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Sharp bound on the truncated metric dimension of trees

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ABSTRACT

A k-truncated resolving set of a graph is a subset $S \subseteq V$ of its vertex set such that the vector $(d_k(s,v))_{s\in S}$ is distinct for each vertex $v \in V$ where $d_k(x,y) = \min\{d(x,y), k+1\}$ is the graph distance truncated at $k+1$. We think of elements of a k-truncated resolving set as sensors that can measure up to distance k. The k-truncated metric dimension (Tmd_k) of a graph G is the minimum cardinality of a k-truncated resolving set of G. We give a sharp lower bound on Tmd_k for any tree T in terms of its number of vertices $|T|$ and the measuring radius k. Our result is that $\mathrm{Tmd}_k(T) \ge |T| \cdot 3/(k^2+4k+3+\mathbb{1}\{k \equiv 1 \pmod 3\}) + c_k$, disproving earlier conjectures by Frongillo et al. that suspected $|T|/(\lfloor k^2/4 \rfloor + 2k) + c'_k$ as general lower bound, where c_k, c'_k are k-dependent constants. We provide a construction for trees with the largest number of vertices with a given Tmd_k value. The proof that our optimal construction cannot be improved relies on edge-rewiring procedures of arbitrary (suboptimal) trees with arbitrary resolving sets, which reveal the *structure* of how small subsets of sensors measure and resolve certain areas in the tree that we call the attraction of those sensors. The notion of 'attraction of sensors' might be useful in other contexts beyond trees to solve related problems. We also provide an improved lower bound on Tmd_k of arbitrary trees that takes into account the structural properties of the tree, in particular, the number and length of simple paths of degree-two vertices terminating in leaf vertices. This bound complements the result of the above-mentioned work of Frongillo et al., where only trees *without* degree-two vertices were considered, except the simple case of a single path.

1. Introduction

The *metric dimension* of graphs is a combinatorial notion first introduced by Slater [34] in 1975, and independently by Harary and Melter [24] one year later. It is the optimal value of a source detection problem described as follows. Let $G = (V, E)$ be a simple, undirected graph, and let d denote the graph distance on its vertex set V, with the convention that $d(x, x) = 0$. We call a subset of vertices $S \subseteq V$ *resolving*, if the vector of graph distances $(d(s,v))_{s\in S}$ is distinct for each vertex $v \in V$. In other words, we imagine that the vertices in S are *sensors* that can measure their distances from each vertex in the graph, and call S a resolving set if each vertex in V can be (uniquely) identified by the measurements of the sensors. Then the metric dimension of G is the smallest cardinality of such a resolving set. This model of source detection is motivated by the problem of finding the unknown source of an infection in a network, based on the measured infection time of a certain subset of individuals, as in [32,38].

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In this paper we consider a modified version of the above problem, where we replace the graph distance d above by its truncated form $d_k(\cdot,\cdot) = \min\{d(\cdot,\cdot), k+1\}$, with k being an integer parameter. This corresponds to limiting the radius of measurement of each sensor vertex to k, i.e., not allowing the sensor to distinguish between vertices that are further away than k. A set $S \subseteq V$ is called a *k-truncated resolving set* if S is a resolving set under the metric induced by $d_k(\cdot,\cdot)$, and additionally for every $v \in V$ there is some $s \in S$ with $d_k(s,v) \le k$. The smallest cardinality of such a set is the *k-truncated metric dimension* of G, denoted by $\mathrm{Tmd}_k(G)$. In this paper we study the truncated metric dimension of *trees* (cycle-free connected graphs), see Section 2 below for the usual definition. Trees, regarding the (classic) metric dimension are analytically tractable, see the work of Slater [34] which introduced the notion of metric dimension, and studied it first on trees. It is thus a natural question whether trees are also analytically tractable regarding the truncated version of the metric dimension. Our main result is a worst-case lower bound on $\mathrm{Tmd}_k(G)$ based on the number of vertices in G when G is a tree.

Theorem 1.1 *(Lower bound on the k-truncated metric dimension). Let T be any tree on $n \ge 1$ vertices. Then for all $k \ge 1$,*

$$\mathrm{Tmd}_k(T) \ge \left\lceil \frac{3n + k^2 + k + \mathbb{1}\{k \equiv 1 \pmod 3\}}{k^2 + 4k + 3 + \mathbb{1}\{k \equiv 1 \pmod 3\}} \right\rceil.$$

These bounds are sharp, in the sense that for any positive integers n and k there exists a tree T on n vertices, which satisfies the respective bound. We prove Theorem 1.1 by identifying the size (number of vertices) of the largest tree with a given Tmd_k (see Proposition 2.4 below), which turns out to be $\Theta(mk^2/3)$. We also provide a construction of trees of optimal size, and from our proof it follows that the optimal tree is non-unique. In fact, for each m, the number of largest-size (i.e., optimal) trees that can be resolved by m sensors is at least as large as the number of non-isomorphic unlabeled trees on m vertices.

The largest part of our paper is devoted to the proof that no tree on n vertices can be measured by less sensors than the lower bound in Theorem 1.1. In principle, there could be two proof strategies to show such a lower bound. Either one 'spares' a sensor on any suboptimal tree, i.e., one shows that the tree can be resolved by fewer sensors if it does not follow the optimal construction. This strategy however, does not work, since it is not hard to construct suboptimal trees and resolving sets where one cannot spare a sensor: such an example is a star-graph (a central vertex connected to $n-1$ leafs). The star-graph needs at least $n-2$ sensors for all k and it is not hard to show that no sensor can be removed.

The second possible strategy to show that a given labeled tree is suboptimal is to keep the sensors in place and add a new vertex to the tree, while still ensuring that every vertex is uniquely resolved. We follow this latter proof strategy. We add a vertex via a series of 'transformations', which can be applied to any tree *not* following the optimal construction. These transformations all preserve Tmd_k and do not decrease the number of vertices. In more detail: given a labeled tree with a resolving set that does not follow the optimal construction, we slightly modify the edge set by rewiring a few edges and possibly adding a few labeled vertices, and show that the obtained (possibly) larger tree is still measured by the same sensor vertices, which violates the assumption that the tree was largest possible. Since the optimal tree is non-unique, these transformations either do not change the number of vertices and result in an optimal construction that we describe, or else, when we could add a vertex, they result in a tree T' on a strictly larger vertex set with $\mathrm{Tmd}_k(T') = m$.

Some notions that we introduce during the proofs, especially what we call *attraction of sensors*, might be useful in other contexts as well, because they uncover the structure of the vertices measured by a subset of sensors and as a result they reveal where optimality may be violated.

Besides giving a worst-case lower bound on $\mathrm{Tmd}_k(T)$ in Theorem 1.1, we also provide a sharper lower bound for certain suboptimal trees, which takes into account the *structural properties* of the particular tree in question (see Theorem 2.9). This bound is obtained by providing a locally optimal placement of sensors around certain degree-two paths terminating in leaves of the tree, and then applying Theorem 1.1 for the rest of the graph. In comparison to [22], which identifies the k-truncated metric dimension of a certain subclass of trees *without* degree-two vertices, our lower bound builds on exactly these degree-two structures. While our lower bound might be suboptimal on most trees, there are trees (with leaf-paths) on which it provides sharp bounds, hence in some sense the bound cannot be improved, at least not in full generality.

1.1. Related work and open questions

Algorithmic aspects. The question of finding the metric dimension of graphs has been extensively studied from an algorithmic point of view. The problem on general graphs is NP-hard [29], and can only be approximated up to a factor $\log n$ [3,25]. For parameter values $k \ge 2$, k-truncated metric dimension of general graphs is also an NP-hard problem [13,21]. For trees, on the other hand, [29] provides a simple linear-time algorithm for the computation of the metric dimension, writing it as the difference between the number of leaves and the number of vertices that have degree at least 3 and are the endpoints of at least one simple path in the tree (which we will call a *leaf-path*). This idea is closely related to our approach to the improved lower bound in Theorem 2.9. As for the k-truncated metric dimension for $k = 1$, known as the location-domination number, the early work of Slater [35,36] shows that the location-domination number can be computed in linear time on trees, and gives a lower bound on its value, which was later improved by [4], and further improved by [37]. We are unaware of such a linear-time algorithm for the k-truncated metric dimension on trees for general $k \ge 2$.

Graph theoretical aspects. Many aspects of the metric dimension have been studied from the graph theoretic point of view as well, including bounds in terms of the diameter of the graph [8,26], bounds for certain Cayley graphs [18], Cartesian products of graphs [7], and the metric dimension of infinite graphs [6], and wheel graphs [5,33].

Modifications of the metric dimension. We also mention a few results on some modified versions of the metric dimension problem. A more general version of the problem is the k-metric dimension, where every pair of vertices needs to be resolved by at least k sensors (and where this notation k is used differently from our paper), has been studied in [11, 12,14–16]. A further variant of this circle of problems is the *fractional metric dimension*, introduced in [10], which is the linear programming relaxation of the integer programming problem encoding the identification of the metric dimension of a graph. Further work on this concept includes [1,2,20,19].

Slightly less related to our work are the concepts of r-*identifying codes* and r-*locating-dominating codes*, where a sensor can measure up to distance r but cannot distinguish between vertices within this radius (in the case of the identifying code), except itself (in the case of r-locating-dominating codes). The authors of [9] provide the sizes of the optimal r-identifying codes of paths and cycles, which asymptotically coincide with the optimal size of a 1-identifying code, i.e., it has density $1/2$. Similar results for the r-locating-dominating code for cycles in [17] show that the optimal code in this case also has the same asymptotic density, $1/3$, as in the $r = 1$ case. For other work on this topic, see the references within [9].

Applied motivation. The particular version of the model that we study in this paper, the truncated metric dimension, is inspired by the scenario where the sensors can only make noisy measurements in a source detection problem: the noise accumulates over distance, and above a certain threshold the measurements become unreliable, as in [32,38,31]. In particular, if a Gaussian noise with variance $\varepsilon \ll 1$ is added along each edge, then detecting the source of infection from any given source breaks down roughly at distance of order $1/\varepsilon$. Hence, the metric dimension problem with noisy observations roughly corresponds to the truncated metric dimension problem with truncation radius around $1/\varepsilon$ (see reference [31]). Works where the k-truncated metric dimension has been introduced include [13,39,22] and [23], and for $k = 1$ the truncated metric dimension is equivalent to the locating-dominating code problem, introduced by Slater earlier in [36].

Open questions. Next, we mention questions that our paper leaves open. The question of (algorithmically) quickly finding an optimal arrangement of sensors stays open on trees for general $k \geq 2$. Our proof of Theorem 2.9 gives partial answer by finding the optimal placement on certain leaf-paths, parts of the tree that are simple paths leading to a leaf vertex. For the rest of the tree, however, only the size-dependent lower bound in Theorem 1.1 is used.

For the probability community large random trees (of various distributions) are important for several reasons. Some of the most commonly used non-spatial, sparse (i.e., finite average degree) random graph models are locally tree-like, e.g. the Erdős–Rényi random graph, the configuration model, preferential attachment models, etc. Moreover, combinatorial problems on trees are often better understood. See e.g. the surveys of [28,27], where the authors establish law of large numbers for various combinatorial problems on classes of random tree models, including random binary search trees, m-ary search trees, preferential attachment trees and conditioned Galton–Watson trees. The optimal resolving set – for the classic metric dimension – of a tree is local in the following sense: whether a vertex is a sensor or not is entirely determined by the structure of the subtree of the vertex pointing away from the root. This then implies law of large numbers for the metric dimension of a large class of random trees [30], including the above-mentioned ones. We conjecture that the truncated metric dimension of a tree is also a local property, and expect law of large numbers to hold for at least the random trees treated in [28].

The rest of the paper is structured as follows. In Section 2 we state our results after providing the necessary notation and definitions, and give a short sketch of our proofs. In Section 3 we introduce further definitions and concepts that will be used throughout the paper. Sections 4, 5 and 6 contain the three transformations that form the three main steps of the proof of Theorem 1.1. In Section 7 we complete the proof of Theorem 1.1, while also giving a construction for trees of optimal size. Finally, in Section 8 we prove the structural lower bound, Theorem 2.9.

2. Main results

We will start by fixing the notation that we will use throughout the paper. For a set V let $V^{(2)}$ denote the 2-element subsets of V. A (simple, undirected) *graph* G is an ordered pair (V, E) where V is a set of *vertices* and $E \subseteq V^{(2)}$ is a set of *edges*. We say that $e \in E$ *connects* its two vertices in the graph. We will sometimes write $V(G)$ and $E(G)$ for the vertex and edge sets, respectively, of a given graph G. Somewhat abusing the notation we will sometimes write $v \in G$ instead of $v \in V(G)$ for a vertex of G, as is standard. The *size* of a graph G, denoted by $|G|$, is the cardinality of its vertex set. A *subgraph* of G is a pair (V', E') such that $V' \subseteq V$ and $E' \subseteq E \cap V'^{(2)}$. A subgraph *induced* by $V_1 \subseteq V$ in G is the pair (V_1, E_1) where $E_1 = E \cap V_1^{(2)}$. A *path* is a graph P such that $V(P) = \{v_i\}_{i=1}^{k+1}$, for some $k \geq 1$, and $E(P) = \{\{v_i, v_{i+1}\}\}_{i=1}^{k}$. We call v_1, v_{k+1} the end vertices of the path. A *cycle* is a graph C such that $V(C) = \{v_i\}_{i=1}^{k}$, $k \geq 3$, and $E(C) = \cup_{i=1}^{k-1}\{\{v_i, v_{i+1}\}\} \cup \{\{v_k, v_1\}\}$. The *length* of a path P or a cycle C is the number of edges in it. A graph G is *connected* if for any pair $u, v \in V(G)$ there is a path in G (as a subgraph) that contains both u and v. We call a graph a *forest* if it has no cycles in it, and we call a connected forest a *tree*. The degree of a vertex v, denoted by $\deg(v)$, is the number of edges containing v. A vertex of degree one is called a *leaf*. We call a path P a *leaf-path* in G if P is the induced subgraph of G on the vertices $V(P)$, and, with respect to G, one end vertex of P is a leaf, the other has degree at least 3, and all its other vertices have degree 2. The *distance* between two vertices u, v in G, denoted by $d_G(u, v)$, is the length of the shortest path between u and v in G, with the convention that $d_G(u,u)=0$. Here we omit the subscript when the underlying graph is clear from the context. Finally, we denote by $k \wedge l = \min\{k,l\}$, and $d_k(x,y) := d(x,y) \wedge (k+1)$.

Next we define the main topic of this paper.

Definition 2.1. Let $G=(V,E)$ be an arbitrary simple, undirected graph, and fix an integer threshold $k \geq 1$. We say that a vertex s *resolves* (or equivalently, *distinguishes*) a pair of vertices $x,y \in V$ if $d_k(s,x) \neq d_k(s,y)$. A *k-truncated resolving set* is a subset S of V such that for every pair of vertices $x,y \in V$ there is some vertex $s \in S$ such that s resolves x,y, and for every $x \in V$ there is some $s \in S$ such that $d_k(s,x) \leq k$. The k-truncated metric dimension of G, denoted by $\mathrm{Tmd}_k(G)$, is the smallest integer m such that there exists a k-truncated resolving set S for G with $|S|=m$.

We call the elements of a k-truncated resolving set *sensors*. We say that a vertex x is *measured by* a sensor s if $d_k(s,x) \leq k$.

Our main result, Theorem 1.1, readily implies the result of Slater [35] about the 1-truncated metric dimension, which is identical to the locating-dominating number of the tree.

Corollary 2.2 *(Lower bound on the locating-dominating number [35]). Let T be any tree on n vertices. Then*

$$\mathrm{Tmd}_1(T) \geq \left\lceil \frac{n+1}{3} \right\rceil.$$

To prove Theorem 1.1 we will identify the largest trees with a given k-truncated metric dimension. To state that result we introduce some notation first.

Definition 2.3. Fix $k \geq 1$. We denote the set of trees with k-truncated metric dimension m by $\mathcal{T}_m = \mathcal{T}_m(k)$, and $\mathcal{T}_m^\star = \mathcal{T}_m^\star(k)$ denotes the set of trees with the largest possible size within $\mathcal{T}_m$:

$$\mathcal{T}_m := \{T \text{ tree} : \mathrm{Tmd}_k(T) = m\},$$
$$\mathcal{T}_m^\star := \{T^\star \in \mathcal{T}_m : |T^\star| = \max_{T \in \mathcal{T}_m}(|T|)\}.$$

We then identify the maximal size of a tree that can be measured by m sensors.

Proposition 2.4. *For all $k \geq 1$, and any $T^\star \in \mathcal{T}_m^\star$,*

$$\begin{aligned}|T^\star| &= m \cdot \frac{k^2+4k+3+\mathbb{1}\{k \equiv 1 \pmod 3\}}{3} - \frac{k^2+k+\mathbb{1}\{k \equiv 1 \pmod 3\}}{3} \\ &= (k+1)m + (m-1)(k^2+k+\mathbb{1}\{k \equiv 1 \pmod 3\})/3.\end{aligned} \tag{1}$$

We give the proof of Proposition 2.4 in Section 7 after establishing preliminary results in Sections 3, 4, 5 and 6. The idea of the proof is the following. We start with a tree with a k-truncated resolving set S on it, such that $|S|=m$. We show that the edges of this tree can be rewired by a series of transformations that do not decrease the number of vertices, keep S a k-truncated resolving set, and result in a construction that can be easily optimized to give an element of $\mathcal{T}_m^\star$. We discuss the three required transformations in Sections 4, 5 and 6, respectively.

Theorem 1.1 is then immediate from Proposition 2.4.

Proof of Theorem 1.1. Let T be a tree on n vertices for which $\mathrm{Tmd}_k(T)=m$. Then, by Definition 2.3, we have that $n \leq |T^\star|$ for any $T^\star \in \mathcal{T}_m^\star$. Combining this with (1) of Proposition 2.4 and rearranging yields Theorem 1.1. □

We can contrast Theorem 1.1 and Proposition 2.4 to the results of Frongillo et al. [40] (updated recently by the authors [22]). The authors of that paper conjecture that the leading coefficient of m in Proposition 2.4 should be $\lfloor k^2/4 \rfloor$, while we find that it is, in fact, $(k^2+4k+4)/3$ when $k \equiv 1 \pmod 3$ and $(k^2+4k+3)/3$ otherwise, which is higher. The underestimation of the size of the optimal tree comes from assuming that the distance between two neighboring sensors in the tree is exactly k for all k, when in fact this is a parameter that can be optimized and is the nearest integer to $(2k+1)/3$.

Remark 2.5. Recall from Definition 2.1 that we require for a k-truncated resolving set S that every vertex $v \in V$ be measured by at least one sensor in S. We use this convention to make the presentation of our results slightly more convenient. Omitting this requirement would simply add one extra vertex to the optimally-sized trees in $\mathcal{T}_m^\star$ that is not measured by any sensor, and would change the bounds in Theorem 1.1 accordingly.

Next, we give an improved structural lower bound on the truncated metric dimension of suboptimal trees as well. The idea is that having multiple leaf-paths emanating from a common vertex v is very costly in terms of how many sensors are needed to identify them. Since sensors that are not part of such leaf-paths can only measure such paths via v, they cannot distinguish vertices at equal distance from v located on two different leaf-paths. Thus, we can compute how many sensors such a system of leaf-paths minimally requires. For the 'rest' of the tree, we then essentially use the optimal bound that we developed in Theorem 1.1. Our lower bound is valid for any tree, but gives fairly sharp lower bounds only for trees that have relatively large number of leaf-paths. To be able to state the result, we start with some definitions.

Definition 2.6 *(Leaf paths and support vertices).* We will write $\mathcal{L}_v = \{\mathcal{P}_j^{(v)}\}$ for the collection of leaf-paths starting at a vertex v with $\deg v \geq 3$, and denote their number by $L_v = |\mathcal{L}_v|$. The length (number of edges) of a leaf-path $\mathcal{P}_j^{(v)}$ will be denoted by $\ell(\mathcal{P}_j^{(v)}) = \ell(v, j)$. Define $F_T = \{v \in T : \deg(v) \geq 3, L_v \geq 2\}$ to be the set of *s*upport vertices of T.

Definition 2.7. Fix $k \geq 1$. For an integer $\ell \geq 1$, let q and r be the non-negative integers such that $\ell = q(3k+2) + r$, and $r \leq 3k+1$. Define the *u*pper and *l*ower complexity, respectively, of a path of length ℓ to be

$$\bar{c}(\ell) := 2q + \mathbb{1}\{r \geq 1\} + \mathbb{1}\{r \geq 2k+2\}, \quad \text{and}$$
$$\underline{c}(\ell) := 2q + \mathbb{1}\{r \geq k+1\} + \mathbb{1}\{r \geq 2k+2\}.$$

The next lemma identifies how many sensors a system of leaf-paths minimally requires. We place the lemma here since it might be useful also for algorithmic aspects. In the proof, below in Section 8, we also provide the location of the sensors mentioned in the lemma.

Lemma 2.8. *If S is a k-truncated resolving set on T, and $v \in F_T$, then all but at most one of the vertex sets $V(P_j^{(v)}) \setminus \{v\}$ for $P_j^{(v)} \in \mathcal{L}_v$ contain at least $\bar{c}(\ell(v, j))$ sensors in S, while $V(P^\star) \setminus \{v\}$ for the remaining path $P^\star \in \mathcal{L}_v$ contains at least $\underline{c}(\ell(P^\star))$ sensors in S.*

To determine which path shall be the special path $P^\star$ in Lemma 2.8, for a lower bound we subtract the difference between the upper and the lower complexity for each path $P_j^{(v)}$ (since this is the number of sensors 'spared' by choosing $P_j^{(v)}$ to be the special path $P^\star$) and maximize it over paths in $\mathcal{L}_v$.

Then the minimal number of sensors that need to be placed on $\cup_{j \leq L_v} V(P_j^{(v)}) \setminus \{v\}$ for some $v \in F_T$ is at least

$$R(\mathcal{L}_v) = \sum_{j=1}^{L_v} \bar{c}(\ell(v, j)) - \max_{1 \leq j \leq L_v} \left\{ \bar{c}(\ell(v, j)) - \underline{c}(\ell(v, j)) \right\}. \tag{2}$$

As a combination of Theorem 1.1 and Lemma 2.8, and (2) we get the general lower bound on the k-truncated metric dimension of trees:

Theorem 2.9. *Let T be a tree with n vertices and fix $k \geq 1$. Then*

$$\mathrm{Tmd}_k(T) \geq \left\lceil \frac{3n - 3\sum_{v \in F_T} \sum_{j=1}^{L_v} \ell(v, j) + k^2 + k + \mathbb{1}\{k \equiv 1 \pmod 3\}}{k^2 + 4k + 3 + \mathbb{1}\{k \equiv 1 \pmod 3\}} \right\rceil + \sum_{v \in F_T} R(\mathcal{L}_v) - |F_T|. \tag{3}$$

Observe that the sum $\sum_{v \in F_T} \sum_{j=1}^{L_v} \ell(v, j)$ is the total number of vertices that are on leaf-paths emanating from support vertices.

To obtain the structural lower bound in Theorem 2.9, we show that each system of leaf-paths connecting to the same support vertex needs at least as many sensors as in (2). Once these leaf-paths are resolved, the sensors on them can measure some vertices in the rest of the tree, and resolve vertices there, but they cannot measure further than their support vertex would, if it was a sensor. Combining this argument with the lower bound on the number of sensors on the rest of the tree from Theorem 1.1 allows us to find a lower bound: adding the number of sensors on leaf-paths to the number of sensors the skeleton would need if it was an optimal tree, and then finally subtracting the number of support vertices. We provide the proof in Section 8.

3. The attraction of sensors

Before the proofs we first introduce some further notions, that not only will be crucial in our proofs, but we believe they could be useful in other contexts as well. As it will become clear later, the construction of the optimal trees is centered around the paths between sensors and the structure on 'how' they measure vertices with respect to other sensors, that we call *direct* measuring, and a related notion of *attraction* of sensors below.

Definition 3.1 *(Paths).* For a tree $T=(V,E)$ and any pair of distinct vertices $x,y\in V$ we denote the unique path in T between x and y by $\mathcal{P}_T(x,y)$, its vertex set by $V(\mathcal{P}_T(x,y))$, and its edge set by $E(\mathcal{P}_T(x,y))$. We will omit the subscripts when the underlying graph is clear from the context.

Definition 3.2 *(Weak and strong sensor paths).* Given a tree $T=(V,E)$, a set of sensors $S\subseteq V$ and a distinct pair $s_1,s_2\in S$, we call $\mathcal{P}_T(s_1,s_2)$ a *sensor path* if it does not contain any other sensors beside s_1 and s_2. $\mathcal{P}_T(s_1,s_2)$ is called a *strong sensor path* if it is a sensor path and $|E(\mathcal{P}_T(s_1,s_2))|\le k+1$. A sensor path that is not strong is called *weak*.

Definition 3.3 *(Measuring and direct measuring).* Given a tree $T=(V,E)$ and a set of sensors $S\subseteq V$, we say that a sensor s *measures* a vertex x in T if $d_T(s,x)\le k$. In this case we further say that s *directly measures* x, if it also holds that $s'\notin V(\mathcal{P}_T(s,x))$ for all $s'\in S\setminus\{s\}$.

We will introduce a concept that we call *minimally resolving* that will be crucial in our proofs. To ease the reader into the fairly non-obvious definition, we start with a simpler definition that is more intuitive:

Definition 3.4 *(Resolved within a subset of sensors).* Let $T=(V,E)$ be a tree with a k-truncated resolving set S. We say that a vertex $x\in V\setminus S$ is *resolved within* the sensors $S'=\{s_1,\dots,s_r\}\subseteq S$ in T if
(i) $d(s_i,x)\le k$ for at least one $i=1,\dots,r$,
(ii) there is *no* sensor in $S\setminus S'$ that directly measures x,
(iii) for all $y\in V$ for which (ii) holds, there is some $s_i\in S'$ that resolves x and y.

Note that (ii) is equivalent to the following: every sensor $s^\star$ in $S\setminus S'$ either does not measure x or has $d(s^\star,x)=d(s^\star,s_i)+d(s_i,x)$ for some $i\le r$. Heuristically, x is resolved within S' if all sensors not in S' can only measure x via a path crossing some sensor in S', and x is distinguished by S' from all other vertices having the same property. Observe that in (iii), the condition that (ii) holds implies that (i) also holds for y. Indeed, if (ii) holds for y but (i) does not, then either there is a sensor $s^\star\notin S'$ measuring y, such that $d(s',y)\le d(s^\star,y)\le k$ for some $s'\in S'$ by (ii), which is a contradiction, or y cannot be measured by any sensor, and we assumed throughout the paper that we only consider trees where such a vertex is not present in T, a contradiction again.

Definition 3.5 *('Resolved-within area' of a set of sensors).* Let $T=(V,E)$ be a tree with a k-truncated resolving set $S\subseteq V$. The *resolved-within area* of a set of sensors $S'\subseteq S$ is

$$M_T(S')=\{x\in V\setminus S: x \text{ is resolved within } S'\}.$$

It is not hard to see that the 'resolved-within' area is monotone under containment, i.e., when $B\subset S'$, then $M_T(B)\subseteq M_T(S')$, and if S is a k-truncated resolving set for T, then $M_T(S)=V\setminus S$.

The next definition decomposes $M_T(S')$ into *disjoint subsets*: heuristically speaking, starting from the set of single sensors, and increasing the set-size gradually, for a vertex x it finds the minimal set of sensors needed that can distinguish x from all other vertices via direct measuring.

Definition 3.6 *(Minimally resolving).* Let $T=(V,E)$ be a tree with a k-truncated resolving set S. A subset of sensors $S'\subseteq S$, $S'=\{s_1,\dots,s_r\}$ is said to *minimally resolve* a vertex x in T if
(i) $d(s_i,x)\le k$ for all $i=1,\dots,r$,
(ii) there is *no* sensor in $S\setminus S'$ that directly measures x,
(iii) for all $y\in V$ for which (ii) holds, there is some $s_i\in S'$ that resolves x and y,
(iv) all $s_i\in S'$ directly measure x, i.e., $d(s_i,x)\neq d(s_j,x)+d(s_i,s_j)$ for $i,j=1,\dots,r$, $i\neq j$.

For a single sensor s, Definitions 3.4 and 3.6 are identical. For more sensors, the difference between Definition 3.5 and 3.6 is in parts (i) and the new criterion (iv): heuristically, a set of sensors minimally resolve a vertex x if they *all directly measure it*, (i.e., the shortest paths leading to the vertex x do not contain fully each other), and the set is minimal in the sense that no other sensor directly measures x by part (ii). Part (iii), similarly to Definition 3.5(iii), ensures that an $x\in V$ for which all the conditions hold is distinguished from all vertices y in the resolved-within area of S'.

We call the vertices that are minimally resolved by a set of sensors the attraction of the sensor set S', and this is our next definition.

Definition 3.7 *(Attraction of a set of sensors).* Let $T=(V,E)$ be a tree with a k-truncated resolving set $S\subseteq V$. The *attraction* of a set of r sensors $S'=\{s_1,\dots,s_r\}\subseteq S$ is

$$A_T(s_1,\dots,s_r)=\{v\in V\setminus S: v \text{ is minimally resolved by } \{s_1,\dots,s_r\}\}.$$

We will omit the subscript T when the underlying graph is clear from the context.

It is not hard to see that $A_T(s) = M_T(s)$ for a single sensor $s \in S$, $A_T(S')$ and $A_T(S'')$ are disjoint whenever $S' \neq S''$, and that

$$M_T(S') = \bigcup_{B \subseteq S'} A_T(B).$$

Heuristically, if a path between a sensor and a vertex does not contain any sensor from S' then none of the vertices of this path can be in the attraction of S'. More formally, we will use the following straightforward claim in our proofs:

Claim 3.8. *For a tree $T = (V, E)$ with k-truncated resolving set S, let $S' \subseteq S$ be any subset of sensors, and let $s \in S \setminus S'$. Assume that $x \in V$ is measured by s, and $V(\mathcal{P}_T(s, x))$ is disjoint from S'. Then $V(\mathcal{P}_T(s, x))$ is also disjoint from $A_T(S')$.*

Proof. Assume that $y \in V(\mathcal{P}_T(x, s))$. Since x is measured by s, we have $d_T(s, y) \le d_T(s, x) \le k$. Hence, y is also measured by s. Since $s \notin S'$, and $V(\mathcal{P}_T(s, y))$ does not contain any sensor from S', it follows that $y \notin A_T(S')$, otherwise Definition 3.6(ii) would be violated. □

Before we continue, we make a few claims, and a few definitions about the structure of the attraction of one or two sensors.

Claim 3.9 *(Size of the attraction of a single sensor). The size of $A_T(s)$, for any sensor $s \in S$, is at most k.*

Proof. It follows simply from the fact that for each distance $1 \le j \le k$, there can only be a single vertex at graph distance j from s belonging to $A_T(s)$. Suppose to the contrary that for some j there are two vertices $x, x' \in A_T(s)$ with $d(s, x) = d(s, x') = j$. Then s cannot distinguish between x, x', hence Definition 3.6 (iii) is violated. □

Claim 3.10 *(Structure of the attraction of a pair of sensors). If $v \in V$ belongs to $A_T(s_1, s_2)$ for a pair of sensors $s_1, s_2 \in S$, then either $v \in V(\mathcal{P}_T(s_1, s_2)) \setminus \{s_1, s_2\}$ or v is connected by a path to a vertex in $V(\mathcal{P}_T(s_1, s_2)) \setminus \{s_1, s_2\}$ which does not contain any sensor.*

Proof. First, it is not possible that $\mathcal{P}(v, s_1)$ fully contains $\mathcal{P}(v, s_2)$, or vice versa, otherwise Definition 3.6 (iv) would be violated. Hence, $\mathcal{P}(v, s_1)$ and $\mathcal{P}(v, s_2)$ both intersect $V(\mathcal{P}_T(s_1, s_2)) \setminus \{s_1, s_2\}$. Therefore, there are two possibilities for the location of v. Either $v \in V(\mathcal{P}_T(s_1, s_2)) \setminus \{s_1, s_2\}$ or v is connected by a path to a vertex in $V(\mathcal{P}_T(s_1, s_2)) \setminus \{s_1, s_2\}$ that does not contain either s_1 or s_2. Further, there cannot be any third sensor on this path, otherwise $\{s_1, s_2\}$ would not minimally resolve v by Definition 3.6 (ii). □

Based on Claim 3.10, we more generally define a type and a height of a vertex with respect to a pair of sensors s, s', which we will use in Sections 4–6.

Definition 3.11 *(Type and height with respect to a sensor pair).* Consider a tree T with a k-truncated resolving set S, and let $s, s' \in S$ and $v \notin \{s, s'\}$. We define

$$\mathrm{typ}_{s,s'}(x) := \big(d(x, s) - d(x, s') + d(s, s')\big)/2, \tag{4}$$

and we say that x is of type j ($j = \mathrm{typ}_{s,s'}(x)$) with respect to s, s'. We further define

$$\mathrm{hgt}_{s,s'}(x) := d_T(x, s) - \mathrm{typ}_{s,s'}(x), \tag{5}$$

and we say that it is the height of the vertex x with respect to s, s'.

A short interpretation: $x \in V$ is of type 0 if the shortest path $\mathcal{P}_T(x, s')$ from x to s' fully contains $\mathcal{P}_T(x, s)$, and x is of type $d(s, s')$ if the situation is reversed. For $1 \le j \le d(s, s') - 1$, $x \in V$ is of *type* j if the closest vertex to x on the path $\mathcal{P}_T(s, s')$ is of distance j from s. Similarly, the *height* of a vertex $x \in V$ with respect to s, s' is the distance of x from the path $\mathcal{P}_T(s, s')$.

Based on these definitions, the following claims are direct consequences.

Claim 3.12 *(Type-difference). Consider a tree T with a k-truncated resolving set S, and let $s, s' \in S$ and $x, x' \notin \{s, s'\}$ satisfying that x is measured by s and x' is measured by s'. Then, if additionally $\mathrm{typ}_{s,s'}(x) \neq \mathrm{typ}_{s,s'}(x')$ then the set $\{x, x'\}$ is resolved by $\{s, s'\}$.*

Proof. First, if x is measured only by s and not by s', then $d_k(s', x) = k + 1 \ge d_k(s', x')$, hence s' resolves the pair x, x'. Analogously, if x' is measured only by s' but not by s then s resolves the pair x, x'. The only remaining case is when both vertices are measured by both sensors. In this case, since x, x' have different types, by (4)

$$d_k(x,s) - d_k(x,s') = d(x,s) - d(x,s') \neq d(x',s) - d(x',s') = d_k(x',s) - d_k(x',s'),$$

hence, at least one of s, s' resolves x, x'. □

Claim 3.13. *Consider a tree T with a k-truncated resolving set S, and let $s, s' \in S$. Then the following is true.*

(i) *There are no two vertices $x, y \in A_T(s,s')$ with $\mathrm{typ}_{s,s'}(x) = \mathrm{typ}_{s,s'}(y)$ and $\mathrm{hgt}_{s,s'}(x) = \mathrm{hgt}_{s,s'}(y)$.*
(ii) *For any $1 \le j \le d(s,s') - 1$ there are at most $k - \max\{j, d(s,s') - j\} + 1$ vertices in $A_T(s,s')$ with type j (with respect to s, s').*

Proof. Part (i) is a consequence of the fact that $(d_T(x,s), d_T(x,s'))$ is a one-to-one function of $(\mathrm{typ}_{s,s'}(x), \mathrm{hgt}_{s,s'}(x))$. Hence, if we had two vertices $x, y \in A_T(s,s')$ with $\mathrm{typ}_{s,s'}(x) = \mathrm{typ}_{s,s'}(y)$ and $\mathrm{hgt}_{s,s'}(x) = \mathrm{hgt}_{s,s'}(y)$, then no sensor would resolve them. To show part (ii), notice that the possible pairs of distances of a type j vertex from s and s', respectively, are $(j, d(s,s') - j), (j+1, d(s,s') - j + 1), \ldots$ with the pair of largest distances being either $(k, d(s,s') + k - 2j)$ or $(2j + k - d(s,s'), k)$, depending on whether $j \ge (d(s,s'))/2$ or not (these pairs of distances correspond to heights $0, 1, \ldots, k - \max\{j, d(s,s') - j\}$). □

In the setting of part (ii) of Claim 3.13, observe that it could happen that a type j vertex is not in $A_T(s,s')$ for some $j \in \{1, \ldots, d(s,s') - 1\}$ even if its height is at most $k - \max\{j, d(s,s') - j\}$, namely, when that vertex is directly measured by a third sensor s''.

The next three sections are structured as follows. In each of the Sections 4, 5, 6, we introduce a different rewiring procedure called Transformation A, B, C, respectively. Then, we gather some necessary preliminary claims about them, before we state and prove their main properties. Finally we use these transformations to obtain a modified version of our tree, which we can use to proceed further towards a structure that we will be able to optimize in Section 7.

4. Transformation A: moving the attractions of single sensors into leaf-paths

To prove Proposition 2.4, we take any tree T with a k-truncated resolving set S on it, and start by rewiring its edges such that attractions of single sensors become leaf-paths. This will be achieved by the repeated application of Transformation A, introduced in this section. After defining it, we prove several properties of the transformation before proving Lemma 4.4, the main result of this section. It states that the tree T can be transformed in a way (using Transformation A) such that its vertex set does not change, S remains a k-truncated resolving set, and every sensor has its attraction contained in a leaf-path starting from that sensor. This will make further transformations possible that can increase the size of the tree as well.

Recall from Claim 3.9 that $|A_T(s)| \le k$ for any sensor s.

Definition 4.1 *(Transformation A).* Given a tree $T = (V, E)$ with a k-truncated resolving set $S \subseteq V$ and some fixed $s \in S$, let $A_T(s) = \{v_1, \ldots, v_\ell\}$ for some $\ell \le k$. Denote the connected components of T spanned on $V \setminus A_T(s)$ by $\widetilde{T}_0, \widetilde{T}_1, \ldots, \widetilde{T}_r$ for some r, where $\widetilde{T}_0$ contains s. For each $1 \le i \le r$ let x_i be the (unique) vertex in $\widetilde{T}_i$ that is closest to s in T. Define the edge sets

$$E_1 := \{\{u, v\} \in E(T) : u \in A_T(s) \text{ or } v \in A_T(s)\}, \tag{6}$$

$$E_2 := \left\{\{s, v_1\} \cup \left(\cup_{i=1}^{\ell-1} \{v_i, v_{i+1}\}\right)\right\}, \tag{7}$$

$$E_3 := \{\{s, x_i\} : 1 \le i \le r\}. \tag{8}$$

Then define $\mathrm{tr}_A(T, S, s) = (V, E')$ where $E' = (E \setminus E_1) \cup E_2 \cup E_3$.

For an example of Transformation A see Fig. 1.

A couple of comments on this definition: E_1 is the set of edges that are adjacent to the vertices in $A_T(s)$ in T. After removing E_1, E_2 rewires $A_T(s)$ into a leaf-path emanating from s ending at v_ℓ. E_3 connects the components on T spanned on $V \setminus A_T(s)$ back together, by connecting s to the (originally) closest vertex x_i in each of the other components $\widetilde{T}_i$. Note that the vertices x_i in the above definition are indeed well-defined, since if some $\widetilde{T}_i$ had two closest vertices to s in T, then they would lie on a cycle in T.

We observe that $\mathrm{tr}_A(T, S, s)$ is indeed a tree, i.e., connected, since the addition of the edge set E_3 to $\widetilde{T}$ adds exactly one connection between the components $\widetilde{T}_0$ and $\widetilde{T}_i$ for each $i = 1, 2, \ldots, r$, and all the vertices in $A_T(s)$ are connected to s via a leaf-path. See Fig. 1 for an illustration.

We start with a basic property related to tr_A, ensuring that sensors in the components $\widetilde{T}_1, \ldots \widetilde{T}_r$ can only measure vertices within their own component.

Claim 4.2 *(No 'communication' between different subtrees). Consider the notation of Definition 4.1, and let $T' := \mathrm{tr}_A(T, S, s)$.*

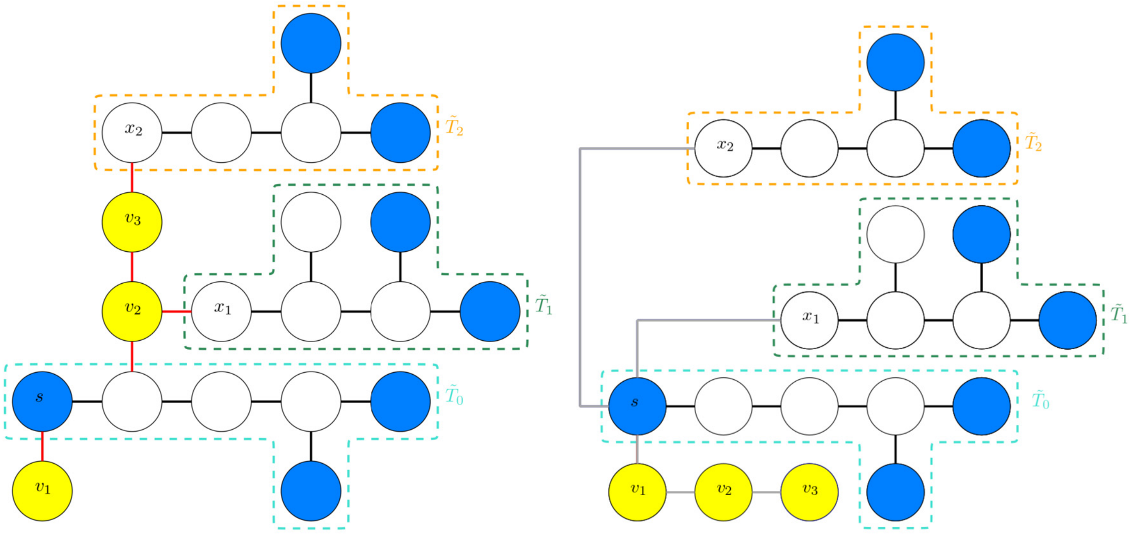

Fig. 1. An example of transformation A with T on the left and $T' = \mathrm{tr}_A(T, S, s)$ on the right. Here $k = 3$. The blue vertices are the sensors in S, and the yellow vertices are in $A_T(s)$. The red edges belong to E_1, and are deleted by the transformation. The grey edges belong to $E_2 \cup E_3$, and are added by the transformation. The subtrees $\widetilde{T}_0, \widetilde{T}_1$ and $\widetilde{T}_2$ are also highlighted. As an illustration of Claim 4.2, we can observe that sensors in $\widetilde{T}_1$ and in $\widetilde{T}_2$ only measure vertices inside their own subtrees, whereas s measures the vertex x_1 in $\widetilde{T}_1$. Furthermore, s is necessary in both T and in T' to distinguish x_1 from the other leaf vertex at distance 2 from it in $\widetilde{T}_1$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

(i) *Let $s^\star$ be any sensor in $\widetilde{T}_i$ for some $i \in \{1, \dots, r\}$. Then, for all vertices $y \notin \widetilde{T}_i$, $d_T(s^\star, y) \geq k+1$ and $d_{T'}(s^\star, y) \geq k+1$ both hold.*
(ii) *Let $s^\star \neq s$ be any sensor in $\widetilde{T}_0$. Then, for any $y \in \widetilde{T}_i$, $i \geq 1$, either $d_T(s^\star, y) \geq k+1$, or the shortest path from $s^\star$ to y in T contains s.*

Proof. Proof of (i): Consider a vertex $y \notin \widetilde{T}_i$. Using the notation of Definition 4.1, the vertex closest to s within $\widetilde{T}_i$ on the path $\mathcal{P}_T(s^\star, s)$ is x_i. Since $x_i \in \widetilde{T}_i$, the edges of $\mathcal{P}(s^\star, x_i)$ are present in both T and T'. Moreover, x_i has a neighboring vertex $v \in A_T(s)$ in T, by the definition of tr_A, see the comments below the definition. v cannot be measured (in T) by any sensor in $\widetilde{T}_i$, otherwise that sensor would measure v via a path not containing s, and Definition 3.6(iii) would be violated as $v \in A_T(s)$. Hence, $d_T(s^\star, v) \geq k+1$, and

$$d_T(s^\star, x_i) \geq d_T(s^\star, v) - 1 \geq k+1-1 = k.$$

These two facts will imply both conclusions as we argue next.

First, as $y \notin \widetilde{T}_i$, the path $\mathcal{P}_T(s^\star, y)$ contains $v \in A_T(s)$, and we also have that $\mathcal{P}_T(s^\star, v)$ does not contain s. Hence, if $s^\star$ measures y in T, then it also measures v in T, contradicting Claim 3.8, as $v \in A_T(s)$. Hence, $d_T(s^\star, y) \geq k+1$ holds.

Next, we will show that $d_{T'}(s^\star, y) \geq k+1$ as well. Since $y \notin \widetilde{T}_i$ the path $\mathcal{P}_{T'}(s^\star, y)$ contains s, since the only connection between $\widetilde{T}_i$ and $V \setminus V(\widetilde{T}_i)$ is the edge (s, x_i) in T'. This implies that

$$d_{T'}(s^\star, y) \geq d_{T'}(s^\star, s) = d_{T'}(s^\star, x_i) + 1 = d_T(s^\star, x_i) + 1 \geq k+1.$$

Proof of (ii): Assume that $d_T(s^\star, y) \leq k$. Then, since $s^\star \in \widetilde{T}_0$ and $y \in \widetilde{T}_i$, $i \geq 1$, the path $\mathcal{P}_T(s^\star, y)$ contains a vertex $v \in A_T(s)$. Hence, by Claim 3.8, $\mathcal{P}_T(s^\star, y)$ has to contain s, otherwise Definition 3.6 (ii) would be violated for v belonging to $A_T(s)$. □

Building on the above preliminary facts, we next state and prove the main effects of applying Transformation A.

Lemma 4.3 *(Properties of Transformation A). Let $T = (V, E)$ be a tree with a k-truncated resolving set $S \subseteq V$. Fix some $s \in S$, and consider $T' := \mathrm{tr}_A(T, S, s)$. Then the following hold:*

(i) *S remains a k-truncated resolving set for the tree T'.*
(ii) *$A_{T'}(s) = A_T(s)$, and $A_{T'}(s)$ forms a leaf-path emanating from s.*
(iii) *For any sensor $s^\star \in S \setminus \{s\}$, if $A_T(s^\star)$ is a leaf-path emanating from $s^\star$ in T, then $A_{T'}(s^\star) = A_T(s^\star)$ is still a leaf-path emanating from $s^\star$ in T'.*
(iv) *For any sensor $s^\star \in S \setminus \{s\}$, if $A_T(s^\star)$ is not a leaf-path emanating from $s^\star$, then $A_{T'}(s^\star) \subseteq A_T(s^\star)$.*

Observe that (iii) ensures that attraction of sensors that are already leaf-paths are left untouched by tr_A, while (iv) ensures that for sensors with attraction that are not (entirely) leaf-paths, tr_A potentially decreases the number of vertices in the attraction, but never adds new vertices to it.

Proof of Lemma 4.3. Proof of (i): Let $x, y \in V$ be a pair of distinct vertices. We shall prove that there is a sensor in S that resolves them in T'. We will use the notations of Definition 4.1. We will do a case-distinction analysis with respect to the location of x and y in the components $\widetilde{T}_i, i \geq 0$ described in the transformation. We start with cases when neither of the vertices are in $A_T(s)$:

Case 1: Assume that $x \in \widetilde{T}_i$ and $y \in \widetilde{T}_j$ for some $i \geq 1$, $j \geq 0$, $i \neq j$. Then, since $x \notin A_T(s)$, there is a sensor $s' \in S \setminus \{s\}$ that measures x such that $\mathcal{P}_T(s', x)$ does not include s. Then, by Claim 4.2(i)–(ii), $s' \in \widetilde{T}_i$. Therefore, the edges of $\mathcal{P}_T(s', x)$ are unchanged in T', so s' still measures x in T'. However, it does not measure $y \in \widetilde{T}_j$ in T' by Claim 4.2(i). Hence, s' resolves x and y in T'.

Case 2: Now assume that $x, y \in \widetilde{T}_i$ for some $i \geq 1$. Let $s' \in S$ be a sensor that resolves x and y in T. Then s' has to measure at least one of x and y in T, hence $s' \notin \widetilde{T}_j$ for $j \geq 1$, $j \neq i$ by Claim 4.2(i). There are two (sub)cases: either $s' \in \widetilde{T}_i$, or $s' \in \widetilde{T}_0$. First we consider $s' \in \widetilde{T}_i$. Then the paths $\mathcal{P}_T(s', x)$, $\mathcal{P}_T(s', y)$ do not contain any vertex in $A_T(s)$ (since both endpoints of both paths belong to $\widetilde{T}_i$). Therefore, the edges of $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ are all still present in T', and s' resolves x and y in T'.

If $s' \in \widetilde{T}_0$, then either $s' = s$ or $s' \neq s$. We start with the case $s' = s$. Recall x_i from Definition 4.1. Then $\{s, x_i\} \in E_3$ (an edge added to create T'). Since every path $\mathcal{P}_T(s, v)$, $v \in \widetilde{T}_i$ starts with the segment $\mathcal{P}_T(s, x_i)$ in T, that we replaced with the single edge $\{s, x_i\}$ to obtain $\mathcal{P}_{T'}(s, v)$, the following holds for all $v \in \widetilde{T}_i$:

$$d_{T'}(s, v) = d_T(s, v) - d_T(s, x_i) + 1.$$

Hence, the difference of the distances does not change:

$$d_{T'}(s, x) - d_{T'}(s, y) = d_T(s, x) - d_T(s, y) \neq 0,$$

the latter being nonzero by the assumption that s resolves x, y in T. Since these distances in T' are less than in T, s still resolves x, y in T'.

The last possibility is that $s' \in \widetilde{T}_0 \setminus \{s\}$. Then s' has to measure at least one of x and y in T, say it measures x. Then, by Claim 4.2(ii), $\mathcal{P}_T(s', x)$ contains s. On the path $\mathcal{P}_T(s, x)$, there has to be at least one vertex in $A_T(s)$ (since $x \in \widetilde{T}_i$), let the closest one to x be u. Then, since $\widetilde{T}_i$ is a connected component in $V \setminus A_T(s)$, and $y \in \widetilde{T}_i$ (and T is a tree), u is also on the path $\mathcal{P}_T(s', y)$. Hence, $\mathcal{P}_T(s', s) \subseteq \mathcal{P}_T(s', u) \subseteq \mathcal{P}_T(s', x) \cap \mathcal{P}_T(s', y)$. This implies that

$$d_T(s, x) = d_T(s', x) - d_T(s', s), \quad \text{and} \quad d_T(s, y) = d_T(s', y) - d_T(s', s).$$

Therefore, if s' resolves x and y in T, then s also resolves them in T, and then the reasoning of the previous paragraph applies, and so s resolves x, y also in T'.

Case 3: Next, suppose that $x, y \in \widetilde{T}_0$, and let s' be a sensor that resolves them in T. Then s' measures at least one of x and y in T, which, by Claim 4.2(i) is only possible if $s' \in \widetilde{T}_0$. (Here s' may or may not be s.) This implies that the edges of the paths $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ stay intact in T', thus they are still distinguished by s' in T'.

We continue with cases for which at least one of $x, y \in A_T(s)$:

Case 4: If x and y are both in $A_T(s)$, then in T' they will both be part of a single leaf-path (of length at most k) emanating from s, hence s will resolve them in T'.

Case 5: The next case is when $x \in A_T(s)$ and $y \in \widetilde{T}_i$ for some $i \geq 1$. Since $y \notin A_T(s)$, there has to be a sensor s' that measures y such that $\mathcal{P}_T(s', y)$ does not contain s. Further, since $y \in \widetilde{T}_i$ and by Claim 4.2(ii), this is only possible if $s' \in \widetilde{T}_i$. Then the vertices of the path $\mathcal{P}_T(s', y)$ remain unchanged in T', so s' still measures y in T'. However, it cannot measure $x \in A_T(s)$ in T' by Claim 4.2(i). Thus, s' distinguishes x and y in T'.

Case 6: The final case is when $x \in A_T(s)$ and $y \in \widetilde{T}_0$. Then there exists a sensor $s' \in S \setminus \{s\}$ which measures y such that the path $\mathcal{P}_T(s', y)$ does not contain s. By Claim 4.2 (ii), $s' \in \widetilde{T}_0$. Then there are two possible subcases.

First assume that $V(\mathcal{P}_T(s, s')) \cap V(\mathcal{P}_T(s', y)) = \{s'\}$, i.e., s, s', y all lie on a path in $\widetilde{T}_0$ in this order. Then in T', x is added to a leaf-path emanating from s, and the edges of $\mathcal{P}_T(s, s')$, $\mathcal{P}(s', y)$ are unchanged, hence x, s, s', y will all lie on a single path in T' in this order. Then, since x is measured by s and y is measured by s', and $\mathrm{typ}_{s,s'}(x) = 0$ while $\mathrm{typ}_{s,s'}(y) = d_{T'}(s, s')$, by Claim 3.12, at least one of s, s' resolves x, y.

Next, assume that $V(\mathcal{P}_T(s, s')) \cap V(\mathcal{P}_T(s', y)) \neq \{s'\}$. Then, in fact, $V(\mathcal{P}_T(s, s')) \cap V(\mathcal{P}_T(s', y)) = V(\mathcal{P}_T(s', v))$ for some $v \neq s'$, i.e., $\mathrm{typ}_{s,s'}(y) = \mathrm{typ}_{s,s'}(v) \neq 0$. For this we also know that $v \neq s$, since $\mathcal{P}_T(s', y)$ does not contain s. Observe that the edges of the paths $\mathcal{P}_T(s, v)$, $\mathcal{P}_T(s', v)$, $\mathcal{P}_T(y, v)$ are unchanged in T'. We also know that $\mathrm{typ}_{s,s'}(x) = 0$ in T' since x is in a leaf-path emanating from s in T'. Hence, Claim 3.12 applies with $y := x'$ so at least one of s, s' resolves x, y.

Proof of (ii): Using the notation of Definition 4.1, let $A_T(s) = \{v_1, \ldots, v_\ell\}$. We claim that $A_{T'}(s) = \{v_1, \ldots, v_\ell\}$. Indeed, $\{v_1, \ldots, v_\ell\} \subseteq A_{T'}(s)$ since all these vertices are measured by s in T', and $(s, v_1, \ldots, v_\ell)$ forms a leaf-path in T', so the shortest path from every other sensor $s^\star \in S$ to any v_i contains the sensor s and hence Definition 3.6 (iii) would be violated otherwise.

In order to show that $A_{T'}(s) \subseteq \{v_1, \ldots, v_\ell\}$, consider a vertex $x \in V \setminus (A_T(s) \cup S)$. Since $x \notin A_T(s)$, there exists (at least one) $s^\star \in S \setminus \{s\}$ that directly measures x. We will show that in this case

$$d_{T'}(s^\star, x) \leq k \quad \text{and} \quad s \notin V(\mathcal{P}_{T'}(s^\star, x)), \tag{9}$$

hence x cannot be in $A_{T'}(s)$, either. Consider the path $\mathcal{P}_T(s^\star, x)$. Since it has length at most k, and it does not contain s, none of its vertices can be in $A_T(s)$ by Claim 3.8. It follows that $\mathrm{tr}_A(T, S, s)$ does not remove any of the edges in $\mathcal{P}_T(s^\star, x)$, since none of them are adjacent to any vertex in $A_T(s)$. Hence, all the edges of $\mathcal{P}_T(s^\star, x)$ are present in T'. As a result, (9) holds and $x \notin A_{T'}(s)$.

Finally, $A_{T'}(s) = A_T(s)$ forms a leaf-path emanating from s in T', because E_2 forms exactly that leaf-path in (7), finishing the proof of (ii).

Proof of (iii): Since the vertices of $A_T(s^\star)$ are all on a leaf-path emanating from $s^\star$, the shortest path between any of them and s contains $s^\star$, hence, these vertices, (including $s^\star$), cannot belong to $A_T(s)$. Therefore, when executing Transformation A at s, all the edges between vertices of $A_T(s^\star) \cup \{s^\star\}$ remain intact. Moreover, $d_T(s, s^\star) \le d_T(s, x)$ for any $x \in A_T(s^\star)$, so such x is not the closest vertex in the component to s, hence in the edge set E_3 of (8) none of the edges connect to any $x \in A_T(s^\star)$. This shows that $A_T(s^\star)$ still forms a leaf-path in T' emanating from $s^\star$, and $A_T(s^\star) \subseteq A_{T'}(s^\star)$. The fact that $A_{T'}(s^\star) \subseteq A_T(s^\star)$ is proved in the same way as part (iv) below.

Proof of (iv): For an indirect proof, let us assume that there is a vertex $x \in A_{T'}(s^\star)$, such that $x \notin A_T(s^\star)$, i.e., a new vertex is added to the attraction of sensor $s^\star$ because of the transformation. We observe that $x \in A_{T'}(s^\star)$ implies by Definition 3.6(i) that the path $\mathcal{P}_{T'}(s^\star, x)$ has length at most k, and it does not contain s, or any other sensor besides $s^\star$, by Definition 3.6(ii) and (iii). This means that the edges of the path $\mathcal{P}_{T'}(s^\star, x)$ can be neither in E_3 nor in E_2. That is, all edges of $\mathcal{P}_{T'}(s^\star, x)$ are also present in T. If despite this $x \notin A_T(s^\star)$, it is only possible if there is a sensor $s' \in S \setminus \{s^\star\}$ such that the path $\mathcal{P}_T(s', x)$ has length at most k, and it contains no sensors besides s' in T. Then there are the following two possibilities.

If $s' \ne s$, then none of the vertices in $\mathcal{P}_T(s', x)$ can belong to $A_T(s)$ because the path $\mathcal{P}_T(s', x)$ does not contain s by assumption. Hence none of the edges of $\mathcal{P}_T(s', x)$ is in E_1, i.e., the transformation leaves these edges untouched. Therefore, $\mathcal{P}_T(s', x)$ is still a path in T' (of length at most k), contradicting the assumption that $x \in A_{T'}(s^\star)$, since both $s^\star$ and s' directly measure x in T'.

If $s' = s$, and none of the vertices of $\mathcal{P}_T(s, x)$ is in $A_T(s)$, then $E(\mathcal{P}_T(s, x))$ is disjoint from E_1, hence $\mathcal{P}_T(s, x)$ is still a path in T' (of length at most k). Since $s^\star \notin V(\mathcal{P}_T(s, x))$, this contradicts the assumption that $x \in A_{T'}(s^\star)$.

The only remaining case is that $s' = s$, and at least one vertex in $V(\mathcal{P}_T(s, x))$ belongs to $A_T(s)$. This means that s and x get disconnected in $\widetilde{T} = (V, E \setminus E_1)$, i.e., we may assume $x \in \widetilde{T}_i$ for some $i \ge 1$. In this case, the edge set E_3 contains an edge from s to x_i (see Definition 4.1. This vertex x_i must lie on the path $\mathcal{P}_T(x, s)$, otherwise there would be a cycle in T. Hence, $V(\mathcal{P}_{T'}(s, x)) \subseteq V(\mathcal{P}_T(s, x))$. Therefore, the path $\mathcal{P}_{T'}(s, x)$ is of length at most k, and it does not contain any other sensor besides s, contradicting the assumption that $x \in A_{T'}(s^\star)$. This shows that $A_{T'}(s^\star) \subseteq A_T(s^\star)$ for all $S \setminus \{s\}$, and finishes the proof of (iv). □

Now we are in a position to state and prove the main result of this section. Lemma 4.3 shows how we can use Transformation A to achieve the desired modification of the tree, described as follows.

Lemma 4.4. *Let $T = (V, E)$ be a tree on n vertices with k-truncated resolving set $S \subseteq V$. For any $s \in S$ let $A_T(s) = \{v_1, v_2, \dots, v_{\ell(s)}\}$ for some $\ell(s) \le k$. Then there exists a tree $\widehat{T} = (V, \widehat{E})$ on the same vertex set in which S is also a k-truncated resolving set, and in which, for each $s \in S$, $A_{\widehat{T}}(s) \subseteq A_T(s)$ with $A_{\widehat{T}}(s)$ being a* leaf-path *emanating from s.*

Remark 4.5. A consequence of the proof of Lemma 4.4 is that if $|A_T(s)|$ is less than k for some $s \in S$, then T cannot be optimal, since we can add at least one extra vertex to the tree to this leaf-path and the new tree is still resolved.

Proof of Lemma 4.4. Let $S = \{s_1, s_2, \dots, s_m\}$. Let $T_0 = T$, and then let us iteratively define $T_i := \mathrm{tr}_A(T_{i-1}, S, s_i)$ for $1 \le i \le m$. We prove that $\widehat{T} = T_m$ satisfies the conditions of Lemma 4.4. Indeed, for T_1 it is true that $A_{T_1}(s_1) = A_T(s_1)$ is a leaf-path emanating from s_1 by Lemma 4.3(ii), for all other sensors s_j, $j \ge 2$, $A_{T_1}(s_j) \subseteq A_T(s_j)$, and S is still a k-truncated resolving set for T_1. Inductively we can then assume that in T_i, the vertices of $A_T(s_1), \dots, A_T(s_i)$ already form leaf-paths emanating from $s_1, \dots, s_i$, respectively, $A_{T_i}(s_j) \subseteq A_T(s_j)$ for all $j \ge 1$, and S is a k-truncated resolving set in T_i. Then $T_{i+1} = \mathrm{tr}_A(T_i, S, s_{i+1})$ moves the vertices $A_{T_i}(s_{i+1}) = A_T(s_{i+1})$ into a leaf-path emanating from s_{i+1}, and leaves the attraction of sensors $j \le i$ intact, i.e., $A_{T_{i+1}}(s_j) = A_{T_i}(s_j)$ is a leaf-path emanating from s_j for all $j \le i$ by Lemma 4.3 (iii). And, for $j \ge i+1$ it holds that $A_{T_{i+1}}(s_j) \subseteq A_{T_i}(s_j) \subseteq A_T(s_j)$ by Lemma 4.3 (iv) and the inductive assumption. This means that in T_{i+1} already $A_T(s_1), \dots, A_T(s_{i+1})$ are all leaf-paths emanating from their respective sensors. Finally S still resolves T_{i+1}, hence the induction can be advanced. When $i = m$, the attraction of each sensor has been transformed into a leaf-path, and S still resolves T_m, finishing the proof. □

Lemma 4.4 shows us that it is sufficient to consider optimal trees that have the attractions of single sensors contained in leaf-paths attached to the corresponding sensors. Next we analyze the arrangement of the attraction of *pairs* of sensors in an optimal tree.

5. Transformation B: shortening too long sensor paths

Let us consider now a tree T with k-truncated resolving set S on it, for which Transformation A has already been repetitively applied, as in Lemma 4.4. Recall from Definition 3.2 that sensor paths are strong (respectively, weak) if their length is at most $k+1$ (respectively, at least $k+2$). Our next step is to obtain another tree from T, which has an overlapping pair of strong sensor paths, while keeping the properties that S is a k-truncated resolving set and that attractions of single sensors are contained in leaf-paths attached to the sensor. We will show that it is possible to do via the repeated application of another edge-rewiring process, Transformation B, as long as T has at least one weak sensor path. The transformation will shorten weak sensor paths with each application, until it finally produces a tree with overlapping strong sensor paths. In the case when T already has such a pair of sensor paths, we skip over Transformation B and continue with Transformation C, introduced in the next section. In the case when T does not have such a pair of sensor paths, but it also does not have any weak sensor paths, we move directly to Section 7, where we find the maximal size of such a tree.

In this section, we first state the conditions under which we will apply Transformation B, then we define this transformation. Next, after some preliminary observations, we prove the main properties of Transformation B in Lemma 5.7. Finally, we state and prove Lemma 5.8, the main result of this section, which makes use of this transformation to achieve our above-mentioned purpose.

Condition 5.1. Let $T=(V,E)$ be a tree on n vertices with a k-truncated resolving set S. Suppose that the following hold:

(i) for all sensors $s\in S$ the attraction $A_T(s)$ is contained in a single leaf-path starting from s,
(ii) any pair of strong sensor paths are disjoint, possibly except for their endpoints, and
(iii) there is at least one weak sensor path in T, and $\mathcal{P}_T(s_0,s_1)$ is a longest one.

Claim 5.2. *Condition 5.1(ii) above is equivalent to the following:*

(ii)' *there are no two strong sensor paths in T that share an edge.*

Proof. Assume $\mathcal{P}_T(s_1,s_2)$ and $\mathcal{P}_T(s_3,s_4)$ are two distinct strong sensor paths that do not share an edge, but do share a vertex v that is not the endpoint of either of them. If the s_i-s are not all different, say $s_4=s_2$, then let w be the vertex neighboring s_2 on the path $\mathcal{P}_T(s_1,s_2)$. Then $\mathcal{P}_T(s_1,s_2)$ and $\mathcal{P}_T(s_2,s_3)$ share the edge $\{w,s_2\}$, a contradiction. Hence, s_1,s_2,s_3,s_4 are indeed all different. Let s_1',s_2',s_3',s_4' be a relabeling of s_1,s_2,s_3,s_4 such that s_i' $(i=1,2,3,4)$, is the i-th closest sensor to v among s_1,s_2,s_3,s_4, breaking ties arbitrarily. Let u be the vertex neighboring v on $\mathcal{P}_T(s_1',v)$. Then $\mathcal{P}_T(s_1',s_2')$ and $\mathcal{P}_T(s_1',s_3')$ share the edge $\{u,v\}$, and both are strong sensor paths, since

$$\begin{aligned}\max\{d_T(s_1',s_2'),d_T(s_1',s_3')\}&=\max\{d_T(s_1',v)+d_T(v,s_2'),d_T(s_1',v)+d_T(v,s_3')\}\\&\le\max\{d_T(s_1,v)+d_T(v,s_2),d_T(s_3,v)+d_T(v,s_4)\}\\&=\max\{d_T(s_1,s_2),d_T(s_3,s_4)\}\le k+1.\quad\square\end{aligned}$$

Building on Claim 5.2 we will use Conditions 5.1(ii) and (ii)' interchangeably later in this paper.

To introduce Transformation B, recall Definition 3.7 and Claim 3.10 about the structure of the attraction of two sensors, as well as Definition 3.11 about the types and heights of a vertex x with respect to two sensors s,s'.

Definition 5.3 *(Transformation B).* Let $T=(V,E)$ be a tree with a k-truncated resolving set $S\subseteq V$ and $s_0,s_1\in S$ such that Conditions 5.1(i)–(iii) hold. Let $w_1\in V(\mathcal{P}_T(s_0,s_1))$ be the vertex for which $\{s_0,w_1\}\in E$. Since $w_1\notin A_T(s)$ and Condition 5.1(ii) holds, there is a unique other sensor $s_0'\neq s_1$ that directly measures w_1. Let $w_1,w_2,\dots,w_{d_T(s_0,s_0')-1}$ be the vertices in $V(\mathcal{P}_T(s_0,s_0'))\setminus\{s_0,s_0'\}$ in order of increasing distance from s_0, and let

$$V(\mathcal{P}_T(s_0,s_1))\cap V(\mathcal{P}_T(s_0,s_0'))=\{w_1,w_2,\dots,w_q\}.$$

Furthermore, let u_{q+1} be the vertex of $\mathcal{P}_T(w_q,s_1)$ for which $\{w_q,u_{q+1}\}\in E(\mathcal{P}_T(w_q,s_1))$. Now define the vertex sets

$$\begin{aligned}V_1&:=\{v\in A_T(s_0,s_0'):\ \mathrm{typ}_{s_0,s_0'}(v)=q,\ \mathrm{hgt}_{s_0,s_0'}(v)\ge 1,\ u_{q+1}\in V(\mathcal{P}_T(w_q,v))\},\\V_2&:=\{v\in V:\ \mathrm{typ}_{s_0,s_0'}(v)=q,\ \mathrm{hgt}_{s_0,s_0'}(v)\ge 1,\ u_{q+1}\notin V(\mathcal{P}_T(w_q,v))\}\end{aligned}$$

with $V_1=:\{v^{(1)},v^{(2)},\dots,v^{(|V_1|)}\}$ and $V_2=:\{v^{(|V_1|+1)},v^{(|V_1|+2)},\dots,v^{(|V_1|+|V_2|)}\}$, and the edge sets

$$\begin{aligned}E_1&:=\{\{u,v\}\in E: u\in V_1\cup V_2 \text{ or } v\in V_1\cup V_2\}\cup\{w_q,u_{q+1}\},\\E_2&:=\{\{w_q,v^{(1)}\}\}\bigcup\left(\cup_{i=1}^{|V_1|+|V_2|-1}\{\{v^{(i)},v^{(i+1)}\}\}\right).\end{aligned}$$

Consider the subgraph $\widetilde{T}=(V\setminus(V_1\cup V_2),E\setminus E_1)$, and denote its connected components by $\widetilde{T}_0,\widetilde{T}_1,\dots,\widetilde{T}_r$ where $\widetilde{T}_0$ contains s_0 (and s_0'), and $\widetilde{T}_1$ contains s_1. For each $1\le i\le r$ let x_i be the vertex in $\widetilde{T}_i$ that is closest to s_0 in T. Define the third edge set

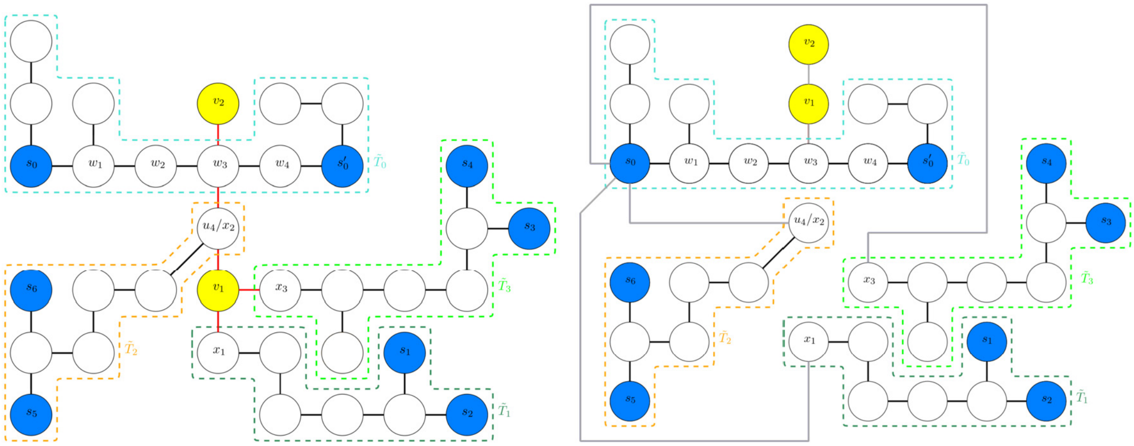

Fig. 2. An example of transformation B with T on the left and $T' = \mathrm{tr}_B(T, S, s_0, s_0')$ on the right. Here $k = 5$, and $q = 3$. The blue vertices are the sensors in S, and the yellow vertices are in $V_1 \cup V_2$: $v_1 \in V_1$, $v_2 \in V_2$. The red edges belong to E_1, and are deleted by the transformation. The grey edges belong to $E_2 \cup E_3$, and are added by the transformation. The subtrees $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2$ and $\widetilde{T}_3$ are also highlighted. Note that x_3 and the leaf vertex at distance two from x_3 within $\widetilde{T}_3$ are only resolved by s_0' in T, and by s_0 in T'. While the sensors s_0, s_0' may measure vertices in the subtrees $\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3$, any sensor in $\widetilde{T}_i, i \geq 1$ can measure vertices only within its own subtree and possibly in $\widetilde{T}_0$ (this latter case is not depicted in the picture, but could happen e.g. if the sensors s_5, s_6 were closer to u_4).

$$E_3 := \{\{s_0, x_i\} : 1 \leq i \leq r\}.$$

Finally, define $\mathrm{tr}_B(T, S, s_0, s_1) = (V, E')$ where $E' = (E \setminus E_1) \cup E_2 \cup E_3$.

For an example of Transformation B see Fig. 2.

A couple of comments on this definition: s_0' cannot be equal to s_1, since $d(s_0, s_1) \geq k + 2$ by the assumption that $\mathcal{P}_T(s_0, s_1)$ is a weak sensor path. The sensor s_0' is indeed unique, since if there was another sensor s_0'' also directly measuring w_1 in T, then the two sensor paths $\mathcal{P}_T(s_0, s_0'), \mathcal{P}_T(s_0, s_0'')$ would both be strong and violate Condition 5.1(ii). This uniqueness of the sensor s_0' means that $w_1 \in A_T(s_0, s_0')$. A similar argument shows that $w_{d_T(s_0,s_0')-1}$ is also in $A_T(s_0, s_0')$.

E_1 is the set of edges that are adjacent to the vertices in $V_1 \cup V_2$ in T, plus the edge $\{w_q, u_{q+1}\}$ (in case $u_{q+1} \notin V_1$, that is, $u_{q+1} \notin A_T(s_0, s_0')$). The point of removing E_1 from the graph is to rewire the edges (with the addition of E_2 and E_3) such that the path between s_0 and s_1 becomes shorter while ensuring that the vertices of $A_T(s_0, s_0')$ are still identified by the sensors. The removal of E_1, and then the addition of E_2 rewires $A_T(s_0, s_0')$ as follows: those vertices in $A_T(s_0, s_0')$ that are not of type q stay 'at their place' (relative to s_0 and s_0'). Those that are type q, are rewired into a single leaf-path emanating from w_q. We will show that V_2 must be already a leaf-path in T, and thus we essentially append V_1 to the end of this path. We will show in Claim 5.6 that this leaf-path contains at most $k - q$ vertices besides w_q, which ensures that all of them will be measured by both s_0 and s_0' after the transformation. After all this, E_3 connects the components of $(V, (E \setminus E_1) \cup E_2)$ back together, by connecting s_0 to the (originally) closest vertex x_i in each of the other components $\widetilde{T}_i$. Note that the vertices x_i in the above definition are indeed well-defined, since if some $\widetilde{T}_i$ had two closest vertices to s_0 in T, then they would lie on a cycle in T, contradicting the tree property, similarly as before for Transformation A.

We observe that $\mathrm{tr}_B(T, S, s_0, s_1)$ is indeed a tree, i.e., connected, since the addition of the edge set E_3 to $\widetilde{T}$ adds exactly one connection between the components $\widetilde{T}_0$ and $\widetilde{T}_i$ for each $i = 1, 2, \ldots, r$, and the addition of E_2 adds the vertices of $V_1 \cup V_2$ to $\widetilde{T}_0$ as a single leaf-path. See Fig. 2 for an illustration.

In order to proceed with description of Transformation B, we will first prove the following structural property. Recall the type of vertices with respect to two sensors from Definition 3.11.

Claim 5.4 *(Location of sensors). Consider the notation of Definition 5.3. Then in T for any sensor $s^\star \in S$ one of the following three possibilities holds:*

(i) $\mathrm{typ}_{s_0,s_0'}(s^\star) = 0$,
(ii) $\mathrm{typ}_{s_0,s_0'}(s^\star) = d_T(s_0, s_0')$ *or*
(iii) $\mathrm{typ}_{s_0,s_0'}(s^\star) = q$ *and* $\{w_q, u_{q+1}\} \in E(\mathcal{P}_T(s_0, s^\star))$.

Proof. First, we prove that $\mathrm{typ}_{s_0,s_0'}(s^\star) \notin \{0, q, d_T(s_0, s_0')\}$ cannot hold. Assume indirectly that it does hold, and $1 \leq \mathrm{typ}_{s_0,s_0'}(s^\star) \leq q - 1$. We can assume that $\mathrm{hgt}_{s_0,s_0'}(s^\star)$ is minimal among the sensors of the same type, hence, there is no sensor besides $s^\star$ on the path $\mathcal{P}_T(s^\star, w_{\mathrm{typ}(s^\star)})$. By the comments after Definition 5.3 we have $w_{-1} := w_{d_T(s_0,s_0')-1} \in A_T(s_0, s_0')$. This implies that $s^\star$ cannot measure w_{-1}, hence

$$d_T(s^\star, w_{-1}) \geq k+1 > d_T(s_0, w_{-1}). \tag{10}$$

As w_q lies on both paths $\mathcal{P}_T(s_0, w_{-1})$ and $\mathcal{P}_T(s^\star, w_{-1})$, (10) gives

$$d_T(s^\star, w_q) > d_T(s_0, w_q). \tag{11}$$

Furthermore, as w_q lies on both paths $\mathcal{P}_T(s_0, s_1)$ and $\mathcal{P}_T(s^\star, s_1)$, (11) implies that $\mathcal{P}_T(s^\star, s_1)$ is a longer sensor path than $\mathcal{P}_T(s_0, s_1)$, contradicting Condition 5.1 (iii). Therefore, $1 \leq \mathrm{typ}_{s_0,s_0'}(s^\star) \leq q-1$ cannot hold for any sensor $s^\star$.

The proof that $q+1 \leq \mathrm{typ}_{s_0,s_0'}(s^\star) \leq d_T(s_0, s_0') - 1$ cannot hold either for any sensor $s^\star$ is analogous to the above, reversing the roles of s_0 and s_0', and those of w_1 and w_{-1}.

We are left to prove the fact that if $\mathrm{typ}_{s_0,s_0'}(s^\star) = q$, then $\{w_q, u_{q+1}\} \in E(\mathcal{P}_T(s_0, s^\star))$. Assume to the contrary that there exists a sensor $s^\star$ with $\mathrm{typ}_{s_0,s_0'}(s^\star) = q$, and $\{w_q, u_{q+1}\} \notin E(\mathcal{P}_T(s_0, s^\star))$, moreover, $d_T(w_q, s^\star)$ is minimal among these sensors. Now if $d_T(w_q, s^\star) \leq q$, then

$$d_T(s^\star, s_0') = d_T(s^\star, w_q) + d_T(w_q, s_0') \leq d_T(s_0, w_q) + d_T(w_q, s_0') = d_T(s_0, s_0'),$$

implying that $\mathcal{P}_T(s^\star, s_0')$ is a strong sensor path sharing an edge with $\mathcal{P}_T(s_0, s_0')$, contradicting Condition 5.1 (ii). On the other hand, if $d_T(w_q, s^\star) > q$, then

$$d_T(s^\star, s_1) = d_T(s^\star, w_q) + d_T(w_q, s_1) > d_T(s_0, w_q) + d_T(w_q, s_1) = d_T(s_0, s_1), \tag{12}$$

as $\{w_q, u_{q+1}\} \in E(\mathcal{P}_T(s_0, s_1))$ and $\{w_q, u_{q+1}\} \notin E(\mathcal{P}_T(s_0, s^\star))$ by the assumptions, implying that $\{w_q, u_{q+1}\} \in E(s^\star, s_1)$. The consequence of (12) is that $\mathcal{P}_T(s^\star, s_1)$ is a longer sensor path than $\mathcal{P}_T(s_0, s_1)$, contradicting Condition 5.1. □

Notice that Claim 5.4 implies, in particular, that for any $v \in V_1$ in Definition 5.3, $\{w_q, u_{q+1}\} \in E(\mathcal{P}_T(s_0, v))$. In fact,

$$V_1 = \{v \in A_T(s_0, s_0') : u_{q+1} \in V(\mathcal{P}_T(w_q, v))\}, \tag{13}$$

and thus $A_T(s_0, s_0') \setminus V_1 \subseteq \widetilde{T}_0$.

Next, similarly to Transformation A, we prove a 'no communication' lemma for the graph components in tr_B.

Claim 5.5 *(No 'communication' between different subtrees). Consider the notation of Definition 5.3, and let $T' := \mathrm{tr}_B(T, S, s_0, s_1)$.*

(i) *Let $s^\star$ be any sensor in $\widetilde{T}_i$ for some $i \geq 1$. Then, for all vertices $y \in \widetilde{T}_j$, $j \geq 1$, $j \neq i$, we have that $d_T(s^\star, y) \geq k+1$, and $s_0 \in V(\mathcal{P}_{T'}(s^\star, y))$.*
(ii) *Let $s^\star \notin \{s_0, s_0'\}$ be a sensor in $\widetilde{T}_0$. Then, for any $y \in V \setminus \widetilde{T}_0$ the path $\mathcal{P}_T(s^\star, y)$ contains s_0 or s_0'.*

Proof. (i) Since $s^\star$ and y are in different components ($\widetilde{T}_i$ and $\widetilde{T}_j$), $s_0 \in V(\mathcal{P}_{T'}(s^\star, y))$ follows from the construction of the edge set E_3. Now consider the paths $\mathcal{P}_{T'}(s^\star, s_0)$ and $\mathcal{P}_{T'}(y, s_0)$. By the construction of the edge set E_3 again, the vertices on these two paths neighboring s_0 are x_i and x_j, respectively. Since $x_i \neq x_j$, at least one of these two is not equal to u_{q+1}. Without loss of generality, assume that $x_i \neq u_{q+1}$ (the proof of the other case is analogous). This implies that on the path $\mathcal{P}_T(x_i, s_0)$ the edge incident to x_i was removed as part of E_1 because its other endpoint, say v, belonged to V_1 (and not because it was the edge $\{w_q, u_{q+1}\}$). Hence, $v \in A_T(s_0, s_0')$, implying that $d_T(s^\star, v) \geq k+1$. Consequently,

$$d_T(s^\star, x_i) \geq d_T(s^\star, v) - 1 \geq k+1-1 = k,$$

and thus

$$d_{T'}(s^\star, y) \geq d_{T'}(s^\star, s_0) = d_T(s^\star, x_i) + 1 \geq k+1,$$

since the path $\mathcal{P}_T(s^\star, x_i)$ remains untouched by the transformation.

To prove part (ii) notice that by Definition 5.3, $\mathrm{typ}_{s_0,s_0'}(y) = q$. On the other hand, Claim 5.4 implies that $\mathrm{typ}_{s_0,s_0'}(s^\star)$ is either 0 or $d_T(s_0, s_0')$ (as option (iii) of Claim 5.4 would contradict with $s^\star \in \widetilde{T}_0$). In the former case $\mathcal{P}_T(s^\star, y)$ contains $\mathcal{P}_T(s_0, w_q)$ as a sub-path, and in the latter it contains $\mathcal{P}_T(s_0', w_q)$ as a sub-path, finishing the proof. □

We continue by showing that there cannot be too many vertices in $V_1 \cup V_2$.

Claim 5.6. *Consider the notation of Definition 5.3. Then the following hold in T:*

(i) *for every $h \in \{1, 2, \ldots, k-q\}$, $|\{v \in V_1 \cup V_2 : \mathrm{hgt}_{s_0,s_0'}(v) = h\}| \in \{0, 1\}$, and*
(ii) *for every $h > k-q$, $|\{v \in V_1 \cup V_2 : \mathrm{hgt}_{s_0,s_0'}(v) = h\}| = 0$.*

Consequently, $|V_1 \cup V_2| \leq k-q$, and every vertex in $V_1 \cup V_2$ is directly measured by both s_0 and s_0' in $T' = \mathrm{tr}_B(T, S, s_0, s_1)$.

Proof. In the following, 'type' and 'height' will always refer to type and height in T with respect to s_0, s_0'. First, Claim 3.13 implies that there cannot be two distinct vertices $x, y \in V_1$ with the same height, since both x and y are of type q, and both belong to $A_T(s_0, s_0')$. Next, by Claim 5.4, for every sensor $s \in S$ and for every $v \in V_2$ it holds that $w_q \in V(\mathcal{P}_T(s, v))$. Hence, for any $h \geq 1$ if there were at least two vertices in V_2 with the same height (hence, the same distance from w_q), then they would not be distinguished from each other in T by any sensor.

Note that we must have $q = d_T(s_0, w_q) \geq d_T(s_0', w_q)$ as otherwise $\mathcal{P}_T(s_0', s_1)$ would be a longer sensor path than $\mathcal{P}_T(s_0, s_1)$, violating Condition 5.1(iii). This implies, by the discussion before Definition 5.3, that the maximal height of a type-q vertex in $A_T(s_0, s_0')$ is $k - q$. Consequently, for any $v \in V_1$, $\mathrm{hgt}_{s_0,s_0'}(v) \leq k - q$. To show the same for the vertices of V_2 fix some $v \in V_2$ and assume that there exists a sensor $s \in S \setminus \{s_0, s_0'\}$ that directly measures v. Then s also directly measures w_q (by Claim 5.4). If $d_T(s, w_q) \leq q$ held, then $\mathcal{P}_T(s, s_0')$ would be a sensor path that is at most as long as $\mathcal{P}_T(s_0, s_0')$, meaning that it would be a strong sensor path. $\mathcal{P}_T(s, s_0')$ and $\mathcal{P}_T(s_0, s_0')$ would then form a pair of strong sensor paths sharing an edge, contradicting Condition 5.1(ii). Hence, $d_T(s, w_q) > q = \max\{d_T(s_0, w_q), d_T(s_0', w_q)\}$. Since $w_q \in V(\mathcal{P}_T(s, v))$, this implies that if s directly measures v, then so do both s_0 and s_0'. Hence, regardless of the locations of the other sensors, s_0 and s_0' both measure directly every vertex in V_2, implying that their height can be at most $k - q$.

Finally, we have to show that if $x \in V_1$ and $y \in V_2$, then $\mathrm{hgt}_{s_0,s_0'}(x) = \mathrm{hgt}_{s_0,s_0'}(y)$ cannot hold. Assume that it does hold. Then $x \in A_T(s_0, s_0')$ implies that $y \notin A_T(s_0, s_0')$ (otherwise they would not be distinguished). That is, there exists a sensor $s_y \in S \setminus \{s_0, s_0'\}$ that directly measures y. As before, $w_q \in V(\mathcal{P}_T(s_y, y))$ holds by Claim 5.4. This implies that

$$d_T(s_y, x) \leq d_T(s_y, w_q) + d_T(w_q, x) = d_T(s_y, w_q) + d_T(w_q, y) = d_T(s_y, y) \leq k, \tag{14}$$

where in (14) we used that $\mathrm{hgt}_{s_0,s_0'}(x) = \mathrm{hgt}_{s_0,s_0'}(y)$, combined with $\mathrm{typ}_{s_0,s_0'}(x) = \mathrm{typ}_{s_0,s_0'}(y) = q$. Hence, s_y also measures x. By $s_0, s_0' \notin V(\mathcal{P}_T(s_y, x))$, this means that s_y either directly measures x, or there is a sensor $s' \in S \setminus \{s_0, s_0'\}$ on the path $\mathcal{P}_T(s_y, x)$ that directly measures x, contradicting the assumption that $x \in A_T(s_0, s_0')$.

Combining all the above finishes the proof of (i) and (ii). It then follows that $|V_1 \cup V_2| \leq k - q$. Since $q = d_{T'}(s_0, w_q) \geq d_{T'}(s_0', w_q)$, and the vertices of $V_1 \cup V_2$ form a single leaf-path emanating from w_q in T', this implies that both s_0 and s_0' directly measure every vertex in $V_1 \cup V_2$ in T'. □

Having proved the necessary preliminaries about Transformation B, we now state and prove its main properties.

Lemma 5.7 *(Properties of Transformation B). Let $T = (V, E)$ be a tree with a k-truncated resolving set $S \subseteq V$ and $s_0, s_1 \in S$, for which Condition 5.1(i)–(iii) hold, and consider $T' := \mathrm{tr}_B(T, S, s_0, s_1)$. Then the following hold:*

(i) *S remains a k-truncated resolving set for T',*
(ii) *for each $s \in S$, $A_{T'}(s) = A_T(s)$, and its vertices still form a leaf-path in T' emanating from s,*
(iii) *for each pair of sensors $s, s' \in S$, if $\mathcal{P}_{T'}(s, s')$ is a sensor path, then $\mathcal{P}_T(s, s')$ was also a sensor path (in T), and $d_{T'}(s, s') \leq d_T(s, s')$,*
(iv) *$\mathcal{P}_{T'}(s_0, s_1)$ is still a sensor path, and is strictly shorter than $\mathcal{P}_T(s_0, s_1)$,*
(v) *if $\mathcal{P}_{T'}(s_0, s_1)$ is a strong sensor path, then T' has a pair of strong sensor paths that share an edge.*

Proof. Proof of (i): Let $x, y \in V$ be a pair of distinct vertices. We shall prove that there is a sensor in S that resolves them in T', similarly to the proof of Lemma 4.3(i). We will use the notations of Definition 5.3. We will do a case-distinction analysis with respect to the location of x and y in the components $\widetilde{T}_i, i \geq 0$ or in V_1 described in the transformation. The numbering of the cases is consistent with that in the proof of Lemma 4.3(i).

Case 1: Assume that $x \in \widetilde{T}_i$ for some $i \geq 1$, and that $y \in \widetilde{T}_j$ for some $j \geq 0$, $j \neq i$. Then, since $x \in V \setminus (A_T(s_0) \cup A_T(s_0') \cup A_T(s_0, s_0'))$, there is a sensor $s' \in S \setminus \{s_0, s_0'\}$ that directly measures x. Then, by Claim 5.5(i)–(ii), $s' \in \widetilde{T}_i$. Therefore, the edges of $\mathcal{P}_T(s', x)$ are unchanged in T', so s' still directly measures x in T'. Then, either s' distinguishes x and y in T', or

$$d_{T'}(s', y) = d_{T'}(s', x) \leq k. \tag{15}$$

Assume that we have this latter case. Now Claim 5.5(i) implies that $s_0 \in V(\mathcal{P}_{T'}(s', y))$ (this is also true if $y \in \widetilde{T}_0$ by the construction). Then $d_{T'}(s_0, y) \leq d_{T'}(s', y) \leq k$, so s_0 also measures y in T'. We will prove that in this case s_0 distinguishes x and y in T'. Assume that this is not the case, and in fact

$$d_{T'}(s_0, x) = d_{T'}(s_0, y) \leq k. \tag{16}$$

Then (15) and (16) imply that

$$d_{T'}(s', y) - d_{T'}(s_0, y) = d_{T'}(s', x) - d_{T'}(s_0, x). \tag{17}$$

Since $s_0 \in V(\mathcal{P}_{T'}(s', y))$, the left-hand side of (17) is equal to $d_{T'}(s', s_0)$. However, this could only be equal to the right-hand side of (17) if $s_0 \in V(\mathcal{P}_{T'}(s', x))$ held, which cannot be the case, since $\mathcal{P}_{T'}(s', x)$ is fully contained in $\widetilde{T}_i$. This contradiction finishes the proof of Case 1.

Case 2: Now assume that $x, y \in \widetilde{T}_i$ for some $i \geq 1$. Let s' be a sensor that distinguishes them in T. If $s' \in \widetilde{T}_i$, then the paths $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ remain unchanged by the transformation, hence s' still distinguishes x and y in T' and we are done. If $s' \notin \widetilde{T}_i$, then $s' \in \widetilde{T}_0$ by Claim 5.5(i). Then, since $x, y \in \widetilde{T}_i$, the paths $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ both contain x_i (the closest vertex to w_q in $\widetilde{T}_i$). On the other hand, the paths $\mathcal{P}_T(x_i, x)$ and $\mathcal{P}_T(x_i, y)$ fully belong to $\widetilde{T}_i$, so they are unchanged by the transformation, while the edge $\{s_0, x_i\}$ is added when creating T'. Combining these facts gives

$$d_{T'}(s_0, x) = 1 + d_{T'}(x_i, x) = 1 + d_T(x_i, x) \leq d_T(s', x_i) + d_T(x_i, x) = d_T(s', x), \tag{18}$$

$$d_{T'}(s_0, y) = 1 + d_{T'}(x_i, y) = 1 + d_T(x_i, y) \leq d_T(s', x_i) + d_T(x_i, y) = d_T(s', y), \tag{19}$$

and we assumed that the right-hand side of both sides is at most k, so s_0 measures both x and y in T', and further, since we replaced the segment $\mathcal{P}_T(s_0, x_i)$ by a single edge in T',

$$d_{T'}(s_0, x) - d_{T'}(s_0, y) = d_T(x_i, x) - d_T(x_i, y) = d_T(s', x) - d_T(s', y). \tag{20}$$

The combination of (18), (19) and (20) implies that if s' distinguished x and y in T, then s_0 distinguishes them in T', finishing the proof.

Case 3: Next, assume that $x, y \in \widetilde{T}_0$. Then there is a sensor s' that resolves them in T. If $s' \in \widetilde{T}_0$, then the paths $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ completely lie in $\widetilde{T}_0$, so they stay intact during the transformation. Hence, s' still resolves x, y in T' and we are done. Now assume that $s' \in \widetilde{T}_i$ for some $i \geq 1$. We will show that in this case either s_0 or s_0' resolves x, y in T'.

By Claim 5.4, $\mathrm{typ}_{s_0,s_0'}(s') = q$ in T, and $\{w_q, u_{q+1}\} \in E(\mathcal{P}_T(w_q, s'))$. Hence, both $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ contain w_q. Without loss of generality we can assume that there is no sensor besides s' on the path $\mathcal{P}_T(s', w_q)$ (if there was one, we could relabel that to s'). Now if $d_T(s', w_q) \leq \max\{d_T(s_0, w_q), d_T(s_0', w_q)\}$ then two of $\mathcal{P}_T(s_0, s_0')$, $\mathcal{P}_T(s_0, s')$ and $\mathcal{P}_T(s_0', s')$ would form a pair strong sensor paths that share an edge, contradicting Condition 5.1(ii). Hence, $d_T(s', w_q) > \max\{d_T(s_0, w_q), d_T(s_0', w_q)\}$, thus

$$d_T(s_0, x) \leq d_T(s_0, w_q) + d_T(w_q, x) < d_T(s', w_q) + d_T(w_q, x) = d_T(s', x), \tag{21}$$

and the same holds for s_0' in place of s_0. This implies that if s' measures x in T, then so do s_0 and s_0', and the same reasoning holds for y.

We know that s' measures at least one of x and y in T, say x. Then, by (21) both s_0 and s_0' also measure x in T. Since $s_0, s_0', x, y \in \widetilde{T}_0$, the paths $\mathcal{P}_T(s_0, x)$, $\mathcal{P}_T(s_0', x)$ remain unchanged by the transformation, hence, s_0 and s_0' both measure x in T' too. Now assume that neither s_0 nor s_0' distinguishes x, y in T'. Recalling Definition 3.11, this means that

$$\mathrm{typ}_{s_0,s_0'}(x) = \mathrm{typ}_{s_0,s_0'}(y), \qquad \mathrm{hgt}_{s_0,s_0'}(x) = \mathrm{hgt}_{s_0,s_0'}(y),$$

in T', and thus also in T, otherwise at least one of s_0, s_0' would resolve x, y. In particular this also implies that $d_T(w_q, x) = d_T(w_q, y)$. Then

$$d_T(s', x) = d_T(s', w_q) + d_T(w_q, x) = d_T(s', w_q) + d_T(w_q, y) = d_T(s', y)$$

by $w_q \in V(\mathcal{P}_T(s', x)) \cap V(\mathcal{P}_T(s', y))$. This contradicts the assumption that s' resolved x, y in T, thus finishing the proof.

Case 4: Assume that $x, y \in V_1 \cup V_2$. Then, by the construction of the edge set E_2, x and y will both lie on a leaf-path in T' emanating from w_q. Hence, $\mathrm{typ}_{s_0,s_0'}(x) = \mathrm{typ}_{s_0,s_0'}(y)$ and $\mathrm{hgt}_{s_0,s_0'}(x) \neq \mathrm{hgt}_{s_0,s_0'}(y)$ in T', so x and y are distinguished by s_0 or s_0' (or both) as long as at least one of them is measured by at least one of s_0 and s_0'. But in fact both x and y are measured by both s_0 and s_0' in T' by Claim 5.6, finishing the proof.

Case 5: Next, assume that $x \in \widetilde{T}_i$ for some $i \geq 1$, and $y \in V_1 \cup V_2$. The proof in this case is identical to that of Case 1.

Case 6: Finally, assume that $x \in \widetilde{T}_0$ and $y \in V_1 \cup V_2$. Then, by the construction of T', and by Claim 5.6, in T' we have $\mathrm{typ}_{s_0,s_0'}(y) = q$ and $1 \leq \mathrm{hgt}_{s_0,s_0'}(y) \leq k - q$, while for x either $x = w_q$ or $\mathrm{typ}_{s_0,s_0'}(x) \neq q$. This implies that x, y are resolved by $\{s_0, s_0'\}$ in T', either by Claim 3.12 or by the fact that the height is not identical.

Proof of (ii): Assume that for a sensor $s \in S$, the vertices of $A_T(s)$ form a leaf-path emanating from s in T. Then none of the vertices in $A_T(s)$ are adjacent to a vertex in $V_1 \subseteq A_T(s_0, s_0')$. Also, any vertex in $A_T(s)$ can only be adjacent to a type-q vertex in T (with respect to s_0, s_0') if $\mathrm{typ}_{s_0,s_0'}(s) = q$. In this case, by Claim 5.4, $\{w_q, u_{q+1}\} \in E(\mathcal{P}_T(w_q, s)) \subseteq E(\mathcal{P}_T(w_q, v))$ for any $v \in A_T(s)$. Thus, none of the vertices in $A_T(s)$ are adjacent to a vertex in V_2 either. Hence, the removal of the edge set E_1 does not change this leaf-path. The addition of the edge set E_2 also does not add an edge adjacent to any of the vertices in $A_T(s)$. After these steps, if any one of the vertices in $A_T(s) \cup \{s\}$ is in $\widetilde{T}_i$ for some $i \geq 1$, then all of $A_T(s) \cup \{s\}$ are in $\widetilde{T}_i$, and s was closer to s_0 in T than any vertex in $A_T(s)$. Hence, the addition of the new edge between s_0 and $\widetilde{T}_i$ in E_3 will again not add an edge adjacent to any vertex in $A_T(s)$. This proves that $A_T(s) \subseteq A_{T'}(s)$, and the vertices of $A_T(s)$ still form a leaf-path emanating from s in T'.

Next, we have to show that there is no new vertex in $A_{T'}(s)$ compared to $A_T(s)$ for any sensor s. Assume that $x \in V \setminus (S \cup (\cup_{s \in S} A_T(s)))$. To prove that $x \notin A_{T'}(s)$ for any $s \in S$, we distinguish the following cases.

Case 1: Assume that $\mathrm{typ}_{s_0,s_0'}(x) = 0$ in T. Then any sensor $s^\star \in S$ that directly measures x in T also has $\mathrm{typ}_{s_0,s_0'}(s^\star) = 0$ (otherwise $s_0 \in V(\mathcal{P}_T(s^\star, x))$ would hold). Hence, the path $\mathcal{P}_T(s^\star, x)$ remains unchanged by the transformation. Since $x \in$

$V \setminus (S \cup (\cup_{s \in S} A_T(s)))$, there are at least two sensors $s_1^\star, s_2^\star$ that directly measure x in T, hence, by the above reasoning applied twice, they both measure x directly in T' too, proving that $x \notin A_{T'}(s)$ for any $s \in S$.

Case 2: Assume that $\mathrm{typ}_{s_0,s_0'}(x) = d_T(s_0, s_0')$ in T. The proof in this case is identical to that of Case 1 with s_0' taking the role of s_0, and type $d_T(s_0, s_0')$ taking that of type 0.

Case 3: Assume that $\mathrm{typ}_{s_0,s_0'}(x) \notin \{0, q, d_T(s_0, s_0')\}$ in T. Then by Definition 5.3 the paths $\mathcal{P}_T(s_0, x)$, $\mathcal{P}_T(s_0', x)$ remain untouched by the transformation. We will show that this implies that both s_0 and s_0' directly measure x in T', and as a result $x \notin \cup_{s \in S} A_{T'}(s)$. Assume that $x \notin \cup_{s \in S} A_{T'}(s) \cup S$. Then there is at least two sensors $s_1^\star, s_2^\star$ directly measuring x in T. We shall show that s_0, s_0' can have these roles. If we immediately know that s_0, s_0' both directly measure x, we are done. Suppose now that we only know that there is an $s^\star \in S \setminus \{s_0, s_0'\}$ that directly measures x.

By Claim 5.4, for every sensor $s \in S$, $\mathrm{typ}_{s_0,s_0'}(s) \in \{0, q, d_T(s_0, s_0')\}$. Since $w_q \notin S$, and $\mathrm{typ}_{s_0,s_0'}(x) \neq \{0, q, d_T(s_0, s_0')\}$ by assumption, this means that there is no sensor on the paths $\mathcal{P}_T(s_0, x)$, $\mathcal{P}_T(s_0', x)$ besides s_0, s_0', respectively. Furthermore, for every sensor $s^\star \in S$ that directly measures x, $w_q \in V(\mathcal{P}_T(s^\star, x))$, so $s^\star$ also directly measures w_q. Now we use the fact that in this case $d_T(s^\star, w_q) > \max\{d_T(s_0, w_q), d_T(s_0', w_q)\}$, as in the proof of Claim 5.6, since otherwise two of the paths $\mathcal{P}_T(s^\star, s_0)$, $\mathcal{P}_T(s^\star, s_0')$ and $\mathcal{P}_T(s_0, s_0')$ would form a pair of strong sensor paths sharing an edge, contradicting Condition 5.1. Hence,

$$d_T(s_0, x) \le d_T(s_0, w_q) + d_T(w_q, x) < d_T(s^\star, w_q) + d_T(w_q, x) = d_T(s^\star, x) \le k,$$

and the same holds for s_0' in place of s_0. Therefore, both s_0 and s_0' directly measure x in T. By the fact that $\mathrm{typ}_{s_0,s_0'}(x) \neq q$, the paths $\mathcal{P}_T(s_0, x), \mathcal{P}_T(s_0', x)$ are untouched by transformation B, s_0, s_0' both directly measure x in T' as well, showing that $x \notin A_{T'}(s)$ for any sensor $s \in S$.

Case 4: Assume that $x \in V_1 \cup V_2$. Then x will lie on a leaf-path in T' emanating from w_q, which is of length at most $k - q$ by Claim 5.6. Hence, x will be directly measured by both s_0 and s_0' in T', implying that $x \notin A_{T'}(s)$ for any $s \in S$.

Case 5: The last case is when $\mathrm{typ}_{s_0,s_0'}(x) = q$ in T and $x \notin V_1 \cup V_2$. Then, by Definition 5.3, $x \in \widetilde{T}_i$ for some $i \ge 1$, and $x \notin A_T(s_0, s_0')$. This latter fact implies that there exist two distinct sensors $s_1^\star, s_2^\star$ that directly measure x in T and $\{s_1^\star, s_2^\star\} \neq \{s_0, s_0'\}$. If both $s_1^\star, s_2^\star \in \widetilde{T}_i$, then the paths $\mathcal{P}_T(s_1^\star, x)$, $\mathcal{P}_T(s_2^\star, x)$ remain unchanged by the transformation, hence, both $s_1^\star$ and $s_2^\star$ still directly measure x in T', finishing the proof. If either $s_1^\star$ or $s_2^\star$ is not in $\widetilde{T}_i$, then it has to be in $\{s_0, s_0'\}$ by Claim 5.5(i)–(ii). Since $\{s_1^\star, s_2^\star\} \neq \{s_0, s_0'\}$, we can assume in this case that $s_1^\star \in \{s_0, s_0'\}$ and $s_2^\star \in \widetilde{T}_i$. Similarly as above, $s_2^\star$ then directly measures x in T'. We will finish the proof by showing that the fact that either s_0 or s_0' directly measures x in T implies that s_0 directly measures x in T'. Since x_i is the closest vertex of $\widetilde{T}_i$ to s_0 in T, and the edge $\{s_0, x_i\}$ is added in T' by the transformation, we have

$$d_{T'}(s_0, x) = 1 + d_{T'}(x_i, x) = 1 + d_T(x_i, x),$$

since the path $\mathcal{P}_T(x_i, x)$ remains unchanged by the transformation. Hence,

$$d_{T'}(s_0, x) \le \min\{d_T(s_0, x_i), d_T(s_0', x_i)\} + d_T(x_i, x) = \min\{d_T(s_0, x), d_T(s_0', x)\} \le k,$$

as x_i lies on both paths $\mathcal{P}_T(s_0, x)$ and $\mathcal{P}_T(s_0', x)$. This shows that s_0 indeed measures x in T'. The fact that s_0 directly measures x in T' follows from the fact that $\mathcal{P}_{T'}(x_i, x)$ does not contain any sensors, since it is a subpath of both $\mathcal{P}_T(s_0, x)$ and $\mathcal{P}_T(s_0', x)$, and one of these did not contain any internal sensors by the assumption. This finishes the proof that there are indeed at least two sensors that directly measure x in T', and thus $x \notin A_{T'}(s)$ for any $s \in S$.

Proof of (iii): If $\mathcal{P}_{T'}(s, s')$ is a sensor path, then there are two possible cases. First, if $s, s' \in \widetilde{T}_i$ for some $i \ge 0$, then $\mathcal{P}_{T'}(s, s')$ is the same as $\mathcal{P}_T(s, s')$ (its edges remain unchanged by the transformation), hence, $\mathcal{P}_T(s, s')$ is also a sensor path with the same length. Second, if, say, $s' = s_0$, and $s \in \widetilde{T}_i$ for some $i \ge 1$, then the path $\mathcal{P}_{T'}(s_0, s)$ consists of the sub-path $\mathcal{P}_T(x_i, s)$ of $\mathcal{P}_T(s_0, s)$ and the edge $\{s_0, x_i\}$, where recall that x_i is the closest vertex of $\widetilde{T}_i$ to s_0 in T. If $\mathcal{P}_{T'}(s_0, s)$ is a sensor path, then there is no sensor beside s on $\mathcal{P}_{T'}(x_i, s) = \mathcal{P}_T(x_i, s)$. On the other hand, if y is the vertex neighboring x_i on the path $\mathcal{P}_T(s_0, x_i)$, then either $y \in A_T(s_0, s_0')$ or $y = w_q$, both implying that s_0 directly measures y in T, that is, there is no other sensor between them. Consequently, $\mathcal{P}_T(s_0, s)$ is indeed a sensor path, and its length is more than that of $\mathcal{P}_{T'}(s_0, s)$, as $y \in V(\mathcal{P}_T(s_0, s)) \setminus V(\mathcal{P}_{T'}(s_0, s))$.

There are indeed no more cases for a sensor path $\mathcal{P}_{T'}(s, s')$, as the construction in Definition 5.3 ensures that if s and s' are in different components among $\widetilde{T}_0, \widetilde{T}_1, \ldots$, then $\mathcal{P}_{T'}(s, s')$ contains s_0.

Proof of (iv): By Definition 5.3, x_1 is the closest vertex to s_0 in T in the subtree $\widetilde{T}_1$. Since T is a tree, and $s_1 \in \widetilde{T}_1$, x_1 lies on the path $\mathcal{P}_T(s_0, s_1)$. This implies that $\mathcal{P}_T(x_1, s_1) \subseteq \mathcal{P}_T(s_0, s_1)$, and the edges of $\mathcal{P}_T(x_1, s_1)$ stay intact in T'. In particular, $\mathcal{P}_{T'}(x_1, s_1)$ does not contain another sensor besides s_1. Since $\{s_0, x_1\}$ is an edge in T', this proves that $\mathcal{P}_{T'}(s_0, s_1)$ is indeed a sensor path in T'.

Next, we will prove that $\mathcal{P}_{T'}(s_0, s_1)$ is strictly shorter than $\mathcal{P}_T(s_0, s_1)$. Since $x_1 \in \widetilde{T}_1$, and $w_1 \in \widetilde{T}_0$, it holds that $x_1 \neq w_1$, furthermore, $w_1 \in V(\mathcal{P}_T(s_0, x_1)) \setminus \{s_0, x_1\}$. Hence, $|E(\mathcal{P}_T(s_0, x_1))| \ge 2$, while $\{s_0, x_1\}$ is a single edge in T'. Since the edges of $\mathcal{P}_T(x_1, s_1)$ remain unchanged in T',

$$\mathcal{P}_T(s_0, s_1) = \mathcal{P}_T(s_0, x_1) \cup \mathcal{P}_T(x_1, s_1) \quad \text{and} \quad \mathcal{P}_{T'}(s_0, s_1) = \{s_0, x_1\} \cup \mathcal{P}_T(x_1, s_1),$$

this finishes the proof.

Proof of (v): Let z be the vertex next to s_1 on the path $\mathcal{P}_T(s_0, s_1)$. Since $\mathcal{P}_T(s_0, s_1)$ is a weak sensor path, s_0 cannot measure z in T. On the other hand, s_0' cannot measure z in T either, as we will now show. Assume s_0' measures z in T. Then we have

$$d_T(s_0', s_1) \le d_T(s_0', z) + d_T(z, s_1) \le k+1. \tag{22}$$

We also have $V(\mathcal{P}_T(s_0', s_1)) = V(\mathcal{P}_T(s_0', w_q)) \cup V(\mathcal{P}_T(w_q, s_1))$, and $V(\mathcal{P}_T(w_q, s_1)) \subseteq V(s_0, s_1)$. Since $\mathcal{P}_T(s_0, s_1)$ and $\mathcal{P}_T(s_0, s_0')$ are both sensor paths, it follows that $\mathcal{P}_T(s_0', s_1)$ is also a sensor path. Hence, (22) shows that $\mathcal{P}_T(s_0', s_1)$ is a strong sensor path. Thus $\mathcal{P}_T(s_0, s_0')$ and $\mathcal{P}_T(s_0', s_1)$ are a pair of strong sensor paths sharing an edge. This contradicts with Condition 5.1(ii), showing that neither s_0 nor s_0' can measure z in T. But $z \in \mathcal{P}_T(s_0, s_1)$, so by Condition 5.1(i), $z \notin A_T(s_1)$, i.e., there has to be another sensor besides s_1 that directly measures z, say s_2. This, similarly to (15), shows that $\mathcal{P}_T(s_2, s_1)$ is a strong sensor path. This, in particular, implies that $s_2 \in \widetilde{T}_1$, and the edges of $\mathcal{P}_T(s_2, s_1)$ remain unchanged in T'. Now, if after the transformation, $\mathcal{P}_{T'}(s_0, s_1)$ becomes a strong sensor path, then $s_2 \notin \mathcal{P}_{T'}(s_0, s_1)$ shows that $\mathcal{P}_{T'}(s_0, s_2)$ and $\mathcal{P}_{T'}(s_2, s_1)$ form a pair of strong sensor paths that share an edge, thus finishing the proof of part (v). □

Finally, we can use the main properties of Transformation B to achieve the main result of this section: a modification of the tree that has overlapping strong sensor paths.

Lemma 5.8. *Let $T = (V, E)$ be a tree with a k-truncated resolving set S. Suppose that Condition 5.1(i)–(iii) hold for T. Then there is another tree $\widehat{T} = (V, \widehat{E})$ on the same vertex set such that the following hold:*

(i) *S is still a k-truncated resolving set in $\widehat{T}$,*
(ii) *for each $s \in S$, $A_{\widehat{T}}(s) = A_T(s)$, and its vertices still form a leaf-path in $\widehat{T}$ emanating from s, and*
(iii) *there is at least one pair of strong sensor paths in $\widehat{T}$ that share an edge.*

Proof. Let $T_0 = T$, and then let us iteratively define $T_i := \mathrm{tr}_B\big(T_{i-1}, S, s_0^{(i-1)}, s_1^{(i-1)}\big)$ for $i \ge 1$, as long as T_{i-1} does not have a pair of strong sensor paths that share an edge, and where $s_0^{(i-1)}$, $s_1^{(i-1)}$ are the endpoints of one of the longest weak sensor paths in T_{i-1}. Let $\widehat{T} = T_{i_{\max}}$ where $i_{\max}$ is the first index i in this procedure for which there is a pair strong sensor paths in T_i that share an edge. We prove that this procedure is well-defined, and $\widehat{T} = T_{i_{\max}}$ satisfies the conditions of Lemma 5.8.

For an inductive proof, assume that S is a k-truncated resolving set in T_{i-1}, $1 \le i \le i_{\max}$, and Condition 5.1(i)–(iii) all hold for T_{i-1}. (These indeed hold for $i = 1$.) Then $T_i := \mathrm{tr}_B\big(T_{i-1}, S, s_0^{(i-1)}, s_1^{(i-1)}\big)$ is well-defined.

By Lemma 5.7(i), if S was a k-truncated resolving set in T_{i-1}, then it will remain so in T_i. By Lemma 5.7(ii), if Condition 5.1(i) held for T_{i-1}, then it will also hold in T_i. Furthermore, for every sensor $s \in S$, $A_{T_i}(s) = A_{T_{i-1}}(s)$, and its vertices still form a leaf-path emanating from s in T_i. Condition 5.1(ii) holds for T_i by assumption when $i \le i_{\max} - 1$. By Lemma 5.7(iv)–(v) either $\mathcal{P}_{T_i}(s_0^{(i-1)}, s_1^{(i-1)})$ is still a weak sensor path in T_i, and then Condition 5.1(iii) holds for T_i, or $\mathcal{P}_{T_i}(s_0^{(i-1)}, s_1^{(i-1)})$ is a strong sensor path in T_i, and then T_i has a pair of strong sensor paths sharing an edge, meaning $i = i_{\max}$. This finishes the proof that $T_i := \mathrm{tr}_B\big(T_i, S, s_0^{(i-1)}, s_1^{(i-1)}\big)$ is indeed well-defined for $i \le i_{\max}$, and inductively shows that parts (i), (ii) of Lemma 5.8 hold for T_i for any $i \le i_{\max}$. We are only left to show that the procedure finishes in finitely many steps, that is, $i_{\max} < \infty$. In that case, by assumption part (iii) of Lemma 5.8 also holds for $\widehat{T} = T_{i_{\max}}$.

By Lemma 5.7(iii), for $i \le i_{\max}$, each sensor path in T_{i-1} either stops being a sensor path in T_i, or otherwise its length does not increase. Also, new sensor paths cannot emerge in T_i compared to T_{i-1}. Moreover, by part (iv) of the same lemma, the length of at least one sensor path strictly decreases from T_{i-1} to T_i. Therefore, if Σ_i is the sum of the lengths of all sensors paths in T_i, then $\Sigma_i \le \Sigma_{i-1} - 1$ for every $i \le i_{\max}$. But $\Sigma_i \ge 0$ has to hold for every i, which implies that $i_{\max} < \infty$, finishing the proof. □

6. Transformation C: overlapping short sensor paths are suboptimal

Having applied Transformations A and B repeatedly, as in Lemmas 4.4 and 5.8 in the previous two sections, we can now assume that we have a tree T with k-truncated resolving set S such that one of the following two cases holds. In the first case T has a special structure, such that (i) attractions of single sensors are contained in a leaf-path attached to that sensor, (ii) all the strong sensor paths in T are disjoint, possibly except for their endpoints, and (iii) T does not have any weak sensor path. In this case, we skip the arguments in this section, and move directly to Section 7 to find the maximal size of such a tree. The other case is when T has a pair overlapping strong sensor paths. In this section we prove that such a tree cannot be optimal. We achieve this by introducing a third rewiring procedure, Transformation C, which separates these two overlapping sensor paths, while adding a new vertex to T, such that S is still a k-truncated resolving set on this larger graph.

The structure of this section is as follows. First, we state the conditions under which Transformation C will be applied, fixing the notation, before introducing the transformation. After that, we first observe preliminary facts about Transformation

C, then state and prove its main properties in Lemma 6.4. Finally, we state and prove Lemma 6.5, the main result of this section, which shows that an optimal tree cannot have overlapping strong sensor paths.

To help the reader we try to make the notation as similar to Transformations A and B as possible throughout this section.

Condition 6.1. Let $T=(V,E)$ be a tree with a k-truncated resolving set $S\subseteq V$. Assume that T has at least one pair of strong sensor paths that share an edge. We assume that $\mathcal{P}_T(s_0,s_0')$ is one of the shortest among the strong sensor paths that share an edge with another strong sensor path, and that $\mathcal{P}_T(s_0,s_1)$ is one of the shortest among the strong sensor paths that share an edge with $\mathcal{P}_T(s_0,s_0')$. With $|V(\mathcal{P}_T(s_0,s_0'))\cap V(\mathcal{P}_T(s_0,s_1))|=q\le k-1$, we denote the vertices on these two paths as follows:

$$\begin{aligned}V(\mathcal{P}_T(s_0,s_0'))&=\{s_0,w_1,\dots,w_q,w_{q+1},\dots,w_{d(s_0,s_0')-1},s_0'\},\\ V(\mathcal{P}_T(s_0,s_1))&=\{s_0,w_1,\dots,w_q,u_{q+1},\dots,u_{d(s_0,s_1)-1},s_1\},\end{aligned}\tag{23}$$

with $q\le d_T(s_0,s_0')-q\le d_T(s_0,s_1)-q$, and each sensor $s^\star\in S\setminus\{s_0,s_0',s_1\}$ that directly measures w_q has $d_T(s^\star,w_q)\ge d_T(s_0,s_1)-q$.

The last statement is indeed true since the assumptions that $\mathcal{P}_T(s_0,s_0')$ is the shortest strong sensor path among the ones sharing an edge with another strong one, and that $\mathcal{P}_T(s_0,s_1)$ is at most as long as $\mathcal{P}_T(s_0',s_1)$ imply that $d_T(s_0,w_q)\le d_T(s_0',w_q)\le d_T(s_1,w_q)$. I.e., among the sensors $s\in S$ for which there is no other sensor on the path $\mathcal{P}_T(s,w_q)$, the closest one to w_q is s_0, the second closest one is s_0', and the third closest one is s_1 (ties are allowed). Moreover, all three of s_0,s_0',s_1 (directly) measure w_q, since $\mathcal{P}_T(s_0,s_0')$ and $\mathcal{P}_T(s_0,s_1)$ are strong sensor paths.

Note that it is possible that $w_{q+1}=s_0'$ or $u_{q+1}=s_1$, but only if $q=1$.

In order to prove that overlapping strong sensor paths (as in Condition 6.1) make T suboptimal, we introduce *Transformation C* next. Heuristically speaking, we will do the following. We 'separate' the overlapping paths $\mathcal{P}_T(s_0,s_0')$ and $\mathcal{P}_T(s_0,s_1)$ by keeping the former one intact, while in $\mathcal{P}_T(s_0,s_1)$ we replace the segment $\mathcal{P}_T(s_0,u_{q+1})$ by a new path $(s_0,v^\star,u_{q+1})$ with a new vertex $v^\star$, while cutting the edge $\{w_q,u_{q+1}\}$. This way we increase the number of vertices in the graph while not increasing the length of either $\mathcal{P}_T(s_0,s_0')$ or $\mathcal{P}_T(s_0,s_1)$. Now the vertices that were measured in T by s_0,s_0' 'through' w_q might not be distinguished from some other vertices anymore, since we cut the edge $\{w_q,u_{q+1}\}$. To solve this problem we make some further changes in the graph. We *pretend* that w_q is a sensor, but with a smaller measuring radius $k-(d_T(s_0,s_0')-q)$. This is the distance up to which s_0' measured vertices through w_q in T. We then 'cut out' the attraction of w_q (vertices that are only directly measured by w_q, if it were the above-mentioned sensor) from the tree, and move it to a leaf-path emanating from w_q, similarly to transformation A. We obtain then a forest. Then we connect each connected component of this forest to form a new tree by connecting s_0 to the originally closest vertex in every other component, again similarly to transformation A. Formally, the transformation is as follows.

Definition 6.2 *(Transformation C).* Let $T=(V,E)$ be a tree with a k-truncated resolving set $S\subseteq V$ satisfying Condition 5.1(i), and $s_0,s_0',s_1\in S$, $w_q\in V$ satisfying the setting of Condition 6.1. Let, for some $\ell\in\{0,1,\dots,k-(d_T(s_0,s_0')-q)\}$,

$$A_T^\star(w_q):=\big\{x\in V\setminus S:\forall s^\star\in S:d(s^\star,x)\ge k+1\text{ or }w_q\in V(\mathcal{P}_T(s^\star,x))\big\}=:\{v_1,\dots,v_\ell\},\tag{24}$$

where $d_T(w_q,v_i)\le d_T(w_q,v_j)$ when $i\le j$. Then define the following edge sets.

$$E_1:=\{\{w_q,u_{q+1}\}\}\cup\big\{\{x,y\}\in E(T):x\in A_T^\star(w_q)\text{ or }y\in A_T^\star(w_q)\big\},\tag{25}$$

$$E_2:=\{\{w_q,v_1\}\}\bigcup(\cup_{i=1}^{\ell-1}\{\{v_i,v_{i+1}\}\}).\tag{26}$$

Let $\widetilde{T}_0,\widetilde{T}_1,\dots,\widetilde{T}_r$ be the connected components of $\widetilde{T}=(V\setminus A_T^\star(w_q),E\setminus E_1)$, with $\widetilde{T}_0$ containing s_0 (and the whole path $\mathcal{P}_T(s_0,s_0')$), and $\widetilde{T}_1$ containing s_1. Let x_i be the unique closest vertex of $\widetilde{T}_i$ to w_q in T, for $i\in\{2,3,\dots,r\}$, and let $v^\star$ be a new vertex, which is not in V. Then we also define the edge set

$$E_3:=\Big(\{\{s_0,v^\star\}\}\cup\{\{v^\star,u_{q+1}\}\}\Big)\bigcup\Big(\cup_{i=1}^r\{\{s_0,x_i\}\}\Big).\tag{27}$$

Then define $\mathrm{tr}_C(T,S,s_0,s_0',s_1):=(V',E')$ where $V':=V\cup\{v^\star\}$, and $E':=(E\setminus E_1)\cup E_2\cup E_3$.

For an example of Transformation C see Fig. 3.

We make a couple of comments on this definition. Since $\mathcal{P}_T(s_0,s_0')$ and $\mathcal{P}_T(s_0,s_1)$ are both strong sensor paths, all of their vertices are measured directly by both endpoints of the path. Hence, $V(\mathcal{P}_T(s_0,s_0'))\cup V(\mathcal{P}_T(s_0,s_1))\subseteq V\setminus A_T^\star(w_q)$, and in fact, $V(\mathcal{P}_T(s_0,s_0'))\subseteq\widetilde{T}_0$, and $V(\mathcal{P}_T(u_{q+1},s_1))\subseteq\widetilde{T}_1$. (Note that $\widetilde{T}_1$ is the only component $\widetilde{T}_i$ that was not separated from $\widetilde{T}_0$ by a vertex in $A_T^\star(w_q)$, but by the additional cut that we made at the edge $\{w_q,u_{q+1}\}$.) Next, $|A_T^\star(w_q)|\le k-(d_T(s_0,s_0')-q)$, as we will now show. By the definition of $A_T^\star$, for every sensor s that measures a vertex $x\in A_T^\star$, $w_q\in V(\mathcal{P}_T(s,x))$. Hence, if $x,y\in A_T^\star(w_q)$ had the same distance from w_q, then they would have the same distance from every sensor that measures at

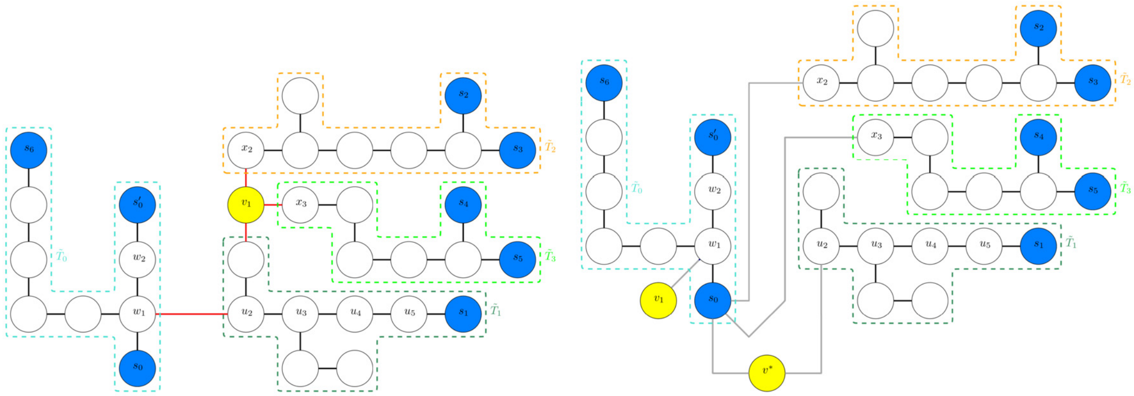

Fig. 3. An example of transformation C with T on the left and $T' = \mathrm{tr}_C(T, S, s_0, s'_0, s_1)$ on the right. Here $k = 5$. The blue vertices are the sensors in S, and the yellow vertices are in $A^\star_T(w_q) \cup \{v^\star\}$. The red edges belong to E_1, and are deleted by the transformation. The grey edges belong to $E_2 \cup E_3$, and are added by the transformation. The subtrees $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2$ and $\widetilde{T}_3$ are also highlighted. Observe that vertex x_2 and the leaf vertex at distance two from x_2 in $\widetilde{T}_2$ are only resolved by s_0 both in T and in T'. In T, the sensors in $\widetilde{T}_0$ may measure vertices in any subtree, while s_1 may only measure vertices in $\widetilde{T}_1$ and possibly in $\widetilde{T}_0$. Sensors in $\widetilde{T}_2$ and $\widetilde{T}_3$ can only measure vertices within their own subtrees.

least one of them, contradicting the fact that S is a k-truncated resolving set in T. On the other hand, the largest distance a vertex in $A^\star_T$ can be from w_q is $k - (d_T(s_0, s'_0) - q)$, otherwise s'_0 would not measure it, meaning that only s_0 could measure it directly (by the remarks after Condition 6.1), contradicting Condition 5.1(i).

Next, similarly to Transformations A and B, we prove a 'no communication' lemma for the graph components in tr_C.

Claim 6.3 *(No 'communication' between different subtrees). Consider the setting and notation of Definition 6.2 and let* $T' = \mathrm{tr}_C(T, S, s_0, s'_0, s_1)$.

(i) *Let $s^\star$ be any sensor in $\widetilde{T}_i$ for some $i \in \{2, 3, \dots, r\}$. Then, for all vertices $y \notin \widetilde{T}_i$, $d_T(s^\star, y) \ge k+1$ and $d_{T'}(s^\star, y) \ge k+1$ both hold.*
(ii) *Let $s^\star$ be any sensor in $\widetilde{T}_0 \cup \widetilde{T}_1$. Then, for any $y \in V \setminus (\widetilde{T}_0 \cup \widetilde{T}_1)$, either $d_T(s^\star, y) \ge k+1$, or the path $\mathcal{P}_T(s^\star, y)$ contains w_q.*
(iii) *Let $s^\star$ be any sensor in $\widetilde{T}_0$. Then, for any $y \in \widetilde{T}_1$, it holds that $\{w_q, u_{q+1}\} \in V(\mathcal{P}_T(s^\star, y))$. The same is true if $s^\star \in \widetilde{T}_1$ and $y \in \widetilde{T}_0$.*

Proof. The proofs parts (i)–(ii) are completely analogous to those of Claim 4.2(i)–(ii), with $s, A_T(s)$ there replaced by $w_q, A^\star_T(w_q)$ here. Part (iii) is immediate by the fact that the only edge connecting vertices of $\widetilde{T}_0$ and $\widetilde{T}_1$ in T is $\{w_q, u_{q+1}\}$. □

We continue by proving the main properties of Transformation C.

Lemma 6.4. *Let $T = (V, E)$ be a tree with a k-truncated resolving set $S \subseteq V$ such that Condition 5.1(i) and Condition 6.1 hold. Then S is still a k-truncated resolving set in* $\mathrm{tr}_C(T, S, s_0, s'_0, s_1)$.

Proof. We will prove that for any pair of vertices $x, y \in V' \setminus S$ there is a sensor in S that resolves them in $T' = \mathrm{tr}_C(T, S, s_0, s'_0, s_1)$, similarly to the proofs of Lemma 4.3(i) and Lemma 5.7(i). We will use the notation of Condition 6.1 and Definition 6.2. We will do a case-distinction analysis with respect to the location of x and y in the components $\widetilde{T}_i$, $i \ge 0$, and in the vertex sets $A^\star_T(w_q)$ and $\{v^\star\}$. The numbering of the cases is consistent with those in the proofs of Lemma 4.3(i) and 5.7(i).

Case 1a: Assume that $x \in \widetilde{T}_i$ and $y \in \widetilde{T}_j$ for some $i \ge 2$, $j \ge 0$, $j \ne i$. Then, since $x \notin A^\star_T(w_q)$, there is a sensor $s' \in S$ that measures x such that $\mathcal{P}_T(s', x)$ does not contain w_q. Then, by Claim 6.3(i)–(ii), $s' \in \widetilde{T}_i$. Therefore, the edges of $\mathcal{P}_T(s', x)$ are unchanged in T', so s' still measures x in T'. However, it does not measure $y \in \widetilde{T}_j$ in T' by Claim 6.3(i). Hence, s' resolves x and y in T'.

Case 1b: Assume that $x \in \widetilde{T}_1$ and $y \in \widetilde{T}_0$. Since $x \notin A^\star_T(w_q)$, there has to exist a sensor $s(x) \in S$ that measures x in T such that $w_q \notin V(\mathcal{P}_T(s(x), x))$. By Claim 6.3(i), $s(x) \notin \cup_{i\ge 2} V(\widetilde{T}_i)$, and $s(x)$ also cannot be in $\widetilde{T}_0$, since then $w_q \in V(\mathcal{P}_T(s(x), x))$ would be the case. Hence, $s(x) \in \widetilde{T}_1$. By the similar reasoning, there has to exist a sensor $s(y) \in S$ such that $s(y)$ measures y, and $w_q \notin V(\mathcal{P}_T(s(y), y))$, and hence $s(y) \in \widetilde{T}_0$. It follows that the paths $\mathcal{P}_T(s(x), x)$ and $\mathcal{P}_T(s(y), y)$ are unchanged by the transformation, hence, $s(x)$ still measures x in T', and $s(y)$ still measures y in T'. This implies that either $s(x)$ or $s(y)$ resolves x, y in T' as follows. For an indirect proof assume that neither $s(x)$ nor $s(y)$ resolves x, y in T'. Then, this assumption implies that $s(x)$ measures both x and y in T' with $d_{T'}(s(x), x) = d_{T'}(s(x), y)$, and the same holds for $s(y)$. It then follows that $\mathrm{typ}_{s(x),s(y)}(x) = \mathrm{typ}_{s(x),s(y)}(y)$ in T'. But this cannot be the case, as $x, s(x) \in \widetilde{T}_1$ and $y, s(y) \in \widetilde{T}_0$ together imply that

$$\mathrm{typ}_{s(x),s(y)}(x) < \mathrm{typ}_{s(x),s(y)}(v^\star) < \mathrm{typ}_{s(x),s(y)}(y)$$

in T', finishing the proof.

Case 2: Now assume that $x, y \in \widetilde{T}_i$ for some $i \geq 2$. Let $s' \in S$ be a sensor that resolves x and y in T. Then s' has to measure at least one of x and y in T, hence $s' \notin \widetilde{T}_j$ for $j \geq 2$, $j \neq i$ by Claim 6.3(i). There are two (sub)cases: either $s' \in \widetilde{T}_i$, or $s' \in \widetilde{T}_0 \cup \widetilde{T}_1$. First we consider $s' \in \widetilde{T}_i$. Then the edges of both $\mathcal{P}_T(s', x)$ and $\mathcal{P}_T(s', y)$ are all still present in T', and s' resolves x and y in T'.

For the other case, we assume that $s' \in \widetilde{T}_0 \cup \widetilde{T}_1$. We will prove that s_0 also resolves x, y in T in this case, and as a result s_0 will also resolve them in T'. First, we know that s' measures at least one of x and y, say it measures x. Then, since $x \in \widetilde{T}_i$ for some $i \geq 2$, by Claim 6.3(ii), $\mathcal{P}_T(s', x)$ contains w_q. On the path $\mathcal{P}_T(w_q, x)$ there has to be at least one vertex in $A^\star_T(w_q)$, let the closest one to x be u (u is unique, otherwise there would be a cycle in T). Then, since $\widetilde{T}_i$ is a connected component in $V \setminus A^\star_T(w_q)$, and $y \in \widetilde{T}_i$, u is also on the path $\mathcal{P}_T(w_q, y)$. Hence, $V(\mathcal{P}_T(s', w_q)) \subseteq V(\mathcal{P}_T(s', u)) \subseteq V(\mathcal{P}_T(s', x)) \cap V(\mathcal{P}_T(s', y))$. This implies that w_q is also contained in the path $\mathcal{P}_T(s', y)$. On the other hand, by the discussion after Condition 6.1, we have that $d_T(s_0, w_q) \leq d_T(s', w_q)$, hence

$$d_T(s_0, x) \leq d_T(s_0, w_q) + d_T(w_q, x) \leq d_T(s', w_q) + d_T(w_q, x) = d_T(s', x) \leq k,$$

and thus s_0 measures x in T. Then, by Claim 6.3(ii), the path $\mathcal{P}_T(s_0, x)$ contains w_q. Then, by the same reasoning as above (changing s' to s_0), we get that $\mathcal{P}_T(s_0, y)$ also contains w_q. Consequently,

$$d_T(s_0, x) - d_T(s_0, y) = d_T(w_q, x) - d_T(w_q, y) = d_T(s', x) - d_T(s', y),$$

proving that s_0 indeed also resolves x, y in T if s' resolves them in T.

Next, we will prove that s_0 then also resolves x, y in T'. Recall $x_i = \arg\min_{v \in \widetilde{T}_i} d_T(w_q, v)$ from Definition 6.2. To obtain T', we cut the edges adjacent to $A^\star_T(w_q)$ and replaced them by $\{s_0, x_i\} \in E_3$ (an edge added when creating T'). Since every path $\mathcal{P}_T(s_0, v)$, $v \in \widetilde{T}_i$, starts with the segment $\mathcal{P}_T(s_0, x_i)$ in T, which we replaced with the single edge $\{s_0, x_i\}$ to obtain $\mathcal{P}_{T'}(s_0, v)$, the following holds for all $v \in \widetilde{T}_i$ (for any $i \geq 2$):

$$d_{T'}(s_0, v) = d_T(s_0, v) - d_T(s_0, x_i) + 1 \leq d_T(s_0, v). \tag{28}$$

Hence,

$$d_{T'}(s_0, x) - d_{T'}(s_0, y) = d_T(s_0, x) - d_T(s_0, y), \tag{29}$$

and these distances in T' are no longer than in T. So, if s_0 resolved x any y in T, then it still resolves them in T'. This finishes the proof of Case 2.

Case 3a: Next, suppose that $x, y \in \widetilde{T}_0$, and let $s^\star$ be a sensor that resolves them in T. If $s^\star \in \widetilde{T}_0$, then the paths $\mathcal{P}_T(s^\star, x)$ and $\mathcal{P}_T(s^\star, y)$ remain unchanged by the transformation, and thus $s^\star$ still resolves x, y in T'. Now assume that $s^\star \notin \widetilde{T}_0$. By Claim 6.3(i), $s^\star \in \widetilde{T}_1$ has to hold. We will prove that in this case either s_0 or s'_0 will resolve x, y in T'. First, since $s^\star \in \widetilde{T}_1$ and $x, y \in \widetilde{T}_0$, it has to hold that $w_q \in V(\mathcal{P}_T(s^\star, x)) \cap V(\mathcal{P}_T(s^\star, y))$ by Claim 6.3(iii). Since $s^\star$ measures at least one of x, y, say it measures x, we have

$$\begin{aligned} k \geq d_T(s^\star, x) &= d_T(s^\star, w_q) + d_T(w_q, x) \\ &\geq \max\{d_T(s_0, w_q), d_T(s'_0, w_q)\} + d_T(w_q, x) \geq \max\{d_T(s_0, x), d_T(s'_0, x)\}, \end{aligned}$$

where in the first inequality we used the argument after Condition 6.1. Hence, both s_0 and s'_0 measure x in T'. Now assume indirectly that neither s_0 nor s'_0 resolves x, y in T', then

$$\mathrm{typ}_{s_0,s'_0}(x) = \mathrm{typ}_{s_0,s'_0}(y) =: t, \qquad \mathrm{hgt}_{s_0,s'_0}(x) = \mathrm{hgt}_{s_0,s'_0}(y) =: h \tag{30}$$

in T'. Then the same hold in T as all of $s_0, s'_0, x, y \in \widetilde{T}_0$, hence the paths between them are all unchanged by the transformation. Then (30) implies that

$$d_T(s^\star, x) = d_T(s^\star, w_q) + |q - t| + h = d_T(s^\star, y),$$

contradicting the fact that $s^\star$ resolved x, y in T. This finishes the proof.

Case 3b: Now assume that $x, y \in \widetilde{T}_1$, and let $s^\star$ be a sensor that resolves them in T. If $s^\star \in \widetilde{T}_1$, then the paths $\mathcal{P}_T(s^\star, x)$ and $\mathcal{P}_T(s^\star, y)$ remain unchanged by the transformation, and thus $s^\star$ still resolves x, y in T' and we are done. Now assume that $s^\star \notin \widetilde{T}_1$. By Claim 6.3(i), $s^\star \in \widetilde{T}_0$ has to hold. Then, by part (iii) of the same claim, $u_{q+1} \in V(\mathcal{P}_T(s^\star, x)) \cap V(\mathcal{P}_T(s^\star, y))$. We will show that this implies that in T' s_0 will resolve x, y. Recall that in T', the length-2 path $(s_0, v^\star, u_{q+1})$ connects s_0 to T_1. Hence for any $v \in \widetilde{T}_1$,

$$d_{T'}(s_0, v) = d_{T'}(s_0, u_{q+1}) + d_{T'}(u_{q+1}, v) = 2 + d_T(u_{q+1}, v), \tag{31}$$

and since $d_T(s^\star, u_{q+1}) \geq 2$ for all $s^\star \in S$,

$$k \geq d_T(s^\star, v) = d_T(s^\star, u_{q+1}) + d_T(u_{q+1}, v) \geq 2 + d_T(u_{q+1}, v). \tag{32}$$

The combination of (31) and (32) with $v = x$ and $v = y$, respectively, shows that s_0 measures both x and y in T'. Since $d_T(s^\star, x) \neq d_T(s^\star, y)$, (32) applied twice for $v = x$ and $v = y$ shows that $d_T(u_{q+1}, x) \neq d_T(u_{q+1}, y)$, showing in turn by (31) that $d_{T'}(s_0, x) \neq d_{T'}(s_0, y)$. This finishes the proof that s_0 resolves x, y in T'.

Case 4: Assume that $x, y \in A^\star_T(w_q)$. In T' they will then both lie on a single leaf-path emanating from w_q. By the remarks after Definition 6.2, x and y are at different distances from w_q in T', and both are measured by both s_0 and s'_0, implying that both s_0 and s'_0 resolve them in T'.

Case 5: Assume that $x \in \widetilde{T}_i$ for some $i \geq 2$, and $y \in A^\star_T(w_q)$. The proof in this case is exactly the same as in Case 1a.

Case 6a: Assume that $x \in \widetilde{T}_0$ and $y \in A^\star_T(w_q)$. As noted before, y will be measured by both s_0 and s'_0 in T'. Assume that neither of these two sensors resolve x, y in T'. This then implies that $d_{T'}(s_0, x) = d_{T'}(s_0, y) \leq k$, and the same holds for s'_0 in place of s_0. It then follows that

$$\mathrm{typ}_{s_0,s'_0}(x) = \mathrm{typ}_{s_0,s'_0}(y) = q, \qquad \mathrm{hgt}_{s_0,s'_0}(x) = \mathrm{hgt}_{s_0,s'_0}(y) =: h \tag{33}$$

in T'. (Such a scenario can be seen on the right picture of Fig. 3, with $q = h = 1$, with $y = v_1$ and x being the vertex right above v_1, measured by s_6.) Since $x \notin A^\star_T(w_q)$, there has to be a sensor $s(x) \in S \setminus \{s_0, s'_0\}$ such that $s(x)$ measures x in T, and $w_q \notin V(\mathcal{P}_T(s(x), x))$. By (33) and since $h \geq 1$, by Claim 6.3(i),(iii), this implies that $s(x) \in \widetilde{T}_0$, moreover, $\mathrm{typ}_{s_0,s'_0}(s(x)) = q$ in both T and T' (another type would mean that the path $\mathcal{P}_T(s(x), x)$ passes through w_q, since $\mathrm{typ}_{s_0,s'_0}(x) = q = \mathrm{typ}_{s_0,s'_0}(w_q)$, a contradiction with $x \notin A^\star_T(w_q)$). Let the vertex of the path $\mathcal{P}_T(s(x), x)$ that is closest to w_q be z (on Fig. 3 z and x coincide, but this is not necessarily the case if $h \geq 2$). Then $z \neq w_q$ by $w_q \notin V(\mathcal{P}_T(s(x), x))$. Notice that the edges of the paths $\mathcal{P}_T(s(x), x)$ and $\mathcal{P}_T(s(x), w_q)$ are unchanged by the transformation, as these paths entirely lie in $\widetilde{T}_0$. Also note that $z \in V(\mathcal{P}_T(w_q, x))$ and then $d_T(z, x) + d_T(z, w_q) = h$. With this, we can write the following:

$$\begin{aligned} d_{T'}(s(x), y) &= d_{T'}(s(x), z) + d_{T'}(z, w_q) + d_{T'}(w_q, y) \\ &= d_T(s(x), z) + d_T(z, w_q) + h \\ &> d_T(s(x), z) + d_T(z, x) = d_{T'}(s(x), x), \end{aligned}$$

where in the last line we used that $d_T(z, x) < d_T(w_q, x) = h$. This proves that $s(x)$ resolves x, y in T'.

Case 6b: Assume that $x \in \widetilde{T}_1$ and $y \in A^\star_T(w_q)$. We will prove that in this case either s_0 or s'_0 will resolve x, y in T'. Assume indirectly that it is not the case. Then (33) of Case 6a applies for the same reason, which is a contradiction, since $\mathrm{typ}_{s_0,s'_0}(x) = 0$ in T' (as the only the pair of edges $\{s_0, v^\star\} \cup \{v^\star, u_{q+1}\}$ connects $\widetilde{T}_0$ with $\widetilde{T}_1$ in T').

Case 7: Finally, assume that $x = v^\star \neq y$. Then we prove that either s_0 or s_1 will resolve x, y in T'. Assume indirectly that this does not hold. Since both s_0 and s_1 measure $v^\star$ in T' (as $d_{T'}(s_1, v^\star) = d_T(s_1, w_q) \leq k$), this implies that $d_{T'}(s_0, y) = d_{T'}(s_0, v^\star) \leq k$, and the same holds for s_1 in place of s_0. Consequently, $\mathrm{typ}_{s_0,s_1}(y) = \mathrm{typ}_{s_0,s_1}(v^\star)$ in T'. But this is impossible, as $\mathrm{typ}_{s_0,s_1}(v^\star) = 1$, and $v^\star$ is the only such type-1 vertex in T', as it has no other neighbors than the two on the path $\mathcal{P}_{T'}(s_0, s_1)$. This contradiction finishes the proof. □

Finally, we are in a position to prove the main result of this section. Recall Definition 2.3 introducing $T \notin \mathcal{T}^\star_m$ as the set of largest trees with $\mathrm{Tmd}_k(T) = m$.

Lemma 6.5. *Let $T = (V, E)$ be a tree with a k-truncated resolving set S where $|S| = m$. Assume that for all $s \in S$, $A_T(s)$ is contained in a single leaf-path starting from s. Then, if T contains a pair of strong sensor paths $\mathcal{P}_T(s_0, s'_0)$ and $\mathcal{P}_T(s_0, s_1)$ that share an edge, then $T \notin \mathcal{T}^\star_m$.*

Proof. Suppose that T satisfies Condition 5.1(i) and contains a pair of strong sensor paths that share an edge. Then by choosing the shortest among the sensor paths that share an edge with another sensor path, and then choosing the shortest among those that overlap with the first path we can identify s_0, s'_0, s_1 and assume that Condition 6.1 holds. From now on we will use the notation therein. In this case, Lemma 6.4 implies that S is a k-truncated resolving set in $T' = \mathrm{tr}_C(T, S, s_0, s'_0, s_1)$, whereas T' has one more vertex than T, proving that $T \notin \mathcal{T}^\star_m$. □

Corollary 6.6. *Let $T = (V, E)$ be a tree with a k-truncated resolving set S where $|S| = m$. Assume that for all $s \in S$, $A_T(s)$ is contained in a single leaf-path starting from s. If (T, S) either has a weak sensor path or a pair of strong sensors paths that share an edge, then $T \notin \mathcal{T}^\star_m$.*

Proof. If (T, S) has a pair of strong sensors paths that share an edge, then Lemma 6.5 is immediately applicable, yielding that $T \notin \mathcal{T}^\star_m$. In case (T, S) does not have a pair of overlapping strong sensor paths, but has a weak sensor path, then notice that the application of Lemma 5.8 for (T, S) results in another tree $\widehat{T}$ on the same vertex set, and with S still being a k-truncated resolving set on $\widehat{T}$, for which Lemma 6.5 can be applied, and so again $T \notin \mathcal{T}^\star_m$. □

7. The size of the optimal tree

In this section, we establish the maximal size of a tree T with a given k-truncated metric dimension. Using Lemma 6.5 and Corollary 6.6 will allow us to restrict to the case where (i) attractions of single sensors are contained in a leaf-path attached to that sensor, (ii) all the strong sensor paths in T are disjoint, possibly except for their endpoints, and (iii) T does not have any weak sensor path. Recall $\mathcal{T}_m^\star$, the set of trees with maximal number of vertices that can be resolved using a sensor set of size m.

Definition 7.1. Let $\mathcal{T}_m^{\star\star} \subseteq \mathcal{T}_m^\star$ be the set of trees T such that there is a k-truncated resolving set $S(T)$ on T for which $|S(T)| = m$, and for which Condition 5.1(i) holds. In case $S(T)$ is not unique we fix an arbitrary such choice.

Notice that $\mathcal{T}_m^{\star\star} \neq \emptyset$, since the application of Lemma 4.4 for any $T \in \mathcal{T}_m^\star$ results in a tree $\widehat{T} \in \mathcal{T}_m^{\star\star}$. Hence, giving the size of any tree in $\mathcal{T}_m^\star$ is equivalent to giving the size of any tree in $\mathcal{T}_m^{\star\star}$.

Lemma 7.2 *(Number of sensor paths). Let $T = (V, E) \in \mathcal{T}_m^{\star\star}$ and consider the sensor set $S = S(T)$ on it. Then T has $m-1$ sensor paths.*

Proof. We 'renormalize' the tree $T \in \mathcal{T}_m^{\star\star}$: we contract every sensor path to be a single edge and delete all vertices that are not sensors. This gives us $H = (V_2, E_2)$ with $V_2 = S$ and $\{s_1, s_2\} \in E_2$ if $s_1, s_2 \in S$ and there is a sensor path between s_1 and s_2 in T.

We now prove that H is a tree. H is connected: if there were any s_1, s_2 in H with no path between them, then there would also not be a path between s_1, s_2 in T, which contradicts with T being a tree. Then, assume H has a cycle $(s_1, s_2, \dots, s_n, s_1)$. Then the union of the sensor paths between these consecutive s_i's would form a cycle in T, as these sensor paths in T must be disjoint by Lemma 6.5 and Corollary 6.6. But T having a cycle contradicts with T being a tree, so H cannot have a cycle. Hence, H is a tree as it is a connected graph without cycles. As H is a tree on m vertices, it has $m-1$ edges, meaning that there are $m-1$ sensor paths in T. □

Lemma 7.2 tells us that T has $m-1$ sensor paths, arranged in a tree-structure H. Now we optimize the number of vertices that can be identified by each of these sensor paths.

Lemma 7.3 *(Number of vertices on each sensor path). Consider a tree $T \in \mathcal{T}_m^{\star\star}$ with the k-truncated resolving set $S = S(T)$ for some $m \geq 1$. The maximal number of vertices in $A_T(s_0, s_1)$ of two neighboring sensors s_0, s_1 is $(k^2 + k + 1)/3$ if $k \equiv 1 \pmod 3$ and $(k^2 + k)/3$ otherwise.*

Proof. By Lemma 6.5 and Corollary 6.6 we know that all sensor paths in T are disjoint and strong, i.e., they have at most $k+1$ edges. Consider the sensor path between two neighboring sensors $s_0, s_1 \in S$. Using Definition 3.3, disjointness of sensor paths implies that the vertices in $V(\mathcal{P}_T(s_0, s_1)) \setminus \{s_0, s_1\}$ are not directly measured by any sensor in $S \setminus \{s_0, s_1\}$. Hence, each vertex on the sensor path $\mathcal{P}_T(s_0, s_1)$ belongs to $A_T(s_0, s_1)$. Recall now the types and heights of vertices from Definition 3.11. Using Claim 3.10, and the observation before Definition 5.3, all vertices in $A_T(s_0, s_1)$ must have types between 1 and $d_T(s_0, s_1) - 1$ with respect to s_0, s_1, and these indeed all belong to $A_T(s_0, s_1)$. Further, again by the observation before Definition 5.3, they all must have different (type, height) vectors.

Denote by $d := |V(\mathcal{P}_T(s_0, s_1))| - 2$ the number of vertices between the sensors of the sensor path, so the distance between the sensors is $d+1$. By the observation before Definition 5.3, the maximal number of different height values that can belong to type-i vertices is $\min\{k-i, k-(d+1-i)\} + 1$, since both s_0 and s_1 have to measure these vertices, the plus one is because of the vertex with height 0.

So the maximum number of vertices in $A_T(s_0, s_1)$ is

$$\max_{d \leq k+1} (|A_T(s_0, s_1)|) = \max_{d \leq k+1} \sum_{i=1}^{d} \Big(1 + \min\{k-i, k-(d+1-i)\}\Big). \tag{34}$$

The inner sum (denote it by $\mathrm{Sum}(d)$) can be simplified, namely if d is even:

$$\mathrm{Sum}_e(d) = d + 2 \sum_{i=d/2+1}^{d} (k-i) = dk - \frac{3d^2}{4} + \frac{d}{2}, \tag{35}$$

and if d is odd, we have:

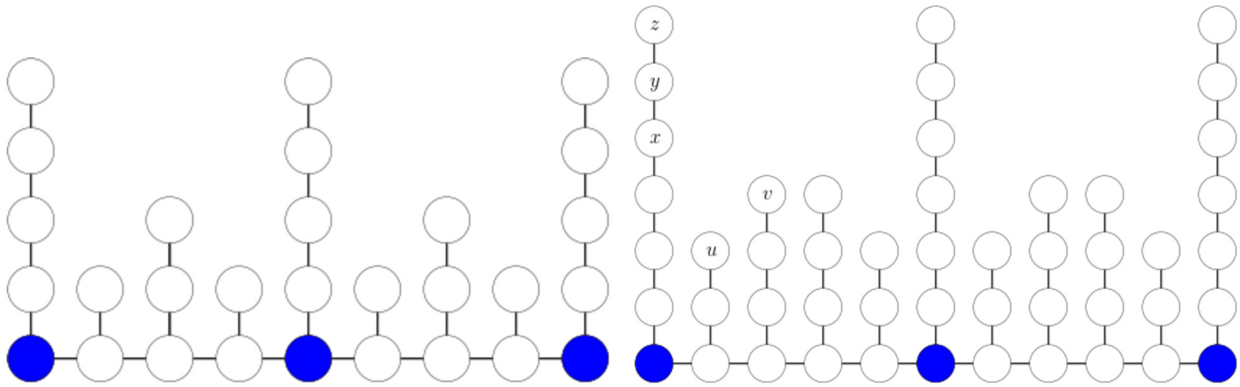

Fig. 4. Two examples for the optimal construction from the proof of Proposition 2.4. In both cases, the number of sensors (blue vertices) is $m = 3$, while the measuring radius is $k = 4$ for the figure on the left, and $k = 6$ for the figure on the right. In the case of latter, if we disconnect the vertices x, y, z from their current locations, and instead connect x to u, then y to x, and z to v, then we get another construction with optimal size (that is, a graph in $\mathcal{T}_3^\star$), but Condition 5.1(i) will be violated by the leftmost sensor, hence this graph will not be in $\mathcal{T}_3^{\star\star}$.

$$\begin{aligned} \mathrm{Sum}_o(d) &= d + (k - \frac{d+1}{2}) + 2\sum_{i=1}^{(d-1)/2}(k-(d+1-i)) \\ &= dk - \frac{3d^2}{4} + \frac{d}{2} + \frac{1}{4}. \end{aligned} \tag{36}$$

Observe that $\mathrm{Sum}_\Box(d)$ is a concave parabola of d for both $\Box = o, e$. In both cases the continuous maximizer results in $d = (2k+1)/3$. Because the formulas are quadratic, the maximal integer value of the formula is found by rounding $(2k+1)/3$ to the closest integer.

Here, we distinguish three cases depending on the value of $(k \mod 3)$.

1. If $k \equiv 0 \pmod 3$, then the closest integer to $(2k+1)/3$ is $d = 2k/3$. This value is always even, so we use $d_\star = 2k/3$ in (35), which gives $(k^2+k)/3$.
2. If $k \equiv 1 \pmod 3$, then the closest integer to $(2k+1)/3$ is $(2k+1)/3$. This value is always odd, so we substitute $d_\star = (2k+1)/3$ into (36), which gives $(k^2+k+1)/3$.
3. If $k \equiv 2 \pmod 3$, then the closest integer to $(2k+1)/3$ is $(2k+2)/3$. This value is always even, so we substitute $d_\star = (2k+2)/3$ into (35), which gives $(k^2+k)/3$.

Observe that the optimizer $d_\star \le k+1$ holds in all cases and for all k, even if we would drop the restriction of $d \le k+1$ in (34). (This means that in principle one could allow weak sensor paths in the optimization but one would not gain extra vertices on them.) □

We can now prove Proposition 2.4.

Proof of Proposition 2.4. By the remark above Lemma 7.2 it is sufficient to restrict to $T^\star \in \mathcal{T}_m^{\star\star}$ with the corresponding k-truncated resolving set $S = S(T^\star)$. Any such $T^\star$ has m sensors that have their attraction in a leaf-path of length k attached to the sensors themselves, accounting for $(k+1)m$ vertices. T also has $m-1$ sensor paths, each carrying the maximal possible size of the attraction of two neighboring sensors, which is $(k^2+k+1)/3$ vertices if $k \equiv 1 \pmod 3$ and $(k^2+k)/3$ otherwise by Lemma 7.3. In total, this means $|T^\star| = (k+1)m + (m-1)(k^2+k+1)/3$ if $k \equiv 1 \pmod 3$ and $|T^\star| = (k+1)m + (m-1)(k^2+k)/3$ otherwise. □

See Fig. 4 for two examples of the optimal construction described in the proof.

Remark 7.4. In our proof, we only constructed the optimal trees in $\mathcal{T}_m^{\star\star}$ of Definition 7.1. There are trees of optimal size that do not satisfy Condition 5.1(i), that is, they belong to $\mathcal{T}_m^\star \setminus \mathcal{T}_m^{\star\star}$. An example of how such a tree can be obtained from our constructions is illustrated in Fig. 4. However, it follows from our proofs that the repetitive application of Transformation A on any tree in $\mathcal{T}_m^\star \setminus \mathcal{T}_m^{\star\star}$ will result in a tree in $\mathcal{T}_m^{\star\star}$, which follows the construction that we described.

8. Improved lower bound based on leaves

Recall the notation $\mathcal{L}_v = \{\mathcal{P}_j^{(v)}\}$ for the collection of leaf-paths starting at a vertex v, and their number $L_v = |\mathcal{L}_v|$, and that the length of a leaf-path $\mathcal{P}_j^{(v)}$ is denoted by $\ell(\mathcal{P}_j^{(v)}) = \ell(v, j)$. Furthermore, we will denote the vertices of $\mathcal{P}_j^{(v)}$ other than v by $x_1^{(v,j)}, x_2^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}$, in order of increasing distance from v. For a generic $\mathcal{P} \in \mathcal{L}_v$ we will also denote its vertices other than v in order by $x_1^{(v)}, x_2^{(v)}, \dots, x_{\ell(\mathcal{P})}^{(v)}$. Recall the support vertices F_T from Definition 2.6 and the upper and lower complexities $\bar{c}(\ell), \underline{c}(\ell)$ from Definition 2.7.

Proof of Lemma 2.8. Let us give some heuristics first: if the length of the leaf-path is at least $3k + 2$, we can place two sensors at distance k and at distance $2k + 1$ from the end-vertex (the leaf) of the leaf-path. These two sensors then resolve the last section containing $3k + 2$ vertices. We can then 'cut this section off' and iterate the procedure until the length of the remaining leaf-path is strictly shorter than $3k + 2$. Then we treat the remaining short leaf-paths together, and one of them will be the special path $P^\star$ that might need one less sensor since it could be measured via a sensor through v. To show that this procedure is optimal, we prove by induction.

The base case is the following: all $P_j^{(v)} \in \mathcal{L}_v$ have length at most $3k + 1$. Let us first assume that this base case holds. We distinguish three subcases.

(1) If $P_j^{(v)} \in \mathcal{L}_v$ has length $\ell(v, j) \in [2k + 2, 3k + 1]$, then the end vertex $x_{\ell(v,j)}^{(v,j)}$ needs to be measured by a sensor s in $\{x_{\ell(v,j)-k}^{(v,j)}, x_{\ell(v,j)-k+1}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}\}$, say $s = x_i^{(v,j)}$. Furthermore, s cannot distinguish between $x_{i-1}^{(v,j)}$ and $x_{i+1}^{(v,j)}$ unless there is another sensor among $x_{i-k-1}^{(v,j)}, x_{i-k}^{(v,j)}, \dots, x_\ell^{(v,j)}$ (note that $i - k - 1 \geq \ell(v, j) - 2k - 1 \geq 1$). Hence, we do need at least two sensors in $V(P_j^{(v)}) \setminus \{v\}$, which is exactly (both) $\bar{c}(\ell), \underline{c}(\ell)$ for $\ell \in [2k + 2, 3k + 1]$.

(2) If $P_j^{(v)} \in \mathcal{L}_v$ has length $\ell(v, j) \in [k + 1, 2k + 1]$, then the end vertex $x_{\ell(v,j)}^{(v,j)}$ again has to be measured by a sensor in $\{x_{\ell(v,j)-k}^{(v,j)}, x_{\ell(v,j)-k+1}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}\}$ (note that $\ell(v, j) - k \geq 1$), so $V(P_j^{(v)}) \setminus \{v\}$ needs to contain at least one sensor, which gives (both) $\bar{c}(\ell), \underline{c}(\ell)$ for $\ell \in [k + 1, 2k + 1]$.

(3) Now consider all $P_j^{(v)} \in \mathcal{L}_v$ that have length $\ell(v, j) \in [1, k]$, and call these *short leaf-paths*. In order to distinguish between the vertices $\{x_1^{(v,j)}\}_j$ of all short leaf-paths $\{P_j^{(v)}\}_j$, all but at most one of them need to contain a sensor that is not at v: this gives $\bar{c}(\ell)$ for all but one short leaf-paths, and gives $\underline{c}(\ell)$ for a single short leaf-path.

This finishes the proof for the base case.

For the inductive step, assume that some $P_j^{(v)} \in \mathcal{L}_v$ has length $\ell(v, j) \geq 3k + 2$. Then, similarly to the first case above, $x_{\ell(v,j)}^{(v,j)}$ can only be measured by a sensor in $\{x_{\ell(v,j)-k}^{(v,j)}, x_{\ell(v,j)-k+1}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}\}$, say $s = x_i^{(v,j)}$. Furthermore, s cannot distinguish between $x_{i-1}^{(v,j)}$ and $x_{i+1}^{(v,j)}$ unless there is another sensor s' among $x_{i-k-1}^{(v,j)}, x_{i-k}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}$. Here $i - k - 1 \geq \ell(v, j) - 2k - 1$. Then, all the vertices that either s or s' can measure belong to $\{x_{\ell(v,j)-3k-1}^{(v,j)}, x_{\ell(v,j)-3k}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}\}$. Hence, the rest of the leaf-paths, that is, $\cup_{k=1}^{L_v} V(P_k^{(v)}) \setminus \{v, x_{\ell(v,j)-3k-1}^{(v,j)}, x_{\ell(v,j)-3k}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}\}$ need at least as many sensors as they would need in the graph $T \setminus \{x_{\ell(v,j)-3k-1}^{(v,j)}, x_{\ell(v,j)-3k}^{(v,j)}, \dots, x_{\ell(v,j)}^{(v,j)}\}$. Thus, a leaf-path needs an extra 2 sensors at every multiple of $3k + 2$, and this is exactly what both $\bar{c}(\ell)$ and $\underline{c}(\ell)$ express. This provides the induction step, and finishes the proof. □

Proof of Theorem 2.9. Assume first that $k \equiv 1 \pmod 3$, and let

$$B_T := \left\lceil \frac{3n - 3\sum_{v \in F_T} \sum_{j=1}^{L_v} \ell(v, j) + k^2 + k + 1}{k^2 + 4k + 4} \right\rceil.$$

For an indirect proof, assume that (3) does not hold, and in fact there exists a k-truncated resolving set $S^\star$ for T such that

$$|S^\star| \leq B_T - 1 + \sum_{v \in F_T} R(\mathcal{L}_v) - |F_T|. \tag{37}$$

Let

$$V_{LP} := \bigcup_{v \in F_T} \bigcup_{j=1}^{L_v} V\left(P_j^{(v)}\right) \setminus \{v\},$$

the union of vertices in leaf-paths starting at support vertices, and let $T' := T \setminus V_{LP}$ be the 'trimmed' version of T, when the leaf-paths emanating from the support vertices are removed, but the support vertices still belong to T'. Observe that T' is indeed a tree, i.e., connected, since we only removed leaf-paths ending at leafs. By Lemma 2.8, and (2),

$$\left|S^\star \cap V_{LP}\right| \geq \sum_{v\in F_T} R(\mathcal{L}_v).$$

Hence, since T' and V_{LP} are on disjoint vertex-sets, by (37),

$$|S^\star \cap V(T')| \leq B_T - 1 - |F_T|. \tag{38}$$

Consider now the new sensor set $\widetilde{S} = S^\star \cup F_T$. Since $S^\star$ is a k-truncated resolving set for T, so is $\widetilde{S}$. Moreover, since $F_T \subseteq \widetilde{S}$, none of the sensors in $\widetilde{S} \cap V_{LP}$ directly measures any vertex in $T' \setminus \widetilde{S}$, in the sense of Definition 3.3. This also means that if some sensor $s \in (V(\mathcal{L}_v) \setminus \{v\}) \cap S^\star$ resolves two vertices $x, y \in T'$, then so does $v \in F_T \cap \widetilde{S}$. It then follows that $\widetilde{S} \cap V(T')$ is a k-truncated resolving set for T'. Moreover, since $\widetilde{S} = S^\star \cup F_T$, by (38),

$$|\widetilde{S} \cap V(T')| \leq |\widetilde{S}^\star \cup V(T')| + |F_T| \leq B_T - 1.$$

However, Theorem 1.1 implies that $\mathrm{Tmd}_k(T') \geq B_T$, as $|V(T')| = n - \sum_{v\in F_T}\sum_{j=1}^{L_v} \ell(v, j)$. This contradiction finishes the proof when $k \equiv 1 \pmod 3$. The proof in the case $k \not\equiv 1 \pmod 3$ is completely analogous. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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