left(\tfrac{x}{n}\right)\right) \tag{5.80}$$

Remark 5.5.1. *Notice that the contribution coming from the second term in the RHS of* (5.79) *does not appear in the case* $k=1$.

First of all we prove that

$$\sum_{r\in\mathscr{R}} p(r)\Delta_n^r\varphi(x) = \tfrac{\chi}{2}\cdot\Delta\varphi\left(\tfrac{x}{n}\right)+\tfrac{1}{n}\psi_n\left(\tfrac{x}{n}\right) \tag{5.81}$$

for a suitable $\psi_n \in S(\mathbb{R})$ such that

$$\sup_n \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\psi_n\left(\tfrac{x}{n}\right) < \infty. \tag{5.82}$$

To prove this we use the Taylor expansion:

$$\varphi\left(\tfrac{x+r}{n}\right) - \varphi\left(\tfrac{x}{n}\right) = \frac{1}{n}\sum_{j=1}^{d} r_j \cdot \frac{\partial \varphi}{\partial x_j}\left(\tfrac{x}{n}\right) + \frac{1}{2n^2}\sum_{j,\ell=1}^{d} r_j r_\ell \frac{\partial^2 \varphi}{\partial x_j \partial x_\ell}\left(\tfrac{x}{n}\right) + \dots \tag{5.83}$$

and then

$$\begin{aligned} & n^2 \sum_{r\in\mathscr{R}} p(r)\left(\varphi\left(\tfrac{x+r}{n}\right) - \varphi\left(\tfrac{x}{n}\right)\right) \\ & = n\sum_{j=1}^{d}\left(\sum_{r\in\mathscr{R}} r_j p(r)\right)\cdot \frac{\partial\varphi}{\partial x_j}\left(\tfrac{x}{n}\right) \\ & +\frac{1}{2}\sum_{j,\ell=1}^{d}\left(\sum_{r\in\mathscr{R}} r_j r_\ell p(r)\right)\frac{\partial^2\varphi}{\partial x_j\partial x_\ell}\left(\tfrac{x}{n}\right) + \dots \end{aligned}$$

for some ψ_n satisfying (5.82). From the assumption (5.2), it follows that:

$$\sum_{r_j=-R}^{R} r_j p(r) = 0 \tag{5.84}$$

thus, from the fact that $\mathscr{R} = [-R,R]^d \cap \mathbb{Z}^d$ we have

$$\sum_{r\in\mathscr{R}} r_j p(r) = 0 \qquad \text{and} \qquad \sum_{r\in\mathscr{R}} r_j r_\ell p(r) = 0 \quad \text{for } j \neq \ell \tag{5.85}$$

as a consequence,

$$\begin{aligned} \sum_{r\in\mathscr{R}} p(r)\Delta_n^r\varphi(x) &= \frac{1}{2}\sum_{\ell=1}^{d}\left(\sum_{r\in\mathscr{R}} r_\ell^2 p(r)\right)\frac{\partial^2\varphi}{\partial x_\ell^2}\left(\tfrac{x}{n}\right) + \tfrac{1}{n}\psi_n\left(\tfrac{x}{n}\right) \\ &= \tfrac{\chi}{2}\cdot\sum_{\ell=1}^{d}\frac{\partial^2\varphi}{\partial x_\ell^2}\left(\tfrac{x}{n}\right) + \tfrac{1}{n}\psi_n\left(\tfrac{x}{n}\right) \end{aligned}$$

from which (5.81) follows.

Now we have

$$\begin{aligned} n^2[\mathscr{L}^{(k)}\Phi_n](\xi) &= \alpha\sum_{x\in\mathbb{Z}^d}\Phi_n(\xi-\delta_x)\cdot\xi_x\cdot\left(\tfrac{\chi}{2}\cdot\Delta\varphi\left(\tfrac{x}{n}\right) + \tfrac{1}{n}\psi_n\left(\tfrac{x}{n}\right)\right) \\ &+ E_n(\varphi,\xi) \end{aligned}$$

with

$$E_n(\varphi,\xi) \quad := \quad \sigma \sum_{x\in\mathbb{Z}^d} \Phi_n(\xi-\delta_x) \sum_{r\in\mathscr{R}} p(r)\cdot \xi_x\xi_{x+r}\cdot n^2 \left(\varphi\left(\tfrac{x+r}{n}\right) - \varphi\left(\tfrac{x}{n}\right)\right) \tag{5.86}$$

Then we have

$$\begin{aligned} &\mathscr{L}\mathscr{Y}^{(n,k)}(\Phi,\eta) - \frac{1}{n^{kd/2}} \sum_{\xi\in\Omega_k} E_n(\varphi,\xi)\cdot \mathfrak{D}(\xi,\eta) \\ &= \frac{\alpha}{n^{kd/2}} \sum_{\xi\in\Omega_k} \mathfrak{D}(\xi,\eta) \sum_{x\in\mathbb{Z}^d} \Phi_n(\xi-\delta_x)\cdot\xi_x\cdot \left(\tfrac{\chi}{2}\cdot\Delta\varphi\left(\tfrac{x}{n}\right) + \tfrac{1}{n}\psi_n\left(\tfrac{x}{n}\right)\right). \end{aligned}$$

It is now convenient to pass to the coordinate notation to treat sums of the type:

$$\sum_{\xi\in\Omega_k} \mathfrak{D}(\xi,\eta) \sum_{x\in\mathbb{Z}^d} \Phi_n(\xi-\delta_x)\cdot\xi_x\cdot\psi(\tfrac{x}{n})$$

for some $\psi\in S(\mathbb{R}^d)$. First of all we notice that summing over $\xi\in\Omega_k$ is the same as summing over $\mathbf{x}\in\mathbb{Z}^{kd}$:

$$\begin{aligned} &\sum_{\xi\in\Omega_k} \mathfrak{D}(\xi,\eta) \sum_{x\in\mathbb{Z}^d} \Phi_n(\xi-\delta_x)\cdot\xi_x\cdot\psi(\tfrac{x}{n}) \\ &= \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \frac{1}{N(\xi(\mathbf{x}))}\cdot \mathfrak{D}(\xi(\mathbf{x}),\eta) \sum_{i=1}^{k} \Phi_n(\xi(\mathbf{x})-\delta_{x_i})\cdot\psi(\tfrac{x_i}{n}) \\ &= \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \frac{\Lambda(\xi(\mathbf{x}))}{N(\xi(\mathbf{x}))}\cdot D(\xi(\mathbf{x}),\eta) \sum_{i=1}^{k} \psi(\tfrac{x_i}{n}) \prod_{\substack{\ell=1\\ \ell\neq i}}^{k} \varphi(\tfrac{x_\ell}{n}) \\ &= k \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \Pi(\mathbf{x})\cdot D(\xi(\mathbf{x}),\eta) \prod_{\ell=1}^{k-1} \varphi(\tfrac{x_\ell}{n})\cdot\psi(\tfrac{x_k}{n}) \\ &= kn^{kd/2}\ \mathscr{X}^{(n,k)}(\varphi^{(k-1)}\otimes\psi,\eta) \end{aligned}$$

where the last identity follows using the expression of the field acting on more general (i.e., non-symmetric) test functions (5.67). Then, substituting in (6.56) we get

$$\begin{aligned} &\mathscr{L}\mathscr{Y}^{(n,k)}(\Phi,\eta) - \frac{1}{n^{kd/2}} \sum_{\xi\in\Omega_k} E_n(\varphi,\xi)\cdot\mathfrak{D}(\xi,\eta) \\ &= \alpha k \mathscr{X}^{(n,k)}\left(\varphi^{(k-1)}\otimes(\tfrac{\chi}{2}\Delta\varphi + \tfrac{1}{n}\psi_n),\eta\right) \end{aligned}$$

where we used the fact that φ is uniformly bounded on $\mathbb{Z}$. From this we can see that it is possible to close the equation for the second-order fluctuation field, modulo an error term that we define as follows

$$\mathscr{E}^{(n,k)}(\varphi,\eta) := \mathscr{L}\mathscr{Y}^{(n,k)}(\Phi,\eta) - \alpha k \cdot \tfrac{\chi}{2} \cdot \mathscr{X}^{(n,k)}(\varphi^{(k-1)} \otimes \Delta\varphi, \eta) \tag{5.87}$$

Then we have

$$\mathscr{E}^{(n,k)}(\varphi,\eta) = \mathscr{E}_1^{(n,k)}(\varphi,\eta) + \mathscr{E}_2^{(n,k)}(\varphi,\eta) \tag{5.88}$$

with

$$\mathscr{E}_1^{(n,k)}(\varphi,\eta) := \frac{\alpha k}{n}\ \mathscr{X}^{(n,k)}\left(\varphi^{(k-1)} \otimes \psi_n, \eta\right)$$

and

$$\mathscr{E}_2^{(n,k)}(\varphi,\eta) := \frac{1}{n^{kd/2}} \sum_{\xi\in\Omega_k} E_n(\varphi,\xi)\mathfrak{D}(\xi,\eta) \tag{5.89}$$

that has to be estimated. Analogously to the previous computation we have

$$\begin{aligned}
E_n(\varphi,\xi(\mathbf{x})) &= \sigma n^2 \sum_{i=1}^{k} \Big(\prod_{\substack{\ell=1\\ \ell\neq i}}^{k} \varphi(\tfrac{x_\ell}{n}) \Big) \cdot \sum_{r\in\mathscr{R}} p(r) \left(\sum_{j=1}^{k} \mathbf{1}_{x_j = x_i + r} \right) \\
&\times \ \big(\varphi(\tfrac{x_i+r}{n}) - \varphi(\tfrac{x_i}{n})\big) \\
&= \sigma n^2 \sum_{i=1}^{k} \Big(\prod_{\substack{\ell=1\\ \ell\neq i}}^{k} \varphi(\tfrac{x_\ell}{n}) \Big) \cdot \sum_{j=1}^{k} p(x_j - x_i) \big(\varphi(\tfrac{x_j}{n}) - \varphi(\tfrac{x_i}{n})\big) \\
&= \sigma n^2 \sum_{i,j=1}^{k} \Big(\prod_{\substack{\ell=1\\ \ell\neq i,j}}^{k} \varphi(\tfrac{x_\ell}{n}) \Big) \cdot p(x_j - x_i)\varphi(\tfrac{x_j}{n}) \big(\varphi(\tfrac{x_j}{n}) - \varphi(\tfrac{x_i}{n})\big) \\
&= \sigma \sum_{\substack{\{i,j\}\\ 1\le i,j\le k}} \Big(\prod_{\substack{\ell=1\\ \ell\neq i,j}}^{k} \varphi(\tfrac{x_\ell}{n}) \Big) \cdot p(x_j - x_i) n^2 \big(\varphi(\tfrac{x_j}{n}) - \varphi(\tfrac{x_i}{n})\big)^2
\end{aligned}$$

where in the last step we used the symmetry of $p(\cdot)$. Then

$$
\begin{aligned}
&\mathscr{E}_2^{(n,k)}(\varphi,\eta) = \\
&= \frac{1}{n^{kd/2}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \Pi(\mathbf{x}) \cdot D(\xi(\mathbf{x}),\eta) \cdot E_n(\varphi,\xi(\mathbf{x})) \\
&= \frac{\sigma}{n^{kd/2}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \Pi(\mathbf{x}) \cdot D(\xi(\mathbf{x}),\eta) \cdot \sum_{\substack{\{i,j\}\\ 1\le i,j\le k}} \Big(\prod_{\substack{\ell=1\\ \ell\neq i,j}}^{k} \varphi(\tfrac{x_\ell}{n}) \Big) \cdot p(x_j - x_i) \\
\times\quad & n^2 \left(\varphi(\tfrac{x_j}{n}) - \varphi(\tfrac{x_i}{n})\right)^2 \\
&= \frac{k(k-1)\sigma}{2n^{kd/2}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \Pi(\mathbf{x}) \cdot D(\xi(\mathbf{x}),\eta) \cdot \Big(\prod_{\ell=1}^{k-2} \varphi(\tfrac{x_\ell}{n}) \Big) \cdot p(x_k - x_{k-1}) \\
\times\quad & n^2 \left(\varphi(\tfrac{x_{k-1}}{n}) - \varphi(\tfrac{x_k}{n})\right)^2 .
\end{aligned}
$$

Hence we have

$$
\mathscr{E}^{(n,k)}(\varphi,\eta) = \frac{k}{n^{kd/2}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \Pi(\mathbf{x}) \cdot D(\xi(\mathbf{x}),\eta) \cdot \Psi_n(\mathbf{x}) \tag{5.90}
$$

with

$$
\begin{aligned}
\Psi_n(\mathbf{x}) \quad &:= \quad \varphi^{(k-2)}(x_1,\ldots,x_{k-2}) \otimes \left(\tfrac{\alpha}{n}\, \varphi(x_{k-1}) \cdot \psi_n(\tfrac{x_k}{n}) \right. \\
&+ \quad \left. \tfrac{\sigma(k-1)}{2} p(x_k - x_{k-1}) n^2 \left(\varphi(\tfrac{x_{k-1}}{n}) - \varphi(\tfrac{x_k}{n})\right)^2 \right).
\end{aligned} \tag{5.91}
$$

It remains to show that the $L^2(\mathbb{P}_n)$ norm of $\mathscr{E}^{(n,k)}(\varphi,\eta(n^2t))$ vanishes in the limit as n goes to infinity. This is done in the following lemma:

Lemma 5.5.1. *Let $\mathscr{E}^{(n,k)}(\varphi,\eta)$ be given by (5.87), then, for every test function $\varphi \in \hat{S}(\mathbb{R}^d)$ there exists $C > 0$ such that, for all $t \geq 0$ and $n \in \mathbb{N}$,*

$$
\mathbb{E}_n \left[\left(\int_0^t \mathscr{E}^{(n,k)}(\varphi,\eta(n^2 s)) ds \right)^2 \right] \leq C \cdot \frac{t^2}{n}. \tag{5.92}
$$

Proof. Using the fact that φ is bounded and that $p(\cdot)$ has finite range we can conclude that there exists an $M > 0$ such that

$$
\sup_n \sup_{\mathbf{x}\in\mathbb{Z}^{kd}} |\Psi_n(\mathbf{x})| \leq M. \tag{5.93}
$$

We recall here that the duality function is parametrized by the density parameter ρ, i.e. $D(\cdot,\cdot) = D_\rho(\cdot,\cdot)$ and that $\{D_\rho(\xi,\cdot), \xi \in \Omega\}$ is a family of products of polynomials that are orthogonal with respect to the reversible measure ν_ρ. From the stationarity of ν_ρ we have

$$\mathbb{E}_n\left[\left(\int_0^t \mathscr{E}^{(n,k)}(\varphi,\eta(n^2 s))ds\right)^2\right]$$
$$= \int_0^t\int_0^t \mathbb{E}_n\left[\mathscr{E}^{(n,k)}(\varphi,\eta_{n^2 s})\mathscr{E}^{(n,k)}(\varphi,\eta_{n^2 u})\right] du\, ds$$
$$= 2\int_0^t\int_0^s\int \mathbb{E}_\eta\left[\mathscr{E}^{(n,k)}(\varphi,\eta_{n^2(s-u)})\right]\mathscr{E}^{(n,k)}(\varphi,\eta)\nu_\rho(d\eta)du\, ds. \tag{5.94}$$

The fact that we can exchange expectations and integral is a consequence of Proposition 5.5.1 in Section 5.5.2.2, which does not use any results of the current section.

Let us denote by $V_n(\varphi)$ the integrand in (5.94), then, using (5.18), we have

$$\begin{aligned}
V_n(\varphi) &= \frac{1}{n^{kd}}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^{kd}}\Psi_n(\mathbf{x})\Psi_n(\mathbf{y})\cdot\Pi(\mathbf{x})\Pi(\mathbf{y})\\
&\times \int \mathbb{E}_\eta\left[D_\rho(\xi(\mathbf{x}),\eta_{n^2(s-u)})\right] D_\rho(\xi(\mathbf{y}),\eta)\nu_\rho(d\eta)\\
&= \frac{1}{n^{kd}}\sum_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^{kd}}\Psi_n(\mathbf{x})\Psi_n(\mathbf{y})\cdot\Pi(\mathbf{x})\Pi(\mathbf{y})\\
&\times \frac{1}{\mu_\rho(\xi(\mathbf{y}))}\cdot p_{n^2(s-u)}(\xi(\mathbf{x}),\xi(\mathbf{y}))\\
&= \frac{c}{n^{kd}}\sum_{\mathbf{x}\in\mathbb{Z}^{kd}}\Psi_n(\mathbf{x})\cdot\Pi(\mathbf{x})\\
&\times \sum_{\mathbf{y}\in\mathbb{Z}^{kd}}\frac{1}{N(\xi(\mathbf{y}))}\cdot\Psi_n(\mathbf{y})\cdot p_{n^2(s-u)}(\xi(\mathbf{x}),\xi(\mathbf{y}))\\
&\le \frac{cM}{n^{kd}}\sum_{\mathbf{x}\in\mathbb{Z}^{kd}}\mid\Psi_n(\mathbf{x})\mid\cdot\Pi(\mathbf{x})\sum_{\mathbf{y}\in\mathbb{Z}^{kd}}\frac{1}{N(\xi(\mathbf{y}))}\cdot p_{n^2(s-u)}(\xi(\mathbf{x}),\xi(\mathbf{y}))\\
&= \frac{cM}{n^{kd}}\sum_{\mathbf{x}\in\mathbb{Z}^{kd}}\mid\Psi_n(\mathbf{x})\mid\cdot\Pi(\mathbf{x})\sum_{\xi'\in\Omega_k}\cdot p_{n^2(s-u)}(\xi(\mathbf{x}),\xi')\\
&\le \frac{c'M}{n^{kd}}\sum_{\mathbf{x}\in\mathbb{Z}^{kd}}\mid\Psi_n(\mathbf{x})\mid
\end{aligned} \tag{5.95}$$

where we used (5.18) in the second identity, (5.57) and (5.20) in the third identity (with $c = c(k, \rho)$) and (5.93) in the fourth line. From (5.91) we have

$$\begin{aligned}
&\frac{1}{n^{kd}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \mid \Psi_n(\mathbf{x}) \mid \\
&\leq \frac{\alpha}{n^{kd+1}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \mid \psi_n \mid \left(\tfrac{x_k}{n}\right) \cdot \prod_{\ell=1}^{k-1} \mid \varphi(\tfrac{x_\ell}{n}) \mid \qquad (5.96)\\
&+\frac{\sigma(k-1)}{2n^{kd}} \sum_{\mathbf{x}\in\mathbb{Z}^{kd}} \prod_{\ell=3}^{k} \mid \varphi(\tfrac{x_\ell}{n}) \mid \cdot p(x_2 - x_1) n^2 \left(\varphi(\tfrac{x_2}{n}) - \varphi(\tfrac{x_1}{n})\right)^2 .
\end{aligned}$$

Using (5.82), we have that the first term in the r.h.s. of (5.96) is bounded by a constant times n^{-1}. For what concerns the second term, we have:

$$\begin{aligned}
&\frac{\sigma(k-1)}{2n^{(k-2)d}} \left(\prod_{\ell=3}^{k} \sum_{x_\ell\in\mathbb{Z}^d} \varphi(\tfrac{x_\ell}{n})\right)\\
&\quad\times \frac{1}{n^{2d}} \sum_{x_1,x_2\in\mathbb{Z}^d} p(x_2-x_1)n^2 \left(\varphi\left(\tfrac{x_2}{n}\right) - \varphi\left(\tfrac{x_1}{n}\right)\right)^2\\
&\quad\leq \frac{c}{n^{2d}} \sum_{x_1,x_2\in\mathbb{Z}^d} p(x_2-x_1)n^2 \left(\varphi\left(\tfrac{x_2}{n}\right) - \varphi\left(\tfrac{x_1}{n}\right)\right)^2
\end{aligned}$$

Now, from the Taylor expansion (5.83) we know that there exists a sequence of functions where, using the fact that the range of $p(\cdot)$ is $\mathscr{R} = [-R, R]^d$, and the Taylor expansion (5.83) we have that there exists a smooth function $\widetilde{\psi} \in S(\mathbb{R}^d)$ such that, for all $x \in \mathbb{Z}^d$,

$$\sup_{r\in\mathbb{R}}\{n^2 \left(\varphi\left(\tfrac{x+r}{n}\right) - \varphi\left(\tfrac{x}{n}\right)\right)^2\} \leq \widetilde{\psi}\left(\tfrac{x}{n}\right) \qquad (5.97)$$

as a consequence we obtain the upper bound

$$\begin{aligned}
&\frac{1}{n^{2d}} \sum_{x_1,x_2\in\mathbb{Z}^d} p(x_2-x_1)n^2 \left(\varphi(\tfrac{x_2}{n}) - \varphi(\tfrac{x_1}{n})\right)^2\\
&= \frac{1}{n^{2d}} \sum_{x\in\mathbb{Z}^d} \sum_{r\in\mathscr{R}} p(r) n^2 \left(\varphi\left(\tfrac{x+r}{n}\right) - \varphi\left(\tfrac{x}{n}\right)\right)^2\\
&\leq \frac{1}{n^{2d}} \sum_{r\in\mathscr{R}} \sum_{x\in\mathbb{Z}^d} p(r) \cdot \widetilde{\psi}\left(\tfrac{x}{n}\right) \leq \frac{c}{n^{2d}} \sum_{x\in\mathbb{Z}^d} \widetilde{\psi}\left(\tfrac{x}{n}\right) \leq \frac{c'}{n^d} \qquad (5.98)
\end{aligned}$$

where the inequality holds for a suitable $c' > 0$. In conclusion we have that there exists a constant $C > 0$ such that

$$V_n(\varphi) \quad \leq \quad \frac{C}{n} \qquad (5.99)$$

from which the statement follows. □

As a consequence of Lemma 5.5.1 we can close the drift term, i.e.

$$\begin{aligned}\mathscr{L}\mathscr{Y}^{(n,k)}(\Phi,\eta) &= \alpha k \cdot \tfrac{\chi}{2} \cdot \mathscr{X}^{(n,k)}(\varphi^{(k-1)} \otimes \Delta\varphi, \eta) + \mathscr{E}^{(n,k)}(\varphi,\eta) \\ &= \alpha k \cdot \tfrac{\chi}{2} \cdot \mathscr{X}^{(n,k)}(\varphi^{(k-1)} \otimes \Delta\varphi, \eta) + O(n^{-1})\end{aligned} \tag{5.100}$$

5.5.2 Closing the equation for the carré-du-champ

In this section we will show that the integrand in the RHS of equation (5.75) can be expressed in terms of the $(k-1)$th-order fluctuation field $\mathscr{Y}^{(n,k-1)}$. To achieve this we consider the expression for the carré-du-champ given by (A.27) in the Appendix. For the case of our kth-order fluctuation field this becomes

$$\begin{aligned}n^2\Gamma\mathscr{Y}^{(n,k)}(\Phi,\eta) &= \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\eta_x(\alpha+\sigma\eta_{x+r}) \\ &\times \left[n^{d/2+1}\left(\mathscr{Y}^{(n,k)}(\Phi,\eta^{x,x+r}) - \mathscr{Y}^{(n,k)}(\Phi,\eta)\right)\right]^2\end{aligned} \tag{5.101}$$

Notice that here we multiplied by a factor $n^{d/2+1}$ the term which is squared in order to cancel the n^2 in front of the carré-du-champ and get a general factor n^{-d} in front of the sum.

In the next section we find some recursion relations for duality polynomials. The main application of these relations consists in allowing us to rewrite any polynomial depending on $\eta^{x,x+r}$ in terms of polynomials depending on the unmodified η.

5.5.2.1 Recursion relation for duality polynomials

In this section we obtain a recurrence relation for the single-site orthogonal polynomials. Before giving the result it is convenient to summarize the expression for the self-duality generating function by defining the function

$$f_\sigma(t,n) := \sum_{m=0}^{\infty} \mathfrak{d}(m,n)\cdot t^m \tag{5.102}$$

then f_σ can be written in the form

$$f_\sigma(t,n) = e_\sigma(t)\cdot h_\sigma(t)^n, \quad h_\sigma(t) = \frac{1+c_\sigma b_\sigma t}{1-\sigma^2 t}, \quad e_\sigma(t) = \begin{cases} e^{-t} & \text{if} \quad \sigma = 0 \\ (1-t)^{-\sigma\alpha} & \text{if} \quad \sigma = \pm 1\end{cases} \tag{5.103}$$

with c_σ given by (5.40), and b_σ is given as follows:

$$b_\sigma = \begin{cases} 1 & \text{if} \quad \sigma = 0 \\ \alpha + \rho & \text{if} \quad \sigma = 1 \\ \alpha - \rho & \text{if} \quad \sigma = -1 \end{cases} . \tag{5.104}$$

Then we define the functions $g_\sigma, \tilde{g}_\sigma : \mathbb{N} \to \mathbb{R}$ given by

$$g_\sigma(m) := \frac{1}{m!} \frac{d^m}{dt^m} h_\sigma(t) \Big|_{t=0} \quad \text{and} \quad \tilde{g}_\sigma(m) := \frac{1}{m!} \frac{d^m}{dt^m} \frac{1}{h_\sigma(t)} \Big|_{t=0} \quad \text{for } m \geq 1$$

$$\text{and} \quad g_\sigma(0) = \tilde{g}_\sigma(0) := 1 \tag{5.105}$$

that are exactly computable:

$$g_\sigma(m) = \begin{cases} \frac{1}{\rho} \cdot \mathbf{1}_{m=1} & \sigma = 0 \\ -\frac{\alpha}{\rho} & \sigma = +1 \\ \frac{\alpha}{\rho} & \sigma = -1 \end{cases}$$

$$\tilde{g}_\sigma(m) = \begin{cases} \left(\frac{1}{\rho}\right)^m & \sigma = 0 \\ \left(-\frac{1}{\rho}\right)^{m-1} \cdot (\alpha + \rho)^{m-1} \left(-\frac{\alpha}{\rho}\right) & \sigma = +1 \\ \left(\frac{1}{\rho}\right)^{m-1} \cdot (\alpha - \rho)^{m-1} \left(\frac{\alpha}{\rho}\right) & \sigma = -1 \end{cases}$$

for $m \geq 1$, that can be rewritten as

$$\tilde{g}_\sigma(1) = c_\sigma = g_\sigma(1), \tag{5.106}$$

and

$$g_\sigma(m) = \left(c_\sigma b_\sigma + \sigma^2\right) \cdot \sigma^{2m-2} \ , \ \tilde{g}_\sigma(m) = c_\sigma^{m-1} b_\sigma^{m-1} \left(c_\sigma b_\sigma + \sigma^2\right) \quad \text{for } m \geq 2. \tag{5.107}$$

We have the following result.

Theorem 5.5.1. *For any $m, n \in \mathbb{N}$ we have*

$$\mathfrak{d}(m, n+1) = \sum_{j=0}^{m} g(m-j) \cdot \mathfrak{d}(j, n) \tag{5.108}$$

and

$$\mathfrak{d}(m, n-1) = \sum_{j=0}^{m} \tilde{g}(m-j) \cdot \mathfrak{d}(j, n) \tag{5.109}$$

with $g, \tilde{g} : \mathbb{N} \to \mathbb{R}$ as in (5.105)-(5.107).

Proof. From (5.103) we have that

$$f(t, n+1) = f(t, n)h(t) \tag{5.110}$$

then, from the generating function definition (5.102), we deduce that

$$\mathfrak{d}(m, n) = \frac{1}{m!} \cdot \frac{d^m}{dt^m} f(t, n)\Big|_{t=0} \tag{5.111}$$

hence, the recurrence relation (5.110) and an application of Leibniz product rule for differentiation in the RHS above give

$$\begin{aligned}
\mathfrak{d}(m, n+1) &= \frac{1}{m!} \cdot \sum_{j=0}^{m} \binom{m}{j} \frac{d^j}{dt^j} f(t, n)\Big|_{t=0} \cdot \frac{d^{m-j}}{dt^{m-j}} h(t)\Big|_{t=0} \\
&= \frac{1}{m!} \cdot \sum_{j=0}^{m} \binom{m}{j} j! \cdot \mathfrak{d}(j, n) \cdot \frac{d^{m-j}}{dt^{m-j}} h(t)\Big|_{t=0} \\
&= \sum_{j=0}^{m} \frac{1}{(m-j)!} \cdot \frac{d^{m-j}}{dt^{m-j}} h(t)\Big|_{t=0} \cdot \mathfrak{d}(j, n) \\
&= \sum_{j=0}^{m} g(m-j) \cdot \mathfrak{d}(j, n)
\end{aligned}$$

where in the second equality we used (5.111). This concludes the proof of (5.108). Equation (5.109) can be proved from the same reasoning, with the difference that we now have the inverse relation

$$f(t, n-1) = f(t, n) \cdot \frac{1}{h(t)}. \tag{5.112}$$

This change results, after the application of Leibniz rule, in the relation

$$\begin{aligned}
\mathfrak{d}(m, n-1) &= \frac{1}{m!} \cdot \sum_{j=0}^{m} \binom{m}{j} j! \cdot \mathfrak{d}(j, n) \cdot \frac{d^{m-j}}{dt^{m-j}} \frac{1}{h(t)}\Big|_{t=0} \\
&= \sum_{j=0}^{m} \tilde{g}(m-j) \cdot \mathfrak{d}(j, n)
\end{aligned}$$

that concludes the proof. □

5.5.2.2 Controlling the moments of the fields

The objective of this section is to take advantage of the ergodic properties of our process to introduce a result that will allow us to make multiple replacements, in the appropriate sense, inside the expression of the carré-du-champ given in (5.101). Let us start first with a uniform estimate for moments of the fields $\mathscr{Y}^{(n,l)}(\Phi, \eta)$.

Proposition 5.5.1. *Let $l, m \in \mathbb{N}$, then we have*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\nu_\rho}\left[\mathscr{Y}^{(n,l)}(\Phi,\eta)^m\right] \leq C(\rho,\varphi) \tag{5.113}$$

Proof. As claimed in the statement of the proposition, this result holds for any finite natural number m. Nevertheless for simplicity we will only show how to obtain the estimates for $m \in \{2, 4\}$ (which indeed are the only two uses that we make of this result). Let us start with the simplest non-trivial case, $m = 2$, for which the result comes directly from orthogonality

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\left[\mathscr{Y}^{(n,l)}(\Phi,\eta)^2\right] \\
&= n^{-ld} \sum_{\xi,\xi'\in\Omega_l} \Phi_n(\xi)\Phi_n(\xi')\Lambda(\xi)\Lambda(\xi')\mathbb{E}_{\nu_\rho}\left[D(\xi,\eta)D(\xi',\eta)\right] && (5.114)\\
&= n^{-ld} \sum_{\xi\in\Omega_l} \Phi_n(\xi)^2\Lambda(\xi)^2\frac{1}{\mu_\rho(\xi)} && (5.115)\\
&\leq K\cdot n^{-ld} \sum_{\xi\in\Omega_l} \Phi_n(\xi)^2 < \infty && (5.116)
\end{aligned}$$

where in the second line we used (5.16) and K is given by

$$K = \sup_{\xi\in\Omega_k} \frac{\Lambda(\xi)^2}{\mu_\rho(\xi)}.$$

Notice that the previous estimate was possible due the fact that orthogonality, in the form of expression (5.16), allowed us to reduce the summation in the RHS of (5.114) from a $2ld$-dimensional sum to an ld-dimensional sum in (5.115).

For the case $m = 4$ we have

$$\mathbb{E}_{\nu_\rho}\left[\mathscr{Y}^{(n,l)}(\Phi,\eta)^4\right] = n^{-2ld} \sum_{\xi^{(j)}\in\Omega_l} \prod_{j=1}^{4} \Phi_n(\xi^{(j)})\cdot\Lambda(\xi^{(j)})\cdot \mathbb{E}_{\nu_\rho}\left[D(\xi^{(1)},\eta)D(\xi^{(2)},\eta)D(\xi^{(3)},\eta)D(\xi^{(4)},\eta)\right] \tag{5.117}$$

For this case the sum in the RHS of (5.117) is $4ld$-dimensional. Given the factor n^{-2ld} in front of the RHS, in order to obtain a uniform estimate, we would like this summation to be $2ld$-dimensional instead. In order to see that this is indeed the case, we analyze the non-zero contribution coming from

$$\mathbb{E}_{\nu_\rho}\left[D(\xi^{(1)},\eta)D(\xi^{(2)},\eta)D(\xi^{(3)},\eta)D(\xi^{(4)},\eta)\right]$$

By the product nature of the measure ν_ρ and the duality polynomials we have

$$\begin{aligned}\mathbb{E}_{\nu_\rho}&\left[D(\xi^{(1)},\eta)D(\xi^{(2)},\eta)D(\xi^{(3)},\eta)D(\xi^{(4)},\eta)\right]\\&=\prod_{x\in\mathbb{Z}^d}\mathbb{E}_{\nu_\rho}\left[d(\xi_x^{(1)},\eta)d(\xi_x^{(2)},\eta)d(\xi_x^{(3)},\eta)d(\xi_x^{(4)},\eta)\right]\end{aligned} \tag{5.118}$$

Notice that for every x for which $\xi_x^{(j)}=0$ for all $j\in\{1,2,3,4\}$, the corresponding contribution in the RHS of (5.118) is equal to 1 and therefore negligible. This is precisely the reason why the summation in the RHS of (5.117) is at most $4ld$-dimensional. We have indeed that the maximum number of $x\in\mathbb{Z}^d$ contributing to the product in the RHS of (5.118) is at most $4l$, i.e. one for each of the $4l$ particles that all the $\xi^{(j)}$ have in total. In reality we can see that there are less x's giving a non-zero contribution. In order to see this, consider an $x\in\mathbb{Z}^d$ such that there exists a unique $j\in\{1,2,3,4\}$ for which $\xi_x^{(j)}\neq 0$. In this case, because of the zero mean of the single-site duality function we have

$$\mathbb{E}_{\nu_\rho}\left[d(\xi_x^{(1)},\eta)d(\xi_x^{(2)},\eta)d(\xi_x^{(3)},\eta)d(\xi_x^{(4)},\eta)\right]=0 \tag{5.119}$$

this means that whenever $x\in\mathbb{Z}^d$ is such that there exists a $j\in\{1,2,3,4\}$ for which $\xi_x^{(j)}\neq 0$, there must be another $j'\in\{1,2,3,4\}$ for which $\xi_x^{(j')}\neq 0$. In other words we only have a possibility of $2l$ particles to distribute freely, and hence the summation in the RHS of (5.117) is at most $2ld$-dimensional. □

Proposition 5.5.2. *Let $f:\mathbb{R}^d\to\mathbb{R}$ be a test function, and let $\{M_n:\Omega\times\mathbb{R}\to\mathbb{R}:n\in\mathbb{N}\}$ be a sequence of uniformly bounded cylindrical functions of the form*

$$M_n(\eta,x)=f(x/n)\prod_{j\in\mathbb{N}}d(b_j,\eta_x) \tag{5.120}$$

where only a finite number of b_j are different from zero. Let also $\{a_n:n\in\mathbb{N}\}$ be a sequence of real numbers converging to 0, we then have

$$\lim_{n\to\infty}\mathbb{E}_n\left[\left(\int_0^t\frac{a_n}{n^d}\sum_{x\in\mathbb{Z}^d}c(\eta(n^2s),x)\cdot\mathscr{Y}^{(n,l)}(\Phi,\eta(n^2s))^m\,ds\right)^2\right]=0 \tag{5.121}$$

for all $l\in\{1,2,\dots,k-1\}$, $m\in\mathbb{N}$, and where

$$c(\eta,x)=\sum_{r\in\mathscr{R}}p(r)\eta_x(\alpha+\sigma\eta_{x+r})M_n(\eta,x) \tag{5.122}$$

Proof. By Cauchy-Schwarz we have

$$\begin{aligned}
&\mathbb{E}_n\left[\left(\int_0^t \frac{a_n}{n^d}\sum_{x\in\mathbb{Z}^d} c(\eta(n^2s),x)\cdot \mathscr{Y}^{(n,l)}(\Phi,\eta(n^2s))^m\,ds\right)^2\right]\\
\le\ & \frac{a_n^2 t}{n^{2d}}\int_0^t \mathbb{E}_n\left[\mathscr{Y}^{(n,l)}(\Phi,\eta(n^2s))^{2m}\cdot\left(\sum_{x\in\mathbb{Z}^d} c(\eta(n^2s),x)\right)^2\right]ds\\
=\ & \frac{a_n^2 t^2}{n^{2d}}\mathbb{E}_n\left[\mathscr{Y}^{(n,l)}(\Phi,\eta)^{2m}\cdot\left(\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\eta_x(\alpha+\sigma\eta_{x+r})\cdot M_n(\eta_x)\cdot\right)^2\right]\\
=\ & \frac{a_n^2 t^2}{n^{2d}}\sum_{x,y\in\mathbb{Z}^d}\sum_{r_1,r_2\in\mathscr{R}} p(r_1)\cdot p(r_2)\\
\times\ & \mathbb{E}_n\left[M_n(\eta_x)\cdot M_n(\eta_y)\cdot\mathscr{Y}^{(n,l)}(\Phi,\eta)^{2m}\right]\\
\le\ & \frac{a_n^2 t^2}{n^{2d}}\sum_{x,y\in\mathbb{Z}^d}\sum_{r_1,r_2\in\mathscr{R}} p(r_1)\cdot p(r_2)\\
\times\ & \sqrt{\mathbb{E}_n\left[M_n(\eta_x)^2\cdot M_n(\eta_y)^2\right]}\cdot\sqrt{\mathbb{E}_n\left[\cdot\mathscr{Y}^{(n,l)}(\Phi,\eta)^{4m}\right]}\\
\le\ & Kt^2a_n^2
\end{aligned}\tag{5.123}$$

where in the last line we used Proposition 5.5.1, the boundedness of the single-site duality polynomials $d(b_j,\eta_x)$ and the smoothness of f in the representation (5.120). The result then follows from the convergence $a_n\to 0$. □

5.5.2.3 The gradient of the fluctuation fields

Our goal for this section is to rewrite the square inside the RHS of (5.101) in terms of lower-order fluctuation fields. We will see that this can be expressed, in agreement with (5.73) , only in terms of the field of order $k-1$. Let us then denote by $\nabla_d^{i,i+r}$ the d-dimensional gradient

$$\nabla_d^{i,i+r}\mathscr{Y}^{(n,k)}(\Phi,\eta)=n^{d/2+1}\left(\mathscr{Y}^{(n,k)}(\Phi,\eta^{i,i+r})-\mathscr{Y}^{(n,k)}(\Phi,\eta)\right).\tag{5.124}$$

Notice that, by linearity of the k-th-order field, we have

$$\nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta)\ :=\ n^{-\frac{(k-1)d}{2}+1}\sum_{\xi\in\Omega_k}\Phi_n(\xi)\left[\mathfrak{D}(\xi,\eta^{i,j})-\mathfrak{D}(\xi,\eta)\right]\tag{5.125}$$

with $\mathfrak{D}(\cdot,\cdot)$ as in (5.47). We define now, for $i,j \in \mathbb{Z}^d$, $\ell \leq k$, the auxiliary field

$$\mathscr{Z}_{i,j}^{(n,k,\ell)}(\Phi,\eta) := n^{-kd/2} \sum_{\xi\in\Omega_k} \mathbf{1}_{\xi_i+\xi_j=\ell} \cdot \Phi_n(\xi)\mathfrak{D}(\xi,\eta) \tag{5.126}$$

then we have the following formula for the gradient of the fluctuation field.

Proposition 5.5.3. *We have the following relation*

$$\begin{aligned}&\nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta)\\ &= \sum_{s=1}^{k} n^{-\frac{(s-1)d}{2}} \cdot \sum_{m=1}^{s} n\left(\varphi(\tfrac{j}{n})^m - \varphi(\tfrac{i}{n})^m\right) \cdot g(m) \cdot \mathscr{Z}_{i,j}^{(n,k-m,s-m)}(\varphi,\eta-\delta_i)\end{aligned}$$

Proof. Using the product nature of the polynomials $\mathfrak{D}(\cdot,\eta)$ and of $\Phi_n(\cdot)$ we get

$$\nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta) = n\sum_{s=1}^{k} n^{-\frac{(s-1)d}{2}} \cdot \mathscr{Z}_{i,j}^{(n,k-s,0)}(\varphi,\eta) \cdot \sum_{a=0}^{s} Y_{i,j}^{(n,a,s-a)}(\varphi,\eta) \tag{5.127}$$

and

$$\begin{aligned}&Y_{i,j}^{(n,a,b)}(\varphi,\eta)\\ &:= \varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^b\left\{\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(b,\eta_j+1) - \mathfrak{d}(a,\eta_i)\mathfrak{d}(b,\eta_j)\right\}\\ &= \varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^b\Big\{\mathfrak{d}(a,\eta_i-1)\left[\mathfrak{d}(b,\eta_j+1)-\mathfrak{d}(b,\eta_j)\right]\\ &+ \mathfrak{d}(b,\eta_j)\left[\mathfrak{d}(a,\eta_i-1)-\mathfrak{d}(a,\eta_i)\right]\Big\}\end{aligned}$$

hence, using (5.108) we get

$$\begin{aligned}&\sum_{a=0}^{s} Y_{i,j}^{(n,a,s-a)}(\varphi,\eta)\\ &= \sum_{a=0}^{s-1}\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{s-a}\mathfrak{d}(a,\eta_i-1)\left[\mathfrak{d}(s-a,\eta_j+1)-\mathfrak{d}(s-a,\eta_j)\right]\\ -&\sum_{b=0}^{s-1}\varphi(\tfrac{i}{n})^{s-b}\varphi(\tfrac{j}{n})^{b}\mathfrak{d}(b,\eta_j)\left[\mathfrak{d}(s-b,\eta_i)-\mathfrak{d}(s-b,\eta_i-1)\right]\\ =&\sum_{a=0}^{s-1}\sum_{\kappa=0}^{s-a-1}\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{s-a}\cdot g(s-a-\kappa)\cdot\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\kappa,\eta_j)\\ -&\sum_{b=0}^{s-1}\sum_{m=0}^{s-b-1}\varphi(\tfrac{i}{n})^{s-b}\varphi(\tfrac{j}{n})^{b}\cdot g(s-b-m)\cdot\mathfrak{d}(m,\eta_i-1)\mathfrak{d}(b,\eta_j)\end{aligned}$$

Now, calling $b=\kappa$ and $m=a$, we get

$$\begin{aligned}
&\sum_{a=0}^{s-1}\sum_{\kappa=0}^{s-a-1}\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{s-a}\cdot g(s-a-\kappa)\cdot\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\kappa,\eta_j)\\
&-\sum_{\kappa=0}^{s-1}\sum_{a=0}^{s-\kappa-1}\varphi(\tfrac{i}{n})^{s-\kappa}\varphi(\tfrac{j}{n})^{\kappa}\cdot g(s-a-\kappa)\cdot\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\kappa,\eta_j)\\
&=\sum_{a=0}^{s-1}\sum_{\ell=a}^{s-1}\Big(\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{s-a}-\varphi(\tfrac{i}{n})^{s+a-\ell}\varphi(\tfrac{j}{n})^{\ell-a}\Big)\cdot g(s-\ell)\\
&\times\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\ell-a,\eta_j)\\
&=\sum_{a=0}^{s-1}\sum_{\ell=a}^{s-1}\Big(\varphi(\tfrac{j}{n})^{s-\ell}-\varphi(\tfrac{i}{n})^{s-\ell}\Big)\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{\ell-a}\cdot g(s-\ell)\\
&\times\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\ell-a,\eta_j)\\
&=\sum_{\ell=0}^{s-1}\Big(\varphi(\tfrac{j}{n})^{s-\ell}-\varphi(\tfrac{i}{n})^{s-\ell}\Big)\cdot g(s-\ell)\sum_{a=0}^{\ell}\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{\ell-a}\\
&\times\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\ell-a,\eta_j)
\end{aligned}$$

where the first identity follows from the change of variable $\ell=\kappa+a$. Then

$$\begin{aligned}
&\mathscr{Z}_{i,j}^{(n,k-s)}(\Phi,\eta)\cdot\sum_{a=0}^{s}Y_{i,j}^{(n,a,s-a)}(\varphi,\eta)\\
&=\sum_{\ell=0}^{s-1}\Big(\varphi(\tfrac{j}{n})^{s-\ell}-\varphi(\tfrac{i}{n})^{s-\ell}\Big)\cdot g(s-\ell)\cdot\mathscr{Z}_{i,j}^{(n,k-s,0)}(\Phi,\eta)\\
&\times\sum_{a=0}^{\ell}\varphi(\tfrac{i}{n})^a\varphi(\tfrac{j}{n})^{\ell-a}\cdot\mathfrak{d}(a,\eta_i-1)\mathfrak{d}(\ell-a,\eta_j)\\
&=\sum_{\ell=0}^{s-1}\Big(\varphi(\tfrac{j}{n})^{s-\ell}-\varphi(\tfrac{i}{n})^{s-\ell}\Big)\cdot g(s-\ell)\cdot\mathscr{Z}_{i,j}^{(n,k-(s-\ell),\ell)}(\Phi,\eta-\delta_i)
\end{aligned}\tag{5.128}$$

Then

$$\begin{aligned}
\nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta) &= n\sum_{s=1}^{k}n^{-\frac{(s-1)d}{2}}\cdot\sum_{\ell=0}^{s-1}\Big(\varphi(\tfrac{j}{n})^{s-\ell}-\varphi(\tfrac{i}{n})^{s-\ell}\Big)\cdot g(s-\ell)\\
&\times\ \mathscr{Z}_{i,j}^{(n,k-(s-\ell),\ell)}(\Phi,\eta-\delta_i)\\
&= \sum_{s=1}^{k}n^{-\frac{(s-1)d}{2}}\cdot\sum_{m=1}^{s}n\Big(\varphi(\tfrac{j}{n})^{m}-\varphi(\tfrac{i}{n})^{m}\Big)\cdot g(m)\\
&\times\ \mathscr{Z}_{i,j}^{(n,k-m,s-m)}(\Phi,\eta-\delta_i).
\end{aligned}$$

This concludes the proof. □

The advantage that Proposition 5.5.3 gives us is that we now have an expression in terms of the auxiliary field (5.126):

$$\begin{aligned}\nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta) &= \sum_{s=1}^{k} n^{-\frac{(s-1)d}{2}} \cdot \sum_{m=1}^{s} n\left(\varphi(\tfrac{j}{n})^m - \varphi(\tfrac{i}{n})^m\right)\cdot g(m) \\ &\times \mathscr{Z}_{i,j}^{(n,k-m,s-m)}(\Phi,\eta-\delta_i) \\ &= n\left(\varphi(\tfrac{j}{n}) - \varphi(\tfrac{i}{n})\right)\cdot g(1)\cdot \mathscr{Z}_{i,j}^{(n,k-1,0)}(\Phi,\eta-\delta_i) \\ &+ \sum_{s=2}^{k} n^{-\frac{(s-1)d}{2}} \cdot \sum_{m=1}^{s} n\left(\varphi(\tfrac{j}{n})^m - \varphi(\tfrac{i}{n})^m\right)\cdot g(m) \\ &\times \mathscr{Z}_{i,j}^{(n,k-m,s-m)}(\Phi,\eta-\delta_i). \end{aligned} \tag{5.129}$$

Recall that we claimed that we are able to close the carré-du-champ by using an expression depending only on the field of order $k-1$. In order to achieve this it remains to:

1. replace the first term in the RHS of (5.129) by some expressions depending on the field of order $k-1$;

2. show that the second term in the RHS of (5.129) vanishes as $n \to \infty$.

We will achieve this in several steps, the first one being the proof of the following proposition.

Proposition 5.5.4. *For all $k \in \mathbb{N}$ we have*

$$\lim_{n\to\infty} \mathbb{E}_n \left[\left(\int_0^t \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\eta_x(n^2 s)(\alpha+\sigma\eta_{x+r}(n^2 s)) \cdot \left(\mathscr{Z}_{x,x+r}^{(n,k,0)}(\Phi,\eta(n^2 s)) - \mathscr{Y}^{(n,k)}(\Phi,\eta(n^2 s))\right)^2 ds\right)^2\right] = 0. \tag{5.130}$$

Proof. Notice that for any fixed x we have

$$\mathscr{Y}^{(n,k)}(\Phi,\eta(n^2 s)) = \sum_{l=0}^{k} \mathscr{Z}_{x,x+r}^{(n,k,l)}(\Phi,\eta(n^2 s)) \tag{5.131}$$

which implies

$$\begin{aligned}\Big(\mathscr{Z}^{(n,k,0)}_{x,x+r}(\Phi,\eta(n^2s)) - \mathscr{Y}^{(n,k)}(\Phi,\eta(n^2s))\Big)^2 \\ = \left(\sum_{l=1}^{k} \mathscr{Z}^{(n,k,l)}_{x,x+r}(\Phi,\eta(n^2s))\right)^2 \\ \le k\sum_{l=1}^{k} \mathscr{Z}^{(n,k,l)}_{x,x+r}(\Phi,\eta(n^2s))^2.\end{aligned}$$

Moreover, we can also estimate each $\mathscr{Z}^{(n,k,l)}_{x,x+r}(\Phi,\eta(n^2s))$ in terms of the coordinates field $\mathscr{X}^{(n,k-l)}$ given by (5.66) as follows:

$$\mathscr{Z}^{(n,k,l)}_{x,x+r}(\Phi,\eta)^2 \le n^{-ld/2}\, M_n(\eta,l)\cdot \mathscr{X}^{(n,k-l)}(\varphi^{(k-l)},\eta)^2 \tag{5.132}$$

where M_n is made of terms of the form (5.120), i.e.

$$M_n(\eta,l) = \sum_{\xi_x=0}^{l} \Phi(\xi_x\delta_x + (l-\xi_x)\delta_{x+r})\cdot d(\xi_x,\eta_x)\cdot d(l-\xi_x,\eta_{x+r}). \tag{5.133}$$

Thanks to Proposition 5.5.2 we conclude the proof. □

For what concerns the second step, let us denote by $\widehat{G}^{(n,k)}_{i,j}(\Phi,\eta)$ the second term in the RHS of (5.129), i.e.

$$\begin{aligned}&\widehat{G}^{(n,k)}_{i,j}(\Phi,\eta)\\ &:= \sum_{s=2}^{k} n^{-\frac{(s-1)d}{2}}\cdot\sum_{m=1}^{s} n\left(\varphi(\tfrac{j}{n})^m - \varphi(\tfrac{i}{n})^m\right)\cdot g(m)\cdot \mathscr{Z}^{(n,k-m,s-m)}_{i,j}(\Phi,\eta-\delta_i)\end{aligned}$$

We have the following result supporting our claim:

Proposition 5.5.5. *Under the inductive hypothesis 5.4.1 we have*

$$\lim_{n\to\infty}\mathbb{E}_n\left[\int_0^t \left(\frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\eta_x(\alpha+\sigma\eta_{x+r})\cdot \widehat{G}^{(n,k)}_{x,x+r}(\varphi,\eta(n^2s))^2\, ds\right)^2\right] = 0. \tag{5.134}$$

Proof. After expanding $\widehat{G}^{(n,k)}_{x,x+r}(\varphi,\eta(n^2s))^2$, the statement follows from applying multiple times Propositions 5.5.4 and 5.5.2. □

Proposition 5.5.6. *Let*

$$G_{i,j}^{(n,k)}(\varphi,\eta) := \nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta) + c_\sigma \langle j-i, \nabla\varphi(\tfrac{i}{n})\rangle \cdot \mathscr{Y}^{(n,k-1)}(\Phi,\eta)$$

then, under the inductive hypothesis 5.4.1, we have

$$\lim_{n\to\infty} \mathbb{E}_n \left[\int_0^t \left(\frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \sum_{r\in\mathscr{R}} p(r)\eta_x(\alpha+\sigma\eta_{x+r}) \cdot G_{x,x+r}^{(n,k)}(\varphi,\eta(n^2 s))^2 \, ds \right)^2 \right] = 0. \tag{5.135}$$

Proof. Due to the fact that

$$\mathscr{Z}_{i,j}^{(n,k-1,0)}(\Phi,\eta-\delta_i) = \mathscr{Z}_{i,j}^{(n,k-1,0)}(\Phi,\eta) \tag{5.136}$$

if we isolate the term $s=1$ in (5.127) we obtain

$$\nabla^{i,j}\mathscr{Y}^{(n,k)}(\Phi,\eta) = -c_\sigma n \left(\varphi(\tfrac{j}{n}) - \varphi(\tfrac{i}{n})\right) \cdot \mathscr{Z}_{i,j}^{(n,k-1,0)}(\Phi,\eta) + \widehat{G}_{i,j}^{(n,k)}(\Phi,\eta) \tag{5.137}$$

then the statement follows from Proposition 5.5.4 and Proposition 5.5.5. □

5.5.2.4 Conclusion

From (5.101) and (5.135) we have

$$\begin{aligned}
&n^2\Gamma\mathscr{Y}^{(n,k)}(\Phi,\eta)\\
&= \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\eta_x(\alpha+\sigma\eta_{x+r})\\
&\cdot\left(c_\sigma\langle r,\nabla\varphi(\tfrac{x}{n})\rangle\cdot\mathscr{Y}^{(n,k-1)}(\Phi,\eta) - G_{x,x+r}^{(n,k)}(\varphi,\eta)\right)^2\\
&= \frac{c_\sigma^2}{n^d}\left(\mathscr{Y}^{(n,k-1)}(\Phi,\eta)\right)^2\cdot\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} |\langle r,\nabla\varphi(\tfrac{x}{n})\rangle|^2\, p(r)\eta_x(\alpha+\sigma\eta_{x+r})\\
&+\mathscr{G}_1^{(n,k)}(\Phi,\eta)
\end{aligned}$$

with

$$\begin{aligned}
\mathscr{G}_1^{(n,k)}(\Phi,\eta) &:= \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}} p(r)\eta_x(\alpha+\sigma\eta_{x+r})\cdot G_{x,x+r}^{(n,k)}(\varphi,\eta)\\
&\quad\cdot\left(G_{x,x+r}^{(n,k)}(\varphi,\eta) - 2c_\sigma\langle r,\nabla\varphi(\tfrac{x}{n})\rangle\mathscr{Y}^{(n,k-1)}(\Phi,\eta)\right)
\end{aligned}$$

Then we can write

$$\begin{aligned} & n^2\Gamma\mathscr{Y}^{(n,k)}(\Phi,\eta) \\ & = \rho(\alpha+\sigma\rho)\,\frac{c_\sigma^2}{n^d}\left(\mathscr{Y}^{(n,k-1)}(\Phi,\eta)\right)^2\cdot\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}}|\langle r,\nabla\varphi(\tfrac{x}{n})\rangle|^2\,p(r) \\ & +\mathscr{G}_1^{(n,k)}(\Phi,\eta)+\mathscr{G}_2^{(n,k)}(\varphi,\eta) \end{aligned} \tag{5.138}$$

with

$$\begin{aligned} & \mathscr{G}_2^{(n,k)}(\varphi,\eta) \\ & := \frac{c_\sigma^2}{n^d}\left(\mathscr{Y}^{(n,k-1)}(\Phi,\eta)\right)^2\cdot\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}}|\langle r,\nabla\varphi(\tfrac{x}{n})\rangle|^2\,p(r) \\ & \cdot\left\{\alpha(\eta_x-\rho)+\sigma(\eta_x\eta_{x+r}-\rho^2)\right\}. \end{aligned} \tag{5.139}$$

We first estimate the term due to the error $\mathscr{G}_1^{(n,k)}(\Phi,\eta)$.

PROPOSITION 5.5.7. *For every $t>0$ and every test function $\varphi\in S(\mathbb{R})$ there exists $C>0$ such that, for all $n\in\mathbb{N}$,*

$$\lim_{n\to\infty}\mathbb{E}_n\left[\left(\int_0^t \mathscr{G}_1^{(n,k)}(\Phi,\eta(n^2s))ds\right)^2\right]=0 \tag{5.140}$$

PROOF. It follows from Proposition 5.5.5 and the convergence, by the inductive hypothesis, of $\mathscr{Y}^{(n,k-1)}(\varphi,\eta)$. □

The two following propositions allow us to estimate the error $\mathscr{G}_2^{(n,k)}$ and then to perform the replacement in (5.146).

LEMMA 5.5.2. *For every $t>0$ and every test function $\varphi\in S(\mathbb{R})$ there exists $C>0$ such that, for all $n\in\mathbb{N}$,*

$$\lim_{n\to\infty}\mathbb{E}_n\left[\left(\int_0^t\frac{1}{n^d}\sum_{x\in\mathbb{Z}^d}\sum_{r\in\mathscr{R}}|\langle r,\nabla\varphi(\tfrac{x}{n})\rangle|^2\,p(r)(\eta_x(n^2s)-\rho)ds\right)^2\right]=0. \tag{5.141}$$

PROOF. From (5.39) we can write the integrand in (5.141) as

$$\frac{1}{n^{d/2}}\mathscr{Y}_s^{(n,1)}(\Psi),\quad\text{with}\quad\Psi(\xi):=\prod_{x\in\mathbb{Z}^d}\psi(x)^{\xi_x} \tag{5.142}$$

and

$$\psi(x):=\sum_{r\in\mathscr{R}}|\langle r,\nabla\varphi(x)\rangle|^2\,p(r) \tag{5.143}$$

Then the statement follows from the convergence of $\mathscr{Y}_s^{(n,1)}(\Psi)$ and the extra factor $\frac{1}{n^{d/2}}$. $\square$

Similarly, another replacement is necessary on the second term of the RHS of (5.139).

Lemma 5.5.3. *For every $t > 0$ and every test function φ we have*

$$\lim_{n\to\infty} \mathbb{E}_n \left[\left(\int_0^t \frac{1}{n^d} \sum_{x\in\mathbb{Z}^d} \sum_{r\in\mathscr{R}} |\langle r, \nabla\varphi(\tfrac{x}{n})\rangle|^2 \, p(r) \left(\eta_x(n^2 s) - \rho\right) \left(\eta_{x+r}(n^2 s) - \rho\right) ds \right)^2 \right] = 0. \tag{5.144}$$

Proof. The proof of this lemma is done in the same spirit as Proposition 5.5.2. $\square$

Proposition 5.5.8. *For every $t > 0$ and every test function $\varphi \in S(\mathbb{R})$ there exists $C > 0$ such that, for all $n \in \mathbb{N}$,*

$$\lim_{n\to\infty} \mathbb{E}_n \left[\left(\int_0^t \mathscr{G}_2^{(n,k)}(\Phi, \eta(n^2 s)) ds \right)^2 \right] = 0. \tag{5.145}$$

Proof. It follows from Lemma 5.5.2, Lemma 5.5.3 and the convergence, by the inductive hypothesis, of $\mathscr{Y}^{(n,k-1)}(\varphi,\eta)$. $\square$

From Propositions 5.5.7 and 5.5.8 we can write

$$\begin{aligned} & n^2 \Gamma \mathscr{Y}^{(n,k)}(\Phi,\eta) = \\ & = \rho(\alpha+\sigma\rho) \frac{c_\sigma^2}{n^d} \left(\mathscr{Y}^{(n,k-1)}(\Phi,\eta) \right)^2 \cdot \sum_{x\in\mathbb{Z}^d} \sum_{r\in\mathscr{R}} |\langle r, \nabla\varphi(\tfrac{x}{n})\rangle|^2 \, p(r) \\ & + \mathscr{G}^{(n,k)}(\Phi,\eta) \end{aligned} \tag{5.146}$$

where the term $\mathscr{G}^{(n,k)}(\Phi,\eta)$ is a vanishing error:

$$\lim_{n\to\infty} \mathbb{E}_n \left[\left(\int_0^t \mathscr{G}^{(n,k)}(\Phi, \eta(n^2 s)) ds \right)^2 \right] = 0.$$

Therefore we conclude that the proposed (remember that at this point we do not know if the limiting object is indeed a martingale) predictable quadratic variation of our limiting martingale is given by

$$c_\sigma^2 \chi \rho(\alpha+\sigma\rho) t \left(\mathscr{X}_s^{(k-1)}(\varphi^{(k-1)}) \right)^2 \int_{\mathbb{R}^d} \left\| \nabla\varphi(x) \right\|^2 dx. \tag{5.147}$$

Arrived at this point we can conclude that if $\{M_t^{(n,k)}(\Phi) : t \in [0,T]\}$ has a limit as $n \to \infty$, and if the limit is a square-integrable martingale then its quadratic variation is given by (5.147). In what follows we will show tightness and uniform integrability, i.e. we will prove that $\{M_t^{(n,k)}(\Phi) : t \in [0,T]\}$ converges to $\{M_t^{(k)}(\Phi) : t \in [0,T]\}$ and that $\{M_t^{(k)}(\Phi) : t \in [0,T]\}$ is indeed a martingale.

5.5.3 Tightness

In this section we prove tightness for the family of laws $\{Q_n^{(k)}\}_{n\in\mathbb{N}}$, induced by $\{\mathscr{X}^{(n,k)}(\cdot,t)\}_{t\geq 0}$ on $D([0,\infty), S'(\mathbb{R}^k))$. From the Dynkin formula we know that

$$M_n'(t,\varphi^{(k)}) = \mathscr{X}_t^{(n,k)}(\varphi^{(k)}) - n^2 \int_0^t \mathscr{L}\mathscr{X}_s^{(n,k)}(\varphi^{(k)})ds \tag{5.148}$$

and

$$N_n'(t,\varphi^{(k)}) = M_n'(t,\varphi^{(k)})^2 - n^2 \int_0^t \Gamma\mathscr{X}_s^{(n,k)}(\varphi^{(k)})ds \tag{5.149}$$

are martingales. Theorem 2.3 in [38], which we include in Appendix A.3, allows us to reduce the proof the tightness of $\{Q_n^{(k)}\}_{n\in\mathbb{N}}$ to the verification of conditions (A.30)-(A.32). We verify these conditions in Proposition 5.5.9, Proposition 5.5.10 and Proposition 5.5.11 below.

5.5.3.1 The γ_1 term

The following Proposition shows that conditions (A.30) and (A.31) hold true.

Proposition 5.5.9. *For any $\varphi^{(k)} \in S(\mathbb{R}^{kd})$ and $t_0 \geq 0$ we have:*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq t\leq t_0} \mathbb{E}_n\left[\left(\mathscr{X}_t^{(n,k)}(\varphi^{(k)})\right)^2\right] < \infty \tag{5.150}$$

and

$$\sup_{n\in\mathbb{N}} \sup_{0\leq t\leq t_0} \mathbb{E}_n\left[\left(n^2\mathscr{L}\mathscr{X}_t^{(n,k)}(\varphi^{(k)})\right)^2\right] < \infty. \tag{5.151}$$

Proof. We start with the proof (5.151) which is more involved. Thanks to stationarity, the expectation does not depend on time, and then we can ignore the supremum over time in (5.151). From (5.100) we already have an expression for the integrand of (5.151):

$$n^2\mathscr{L}\mathscr{X}^{(n,k)}(\varphi^{(k)},\eta) = \alpha k \cdot \tfrac{\chi}{2} \cdot \mathscr{X}^{(n,k)}(\varphi^{(k-1)} \otimes \Delta\varphi, \eta) + O(n^{-1}) \tag{5.152}$$

recall that here again we are using the fact that the field $\mathscr{X}^{(n,k)}$ can be also thought as acting on general (not necessarily symmetric) test functions. Because of stationarity it is enough to estimate

$$\mathbb{E}_{\nu_\rho}\left[\left(\mathscr{X}^{(n,k)}(\varphi^{(k-1)} \otimes \Delta\varphi, \eta)\right)^2\right]. \tag{5.153}$$

Then the desired bound is obtained by applying Proposition 5.5.1. In the same spirit we can use Proposition 5.5.1 to bound (5.150). □

5.5.3.2 The γ_2 term

Similarly to the previous section, here we prove the following proposition in order to verify the condition (A.31) for γ_2.

PROPOSITION 5.5.10. *For any $\varphi^{(k)} \in S(\mathbb{R}^{kd})$ and $t_0 \geq 0$ we have:*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq t\leq t_0} \mathbb{E}_n \left[\left(n^2 \Gamma \mathscr{X}_t^{(n,k)}(\varphi^{(k)}) \right)^2 \right] < \infty. \tag{5.154}$$

PROOF. Thanks to stationarity we can neglect the supremum over time. Recall that in (5.146) we have obtained an expression for the integrand on (5.154)

$$\begin{aligned} & n^2 \Gamma \mathscr{X}^{(n,k)}(\varphi^{(k)}, \eta) \\ & = \rho(\alpha + \sigma\rho) \, \frac{k^2 c_\sigma^2}{n^d} \left(\mathscr{Y}^{(n,k-1)}(\Phi, \eta) \right)^2 \cdot \sum_{x\in\mathbb{Z}^d} \sum_{r\in\mathscr{R}} |\langle r, \nabla\varphi(\tfrac{x}{n})\rangle|^2 \, p(r) \\ & + O(n^{-1}) \end{aligned}$$

taking the square of which we obtain

$$\begin{aligned} & \mathbb{E}_{\nu_\rho} \left[(n^2 \Gamma \mathscr{X}^{(n,k)}(\varphi^{(k)}, \eta))^2 \right] \\ = \; & \rho^2 (\alpha + \sigma\rho)^2 \, \frac{k^4 c_\sigma^4}{n^{2d}} \cdot \sum_{x,y\in\mathbb{Z}^d} \sum_{r_1,r_2\in\mathscr{R}} |\langle r_1, \nabla\varphi(\tfrac{x}{n})\rangle|^2 \cdot |\langle r_2, \nabla\varphi(\tfrac{y}{n})\rangle|^2 \\ \times \; & p(r_1) \cdot p(r_2) \cdot \mathbb{E}_{\nu_\rho} \left[\left(\mathscr{Y}^{(n,k-1)}(\Phi, \eta) \right)^4 \right]. \end{aligned} \tag{5.155}$$

Notice that the first factor on the RHS of (5.155) can be controlled by using the compact support of φ and the factor $\frac{1}{n^{2d}}$. It is then sufficient to estimate

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\nu_\rho} \left[\left(\mathscr{Y}^{(n,k-1)}(\Phi, \eta) \right)^4 \right] \tag{5.156}$$

then Proposition 5.5.1 finishes the proof. □

5.5.3.3 Modulus of continuity

In this section we show that condition (2.5) of Theorem 2.3 in [38] is satisfied.

Proposition 5.5.11. *For every $\varphi^{(k)} \in S(\mathbb{R}^{kd})$ there exists a sequence $\delta(t, \varphi, n)$ converging to zero as $n \to 0$ such that:*

$$\lim_{n\to\infty} \mathbb{P}_n(\sup_{0\leq t\leq T} |\mathscr{X}_t^{(n,k)}(\varphi^{(k)}, \eta) - \mathscr{X}_{t-}^{(n,k)}(\varphi^{(k)}, \eta)| \geq \delta(t, \varphi, n)) = 0. \quad (5.157)$$

Proof. We know that the jumps of the process $\{\eta(t) : t \geq 0\}$ are determined by exponential clocks. This implies that for any $\epsilon > 0$ the probability of having more than one jump in the interval $(t, t+\epsilon]$ is of the order $o(\epsilon)$. Hence for C, a positive constant that depends on the model parameters, we have

$$\sup_{0\leq t\leq T} |\mathscr{X}_t^{(n,k)}(\varphi^{(k)}) - \mathscr{X}_{t-}^{(n,k)}(\varphi^{(k)})| \leq C\frac{\|\varphi\|_\infty}{n^{kd/2}} \quad (5.158)$$

with probability $1 - o(\epsilon)$.

Taking the sequence $\{\delta(t, \varphi, n)\}_{n\geq 1}$ given by

$$\delta(t, \varphi, n) = C\frac{\|\varphi\|_\infty + 1}{n^{1/2}} \quad (5.159)$$

finishes the proof. □

Remark 5.5.2. *Notice that Proposition 5.5.11 implies, in particular, that the law induced by $\mathscr{Y}_t^{(n,k)}$ is concentrated on continuous paths.*

5.5.4 Characterization of limit points

At this point we can only say that the sequence $\{M_t^{(n,k)}(\cdot) : t \in [0, T]\}$ converges weakly to the process $\{M_t^{(k)}(\cdot) : t \in [0, T]\}$ satisfying expressions (5.72) and (5.73). Nevertheless, we would like to support the claim, given in Theorem 5.4.1, that the limiting process $\{M_t^{(k)}(\cdot) : t \in [0, T]\}$ is indeed a martingale with the proposed predictable quadratic variation given by (5.147). At this aim we prove the following result.

Proposition 5.5.12. *The sequence $\{M_t^{(n,k)}(\cdot) : t \in [0, T]\}$ is uniformly integrable.*

Proof. By standard arguments it is enough to provide a uniform $L^p(\mathbb{P}_n)$ bound for $p > 1$. Notice that, thanks to the martingale decomposition (5.74), and the same type of arguments used in the proofs of Propositions 5.5.9 and 5.5.10, we can indeed find the desired bounds for $p = 2$. □

The same type of reasoning used in Proposition 5.5.12 gives us the following result.

Proposition 5.5.13. *The sequence* $\{N_t^{(n,k)}(\cdot) : t \in [0,T]\}$ *is uniformly integrable.*

Combining Propositions 5.5.12 and 5.5.13 we show that any limit point of the sequence $\{M_t^{(n,k)}(\cdot) : t \in [0,T]\}$ satisfies the recursive martingale problem (5.72)-(5.73).

5.5.5 Uniqueness

It remains to show uniqueness of the solution of the martingale problem (5.72)-(5.73). First notice that by the Duhamel formula, from (5.72), we can deduce

$$\mathscr{X}_t^{(k)}(\varphi^{(k)}) = \mathscr{X}_0^{(k)}(S_t^{(k)}\varphi^{(k)}) + \int_0^t dM_s^{(k)}(S_{t-s}^{(k)}\varphi^{(k)}) \tag{5.160}$$

where $S_t^{(k)}$ is the semigroup associated to the kd-dimensional Laplacian (or to the kd-dimensional Brownian motion).

Remark 5.5.3. *Notice that there is not ambiguity in using* (5.72)-(5.73) *with test functions of the form* $S_t^{(k)}\varphi^{(k)}$*. From the fact that the d-dimensional Brownian semigroup leaves invariant the space* $S(\mathbb{R}^d)$*, we can deduce that the kd-dimensional Brownian semigroup keeps both the symmetry and the Schwartz space nature of the test function* $\varphi^{(k)}$*. More precisely:*

$$S_t^{(k)}\varphi^{(k)} = (S_t^{(1)}\varphi)^{(k)} \tag{5.161}$$

where $S_t^{(1)}$ *denotes the semigroup of a d-dimensional Brwonian motion.*

Remark 5.5.4. *In the RHS of equation* (5.160)*, the integral term*

$$\int_0^t dM_s^{(k)}(S_{t-s}^{(k)}\varphi^{(k)}) \tag{5.162}$$

should be interpreted as a martingale with quadratic variation

$$\int_0^t \int_{\mathbb{R}^d} \left\| \nabla S_{t-s}^{(1)}\varphi(x) \right\|^2 \mathscr{X}_s^{(k-1)}(S_{t-s}^{(k-1)}\varphi^{(k-1)})\, ds. \tag{5.163}$$

Given the distribution of $\mathscr{X}_0^{(k)}$ and the well-definedness of $M_t^{(k)}$, the RHS of (5.160) uniquely determines the finite-dimensional distributions of $\mathscr{X}_t^{(k)}$. Then, by the continuity of $\mathscr{X}_t^{(k)}$, we conclude the uniqueness of limiting point. We refer to [47] for more details on how to proceed for the case $k = 2$.

Part II

Condensation

Chapter 6

Condensation of SIP particles and sticky Brownian motion

The symmetric inclusion process (SIP) is an interacting particle system where a single particle performs symmetric continuous-time random walks on the lattice $\mathbb{Z}$, with rates $\alpha p(i,j) = \alpha p(j,i)$ ($\alpha > 0$) and where particles interact by attracting each other (see below for the precise definition) at rate $p(i,j)\eta_i\eta_j$, where η_i is the number of particles at site i. The parameter α regulates the relative strength of diffusion w.r.t. attraction between particles. The symmetric inclusion process is self-dual, and many results on its macroscopic behavior can be obtained via this property. Self-duality implies that the expectation of the number of particles can be understood from one dual particle. In particular, because one dual particle scales to Brownian motion in the diffusive scaling, the hydrodynamic limit of SIP is the heat equation. The next step is to understand the variance of the density field, which requires two dual particles.

It is well-known that in the regime $\alpha \to 0$ the SIP manifests condensation (the attractive interaction dominates), and via the self-duality of SIP more information can be obtained about this condensation process than for a generic process (such as zero-range processes). Two of my coauthors of the joint paper, on which this chapter is based have obtained an explicit formula for the Fourier-Laplace transform of two particle transition probabilities for interacting particle systems such as the simple symmetric exclusion and the simple symmetric inclusion process, where simple refers to nearest neighbor in dimension 1. From this formula, the authors were able to extract information about the variance of the time-dependent density field starting from a homogeneous product measure. With the help of duality this reduces to the study of the scaling behavior of two dual particles. In particular, for the inclusion process in the condensation regime, from the study of the scaling behavior of the time-dependent variance of the density field, one can extract information about the coarsening process. It turned out that the scaling limit of two particles is in that case a pair of sticky Brownian motions. From this one can infer the qualitative picture that in the condensation regime, when started from a homogeneous product measure, large piles of particles are formed which move as Brownian motion, and interact with each other as sticky Brownian motions.

The whole analysis in [18] is based on the exact formula for the Fourier-Laplace transform of the transition probabilities of two SIP particles as mentioned above. This exact computation is based on the fact that the underlying random walk is nearest neighbor, and therefore the results are restricted to that case. However, we expect that for the SIP in the condensation regime, sticky Brownian motion appears as a scaling limit in much larger generality in dimension 1. The exact formula in [18] yields convergence of semigroups, and therefore convergence of finite-dimensional distributions. However, because of the rescaling in the condensation regime, one cannot expect convergence of generators, but rather a convergence result in the spirit of slow-fast systems, i.e., of the type gamma convergence. Moreover, the difference of two SIP-particles is not simply a random walk slowed down when it is the origin as in e.g. [1]. Instead, it is a random walk which is pulled towards the origin when it is close to it, which only in the scaling limit leads to a slow-down at the origin, i.e., sticky Brownian motion.

In this chapter, we obtain a precise scaling behavior of the variance of the density field in the condensation regime. We find the explicit scaling form for this variance in real time (as opposed to the Laplace transformed result in [18]), thus giving more insight in the coarsening process when initially started from a homogeneous product measure of density ρ. This is the first rigorous result on coarsening dynamics in interacting particle systems directly on infinite lattices, for a general class of underlying random walks. There exist important results on condensation either heuristically on the infinite lattice or rigorous but constrained to finite lattices. For example [14] heuristically discusses on infinite lattices the effective motion of clusters in the coarsening process for the TASIP; or the work [21] which based on heuristic mean-field arguments studies the coarsening regime for the explosive condensation model. On the other hand, on finite lattices via martingale techniques [8] studies the evolution of a condensing zero-range process. In the context of the SIP, the authors of [49], on a finite lattice, showed the emergence of condensates as the parameter $\alpha \to 0$ and rigorously characterize their dynamics. We also mention the recent work [55] where the structure of the condensed phase in SIP is analyzed in stationarity, in the thermodynamic limit. More recently in [57], condensation was proven for a large class of inclusion processes for which there is no explicit form of the invariant measures. The work in [57] also derived rigorous results on the metastable behavior of non-reversible inclusion processes.

Our main result is obtained by proving that the difference of two SIP particles converges to a two-sided sticky Brownian motion in the sense of Mosco convergence of Dirichlet forms, originally introduced in [76] and extended to the case of varying state spaces in [66]. Because this notion of convergence implies convergence of semigroups in the L^2-space of the reversible measure, which is $dx + \gamma\delta_0$ for the sticky Brownian motion with stickiness parameter $\gamma > 0$, the convergence of semigroups also implies that of transition probabilities of the form $p_t(x, 0)$. This, together with self-duality, helps to explicitly obtain the limiting variance of the fluctuation field. Technically speaking, the main difficulty in our approach is that we have to define carefully how to transform functions defined on the discretized rescaled lattices into functions on the continuous limit space in order to obtain convergence of the relevant Hilbert spaces, and at the same time obtain the second condition of Mosco convergence. Mosco convergence is a weak form of convergence which is not frequently used in the probabilistic context. In our context it is however exactly the form of convergence which we need to study the variance of the density field. As already mentioned before, as it is strongly related to gamma-convergence, it is also a natural form of convergence in a setting reminiscent of slow-fast systems.

The rest of this chapter is organized as follows. In Section 6.1 we deal with some preliminary notions; we introduce both the inclusion and the difference process in terms of their infinitesimal generators. In this section we also introduce the concept of duality and describe the appropriate regime in which condensation manifests itself. Our main result is stated in Section 6.2, were we present some non-trivial information about the variance of the time-dependent density field in the condensation regime and provide some heuristics for the dynamics described by this result. In Section 6.3, we present the proof of our main result and also show that the finite-range difference process converges in the sense of Mosco convergence of Dirichlet forms to the two-sided sticky Brownian motion.

As supplementary material in Appendix B, we refer for basic notions on Dirichlet forms and the construction via stochastic time changes of Dirichlet forms the two-sided sticky Brownian motion at zero. We also refer to Appendix C.2, where we deal with the convergence of independent random walkers to standard Brownian motion. This last result, despite being basic becomes a cornerstone for our results of Section 6.2.

6.1 Preliminaries

6.1.1 The Model: inclusion process

The Symmetric Inclusion Process (SIP) is an interacting particle system where particles randomly hop on the lattice $\mathbb{Z}$ with attractive interaction and no restrictions on the number of particles per site. Configurations are denoted by

η and are elements of $\Omega = \mathbb{N}^{\mathbb{Z}}$ (where $\mathbb{N}$ denotes the set of natural numbers including zero). We denote by η_x the number of particles at position $x \in \mathbb{Z}$ in the configuration $\eta \in \Omega$. The generator working on local functions $f : \Omega \to \mathbb{R}$ is of the type

$$\mathscr{L} f(\eta) = \sum_{i,j\in\mathbb{Z}} p(j-i)\eta_i(\alpha+\eta_j)(f(\eta^{ij}) - f(\eta)) \tag{6.1}$$

where η^{ij} denotes the configuration obtained from η by removing a particle from i and putting it at j. For the associated Markov process on Ω, we use the notation $\{\eta(t) : t \geq 0\}$, i.e., $\eta_x(t)$ denotes the number of particles at time t at location $x \in \mathbb{Z}$. Additionally, we assume that the function $p : \mathbb{R} \to [0,\infty)$ satisfies the following properties

1. Symmetry: $p(r) = p(-r)$ for all $r \in \mathbb{R}$
2. Finite-range: there exists $R > 0$ such that: $p(r) = 0$ for all $|r| > R$.
3. Irreducibility: for all pair of points $x, y \in \mathbb{Z}$ there exists $m \in \mathbb{N}$ and points $x = i_1, i_2, \ldots, i_{m-1}, i_m = y$, such that $\prod_{j=1}^{m-1} p(i_{j+1} - i_j) > 0$.

It is known that these particle systems have a one-parameter family of homogeneous (w.r.t. translations), reversible and ergodic product measures $\mu_\rho, \rho > 0$ with marginals

$$\mu_\rho(\eta_i = m) = \frac{\alpha^\alpha \rho^m}{(\alpha+\rho)^{\alpha+m}} \frac{\Gamma(\alpha+m)}{\Gamma(m+1)\Gamma(\alpha)}$$

This family of measures is indexed by the density of particles, i.e.,

$$\int \eta_0 d\mu_\rho = \rho$$

Remark 6.1.1. *Notice that for these systems the initial configuration has to be chosen in a subset of configurations such that the process $\{\eta(t) : t \geq 0\}$ is well-defined. A possible such subset is the set of tempered configurations. This is the set of configurations η such that there exist $C, \beta \in \mathbb{R}$ that satisfy $|\eta(x)| \leq C|x|^\beta$ for all $x \in \mathbb{R}$. We denote this set (with slight abuse of notation) still by Ω, because we will always start the process from such configurations, and this set has $\mu_{\bar\rho}$ measure 1 for all ρ. Since we are working mostly in $L^2(\mu_{\bar\rho})$ spaces, this is not a restriction.*

6.1.2 Self-duality

Let us denote by $\Omega_f \subseteq \Omega$ the set of configurations with a finite number of particles. We then have the following definition:

Definition 6.1.1. *We say that the process* $\{\eta_t : t \geq 0\}$ *is self-dual with self-duality function* $D : \Omega_f \times \Omega \to \mathbb{R}$ *if*

$$\mathbb{E}_\eta \big[D(\xi, \eta_t)\big] = \mathbb{E}_\xi \big[D(\xi_t, \eta)\big] \tag{6.2}$$

for all $t \geq 0$ *and* $\xi \in \Omega_f, \eta \in \Omega$.

In the definition above $\mathbb{E}_\eta$ and $\mathbb{E}_\xi$ denote expectation when the processes $\{\eta_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$ are initialized from the configuration η and ξ respectively . Additionally we require the duality functions to be of factorized form, i.e.,

$$D(\xi, \eta) = \prod_{i \in \mathbb{Z}} d(\xi_i, \eta_i) \tag{6.3}$$

where the single-site duality function $d(k, \cdot)$ is a polynomial of degree k, more precisely

$$d(k, m) = \frac{m!\Gamma(\alpha)}{(m-k)!\Gamma(\alpha+k)} 1\!\!1_{\{k \leq m\}} \tag{6.4}$$

One important consequence of the fact that a process enjoys the self-duality property is that the dynamics of k particles provides relevant information about the time-dependent correlation functions of degree k. As an example we now state the following proposition, Proposition 5.1 in [18], which provides evidence for the case of two particles

Proposition 6.1.1. *Let* $\{\eta(t) : t \geq 0\}$ *be a process with generator* (6.1), *then*

$$\int \mathbb{E}_\eta \left(\eta_t(x) - \rho\right) \left(\eta_t(y) - \rho\right) \nu(d\eta) \tag{6.5}$$

$$= \left(1 + \frac{1}{\alpha} 1\!\!1_{\{x=y\}}\right) \left(\frac{\alpha\sigma}{\alpha+1} - \rho^2\right) \mathbb{E}_{x,y} \left[1\!\!1_{\{X_t = Y_t\}}\right] + 1\!\!1_{\{x=y\}} \left(\frac{\rho^2}{\alpha} + \rho\right)$$

where ν *is assumed to be a homogeneous product measure with* ρ *and* σ *given by*

$$\rho := \int \eta_x \nu(d\eta) \qquad \text{and} \qquad \sigma := \int \eta_x(\eta_x - 1)\nu(d\eta) \tag{6.6}$$

and X_t *and* Y_t *denote the positions at time* $t > 0$ *of two dual particles started at* x *and* y *respectively and* $\mathbb{E}_{x,y}$ *the corresponding expectation.*

Proof. We refer to [18] for the proof. □

Remark 6.1.2. *Notice that Proposition 6.1.1 shows that the two-point correlation functions depend on the two-particle dynamics via the indicator function* $1\!\!1_{\{X_t=Y_t\}}$. *More precisely, these correlations can be expressed in terms of the difference of the positions of two dual particles and the model parameters.*

Motivated by Remark 6.1.2, and for reasons that will become clear later, we will study in the next section the stochastic process obtained from the generator (6.1) by following the evolution in time of the difference of the positions of two dual particles.

6.1.3 The difference process

We are interested in a process obtained from the dynamics of the process $\{\eta(t) : t \geq 0\}$ with generator (6.1) initialized originally with two labeled particles. More precisely, if we denoted by $(x_1(t), x_2(t))$ the particle positions at time $t \geq 0$, from the generator (6.1) we can deduce the generator for the evolution of these two particles; this is, for $f : \mathbb{Z}^2 \to \mathbb{R}$ and $\mathbf{x} \in \mathbb{Z}^2$ we have

$$Lf(\mathbf{x}) = \sum_{i=1}^{2} \sum_{r} p(r) \Big(\alpha + \sum_{j=1}^{2} \mathbb{1}_{x^i + r = x^j} \Big) \Big(f(\mathbf{x}^{i,r}) - f(\mathbf{x}) \Big)$$

where $\mathbf{x}^{i,r}$ results from changing the position of particle i from the site x^i to the site $x^i + r$.

Given this dynamics, we are interested in the process given by the difference

$$w(t) := x^2(t) - x^1(t), \qquad t \geq 0. \tag{6.7}$$

Notice that the labels of the particles are fixed at time zero and do not vary thereafter. This process was studied for the first time in [77] and later on [18], but in contrast to [18], we do not restrict ourselves to the nearest-neighbor case, hence any time a particle moves the value of $w(t)$ can change by r units, with $r \in A := [-R, R] \cap \mathbb{Z} \setminus \{0\}$.

Using the symmetry and translation invariance properties of the transition function we obtain the following operator as generator for the difference process

$$(Lf)(w) = \sum_{r \in A} 2p(r) \left(\alpha + \mathbb{1}_{r=-w} \right) \big[f(w + r) - f(w) \big] \tag{6.8}$$

where we used that $p(0) = 0$ and $p(-r) = p(r)$.

Let μ denote the discrete counting measure and δ_0 the Dirac measure at the origin, then we have the following result:

Proposition 6.1.2. *The difference process is reversible with respect to the measure ν_α given by*

$$\nu_\alpha := \mu + \frac{\delta_0}{\alpha}, \qquad i.e. \qquad \nu_\alpha(w) = \begin{cases} 1 + \frac{1}{\alpha} & \text{if} \quad w = 0 \\ 1 & \text{if} \quad w \neq 0 \end{cases} \tag{6.9}$$

Proof. By detailed balance, see for example Proposition 4.3 in [58], we obtain that any reversible measure should satisfy the following:

$$\nu_\alpha(w) = \frac{(\alpha + \mathbb{1}_{w=0})}{(\alpha + \mathbb{1}_{r=-w})} \, \nu_\alpha(w + r) \tag{6.10}$$

where, due to the symmetry of the transition function, we have cancelled the factor $\frac{p(-r)}{p(r)}$. In order to verify that ν_α satisfies (6.10) we have to consider three possible cases: Firstly $w \notin \{0, -r\}$, secondly $w = 0$ and finally $w = -r$. For $w \notin \{0, -r\}$, (6.10) reads $\nu_\alpha(w) = \nu_\alpha(w + r)$ which is clearly satisfied by (6.9). For $w = 0$ and for $w = -r$, (6.10) reads $\nu_\alpha(0) = (1 + \frac{1}{\alpha})\nu_\alpha(r)$ which is also satisfied by (6.9). □

Remark 6.1.3. *Notice that in the case of a symmetric transition function the reversible measures ν_α are independent of the range of the transition function.*

6.1.4 Condensation and Coarsening

6.1.4.1 The sticky regime

It has been shown in [48] that the inclusion process with generator (6.1) can exhibit a condensation transition in the limit of a vanishing diffusion parameter α. The parameter α controls the rate at which particles perform random walks, hence in the limit $\alpha \to 0$ the interaction due to inclusion becomes dominant which leads to condensation. The type of condensation in the SIP is different from other particle systems such as zero-range processes, see [50] and [34] for example, because in the SIP the critical density is zero.

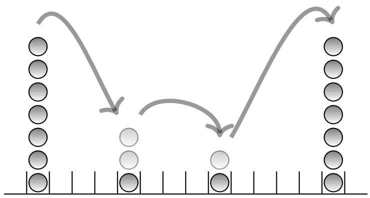

Figure 6.1: Condensate in ZRP

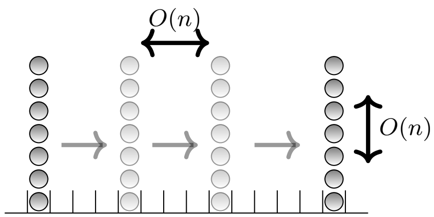

Figure 6.2: Condensate in SIP

Moreover, as depicted in Figures 6.1 and 6.2 the dynamics of the condensates in the SIP and ZRP are different. In the case of the ZRP (Figure 6.1) the condensate moves by first detaching some particles from it, later these free particles start to wander around to finally form a new condensate possibly at a distance much further than the size of the original condensate. On the other hand, for the SIP (Figure 6.2) the size of a typical condensate is of order ($O(n)$) and all the particles in this condensate move together a distance of the order the size of the condensate (again $O(n)$).

In the symmetric inclusion process we can achieve the condensation regime by rescaling the parameter α, i.e. making it of order $1/n$. If on top of that rescaling we also rescale space by $1/n$ and accelerate time with a factor of order n^3 then we enter the sticky regime introduced in [18]. More precisely, for $\gamma > 0$, we speed up time by a factor $n^3\gamma/\sqrt{2}$, scale space by $1/n$ and rescale the parameter α by $\frac{1}{\sqrt{2}\gamma n}$; in this case the generator (6.1) becomes

$$\mathscr{L}_n f(\eta) = \frac{n^3\gamma}{\sqrt{2}} \sum_{i,j\in\frac{1}{n}\mathbb{Z}} p(j-i)\eta_i(\tfrac{1}{\sqrt{2}\gamma n} + \eta_j)(f(\eta^{ij}) - f(\eta)) \tag{6.11}$$

Notice that by splitting the generator (6.11) as follows:

$$\mathscr{L}_n f(\eta) = \mathscr{L}_n^{\mathrm{IRW}} f(\eta) + \mathscr{L}_n^{\mathrm{SIP}} f(\eta)$$

where

$$\mathscr{L}_n^{\mathrm{IRW}} f(\eta) = \frac{n^2}{2} \sum_{i,j\in\frac{1}{n}\mathbb{Z}} p(j-i)\eta_i(f(\eta^{ij}) - f(\eta)) \tag{6.12}$$

and

$$\mathscr{L}_n^{\mathrm{SIP}} f(\eta) = \frac{n^3\gamma}{\sqrt{2}} \sum_{i,j\in\frac{1}{n}\mathbb{Z}} p(j-i)\eta_i\eta_j(f(\eta^{ij}) - f(\eta)) \tag{6.13}$$

We can indeed see two forces competing with each other. On the one hand, with a multiplicative factor of $\frac{n^2}{2}$ we see the diffusive action of the generator (6.12). While on the other hand, at a much larger factor $\frac{n^3\gamma}{\sqrt{2}}$ we see the action of the infinitesimal operator (6.13) making particles condense. Therefore the sum of the two generators has the flavor of a slow-fast system. This gives us the hint that, for the associated process, we cannot expect convergence of the generators. Instead, as it will become clear later, we will work with Dirichlet forms.

6.1.4.2 Coarsening and the density fluctuation field

It was found in [48] that in the condensation regime (when started from a homogeneous product measure with density $\rho > 0$) sites are either empty with very high probability, or contain a large number of particles to match the fixed expected value of the density. We also know that in this regime the variance of the particle number is of order n and hence a rigorous hydrodynamical description of the coarsening process, by means of standard techniques, becomes inaccessible. Nevertheless, as it was already hinted in [18] at the level of the Fourier-Laplace transform, a rigorous description at the level of fluctuations might be possible. Therefore we introduce the fluctuation field in the the condensive time scaling:

$$\mathscr{Y}_n(\eta,\varphi,t) = \frac{1}{n}\sum_{x\in\mathbb{Z}} \varphi(x/n)\left(\eta_{s(n,t)}(x) - \rho\right) \quad \text{with} \quad s(n,t) := \tfrac{\gamma n^3 t}{\sqrt{2}} \tag{6.14}$$

defined for any φ in the space of Schwartz functions:

$$\mathscr{S}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} |x^\alpha D^\beta \varphi| < \infty, \forall \alpha, \beta \in \mathbb{N}\}. \tag{6.15}$$

Remark 6.1.4. *Notice that the scaling in* (6.14) *differs from the standard setting of fluctuation fields, given for example in Chapter 11 of [58]. In our setting, due to the exploding variances it is necessary to re-scale the fields by an additional factor of* $\frac{1}{\sqrt{n}}$.

6.2 Main result: time dependent variances of the density field

Let us initialize the nearest-neighbor SIP configuration process from a spatially homogeneous product measure ν parametrized by its mean ρ and such that

$$\mathbb{E}_\nu[\eta(x)^2] < \infty. \tag{6.16}$$

We have the following result concerning the time-dependent variances of the density field (6.14):

Theorem 6.2.1. *Let* $\{\eta_{\alpha(N,t)} : t \geq 0\}$ *be the time-rescaled inclusion process, with infinistesimal generator* (6.11)*, in configuration space. Consider the fluctuation field* $\mathscr{Y}_n(\eta, \varphi, t)$ *given by* (6.14)*. Let* ν_ρ *be an initial homogeneous product measure parametrized by its mean* ρ *and satisfying* (6.16) *holds. Then the limiting time dependent variance of the density field is given by:*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\nu \left[\mathscr{Y}_n(\eta, \varphi, t)^2\right] \\
&= -\sqrt{2}\gamma^2\rho^2\, e^{4\gamma^2 t} \int_{\mathbb{R}^2} \varphi(x)\varphi(y)\, e^{2\sqrt{2}\gamma|x-y|}\, \mathrm{erf}(2\gamma\sqrt{t} + \tfrac{|x-y|}{\sqrt{2t}})\, dx\, dy \\
&\quad + \sqrt{2}\gamma\rho^2 \left(1 - e^{4\gamma^2 t}\, \mathrm{erf}(2\gamma\sqrt{t})\right) \int_{\mathbb{R}} \varphi(x)^2\, dx
\end{aligned} \tag{6.17}$$

where the error function is:

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy.$$

Heuristics of the coarsening process

In this section we give some intuition about the limiting behavior of the density field, as found in Theorem 6.2.1. More concretely, we show that Theorem 6.2.1 is consistent with the following "coarsening picture". Under the condensation regime, and started from an initial homogeneous product measure ν with density

ρ, over time large piles are created which are typically at distances of order n and of size ρn. The location of these piles evolves on the appropriate time scale according to a diffusion process. If we focus on two piles, this diffusion process is of the form $(X(t), Y(t))$ where $X(t) - Y(t)$ is a sticky Brownian motion $B^{\mathrm{sbm}}(t)$, and where the sum $X(t) + Y(t)$ is an independent Brownian motion $\overline{B}(t)$, time-changed via the local time inverse at the origin $\tau(t)$ of the sticky Brownian motion $B^{\mathrm{sbm}}(t)$ via $X(t) + Y(t) = \overline{B}(2t - \tau(t))$.

In the following we denote by $p_t^{\mathrm{sbm}}(x, dy)$ the transition kernel of a Sticky Brownian motion with stickiness parameter $\sqrt{2}\gamma$. This kernel consists of a first term that is absolutely continuous w.r.t. the Lebesgue measure and a second term that is a Dirac-delta at the origin times the probability mass function at zero. With a slight abuse of notation we will denote by

$$p_t^{\mathrm{sbm}}(x, dy) = p_t^{\mathrm{sbm}}(x, y)\, dy + p_t^{\mathrm{sbm}}(x, 0) \cdot \delta_0(dy) \tag{6.18}$$

where $p_t^{\mathrm{sbm}}(x, y)$ for $y \neq 0$ denotes a probability density to arrive at y at time t when started from x , and for $y = 0$ the probability to arrive at zero when started at x. See equation (2.15) in [54] for an explicit formula for (6.18).

Let us now make this heuristics more precise. Define the non-centered field

$$\mathscr{Z}_n(\eta, \varphi, t) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \varphi(\tfrac{x}{n}) \eta_{s(n,t)}(x) \tag{6.19}$$

then one has, using that at every time $t > 0$, and $x \in \mathbb{Z}^d$, $\mathbb{E}_\nu(\eta_t(x)) = \rho$:

$$\lim_{n\to\infty} \mathbb{E}_\nu \left[\mathscr{Z}_n(\eta, \varphi, t)\right] = \rho \int_{\mathbb{R}} \varphi(x)\, dx \tag{6.20}$$

and

$$\lim_{n\to\infty} \left(\mathbb{E}_\nu \left[\mathscr{Z}_n(\eta, \varphi, t)^2\right] - \mathbb{E}_\nu \left[\mathscr{Y}_n(\eta, \varphi, t)^2\right] \right) = \rho^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x)\varphi(y)\, dx\, dy.$$

As we will see later in the proof of our main theorem, the RHS of (6.17) can be written as

$$-\frac{\rho^2}{2} \int_{\mathbb{R}^2} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2}) p_t^{\mathrm{sbm}}(v, 0)\, dv\, du - \left(\sqrt{2}\gamma\rho^2 p_t^{\mathrm{sbm}}(0,0) - \sqrt{2}\gamma\rho^2\right) \int_{\mathbb{R}} \varphi(u)^2\, du,$$

and hence, we have that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\nu \left[\mathscr{X}_n(\eta,\varphi,t)^2\right] = \rho^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\varphi(y)\,dx\,dy + \sqrt{2}\gamma\rho^2 \int_{\mathbb{R}} \varphi(u)^2\,du \\
&-\frac{\rho^2}{2}\int_{\mathbb{R}^2} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2})p_t^{\mathrm{sbm}}(v,0)\,dv\,du - \sqrt{2}\gamma\rho^2 p_t^{\mathrm{sbm}}(0,0)\int_{\mathbb{R}}\varphi(u)^2\,du \\
&= \frac{\rho^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi\left(\tfrac{u+v}{2}\right)\varphi\left(\tfrac{u-v}{2}\right)\left(1-p_t^{\mathrm{sbm}}(v,0)\right)\left(dv+\sqrt{2}\gamma\delta_0(dv)\right)du \\
&= \frac{\rho^2}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \mathbb{E}_v^{\mathrm{sbm}}\left(\varphi(\tfrac{u+v_t}{2})\varphi(\tfrac{u-v_t}{2})\right)\left(1-\mathbb{1}_{\{0\}}(v)\right)\left(dv+\sqrt{2}\gamma\delta_0(dv)\right)\right)du \\
&= \frac{\rho^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_v^{\mathrm{sbm}}\left(\varphi(\tfrac{u+v_t}{2})\varphi(\tfrac{u-v_t}{2})\right)dv\,du \\
&= \frac{\rho^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(\tfrac{u+z}{2})\varphi(\tfrac{u-z}{2})\,p_t^{\mathrm{sbm}}(v,dz)\,dv\,du \\
&= \rho^2\int_{\mathbb{R}} dv \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\varphi(y)\cdot \bar{p}_t^{\mathrm{sbm}}(v;dx,dy) \qquad (6.21)
\end{aligned}$$

where

$$\bar{p}_t^{\mathrm{sbm}}(v;dx,dy) := p_t^{\mathrm{sbm}}(v,x-y)\,dx\,dy + p_t^{\mathrm{sbm}}(v,0)\,dx\,\delta_x(dy). \qquad (6.22)$$

In the second line we used the change of variables $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$.

We now want to describe a "macroscopic" time-dependent random field $\mathscr{X}(\varphi,t)$ that is consistent with the limiting expectation and second moment computed in (6.20) and (6.21). This macroscopic field describes intuitively the positions of the piles formed from the initial homogeneous background.

For any fixed $k\in\mathbb{N}$ we define the family of $\mathbb{R}^k$-valued diffusion processes $\{X^{\mathbf{x}}(t),\, t\geq 0\}_{\mathbf{x}\in\mathbb{R}^k}$ together on a common probability space Ω. Here $\mathbf{x}=(x_1,\ldots,x_k)$ is the vector of initial positions: $X^{\mathbf{x}}(0)=\mathbf{x}$. Then we will denote by $X_i^{\mathbf{x}}(t)$, $i=1,\ldots,k$, the i-th component of $X^{\mathbf{x}}(t)=(X_1^{\mathbf{x}}(t),\ldots,X_k^{\mathbf{x}}(t))$ that is defined as the trajectory started from x_i, i.e. the i-th component of $\mathbf{x}$. Then for any fixed $\omega\in\Omega$, we define the define the macroscopic field $\mathscr{X}^{(k)}(\cdot,t)(\omega)$ working on test functions $\varphi:\mathbb{R}\to\mathbb{R}$ as follows:

$$\mathscr{X}^{(k)}(\varphi,t)(\omega) = \frac{\rho}{k}\sum_{i=1}^{k}\int_{\mathbb{R}} \varphi(X_i^{\mathbf{x}}(t)(\omega))dx_i. \qquad (6.23)$$

We want to find the conditions on the probability law of the trajectories $\{X_i^{\mathbf{x}}(t),\, t\geq 0\}$ and on their couplings that make the macroscopic field $\mathscr{X}(\varphi,t)$ compatible with the limiting expectation (6.20) and second moment (6.21) of the microscopic field. We will see that, in order to achieve this it is sufficient to define the law of the one-component $\{X_i^{\mathbf{x}}(t),\, t\geq 0\}$ and two-components $\{(X_i^{\mathbf{x}}(t), X_i^{\mathbf{x}}(t)),\, t\geq 0\}$ marginals.

We assume that the family of processes $\{X^{\mathbf{x}}(t),\, t\geq 0\}_{\mathbf{x}\in\mathbb{R}^k}$ is such that, for all $\mathbf{x}=(x_1,\dots,x_k)$,

a) for all $i=1,\dots,k$, the marginal $X_i^{\mathbf{x}}(t)$ is a Brownian motion with diffusion constant $\chi/2$ started from x_i.

b) for all $i,j=1,\dots,k$, the pair $\{(X_i^{\mathbf{x}}(t), X_j^{\mathbf{x}}(t)),\, t\geq 0\}$ is a couple of sticky Brownian motions starting from (x_i,x_j), i.e. at any fixed time $t\geq 0$ it is distributed in such a way that the difference-sum process is given by

$$(X_i^{\mathbf{x}}(t)-X_j^{\mathbf{x}}(t)), X_i^{\mathbf{x}}(t)+X_j^{\mathbf{x}}(t)) = (B^{\mathrm{sbm},x_i-x_j}(t), \bar{B}^{x_i+x_j}(2t-\tau(t))). \tag{6.24}$$

Here $B^{\mathrm{sbm},x_i-x_j}(t)$ is a sticky Brownian motion with stickiness at 0, stickiness parameter $\sqrt{2}\gamma$, and diffusion constant χ, started from x_i-x_j and where $\tau(t)$ is the corresponding local time-change defined in (B.12), and $\bar{B}^{x_i+x_j}(2t-\tau(t))$ is another Brownian motion and diffusion constant χ, independent from $B^{\mathrm{sbm}}(t)$ started from x_i+x_j.

Remark 6.2.1. *For an example of a coupling satisfying requirements a) and b) above, we refer the reader to the family of processes introduced in [53].*

We will see that, for any fixed k, the field $\mathscr{X}^{(k)}(\varphi,t)$ reproduces correctly the first and second moments of (6.20) and (6.21).

For the expectation we have, using item a) above

$$\begin{aligned}\mathbb{E}[\mathscr{X}^{(k)}(\varphi,t)] &= \frac{\rho}{k}\sum_{i=1}^k \int_{\mathbb{R}} \mathbb{E}[\varphi(X^{\mathbf{x}}(t))]dx_i \\ &= \rho\int_{\mathbb{R}}\varphi(x)\int_{\mathbb{R}} p_t^{\mathrm{bm}}(x_i,x)\,dx_i\,dx = \rho\int_{\mathbb{R}}\varphi(x)\,dx \end{aligned}\tag{6.25}$$

where the last identity follows from the symmetry: $p_t^{\mathrm{bm}}(x_i,x)=p_t^{\mathrm{bm}}(x,x_i)$. Notice that indeed the RHS of (6.25) coincides with (6.20).

On the other hand, for the second moment, using item b) above

$$\mathbb{E}[\mathscr{X}^{(k)}(\varphi,t)^2] = \frac{\rho^2}{k^2}\sum_{i,j=1}^k \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[\varphi(X_i^{\mathbf{x}}(t))\varphi(X_j^{\mathbf{y}}(t))]dx_i dy_j. \tag{6.26}$$

Then, from our assumptions,

$$\mathbb{E}[\varphi(X_i^{\mathbf{x}}(t))\varphi(X_j^{\mathbf{y}}(t))] = \int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x)\varphi(y)p_t(x_i,y_j;dx,dy).$$

Here $p_t(x_i, y_j; dx, dy)$ is the transition probability kernel of the pair $(X_i^{\mathbf{x}}(t), X_j^{\mathbf{y}}(t))$. Denoting now by $\tilde{p}_t(v_0, u_0; dv, du)$ the transition probability kernel of the pair $(X_i^{\mathbf{x}}(t) - X_j^{\mathbf{y}}(t), X_i^{\mathbf{x}}(t) + X_j^{\mathbf{y}}(t))$, and by π_t the probability measure of the time change $\tau(t)$, at time t, we have

$$\begin{aligned}\tilde{p}_t(v_0, u_0; dv, du) &= \int_{\mathbb{R}} \tilde{p}_t(v_0, u_0; dv, du \,|s)\, \pi_t(ds) \\ &= \int_{\mathbb{R}} \tilde{p}_t^{(1)}(v_0, dv \,|s)\, \tilde{p}_t^{(2)}(u_0, du \,|s)\, \pi_t(ds)\end{aligned}$$

(where $\tilde{p}_t^{(i)}(\cdot, \cdot|s)$ for $i = 1, 2$, are resp. the transition probability density functions of the Brownian motions $B(t)$ and $\bar{B}(t)$ conditioned on s) as, from (6.24), the difference and sum processes are independent conditioned on the realization of $s = \tau(t)$. Now we have that

$$\int_{\mathbb{R}} \tilde{p}_t^{(1)}(v_0, dv \,|s)\pi_t(ds) = p_t^{\mathrm{sbm}}(v_0, dv) \quad \text{and} \quad \tilde{p}_t^{(2)}(u_0, du \,|s) = p_{2t-s}^{\mathrm{bm}}(u_0, du)$$

hence

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{p}_t(v_0, u_0; dv, du)\, dv_0\, du_0 \\ =\;& \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tilde{p}_t^{(1)}(v_0, dv \,|s)\, dv_0\right) \cdot \left(\int_{\mathbb{R}} p_{2t-s}^{\mathrm{bm}}(u_0, du)\, du_0\right) \pi_t(ds) \\ =\;& \int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{p}_t^{(1)}(v_0, dv \,|s)\pi_t(ds)\, dv_0 = \int_{\mathbb{R}} p_t^{\mathrm{sbm}}(v_0, dv)\, dv_0 \qquad (6.27)\end{aligned}$$

where the second identity follows from the symmetry of $p^{\mathrm{bm}}(\cdot, \cdot)$. Then, from the change of variables $v_0 := x_i - y_j$, $u_0 = x_i + y_j$, and $v = x - y$, $u = x + y$, and since $dv_0\, du_0 = 2dx_i\, dy_j$, it follows that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} p_t(x_i, y_j; dx, dy)\, dx_i\, dy_j = \int_{\mathbb{R}} \bar{p}_t^{\mathrm{sbm}}(v_0; dx, dy)\, dv_0. \qquad (6.28)$$

As a consequence

$$\mathbb{E}[(\mathscr{X}^{(k)}(\varphi, t))^2] = \rho^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\varphi(y) \int_{\mathbb{R}} \bar{p}_t^{\mathrm{sbm}}(v; dx, dy)dv,$$

which is exactly the same expression as (6.21).

Remark 6.2.2. *In order to match the first two moments of the limiting density field, it suffices to take in* (6.23) *any $k \geq 2$. We believe that in order to match all moments up to order m we need $k \geq m$, and so the limiting field would correspond to taking the limit $k \to \infty$. However, because in the current paper we can only deal with two particles, we cannot say more about higher moments.*

6.3 Proof of main result

Our main theorem, Theorem 6.2.1, is a consequence of self-duality and Theorem 6.3.1 below concerning the convergence in the Mosco sense of the sequence of Dirichlet forms associated to the difference process to the Dirichlet form corresponding to the so-called two-sided sticky Brownian motion (see the Appendix for details on this process). Before stating Theorem 6.3.1 let us introduce the relevant setting for this convergence:

The convergence of the difference process to sticky Brownian motion takes place in the sticky regime introduced earlier in Section 6.1.4.1. In this regime the corresponding scaled difference process is given by:

$$w_n(t) := \frac{1}{n}\, w\left(\frac{n^3\gamma}{\sqrt{2}}\, t\right) \qquad \text{with inclusion-parameter} \quad \alpha_n := \frac{1}{\sqrt{2}\gamma n}$$

with infinitesimal generator

$$(L_n f)(w) = \frac{n^3\gamma}{\sqrt{2}} \sum_{r\in A_n} 2p_n(r)\left(\frac{1}{\sqrt{2}n\gamma} + \mathbb{1}_{r=-w}\right)\left[f(w+r) - f(w)\right] \qquad (6.29)$$

for $w \in \frac{1}{n}\mathbb{Z}$, with

$$p_n(r) := p(nr) \qquad \text{and} \qquad A_n := \frac{1}{n}\{-R, -R+1, \ldots, R-1, R\} \setminus \{0\}. \quad (6.30)$$

Notice that by Proposition 6.1.2 the difference processes are reversible with respect to the measures $\nu_{\gamma,n}$ given by

$$\nu_{\gamma,n} = \mu_n + \sqrt{2}\gamma\delta_0 \qquad (6.31)$$

and by (B.1) the corresponding sequence of Dirichlet forms is given by

$$\mathscr{E}_n(f) = -\sum_{w\in\frac{1}{n}\mathbb{Z}} f(w) \sum_{r\in A_n} 2p_n(r)\left(\tfrac{n^2}{2} + \tfrac{n^3\gamma}{\sqrt{2}}\mathbb{1}_{r=-w}\right)\left(f(w+r) - f(w)\right)\, \nu_{\gamma,n}(w) \quad (6.32)$$

Remark 6.3.1. *The choice of the reversible measures $\nu_{\gamma,n}$ determines the sequence of approximating Hilbert spaces given by $H_n^{sip} := L^2(\frac{1}{n}\mathbb{Z}, \nu_{\gamma,n})$, $n \in \mathbb{N}$. Here for $f, g \in H_n^{sip}$ their inner product is given by*

$$\langle f, g\rangle_{H_n^{sip}} = \sum_{w\in\frac{1}{n}\mathbb{Z}} f(w)g(w)\ \nu_{\gamma,n}(w) = \langle f, g\rangle_{H_n^{rw}} + \sqrt{2}\gamma f(0)g(0) \quad (6.33)$$

where

$$\langle f, g\rangle_{H_n^{rw}} = \frac{1}{n}\sum_{w\in\frac{1}{n}\mathbb{Z}} f(w)g(w)$$

is the inner product of Section C.2.

On the other hand, the two sided sticky Brownian motion with sticky parameter $\gamma > 0$ can be described in terms of the Dirichlet form $\big(\mathscr{E}_{\mathrm{sbm}}, D(\mathscr{E}_{\mathrm{sbm}})\big)$ given by

$$\mathscr{E}_{\mathrm{sbm}}(f) = \frac{\chi}{2}\int_{\mathbb{R}} \mathbb{1}_{\{x\neq 0\}}(x) f'(x)^2 dx, \qquad \chi = \sum_{r=1}^{R} r^2\, p(r) \tag{6.34}$$

whose domain is

$$D(\mathscr{E}_{\mathrm{sbm}}) = H^1(\mathbb{R}) \cap L^2(\mathbb{R}, \bar{\nu}) \qquad \text{with} \qquad \bar{\nu} = dx + \sqrt{2}\gamma\delta_0. \tag{6.35}$$

Convergence of Hilbert spaces

As we already mentioned in Remark 6.3.1, by choosing the reversible measures $\nu_{\gamma,N}$ we have determined the convergent sequence of Hilbert spaces and, as a consequence, we have also set the limiting Hilbert space H^{sbm} to be $L^2(\mathbb{R}, \bar{\nu})$ with $\bar{\nu}$ as in (6.35). Notice that from the regularity of this measure, by Theorem 13.21 in [51] and standard arguments, we know that the set $C_k^\infty(\mathbb{R})$ of smooth compactly supported test functions is dense in $L^2(\mathbb{R}, \bar{\nu})$. Moreover the set

$$C^0(\mathbb{R} \setminus \{0\}) := \{f + \lambda \mathbb{1}_{\{0\}} : f \in C_k^\infty(\mathbb{R}), \lambda \in \mathbb{R}\}, \tag{6.36}$$

denoting the set of all continuous functions on $\mathbb{R} \setminus \{0\}$ with finite value at 0, is also dense in $L^2(\mathbb{R}, \bar{\nu})$.

Before stating our convergence result, we have to define the right "embedding" operators $\{\Phi_n\}_{n\geq 1}$, cf. Definition B.4.1 , to not only guarantee convergence of Hilbert spaces $H_n \to H$, but Mosco convergence as well. We define these operators as follows:

$$\{\Phi_n : C^0(\mathbb{R} \setminus \{0\}) \to H_n^{\mathrm{sip}}\}_n \qquad \text{defined by} \qquad \Phi_n f = f\,|_{\frac{1}{n}\mathbb{Z}}\,. \tag{6.37}$$

Proposition 6.3.1. *The sequence of spaces $H_N^{sip} = L^2(\frac{1}{n}\mathbb{Z}, \nu_{\gamma,n})$, $n \in \mathbb{N}$, converges, in the sense of Definition B.4.1, to the space $H^{sbm} = L^2(\mathbb{R}, \bar{\nu})$.*

Proof. The statement follows from the definition of $\{\Phi_n\}_{n\geq 1}$. □

Mosco convergence of the difference process

In the context described above, we have the following theorem:

Theorem 6.3.1. *The sequence of Dirichlet forms $\{\mathscr{E}_n, D(\mathscr{E}_n)\}_{n\geq 1}$ given by (6.32) converges in the Mosco sense to the form $\big(\mathscr{E}_{sbm}, D(\mathscr{E}_{sbm})\big)$ given by (6.34) and (6.35). As a consequence, if we denote by $T_n(t)$ and T_t the semigroups associated to the difference process $w_n(t)$ and the sticky Brownian motion B_t^{sbm}, we have that $T_n(t) \to T_t$ strongly in the sense of Definition B.4.4.*

In the following section we will show how to use this result to prove Theorem 6.2.1. The proof of Theorem 6.3.1 will be postpone to Section 6.3.2.

6.3.1 Proof of main theorem: Theorem 6.2.1

We denote by $T_n(t)$ and T_t the semigroups associated to the difference process $w_n(t)$ and the sticky Brownian motion B_t^{sbm}. Because of our result on Mosco convergence and thanks to Theorem B.4.1 we know that the sequence of semigroups $\{T_n(t)\}_{n\geq 1}$ converges strongly to T_t in the H_n^{sip} Hilbert convergence sense. We will see that this implies the convergence of the probability mass function at 0.

Proposition 6.3.2. *For all $t > 0$ denote by $p_t^n(w,0)$ the transition function that the difference process starting from $w \in \frac{1}{n}\mathbb{Z}$ finishes at 0 at time t. Then the sequence $p_t^n(\cdot,0)$ converges strongly to $p_t^{sbm}(\cdot,0)$ with respect to H_n^{sip} Hilbert convergence.*

Proof. From the fact that $\{T_n(t)\}_{n\geq 1}$ converges strongly to T_t, we have that for all f_n strongly converging to f, the sequence $\{T_n(t)f_n\}_{n\geq 1} \in H_n^{\mathrm{sip}}$ converges strongly to $T_t f$. In particular, for $f_n = \mathbb{1}_{\{0\}}$ we have that the sequence

$$T_n(t)f_n(w) = \mathbb{E}_w^n \mathbb{1}_{\{0\}}(w_t) = p_t^n(w,0), \tag{6.38}$$

converges strongly to

$$T_t f(w) = \mathbb{E}_w^{\mathrm{sbm}} \mathbb{1}_{\{0\}}(w_t) = p_t^{\mathrm{sbm}}(w,0) \tag{6.39}$$

where $\mathbb{E}_w^{\mathrm{sbm}}$ denotes expectation with respect to the sticky Brownian motion started at w. □

Remark 6.3.2. *Despite the fact that Proposition 6.3.2 is not a point-wise statement, we can still say something more relevant when we start our process at the point zero:*

$$\lim_{n\to\infty} p_t^n(0,0) = p_t^{sbm}(0,0). \tag{6.40}$$

The reason is that we can see $p_t^n(w,0)$ as a weakly converging sequence and used again the fact that $f_n = \mathbb{1}_{\{0\}}$ converges strongly.

Proof. Theorem 6.2.1
Let ρ and σ be given by (6.6), then we can write

$$\begin{aligned}&\mathbb{E}_\nu\left[\mathscr{Y}_n(\eta,\varphi,t)^2\right]\\ &= \frac{1}{n^2}\sum_{x,y\in\mathbb{Z}} \varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n}) \int \mathbb{E}_\eta\left(\eta_{s(n,t)}(x)-\rho\right)\left(\eta_{s(n,t)}(y)-\rho\right)\nu(d\eta)\end{aligned}$$

where, from Proposition 5.1 in [18], using self-duality, we can simplify the integral above as

$$\begin{aligned}
&\int \mathbb{E}_\eta \left(\eta_{s(n,t)}(x) - \rho\right)\left(\eta_{s(n,t)}(y) - \rho\right)\nu(d\eta) \\
&= \left(1 + \frac{1}{\alpha_n}\mathbb{1}_{\{x=y\}}\right)\left(\frac{\alpha_n \sigma}{\alpha_n + 1} - \rho^2\right)\mathbb{E}_{x,y}\mathbb{1}_{\{X_{s(n,t)}=Y_{s(n,t)}\}} \\
&+\mathbb{1}_{\{x=y\}}\left(\frac{\rho^2}{\alpha_n} + \rho\right). \qquad (6.41)
\end{aligned}$$

Notice that the expectation in the RHS of (6.41) can be re-written in terms of our difference process as follows:

$$\mathbb{E}_{x,y}\left[\mathbb{1}_{\{X_{s(n,t)}=Y_{s(n,t)}\}}\right] = p_{s(n,t)}(x-y,0) \qquad (6.42)$$

where $p_{s(n,t)}$ is the transition function p_t^n under the space-time rescaling defined in (6.14), since under the condensation regime we have, as in Section 6.1.4.1, $\alpha_n = \frac{1}{\sqrt{2}\gamma n}$. We then obtain:

$$\begin{aligned}
&\mathbb{E}_\nu\left[\mathscr{Y}_n(\eta,\varphi,t)^2\right] \\
&= \frac{1}{n^2}\sum_{x,y\in\mathbb{Z}}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})\left(1 + \sqrt{2}\gamma n\mathbb{1}_{\{x=y\}}\right)\left(\frac{\sigma}{1+\sqrt{2}\gamma n} - \rho^2\right)p_{s(n,t)}(x-y,0) \\
&+\frac{1}{n^2}\sum_{x\in\mathbb{Z}}\varphi(\tfrac{x}{n})\varphi(\tfrac{x}{n})\left(\sqrt{2}\gamma n\rho^2 + \rho\right). \qquad (6.43)
\end{aligned}$$

At this point we have 3 non-vanishing contributions:

$$\begin{aligned}
C_n^{(1)} &:= \frac{\rho^2}{n^2}\sum_{x,y\in\mathbb{Z}}\varphi(\tfrac{x}{n})\varphi(\tfrac{y}{n})p_{s(n,t)}(x-y,0), \\
C_n^{(2)} &:= \frac{\sqrt{2}\gamma\rho^2}{n}\sum_{x\in\mathbb{Z}}\left(\varphi(\tfrac{x}{n})\right)^2 p_{s(n,t)}(0,0) \\
C_n^{(3)} &:= \frac{\sqrt{2}\gamma\rho^2}{n}\sum_{x\in\mathbb{Z}}\left(\varphi(\tfrac{x}{n})\right)^2
\end{aligned}$$

where we already know:

$$\lim_{n\to\infty} C_n^{(3)} = \sqrt{2}\gamma\rho^2\int_{\mathbb{R}}\varphi(v)^2 dv \qquad (6.44)$$

and, by Remark 6.3.2,

$$\lim_{n\to\infty} C_n^{(2)} = \sqrt{2}\gamma\rho^2 p_t^{\mathrm{sbm}}(0,0)\int_{\mathbb{R}}\varphi(v)^2 dv. \qquad (6.45)$$

To analyze the first contribution we use the change of variables $u = x + y$, $v = x - y$ from which we obtain:

$$C_n^{(1)} = \frac{\rho^2}{n^2} \sum_{\substack{u,v \in \frac{1}{n}\mathbb{Z} \\ u \equiv v \mod 2}} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2})\, p_{s(n,t)}(v,0). \tag{6.46}$$

Hence by (6.33), $C_n^{(1)}$ can be re-written as

$$C_n^{(1)} = \Big\langle F_n(\cdot), p_{s(n,t)}(\cdot,0) \Big\rangle_{H_n^{\mathrm{sip}}} - \frac{\gamma\rho^2}{\sqrt{2}n} \sum_{u \in \frac{1}{n}\mathbb{Z}} \varphi(\tfrac{u}{2})\varphi(\tfrac{u}{2})\, p_{s(n,t)}(0,0) \tag{6.47}$$

with F_n given by

$$F_n(v) = \frac{\rho^2}{n} \sum_{\substack{u \in \frac{1}{n}\mathbb{Z} \\ u \equiv v \mod 2}} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2}), \qquad \text{for all } v \in \tfrac{1}{n}\mathbb{Z}. \tag{6.48}$$

We then have the following proposition:

Proposition 6.3.3. *The sequence of functions* $\{F_n\}_{n\geq 1} \in H_n^{sip}$*, given by* (6.48)*, converges strongly to* $F \in H^{sbm}$ *given by*

$$F(x) := \frac{\rho^2}{2} \int_{\mathbb{R}} \varphi(\tfrac{y+x}{2})\varphi(\tfrac{y-x}{2})\, dy. \tag{6.49}$$

Proof. For simplicity let us deal with the case $\varphi \in C_k^\infty(\mathbb{R})$. The case where $\varphi \in \mathscr{S}(\mathbb{R}) \setminus C_k^\infty(\mathbb{R})$ can be done by standard approximations using a combination of truncation and convolution with a kernel (see for example the proof of Proposition C.2.1 in the Appendix).

In the language of Definition B.4.2, we set the following sequence of reference functions:

$$\tilde{F}_m(x) := \frac{\rho^2}{2m} \sum_{y \in \mathbb{Z}} \varphi(\tfrac{y}{2m} + \tfrac{x}{2})\varphi(\tfrac{y}{2m} - \tfrac{x}{2}) \tag{6.50}$$

for all $x \in \mathbb{R}$.

Then we have:

$$\begin{aligned}
&\lim_{m\to\infty} \|\tilde{F}_m - F\|^2_{H^{\mathrm{sbm}}} \\
= &\lim_{m\to\infty} \frac{\rho^4}{4} \int_{\mathbb{R}} \left(\frac{1}{m} \sum_{y_1\in\mathbb{Z}} \varphi(\tfrac{y_1}{2m} + \tfrac{x}{2})\varphi(\tfrac{y_1}{2m} - \tfrac{x}{2}) - \int_{\mathbb{R}} \varphi(\tfrac{y_2+x}{2})\varphi(\tfrac{y_2-x}{2})\, dy_2 \right)^2 \nu_\gamma(dx) \\
= &\frac{\rho^4}{4} \int_{\mathbb{R}} \left[\lim_{m\to\infty} \frac{1}{m^2} \sum_{y_1,y_2\in\mathbb{Z}} \varphi(\tfrac{y_1}{2m} + \tfrac{x}{2})\varphi(\tfrac{y_1}{2m} - \tfrac{x}{2})\varphi(\tfrac{y_2}{2m} + \tfrac{x}{2})\varphi(\tfrac{y_2}{2m} - \tfrac{x}{2}) \right] dx \\
+ &\frac{\rho^4}{4} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \varphi(\tfrac{y_1+x}{2})\varphi(\tfrac{y_1-x}{2})\, dy_1 \int_{\mathbb{R}} \varphi(\tfrac{y_2+x}{2})\varphi(\tfrac{y_2-x}{2})\, dy_2 \right] dx \\
- &\frac{\rho^4}{2} \int_{\mathbb{R}} \left[\lim_{m\to\infty} \frac{1}{m} \sum_{y_1\in\mathbb{Z}} \varphi(\tfrac{y_1}{2m} + \tfrac{x}{2})\varphi(\tfrac{y_1}{2m} - \tfrac{x}{2}) \int_{\mathbb{R}} \varphi(\tfrac{y_2+x}{2})\varphi(\tfrac{y_2-x}{2})\, dy_2 \right] dx \\
+ &\frac{\sqrt{2}\gamma\rho^4}{4} \left[\lim_{m\to\infty} \frac{1}{m^2} \sum_{y_1,y_2\in\mathbb{Z}} \varphi(\tfrac{y_1}{2m})^2 \varphi(\tfrac{y_2}{2m})^2 \right] \\
+ &\frac{\sqrt{2}\gamma\rho^4}{4} \left[\int_{\mathbb{R}} \varphi(\tfrac{y_1}{2})^2\, dy_1 \int_{\mathbb{R}} \varphi(\tfrac{y_2}{2})^2\, dy_2 \right] \\
- &\frac{\sqrt{2}\gamma\rho^4}{2} \left[\lim_{m\to\infty} \frac{1}{m} \sum_{y_1\in\mathbb{Z}} \varphi(\tfrac{y_1}{2m})^2 \int_{\mathbb{R}} \varphi(\tfrac{y_2+x}{2})^2\, dy_2 \right] \\
= &\ 0
\end{aligned} \tag{6.51}$$

where in the last line we used the convergence

$$\lim_{m\to\infty} \frac{1}{2m} \sum_{y\in\mathbb{Z}} \varphi(\tfrac{y}{2m} + \tfrac{x}{2})\varphi(\tfrac{y}{2m} - \tfrac{x}{2}) = \frac{1}{2} \int_{\mathbb{R}} \varphi(\tfrac{y+x}{2})\varphi(\tfrac{y-x}{2})\, dy. \tag{6.52}$$

Moreover, a similar expansion (substituting integrals by sums) gives:

$$
\begin{aligned}
&\|\Phi_n \tilde{F}_m - F_n\|^2_{H_n^{\mathrm{sip}}} \\
=\;& \frac{\rho^4}{n} \sum_{x \in \frac{1}{n}\mathbb{Z}} \left[\frac{1}{4m^2} \sum_{y_1, y_2 \in \frac{1}{m}\mathbb{Z}} \varphi(\tfrac{y_1+x}{2})\varphi(\tfrac{y_1-x}{2})\varphi(\tfrac{y_2+x}{2})\varphi(\tfrac{y_2-x}{2}) \right] \\
-\;& \frac{2\rho^4}{n} \sum_{x \in \frac{1}{n}\mathbb{Z}} \left[\frac{1}{2mn} \sum_{y \in \frac{1}{m}\mathbb{Z}} \sum_{\substack{u \in \frac{1}{n}\mathbb{Z} \\ u \equiv x \mod 2}} \varphi(\tfrac{y+x}{2})\varphi(\tfrac{y-x}{2})\varphi(\tfrac{u+x}{2})\varphi(\tfrac{u-x}{2}) \right] \\
+\;& \frac{\rho^4}{n} \sum_{x \in \frac{1}{n}\mathbb{Z}} \left[\frac{1}{n^2} \sum_{\substack{u_1, u_2 \in \frac{1}{n}\mathbb{Z} \\ u_i \equiv x \mod 2}} \varphi(\tfrac{u_1+x}{2})\varphi(\tfrac{u_1-x}{2})\varphi(\tfrac{u_2+x}{2})\varphi(\tfrac{u_2-x}{2}) \right] \\
+\;& \sqrt{2}\gamma\rho^4 \left[\frac{1}{4m^2} \sum_{y_1, y_2 \in \frac{1}{m}\mathbb{Z}} \varphi(\tfrac{y_1}{2})^2 \varphi(\tfrac{y_2}{2})^2 \right] \\
-\;& 2\sqrt{2}\gamma\rho^4 \left[\frac{1}{2mn} \sum_{y \in \frac{1}{m}\mathbb{Z}} \sum_{\substack{u \in \frac{1}{n}\mathbb{Z} \\ u \equiv 0 \mod 2}} \varphi(\tfrac{y}{2})^2 \varphi(\tfrac{u}{2})^2 \right] \\
+\;& \sqrt{2}\gamma\rho^4 \left[\frac{1}{n^2} \sum_{\substack{u_1, u_2 \in \frac{1}{n}\mathbb{Z} \\ u_i \equiv 0 \mod 2}} \varphi(\tfrac{u_1}{2})^2 \varphi(\tfrac{u_2}{2})^2 \right]
\end{aligned} \tag{6.53}
$$

where to conclude

$$
\lim_{m\to\infty} \limsup_{n\to\infty} \|\Phi_n \tilde{F}_m - F_n\|^2_{H_n^{\mathrm{sip}}} = 0
$$

we can use (6.52) and the convergence:

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{u \in \frac{1}{n}\mathbb{Z} \\ u \equiv v \mod 2}} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2}) = \frac{1}{2} \int_{\mathbb{R}} \varphi(\tfrac{y+v}{2})\varphi(\tfrac{y-v}{2})\, dy. \tag{6.54}
$$

□

From the strong convergence $F_n \to F$, Proposition 6.3.2, and Remark 6.3.2 we conclude

$$\lim_{n\to\infty} C_n^{(1)} = \frac{\rho^2}{2}\int_{\mathbb{R}^2} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2})p_t^{\rm sbm}(v,0)\,du\,dv. \tag{6.55}$$

Substituting the limits of the contributions we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\nu\left[\mathscr{Y}_n(\eta,\varphi,t)^2\right]\\
&= -\frac{\rho^2}{2}\int_{\mathbb{R}^2} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2})p_t^{\rm sbm}(v,0)\,dv\,du - \left(\sqrt{2}\gamma\rho^2 p_t^{\rm sbm}(0,0) - \sqrt{2}\gamma\rho^2\right)\int_{\mathbb{R}}\varphi(u)^2\,du\\
&= -\frac{\rho^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2})\mathbb{E}_v^{\rm sbm}\left[\mathbb{1}_{\{0\}}(v_t)\right]\left(dv+\sqrt{2}\gamma\delta_0(dv)\right)du + \sqrt{2}\gamma\rho^2\int_{\mathbb{R}}\varphi(u)^2\,du\\
&= -\frac{\rho^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_v^{\rm sbm}\left[\varphi(\tfrac{u+v_t}{2})\varphi(\tfrac{u-v_t}{2})\right]\mathbb{1}_{\{0\}}(v)\left(dv+\sqrt{2}\gamma\delta_0(dv)\right)du + \sqrt{2}\gamma\rho^2\int_{\mathbb{R}}\varphi(u)^2\,du\\
&= \frac{\sqrt{2}\gamma\rho^2}{2}\int_{\mathbb{R}}\left\{\varphi(\tfrac{u}{2})^2 - \mathbb{E}_0^{\rm sbm}\left[\varphi(\tfrac{u+v_t}{2})\varphi(\tfrac{u-v_t}{2})\right]\right\}du\\
&= \frac{\sqrt{2}\gamma\rho^2}{2}\int_{\mathbb{R}}\left\{\varphi(\tfrac{u}{2})^2 - \int_{\mathbb{R}} p_t^{\rm sbm}(0,dv)\varphi(\tfrac{u+v}{2})\varphi(\tfrac{u-v}{2})\right\}du
\end{aligned} \tag{6.56}$$

where in the third equality we used the reversibility of SBM with respect to the measure $\hat{\nu}(dv) = dv + \sqrt{2}\gamma\delta_0(dv)$. Then, (6.17) follows, after a change of variable, using the expression (2.15) given in [54] for the transition probability measure $p_t^{\rm sbm}(0,dv)$ of the Sticky Brownian motion (with $\theta = \sqrt{2}\gamma$), namely

$$p_t^{\rm sbm}(0,dv) = \sqrt{2}\gamma e^{2\sqrt{2}\gamma|v|+4\gamma^2 t}\,\mathrm{erf}\left(2\gamma\sqrt{t}+\frac{|v|}{\sqrt{2t}}\right)dv + \delta_0(dv)e^{4\gamma^2 t}\ \mathrm{erf}\left(2\gamma\sqrt{t}\right) \tag{6.57}$$

This concludes the proof. □

Remark 6.3.3. *Using the expression of the Laplace transform of $p_t^{sbm}(0,dv)$ given in Section 2.4 of [54], it is possible to verify that the Laplace transform of* (6.17) *(using* (6.56)*) coincides with the expression in Theorem 2.18 of [18].*

6.3.2 Proof of Theorem 6.3.1: Mosco convergence for inclusion dynamics

In this section we prove Theorem 6.3.1; the Mosco convergence of the Dirichlet forms associated to the difference process $\{w_n(t),\, t\geq 0\}$ to the Dirichlet form corresponding to the two-sided sticky Brownian motion $\{B_t^{\rm sbm},\, t\geq 0\}$ given by (6.34) and (6.35).

By Proposition 6.3.1 we have already determined the relevant notions of weak and strong convergence of vectors living in the approximating sequence of Hilbert spaces (the spaces H_n^{sip}). We can then, move directly to the verification of conditions Mosco I and Mosco II in the definition of Mosco convergence. We do this in Section 6.3.2.1 and Section 6.3.2.2 respectively.

6.3.2.1 Mosco I

We will divide our task in two steps. First, we will compare the inclusion Dirichlet form with a random walk Dirichlet form and show that the first one dominates the second one. We will later use this bound and the fact that the random walk Dirichlet form satisfies Mosco I, to prove that Mosco I also holds for the case of inclusion particles.

We consider a random walk on $\mathbb{Z}$ with jump range $A = [-R, R] \cap \mathbb{Z}/\{0\}$. We call again $\{v(t),\ t \geq 0\}$ this process, as in the case of nearest-neighbor RW (which is a special case of this process corresponding to the choice $R = 1$). More generally, in this section we will use the same notation that has been used in Section C.2 for the case $R = 1$, thus we denote by L^{rw} the infinitesimal generator:

$$(L^{rw} f)(v) = \sum_{r \in A} p(r) \left[f(v + r) - f(v) \right], \qquad v \in \mathbb{Z} \tag{6.58}$$

Hence, in the diffusive scaling, the n-infinitesimal generator is given by:

$$\Delta_n g(v) = n^2 \sum_{r \in A_n^+} p_n(r) \left[g(v + r) - 2g(v) + g(v - r) \right], \qquad v \in \tfrac{\mathbb{Z}}{n} \tag{6.59}$$

where $A_n^+ := \{|r| \colon r \in A_n\}$ i.e. the generator of the process $v_n(t) := \frac{1}{n} v(n^2 t)$, $t \geq 0$, and denote by $(\mathscr{R}_n, D(\mathscr{R}_n))$ the associated Dirichlet form.

Comparing RW and SIP Dirichlet forms

The key idea to prove Mosco I is to transfer the difficulties of the SIP nature to independent random walkers. This is done by means of the following observation:

Proposition 6.3.4. *For any $f_n \in H_n^{sip}$ we have*

$$\mathscr{E}_n(f_n) \geq \mathscr{R}_n(f_n) \tag{6.60}$$

Proof. Rearranging (6.32) and using the symmetry of $p(\cdot)$ allows us to write:

$$\mathscr{E}_n(f_n) - \mathscr{R}_n(f_n) = \frac{n^2}{\sqrt{2}} \gamma \sum_{r \in A_n} 2 p_n(r) (f_n(r) - f_n(0))^2 \tag{6.61}$$

and the result follows from the fact that the RHS of this identity is nonnegative. □

Strong and weak convergence in $H_n^{\mathbf{rw}}$ and $H_n^{\mathbf{sip}}$ compared

Proposition 6.3.5. *The sequence $\{h_n = 1\!\!1_{\{0\}}\}_{n\geq 1}$, with $h_n \in H_n^{rw}$, converges strongly to $h = 0 \in H^{bm}$ with respect to H_n^{rw}-Hilbert convergence.*

Proof. In the language of Definition B.4.2 we set $\tilde{h}_m \equiv 0$. With this choice we immediately have

$$\|\hat{h}_m - h\|_{H^{\mathrm{bm}}} = 0 \qquad \text{and} \qquad \|\Phi_n \hat{h}_m - h_n\|^2_{H_n^{\mathrm{rw}}} = \tfrac{1}{n} \tag{6.62}$$

which concludes the proof. □

Proposition 6.3.6. *The sequence $\{h_n = 1\!\!1_{\{0\}}\}_{n\geq 1}$, with $h_n \in H_n^{sip}$, converges strongly to $h = 1\!\!1_{\{0\}} \in H^{sbm}$ with respect to H_n^{sip}-Hilbert convergence.*

Proof. In the language of Definition B.4.2 we set $\tilde{h}_m \equiv 1\!\!1_{\{0\}}$. With this choice we immediately have

$$\|\hat{h}_m - h\|_{H^{\mathrm{sbm}}} = 0 \qquad \text{and} \qquad \|\Phi_n \hat{h}_m - h_n\|_{H_n^{\mathrm{sip}}} = 0 \tag{6.63}$$

which concludes the proof. □

A consequence of Proposition 6.3.6 is that any sequence which is weakly convergent, with respect to H_n^{sip}-Hilbert convergence, converges also at zero.

Proposition 6.3.7. *Let $\{f_n\}_{n\geq 1}$ in $\{H_n^{sip}\}_{n\geq 1}$ be a sequence converging weakly to $f \in H^{sbm}$ with respect to H_n^{sip}-Hilbert convergence, then $\lim_{n\to\infty} f_n(0) = f(0)$.*

Proof. By Proposition 6.3.6 we know that $\{h_n = 1\!\!1_{\{0\}}\}_{n\geq 1}$ converges strongly to $h = 1\!\!1_{\{0\}}$ with respect to H_n^{sip}-Hilbert convergence. This, together with the fact that $\{f_n\}_{n\geq 1}$ converges weakly, implies:

$$\lim_{n\to\infty} \langle f_n, h_n\rangle_{H_n^{\mathrm{sip}}} = \langle f, h\rangle_{H^{\mathrm{sbm}}} = \sqrt{2}\gamma f(0) \tag{6.64}$$

but by (6.33)

$$\langle f_n, h_n\rangle_{H_n^{\mathrm{sip}}} = (\tfrac{1}{n} + \sqrt{2}\gamma) f_n(0) \tag{6.65}$$

which, together with (6.64), implies the statement. □

To further contrast the two notions of convergence, Proposition 6.3.5 has a weaker implication

Proposition 6.3.8. *Let $\{g_n\}_{n\geq 1}$ in $\{H_n^{rw}\}_{n\geq 1}$ be a sequence converging weakly to $g \in H^{bm}$ with respect to H_n^{rw}-Hilbert convergence, then $\lim_{n\to\infty} \frac{1}{n} g_n(0) = 0$.*

Proof. By Proposition 6.3.5 we know that $\{h_n = \mathbb{1}_{\{0\}}\}_{n\geq 1}$ converges strongly to $h = 0$ with respect to H_n^{rw}-Hilbert convergence. This, together with the fact that $\{g_n\}_{n\geq 1}$ converges weakly, implies:

$$\lim_{n\to\infty} \langle g_n, h_n\rangle_{H_n^{\mathrm{rw}}} = 0 \tag{6.66}$$

but we know

$$\langle g_n, h_n\rangle_{H_n^{\mathrm{rw}}} = \frac{1}{n} g_n(0) \tag{6.67}$$

which together with (6.66) concludes the proof. □

From $H_n^{\mathbf{rw}}$ strong convergence to $H_n^{\mathbf{sip}}$ strong convergence

Proposition 6.3.9. *Let $\{g_n\}_{n\geq 1}$ in $\{H_n^{rw}\}_{n\geq 1}$ be a sequence converging strongly to $g \in H^{bm}$ with respect to H_n^{rw}-Hilbert convergence. For all $n \geq 1$ define the sequence*

$$\hat{g}_n = g_n - g_n(0)\mathbb{1}_{\{0\}} \tag{6.68}$$

Then $\{\hat{g}_n\}_{n\geq 0}$ also converges strongly with respect to H_n^{sip}-Hilbert convergence to $\hat{g}$ given by:

$$\hat{g} = g - g(0)\mathbb{1}_{\{0\}} \tag{6.69}$$

Proof. From the strong convergence in the H_n^{rw}-Hilbert convergence sense, we know that there exists a sequence $\tilde{g}_m \in C_k^\infty(\mathbb{R})$ such that

$$\lim_{m\to\infty} \|\tilde{g}_m - g\|_{H^{\mathrm{bm}}} = 0 \tag{6.70}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|\Phi_n \tilde{g}_m - g_n\|_{H_n^{\mathrm{rw}}} = 0 \tag{6.71}$$

for each m we define the function $\hat{g}_m$ given by

$$\hat{g}_m = \tilde{g}_m - \tilde{g}_m(0)\mathbb{1}_{\{0\}}$$

Notice that:

$$\|\hat{g}_m\|^2_{H^{\mathrm{sbm}}} = \|\tilde{g}_m\|^2_{H^{\mathrm{bm}}} < \infty \tag{6.72}$$

and hence we have $\hat{g}_m$ belongs to both $C^0(\mathbb{R}\setminus\{0\})$ and H^{sbm}.

As before, we have the relation:

$$\|\hat{g}_m - \hat{g}\|^2_{H^{\mathrm{sbm}}} = \|\hat{g}_m - \hat{g}\|^2_{H^{\mathrm{bm}}} + \sqrt{2}\gamma(\hat{g}_m(0) - \hat{g}(0))^2 = \|\tilde{g}_m - g\|^2_{H^{\mathrm{bm}}} \tag{6.73}$$

which shows that indeed we have

$$\lim_{m\to\infty} \|\hat{g}_m - \hat{g}\|^2_{H^{\mathrm{sbm}}} = 0 \tag{6.74}$$

For the second requirement of strong convergence we can estimate as follows

$$\|\Phi_n \hat{g}_m - \hat{g}_n\|^2_{H_n^{\mathrm{sip}}} = \frac{1}{n} \sum_{\substack{x \in \frac{1}{n}\mathbb{Z} \\ x \neq 0}} (\Phi_n \tilde{g}_m(x) - g_n(x))^2$$

relation (6.71) allows us to see that the RHS of the equality above vanishes. This, together with (6.74), concludes the proof of the Proposition. □

From H_n^{sip} weak convergence to H_n^{rw} weak convergence

The following proposition says that with respect to weak convergence the implication comes in the opposite direction

Proposition 6.3.10. *Let $\{f_n\}_{n \geq 1}$ in $\{H_n^{sip}\}_{n \geq 1}$ be a sequence converging weakly to $f \in H^{sbm}$ with respect to H_n^{sip}-Hilbert convergence. Then it also converges weakly with respect to H_n^{rw}-Hilbert convergence.*

Proof. Let $\{f_n\}_{n \geq 0}$ in $\{H_n^{\mathrm{sip}}\}_{n \geq 0}$ be as in the Proposition. In order to show that it also converges weakly with respect to H_n^{rw}-Hilbert convergence, we need to show that for any sequence $\{g_n\}_{n \geq 0}$ in $\{H_n^{\mathrm{rw}}\}_{n \geq 0}$ converging strongly to some $g \in H^{\mathrm{bm}}$ we have

$$\lim_{n \to \infty} \langle f_n, g_n \rangle_{H_n^{\mathrm{rw}}} = \langle f, g \rangle_{H^{\mathrm{bm}}} \tag{6.75}$$

Consider such a sequence $\{g_n\}_{n \geq 0}$, by Proposition 6.3.9 we know that the sequence $\{\hat{g}_n\}_{n \geq 1}$ also converges strongly with respect to H_n^{sip}-Hilbert convergence to $\hat{g}$ defined as in (6.69). Then we have:

$$\lim_{n \to \infty} \langle f_n, \hat{g}_n \rangle_{H_n^{\mathrm{sip}}} = \langle f, \hat{g} \rangle_{H^{\mathrm{sbm}}} = \langle f, g \rangle_{H^{\mathrm{bm}}} \tag{6.76}$$

which can be re-written as:

$$\lim_{n \to \infty} \langle f_n, g_n \rangle_{H_n^{\mathrm{rw}}} - \frac{1}{n} f_n(0) g_n(0) = \langle f, g \rangle_{H^{\mathrm{bm}}} \tag{6.77}$$

and together with Propositions 6.3.7 and 6.3.8 implies that:

$$\lim_{n \to \infty} \langle f_n, g_n \rangle_{H_n^{\mathrm{rw}}} = \langle f, g \rangle_{H^{\mathrm{bm}}} \tag{6.78}$$

and the proof is done. □

Conclusion of proof of Mosco I

In order to see that condition Mosco I is satisfied, we combine Proposition 6.3.4, Proposition 6.3.10 and the Mosco convergence of Random Walkers to Brownian motion to obtain that for all $f \in H^{\text{sbm}}$, and all $f_N \in H_n^{\text{sip}}$ converging weakly to f, we have

$$\begin{aligned} \liminf_{n\to\infty} \mathscr{E}_n(f_n) &\geq \liminf_{n\to\infty} \mathscr{R}_n(f_n) \geq \frac{\chi}{2}\int_{\mathbb{R}} f'(x)^2 \, dx \\ &= \frac{\chi}{2}\int_{\mathbb{R}} \mathbb{1}_{\{x\neq 0\}}(x) f'(x)^2 \, dx = \mathscr{E}_{\text{sbm}}(f) \end{aligned}$$

where the last equality comes from equation (B.16) and Remark B.3.3 in the Appendix.

6.3.2.2 Mosco II

We are going to prove that Assumption 2 (in particular (B.39)) is satisfied. We use the set of compactly supported smooth functions $C_k^\infty(\mathbb{R})$, which by the regularity of the measure $dx + \delta_0$ is dense in $H = L^2(dx + \delta_0)$.

The recovering sequence

For every $f \in C_k^\infty(\mathbb{R})$, we need to find a sequence f_N strongly-converging to f and such that

$$\lim_{n\to\infty} \mathscr{E}_n(f_n) = \mathscr{E}(f). \tag{6.79}$$

The obvious choice $f_n = \Phi_n f$ does not work in this case, the reason of this is the emergence in the limit of a non-vanishing term containing $f'(0)$. Nevertheless our candidate is the sequence $\{\Psi_n f\}_{n\geq 1}$ given by

$$(\Psi_n f)(i) = \begin{cases} f(i) & i \in \frac{1}{n}\mathbb{Z} \setminus A_n \\ f(0) & \text{otherwise} \end{cases} \qquad \text{for any } f \in C_k^\infty(\mathbb{R}), \tag{6.80}$$

where A_n is as in (6.30).

Remark 6.3.4. *The sequence $\{\Psi_n f\}_{n\geq 1}$ is chosen in such a way that the SIP part of the Dirichlet form, i.e. the right hand side of* (6.61)*, vanishes at $\Psi_n f$ for all n. See below for the details.*

Our goal is to show that the sequence $\{\Psi_n f\}_{n\geq 1}$ indeed satisfies (6.79). First of all we need to show that $\Psi_n f \to f$ strongly.

Proposition 6.3.11. *For all $f \in C_k^\infty(\mathbb{R}) \subset L^2(dx+\delta_0)$, the sequence $\{\Psi_n f\}_{n\geq 1}$ in H_n^{sip} strongly-converges to f w.r.t. the H_n^{sip}-Hilbert space convergence given.*

Proof. In the language of Definition B.4.2 we set $\tilde{f}_m \equiv f$. Hence the first condition is trivially satisfied:

$$\lim_{m\to\infty} \|\tilde{f}_m - f\|_{H^{\mathrm{sbm}}} = 0. \tag{6.81}$$

Moreover

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|\Phi_n \tilde{f}_m - \Psi_n f\|^2_{H_n^{\mathrm{sip}}} = \limsup_{n\to\infty} \|\Phi_n f - \Psi_n f\|^2_{H_n^{\mathrm{sip}}}$$
$$= \limsup_{n\to\infty} \sum_{i\in\frac{1}{n}\mathbb{Z}} (\Phi_n f(i) - \Psi_n f(i))^2 \nu_{\gamma,n}(i) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i\in A_n} (f(i) - f(0))^2 = 0$$

where we used the boundedness of f and the fact that the cardinality of the set A_n is finite and does not depend on n. □

Preliminary simplifications

To continue the proof of (6.79), the first thing to notice is that the Dirichlet form $\mathscr{E}_n$ evaluated in $\Psi_n f$ can be substantially simplified:

$$\begin{aligned}
&\mathscr{E}_n(\Psi_n f)\\
= \;& -\sum_{i\in\frac{1}{n}\mathbb{Z}} \Psi_n f(i) \sum_{r\in A_n} 2p_n(r)\left(\tfrac{n^2}{2} + \tfrac{n^3\gamma}{\sqrt{2}}\mathbb{1}_{r=-i}\right)(\Psi_n f(i+r) - \Psi_n f(i))\nu_{\gamma,n}(i)\\
= \;& -\sum_{i\in\frac{1}{n}\mathbb{Z}} \Psi_n f(i) \sum_{r\in A_n} p_n(r) n^2 (\Psi_n f(i+r) - \Psi_n f(i))\nu_{\gamma,n}(i) \qquad (6.82)\\
& -\sum_{i\in\frac{1}{n}\mathbb{Z}} \Psi_n f(i) \sum_{r\in A_n} 2p_n(r)\left(\tfrac{n^3\gamma}{\sqrt{2}}\mathbb{1}_{r=-i}\right)(\Psi_n f(i+r) - \Psi_n f(i))\nu_{\gamma,n}(i)
\end{aligned}$$

where, from the observation that for $i = -r$ and $r \in A_n$, via (6.80) we get

$$(\Psi_n f(i+r) - \Psi_n f(i)) = 0, \tag{6.83}$$

and the whole second sum in (6.82) vanishes. Then by (6.31), we are left with

$$\begin{aligned}
\mathscr{E}_n(\Psi_n f) = \;& -n\sum_{r\in A_n} p_n(r) \sum_{i\in\frac{1}{n}\mathbb{Z}} \Psi_n f(i)(\Psi_n f(i+r) - \Psi_n f(i))\\
& -\sqrt{2}\gamma n^2 \sum_{r\in A_n} p_n(r)\Psi_n f(0)(\Psi_n f(r) - \Psi_n f(0)). \qquad (6.84)
\end{aligned}$$

We have again that $(\Psi_n f(r) - \Psi_n f(0)) = 0$ for $r \in A_n$, then our Dirichlet form becomes

$$\mathscr{E}_n(\Psi_n f) = -n\sum_{r\in A_n} p_n(r) \sum_{i\in\frac{1}{n}\mathbb{Z}} \Psi_n f(i)(\Psi_n f(i+r) - \Psi_n f(i))$$

that we split again as follows

$$\mathscr{E}_n(\Psi_n f) = -n \sum_{r\in A_n} p_n(r) \sum_{i\in\frac{1}{n}\mathbb{Z}\setminus A_n} \Psi_n f(i)(\Psi_n f(i+r) - \Psi_n f(i)) - S_n$$

$$\text{with} \qquad S_n = n \sum_{r\in A_n} p_n(r) \sum_{i\in A_n} \Psi_n f(i)(\Psi_n f(i+r) - \Psi_n f(i)). \tag{6.85}$$

The correct limit

First we show that S_n vanishes as $n \to \infty$. For $i \in \frac{1}{n}\mathbb{Z}$, we define the sets

$$A_n^i := A_n - i \qquad \text{and} \qquad A_n^+ = \{|r|\colon r \in A_n\}. \tag{6.86}$$

Notice that for $r \in A_n^i$ we have $(\Psi_n f(i+r) - \Psi_n f(0)) = 0$ and hence

$$\begin{aligned} S_n &= n \sum_{i\in A_n} \sum_{r\in A_n\setminus A_n^i} p_n(r) f(0)(f(i+r) - f(0)) \\ &= n \sum_{i\in A_n^+} \sum_{r\in A_n\setminus A_n^i} p_n(r) f(0)(f(i+r) - 2f(0) + f(-i-r)) \end{aligned}$$

where we used the symmetry of $p(\cdot)$ and the fact that $r \in A_n \setminus A_n^i$ if and only if $-r \in A_n \setminus A_n^{-i}$. We conclude that S_n vanishes by recalling that by a Taylor expansion the factor $(f(i+r) - 2f(0) + f(-i-r))$ is of order n^{-2}.

For what concerns the remaining term in (6.85), we notice that, exploiting the symmetry of the transition function $p(\cdot)$, we can re-arrange it into

$$\mathscr{E}_n(\Psi_n f) + S_n = -n \sum_{r\in A_n^+} p_n(r) \sum_{i\in\frac{1}{n}\mathbb{Z}\setminus A_n} \Psi_n f(i)(\Psi_n f(i+r) - 2\Psi_n f(i) + \Psi_n f(i-r)).$$

Let us define the following set $B_n = \frac{1}{n}\{-2R, -2R+1, \dots, 2R-1, 2R\}$ and split the sum above as follows

$$\begin{aligned} \mathscr{E}_n(\Psi_n f) + S_n = &-n \sum_{r\in A_n^+} p_n(r) \sum_{i\in\frac{1}{n}\mathbb{Z}\setminus B_n} \Psi_n f(i)(\Psi_n f(i+r) - 2\Psi_n f(i) + \Psi_n f(i-r)) \\ &-n \sum_{r\in A_n^+} p_n(r) \sum_{i\in B_n\setminus A_n} \Psi_n f(i)(\Psi_n f(i+r) - 2\Psi_n f(i) + \Psi_n f(i-r)). \end{aligned} \tag{6.87}$$

The above splitting allows to isolate the first term for which we have no issues of the kind $\Psi_n f(i+r) = f(0)$ and hence no complications when Taylor expanding around the points $i \in \frac{1}{n}\mathbb{Z}$.

We now show that the second term in the RHS of (6.87) vanishes as n goes to infinity:

Take a positive $i \in B_n \setminus A_n$, then for $r \in A_n^i$, $\Psi_n f(i+r) = f(0)$.

Remark 6.3.5. *Notice that, for $-i \in B_n \setminus A_n$, the set $A_n^i = -A_n^{-i}$ is such that*

$$\Psi_n f(-i-r) = f(0) \qquad \textit{for all} \quad r \in A_n^i. \tag{6.88}$$

Remark 6.3.6. *We will omit the analysis for $r \notin A_n^i$ because for those terms we can Taylor expand f around the point i and show that the factors containing the discrete Laplacian are of order n^{-2}.*

We now consider the contribution that each pair $(i, -i)$ gives to the second sum in the RHS of (6.87). Let $i \in (B_n \setminus A_n)^+$, then

$$\begin{aligned} C_n(i) \qquad &:= \frac{n}{2} \sum_{r \in A_n^i} p_n(r) \Psi_n f(i) \left[\Psi_n f(i+r) - 2\Psi_n f(i) + \Psi_n f(i-r) \right] \\ &= \frac{n}{2} \sum_{r \in A_n^i} p_n(r) f(i) \left[f(0) - 2f(i) + f(i-r) \right]. \end{aligned} \tag{6.89}$$

Taylor expanding around zero the terms inside the square brackets in the RHS of (6.89) gives

$$C_n(i) \qquad = \frac{n}{2} \sum_{r \in A_n^i} p_n(r) f(i) f'(0) \left[-r - i \right] + O(1/n).$$

Analogously, for the contribution $C_n(-i)$ we obtain

$$C_n(-i) \qquad = \frac{n}{2} \sum_{r \in A_n^i} p_n(r) f(-i) f'(0) \left[i + r \right] + O(1/n).$$

Summing both contributions over all $i > 0$ we obtain

$$\begin{aligned} &\sum_{i \in (B_n \setminus A_n)^+} C_n(i) + C_n(-i) \qquad \qquad (6.90) \\ &= \frac{n}{2} \sum_{i \in (B_n \setminus A_n)^+} \sum_{r \in A_n^i} p_n(r) f'(0) \left(r + i \right) \left[f(-i) - f(i) \right] + O(1/n) = O(1/n) \end{aligned}$$

where we used that the cardinality of the sets A_n^i and $(B_n \setminus A_n)^+$ does not depend on n. Then we can write

$$\begin{aligned} &\mathscr{E}_n(\Psi_n f) \\ &= -\frac{1}{n} \sum_{r \in A_n^+} p_n(r) \sum_{i \in \frac{1}{n}\mathbb{Z} \setminus B_n} n^2 f(i) (f(i+r) - 2f(i) + f(i-r)) + O(1/n), \end{aligned}$$

which indeed by a Taylor expansion gives the limit

$$\begin{aligned} \lim_{n \to \infty} \mathscr{E}_n(\Psi_n f) \quad &= \quad -\frac{\chi}{2} \int_{-\infty}^{0} f(x) f''(x) \; dx - \frac{\chi}{2} \int_{0}^{\infty} f(x) f''(x) \; dx \\ &= \quad \frac{\chi}{2} \int_{\mathbb{R}} \mathbb{1}_{\{x \neq 0\}}(x) f'(x)^2 \; dx, \end{aligned} \tag{6.91}$$

with $\chi = \sum_{r=1}^{R} p(r) r^2$.

This concludes the proof of Mosco II. $\square$

Remark 6.3.7. *Notice that in the second line of* (6.91) *we are using the fact that* $f \in C^\infty(\mathbb{R})$*, and hence* $f'(0^-) = f'(0^+)$*.*

Part III

Perspectives

Chapter 7

Perspectives

7.1 Higher-order fluctuation fields

Concerning chapter 5, we will sketch a direction of future research, to make sense of powers of these fields and relate them to powers of the generalized Ornstein-Uhlenbeck process. The ideas presented here are inspired by the works [4], [5], [46], and [47].

7.1.1 Properties of the quadratic fluctuation field at the diagonal

As an example of what can be done with the limiting fields $\{X_t^{(k)}(\cdot) : t \geq 0\}$, we will specialize to the case of two particles, i.e., to the case $k = 2$. In the context of two particles, the authors of [47] used a martingale characterization (an analogue of our Theorem 5.4.1) of their second-order limiting field $Q_t(\cdot)$ to show Theorem 7.1.1 and Theorem 7.1.2 below.

Let us then denote by $\{Q_t(\cdot) : t \geq 0\}$ the weak limit of the following quadratic fluctuation field introduced in [47]:

$$Q_t^{(n)}(\psi, \eta) = \frac{1}{n} \sum_{\substack{x,y \in \mathbb{T} \\ x \neq y}} \psi(x/n, y/n)\, D(\delta_x + \delta_y, \eta(n^2 t)) \tag{7.1}$$

defined for all $\psi \in S(\mathbb{T}^2)$.

We now introduce some notation needed to make sense of Theorem 7.1.1 and Theorem 7.1.2. For any test function $\varphi \in S(\mathbb{R})$ we denote by $\varphi \otimes \delta$ the distribution in $\mathbb{R}^2$ given by:

$$\langle \varphi \otimes \delta, h \rangle := \int_{\mathbb{R}} f(x) h(x, x)\, dx \tag{7.2}$$

for every $h : \mathbb{R}^2 \to \mathbb{R}$.

We now introduce the notion of approximation to the identity. Fix an arbitrary non-negative test function $i \in S(\mathbb{R})$ with support in $[-1, 1]$ and such that

$$\int_{\mathbb{R}} i(x)\, dx = 1 \tag{7.3}$$

We say that the function i_ϵ is an approximation to the identity if:

$$i_\epsilon(x) = \frac{1}{\epsilon} i(\tfrac{x}{\epsilon}) \tag{7.4}$$

and

$$g(x) \quad = \quad \lim_{\epsilon \to 0} \int_{\mathbb{R}} i_\epsilon(x - y)\, g(y)\, dy \tag{7.5}$$

for all $g : \mathbb{R} \to \mathbb{R}$.

Remark 7.1.1. *Notice that the convolution $(\varphi \otimes \delta) * i_\epsilon \in S(\mathbb{R}^2)$, and that both notions (7.5) and (7.2), can be extended to higher dimensions.*

We then have the following result:

Theorem 7.1.1. *Let i_ϵ denote a two-dimensional approximation of the identity. Let $\{A_t^\epsilon : t \in [0, T]\}$ be the distribution-valued process defined by:*

$$A_t^\epsilon(\varphi) = \int_0^t Q_s((\varphi \otimes \delta) * i_\epsilon)\, ds \tag{7.6}$$

for any test function $\varphi \in S(\mathbb{T})$.

Then $\{A_t^\epsilon : t \in [0, T]\}$ converges weakly in path-space, with respect to the uniform topology, as $\epsilon \to 0$, to a well-defined distribution-valued process $\{A_t : t \in [0, T]\}$.

Furthermore, in the same paper [47] the authors were able to obtain short-time properties of the limiting object $\{A_t : t \in [0, T]\}$. More precisely, they showed the following:

Theorem 7.1.2. *As $\epsilon \to 0$, the process:*

$$\{\epsilon^{3/4} A_{\epsilon t} : t \geq 0\} \tag{7.7}$$

converges in distribution, with respect to the uniform topology, to a stationary Gaussian process $\{\mathscr{B}_t(\varphi); t \geq 0\}$ with covariance given by:

$$\mathbb{E}\left[\mathscr{B}_t(\varphi)\mathscr{B}_s(\varphi)\right] = \frac{4}{3\pi}\left(-1 + \sqrt{2}\right)\{t^{3/2} + s^{3/2} - |t - s|^{3/2}\} \int_{\mathbb{T}} \varphi(x)(-\Delta\varphi(x))\, dx \tag{7.8}$$

It is important to mention that one part of the results of [47] is the following formal relation:

$$Q_t(x,y) = \mathscr{X}_t(x)\cdot\mathscr{X}_t(y) \tag{7.9}$$

where $\{\mathscr{X}_t(\cdot) : t \geq 0\}$ is a distribution-valued process being the weak-limit of the ordinary first-order density fluctuation field. I.e., a generalized Ornstein-Uhlenbeck process.

The way that Theorem 7.1.1 relates to the study of products of distributions is that it gives rigorous meaning to the expression (7.9) evaluated at the diagonal. In other words, the limiting object $\{A_t : t \geq 0\}$ from Theorem 7.1.1 can be intuitively understood as the time-integrated second power of the generalized Ornstein-Uhlenbeck process $\{\mathscr{X}_t(\cdot) : t \geq 0\}$.

Conjecture

Modulo a substitution of the space $\mathbb{T}$ by $\mathbb{R}$, we believe that Theorem 7.1.1 can be extended to the case of k particles. This belief is based on the following result:

Proposition 7.1.1. *For all $T > 0$ we have*

$$\lim_{n\to\infty} \mathbb{E}_{\nu_\rho}\left[\left(\int_0^T \mathscr{X}_s^{(n,2)}(\varphi^{(2)},\eta) - \pi(\delta_0)^2\, Q_s^{(n,2)}(\varphi^{(2)},\eta)\, ds\right)^2\right] = 0 \tag{7.10}$$

for every $\varphi \in S(\mathbb{R}^d)$, where the field $\mathscr{X}_s^{(n,2)}(\cdot,\eta)$ is given in (5.43), and $Q_s^{(n,2)}(\cdot,\eta)$ is defined as:

$$Q_t^{(n,2)}(\psi,\eta) = \frac{1}{n}\sum_{\substack{x,y\in\mathbb{Z}\\ x\neq y}} \psi(x/n, y/n)\, D(\delta_x+\delta_y, \eta(n^2 t)) \tag{7.11}$$

for any $\psi \in S(\mathbb{R}^2)$.

Remark 7.1.2. *Notice that the field $Q_s^{(n,2)}(\cdot,\eta)$ is the natural analogue on $\mathbb{R}^2$ of the field $Q_t^{(n)}(\psi,\eta)$ originally defined on $\mathbb{T}^2$*

Proof. This proposition is a consequence of the second-order Boltzmann-Gibbs principles of Chapter 4 and the following relation:

$$\mathscr{X}_t^{(n,2)}(\varphi^{(2)},\eta) = \pi(\delta_0)^2\, Q_t^{(n,2)}(\varphi^{(2)},\eta) + \frac{1}{n}\sum_{x\in\mathbb{Z}} \varphi(\tfrac{x}{n})^2 D(2\delta_x, \eta_{n^2 t})\pi(2x) \tag{7.12}$$

□

We conclude this section, with the following conjecture which concerns our limiting k-th-order fields of Chapter 5 and what we have described in the previous paragraphs:

Conjecture 7.1.1. *Let i_ϵ denote a k-dimensional approximation of the identity. Let $\{A_t^{(k,\epsilon)} : t \in [0,T]\}$ be the distribution-valued process defined as:*

$$A_t^{(k,\epsilon)}(\varphi) = \int_0^t \mathscr{X}_s^{(k)}((\varphi \otimes \delta) * i_\epsilon)\, ds \tag{7.13}$$

for any test function $\varphi \in S(\mathbb{R})$, where the field $\mathscr{X}_s^{(k)}(\cdot)$ is the weak limit of the field $\mathscr{X}_s^{(n,k)}(\cdot,\eta)$ given in (5.43), and where $(\varphi \otimes \delta) * i_\epsilon$ is given by the k-th dimensional version of (7.5) and (7.2) .

Then $\{A_t^{(k,\epsilon)} : t \in [0,T]\}$ converges in distribution with respect to the uniform topology, as $\epsilon \to 0$, to a well defined distribution-valued process $\{A_t^{(k)} : t \in [0,T]\}$.

Remark 7.1.3. *Analogous to the quadratic case, the field $\mathscr{X}_t^{(k)}(\cdot)$ can be formally understood as:*

$$\mathscr{X}_t^{(k)}(x_1, \ldots, x_k) = \prod_{i=1}^{k} \mathscr{X}_t(x_i). \tag{7.14}$$

Therefore, by plugging in appropriate test functions, Conjecture 7.1.1 opens the possibility to make sense of products, of distribution-valued processes, of the form:

$$\prod_{j=1}^{k} \mathscr{X}_t(\cdot)^{m_j} \tag{7.15}$$

where $m_j \in \mathbb{N}$ and

$$\sum_{j=1}^{k} m_j = k \tag{7.16}$$

7.2 Condensation of SIP particles and SBM

In this section, we talk about two possible future lines of research concerning Chapter 6. For the first direction, we mention in words what needs to be done and one possible way to do it, while for the second one, we already have enough work to immerse ourselves into technical details.

Fluctuations under the condensation regime

The main result of Chapter 6 finds an explicit form for the time-dependent limiting variances of the density fluctuation field in the condensation regime. It is then a natural line of future research to completely characterize the weak limit, if possible, of the fluctuation field in this regime. The first obstacle in this path is then to obtain a suitable characterization of the sticky version of the generalized Ornstein-Uhlenbeck process. This characterization can be obtained, for example, first formally using an SPDE of the same flavor as [63] and later more rigorously by transforming it into the language of martingale problems.

In principle, if successful in the first step, showing the weak convergence of the density fluctuation field should be possible with the help of self-duality and the approach described, for example, in [58].

Mosco convergence of k particles

The second possible direction of this work is the extension of our results to more particles. More precisely, it should be possible to show the convergence, in the sense of Mosco, of the position of k condensively rescaled SIP particles to a limiting process which, by our previous results should correspond to a consistent family of Brownian motions as introduced in [53].

Some things remain to be done to complete this line of research (besides what we include in the following sections). The most important is to show that, indeed, to the limiting Dirichlet form obtained at the end of this section, we can associate a consistent family of Brownian motions as described in [53]. More precisely, we should be able to show that the Dirichlet form is regular and that the associated Markov process has a generator solving the corresponding Martingale problem given in [53].

We now introduce the setting in which this convergence should occur and later sketch how to proceed for the particular case of two particles. Notice that in this case, the positions should converge to a pair of γ-coupled Brownian motions as also introduced in [53].

7.2.1 The k-particles process

Let then x_t^i be the position at time t of the particle with label i. We are interested in the process $\mathbf{x}_t = (x_t^1, x_t^2, \dots x_t^k)$ in $\mathbb{Z}^k$ whose generator is given by:

$$Lf(\mathbf{x}) = \sum_{i=1}^{k} \sum_{r} p(r) \left(\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{k} \mathbb{1}_{x^i + r = x^j} \right) \Big(f(\mathbf{x}^{i,i+r}) - f(\mathbf{x}) \Big) \tag{7.17}$$

where $\mathbf{x}^{i,i+r}$ results from changing the position of particle i from the site x^i to the site $x^i + r$.

Notice that, if we denote by $\{\eta_t\}_{t \geq 0}$ the configuration process, for $w \in \mathbb{Z}$ the way to recover from the positions process the configuration process is given by:

$$\eta_t(w) = \sum_{i=1}^{k} \mathbb{1}_{x_t^i = w} \tag{7.18}$$

This relation becomes very useful for us since, thanks to it, we are able to find reversible measures at the level of the coordinate process $\{\mathbf{x}_t\}_{t \geq 0}$.

Proposition 7.2.1. *The process $\{\mathbf{x}_t\}_{t \geq 0}$ is reversible with respect to the probability measure:*

$$\mu(\mathbf{x}) = \prod_{j \in \mathbb{Z}} \frac{\Gamma(\alpha + \sum_{i=1}^{k} \mathbb{1}_{x_i = j})}{\Gamma(\alpha)} \tag{7.19}$$

Proof. By detailed balance it is enough to verify that μ satisfies the relation:

$$\mu(\mathbf{x}) \left(\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{k} \mathbb{1}_{x^i + r = x^j} \right) = \mu(\mathbf{x}^{i,i+r}) \left(\alpha + \sum_{\substack{j=1 \\ j \neq i}}^{k} \mathbb{1}_{x^i = x^j} \right) \tag{7.20}$$

Please notice that for all $w \notin \{i, i+r\}$ we have

$$\eta^{i,i+r}(w) = \eta(w)$$

and recall the basic property of the Gamma function

$$\Gamma(z+1) = z\Gamma(z)$$

with this in mind, we are ready to proceed:

$$
\begin{aligned}
\mu(\mathbf{x}^{i,i+r}) \quad &= \prod_{j\in\mathbb{Z}} \frac{\Gamma(\alpha+\eta^{i,i+r}(j))}{\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha+\eta^{i,i+r}(x^i))}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\eta^{i,i+r}(x^i+r))}{\Gamma(\alpha)} \\
&\times \prod_{\substack{j\in\mathbb{Z} \\ j\neq x^i, x^i+r}} \frac{\Gamma(\alpha+\eta^{i,i+r}(j))}{\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha+\eta(x^i)-1)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\eta(x^i+r)+1)}{\Gamma(\alpha)} \\
&\times \prod_{\substack{j\in\mathbb{Z} \\ j\neq x^i, x^i+r}} \frac{\Gamma(\alpha+\eta(j))}{\Gamma(\alpha)} \\
&= \frac{\big(\alpha+\eta(x^i+r)\big)}{\big(\alpha+\eta(x^i)-1\big)} \frac{\Gamma(\alpha+\eta(x^i))}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\eta(x^i+r))}{\Gamma(\alpha)} \\
&\times \prod_{\substack{j\in\mathbb{Z} \\ j\neq x^i, x^i+r}} \frac{\Gamma(\alpha+\eta(j))}{\Gamma(\alpha)} \\
&= \frac{\big(\alpha+\eta(x^i+r)\big)}{\big(\alpha+\eta(x^i)-1\big)} \prod_{j\in\mathbb{Z}} \frac{\Gamma(\alpha+\eta(j))}{\Gamma(\alpha)} \\
&= \frac{\left(\alpha+\sum\limits_{\substack{j=1 \\ j\neq i}}^{k} \mathbb{1}_{x^i+r=x^j}\right)}{\left(\alpha+\sum\limits_{\substack{j=1 \\ j\neq i}}^{k} \mathbb{1}_{x^i=x^j}\right)} \mu(\mathbf{x})
\end{aligned}
$$

which concludes the proof. $\square$

From the observation that (7.18) implicitly appears in the measure μ, we have the following:

Proposition 7.2.2. *The measures μ are configuration invariant, i.e.,*

$$\mu(\mathbf{x}) = \mu(\mathbf{y}) \tag{7.21}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^m$ that generate the same configuration η.

Proof. The proof is straightforward and it comes from the following:

$$\mu(\mathbf{x}) = \prod_{j\in\mathbb{Z}} \frac{\Gamma(\alpha+\eta_j)}{\Gamma(\alpha)} = \mu(\mathbf{y}) \tag{7.22}$$

□

Remark 7.2.1. *Notice that, from the fact that if one vector is a permutation of the other one they generate the same configuration, we can conclude that the measures are permutation invariant as well.*

7.2.2 Dirichlet form for the $k = 2$ SIP in coordinate notation

In this section we introduce the Dirichlet form associated to the SIP with 2 particles in coordinate notation. We make use of relation (B.18) to find the Dirichlet form associated to the C-SIP:

$$\mathscr{E}_n^2(f,g) = -n^3\gamma \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) f(\mathbf{x}) \sum_{i=1}^{2} \sum_{r\in A_n} p_n(r) \left(\frac{1}{n\gamma} + \sum_{\substack{j=1\\ j\neq i}}^{2} \mathbb{1}_{x^i+r=x^j} \right) \nabla^{i,i+r} g(\mathbf{x}) \tag{7.23}$$

where

$$\nabla^{i,i+r} g(\mathbf{x}) = g(\mathbf{x}^{i,i+r}) - g(\mathbf{x}) \tag{7.24}$$

and

$$p_n(r) := p(nr) \quad , \quad A_n := \tfrac{1}{n}\{-R, -R+1, \dots, R-1, R\}/\{0\} \tag{7.25}$$

From now and on we are interested in the functional:

$$\mathscr{E}_n^2(f) = \mathscr{E}_n^2(f,f) \tag{7.26}$$

Notice that, due to reversibility of the measure, the Dirichlet form in (7.26) can be written as:

$$\mathscr{E}_n^2(f) = n^3\gamma \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) \sum_{i=1}^{2} \sum_{r\in A_n} p(r) \left(\frac{1}{n\gamma} + \sum_{\substack{j=1\\ j\neq i}}^{2} \mathbb{1}_{x^i+r=x^j} \right) \left(\nabla^{i,i+r} f(\mathbf{x})\right)^2 \tag{7.27}$$

We are now ready to state the main expected result for the case of two particles.

7.3 Mosco convergence of Dirichlet forms

We claim the following theorem that provides us with the limiting Dirichlet form to the 2-particle process. It is important to mention that at this point, we do not know whether this Dirichlet form is regular or not, i.e., if there exists an associated Markov process to it. Nevertheless, we expect that this form corresponds to a pair of γ-coupled Brownian motions.

Theorem 7.3.1. *The sequence of Dirichlet forms* $\{\mathscr{E}_n^2, D(\mathscr{E}_n^2)\}_{n\geq 1}$ *given by* (7.27) *converges in the Mosco sense to the form* $\{\mathscr{E}_{B^{1,2}}, D(\mathscr{E}_{B^{1,2}})\}$ *given by*

$$\mathscr{E}_{B^{1,2}}(f,f) = \frac{\chi}{2}\int_{\mathbb{R}^2} \mathbb{1}_{x\neq y}\,\nabla f(x,y)\cdot\nabla f(x,y)\,dx\,dy, \qquad \chi := \sum_{r=1}^{R} r^2 p(r) \quad (7.28)$$

whose domain is

$$D(\mathscr{E}_{B^{1,2}}) = D(\mathscr{E}_B)\cap L^2(\mathbb{R}^2, d\mathbf{x}+\gamma\delta^{1,2}) \quad (7.29)$$

where $\delta^{1,2}$ *is the lower-dimensional Lebesgue measure concentrated on the hyperplane* $x=y$.

Similar computations can be done for the m-particle case, and a corresponding statement is expected. Once again, this new statement should come with a warning concerning the Dirichlet form's regularity and the existence of the associated Markov process. We stress that for the m-particle case, the candidate process is a consistent family of Brownian motions.

We will sketch the proof of this theorem in three sections; the first one is dedicated to developing all the ingredients corresponding to the convergence of Hilbert spaces, and each of the last two will deal with conditions Mosco I and II of the definition of Mosco convergence.

7.3.1 Convergence of Hilbert Spaces

In this section, we will prove the convergence of the relevant Hilbert spaces related to Theorem 6.3.1. We will also present some results connecting the notions of weak and strong convergence related to inclusion dynamics and the underlying random walkers. These last results will prove themselves to be useful to show condition (B.34) in Section 7.3.2.

Let us then start by defining the relevant Hilbert spaces. First, for the case of SIP particles we have the approximating sequence of Hilbert spaces given by:

$$H_n^{\text{sip}} = L^2(\tfrac{1}{n}\mathbb{Z}^2, \mu_n) \quad (7.30)$$

where the reference measure is given by:

$$\mu_n(\mathbf{x}) = \gamma^2 \prod_{j\in\frac{1}{n}\mathbb{Z}} \frac{\Gamma(\frac{1}{n\gamma} + 1\!\!1_{x_1=j} + 1\!\!1_{x_2=j})}{\Gamma(\frac{1}{n\gamma})} \tag{7.31}$$

Remark 7.3.1. *The factor γ^2 in front of (7.31) is there to make the underlying random walk free of any γ scaling. Notice that this factor does not affect the reversibility of the measures.*

Remark 7.3.2. *Notice that these reversible measures contain all the relevant information with respect to the Hilbert convergence. Indeed, by studying how these measures behave at the diagonal we can predict the limiting Hilbert space.*

Consider then the set $T_n^{1,2}$ given by

$$T_n^{1,2} = \{\mathbf{x} \in \tfrac{1}{n}\mathbb{Z}^2 : \mathbf{x}^1 = \mathbf{x}^2\} \tag{7.32}$$

As we already mentioned in Proposition 7.2.2, the measures μ_n are invariant on these sets, more precisely:

$$\mu_n(\mathbf{x}) = \gamma(\tfrac{1}{n}) + (\tfrac{1}{n})^2 \tag{7.33}$$

for all $\mathbf{x} \in T^{1,2}$.

Even more, from the fact that $T_n^{1,2}$ are $(2-1)$-dimensional subspaces of $\frac{1}{n}\mathbb{Z}^2$ we have the following:

$$\lim_{n\to\infty} \sum_{\mathbf{x}\in T_n^{1,2}} f(\mathbf{x})\mu_n(\mathbf{x}) = \gamma \int_{R^2} f(\mathbf{x})\delta^{1,2} d\mathbf{x} \tag{7.34}$$

Based on this observation we have the following Proposition:

Proposition 7.3.1. *The sequence of spaces $H_n^{sip} = L^2(\frac{1}{n}\mathbb{Z}^2, \mu_n)$, $n \in \mathbb{N}$, converges, in the sense of convergence of Hilbert spaces, to the limiting space $H^{sbm} = L^2(\mathbb{R}^2, d\mathbf{x} + \gamma\delta^{1,2})$.*

Proof. We first define the linear operator $\Phi_n : C_c^\infty(R^2) \to H_n$ that will witness the convergence:

$$\Phi_n f = f \mid_{\frac{1}{n}\mathbb{Z}^2} \tag{7.35}$$

We then take $f \in C_c^\infty(R^2)$ and by the previous observation we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|\Phi_n f\|_{H_n}^2 &= \lim_{n\to\infty} \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} f(\mathbf{x})^2 \mu_n(\mathbf{x}) \\
&= \lim_{n\to\infty} \sum_{\mathbf{x}\notin T} f(\mathbf{x})^2 \mu_n(T) + \sum_{\mathbf{x}\in T} f(\mathbf{x})^2 \mu_n(T) \\
&= \|f\|_H^2
\end{aligned} \tag{7.36}$$

which finishes the proof. □

We now proceed to introduce the Hilbert spaces related to the underlying random walk dynamics. As we may expect, the corresponding Hilbert spaces are:

$$H_n^{\mathrm{rw}} = L^2(\tfrac{1}{n}\mathbb{Z}^2, \nu_n), \qquad H^{\mathrm{bm}} = L^2(\mathbb{R}^2, dx) \tag{7.37}$$

where ν_n is the counting measure properly rescaled to guarantee convergence in the sense of Hilbert spaces with the same operator Φ_n given by (7.35).

Remark 7.3.3. *Notice that, similar to the case of the difference process, the notions of weak and strong convergence corresponding to each case of convergence of Hilbert spaces do not have to correspond to each other. For example, if a sequence of functions converges strongly (resp. weakly) with respect to SIP Hilbert convergence, it does not necessarily converge strongly (resp. weakly) w.r.t RW Hilbert convergence.*

Despite of Remark 7.3.3, Proposition 6.3.9, and Proposition 6.3.10, have something to say in this regard:

Proposition 7.3.2. *Let $\{g_n\}_{n\geq 0}$ in $\{H_n^{rw}\}_{n\geq 0}$ be a sequence converging strongly to $g \in H^{sbm} \cap H^{bm}$ with respect to H_n^{rw}-Hilbert convergence. Then it also converges strongly with respect to H_n^{sip}-Hilbert convergence.*

The following proposition says that, with respect to weak convergence, the implication comes in the opposite direction:

Proposition 7.3.3. *Let $\{f_n\}_{n\geq 0}$ in $\{H_n^{sip}\}_{n\geq 0}$ be a sequence converging weakly to $f \in H^{sbm} \cap H^{bm}$ with respect to H_n^{sip}-Hilbert convergence. Then it also converges weakly with respect to H_n^{rw}-Hilbert convergence.*

We skipped the proofs of Proposition 7.3.2 and Proposition 7.3.3 since they are essentially the same as their respective analogues for the case of the difference process.

7.3.2 Mosco I

Proposition 7.3.3 is the key ingredient on our proof of condition Mosco I (i.e., B.34). We then introduce the form $\mathscr{R}_n^2$ corresponding to the underlying random walk dynamics:

$$\mathscr{R}_n^2(f) = n^2 \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \left(\frac{1}{n}\right)^2 \sum_{i=1}^{2} \sum_{r\in A_n} p_n(r) \left(\nabla^{i,i+r} f(\mathbf{x})\right)^2 \tag{7.38}$$

Remark 7.3.4. *Notice that even though the coordinate SIP process is condensively rescaled, the underlying random walkers turn out to be diffusively scaled.*

It is well known that this rescaled process converges weakly to the corresponding Brownian diffusion in $\mathbb{R}^2$, in particular by Theorem B.4.1 it also converges in the Mosco sense related to H_n^{rw}-Hilbert convergence.

In order to exploit the convergence of the randon walkers we have the following proposition that relates the two Dirichlet forms:

PROPOSITION 7.3.4. *For any $f_n \in H_n$ we have*

$$\mathscr{E}_n^2(f_n) \geq \mathscr{R}_n^2(f_n) \tag{7.39}$$

PROOF. It is a consequence of splitting the form 7.27 according to the two non-negative terms in the transition rates, and by appliying the following inequality:

$$\mu_n(\mathbf{x}) \geq \left(\frac{1}{n}\right)^2 \tag{7.40}$$

□

At this point it is enough to realize that the limiting Dirichlet pairs $\{\mathscr{E}(f), D(\mathscr{E})\}$ and $\{\mathscr{E}_{\text{bm}}(f), D(\mathscr{E}_{\text{bm}})\}$ satisfy the following relation:

$$\mathscr{E}(f) = \mathscr{E}_{\text{bm}}(f) \qquad \forall f \in D(\mathscr{E}) \subset D(\mathscr{E}_{\text{bm}}) \tag{7.41}$$

or, in the language of Dirichlet forms, $\{\mathscr{E}(f), D(\mathscr{E})\}$ is a subspace of $\{\mathscr{E}_{\text{bm}}(f), D(\mathscr{E}_{\text{bm}})\}$.

7.3.3 Mosco II

In this section we prove that the second condition of Mosco (i.e., B.35) convergence is satisfied. For $f \in D(\mathscr{E}_{B^s})$ we consider the strongly convergent sequence $\Phi_n f \in H_n$ and compute:

$$\begin{aligned}
&\lim_{n\to\infty} \mathscr{E}_n^2(f_n) \\
&= \lim_{n\to\infty} \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) \sum_{i=1}^{2} \sum_{r\in A_n} p_n(r) n^2 \left(\nabla^{i,i+r} f(\mathbf{x})\right)^2 \\
&+ \lim_{n\to\infty} n^3\gamma \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) \sum_{i=1}^{2} \sum_{r\in A_n} p_n(r) \sum_{\substack{j=1\\ j\neq i}}^{2} \mathbb{1}_{x^i+r=x^j} \left(\nabla^{i,i+r} f(\mathbf{x})\right)^2
\end{aligned} \tag{7.42}$$

Notice that Taylor expanding around $\mathbf{x}$ we obtain the following estimate

$$n^2 \left(\nabla^{i,i+r} f(\mathbf{x})\right)^2 = (nr)^2 \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2 + O(n^{-3}) \tag{7.43}$$

Recall that χ is given by:

$$\chi = \sum_{r \in A_n} p_n(r)(nr)^2 \tag{7.44}$$

hence we obtain:

$$\begin{aligned}
&\lim_{n\to\infty} \mathscr{E}_n^2(f_n) \\
&= \lim_{n\to\infty} \chi \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) \sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2 \\
&+ \lim_{n\to\infty} \gamma n \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n \mathbf{x}) \sum_{i=1}^{2} \sum_{\substack{j=1\\ j\neq i}}^{2} \sum_{r\in A_n} p_n(r)(nr)^2 \mathbb{1}_{x^i+r=x^j} \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2
\end{aligned} \tag{7.45}$$

where we were allowed to ignore the terms of order $O(n^{-3})$ thanks to the inequality (7.40).

Notice that we also have that:

$$\begin{aligned}
&\lim_{n\to\infty} \gamma n \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) \sum_{i=1}^{2} \sum_{\substack{j=1\\ j\neq i}}^{2} \sum_{r\in A_n} p_n(r)(nr)^2 \mathbb{1}_{x^i+r=x^j} \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2 \\
&= \lim_{n\to\infty} \gamma \frac{1}{n} \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus T^{1,2}} \sum_{i=1}^{2} \sum_{\substack{j=1\\ j\neq i}}^{2} \sum_{r\in A} p(r)(r)^2 \mathbb{1}_{x^i+\frac{r}{n}=x^j} \left(\frac{\partial}{\partial x_i} f(\tfrac{\mathbf{x}}{n})\right)^2 \\
&= 0
\end{aligned} \tag{7.46}$$

which allows us to forget about the second term in (7.45). Hence we have

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{E}_n^2(f_n) &= \lim_{n\to\infty} \chi \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \mu_n(\mathbf{x}) \sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2 \qquad (7.47)\\
&= \lim_{n\to\infty} \chi \frac{1}{n^2} \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2} \sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2 \\
&+ \lim_{n\to\infty} \chi \frac{\gamma}{n} \sum_{\mathbf{x}\in\frac{1}{n}\mathbb{Z}^2\cap T^{1,2}} \sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i} f(\mathbf{x})\right)^2
\end{aligned}$$

and with the boundary condition:

$$\int_{R^2} \left(\frac{\partial}{\partial x_i} f(\mathbf{x}) \right)^2 \delta^{i,j}(d\mathbf{x}) = 0 \tag{7.48}$$

we are able to conclude that indeed we have the correct limit.

Part IV

Appendix

Appendix A

Essentials of Markov Processes

In this section, we recall some of the basic concepts for the study of continuous-time Markov processes. If it is true that it is impossible to cover all the fundamentals in only one section of our work, we have decided to include the basic concepts and results that play an essential role in the context of this thesis.

A.1 Markov Process

A stochastic process $\{X_t\}_{t\geq 0}$ is said to have the Markov property if the conditional probability distribution of future states of the process, given its past, depends only on its present value. In a more formal setting, we have the following definition:

Definition A.1.1. *Let $\mathscr{F}_t = \sigma(X_r : r \leq t)$ be the σ-algebra generated by the random variables X_r with $r \leq t$. Then the process $\{X_t\}_{t\geq 0}$ is said to have the Markov property if for $0 \leq s \leq t$ and all bounded measurable $f : \Omega \to \mathbb{R}$, $\mathbb{P}$-a.s., it holds*

$$E[f(X_t) \mid \mathscr{F}_s] = E[f(X_t) \mid X_s] \tag{A.1}$$

The processes that have this property are also called Markov Processes. The reader that is not familiar with measure theory can think of a σ-algebra $\mathscr{F}_t$ as the set containing all the information of our process up to time t. Therefore, this more formal definition corresponds to the intuitive idea that the future state of the Markov process does not depend on the whole past $\mathscr{F}_s$, but only in its current state X_s. I.e., given the current state, the distribution of future states does not depend on how this current state was reached.

A.1.1 Markov Semigroup

In this section for simplicity we restrict ourselves to Ω being a compact metric space with measurable structure given by the σ-algebra of Borel sets. We also let $C(\Omega, \mathbb{R})$ denote the set of continuous functions equipped with the sup norm:

$$\|f\|_\infty = \sup_{x \in \Omega} \big| f(x) \big| \tag{A.2}$$

For a Markov process we can define a family of linear operators $S_t : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})$, that act on continuous functions $f : \Omega \to \mathbb{R}$, in the following way:

$$S_t f(x) := E[f(X_t) \mid X_0 = x] =: E_x f(X_t). \tag{A.3}$$

In the foreground of explaining what a semigroup is, the following proposition provides us with some of its properties:

Proposition A.1.1. *The semigroup S_t satisfies the following properties:*

a) $S_0 = I$ *, i.e. $S_0 f(x) = f(x)$ for all x*

b) *The map $t \to S_t$ is right continuous.*

c) *For all $t, s > 0$ it holds*

$$S_{t+s} f = S_t(S_s f) = S_s(S_t f) \tag{A.4}$$

d) *Positivity:*

$$S_t f \geq 0$$

for all $f \geq 0$.

e) *Normalization:*

$$S_t 1 = 1$$

f) *Contraction:*

$$\|S_t f\|_\infty \leq \|f\|_\infty$$

where $\|f\|_\infty$ is given by (A.2).

Remark A.1.1. *In general, the notion of semigroup is given in terms of bounded measurable functions endowed with the topology of point-wise bounded convergence. However, it is quite difficult to deal with the poor properties of this space of functions. Hence, we have restricted ourselves to the set of continuous functions $C(\Omega, \mathbb{R})$ and consider only Feller processes. We call a Markov process with metric space Ω, a Feller if for all $f \in C(\Omega, \mathbb{R})$ the time evolved process $S_t f(x)$ is continuous as a function of x.*

We can interpret the semigroups S_t as a family of operators that determines the expected value at time t for any continuous function f on Ω. Because providing a specification of all these expected values fully describes the process $\{X_t\}_{t\geq 0}$. It is common practice that depending on our goal in mind; we can refer to a Markov process in terms of the random variables X_t or its semigroup S_t. Even more, one of the most interesting and useful properties of the semigroups relies on the fact that each semigroup corresponds to a unique! Markov process. Therefore, instead of proving properties about a Markov process, it is sufficient to show that its semigroup holds these properties. An example of such a property is existence. For instance, given a transition semigroup, we can define a transition probability function using indicator functions, and from the transition probability function, we can construct a Markov process using the Kolmogorov extension theorem.

A.1.2 Generators

The generator of a Markov process is an operator L that acts on functions f of the state space. In simple english, if we look at a function of a Markov process and we let the process evolve in an instant of time, the generator will tell us how that function has changed in expectation. Let us now introduce the domain in which this operator will be defined.

$$D(L) = \left\{ f \in C(\Omega) : \lim_{t\to 0} \frac{S_t f - f}{t} \text{ exists} \right\} \tag{A.5}$$

We can now formally define the generator:

Definition A.1.2. *Let S_t be a Markov semigroup and $f \in D(L)$, then its infinitesimal generator is defined by the relation*

$$Lf := \lim_{t\to 0} \frac{S_t f - f}{t} \tag{A.6}$$

In the case of Ω being finite, an immediate consequence of it is that the semigroup S_t and the generator L can be thought of as matrices. In this case property b) in propostition A.1.1 implies the existence of a matrix K such that:

$$S_t = e^{tK} = \sum_{n=0}^{\infty} \frac{t^n}{n!} K^n \tag{A.7}$$

The generator L is then nothing but this matrix K. This simple relation gives us a hint about the existence of a sort of correspondence between Markov generators and semigroups, namely $S_t = e^{tL}$, with the problem that in general e^{tL} cannot be defined in the sense of the Taylor series, and hence more sophisticated versions of the exponential are needed. This is precisely what the Hille-Yosida theorem is about. In the following section we present the Hille-Yosida theorem which generalises this idea to the case of an infinite state space.

A.1.3 Hille-Yosida

Having in mind the need to extend property A.7 to the general case, the Hille-Yosida theorem comes in hand. This theorem states that there is a one to one correspondence between Markov generators and Markov semigroups.

Theorem A.1.1. *There is a one-to-one correspondence between Markov generators and Markov semigroups. This correspondence is given by:*

a) *The domain of the generator is given by*

$$D(L) = \{f \in C(\Omega) : \lim_{t\to 0} \tfrac{S_t f - f}{t} \quad \text{exists}\}$$

and for $f \in D(L)$ we have

$$Lf = \lim_{t\to 0} \frac{S_t f - f}{t}$$

b) *For $f \in C(\Omega)$ and $t \geq 0$*

$$S_t f = \lim_{n\to\infty} (I - \frac{t}{n} L)^{-n} f$$

c) *If $f \in D(L)$, it follows that $S_t f \in D(L)$, and for all $t > 0$,*

$$\frac{d}{dt} S_t f = L S_t f = S_t L f$$

d) *for g continuos and $\lambda \geq 0$, the solution to $f - \lambda L f = g$ is given by:*

$$f = \int_o^\infty e^{-t} S_{\lambda t} g dt$$

Proof. Can be found in [70]. □

In a sort of extension of what we mentioned in section A.1.1, we have an additional object that represents a Markov process. The reason is because of Theorem A.1.1 we have a one to one correspondence between generators and semigroups, which are also related one to one to Markov processes. This relation is particularly useful since sometimes we want to construct a process with a given dynamics, and the generator is the easiest way to describe it in those terms. Expression A.7 and parts b) and c) of theorem A.1.1 coin the correct concept of exponential, relating the generator and semigroup.

A.1.4 Examples

We now present some examples, in the context of Brownian motion, of some well known Markov processes and their infinitesimal generators. We recommend the reader to pay attention on how the behaviour of the underlying Markov process strongly depends not only on the operator itself but also on its domain. We illustrate this with some examples and refer the reader to [10] for more details on the following processes.

One dimensional Brownian motion

The infinitesimal generator is given by

$$Lf(x) = \frac{1}{2}\frac{d^2}{dx^2}f(x) \tag{A.8}$$

with domain

$$D(L) = \{f \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) : f', f'' \in C_0(\mathbb{R})\} \tag{A.9}$$

where $C_0(\mathbb{R})$, and $C^2(\mathbb{R})$, denote the set of vanishing at infinity, and twice differentiable, continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Brownian motion on $[0, \infty)$ absorbed at zero

The infinitesimal generator is given by

$$Lf(x) = \begin{cases} \frac{1}{2}f''(x) & x > 0 \\ \frac{1}{2}f''(0^+) & x = 0 \end{cases} \tag{A.10}$$

with domain

$$D(L) = \{f \in C_0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) : f', f'' \in C_0(\mathbb{R}_+), \quad f(0) = 0, \quad f''(0^+) = 0\} \tag{A.11}$$

Brownian motion on $[0, \infty)$ reflected at zero

The infinitesimal generator is given by

$$Lf(x) = \begin{cases} \frac{1}{2}f''(x) & x > 0 \\ \frac{1}{2}f''(0^+) & x = 0 \end{cases} \tag{A.12}$$

with domain

$$D(L) = \{f \in C_0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) : f', f'' \in C_0(\mathbb{R}_+), \quad f'(0^+) = 0\} \tag{A.13}$$

Brownian motion on $[a, b]$ reflected at a and b

The infinitesimal generator is given by

$$Lf(x) = \begin{cases} \frac{1}{2}f''(x) & a < x > b \\ \frac{1}{2}f''(a^+) & x = a \\ \frac{1}{2}f''(0^-) & x = b \end{cases} \tag{A.14}$$

with domain

$$D(L) = \{f \in C_0([a, b]) \cap C^2([a, b]) : f'' \in C_0([a, b]), \quad f'(a^+) = f'(b^-) = 0\} \tag{A.15}$$

Brownian motion on $\mathbb{R}$ sticky at zero

The infinitesimal generator is given by

$$Lf(x) = \begin{cases} \frac{1}{2}f''(x) & x \neq 0 \\ \frac{1}{2}f''(0^+) = \frac{1}{2}f''(0^-) & x = 0 \end{cases} \tag{A.16}$$

with domain given by:

$$D(L) = \{f \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) : f'' \in C_0(\mathbb{R}), \quad \gamma f''(0^+) = f'(0^+) - f'(0^-)\} \tag{A.17}$$

where $\gamma > 0$, is known as the stickiness parameter.

A.2 The Dynkin Martingale

This section contains two results that are useful in the derivation of hydrodynamic limits. The first result gives a way of characterising Markov processes by a martingale with respect to its natural filtration $\mathscr{F}_t$. Second, it turns out that the martingale has a simple form for its quadratic variation.

Theorem A.2.1. *Let X_t be a Markov process with generator L. For any $f \in D(L)$, then*

$$M_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds \tag{A.18}$$

is an $\mathscr{F}_t$-martingale.

Proof. $\mathscr{F}_t$-adaptivity comes from the fact that X_t is a Markov Process. Now for $0 \leq s \leq t$ we have

$$\begin{aligned} E[M_t - M_s \mid \mathscr{F}_s] &= E[f(X_t) - f(X_s) - \int_s^t Lf(X_u)du \mid \mathscr{F}_s] \\ &= E[f(X_t) - f(X_s) - \int_s^t Lf(X_u)du \mid X_s] \\ &= E[f(X_t) \mid X_s] - f(X_s) - \int_s^t E[Lf(X_u) \mid X_s]du \\ &= S_t f(X_s) - f(X_s) - \int_s^t S_u Lf(X_s)du \\ &= 0 \end{aligned}$$

where the last part comes from integrating from 0 to s part c) of Hille-Yoside. □

Now in the case that L is such that $f \in D(L)$ implies $f^2 \in D(L)$ we have an expression for the quadratic variation of the Martingale M_t.

Lemma A.2.1. *Let L be such that if $f \in D(L)$, then $f^2 \in D(L)$. Then the predictable quadratic variation of the Dynkin martingale previously defined is given by*

$$\langle M_t \rangle = \int_0^t [Lf^2(X_s) - 2f(X_s)Lf(X_s)]ds \tag{A.19}$$

Proof. The idea is to prove that $M_t^2 - \int_0^t [Lf^2(X_s) - 2f(X_s)Lf(X_s)]ds$ is an $\mathscr{F}_t$-martingale. Without compromising the results, here we work with the simpler martingale:

$$M_t = f(X_t) - \int_0^t Lf(X_s)ds \tag{A.20}$$

which has the same quadratic variation than A.18. Then let us compute:

$$\begin{aligned} M_t^2 &= f^2(X_t) - 2f(X_t)\int_0^t Lf(X_s)ds + (\int_0^t Lf(X_s)ds)^2 \\ &= f^2(X_t) - 2M_t(X_t)\int_0^t Lf(X_s)ds - (\int_0^t Lf(X_s)ds)^2 \end{aligned} \tag{A.21}$$

Let us then also differentiate

$$
\begin{aligned}
& d[2M_t(X_t)\int_0^t Lf(X_s)ds + (\int_0^t Lf(X_s)ds)^2] \\
& = 2\int_0^t Lf(X_s)ds dM_t(X_t) \\
& +2Lf(X_t)dt M_t(X_t) + 2Lf(X_t)dt\int_0^t Lf(X_s)ds \\
& = 2\int_0^t Lf(X_s)ds dM_t(X_t) + 2f(X_t)Lf(X_t)dt \\
& -2Lf(X_t)dt\int_0^t Lf(X_s)ds + 2Lf(X_t)dt\int_0^t Lf(X_s)ds \\
& = 2\int_0^t Lf(X_s)ds dM_t(X_t) + 2f(X_t)Lf(X_t)dt \qquad \text{(A.22)}
\end{aligned}
$$

Integration of A.22 and substitution in A.21 gives:

$$
\begin{aligned}
M_t^2 \quad & = f^2(X_t) - 2f(X_t)\int_0^t Lf(X_s)ds + (\int_0^t Lf(X_s)ds)^2 \\
& = f^2(X_t) - 2\int_0^t\int_0^s Lf(X_u)du dM_s(X_s) - \int_0^t 2f(X_s)Lf(X_s)ds \\
& = f^2(X_t) - \int_0^t Lf^2(X_s)ds - 2\int_0^t\int_0^s Lf(X_u)du dM_s(X_s) \\
& + \int_0^t Lf^2(X_s) - 2f(X_s)Lf(X_s)ds. \qquad \text{(A.23)}
\end{aligned}
$$

By Ito's formula, we know that $\int_0^t\int_0^s Lf(X_u)du dM_s(X_s)$ is a martingale , and since $f^2 \in D(L)$ the sum of the first two terms of A.23 is a martingale as well. Which proves that

$$
M_t^2 - \int_0^t \left(Lf^2(X_s) - 2f(X_s)Lf(X_s)\right) ds \qquad \text{(A.24)}
$$

is a martingale as well. Which completes the proof. □

A.2.1 Carré-du-champ

The term inside the time integral for the quadratic variation of the Dynkin martingale is known in the literature as the Carré-du-champ. This is an operator Γ acting on local functions as follows

$$
\Gamma f(X) = Lf^2(X) - 2f(X)Lf(X) \qquad \text{(A.25)}
$$

The following proposition gives an additional expression for the Carré-du-champ for the type of interacting particle systems we consider in this thesis.

Proposition A.2.1. *Consider a Markov process $\{X_t : t \geq 0\}$ with generator*

$$Lf(X) = \sum_{X'} c(X, X') \left(f(X') - f(X)\right) \tag{A.26}$$

the following is an alternative formulation for its carré-du-champ

$$\Gamma(f)(X) = \sum_{X'} c(X, X') \left(f(X') - f(X)\right)^2 . \tag{A.27}$$

Proof. By definition we have

$$\begin{aligned}
\Gamma(f)(X) &= \sum_{X'} c(X, X') \left(f(X')^2 - f(X)^2\right) \\
&- 2f(X) \sum_{X'} c(X, X') \left(f(X') - f(X)\right) \\
&= \sum_{X'} c(X, X') \left(f(X')^2 - f(X)^2\right) \\
&- \sum_{X'} c(X, X') \left(2f(X)f(X') - 2f(X)^2\right) \\
&= \sum_{X'} c(X, X') \left(f(X')^2 - f(X)^2 - 2f(X)f(X') + 2f(X)^2\right) \\
&= \sum_{X'} c(X, X') \left(f(X')^2 - 2f(X)f(X') + f(X)^2\right) \\
&= \sum_{X'} c(X, X') \left(f(X') - f(X)\right)^2
\end{aligned}$$

that concludes the proof. □

A.3 Tightness criterium

In this section we state a well known criterion for tightness of distribution valued processes. This criterion can be originally found in [38] for the one dimensional case. In what follows $S(\mathbb{R}^k)$ denotes the space of rapidly decreasing test functions, and $S'(\mathbb{R}^k)$ its dual, i.e., the space of Schwartz distributions. Additionally $C([0,\infty), S'(\mathbb{R}^k))$ and $D([0,\infty), S'(\mathbb{R}^k))$ denote the spaces of continuous, and right continuous with left limits, paths with values in $S'(\mathbb{R}^k)$.

Theorem A.3.1. *Let $(\Omega, \mathscr{F})$ be a measurable space with right-continuous filtrations $\{\mathscr{F}_t^n\}_{t\geq 0}$ and probability measures $\mathbb{P}_n(\cdot)$, $n \in \mathbb{N}$. Let $\{\mathscr{Y}_t^n\}_{t\geq 0}$ be an*

$\mathscr{F}_t^n$-adapted process with paths in $D([0,\infty), S'(\mathbb{R}^k))$ and let us also suppose that there exists, for each $\varphi \in S(\mathbb{R}^k)$, $\mathscr{F}_t^n$-predictable processes $\gamma_1^n(\cdot,\varphi), \gamma_2^n(\cdot,\varphi)$ such that:

$$M_t^n(\varphi) := \mathscr{Y}_t^n(\varphi) - \int_0^t \gamma_1^n(s,\varphi)ds \tag{A.28}$$

and

$$M_t^n(\varphi)^2 - \int_0^t \gamma_2^n(s,\varphi)ds \tag{A.29}$$

are martingales. Assume further that it holds:

CI: *for $t_0 \geq 0$ and $\varphi \in S(\mathbb{R}^k)$:*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq t\leq t_0} \mathbb{E}_n(\mathscr{Y}_t^n(\varphi)^2) < \infty \tag{A.30}$$

and for $i \in \{1,2\}$:

$$\sup_{n\in\mathbb{N}} \sup_{0\leq t\leq t_0} \mathbb{E}_n(\gamma_i^n(t,\varphi)^2) < \infty; \tag{A.31}$$

CII: *for every $\varphi \in S(\mathbb{R}^k)$ there exists a sequence $\delta(t,\varphi,n)$ converging to zero as $n \to 0$ such that:*

$$\lim_{n\to\infty} \mathbb{P}_n(\sup_{0\leq s\leq t} |\mathscr{Y}_s^n(\varphi) - \mathscr{Y}_{s-}^n(\varphi)| \geq \delta(t,\varphi,n)) = 0 \tag{A.32}$$

then the family of laws $\{Q^n\}_{n\in\mathbb{N}}$, induced by $\{\mathscr{Y}_t^n\}_{t\geq 0}$ on $D([0,\infty), S'(\mathbb{R}^k))$ under $\mathbb{P}_n$, is a tight family and any weak limit point is supported by $C([0,\infty), S'(\mathbb{R}^k))$.

Appendix B

Dirichlet forms

In this chapter we will present some of the basic concepts on the theory of Dirichlet forms. Our intention is to convince the reader that is not so familiar of this theory, of the beauty and advantages that it may bring when applied to IPS.

B.1 Dirichlet forms

A Dirichlet form on a Hilbert space is defined as a symmetric form which is closed and Markovian. The importance of Dirichlet forms in the theory of Markov processes is that the Markovian nature of the first corresponds to the Markovian properties of the associated semigroups and resolvents on the same space. Related to the present work, probably one of the best examples of this connection is the work of Umberto Mosco. In [76] Mosco introduced a type of convergence of quadratic forms, Mosco convergence, which is equivalent to strong convergence of the corresponding semigroups. Before defining this notion of convergence, we recall the precise definition of a Dirichlet form.

Definition B.1.1 (Dirichlet forms)**.** *Let H be a Hilbert space of the form $L^2(E;m)$ for some σ-finite measure space $(E,\mathscr{B}(E),m)$. Let H be endowed with an inner product $\langle\cdot,\cdot\rangle_H$. A Dirichlet form $\mathscr{E}(f,g)$, or $(\mathscr{E},D(\mathscr{E}))$, on H is a symmetric bilinear form such that the following conditions hold*

1. *The domian $D(\mathscr{E})$ is a dense linear subspace of H.*

2. *The form is closed, i.e. the domain $D(\mathscr{E})$ is complete with respect to the metric determined by*

$$\mathscr{E}_1(f,g)=\mathscr{E}(f,g)+\langle f,g\rangle_H.$$

3. *The unit contraction operates on $\mathscr{E}$, i.e. for $f\in D(\mathscr{E})$, if we set $g:=(0\vee f)\wedge 1$ then we have that $g\in D(\mathscr{E})$ and $\mathscr{E}(g,g)\le\mathscr{E}(f,f)$.*

When the third condition is satisfied we say that the form $\mathscr{E}$ is Markovian. We refer the reader to [42] for a comprehensible introduction to the subject of Dirichlet forms. For the purposes of this work, the key property of Dirichlet forms is that there exists a natural correspondence between the set of Dirichlet forms and the set of Markov generators. In other words, to a symmetric Markov process we can always associate a Dirichlet form that is given by:

$$\mathscr{E}(f,g) = -\langle f, Lg\rangle_H \qquad \text{with} \qquad D(\mathscr{E}) = D(\sqrt{-L}) \tag{B.1}$$

where the operator L is the corresponding infinitesimal generator of a symmetric Markov process. As an example of this relation, consider the Brownian motion in $\mathbb{R}$. We know that the associated infinitesimal generator is given by the Laplacian. Hence its Dirichlet form is

$$\mathscr{E}_{\text{bm}}(f,g) = \frac{1}{2}\int_{-\infty}^{\infty} f'(x)g'(x)dx \qquad \text{with domain} \qquad D(\mathscr{E}_{\text{bm}}) = H^1(\mathbb{R}) \tag{B.2}$$

namely the Sobolev space of order 1.

From now on we will mostly deal with the quadratic form $\mathscr{E}(f,f)$ that we can view as a functional defined on the entire Hilbert space H by defining

$$\mathscr{E}(f) = \begin{cases} \mathscr{E}(f,f), & f \in D(\mathscr{E}) \\ \infty, & f \notin D(\mathscr{E}), \end{cases} \qquad f \in H \tag{B.3}$$

which is lower-semicontious if and only if the form $(\mathscr{E}, D(\mathscr{E}))$ is closed.

B.2 Time changes of Dirichlet forms

In this section we present some basic notions in the context time changes of Markov processes and their Dirichlet forms. First, in Section B.2, we introduce the basic notions related to time changes of Markov processes and their Dirichlet forms. Then, in Section B.3, we use the machinery from Section B.2 to compute the Dirichlet form of the two-sided sticky Brownian motion at zero. The content of this section follows [22]. In particular we refer the reader to Chapter 5 and the Appendix of [22] for more details and necessary background.

Let $M = (\Omega, \mathscr{M}, M_t, \zeta, \mathbb{P})$ a right continuous Markov process, on a Lusin space $(E, \mathscr{B}(E))$, where the relevant probability space is given by the triple $(\Omega, \mathscr{M}, \mathbb{P})$, and where for every $\omega \in \Omega$ the random variable $\zeta(\omega)$ denotes the lifetime of the sample path of ω.i.e.,

$$\zeta(\omega) = \inf\{t \geq 0 : M_t(\omega) = \partial\} \tag{B.4}$$

for ∂ the cemetery point.

Furthermore, we assume that for each $t \geq 0$ there exists a map $\theta_t : \Omega \to \Omega$ such that

$$M_s \circ \theta_t = M_{s+t}$$

for every $s \geq 0$. Moreover, we have $\theta_0 \omega = \omega$, and $\theta_\infty \omega = [\partial]$, where $[\partial]$ denotes a specific element of Ω such that $M_t([\partial]) = \partial$.

In addition, we denote by $\{\mathscr{F}_t\}_{t \geq 0}$ the filtration generated by the Markov process M_t, i.e., for $t < \infty$:

$$\mathscr{F}_t = \sigma\{M_s : s \leq t\}.$$

For convenience we extend the parameter t of the filtration to $[0, \infty]$ by setting:

$$\mathscr{F}_\infty = \sigma\{\mathscr{F}_t : t \geq 0\}$$

Definition B.2.1 (PCAF). *A function $A_t(\omega)$ of two variables $t \geq 0$ and $\omega \in \Omega$ is called an additive functional of M_t if there exists $\Lambda \in \mathscr{F}_\infty$ and a μ-inessential set $N \subset E$ with*

$$P_x(\Lambda) = 1 \quad \text{for } x \in E \setminus N \quad \text{and} \quad \theta_t \Lambda \subset \Lambda \quad \text{for } t \geq 0 \tag{B.5}$$

if the following conditions are satisfied:

(i) *For each $t \geq 0$, $A_t \mid_\Lambda$ is $\mathscr{F}_t$-measurable.*

(ii) *For any $\omega \in \Lambda$, $A_\cdot(\omega)$ is right continuous on $[0, \infty)$ has left limits on $(0, \zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, and $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for all $t \geq \zeta(\omega)$.*

(iii) *The additivity property is satisfied, i.e.,*

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\omega) \text{ for all } t, s \geq 0. \tag{B.6}$$

If we denote by $\mathscr{A}_c^+$ the set of all PCAF, it turns out that there exists a one to one correspondence between the set $\mathscr{A}_c^+$ and a special subset of the set of the Borel measures on E. Which we now introduce:

Definition B.2.2 (Smooth measures). *Let ν be a positive measure on $(E, \mathscr{B}(E))$, ν is said to be smooth if*

1. *It does not charge any $\mathscr{E}_M$-polar set.*
2. *There exists a nest $\{F_k\}_{k \geq 1}$ such that $\nu(F_k) < \infty$ for all $k \geq 1$.*

Remark B.2.1. *Notice that all the Dirichlet forms related concepts ($\mathscr{E}_M$-capacity for example) are in terms of the Dirichlet space $(\mathscr{E}_M, D(\mathscr{E}_M))$, which corresponds to the symmetric Markov process M_t.*

We denote by $S(E)$ the set of all smooth measures on E. The correspondence we mentioned above is between $\mathscr{A}_c^+$ and $S(E)$. Formally, this correspondence is given by the following result:

Theorem B.2.1 (PCAF and Smooth measures). *For $A \in \mathscr{A}_c^+$ we denote by ν_A the measure that is in Revuz correspondence with A, i.e. the measure that for any $f \in \mathscr{B}_+(E)$ satisfies:*

$$\int_E f(x)\nu_A(dx) = \lim_{t\downarrow 0} \frac{1}{t} E_\mu[\int_0^t f(M_s) dA_s] \tag{B.7}$$

where the expectation E_μ on the right hand side of (B.7) *is taken over both the position of the starting point of the process M_s, which is selected according to the invariant measure μ, and over the trajectory of the process M_s.*

Then we have the following:

(i) *For any $A \in \mathscr{A}_c^+$, $\nu_A \in S(E)$.*

(ii) *For any $\nu \in S(E)$, there exists $A \in \mathscr{A}_c^+$ satisfying $\nu_A = \nu$ uniquely up to μ-equivalence.*

Proof. This is part of Theorem 4.1.1 in [22] where the proof is included. □

It is known that there exists a one to one correspondence between Markov process and Dirichlet forms [43]. The idea is that given a PCAF A_t we can define a stochastic time-changed process given by the generalized inverse of A_t in terms of its corresponding Dirichlet form. More precisely:

Theorem B.2.2. *Let M_t be a symmetric Markov process with corresponding Dirichlet space given by $(\mathscr{E}_M, D(\mathscr{E}_M))$. Let also A_t be a PCAF whose Revuz measure ν_A has full quasi support. Denote by $\tilde{M}_t$ the time-changed process given by the generalized inverse of A_t. Then we have that its corresponding Dirichlet space $(\mathscr{E}_{\tilde{M}}, D(\mathscr{E}_{\tilde{M}}))$ is given by*

$$\mathscr{E}_{\tilde{M}}(f,g) = \mathscr{E}_M(f,g) \quad \text{and} \quad D(\mathscr{E}_{\tilde{M}}) = D(\mathscr{E}_M) \cap L^2(E, \nu_A). \tag{B.8}$$

Proof. This theorem is just a specialization of Theorem 5.2.2 in [22]. Where the time-changed form is given by

$$\mathscr{E}_{\tilde{M}}(f,g) = \mathscr{E}_M(H_F f, H_F g). \tag{B.9}$$

The specialization consists in the fact that the Revuz measure ν_A has full quasi support, i.e.,

$$H_F h(x) = \mathbb{E}_x[h(M_{\sigma_F}); \sigma_F < \infty] = h(x) \tag{B.10}$$

where $F \subset E$ is the support of ν_A and σ_F is its hitting time. We refer the reader to page 176 of the same reference if more details are needed. □

B.3 Sticky Brownian Motion and its Dirichlet form

In this Appendix we provide some background material on the two-sided sticky Brownian motion in the context of Dirichlet forms. Namely, by means of an example we apply the machinery of Dirichlet forms to the theory of stochastic time changes for Markov processes. The example that we will build at the end of this section plays the role of the limiting process for the difference process. In this appendix we will mostly follow the approach presented in Chapter 5 of [22].

B.3.1 Two-sided sticky Brownian motion

The traditional approach to construct sticky Brownian motion (SBM) on the real line is by means of local times and time changes related to them. Let us say that we are in the one dimensional case and we want to build Brownian motion sticky at zero. We consider then standard Brownian motion $\{B_t\}_{t\geq 0}$ taking values on $\mathbb{R}$ and define its local time at zero by

$$L_t^0 = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{[-\epsilon,\epsilon]}(B_s)ds.$$

Given this local time and for $\gamma > 0$ we consider the functional

$$T_t = t + \gamma L_t^0 \tag{B.11}$$

and denote by τ its generalized inverse, i.e.,

$$\tau(t) = \inf\{s > 0 : T_s > t\}, \tag{B.12}$$

then the process given by the time change

$$B_t^{\text{sbm}} = B_{\tau(t)}, \tag{B.13}$$

is what is known in the literature by two-sided sticky Brownian motion.

Remark B.3.1. *The idea in defining* (B.11) *is that we add some "extra time" at zero and by taking the inverse* (B.12) *via the time change we slow down the new process whenever it is at 0. Notice that the parameter γ controls the factor by which we slow down time.*

As expected, in the context of Dirichlet forms, we can also perfom this kind of stochastic time changes. Our goal for this section is to describe the Dirichlet forms approach to perfom the kind of time changes we are interested in. There are basically two ingredients that we need:

1. A symmetric Markov process M_t with reversible measure μ with support in the state space E.

2. A Positive Continous Additive Functional (PCAF) that, in a sense to be seen later, plays the role of the local time.

Remark B.3.2. *In the same way that the local time L_t^0 implicitly defined the point $\{0\}$ as the "sticky region", the PCAF of the second ingredient above will determine a "sticky region" for our new process.*

Under this setting, it becomes then easier to characterize the time-change of Brownian motion given by the inverse of the functional T_t defined in (B.11). The idea is that under the setting given by one dimensional Brownian motion on the reals. We know that the process $\{B_t\}_{t\geq 0}$ is reversible with respect to the Lebesgue measure dx. On the first hand, the Lebesgue measure dx is in Revuz correspondence with the trivial PCAF $A_t^1 = t$. On the other hand the following computation shows the Revuz correspondence between the PCAF L_t^0 and the Dirac measure at zero δ_0:

$$\begin{aligned}
\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}_{dx}[\int_0^t f(B_s)dL_s^0] &= \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}_{dx}[\int_0^t f(B_s)\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\mathbb{1}_{[-\epsilon,\epsilon]}(B_s)\ ds] \\
&= \lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{R}}\mathbb{E}_{B_0}[f(B_s+x)\mathbb{1}_{[-\epsilon,\epsilon]}(B_s+x)]\ dxds \\
&= \lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{R}^2} f(y+x)\mathbb{1}_{[-\epsilon,\epsilon]}(y+x)\frac{e^{\frac{-y^2}{2s}}}{\sqrt{2\pi s}}\ dydxds \\
&= \lim_{t\downarrow 0}\lim_{\epsilon\downarrow 0}\frac{1}{t}\frac{1}{2\epsilon}\int_0^t\int_{\mathbb{R}}\int_{-\epsilon}^{\epsilon} f(z)\frac{e^{\frac{-(z-x)^2}{2s}}}{\sqrt{2\pi s}}\ dzdxds \\
&= \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f(z)\ dz \\
&= f(0) = \int f(x)\delta_0(dx). \qquad \text{(B.14)}
\end{aligned}$$

Then the measure $\nu = dx + \gamma\delta_0$ is in Revuz correspondence with the PCAF T_t and hence by Theorem B.2.2 the Dirichlet form for one dimensional Sticky Brownian motion $\{B_t^{\text{sbm}}\}_{t\geq 0}$ is given by:

$$\mathscr{E}_{B^{\text{sbm}}}(f,g) = \mathscr{E}_B(f,g) \quad \text{and} \quad D(\mathscr{E}_{B^{\text{sbm}}}) = D(\mathscr{E}_B)\cap L^2(\mathbb{R}, dx+\gamma\delta_0) \qquad \text{(B.15)}$$

where $(\mathscr{E}_B, D(\mathscr{E}_B))$ are given as in (B.2).

In particular for the quadratic functional $\mathscr{E}_{B^{\text{sbm}}}(f)$, given by (B.3), we have:

$$\mathscr{E}_{B^{\text{sbm}}}(f) = \int_{\mathbb{R}} \mathbb{1}_{\{x\neq 0\}}(x) f'(x)^2\, dx \qquad \text{(B.16)}$$

for $f \in H^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx + \gamma\delta_0)$.

Remark B.3.3. *Notice that with an abuse of notation, and taking advantage of the fact that the Lebesgue measure assigns zero mass to the point zero, we can write the equality*

$$\mathscr{E}_{B^{sbm}}(f) = \mathscr{E}_B(f) \tag{B.17}$$

for any $f \in D(\mathscr{E}_{B^{sbm}})$.

B.3.2 Domain of the infinitesimal generator

In this section we will make use of the correspondence between Dirichlet forms and Markov generators to obtain a description of the generator of sticky Brownian motion with parameter γ. Let us then expand a bit on what we mentioned before equation (B.1); this is how the two directions of the correspondence are actually given:

(a) From forms $\mathscr{E}$ to generators L: The correspondence is defined by

$$D(L) \subset D(\mathscr{E}), \quad \mathscr{E}(f,g) = - < Lf, g > \quad \forall f \in D(L),\ g \in D(\mathscr{E}). \tag{B.18}$$

(b) From generators L to forms $\mathscr{E}$: In this case the correspondence is given by

$$D(\mathscr{E}) = D(\sqrt{-L}), \quad \mathscr{E}(f,g) =< \sqrt{-L}f, \sqrt{-L}g > \quad \forall f, g \in D(\mathscr{E}). \tag{B.19}$$

We can think of these relations as the first and second representation theorems for Dirichlet forms in the spirit of Kato [56] for sesquilinear forms. For the particular case of Dirichlet forms, more details and the connection to semigroups and resolvents, can be found on the Appendix of [22].

Remark B.3.4. *Please notice that the time-changed process behaves like Brownian motion on the set* $\mathbb{R} \setminus \{0\}$ *and differently (sticky behavior) when it visits* 0. *Therefore we expect the new generator* $L_{B^{sbm}}$ *to be the same Laplace operator in the region* $\mathbb{R} \setminus \{0\}$ *i.e.*

$$L_{B^{sbm}} f(x) = f''(x) \quad \forall x \in \mathbb{R} \tag{B.20}$$

and some additional restrictions at the point zero.

The idea is to assume that the generator $L_{B^{\mathrm{sbm}}}$ is just the Laplacian at all points, and by using the properties of the time-changed process determine additional constrains at zero.

For $f \in D(\mathscr{E}_{B^{\mathrm{sbm}}})$, thanks to (B.19) we can re-write (B.15) in terms of $L_{B^{\mathrm{sbm}}}$ in the following way:

$$\mathscr{E}_{B^{\mathrm{sbm}}}(f,g) = \int_{\mathbb{R}} \mathbb{1}_{\{x\neq 0\}} g'(x) f'(x) dx \tag{B.21}$$

for all $g \in D(\mathscr{E}_{B^{\mathrm{sbm}}})$.

On the other hand, for $f \in D(L_{B^{\mathrm{sbm}}})$ we have:

$$\begin{aligned} \mathscr{E}_{B^{\mathrm{sbm}}}(f,g) &= -\int_{\mathbb{R}} g(x) L_{B^{\mathrm{sbm}}} f(x) \, \big(dx + \gamma\delta_0(dx)\big) \\ &= -\int_{\mathbb{R}} g(x) f''(x) dx - \gamma g(0) f''(0) \\ &= -\int_{\mathbb{R}\setminus\{0\}} g(x) f''(x) dx - \gamma g(0) f''(0) \end{aligned} \tag{B.22}$$

where in the first line we used (B.18), and in the third line we used the fact the the Lebesgue measure assigns zero mass to the singleton $\{0\}$.

Let us split the first therm on the r.h.s. of (B.22) in two regions:

$$\int_{\mathbb{R}\setminus\{0\}} g(x) f''(x) dx = \int_{x>0} g(x) f''(x) dx + \int_{x<0} g(x) f''(x) dx. \tag{B.23}$$

Integrating by parts in the first integral of the r.h.s. of (B.23) we obtain:

$$\int_{x>0} g(x) f''(x) dx = -g(0) f'(0+) - \int_{x>0} g'(x)\, f'(x) dx \tag{B.24}$$

where

$$f'(0+) = \lim_{h\downarrow 0} \frac{f(h)-f(0)}{h}. \tag{B.25}$$

Similarly we obtain:

$$\int_{x<0} g(x) f''(x) dx = g(0) f'(0-) - \int_{x<0} g'(x)\, f'(x) dx \tag{B.26}$$

therefore, for every $g \in D(\mathscr{E}_{B^{s}})$ we obtain:

$$g(0)\big(\gamma\Delta f(0) - f'(0+) + f'(0-)\big) = 0 \tag{B.27}$$

which gives

$$\gamma f''(0) = f'(0+) - f'(0-) \tag{B.28}$$

for every $f \in D(L_{B^{\mathrm{sbm}}})$.

We then indeed have, from (B.28), that for every $f \in D(L_{B^{\mathrm{sbm}}})$:

$$\begin{aligned} \mathscr{E}_{B^{\mathrm{sbm}}}(f) &= \int_{\mathbb{R}} f(x) \left(-L_{B^{\mathrm{sbm}}} f(x)\right) \left(dx + \gamma\delta_0(dx)\right) dx \\ &= \int_{\mathbb{R}} \mathbb{1}_{\{x\neq 0\}}(x) f'(x)^2 \, dx. \end{aligned} \tag{B.29}$$

Remark B.3.5. *Notice that condition* (B.28) *coindices with what we would expect from the conditions given for two-sided sticky Brownian motion. See for instance Appendix 1 in [10].*

B.4 Mosco convergence

We now introduce the framework to properly define the mode of convergence we are interested in. The idea is that we want to approximate a Dirichlet form on the continuum by a sequence of Dirichlet forms indexed by a scaling parameter N. In this context, the problem with the convergence introduced in [76] is that the approximating sequence of Dirichlet forms does not necessarily live on the same Hilbert space. However, the work in [66] deals with this issue. We also refer to [62] for a more complete understanding and a further generalization to infinite dimensional spaces. In order to introduce this mode of convergence, we first define the concept of convergence of Hilbert spaces.

B.4.1 Convergence of Hilbert spaces

We start with the definition of the notion of convergence of spaces:

Definition B.4.1 (Convergence of Hilbert spaces)**.** *A sequence of Hilbert spaces $\{H_n\}_{n\geq 0}$, converges to a Hilbert space H if there exist a dense subset $C \subseteq H$ and a family of linear maps $\{\Phi_n : C \to H_n\}_n$ such that:*

$$\lim_{n\to\infty} \|\Phi_n f\|_{H_n} = \|f\|_H, \qquad \textit{for all } f \in C \tag{B.30}$$

It is also necessary to introduce the concepts of strong and weak convergence of vectors living on a convergent sequence of Hilbert spaces. Hence in Definitions B.4.2, B.4.3 and B.4.5 we assume that the spaces $\{H_n\}_{n\geq 1}$ converge to the space H, in the sense we just defined, with the dense set $C \subset H$ and the sequence of operators $\{\Phi_n : C \to H_n\}_n$ witnessing the convergence.

Definition B.4.2 (Strong convergence on Hilbert spaces)**.** *A sequence of vectors $\{f_n\}$ with f_n in H_n, is said to strongly-converge to a vector $f \in H$, if there exists a sequence $\{\tilde{f}_m\} \in C$ such that:*

$$\lim_{m\to\infty} \|\tilde{f}_m - f\|_H = 0 \tag{B.31}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|\Phi_n \tilde{f}_m - f_n\|_{H_n} = 0 \tag{B.32}$$

Definition B.4.3 (Weak convergence on Hilbert spaces). *A sequence of vectors $\{f_n\}$ with $f_n \in H_n$, is said to converge weakly to a vector f in a Hilbert space H if*

$$\lim_{n\to\infty} \langle f_n, g_n \rangle_{H_n} = \langle f, g \rangle_H \tag{B.33}$$

for every sequence $\{g_n\}$ strongly convergent to $g \in H$.

Remark B.4.1. *Notice that, as expected, strong convergence implies weak convergence, and, for any $f \in C$, the sequence $\Phi_n f$ strongly-converges to f.*

Given these notions of convergence, we can also introduce related notions of convergence for operators. More precisely, if we denote by $L(H)$ the set of all bounded linear operators in H, we have the following definition

Definition B.4.4 (Convergence of bounded operators on Hilbert spaces). *A sequence of bounded operators $\{T_n\}$ with $T_n \in L(H_n)$, is said to strongly (resp. weakly) converge to an operator T in $L(H)$ if for every strongly (resp. weakly) convergent sequence $\{f_n\}$, $f_n \in H_n$ to $f \in H$ we have that the sequence $\{T_n f_n\}$ strongly (resp. weakly) converges to Tf.*

We are now ready to introduce Mosco convergence.

B.4.2 Definition of Mosco convergence

In this section we assume the Hilbert convergence of a sequence of Hilbert spaces $\{H_n\}_n$ to a space H.

Definition B.4.5 (Mosco convergence). *A sequence of Dirichlet forms $\{(\mathscr{E}_n, D(\mathscr{E}_n))\}_n$ on Hilbert spaces H_n, Mosco-converges to a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ in some Hilbert space H if:*

Mosco I. *For every sequence of $f_n \in H_n$ weakly-converging to f in H*

$$\mathscr{E}(f) \leq \liminf_{n\to\infty} \mathscr{E}_n(f_n) \tag{B.34}$$

Mosco II. *For every $f \in H$, there exists a sequence $f_n \in H_n$ strongly-converging to f in H, such that*

$$\mathscr{E}(f) = \lim_{n\to\infty} \mathscr{E}_n(f_n) \tag{B.35}$$

As we mentioned before, the Markovian properties of the Dirichlet form correspond to the properties of the associated semigroups and resolvents. The following theorem from [66], which relates Mosco convergence with convergence of semigroups and resolvents, is a powerful application of this correspondence and one of the main ingredients of our work:

Theorem B.4.1. *Let $\{(\mathscr{E}_n, D(\mathscr{E}_n))\}_n$ be a sequence of Dirichlet forms on Hilbert spaces H_n and let $(\mathscr{E}, D(\mathscr{E}))$ be a Dirichlet form in some Hilbert space H. The following statements are equivalent:*

1. *$\{(\mathscr{E}_n, D(\mathscr{E}_n))\}_n$ Mosco-converges to $\{(\mathscr{E}, D(\mathscr{E}))\}$.*

2. *The associated sequence of semigroups $\{T_n(t)\}_n$ strongly-converges to the semigroup $T(t)$ for every $t > 0$.*

B.4.3 Mosco convergence and dual forms

The difficulty in proving condition Mosco I lies in the fact that (B.34) has to hold for all weakly convergent sequences, i.e., we cannot choose a particular class of sequences.

In this section we will show how one can avoid this difficulty by passing to the dual form. We prove indeed that Mosco I for the original form is implied by a condition similar to Mosco II for the dual form (Assumption 1).

B.4.3.1 Mosco I

Consider a sequence of Dirichlet forms $(\mathscr{E}_n, D(\mathscr{E}_n))_n$ on Hilbert spaces H_n, and an additional quadratic form $(\mathscr{E}, D(\mathscr{E}))$ on a Hilbert space H. We assume convergence of Hilbert spaces, i.e. that there exists a dense set $C \subset H$ and a sequence of maps $\Phi_n : C \to H_n$ such that $\lim_{n\to\infty} \|\Phi_n f\|_{H_n} = \|f\|_H$. The dual quadratic form is defined via

$$\mathscr{E}^*(f) = \sup_{g\in H} \big(\langle f, g\rangle - \mathscr{E}(g)\big)$$

Notice that from the convexity of the form we can conclude that it is involutive, i.e., $(\mathscr{E}^*)^* = \mathscr{E}$. We now assume that the following holds

Assumption 1. *For all $g \in H$, there exists a sequence $g_n \in H_n$ strongly-converging to g such that*

$$\lim_{n\to\infty} \mathscr{E}_n^*(g_n) = \mathscr{E}^*(g) \tag{B.36}$$

We show now that, under Assumption 1, the first condition of Mosco convergence is satisfied.

Proposition B.4.1. *Assumption 1 implies Mosco I, i.e.*

$$\liminf_{n\to\infty} \mathscr{E}_n(f_n) \geq \mathscr{E}(f) \tag{B.37}$$

for all $f_n \in H_n$ weakly-converging to $f \in H$.

Proof. Let $f_n \to f$ weakly then, by Assumption 1, for any $g \in H$ there exists a sequence $g_n \in H_n$ such that $g_n \to g$ strongly, and (B.36) is satisfied. Fromt the involutive nature of the form, and by Fenchel's inequality, we obtain:

$$\mathscr{E}_n(f_n) = \sup_{h \in H_n} \Big(\langle f_n, h \rangle_{H_n} - \mathscr{E}_n^*(h) \Big) \geq \langle f_n, g_n \rangle_{H_n} - \mathscr{E}_n^*(g_n)$$

by the fact that $f_n \to f$ weakly, $g_n \to g$ strongly, and (B.36) we obtain

$$\liminf_{n\to\infty} \mathscr{E}_n(f_n) \geq \liminf_{n\to\infty} \Big(\langle f_n, g_n \rangle_{H_n} - \mathscr{E}_n^*(g_n) \Big) \geq \langle f, g \rangle_H - \mathscr{E}^*(g)$$

Since this holds for all $g \in H$ we can take the supremum over H,

$$\liminf_{n\to\infty} \mathscr{E}_n(f_n) \geq \sup_{g \in H} \big(\langle f, g \rangle_H - \mathscr{E}^*(g) \big) = \mathscr{E}(f) \tag{B.38}$$

This concludes the proof. □

In other words, in order to prove condition Mosco I all we have to show is that Assumption 1 is satisfied.

B.4.3.2 Mosco II

For the second condition, we recall a result from [3] in which a weaker notion of Mosco convergence is proposed. In this new notion, condition Mosco I is unchanged whereas condition Mosco II is relaxed to functions living in a core of the limiting Dirichlet form. Let us first introduce the concept of core:

Definition B.4.6. *Let $(\mathscr{E}, D(\mathscr{E}))$ and H be as in Definition B.1.1. A set $K \subset D(\mathscr{E}) \cap C_c(E)$ is said to be a core of $(\mathscr{E}, D(\mathscr{E}))$ if it is dense both in $(D(\mathscr{E}), \|\cdot\|_{\mathscr{E}_1})$ and $(C_c(E), \|\cdot\|_\infty)$. Where $C_c(E)$ denotes the set of continuous functions with compact support.*

We now state the weaker notion from [3]:

Assumption 2. *There exists a core $K \subset D(\mathscr{E})$ of $\mathscr{E}$ such that, for every $f \in K$, there exists a sequence $\{f_n\}$ strongly-converging to f, such that*

$$\mathscr{E}(f) = \lim_{n\to\infty} \mathscr{E}_n(f_n). \tag{B.39}$$

Despite of being weaker, the authors were able to prove that this relaxed notion also implies strong convergence of semi-groups. We refer the reader to Section 3 of [3] for details on the proof.

Appendix C

Some results for a system of independent walkers

C.1 Local limit theorems

In this section we state and prove a local central limit theorem for Independent Random Walkers in continuous time. The motivation of this section comes from the fact that, despite it being common knowledge, we were not able to find a reference that includes the proof of such a result. However we do have access to many versions of the discrete case. We state now the version included in [69], since we consider is the most suitable to then jump to the continuous-time case. Theorem C.1.1 below is a direct consequence of Theorem 2.1.1 in the same reference [69].

Theorem C.1.1 (LCLT for Discrete-Time Random Walk). *Let $x \in \mathbb{Z}^d$ and $p_n^{DRW}(\cdot)$ be the probability distribution of a discrete-time random walk in $\mathbb{Z}^d$, then, for any fixed $M \geq 0$ there exists $c = c(M)$ such that*

$$\sup_{|x| \leq M\sqrt{n}} \left| \frac{p_n^{DRW}(x)}{\bar{p}_n(x)} - 1 \right| \leq \frac{c}{n} \tag{C.1}$$

where

$$\bar{p}_t(x) := \frac{\sqrt{d}}{(2\pi t)^{d/2}} \, e^{-\frac{d|x|^2}{2t}} \tag{C.2}$$

The way we generalize this theorem is by means of the following

Theorem C.1.2 (LCLT for Continuous-Time Random Walk). *Let $x \in \mathbb{Z}^d$ and $p_t^{RW}(\cdot)$ be the probability distribution of a continuous-time random walk in $\mathbb{Z}^d$,* *then, for any fixed $M \geq 0$, there exists $c = c(M) > 0$ s.t.*

$$\sup_{|x| \leq M\sqrt{t}} \left| \frac{p_t^{RW}(x)}{\bar{p}_t(x)} - 1 \right| \leq \frac{c}{\sqrt{t}} \tag{C.3}$$

Proof. We can always decompose

$$p_t^{\mathrm{RW}}(x) = \sum_{n=0}^{\infty} P(N_t = n)\, p_n^{\mathrm{DRW}}(x) \tag{C.4}$$

with N_t a Poisson process of rate 1. First by Proposition 2.5.5 in [69] we have

$$P(N_t = n) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(n-t)^2}{2t}} \exp\left\{ \mathscr{O}\left(\frac{1}{\sqrt{t}} + \frac{|n-t|^3}{t^2} \right) \right\} \tag{C.5}$$

Now for $\epsilon > 0$, we assume that

$$\frac{|n-t|}{t} \leq \epsilon$$

After some manipulation we obtain the following relations

$$\frac{1}{n} = \frac{1}{t}\left(1 + \mathscr{O}\left(\frac{|n-t|}{t}\right)\right), \qquad \frac{1}{n^\alpha} = \frac{1}{t^\alpha}\left(1 + \mathscr{O}\left(\frac{|n-t|}{t}\right)\right) \tag{C.6}$$

Combining (C.6) with Theorem C.1.1 we have

$$\begin{aligned} p_n^{\mathrm{DRW}}(x) \quad &= \frac{\sqrt{d}}{(2\pi n)^{d/2}}\, e^{-\frac{d|x|^2}{2n}} \left(1 + \mathscr{O}\left(\frac{1}{n}\right)\right) \\ &= \frac{\sqrt{d}}{(2\pi t)^{d/2}}\, e^{-\frac{d|x|^2}{2n}} \exp\left\{ \mathscr{O}\left(\frac{|x|^2|n-t|}{t^2} \right) \right\} \left(1 + \mathscr{O}\left(\frac{1}{t}\right)\right) \\ &\times \left(1 + \mathscr{O}\left(\frac{|n-t|}{t}\right)\right) \end{aligned} \tag{C.7}$$

Finally, substitution of (C.5) and (C.7) in (C.4) and further manipulations gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} P(N_t = n) p_n^{\mathrm{DRW}}(x) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(n-t)^2}{2t}} \exp\left\{ \mathscr{O}\left(\frac{1}{\sqrt{t}} + \frac{|n-t|^3}{t^2} \right) \right\} \\
&\times \frac{\sqrt{d}}{(2\pi t)^{d/2}}\, e^{-\frac{d|x|^2}{2t}} \exp\left\{ \mathscr{O}\left(\frac{|x|^2 |n-t|}{t^2} \right) \right\} \left(1 + \mathscr{O}\left(\frac{1}{t} \right) \right) \\
&\times \left(1 + \mathscr{O}\left(\frac{|n-t|}{t} \right) \right)
\end{aligned} \tag{C.8}$$

Assuming $|x| \leq M\sqrt{t}$ and using (C.6), we get the following,

$$\exp\left\{ \mathscr{O}\left(\frac{|x|^2 |n-t|}{t^2} \right) \right\} = \exp\left\{ \mathscr{O}\left(\epsilon \right) \right\} \tag{C.9}$$

Hence, more applications of (C.6) give

$$\begin{aligned}
&\sum_{n=0}^{\infty} P(N_t = n) p_n^{\mathrm{DRW}}(x) \\
&= \left(1 + \mathscr{O}\left(\frac{1}{t} \right) \right) \frac{\sqrt{d}}{(2\pi t)^{d/2}} e^{-\frac{d|x|^2}{2t}} \exp\left\{ \mathscr{O}\left(\epsilon \right) \right\} \left(1 + \mathscr{O}\left(\epsilon \right) \right) \\
&\times \exp\left\{ \mathscr{O}\left(\frac{1}{\sqrt{t}} \right) \right\} \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(n-t)^2}{2t}} \exp\left\{ \mathscr{O}\left(\frac{|n-t|^3}{t^2} \right) \right\} \\
&= \bar{p}_t(x) \left(1 + \mathscr{O}\left(\frac{1}{\sqrt{t}} \right) \right)
\end{aligned}$$

□

C.2 Mosco convergence for the Random Walk

In this section, we consider the difference process for the position-coordinates of two particles performing nearest-neighbor symmetric independent random walks. This process, that we denote by $\{v(t), t \geq 0\}$, is itself a random walk in $\mathbb{Z}$ for which convergence to the standard Brownian motion in the diffusive time-scales is well-known. By convergence we mean convergence of generators. In this section we will prove Mosco convergence of Dirichlet forms of $v(t)$.

As we can see in Section 6.3.2, the proof of Mosco-convergence for inclusion walkers strongly relies on the result for independent walkers (in particular for the proof of Mosco I). The choice of considering the independent dynamics case has the purpose to exemplifying the use of the Dirichlet-form approach in a setting simpler than the one of inclusion dynamics.

The generator of $\{v(t),\ t \geq 0\}$ is given by the discrete Laplacian Δ_1:

$$L^{\text{rw}} f(v) = \Delta_1 f(v) = f(v+1) - 2f(v) + f(v-1), \quad v \in \mathbb{Z}. \tag{C.10}$$

that is simply the generator of a random walk in $\mathbb{Z}$. Speeding up time by a factor N^2 and scaling the mesh between the lattice sites by a factor $\frac{1}{N}$ we obtain that the generator of this scaled process is

$$L_n^{\text{rw}} f(v) = \Delta_n f(v) = n^2 \left(f(v + \tfrac{1}{n}) - 2f(v) + f(v - \tfrac{1}{n}) \right), \quad v \in \tfrac{1}{n}\mathbb{Z} \tag{C.11}$$

We denote by $(\mathscr{R}_n, D(\mathscr{R}_n))$ the Dirichlet form associated to the generator (C.11), that is given by

$$\mathscr{R}_n(f) = -\sum_{i \in \mathbb{Z}/n} f(i) \Delta_n f(i) \mu_n(i) \tag{C.12}$$

where μ_n is the discrete counting measure on $\frac{1}{n}\mathbb{Z}$, this is

$$\mu_n(i) = \tfrac{1}{n}, \qquad \text{for all} \quad i \in \tfrac{1}{n}\mathbb{Z} \tag{C.13}$$

which is reversible for the dynamics. We are going to prove the Mosco convergence of the sequence of Dirichlet forms $\{(\mathscr{R}_n, D(\mathscr{R}_n))\}_n$ to the Dirichlet form $(\mathscr{E}_{\text{bm}}, D(\mathscr{E}_{\text{bm}}))$, i.e. the Dirichlet form associated to the standard Brownian motion in $\mathbb{R}$

$$\mathscr{E}_{\text{bm}}(f) = \frac{1}{2} \int_{\mathbb{R}} f'(x)^2 dx. \tag{C.14}$$

Proof of Mosco convergence for RW

Convergence of Hilbert spaces

For the sequence of Hilbert spaces

$$H_n^{\text{rw}} := L^2(\tfrac{1}{n}\mathbb{Z}, \mu_N) \tag{C.15}$$

where μ_n is as in (C.13). It is easy to see that we can guarantee the convergence of $\{H_n^{\mathrm{rw}}\}_{n\geq 1}$ to the Hilbert space

$$H^{\mathrm{bm}} := L^2(\mathbb{R}, dx) \tag{C.16}$$

i.e. the space of Lebesgue square-integrable functions in $\mathbb{R}$, by means of the restriction operators

$$\{\Phi_n : C_k^\infty(\mathbb{R}) \subset H^{\mathrm{bm}} \to H_n^{\mathrm{rw}}\}_n \qquad \text{defined by} \qquad \Phi_n f = f\,|_{\mathbb{Z}/n}\,. \tag{C.17}$$

Remark C.2.1. *The choice of the space of all compactly supported smooth functions $C := C_k^\infty(\mathbb{R})$ as dense set for our Hilbert space turns out to be particularly convenient since it is a core of the Dirichlet form associated to the Brownian motion. As a consequence, we can make use of the same set also for proving that* (B.39) *is satisfied.*

RW: Mosco I

In order to prove that Assumption 1 is satisfied, it is convenient to split the proof in two cases depending whether f belongs or not to the effective domain of $(-\Delta)^{-1/2}$. It is then sufficient to prove Propositions C.2.1 and C.2.2 below:

Proposition C.2.1. *For any $f \in D((-\Delta)^{-1/2})$, there exists a sequence $f_n \in H_n^{sip}$ strongly-converging to f, such that:*

$$\lim_{n\to\infty} \mathscr{R}_n^*(f_n) = \mathscr{E}_{bm}^*(f).$$

Proof. Let us proceed by cases:

Case I: $f \in C_k^\infty(\mathbb{R})$

In this case the approximate sequence f_n is simply given by: where f_n is given as follows:

$$f_n = \Phi_n f \tag{C.18}$$

which converges strongly to f.

Let $G(x)$ be the Green's function of the Laplacian in $\mathbb{R}$, i.e. the fundamental solution to the problem $\Delta G = \delta_0$ that is given by $G(x) = -|x|$. We refer the reader to [40] for more details on Green's functions. Let f be as in the statement, then, by standard variational arguments we know that

$$\begin{aligned}\mathscr{E}_{\mathrm{bm}}^*(f) &= \sup_{g\in D((-\Delta)^{1/2})}\left(\langle g, f\rangle - \frac{1}{2}\left\|(-\Delta)^{1/2}g\right\|^2_{L^2(\mathbb{R})}\right) = \frac{1}{4}\left\|(-\Delta)^{-1/2}f\right\|^2_{L^2(\mathbb{R})}\\ &= -\frac{1}{4}\langle f, G*f\rangle_{L^2(\mathbb{R})} = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)|x-y|\,dxdy.\end{aligned}$$

Analogously, for the discrete case, we can write

$$\mathscr{R}_n^*(\Phi_n f) = -\frac{1}{4}\langle \Phi_n f, \Delta_n^{-1}\Phi_n f\rangle_{H_n^{\mathrm{rw}}} = -\frac{1}{4n}\sum_{i,j\in\frac{1}{n}\mathbb{Z}} \Phi_n f(i)\cdot\Phi_n f(j)\cdot G_n(i-j)$$
$$= -\frac{1}{4n}\sum_{i,j\in\mathbb{Z}} \Phi_n f(\tfrac{i}{n})\cdot\Phi_n f(\tfrac{j}{n})\cdot G_n(\tfrac{i-j}{n})$$

where $G_n(\cdot)$ is the Green's function of the discrete Laplacian Δ_n in $\frac{1}{n}\mathbb{Z}$, i.e. the solution of the discrete problem:

$$\Delta_n G_n = \delta_0 \qquad \text{in } \tfrac{1}{n}\mathbb{Z} \tag{C.19}$$

we refer to Chapter 5 in [69] for more details on discrete Green's functions. Notice that

$$\tfrac{1}{n^2}\, G_1(i) = G_n(\tfrac{i}{n}) \qquad \forall i\in\mathbb{Z}$$

where $G_1(\cdot)$ is the solution of (C.19) for $n=1$. Then we can re-write

$$\mathscr{R}_n^*(\Phi_n f) = -\frac{1}{4n3}\sum_{i,j\in\mathbb{Z}} \Phi_n f(\tfrac{i}{n})\cdot\Phi_n f(\tfrac{j}{n})\cdot G_1(i-j). \tag{C.20}$$

By Theorem 4.4.8 in [69] we have that, for $i\neq j$, there exists $C,\beta>0$ such that

$$G_1(i-j) = -|i-j| + C + O(e^{-\beta|i-j|}).$$

Incorporating the above expression in (C.20) we obtain

$$\begin{aligned}\mathscr{R}_n^*(\Phi_n f) \quad &= \quad \tfrac{1}{4n^3}\sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \Phi_n f(\tfrac{i}{n})\Phi_n f(\tfrac{j}{n})\left(|i-j| + C + O(e^{-\beta|i-j|})\right)\\ &- \quad \tfrac{1}{4n^3}\sum_{i\in\mathbb{Z}}\left(\Phi_n f(\tfrac{i}{n})\right)^2 G_1(0).\end{aligned}$$

Notice that the sum on the diagonal (the second term in the RHS of (C.21)) vanishes as $N\to\infty$. Even more, thanks to the factor n^{-3} in front of the two dimensional sum, we have that

$$\lim_{n\to\infty}\frac{1}{4n^3}\sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \Phi_n f(\tfrac{i}{n})\Phi_n f(\tfrac{j}{n})\left(C + O(e^{-\beta|i-j|})\right) = 0.$$

where we used the smoothness of f and the extra factor $\frac{1}{n}$ in front of the summation.

Then we have

$$\begin{aligned}\lim_{n\to\infty} \mathscr{R}_n^*(\Phi_n f) &= \lim_{n\to\infty} \frac{1}{4n^3} \sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \Phi_n f(\tfrac{i}{n}) \cdot \Phi_n f(\tfrac{j}{n}) \cdot |i-j| \qquad \text{(C.21)}\\ &= \lim_{n\to\infty} \frac{1}{4n^2} \sum_{\substack{i,j\in\mathbb{Z}\\ i\neq j}} \Phi_n f(\tfrac{i}{n}) \cdot \Phi_n f(\tfrac{j}{n}) \cdot |\tfrac{i-j}{n}|\\ &= \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(y)|x-y|\,dx\,dy\\ &= \mathscr{E}_{\mathrm{bm}}^*(f).\end{aligned}$$

Which completes the proof of the first case.

Case II: $f \in D((-\Delta)^{-1/2}) \setminus C_k^\infty(\mathbb{R})$

In this case the sequence f_n is given by: where f_n is given as follows:

$$f_n = \Phi_n \left((f \cdot \mathbb{1}_{\{[-n^a,n^a]\}}) * k_n \right) \tag{C.22}$$

for K_n the following sequence of kernels:

$$K_n(x) = \frac{n^b}{\sqrt{2\pi}} e^{-\frac{|n^b x|^2}{2}} \tag{C.23}$$

with a, b positive real numbers such that $a + b < 1$.

Let us first verify that $\{f_n = \Phi_n \left((f \cdot \mathbb{1}_{\{[-n^a,n^a]\}}) * k_n \right)\}_{n\geq 1}$ converges strongly to f. In the language of Definition B.4.2, we first let $\tilde{f}_m$ to be equal to:

$$\tilde{f}_m = (f \cdot \mathbb{1}_{\{[-m^a,m^a]\}}) * k_m \tag{C.24}$$

Since K_m is an approximation to the identity, given by the Gaussian Kernel, by standard results we have:

$$\lim_{m\to\infty} \left\| \tilde{f}_m - f \right\|_{H^{\mathrm{bm}}}^2 = 0.$$

We also have:

$$\begin{aligned}
&\left\|\Phi_n \tilde{f}_m - f_n\right\|^2_{H_n^{\mathrm{rw}}} \\
&= \frac{1}{n}\sum_{i\in\mathbb{Z}}\left(\int_{\mathbb{R}} f(\tfrac{i}{n}-x)\left(\mathbb{1}_{\{[-m^a,m^a]\}}(\tfrac{i}{n}-x)K_m(x) - \mathbb{1}_{\{[-n^a,n^a]\}}(\tfrac{i}{n}-x)K_n(x)\right)dx\right)^2 \\
&\le \frac{1}{n}\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}} f(\tfrac{i}{n}-x)^2\,dx \int_{\mathbb{R}}\left(\mathbb{1}_{\{[-m^a,m^a]\}}(\tfrac{i}{n}-x)K_m(x) - \mathbb{1}_{\{[-n^a,n^a]\}}(\tfrac{i}{n}-x)K_n(x)\right)^2 dx \\
&\le \frac{C_f}{n}\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}}\left(\mathbb{1}_{\{[-m^a,m^a]\}}(\tfrac{i}{n}-x)K_m(x) - \mathbb{1}_{\{[-n^a,n^a]\}}(\tfrac{i}{n}-x)K_n(x)\right)^2 dx \\
&\le \frac{2C_f}{n}\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_{\{[-m^a,m^a]\}}(\tfrac{i}{n}-x)K_m(x)^2\,dx + \frac{2C_f}{n}\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_{\{[-n^a,n^a]\}}(\tfrac{i}{n}-x)K_n(x)^2\,dx
\end{aligned} \tag{C.25}$$

where the constant C_f is equal to the L^2-norm of f.

Let us deal with the second term in the RHS of (C.25), since the first term is done in a similar way:

$$\begin{aligned}
&\frac{2C_f}{n}\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_{\{[-n^a,n^a]\}}(\tfrac{i}{n}-x)K_n(x)^2\,dx \\
&\le \frac{C'n^a}{n}\int_{\mathbb{R}} K_n(x)^2\,dx = \frac{4C'n^{a+b}}{2\sqrt{\pi}n}\int_{\mathbb{R}} K_n(y)\,dy \\
&\le \frac{Cn^{a+b}}{n}
\end{aligned} \tag{C.26}$$

where in the first inequality we used the finite support of the indicator function, in the equality we used the explicit expression for $K_n(\cdot)$ and the change of variables $y = \sqrt{2}x$. Notice that the constant C' changes values from line to line and incorporates all factors independent from n.

In an analogous way, the first term in the RHS of (C.25) can be bounded as follows:

$$\frac{2C_f}{n}\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}}\mathbb{1}_{\{[-m^a,m^a]\}}(\tfrac{i}{n}-x)K_m(x)^2\,dx \le \frac{Cm^a n^b}{n} \tag{C.27}$$

Recall that in Definition B.4.2 we first take the limit in n and then in m. Hence, strong convergence is then obtained since the RHS of (C.26) vanishes as $n \to \infty$, because $a + b < 1$.

Now we need to verify that indeed we have:

$$\lim_{n\to\infty} \mathscr{R}_n^*(f_n) = \mathscr{E}_{bm}^*(f).$$

Similar to the case where $f \in C_k^\infty(\mathbb{R})$, we have an analogous of expression (C.21), namely:

$$\mathscr{R}_n^*(f_n) = \frac{1}{4n^2}\sum_{i,j\in\mathbb{Z}} \Phi_n\left((f\cdot \mathbb{1}_{\{[-n^a,n^a]\}}) * k_n\right)(\tfrac{i}{n})\cdot \Phi_n\left((f\cdot \mathbb{1}_{\{[-n^a,n^a]\}}) * k_n\right)(\tfrac{j}{n})\cdot |\tfrac{i-j}{n}|$$
$$= \frac{1}{4n^2}\sum_{i,j\in\mathbb{Z}} |\tfrac{i-j}{n}| \int_{-n^a}^{n^a} f(x)\, k_n(\tfrac{i}{n}-x)\, dx \int_{-n^a}^{n^a} f(y)\, k_n(\tfrac{j}{n}-y)\, dy. \tag{C.28}$$

We then want to control the following:

$$|\mathscr{R}_n^*(f_n) - I(f)|$$

where

$$I(f) = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} f(z)f(w)|z-w|\, dz\, dw.$$

In an attempt to use an epsilon over two argument, we then introduce the following

$$I_n(f) = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} |z-w| \int_{-n^a}^{n^a} f(x)\, k_n(w-x)\, dx \int_{-n^a}^{n^a} f(y)\, k_n(z-y)\, dy\, dz\, dw. \tag{C.29}$$

Hence we have:

$$|\mathscr{R}_n^*(f_n) - I(f)| \le |\mathscr{R}_n^*(f_n) - I_n(f)| + |I_n(f) - I(f)| \tag{C.30}$$

where the second term in the RHS of (C.30) can be controlled by the convergence $f_n \to f$, the fact that $f \in D((-\Delta)^{-1/2})$ and dominated convergence.

It is then enough to estimate the following:

$$\begin{aligned}
&|\mathscr{R}_n^*(f_n) - I_n(f)| \\
&\quad\le \tfrac{1}{4}\Big|\frac{1}{n^2}\sum_{i,j\in\mathbb{Z}} |\tfrac{i-j}{n}| \int_{-n^a}^{n^a} f(x)\, k_n(\tfrac{i}{n}-x)\, dx \int_{-n^a}^{n^a} f(y)\, k_n(\tfrac{j}{n}-y)\, dy \\
&\quad - \int_{\mathbb{R}^2} |z-w| \int_{-n^a}^{n^a} f(x)\, k_n(w-x)\, dx \int_{-n^a}^{n^a} f(y)\, k_n(z-y)\, dy\, dz\, dw\Big| \\
&\quad\le \tfrac{1}{4}\int_{\mathbb{R}^2} |f(x)f(y)\hat{G}_n(x,y)|\, dx\, dy
\end{aligned} \tag{C.31}$$

where $\hat{G}_n$ is given by:

$$\begin{aligned}
&\hat{G}_n(x,y) \\
&= \frac{1}{n^2}\sum_{i,j\in\mathbb{Z}} |\tfrac{i-j}{n}|\, k_n(\tfrac{i}{n}-x)\, k_n(\tfrac{j}{n}-y) - \int_{\mathbb{R}^2} |z-w|\, k_n(w-x)\, k_n(z-y)\, dz\, dw
\end{aligned} \tag{C.32}$$

which is controllable since by the smoothness of the kernels K_n the RHS of (C.32) converges to zero, togehter with a combination of Dominated convergence and the fact that $f \in D((-\Delta)^{-1/2})$. This concludes the second case. □

In order to conclude Assumption 1 it remains to consider f such that it does not belong to the domain of $D((-\Delta)^{-1/2})$, this is f such that $\mathscr{E}^*_{\mathrm{bm}}(f) = \infty$.

Proposition C.2.2. *For any $f \in H^{bm} \setminus D((-\Delta)^{-1/2})$ we have*

$$\lim_{n\to\infty} \mathscr{R}^*_n(f_n) = \infty.$$

where f_n is given as follows:

$$f_n = \Phi_n\Big((f \cdot \mathbb{1}_{\{[-n^a,n^a]\}}) * k_n\Big) \tag{C.33}$$

for K_n given in terms of the Gaussian Kernel as follows:

$$K_n(x) = \frac{n^b}{\sqrt{2\pi}} e^{-\frac{|n^b x|^2}{2}} \tag{C.34}$$

with a, b positive real numbers such that $a + b < 1$.

Proof. First, notice that by the same arguments as in the proof of Proposition C.2.1, f_n converges strongly to f.

Let then f be as in the statement, on the one hand we know because $f \notin D((-\Delta)^{-1/2})$:

$$\mathscr{E}^*_{bm}(f) = \frac{1}{8\pi} \int_{\mathbb{R}} \frac{(\widehat{f}(q))^2}{q^2}\, dq = \infty \tag{C.35}$$

where $\widehat{f}$ denotes the Fourier transform of f.

For the discrete setting we have:

$$\mathscr{R}^*_n(f_n) = \frac{1}{4}\langle f_n, \Delta_n^{-1} f_n\rangle_{H_n^{\mathrm{rw}}} = \frac{1}{4n^3} \sum_{x\in\mathbb{Z}} f_n(\tfrac{x}{n}) \cdot \Delta_1^{-1} f_n(\tfrac{x}{n}). \tag{C.36}$$

Let us denote by $\{X_t : t \geq 0\}$ the continuous time random walk on $\mathbb{Z}$ started at x. Then we have that $\Delta_1^{-1} f_N(\frac{x}{n})$ is given by

$$\Delta_1^{-1} f_n(\tfrac{x}{n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\hat{f}_n(k) e^{-ikx}}{2 - 2\cos k} dk \tag{C.37}$$

where

$$\widehat{f}_n(k) = \sum_{x\in\mathbb{Z}} f_n(\tfrac{x}{n}) e^{ikx}$$

Substitution of (C.37) in (C.36) gives:

$$\begin{aligned}\mathscr{R}_n^*(\Phi_n f) &= \frac{1}{8\pi n^3}\int_{-\pi}^{\pi}\frac{\hat{f}_n(k)}{2-2\cos k}\sum_{x\in\mathbb{Z}} f_n(\tfrac{x}{n})e^{-ikx}\,dk \\ &= \frac{1}{8\pi n^3}\int_{-\pi}^{\pi}\frac{\hat{f}_n(k)\hat{f}_n(-k)}{2-2\cos k}\,dk = \frac{1}{8\pi}\int_{-\pi n}^{\pi n}\frac{(\frac{1}{n}\hat{f}_n(\frac{q}{n}))(\frac{1}{n}\hat{f}_n(\frac{-q}{n}))}{n^2(2-2\cos\frac{q}{n})}\,dq.\end{aligned}$$

At this point, in order to get convergence to the limiting dual we have on the one hand the limit:

$$\lim_{n\to\infty} n^2(2-2\cos\tfrac{q}{n}) = q^2.$$

On the other hand, by definition of the Fourier transform for a generic $f \in L^2(\mathbb{R}, dx)$, and arguments analogous to the ones in the proof of Proposition C.2.1 we have:

$$\lim_{n\to\infty}\tfrac{1}{n}\widehat{f}_n(\tfrac{q}{n}) = \widehat{f}(q).$$

By Fatou's Lemma we indeed obtain:

$$\lim_{n\to\infty}\mathscr{R}_n^*(f_n) = \frac{1}{8\pi}\int_{\mathbb{R}}\frac{(\widehat{f}(q))^2}{q^2}\,dq = \infty,$$

which finishes the proof. □

RW: Mosco II

For what concerns the second condition of Mosco convergence, we choose $K := C_k^\infty(\mathbb{R})$ that is a core of $\mathscr{E}_{\text{bm}}$. In this way, for all $f \in C_k^\infty(\mathbb{R})$, we can consider the restrictions $\Phi_n f$ (strongly-convergent to f) and Taylor expand them to prove that:

$$\begin{aligned}\lim_{n\to\infty}\mathscr{R}_n(\Phi_n f) &= -\frac{1}{n}\lim_{n\to\infty}\sum_{i\in\frac{1}{n}\mathbb{Z}}\Phi_n f(i)\Delta_n\Phi_n f(i) \\ &= -\lim_{n\to\infty}\frac{1}{n}\sum_{i\in\frac{1}{n}\mathbb{Z}} f(i)\Delta_n f(i) \\ &= -\lim_{n\to\infty}\frac{1}{2n}\sum_{i\in\mathbb{Z}} f(\tfrac{i}{n})f''(\tfrac{i}{n}) + O(\frac{1}{n}) \\ &= -\frac{1}{2}\int_{\mathbb{R}} f(x)f''(x)dx \\ &= \frac{1}{2}\int_{\mathbb{R}} f'(x)^2dx = \mathscr{E}_{\text{bm}}(f) \qquad \text{(C.38)}\end{aligned}$$

which concludes the proof of Assumption 2. □

Remark C.2.2. *Notice that Theorem 4.48 in [69] also applies for the finite-range case and hence the results concerning Mosco convergence to the corresponding Brownian motion can be extended to the finite-range setting modulus a multiplicative constant depending on the second moment of the transition p.*

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Summary

This thesis is concerned with fluctuations of interacting particle systems that enjoy the property of duality. The main contributions of this work are divided in two main parts. In the first part we study some of the advantages of looking at the density fluctuation field through the lenses of orthogonal self-dualities. In the second part, we made use of self-duality and Mosco convergence of Dirichlet forms to understand the coarsening behaviour of the symmetric inclusion process when the process undergoes a phase transition known as condensation.

In Chapter 4, in the context of independent random walkers, we introduce a quantitative generalization of the Boltzmann-Gibbs principle. This generalization is a consequence of a systematic orthogonal decomposition, in terms of self-duality polynomials, of the fluctuation fields of local functions, where the order of every term can be quantified. Later, still in the context of independent random walkers, we are able to extend the quantitative Boltzmann-Gibbs principles in a non-equilibrium setting. This extension was achieved by using the fact that products of Poisson measures with a slowly varying parameter are preserved under the time evolution of this dynamics, i.e., a strong form of propagation of local equilibrium holds in this non-equilibrium context. Finally, for other interacting particle systems with duality such as the symmetric exclusion process, similar results are obtained, under precise conditions on the n particle dynamics.

In Chapter 5, we settle the basis to further develop the theory of fluctuations by introducing the notion of higher-order fields in the setting of IPS that enjoy the property of orthogonal self-duality. Namely, independent random walkers, symmetric exclusion and inclusion processes. These higher-order fields were derived from the observation that the quantity of interest in the derivation of fluctuation theorems, i.e. the density fluctuation field, can be written in terms of orthogonal self-duality polynomials. In this new hierarchy of fields, the ordinary density fluctuation field corresponds to the field of first order. By means of martingale techniques, we then study the scaling limits of the k-order fields. The limiting k-order field was then characterized in terms of a recursive martingale problem, whose initial solution corresponds to a generalized Ornstein-

Uhlenbeck process. As a consequence of the recursive martingale problem, we can heuristically identified the limiting k-order fields as a tensor product of the generalized Ornstein-Uhlenbeck process.

Finally, in Chapter 6, we first showed the convergence to sticky Brownian motion for the difference of positions of two SIP particles in the sense of Mosco convergence of Dirichlet forms. Because this notion of convergence implies convergence of semigroups in the L^2-space of the reversible measure, which is $dx + \gamma\delta_0$ for the sticky Brownian motion with stickiness parameter $\gamma > 0$, the convergence of semigroups also implies that of transition probabilities of the form $p_t(x, 0)$. This, together with self-duality, helps to explicitly obtain the limiting variance of the fluctuation field. This result provides a way to view the stacks of particles formed during the coarsening process as delta masses in the stochastic flow of kernels that defines a sticky Brownian motion.

Samenvatting

Dit proefschrift gaat over fluctuaties in interacterende deeltjessystemen (IPS) die de dualiteitseigenschap bezitten. De belangrijkste bijdragen van dit werk zijn verdeeld in twee delen. In het eerste deel bestuderen we enkele voordelen van het kijken naar het dichtheidsfluctuatieveld door de lenzen van orthogonale self-dualiteiten. In het tweede deel maken we gebruik van zelfdualiteit en Mosco-convergentie in Dirichlet-vorm om het verruwing gedrag van de symmetrische inclusieproces tijdens condensatie.

In hoofdstuk 4 introduceren we in de context van toevalsbewegingen een kwantitatieve generalisatie van het Boltzmann-Gibbs principe. Deze generalisatie is een gevolg van systematische orthogonale decomposities, in termen van zelf-duale polynomen, van het fluctuatie veld van lokale functies, waar de orde van iedere term gekwantificeerd wordt. Verder, ook in de context van toevalsbewegingen, zijn we in staat om de kwantitatieve Boltzman-Gibbs principes uit te breiden naar een niet-evenwichts omgeving. Deze uitbreiding is mogelijk door gebruik te maken van het feit dat producten van Poisson maten met een traag variërende parameter behouden blijven onder de verandering in de dynamica over de tijd, dat wil zeggen dat een sterke vorm van voortzetting van het lokale evenwicht behouden is in de non-evenwichts context. Tot slot, voor andere IPS, zijn vergelijkbare resultaten behaald onder precieze randvoorwaarden op de deeltjesdynamica.

In hoofdstuk 5, leggen we de basis om de theorie van fluctaties verder te ontwikkelen, door de notie van hogere-orde velden in de context van IPS die de orthogonale (zelf-)dualiteit eigenschap bezitten. Namelijk, onafhankelijke toevalsbewegingen, symmetrische exclusie en inclusie processen. Deze hogere-orde velden zijn afgeleid van de observatie dat de hoeveelheid van belang in de afleiding van de fluctuatie theorema, dat wil zeggen dat de dichtheid fluctuatie veld geschreven kan worden in termen van orthogonale zelf-duale polynomen. In deze nieuwe hierarchie van velden, komen de gewone dichtheid fluctuatie veldne overeen met de eerste orde velden. Door middel van martingaaltechnieken bestuderen we vervolgens de schalende limieten van de k-orde velden. De limiterende k-orde veld wordt gekarakteriseerd door een recursief martingaal probleem, waarbij de

initiele oplossing overeenkomt met een gegeneraliseerd Ornstein-Uhlenbeck proces.

Ten slotte, in hoofdstuk 6, hebben we de convergetnie van een 'sticky' Brownian beweging laten zien voor het verschil in posities van twee SIP-deeltjes in de zin van Mosco-convergentie van Dirichlet vormen. Omdat deze notie van convergentie impliceert convergentie van semigroepen in de L^2-ruimte van reversibele maten, dat is $dx + \gamma\delta_0$ voor de 'sticky' Brownse beweging met stickynessparameter $y > 0$. De convergentie van semigroepen impliceert ook dat de overgangskansen van de vorm $p_t(x, 0)$. Dit, samen met de zelf-dualiteit, draagt bij aan het expliciet verkrijgen van de begrensde variantie van de fluctuatievelden. Dit resultaat biedt een manier van kijken naar stapels deeltjes gevormd gedurende het verruwing proces, zodra deltamassas in de stochastische stroom van 'kernels' die de 'sticky' Brownian beweging definieren.

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Many thanks also go to Dr. Gioia Carinci. Thank you for all your dedicated guidance and support. Thank you for listening to me every time I unexpectedly knocked at your office door to discuss an idea that most of the times was wrong. Thank you for our chats about subjects that perhaps were not completely related to our research. Thank you for your moral support and patience on the diffuclt times. You showed me how to write complex ideas in beautiful and accessible notation. I hope one day I can write mathematics as beautiful as you do.

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Curriculum Vitae

Mario Ayala was born in Tecoman, Colima México. His interest for mathematics was promoted by his participation in the Mexican Mathematical Olympiad and other nation-wide mathematical competitions during his junior-high and high-school studies.

In January of 2007, while attending his Bachelor's degree in mathematics, M.A. started working for the Ministry of Health in the State of Colima. Later, in 2011, M.A. finished his Bachelor's degree in Mathematics at the University of Colima under the supervision of Dr. Carlos M. Hernadez-Suarez. After finishing his Bachelor's degree, M.A. continued working for the Ministry of Health until July 2014. In August 2014, with the support of the Mexican Council of Science and Techonology (CONACYT), M.A. started his Masters degree in Mathematics at Tu Delft. In August 2016, he obtained his Masters Degree with a thesis entitled "Hydrodynamic Limit for the Symmetric Inclusion Process", written under the supervision of Dr. Frank H.J. Redig.

In January of 2017, again with the support of the Mexican Council of Science and Techonology (CONACYT), M,A. started his Ph.D. research under the joint supervision of Prof.dr. Frank H.J. Redig and Dr. Gioia Carinci.

Publications

Submitted

1.- Ayala M., Carinci G. & Redig, F. Higher Order Fluctuation Fields and Orthogonal Duality Polynomials.
arXiv:2004.04812 (2020). In the revision process for *Electronic Journal of Probability.*

2.- Ayala M., Carinci G. & Redig, F. Condensation of SIP Particles and Sticky Brownian Motion. *arXiv:1906.09887* (2019). In the revision process for *Journal of Statistical Physics.*

Published

1.- Ayala M., Carinci G. & Redig, F. Quantitative Boltzmann-Gibbs Principles via Orthogonal Polynomial Duality. *Journal of Statistical Physics* **171(6)**, 980–999 (2018).