# Pricing and Hedging of a Mortgage Portfolio

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Challenge the future

# Pricing and Hedging of a Mortgage Portfolio

by

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# Abstract

Forecasting the prepayments is essential for any financial institution providing mortgages, and it is a crucial step in the hedging of the risk resulting from these unexpected cash flows. The way in which the prepayment rate is predicted impacts on the hedging strategy. For example, if the prepayment model is deterministic only the average prepayments are forecast, and a linear hedge composed of swaps is sufficient. However, in condition of volatile markets the lack of a non-linear hedge can result in losses for the bank. Considering that the there is a correlation between the prepayments and the level of interest rates in the market, we propose a prepayment model which is only based on the refinancing incentive. This way, the prepayments' forecast might be less accurate, but the clear link with the market allows to extend the prepayment model to a stochastic environment. We showed that allowing the notional of a mortgage to be stochastic unveils the non-linear risk embedded in the prepayment option, arising the necessity to include non-linear instruments in the hedging portfolio. The calibration of the refinancing incentive on a data set of more than thirty millions of observations led us to choose the functional form of the prepayments that is able to capture the borrowers' behaviour the most, and it distinguishes the model from full-rational models in which the option to prepay is assumed to be always exercised rationally. Then, the linear and non-linear risks are addressed to a set of tradeable instruments, aiming to build a static hedge. Different combinations of swaps and swaptions are tested, in order to determine which derivatives have the highest replication power. This research can impact considerably the evaluation of the exposure to interest rate risk of mortgage providers, and it can improve the performance of the hedging of the prepayment risk. Moreover, since the linear and non-linear components of the risk embedded in mortgages are distinguished, it can also help in the pricing of the prepayment option, allowing banks to define the mortgage rates with more accuracy.

**Keywords:** Prepayment option, linear risk, non-linear risk, refinancing incentive, swaption, mortgages, Hull-White.

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# 1

# INTRODUCTION

A mortgage is a long-term loan that is secured by a registered good. The nature of this collateral can largely differ, since mortgages could be used, for example, to buy real estates, ships or aircrafts. The two counterparties of a mortgage are the lender, or *mortgagee*, and the borrower, or *mortgagor*. What makes mortgages intriguing to study is the fact that their cash flow is not known with certainty. As we will examine better later, this fact impacts considerably the hedging strategy that any institution follows, and it motivates the whole research. Regardless of the specifics of the underlying asset, in case the mortgagor fails to honour the contractual repayment of the debt, a mortgage contract gives right to the mortgagee to sell the collateral, so that the lent amount can be recouped. However, losses due to default are not overriding. In fact, default losses in the Dutch mortgage market are historically small.

Conducting a brief analysis, we can start from the late Seventies, when housing prices dropped about 30%, but losses above 1% happened rarely [1]. Then, a more stressful scenario is provided by the period 2008-2013, during which the economic crisis challenged even the most solid economies of the world. As reported by the Dutch Banking Association [2], in that time frame The Netherlands experienced a GDP decrease of 3.2%, an unemployment increase of almost 5%, a fall of the housing market and change in tax deductibility that results in a lower affordability of houses. Compared to the level before the crisis, loss rates almost tripled, reaching 0.08% in 2013. Although the increase is dramatic, the absolute level of losses remains very low. As a matter of fact, considered how the Netherlands performed during the crisis, in 2013 Fitch Ratings ranked the default probability of the Dutch mortgages as one of the smallest in Europe, with an expected default rate smaller than the one of France and Germany [3]. Moreover, in 2014 Moody's stated that, unless great shocks in unemployment and interest rates level happened, default rates would only increase slightly in the next year [4]. Finally, moving the focus to the present and using the GDP as benchmark, the Dutch economy has grown by 2.2% in 2016, which is the highest growth rate registered from the beginning of the economic crisis in 2008 [5]. Furthermore, it is also reported that in 2013 residential property sales had nearly halved relative to 2006, but from that moment on the number of transactions increased sharply, leading to breaking the record of 215 thousand (existing) dwellings sold in 2016, a record that dated back to 2006. In addition, according to Fitch [1, 3], Dutch people have a strong cultural aversion to defaulting on debt instruments. Fitch attributes this, among other reasons, to the small dimension of the country and the right of the lender to seize the borrower's salary in the case the borrower defaults. All these considerations lead to the choice of not focusing on default when talking about the losses related to the mortgages faced by the financial institution providing them.

Besides the risk of default, there is a more significant risk that affects mortgages' revenues, namely the prepayment risk. To understand better which kind of risk the prepayments generate, we need to give more details about the rights and options that a mortgage contract comes with. Dutch mortgages usually provide two embedded options, which regard the choice of the interest rate of the mortgage and the possibility to deviate from the scheduled cash flow. Once the borrower receives a quotation from the lender about the cost of the loan (essentially, a quote of the interest rate of the mortgage), there is a period of time during which this offer is valid, called grace period. At the end of the grace period, the borrower has the right to get the smallest interest rate occurred during the whole duration of this time span, which usually is three months. This affects the future revenues of the financial institutions (from now on, the bank) because it loses the higher gain that would have been provided by the higher interest rate risk. This is referred to as *pipeline risk*, and it originates from the first option. The second option is the one generating the prepayment risk, and it deserves to be treated separately.

## 1.1. PREPAYMENT RISK AND RESEARCH MOTIVATION

Mortgages come also with an embedded option that gives the possibility to the mortgagor to redeem part of the debt in advance with respect to the scheduled plan of amortization. When a mortgage is settled, the borrower obtains a precise schedule of payments that has to be followed. These cash flows guarantee that the borrower pays back completely the initial sum, called *notional*, plus an extra amount of money representing his cost of the loan (or, from the other perspective, the profit of the bank). These payments are called *repayments*, and, as we said, they are predetermined. Depending on the frequency and amount of the repayments we can distinguish different typologies of mortgages, as we will analyse later in Section 1.2. On the contrary, *prepayments* are extra payments that the mortgagor can effect during the life of the mortgage, and they consists in deviations from the scheduled cash flows [6]. The reasons behind prepayments are diverse, and they are analysed in Section 1.3. What is important to know is that, in the Netherlands, usually only ten or twenty percent of the notional can be called every year without penalty, and this appears the greatest difference between the Dutch market and the US market, where, usually, all the notional can be payed back at any time. Partial prepayments are also referred to as *curtailments*. However, penalties do not always apply. In fact, most of the Dutch banks give the possibility of prepaying the debt fully without extra costs in case the mortgagor moves to another dwelling. That is why, in periods of economic growth and/or booming of the housing market, prepayments due to movement arise.

The interests represent the profit that a bank generates from producing a mortgage. In general, interests depend on three factors: interest rate, notional and duration of the loan. All these factors affect positively the interests, since a higher interest rate or a higher notional or a higher maturity of the mortgage result in a bigger profit for the bank. Therefore, any prepayment consists a loss, because either the notional on which interests are computed shrinks, or the interest rate decreases, or, in extreme cases, the duration of the loan is reduced. However, a simple loss of earning is not the only risk that prepayments brings.

To understand better why prepayment risk exists, we need to introduce the two main risks that occur whenever money is lent: interest rate risk and liquidity risk. First, a portfolio of mortgages has to be financed, for example through securitization or by borrowing money on the capital market [7]. So, liquidity risk happens when the duration of the funding mismatches the duration of the mortgage, while interest rate risk represents the scenario in which the interest rate paid to fund the (fixed rate) mortgage changes over time. The risks can be handled in basically two ways [8]: the lender can either fund the loan with an asset that has a fixed interest rate and same maturity, or use a series of forward interest rate agreements. The former case can involve a bond with the same maturity of the mortgage, and it helps to cope with both interest rate and liquidity risk. The latter, instead, provides a better hedging of the interest rate risk, but it does not consider liquidity risk.

Independently of the choice, the prepayments are risky events: imagine a 30 years mortgage with interest rate 6%, financed with a 30 years bond at 5%, where 1% spread is applied to cover expenses and profit of the bank. If the interest rate drops after 10 years, for example to 4%, the mortgagor gets a strong incentive to prepay the existing mortgage and to open another contract at 5% for the remaining 20 years, leaving the mortgagor with the obligation to pay 5% on the bond. Moreover, the amount of money received after 10 years will provide a lower return because of the change in the interest rate level. Clearly, there is a loss for the lender, and this is basically due to the position that the bank entered, at the beginning, to limit the interest rate risk. This loss is attributable to prepayment risk. Furthermore, there is a loss of earnings due to the fact that the smaller interest rate produces less interests. In [9] the different effects of prepayment risk are identified:

- **Interest Rate Risk.** Since the bank usually hedgse the changes in interest rates with Interest Rate Swaps, if the mortgage has a fixed interest rate, a prepayment could make the bank pay a fixed leg that is bigger than the interest rates of the new mortgages.
- **Liquidity Risk.** Independently of the rate of the mortgage (fixed or floating), prepayments lead to overestimate the future liquidity resources of the bank. Thus, if the liquidity will be actually required by the circumstances, the bank will have to pay to get it.
- **Mispricing.** First of all, the cost of prepayments could be involved when the price of the loan is given to the client. Second, prepayments change the reference fair value of the mortgage portfolios that is securitised.

All considered, forecasting prepayments and have a better understanding on how to fund and hedge a portfolio of mortgages is essential to any financial institution giving credit, in particular, to banks. Formally, the main goal of the mortgagor is to forecast the Conditional Prepayment Rate (CPR), that is sometimes reported also as Constant Prepayment Rate. One of the earliest definitions of the CPR is found in [10]

**Definition 1** (Conditional Prepayment Rate)**.** The CPR is an annualised rate of prepayment, obtained from a measure called Single Monthly Mortality (SMM) with the following formulas:

$$\begin{aligned} \mathrm{SMM}(t) &= \frac{\text{Unscheduled notional payment at month } t}{\text{Scheduled outstanding at month } t}, \\ \mathrm{CPR}(t) &= 1-(1-\mathrm{SMM}(t))^{12}. \end{aligned} \tag{1.1}$$

At a first glance the CPR might seem unrealistic, because, in reality, if we looked at a single mortgage we would hardly find more than one or two prepayments over the whole duration of the contract. However, when the focus moves to a whole portfolio of mortgages, the CPR is actually an observable phenomenon. This effect will be more clear during the data analysis we will perform in Section 3.2.3. For the moment, we would just like to point out that throughout all this thesis we will assume that we are considering a whole portfolio of mortgages. Thus, even when we will refer to a “mortgage”, this will represent the portfolio. Essentially, we assume that an in-depth data analysis have been performed previously, and that the data have been clustered in order to build a prototypical mortgage on which test the pricing and hedging models.

## 1.2. PREPAYMENT DETERMINANTS

Underlying the decision of prepaying there are several reasons, whose nature differs. Through the literature it is possible to find different specifications about the variables that influence the rate of prepayment the most [1, 6–18]. The choice of the determinants is often influenced by the prepayment data available, nevertheless, there is always agreement about one specific driver: the refinancing incentive.

- The **refinancing incentive** is the main driver in every prepayment model, and, also referring to the example of the previous section, there is a clear explanation. The mortgage rate is a fluctuating variable quoted in the market by the different financial institutions. It depends on the type of mortgage, its maturity, and often on the typology of house that acts as collateral. When a borrower observes a rate that is lower than the current rate he has, a positive incentive of prepaying appears. That is because, theoretically, the drop of the market rate gives the possibility to prepay the old mortgage getting a new loan. This procedure let the mortgagor end up with a lower rate. On the other side, the mortgagor suffers a loss of future interests and a loss caused by the fact that an alternative use of the money received from the prepayment has to be found, and, since the interest rate is currently lower compared to the one at the moment of origination of the mortgage, it is challenging to find an investment that bears more than the funds used for the mortgage itself.

- **Mortgage Age** or **Seasoning**. There is the tendency of not moving houses for the first few months or years from the origination of the mortgage. A reason behind this phenomenon could be the fact that moving houses is an important step for many mortgagors, and many personal factors come into play (attachment to the dwellings, different needs in the components of the family unit...). Moreover, the financial means of the borrower play a key role in the decision of moving and they are unlikely to change radically in a short period. Therefore, the mortgage age is assumed to have a positive correlation with the probability of prepayment.

- **Month of the year** or **Seasoning**. Peaks in the prepayment rate are often observed in the months of January, December and August. This effect is strictly related to the housing turnover, which follows a similar trend. Moreover, salary bonuses usually happen in those periods of the year. Thus, special weight is often dedicated to the prediction of the rate of prepayment during specific months.

- **Burnout**. It is an empirical phenomenon that refers to the slowdown of prepayments despite favorable market scenarios. This effect appears because some people are more keen to react to incentives to refinance. Therefore, if beneficial situations show up again, only the borrowers that are less likely to refinance are left in the pool, and this leads to less prepayments. Essentially, if they were "rational prepayers", they would have prepaid already. So, when burnout is taken into consideration, it reduces the expected prepayments taking into account the history of the incentives.

Beside these well known drivers, also according to the available data, other variables could be included

- **Client age**. Similarly to the mortgage age, the status and salary level of the mortgagor is expected to remain stable within the first months of origination of the mortgage, leading to a higher expectation of prepayments when the borrower gets older.

- **Housing turnover**. A flourish house market incentivizes people to move, and thus, in many occasions, to prepay. However, predicting the house market level in the future inserts other uncertainty, and it is often left aside.

- **Default**. In this case the prepayments happen because the borrower does not have the possibility to honour the mortgage contract anymore. It is a different situation compared to the exercise of the prepayment option.

- **Bank account**. An obvious requirement to prepay is to have enough financial resources to do it. However, forecasting the trend of someone's bank account is nearly impossible, therefore it leads only to theoretical models with limited applicability, like [19].

## 1.3. DIFFERENT TYPOLOGIES OF MORTGAGES

In the introduction we mentioned that the mortgage rate that is offered by a financial institution changes according to various factors and, among them, we find the type of mortgage. Mortgages are classified according to the amortization plan that the notional follows along the duration of the contract. All the other characteristics being equal (duration, initial notional, interest rate), the amortization plan influences the amount of interests raised during the lifetime of the mortgage, and the way that the notional is payed back. The choice of the type of mortgage is personal, and varies according to the necessities of the borrower. Bigger cash flows could be spread throughout the life of the mortgage equally, could be concentrated at the beginning or even condensed in a final payment at the end. The choice is also driven by the reason that led the mortgagor to ask for a loan in the first place, for example if the purchase of the real estate is performed by a family that needs a new house, or by an entrepreneur that is looking for a profitable investment.

Different amortization plans also give rise to different effects on the prepayment rate. Therefore, we analyse the main three typologies of mortgage, giving insights in how the CPR influences the notional and the interests that a bank receives during the life of the contract. More information about other kinds of mortgages could be found in [20], which also provides the general knowledge for the typologies we are interested in.

### INITIAL NOMENCLATURE

Throughout the thesis we will use the following nomenclature:

- $N_0$ is the face value of the loan, namely the amount of money that is initially lent. Since the notional varies in time, we have that $N(t_0) = N_0$.

- The mortgage rate is $K$.

- $t_0$ indicates the moment of origination of the mortgage, while the payment dates are $T_1, ..., T_M$, with $\tau_i = T_{i+1} - T_i$. When not specified, we assume that the payment dates are equally spaced in time, and the rate is converted consistently to the time grid (so $\tau_i = 1$ and if, for example, we assumed yearly payments, the mortgage rate would be annual).

- $I(T_i)$ is the interest payment payed at time $T_i$ and $I$ is the total amount of interests received by the bank at time $T_M$. We should also point out that in a mortgage contract the interests that have to be payed at each time $T_i$ are computed on the notional at time $T_{i-1}$, so

$$I(T_i) = KN(T_{i-1}). \tag{1.2}$$

- $Q(T_i)$ is the principal payment, or notional payment, payed at time $T_i$. It represents the amount of money dedicated to repay the initial amount $N_0$.

- $C(T_i)$ is the installment at time $T_i$. From what we have defined so far, it follows that

$$C(T_i) = Q(T_i) + I(T_i). \tag{1.3}$$

- The prepayment rate is the CPR. However, defining the CPR with a Greek letter should make all the formulas more readable, therefore we will use the convention

$$\text{CPR} \equiv \Lambda.$$

### 1.3.1. BULLET

The bullet is the simplest mortgage. In this type of contract, the borrower receives $N_0$ at the time of settling $t_0$ and the notional is fully redeemed only at the end of the last period, in one single shot (from which the name bullet originates). At the end of each period, only the interest part is payed back to the loaner, so that the notional remains constant until $T_M$,

$$N(T_i) = N_0 \mathbb{1}_{\{T_i < T_M\}}.$$

In this case, the installment serves only to pay the interest part, which takes into account the notional of the loan, the interest rate and the time span that the payment covers

$$C(T_i) = K N_0 \tau_i,$$

implying that the total amount of interests that a loaner receives at the end of the contract is

$$I = \sum_{i=1}^{M} K N_0 \tau_i.$$

Now, we would like to start to take into considerations the effects of prepayments. According to (1.1), which defines the CPR, a prepayment reduces the notional of the mortgage so that, beside the contractual repayments, the notional is reduced further by an amount that corresponds to the magnitude of the prepayment itself. In case of a bullet the repayments are null, so we simply get that the notional at time $T_i$ is

$$N(T_i) = (1-\Lambda)N(T_{i-1}) = (1-\Lambda)^2 N(T_{i-2}) = ... = (1-\Lambda)^i N_0,$$

while the interests beared by the bullet are

$$I = \sum_{i=0}^{M-1} K N(T_i) = K N_0 \sum_{i=0}^{M-1} (1-\Lambda)^i = K N_0 \frac{1-(1-\Lambda)^M}{\Lambda}.$$

Alternatively, we can also reach a continuous formulation, in which we assume that the frequency of payments is raised up to infinite and $T_M \equiv T$. Recalling that

$$\left(1-\frac{\Lambda}{n}\right)^{ni} \xrightarrow{n\to\infty} e^{-\Lambda\cdot i},$$

and that, in any case, at maturity the notional has to be completely redeemed, we obtain

$$\begin{aligned} N(t) &= N_0 e^{-\Lambda\cdot t} \mathbb{1}_{\{t<T\}}, \\ I(t) &= K N(t), \\ I &= \int_0^T I(t)\mathrm{d}t = \frac{K N_0}{\Lambda}\left(1-e^{-\Lambda\cdot T}\right). \end{aligned} \tag{1.4}$$

In Figure 1.1 it is possible to appreciate how prepayments impact on the behaviour of a bullet. In this example, the maturity is supposed to be ten years and the mortgage rate is three percent, in agreement with what the market can offer[1]. On the left we can see the installment that the bank receives in two cases, one without prepayments and the other in case $\Lambda = 12\%$. We should notice that in the first case the installment is constant, since this kind of contract does not involve any repayment. In fact, a payment of $N_0(1+K)$ is performed at maturity to pay the last interest quota

[1]Data from the 11th of April 2018. An updated rate could be found at https://www.rabobank.nl/particulieren/hypotheek/hypotheekrente/?redirect=p-hypotheken-hypotheekrente-senses.

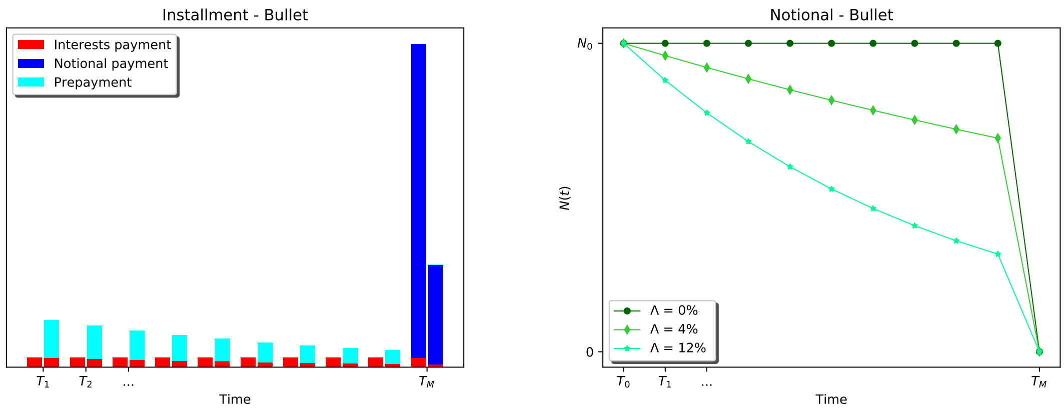



Figure 1.1: bullet with $T_M = 10$ and $K = 3\%$ under different scenarios. Left: the installment composition in case $\Lambda = 0\%$ or $\Lambda = 12\%$. Right: outstanding notional in time under different levels of prepayment.

and to redeem completely the initial amount of the loan. In the other case, we can see how the prepayments (represented by the cyan bars) are added to the contractual payment of the interests. The consequence is that the notional reduces in time, and so do the interests computed on it. Therefore, the final installment is not as large as the one in case no prepayments occurred. In the other graph the effect of $\Lambda$ is shown directly on the remaining notional $N(t)$. Clearly, the higher the prepayment rate, the stronger the decay of the notional in time.

### 1.3.2. ANNUITY

An annuity is a more complicated contract because, contrary to a bullet, it also involves repayments. A repayment $Q(T_i)$ reduced the notional of the same amount, i.e.

$$N(T_{i+1}) = N(T_i) - \Delta T_i Q(T_i) = N(T_i) - \Delta T_i \big(C(T_i) - I(T_i)\big). \tag{1.5}$$

When the first payment is performed one year after the signing of the contract, the annuity is named *ordinary annuity*. Otherwise, if the first payment happened instantly, we would refer to an *annuity due* [20]. Our focus is on the ordinary annuity and therefore, from now on, we will simply call it annuity.

The way of spreading the initial amount over all the installments can vary significantly. For instance, a constant principal repayment is possible, as well as a constant installment. In case of the annuity, the essential characteristic is that the installments are the quantity with a fixed magnitude

$$C(T_i) \equiv C$$

with $i = 1, ..., M$ , presupposing that the composition of the installment itself changes in time. In fact, the interest and principal parts have to be balanced so that their sum is constant at each payment time. Therefore, we expect them to follow opposite trends, because when the notional will be progressively payed back, the interests computed on the latter will diminish. The way to find the correct size of the installment $C$ comes from the evaluation of the present value of the annuity. The main idea to find a present value of a financial instrument is discounting the future cash flow to the date of evaluation. Since the cash flow produced by the annuity relies on the mortgage rate $K$, this is exactly the rate used to discount the payments. Therefore, the present value of the annuity is

$$\mathrm{An}(t_0; K) = \sum_{i=1}^{M} \frac{C}{(1+K)^{T_i}} = \frac{C}{(1+K)} \sum_{i=0}^{M-1} \frac{1}{(1+K)^{T_i}} = \frac{C}{(1+K)} \left( \frac{1 - \frac{1}{(1+K)^{T_M}}}{1 - \frac{1}{1+K}} \right) = \frac{C}{K} \left( 1 - \frac{1}{(1+K)^{T_M}} \right). \tag{1.6}$$

Now, the present value of the annuity is none other than the initial amount of the loan

$$\mathrm{An}(t_0;K) \equiv N_0,$$

therefore the quantity $C$ is obtained straightforwardly from (1.6)

$$C = \frac{KN_0}{1-(1+K)^{-T_M}}. \tag{1.7}$$

Once we know the value of the installment, we can derive the interest rate payment $I(T_i)$ and the principal payment $Q(T_i)$. In fact, $I(T_i)$ is already given from (1.2), so, using (1.3) we get that

$$Q(T_i) = C(T_i) - I(T_i) = \frac{KN_0}{1-(1+K)^{-T_M}} - KN(T_{i-1}). \tag{1.8}$$

We have now all the ingredients to build the cash flow of an annuity in case no prepayments occur.

As we did for the bullet, we would like to formulate the case in which prepayments occur also for the annuity. As we know, prepayments consist in an additional reduction of the notional besides the contractual repayments. In formulas, this means that the notional is not updated according to (1.5) anymore, instead, introducing a prepayment intensity, we get that

$$N(T_{i+1}) = N(T_i) - Q(T_i) - \Lambda N(T_i). \tag{1.9}$$

The effect of the prepayment rate $\Lambda$ at time $T_i$ is also reflected as alteration of both the interest payment $I(T_{i+1})$ and the installment $C(T_{i+1})$. The former case is clear to understand, because at time $T_{i+1}$ the actual notional on which we compute the interest will be different (less) than the expected one. The latter is less immediate, and it finds its reasons in the regulations about prepayments. When a mortgagor decides to prepay, the installment for the remaining dates is rebalanced according to the updated outstanding notional. If this did not happen, the coupon would be too large for the actual capital that still has to be payed. At a certain point in the future, the debt would be fully payed, leading to a premature end of the mortgage. Of course this could also be a possibility, meaning that the borrower could decide to keep paying the old installment in order to get rid of the loan faster. Nevertheless, common practice is to prepay in order to have a smaller coupon in the coming years. That is why we assume that the installment is recomputed maintaining the end date $T_M$ unaltered. Consequently, $C$ becomes time-dependent

$$C(T_i) = \frac{KN(T_i)}{1-(1+K)^{-(T_M-T_i)}}. \tag{1.10}$$

Once we obtained the new installment, we have all the elements to produce a cash flow of an annuity with an intensity of prepayment $\Lambda$ as described by Algorithm 1. Note that the two possibilities with/without prepayments are explicitly distinguished using the formulas we derived, but the case without prepayments is obviously nothing other than a particular case of null prepayment intensity. Examples of annuity payments are shown in Figure 1.2. As we did with the bullet, on the first graph we compare the coupon magnitude in case $\Lambda = 0\%$ or $\Lambda = 12\%$, specifying with different colors the different components. In the chart on the right we explicitly show the impact of different levels of prepayment on the outstanding notional. The larger the $\Lambda$ is, the greater the rate of decay.

#### CONTINUOUS FORMULATION

For sake of completeness, we provide a continuous formulation for an annuity as we did for the bullet. Remind that in this case $T_M \equiv T$. Taking into account that discounting in continuous time

**Algorithm 1** Cash flow of an annuity

**procedure** ANNUITY CASH FLOW
  Define the mortgage rate $K$ and the time grid $T_i$.
  Define the vectors $N$, $I$, $Q$.
  Define the quantity $C$ according to (1.7).
  Initialize $N$ to $N_0$.
  **for** $i \in \{1, ..., M-1\}$ **do**
    Determine $I(T_{i+1})$ according to (1.2).
    Calculate $Q(T_{i+1})$ according to (1.8).
    Update $N(T_{i+1})$ using (1.5). In case of prepayments use (1.9).
    In case of prepayments redefine $C$ using (1.10).

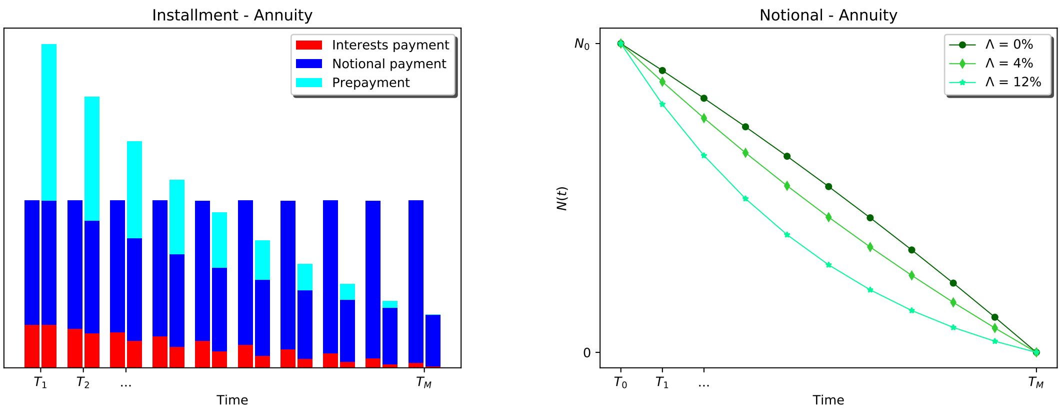



Figure 1.2: annuity with $T_M = 10$ and $K = 3\%$ under different scenarios. Left: the installment composition in case $\Lambda = 0\%$ or $\Lambda = 12\%$. Right: the outstanding notional in time under different levels of prepayment.

is performed with the exponential compounding, and that $\lim_{T_{i+1}\to T_i}\Delta T_i = \mathrm{d}t$, we can first get the installment $C$ from

$$\mathrm{An}(0;K) \equiv N_0 = \int_0^T Ce^{-Kt}\mathrm{d}t = \frac{C}{K}\left(1-e^{-KT}\right) \Longrightarrow C = \frac{KN_0}{1-e^{-KT}}. \tag{1.11}$$

Now, using a continuous form of (1.5) we get the value of the notional in time when $\Lambda = 0\%$

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -C + KN(t) \Longrightarrow N(t) = \frac{C}{K}\left(1-e^{-K(T-t)}\right).$$

On the other hand, when $\Lambda \neq 0\%$ we use (1.9) and we get

$$\begin{aligned}\frac{\mathrm{d}N(t)}{\mathrm{d}t} &= -C(t) + KN(t) - \Lambda\cdot(N(t) - C(t) + KN(t)) = \\ &= -\frac{KN(t)}{1-e^{-K(T-t)}} + KN(t) - \Lambda\cdot\left(N(t) - \frac{KN(t)}{1-e^{-K(T-t)}} + KN(t)\right) \\ &= N(t)\left[\frac{K(\Lambda-1)}{1-e^{-K(T-t)}} + K - \Lambda\cdot(K+1)\right],\end{aligned}$$

with initial condition $N(0) = N_0$. We also proved that the solution to the latter Ordinary Differential Equation is

$$N(t) = N_0 e^{-\Lambda t}\left(\frac{1-e^{-K(T-t)}}{1-e^{-KT}}\right)^{1-\Lambda}. \tag{1.12}$$

### 1.3.3. SAVINGS ACCOUNT

A savings account mortgage is a mortgage that has the same cash flow as an annuity, but it raises the same amount of interests as a bullet. The reason behind creating such an instrument is the fact that the interests payed for a mortgage are deductible from taxes. Since a bullet raises more interests than an annuity, this comes as an advantage for the mortgagor, while from the bank's perspective nothing changes. To achieve this result, the borrower pays every month the amount $I(t) = KN$ (as in a bullet) plus $Q(t) = C - I(t)$ (as in an annuity). $I(t)$ is directly given to the bank, instead $Q(t)$ is deposited in a savings account on which the bank pays the same rate as the mortgage rate $K$. This is a special deposit account because the rate is particularly high and the borrower can neither deposit nor withdraw money. The trick consists in using the interests from the bank account to reduce the mortgage installment. Thus, at time $t$, the borrower pays

$$\begin{aligned}\text{Amount due at time } t &= KN_0 + Q(t) - K\int_0^t Q(u)\mathrm{d}u = \\ &= KN_0 + Ce^{-K(T-t)} - KC\int_0^t e^{-K(T-u)}\mathrm{d}u = \\ &= KN_0 + Ce^{-KT} = \text{ (using the definition of } C) \\ &= KN_0 + \frac{KN_0}{1-e^{-KT}}e^{-KT} \\ &= \frac{KN_0}{1-e^{-KT}} = C.\end{aligned}$$

So, the result is that the mortgagor pays the same installment of an annuity, but gets the same tax deduction as if the mortgage were a bullet. The notional is entirely redeemed at time $T$, using all the money collected in the saving account. This is possible because obviously $\int_0^T Q(t)\mathrm{d}t = N$.

The same reasoning still applies when the borrower prepays with an intensity $\Lambda$. All the prepayments are deposited in the saving accounts and the final cash flow is exactly the same as the one of an annuity with the same prepayments. Therefore, we can conclude that in any case the saving account mortgage is treated as an annuity, since the difference is relevant only for taxation matters.

# 2

# INTEREST RATE INCENTIVE AND CORRELATED MARKET INSTRUMENTS

When dealing with financial instruments, the choice of the quantity that has to be modelled is crucial. The main reason behind the presence of a large number of models is that a comprehensive model which is able to price analytically every instrument being in agreement with market practice does not exist. Moreover, the concerns become even bigger since there are inconsistencies even with well known models used to price derivatives in practice. Take for example the popular and promising family of interest rate models that comes under the name of "Market Models". The popularity comes from the fact that these models are consistent with widely accepted formulas used to price basic derivatives. We are talking about the lognormal forward-Libor model (LFM), which prices caps through Black's cap formula, and the lognormal forward-swap model (LSM), which prices swaptions with Black's swaption formula. Before the introduction of these models, none of the interest rate models were congruous with any of the aforementioned Black's formulas. Instead, Market Models could justify both of them rigorously. Nonetheless, even if both models treated separately agree perfectly with market practice, also with a thorough mathematical formulation, the LFM and the LSM are not compatible themselves [21]. Basically, LFM assumes a lognormal distribution for forward Libor rates, but this makes it impossible to have a lognormal distribution for swap rates, as assumed by LSM. Moreover, Market Models come with other drawbacks. The first one is related to a general "uncertainty principle of modeling" [21]: the more the model fits, the less it explains. Essentially, the capability of Market Models to recover a great number of prices observables in the market makes them unable to give a general evaluation of the condition of the market itself. On the other hand, a model with lower dimensionality is able to fit the market data only under certain conditions, and it will signal problems in case of awkward market prices. Then, it is the practitioner's experience that evaluates whether the market is showing weird phenomena or the model is simply limited. The second aspect that should not be underestimated concerns implementation. The construction of trees becomes more challenging when the number of dimensions increases, thus, this represents an advantage for one-factor models. Furthermore, when talking about Monte Carlo simulations short-rate models manifest other positive aspects. Market Models have a preassigned set of dates, and when a financial product requires an evaluation between them, an interpolation routine is required, giving space to uncertainty. On the other hand, forward and swap rates can be recovered for any expiry, maturity and payment date with many short-rate models. Finally, calibration and Monte Carlo simulation of the short-rate models are less involved compared to Market Models, basically due to the lower number of dimensions that they rely on.

The purpose of this thesis is not the implementation itself, moreover, it is not common to involve stochasticity in the evaluation of the exposure to prepayment risk, therefore a simpler approach is preferred. The focus is providing a framework for the evaluation of the portfolio, which will lead to the construction of a proper hedge. In fact, the novelty lies exactly in setting up a full model, from the interest-rate modelling to the hedge test. Given the framework, a different model for the interest rate as well as a more precise forecast of the Conditional Prepayment Rate would improve the performance of the whole pricing and hedging strategy. However, since every formula in this thesis has been proved and implemented, it has been chosen to dedicate more time to the research rather than to a difficult implementation of a Market Model. Concluding, considering also the useful peculiarities of the short-rate models mentioned previously, the choice of the interest rate model ended on a short-rate model. Among them, the choice has been even more restricted, because, for the development of the prepayment model and the hedging strategy, analytical prices of bonds, European options on bonds, floorlets and swaptions are preferred. Already the first two requirements would steer the choice toward Vasicek, CIR, Hull-White and CIR++ models. Furthermore, since a Normal distribution of the short-rate seems to be preferred lately, probably due to the impossibility of Lognormal or Chi-squared distributions to achieve negative rates, the Vasicek and the Hull-White models suit well. Since an extension to the Vasicek model is provided by the Hull-White model, the latter has been chosen as the model for the short-rate process. Nevertheless, in this chapter also the CIR++ is presented, since it will be used to evaluate the impact of a different distribution of interest rates in the performance of the hedging procedure.

This chapter is organized as follows. First, the essential mathematical framework is developed and the fundamental interest rates are introduced. Then, the focus moves on the Hull-White model and its distributions under different measures, which represent crucial knowledge for the pricing of financial derivatives performed in the next section. After that, similar information is provided for the CIR++ model. Finally, the calibration procedure of the two interest rate models is illustrated, explaining the problems coming from negative rates and, therefore, motivating the existence of two different quotes observable in the market for the implied volatility of the swaptions.

## 2.1. MATHEMATICAL NOTATION AND DEFINITIONS

In this section the mathematical framework is established. In particular, mathematical definitions and theorems applied during the remainder of this thesis are presented and explained. The first concept that is necessary to introduce is the difference between real-world and risk-free measures. These two measures will be extensively used during the calibration of the interest model and of the prepayment function. The distinction is essential since the short-rate calibration is performed under the risk-neutral measure, while the prepayments are observed from a data set and used to establish a relation with the interest rate incentive, resulting in a historical calibration under the real-world measure.

The real-world measure $\mathbb{P}$, together with the universe of all the possible outcomes $\Omega$, determines the probability of the events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The probability measure $\mathbb{P}$ basically maps any achievable event included in the $\sigma-$algebra $\mathcal{F}$ to a probability, i.e. $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$. However, in finance, derivative pricing is performed under the risk-neutral measure $\mathbb{Q}$. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, which means that they agree on the impossible outcomes (they give probability zero to the same events) but they might not agree on the probability of the feasible scenarios. The main feature of the risk-neutral measure is that it allows a universal pricing of any derivative, imposing that a price of an asset is exactly the present value of its expected payoff. In this framework, a challenge could be defining what precisely discounting means. In fact, what is required to do is incorporating all the possible risk-premia that the investors would ask in the real-world in a quantity that fixes a universal risk-free return. This way, it is possible to obtain a market in which all the assets offering the same return will share the same price. The risk-free rate is defined together with the money-savings account as follows

**Definition 2** (Money-market account)**.** $M(t)$ represents the value of a bank account at time $t \geq 0$. The initial condition is assumed to be $M(0) = 1$, thus, the bank account evolves according to the following differential equation:

$$\begin{cases} \mathrm{d}M(t) & = r(t)M(t)\mathrm{d}t, \\ M(0) & = 1, \end{cases}$$

where $r(t)$ is a positive function of time. As a consequence,

$$M(t) = \exp\left\{\int_0^t r(u)\mathrm{d}u\right\}. \tag{2.1}$$

From the definition of risk-free rate and money-savings account, another quantity is derived instantly, the zero-coupon bond. The bond is an essential instrument in the market because, contrary to the short rate, it is actually traded, raising the possibility to use it as numéraire, as explained later in this chapter.

**Definition 3** (Zero-coupon bond)**.** A T-maturity zero-coupon bond is a contract that guarantees its holder the payment of one unit of currency at time $T$, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t,T)$. The fundamental theorem of asset pricing states that the price at time $t$ of any contingent claim with payoff $H(T)$ is given by

$$V(t,r) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\mathrm{d}u}H(T)\middle|\mathcal{F}(t)\right],$$

where the expectation is taken under the risk neutral measure $\mathbb{Q}$. Therefore, the price of a zero-coupon bond at time $t$ with maturity $T$ is given by

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\middle|\mathcal{F}(t)\right],$$

where, clearly, $P(T,T) = 1$ for all $T$.

From this definition it is clear that, since the zero-coupon bond gives the actual price of one unit of money in the future, whenever the actual price of a future cash flow is required, the bond plays a key role. Zero-coupon bonds are also very convenient because, on one hand, starting from bonds all rates can be recovered, while, on the other hand, bonds can be defined using any family of interest rates as starting point. In reality, interest rates are directly quoted in the market, bypassing the bond step. One of the most important rates nowadays is the London Inter-bank Offered Rate (LIBOR), which is obtained by averaging the estimation of the leading banks in London about a short-term loan from other banks. From 2013, the LIBOR is quoted for five different currencies and seven maturities, from overnight to one year. In the framework we are building, as it is usually done, the forward Libor rate is modelled starting from the bond definition.

**Definition 4** (Forward Libor rate)**.** For a given year fraction $\tau_i = T_i - T_{i-1}$ and $P(t,T_i)$ the risk-free zero-coupon bond maturing at time $T_i$, the forward Libor rate is defined as:

$$L(t;T_{i-1},T_i) = \frac{1}{\tau_i}\frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_i)}. \tag{2.2}$$

To develop a theory to model and hedge the prepayment risk, some extra ingredients are required. In particular, swaps, floorlets and swaptions will be largely used to approximate the prepayment rate and address the causes of prepayment to tradeable instruments. Since floorlets and swaptions are not universally priced as swaps with a "model-free" procedure, proper derivations of the price of these derivatives are given under the assumption that the short-rate behaves according to the Hull-White model. Thus, this section concludes postponing the definitions and the pricing of these instruments to the section where the Hull-White model is presented.

**Definition 5** (Swap)**.** A swap is an interest rate derivative that enables to swap a floating interest rate with a fixed rate, or vice versa, with a notional amount $N$. The plain vanilla interest rate swap contains two legs, which are the fixed leg and the floating leg (typically, the Libor rate). If the fixed leg is payed, the swap is called *payer*, otherwise it is called *receiver*. Supposing that the two legs are exchanged at future times $T_{m+1},...,T_n$, and today is indicated as $t$, then the payoff of a (payer) interest rate swap is

$$V(T_m,...,T_n) = \sum_{i=m+1}^{n} \tau_k N(L(T_{i-1};T_{i-1},T_i) - K).$$

Moreover, the price can be determined without any assumption about the underlying model, and it is

$$V_S(t) = N(P(t,T_m) - P(t,T_n)) - NK \sum_{i=m+1}^{n} \tau_i P(t,T_i).$$

**Definition 6** (Swap rate)**.** Standard practice is to determine the strike price $K$ such that the value of the swap at initial time $t$ is zero. This implies that the two parties can enter the contract without any cost. Imposing the price of the swap equal to zero leads to

$$S_{m,n}(t) = \frac{P(t,T_m) - P(t,T_n)}{\sum_{i=m+1}^{n} \tau_i P(t,T_i)}. \tag{2.3}$$

## 2.2. THE HULL-WHITE MODEL

This section is dedicated to the Hull-White model (HW). After introducing the model and its main characteristics, the focus will move to the pricing of derivatives in the Hull-White framework. In particular, the derivatives analysed are the ones used in the next chapter to model and hedge the prepayment risk. Hull and White [22] assumed that the dynamics of the short-rate are

$$\mathrm{d}r(t) = \lambda(\theta(t) - r(t))\,\mathrm{d}t + \eta \mathrm{d}W^{\mathbb{Q}}(t), \tag{2.4}$$

where $\mathrm{d}W^{\mathbb{Q}}(t)$ is the Brownian Motion under the risk-neutral measure $\mathbb{Q}$, $\theta(t)$ is a deterministic function of time and $\lambda,\eta$ are two positive parameters. In this model, $\theta(t)$ represents the long-term average of the short-rate, and it is used to fit the term structure of interest rates observed in the market. The parameter $\lambda$ is usually referred to as speed of mean reversion, and models the rate at which the short-rate converges to $\theta$. Together with $\lambda$, the volatility parameter $\eta$ is calibrated to replicate the swaption implied volatility quoted in the market. Details about the calibration of the three quantities $\theta(t),\lambda,\eta$ will be given in the section dedicated to the calibration of HW. However, in order to reach a correct pricing of swaptions under the Hull-White model, details about the different distributions of the short-rate under different measures need to be shown. The goal is to express the dynamics of $r(t)$ under another measure called the T-forward measure, in which the numéraire is the bond $P(t,T)$. Nevertheless, before performing the change of measure, the distribution under the risk-neutral measure $\mathbb{Q}$ is specified.

#### SOLUTION OF THE SDE AND DISTRIBUTION UNDER THE RISK-NEUTRAL MEASURE

Given the dynamics (2.4), the value of the short rate at time $t$ is

$$r(t) = e^{-\lambda(t-t_0)} r_0 + \lambda \int_{t_0}^{t} \theta(z) e^{-\lambda(t-z)} \mathrm{d}z + \eta \int_{t_0}^{t} e^{-\lambda(t-z)} \mathrm{d}W^{\mathbb{Q}}(z). \tag{2.5}$$

Taking into account that the Itô integral is normally distributed, using Itô isometry[1] we can easily get that $r(t)$ also follows a Normal distribution with mean and variance given by

$$\begin{aligned} \mu_{r,\mathbb{Q}}(t) &= \mathbb{E}\left[r(t)|\mathcal{F}(t_0)\right] = r_0 e^{-\lambda(t-t_0)} + \lambda \int_{t_0}^{t} \theta(z) e^{-\lambda(t-z)} \mathrm{d}z, \\ v^2_{r,\mathbb{Q}}(t) &= \frac{\eta^2}{2\lambda}\left(1 - e^{-2\lambda(t-t_0)}\right). \end{aligned} \tag{2.6}$$

Note that it is explicitly shown that these results are valid under the measure $\mathbb{Q}$ with the use of the proper subscript.

### 2.2.1. THE CHANGE OF MEASURE

The change of measure is a useful toolkit that is extensively used in the pricing procedures. The core of every change of measure is Girsanov's theorem, which allows to describe precisely the dynamics of a stochastic process when this is considered under a different measure. The main idea behind the whole method is using a different quantity as numéraire, in order to simplify the payoff of a derivative and make it possible to reach a closed form solution. However, a numéraire can only be a tradeable asset, otherwise it would not be possible to express the values of the remaining instruments relatively to it. The importance of Girsanov's theorem is huge, since it used to move the focus from the physical measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. Nevertheless, the theorem can be used to move toward any measure, and, for instance, one of the crucial steps of pricing a European option on a zero-coupon bond under the Hull-White dynamics for the short rate is changing measure from $\mathbb{Q}$ to the so called T-forward measure $\mathbb{Q}^T$, in which the numéraire is the zero-coupon bond itself (which, of course, is a tradeable asset). The dynamics of the zero-coupon bond $P(t,T)$ are related to the ones of the short rate $r(t)$ via

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t + P(t,T)\eta B(t,T)\mathrm{d}W^{\mathbb{Q}}(t), \tag{2.7}$$

where the definition of $B(t,T)$ is

$$B(t,T) = \frac{1}{\lambda}\left(e^{-\lambda(T-t)} - 1\right).$$

Once the dynamics of the numéraires of the two measures $\mathbb{Q}$ and $\mathbb{Q}^T$ are known, the computation of the Radon-Nikodym derivative that is necessary to perform the change of measure is straightforward. In fact, Assuming $t_0 < t < T$,

$$\lambda^T_{\mathbb{Q}}(t) := \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \frac{P(t,T)}{P(t_0,T)}\frac{M(t_0)}{M(t)}. \tag{2.8}$$

The dynamics of $\lambda^T_{\mathbb{Q}}(t)$ are obtained with a simple differentiation rule

$$\mathrm{d}\lambda^T_{\mathbb{Q}}(t) = \frac{M(t_0)}{P(t_0,t)}\left(\frac{1}{M(t)}\mathrm{d}P(t,T) - \frac{P(t,T)}{M^2(t)}\mathrm{d}M(t)\right),$$

[1] Particularly useful when it comes to the computation of second moments of random variables $X(t)$, Itô'isometry helps transforming a stochastic integral into a deterministic integral: $\mathbb{E}\left[\left(\int_0^T X(t)\mathrm{d}W(t)\right)^2\right] = \mathbb{E}\left[\int_0^T X^2(t)\mathrm{d}t\right]$.

and, using Definition 2 and Equation (2.7),

$$\begin{aligned} d\lambda_{\mathbb{Q}}^{T}(t) &= \frac{M(t_0)}{P(t_0,t)}\left(\frac{1}{M(t)}r(t)\big(P(t,T)dt + P(t,T)\eta B(t,T)dW^{\mathbb{Q}}(t)\big) - \frac{P(t,T)}{M^2(t)}r(t)M(t)dt\right) \\ &= \frac{M(t_0)}{P(t_0,t)}\frac{P(t,T)\eta B(t,T)dW^{\mathbb{Q}}(t)}{M(t)}. \end{aligned}$$

Finally, using Girsanov's theorem,

$$\begin{aligned} dW^{T}(t) &= dW^{\mathbb{Q}}(t) - \frac{d\lambda_{\mathbb{Q}}^{T}(t)}{\lambda_{\mathbb{Q}}^{T}(t)}dW^{\mathbb{Q}}(t) \\ &= dW^{\mathbb{Q}}(t) - \eta B(t,T)dW^{\mathbb{Q}}(t)\cdot dW^{\mathbb{Q}}(t) \\ &= dW^{\mathbb{Q}}(t) - \eta B(t,T)dt, \end{aligned} \tag{2.9}$$

where the equality $dW^{\mathbb{Q}}(t)\cdot dW^{\mathbb{Q}}(t) = dt$ follows from the well known Ito's multiplication table. Combining (2.4) and (2.9) the dynamics of the Hull-White model can be expressed under the $T$-forward measure:

$$\begin{aligned} dr(t) &= \lambda(\theta(t) - r(t))dt + \eta dW^{\mathbb{Q}}(t) \\ &= \lambda(\hat{\theta}(t) - r(t))dt + \eta dW^{T}(t), \end{aligned} \tag{2.10}$$

where the drift imposed by the Girsanov's theorem in order to maintain the martingality of the Brownian Motion $dW^T$ under $\mathbb{Q}^T$ is reflected in the new long-term average of the short-rate

$$\hat{\theta}(t,T) = \theta(t) + \frac{\eta^2}{\lambda}B(t,T). \tag{2.11}$$

#### DISTRIBUTION OF THE HULL-WHITE MODEL UNDER THE T-FORWARD MEASURE

Once the dynamics of $r(t)$ under the T-forward measure are known, the distribution is achieved straightforwardly. In fact, similarly to (2.6), the result is that $r(t)$ is normally distributed under $\mathbb{Q}^T$, with mean and variance given by

$$\begin{aligned} \mu_{r,T}(t) &= \mathbb{E}\left[r(t)|\mathcal{F}(t_0)\right] = r_0 e^{-\lambda(t-t_0)} + \lambda\int_{t_0}^{t}\hat{\theta}(z,T)e^{-\lambda(t-z)}dz, \\ v_{r,T}^2(t) &= \frac{\eta^2}{2\lambda}\left(1 - e^{-2\lambda(t-t_0)}\right). \end{aligned} \tag{2.12}$$

## 2.3. PRICE OF FINANCIAL INSTRUMENTS UNDER THE HULL-WHITE MODEL

In the previous section the Hull-White model has been introduced, and its distributions under the risk-neutral measure and the T-forward measure have been derived. This section proceeds into the pricing of derivatives under the Hull-White model and it starts showing that this belongs to the affine-diffusion family, a useful property which allows for a closed from price of a zero-coupon bond. From there, more complicated payoffs are priced, such as a European option on the zero-coupon bond, floorlets and swaptions. Not all the details are provided here, and for further clarifications more insight is given in Appendix A.

### 2.3.1. BONDS AND EUROPEAN OPTIONS ON BONDS

One of the main reasons that drove the choice of the interest-rate model toward Hull-White has been the fact that both the price of a zero-coupon bond and the price of a European option on a zero-coupon bond are provided through a closed formula in the Hull-White framework. These advantages come directly from the fact that HW presents an affine diffusion.

**Definition 7** (Affine processes class)**.** Consider the following system of SDEs:

$$\mathrm{d}\mathbf{X}(t) = \mu(\mathbf{X}(t))\mathrm{d}t + \sigma(\mathbf{X}(t))\mathrm{d}\mathbf{W}(t),$$

with independent Brownian motions $\mathbf{W}(t)$. For processes in the affine diffusion class (AD) it is assumed that drift, volatility, and interest rate components are of the affine form, i.e.

$$\begin{aligned}\mu(\mathbf{X}(t)) &= a_0 + a_1\mathbf{X}(t) \text{ for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n},\\ \sigma(\mathbf{X}(t))\sigma(\mathbf{X}(t))^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}(t) \text{ for } (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n},\\ r(\mathbf{X}(t)) &= r_0 + r_1^T\mathbf{X}(t), \text{ for } (r_0, r_1) \in \mathbb{R}^n \times \mathbb{R}^n.\end{aligned}$$

It is easily proved that the Hull-White model belongs to the AD class since, according to Equation (2.4),

$$\begin{aligned}\mu(r(t)) &= \underbrace{\lambda\theta(t)}_{a_0}\underbrace{-\lambda}_{a_1} r(t)\\ \sigma(r(t)) &= \underbrace{\eta^2}_{c_0} + \underbrace{0}_{c_1} r(t)\\ r((r(t)) &= \underbrace{0}_{r_0} + \underbrace{1}_{r_1} r(t).\end{aligned} \tag{2.13}$$

## ZERO-COUPON BOND PRICE UNDER HULL-WHITE

Under the Hull-White dynamics for the short-rate, the price of a zero-coupon bond is given by

**Proposition 1** (Zero-coupon bond price)**.**

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\mathrm{d}u}\middle|\mathcal{F}(t)\right] = e^{A(t,T)+B(t,T)r(t)}, \tag{2.14}$$

with

$$\begin{aligned}B(t,T) &= \frac{1}{\lambda}\left(e^{-\lambda(T-t)} - 1\right)\\ A(t,T) &= \lambda\int_0^{T-t}\theta(T-z)B(T-z,T)\mathrm{d}z + \frac{\eta^2}{4\lambda^3}\left[e^{-2\lambda(T-t)}\left(4e^{\lambda(T-t)} - 1\right) - 3\right] + \frac{\eta^2(T-t)}{2\lambda^2}.\end{aligned} \tag{2.15}$$

*Proof.* Since we showed that Hull-White model belongs to the AD class, thanks to [23] we know that the discounted characteristic function

$$\phi(u, r, t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(z)dz + iur(T)}\middle|\mathcal{F}(t)\right] \text{ for } u \in \mathbb{C},$$

with boundary condition

$$\phi(u, r(t), T, T) = e^{iur(t)}, \tag{2.16}$$

has solution of the form

$$\phi(u, r, t, T) = e^{A(u,t,T)+B(u,t,T)r(t))}.$$

The coefficients $A(u,t,T)$ and $B(u,t,T)$ come from a system of Ordinary Differential Equations (ODEs). Using (2.13), and defining $\tau = T - t$, we have that

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = -r_0 + a_0B(u,\tau) + \frac{1}{2}c_0B^2(u,\tau) = \lambda\underbrace{\theta(t)}_{\theta(T-\tau)}B(u,\tau) + \frac{1}{2}\eta^2B^2(u,\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}B(u,\tau) = -r_1 + a_1B(u,\tau) + \frac{1}{2}c_1B^2(u,\tau) = -1 - \lambda B(u,\tau).\end{cases}$$

Moreover, from (2.16) we get the boundary conditions

$$A(u,T,T) = A(u,0) = 0,$$
$$B(u,T,T) = B(u,0) = iu.$$

We start solving the ODE for $B(u,\tau)$, that is a first-order linear ODE with variable coefficients. For this kind of ODE there is a well known closed formula

$$B(u,\tau) = e^{-\int \lambda \mathrm{d}\tau}\left[-\int e^{\int \lambda \mathrm{d}\tau}\mathrm{d}\tau + C\right] = -\frac{1}{\lambda} + Ce^{-\lambda\tau}.$$

Applying the boundary condition $B(u,0) = iu$ we obtain $C = iu + \frac{1}{\lambda}$, therefore

$$B(u,\tau) = iue^{-\lambda\tau} + \frac{1}{\lambda}\left(e^{-\lambda\tau} - 1\right).$$

Now, the focus is on $A(u,\tau)$. We already know that $A(u,0) = 0$, moreover we notice that the right hand-side of the ODE for $A(u,\tau)$ does not depend on $A(u,\tau)$ itself. Thus, we can simply integrate

$$\begin{aligned}
A(u,\tau) &= \underbrace{\lambda\int_0^\tau \theta(T-z)B(u,z)\mathrm{d}z}_{\Xi} + \frac{1}{2}\eta^2\int_0^\tau B^2(u,z)\mathrm{d}z \\
&= \Xi + \frac{1}{2}\eta^2\int_0^\tau\left[iue^{-\lambda z} + \frac{1}{\lambda}\left(e^{-\lambda z} - 1\right)\right]^2\mathrm{d}z \\
&= \Xi + \frac{1}{2}\eta^2\left[\left(-u^2 + \frac{1}{\lambda^2} + \frac{2iu}{\lambda}\right)\int_0^\tau e^{-2\lambda z}\mathrm{d}z - \left(\frac{2}{\lambda^2} + \frac{2iu}{\lambda}\right)\int_0^\tau e^{-\lambda z}\mathrm{d}z + \frac{\tau}{\lambda}\right] \\
&= \Xi + \frac{1}{2}\eta^2\left[e^{-2\lambda\tau}\left(\frac{u^2}{2\lambda} - \frac{1}{2\lambda^3} - \frac{iu}{\lambda^2}\right) + e^{-\lambda\tau}\left(\frac{2}{\lambda^3} + \frac{2iu}{\lambda^2}\right) - \frac{u^2}{2\lambda} - \frac{3}{2\lambda^3} - \frac{iu}{\lambda^2} + \frac{\tau}{\lambda^2}\right].
\end{aligned}$$

The price of the zero-coupon bond like in (2.14) is then obtained computing $A(0,\tau)$ and $B(0,\tau)$. □

### EU OPTION ON BOND

A European option on a zero-coupon bond is a contract which gives the owner the right, but not the obligation, to buy a zero-coupon bond with a specific maturity at a certain strike price and time in future. The following derivations are valid for a put option, because that will be needed later to price a payer swaption. The price for the call can be obtained either by a similar procedure, or by a put-call parity relation. In this framework, the time horizon is made of three nodes: $t_0$ is the time at which the option is priced, $T$ is the maturity of the option on the bond, and $T_S$ is the maturity of the bond. Therefore, the underlying zero-coupon bond price is $P(T,T_S)$.

**Proposition 2** (European put on a zero-coupon bond)**.**

$$V_{\text{ZCB,Put}}(t_0,T;T_S) = P(t_0,T)e^{A(T,T_S)}\left[\hat{K}\Phi(-d_2) - e^{B(T,T_S)\mu_{r,T}(T) + \frac{1}{2}B^2(T,T_S)v^2_{r,T}(T)}\Phi(-d_1)\right], \tag{2.17}$$

where $\Phi(\cdot)$ indicates the Cumulative Distribution Function (CDF) of a standard Normal distribution.

*Proof.* The initial equation is:

$$\begin{aligned}
V_{\text{ZCB,Put}}(t_0,T;T_S) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}(K - P(T,T_S))^+\middle|\mathcal{F}(t_0)\right] \quad \text{(change measure using (2.8))} \\
&= P(t_0,T)\mathbb{E}^T\left[(K - P(T,T_S))^+\middle|\mathcal{F}(t_0)\right] \quad \text{(bond is expressed with (2.14))} \\
&= P(t_0,T)\mathbb{E}^T\left[\left(K - e^{A(T,T_S)+B(T,T_S)r(T)}\right)^+\middle|\mathcal{F}(t_0)\right] \quad (\hat{K} = Ke^{-A(T,T_S)}) \\
&= P(t_0,T)e^{A(T,T_S)}\mathbb{E}^T\left[\left(\hat{K} - e^{B(T,T_S)r(T)}\right)^+\middle|\mathcal{F}(t_0)\right].
\end{aligned}$$

Now, using Equation (2.12) and the properties of the Normal Distribution, we get that $\hat{X}(T,T_S) = B(T,T_S)r(T)$ is also Normally distributed such that

$$\hat{X}(T,T_S) = B(T,T_S)r(T) \sim \mathcal{N}\left(B(T,T_S)\mu_{r,T}(T), B^2(T,T_S)v^2_{r,T}(T)\right).$$

Thus, we can continue the derivation expressing the value of the European put on a zero-coupon bond as

$$\begin{aligned} V_{\text{ZCB,Put}}(t_0,T;T_S) &= P(t_0,T)e^{A(T,T_S)}\mathbb{E}^T\left[\left(\hat{K}-e^{\hat{X}(T,T_S)}\right)^+\Big|\mathcal{F}(t_0)\right] = \left(\text{using } \frac{\hat{X}(T,T_S)-\mu_{\hat{X}}}{v^2_{\hat{X}}} \sim \mathcal{N}(0,1)\right) \\ &= P(t_0,T)e^{A(T,T_S)}\int_{d_2}^{\infty}\left(\hat{K}-e^{B(T,T_S)\mu_{r,T}(T)+B(T,T_S)v_{r,T}(T)x}\right)f_{\mathcal{N}(0,1)}(x)\mathrm{d}x, \end{aligned}$$

where $d_2$ is obtained by imposing

$$\begin{aligned} \hat{K} &> e^{B(T,T_S)\mu_{r,T}(T)+B(T,T_S)v_{r,T}(T)x} \\ \log(\hat{K}) - B(T,T_S)\mu_{r,T}(T) &> B(T,T_S)v_{r,T}(T)x \quad (\text{note that } B(T,T_S)<0) \\ x &> \frac{\log(\hat{K}) - B(T,T_S)\mu_{r,T}(T)}{B(T,T_S)v_{r,T}(T)} = d_2. \end{aligned}$$

The previous integral is made up of two parts. The first one is

$$\int_{d_2}^{\infty}\hat{K}f_{\mathcal{N}(0,1)}(x)\mathrm{d}x = \hat{K}(1-\Phi(d_2)) = \hat{K}\Phi(-d_2),$$

while the second is

$$\int_{d_2}^{\infty}e^{B(T,T_S)\mu_{r,T}(T)+B(T,T_S)v_{r,T}(T)x}f_{\mathcal{N}(0,1)}(x)\mathrm{d}x = e^{B(T,T_S)\mu_{r,T}(T)+\frac{1}{2}B^2(T,T_S)v^2_{r,T}(T)}\Phi(-d_1),$$

where we set $d_1 = d_2 - B(T,T_S)v_{r,T}(T))$. From here, the result (2.17) follows immediately. □

### 2.3.2. FLOORLETS AND SWAPTIONS

Floorlets and swaptions will be the core of the hedging strategy of the prepayment risk. The fact that they give an option to the holder is particularly useful to model the prepayment option embedded in the mortgage contract and the non-linearity of a mortgage portfolio will be replicated with these instruments. Thus, pricing swaptions becomes essential for two reasons: on one hand, swaptions are used to calibrate the parameters of the Hull-White model, comparing the swaption prices in the market with the ones that the model returns; on the other hand, the replicating portfolio can be evaluated without resorting to Monte Carlo simulations. Swaptions pricing uses EU ZCB options as building blocks, which also motivates the previous derivations.

#### FLOORLET

A floorlet could be seen as a put option on the Libor rate. This contract requires three dates: $t_0$ is the moment of evaluation and $T_{n-1}$ is the expiring date to exercise the option on the Libor $L(T_{n-1};T_{n-1},T_n)$. These particular derivations connect the floorlet's price to the value of a EU call on a ZCB.

**Proposition 3** (Floorlet price)**.**

$$V_{\text{Floorlet}}(t_0;T_{n-1};T_n) = \frac{1}{\hat{K}}V_{\text{ZCB,Call}}(t_0,T_{n-1};T_n,\hat{K}), \tag{2.18}$$

where $(\tau_n K+1) = \frac{1}{\hat{K}}$ and $\tau_n = T_n - T_{n-1}$.

*Proof.*

$$
\begin{aligned}
V_{\text{floorlet}}(t_0;T_{n-1};T_n) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_{n-1})}\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{M(T_{n-1})}{M(T_n)}\tau_n\big(K-L(T_{n-1};T_{n-1},T_n)\big)\Big|\mathcal{F}(T_{n-1})\right]\right)^+\Big|\mathcal{F}(t_0)\right]\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_{n-1})}P(T_{n-1},T_n)\tau_n\big(K-L(T_{n-1};T_{n-1},T_n)\big)^+\Big|\mathcal{F}(t_0)\right] \quad \text{(change of measure)}\\
&= P(t_0,T_{n-1})\mathbb{E}^{T_{n-1}}\left[P(T_{n-1},T_n)\tau_n\big(K-L(T_{n-1};T_{n-1},T_n)\big)^+\Big|\mathcal{F}(t_0)\right] \quad \text{(expand the Libor)}\\
&= P(t_0,T_{n-1})\mathbb{E}^{T_{n-1}}\left[\big(P(T_{n-1},T_n)\cdot(\tau_nK+1)-1\big)^+\Big|\mathcal{F}(t_0)\right]\\
&= P(t_0,T_{n-1})\frac{1}{\hat{K}}\mathbb{E}^{T_{n-1}}\left[\big(P(T_{n-1},T_n)-\hat{K}\big)^+\Big|\mathcal{F}(t_0)\right]\\
&= \frac{1}{\hat{K}}V_{\text{ZCB,Call}}(t_0,T_{n-1};T_n,\hat{K}).
\end{aligned}
$$

□

## SWAPTION

A swaption is a contract which gives the right (but not the commitment) to enter, at a certain time in the future, in a swap with a predetermined strike price $K$. We assume that the swaption maturity (the moment in which the option can be exercised) coincides with the first reset date of the underlying swap. We define $t_0$ as the moment of evaluation of the swaption, $T_m$ as the maturity of the swaption and first reset day of the underlying swap, $T_n$ as last date of payment of the swap.

**Proposition 4** (Payer swaption price)**.**

$$
V_{\text{Swpt,Pay}}(t_0;T_m,T_n) = NP(t_0,T_m)\sum_{k=m+1}^{n} c_k V_{\text{ZCB,Put}}(t_0,T_m;T_k,\widetilde{K}), \tag{2.19}
$$

where $\widetilde{K} = e^{A(T_m,T_k)+B(T_m,T_k)r^*}$ and $r^*$ such that

$$
1-\sum_{k=m+1}^{n} c_k e^{A(T_m,T_k)+B(T_m,T_k)r^*} = 0.
$$

*Proof.* The initial procedure necessary to price the swaption is similar to the one used to price the EU option on a zero-coupon bond

$$
\begin{aligned}
V_{\text{Swpt,Pay}}(t_0;T_m,T_n) &= N\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_m)}\left(\sum_{k=m+1}^{n}\tau_kP(T_m,T_k)\big(L(T_m;T_{k-1},T_k)-K\big)\right)^+\Big|\mathcal{F}(t_0)\right]\\
&= NP(t_0,T_m)\mathbb{E}^{T_m}\left[\left(\sum_{k=m+1}^{n}\tau_kP(T_m,T_k)\big(L(T_m;T_{k-1},T_k)-K\big)\right)^+\Big|\mathcal{F}(t_0)\right]\\
&= NP(t_0,T_m)\mathbb{E}^{T_m}\left[\left(1-P(T_m,T_n)-K\sum_{k=m+1}^{n}\tau_kP(T_m,T_k)\right)^+\Big|\mathcal{F}(t_0)\right]\\
&= NP(t_0,T_m)\mathbb{E}^{T_m}\left[\left(1-\sum_{k=m+1}^{n}c_kP(T_m,T_k)\right)^+\Big|\mathcal{F}(t_0)\right],
\end{aligned}
$$

where $c_i = K\tau_i$ for $i = m+1,\dots,n-1$ and $c_n = 1+K\tau_n$. Using Equation (2.14) and Jamshidian's technique[2] [24], we can express the value of the swaption as

$$\begin{aligned} V_{\text{Swpt,Pay}}(t_0; T_m, T_n) &= NP(t_0, T_m)\mathbb{E}^{T_m}\left[\left(1 - \sum_{k=m+1}^{n} c_k e^{A(T_m,T_k)+B(T_m,T_k)r(T_m)}\right)^+ \middle| \mathcal{F}(t_0)\right] \\ &= NP(t_0, T_m) \sum_{k=m+1}^{n} c_k \mathbb{E}^{T_m}\left[\left(\widetilde{K} - e^{A(T_m,T_k)+B(T_m,T_k)r(T_m)}\right)^+ \middle| \mathcal{F}(t_0)\right]. \end{aligned} \tag{2.20}$$

What remains to do in Equation (2.20) is the evaluation of the put option on the zero-coupon bond, which can be done using Equation (2.17), immediately leading to (2.19) □

## 2.4. CIR++ AND NEGATIVE RATES

The main reason to introduce another interest rate model is to countercheck the results of the hedging procedure. The idea will be to evaluate the model risk behind the choice of the Hull-White model as driver of the short-rate and this will be done trying to see whether the mark-to-model value of the complex derivative representing the mortgage portfolio changes significantly when the distribution of the interest-rate differs from the expected one. In particular, the difference in the distribution of Hull-White and CIR (or CIR++) models is exactly what is needed to verify whether the choice of the Hull-White impacts on the final results. For this reason, using the Vasicek model would have been useless, since it shows the same Normal distribution as HW. On the other hand, the same reasons that led to choose HW are still valid, thus a model with an analytic formula for prices of zero-coupon bonds and bond options is favorable, making the models of Black and Karasinski, Dothan and Exponential Vasicek poor choices. Nevertheless, the issue of negative interest rates remains, and the simple CIR model would be incapable to fit this requirement, while its extension, the CIR++, can capture negative rates and fit the market data. Thus, in this section the CIR and CIR++ models are introduced, together with the formulas necessary to price swaptions, which will be used to calibrate the model.

The CIR model reads

$$\mathrm{d}x(t) = \lambda^{\text{CIR}}\left(\theta^{\text{CIR}} - x(t)\right)\mathrm{d}t + \eta^{\text{CIR}}\mathrm{d}W^{\mathbb{Q}}(t), \quad x(0) = x_0, \tag{2.21}$$

where the roles of the parameters $\Upsilon = \left(\lambda^{\text{CIR}}, \theta^{\text{CIR}}, \eta^{\text{CIR}}\right)$ follow immediately from the theory developed for the Hull-White model. In this case, $x(t)$ represents the short-rate, and the Feller condition $2\lambda^{\text{CIR}}\theta^{\text{CIR}} > \left(\eta^{\text{CIR}}\right)^2$ ensures that the value zero cannot be reached by the process. In fact, the CIR model assumes a non-central chi-squared distribution, which does not allow rates to be negative. However, nowadays, negativity of the interest rates is a feature commonly observed in the markets, therefore a solution is required. Moreover, (2.21) can barely fit the initial bond term structure given by the market, mostly because of the fact that $\theta^{\text{CIR}}$ is maintained constant. Both problems can be solved introducing a deterministic shift, which leads to the formulation commonly known as CIR++ model

$$\begin{aligned} \mathrm{d}x(t) &= \lambda^{\text{CIR}}\left(\theta^{\text{CIR}} - x(t)\right)\mathrm{d}t + \eta^{\text{CIR}}\mathrm{d}W^{\mathbb{Q}}(t), \quad x(0) = x_0, \\ r(t) &= x(t) + \varphi^{\text{CIR}}(t). \end{aligned} \tag{2.22}$$

A proper choice of the function $\varphi^{\text{CIR}}(t)$ helps fitting the term structure and gives freedom to $r(t)$ to access negative values. We define the quantity $h = \sqrt{(\lambda^{\text{CIR}})^2 + 2(\eta^{\text{CIR}})^2}$ which helps to smoothen the notation.

[2] Basically, $\left(K - \sum_i \psi_i(r)\right)^+ = \sum_i \left(\psi_i(r^*) - \psi_i(r)\right)^+$ with $\psi_i$ a monotonic increasing or decreasing sequence of functions, and $r^*$ such that $K - \sum_i \psi_i(r^*) = 0$.

### 2.4.1. ZERO-COUPON BOND, BOND-OPTION AND SWAPTION PRICES UNDER CIR++

Since (2.22) still is an Affine process, the value at time $t$ of a zero-coupon bond is given analytically.

**Proposition 5** (Zero-coupon bond)**.**

$$P(t,T) = \frac{P^M(0,T)A(0,t)e^{-B(0,t)x_0}}{P^M(0,t)A(0,T)e^{-B(0,T)x_0}} A(t,T)e^{-B(t,T)\left(r(t)-\varphi^{\text{CIR}}(t;\Upsilon)\right)}, \tag{2.23}$$

with

$$\begin{aligned} A(t,T) &= \left[\frac{2he^{(\lambda^{\text{CIR}}+h)\frac{(T-t)}{2}}}{2h+(\lambda^{\text{CIR}}+h)\left(e^{(T-t)h}-1\right)}\right]^{\frac{2\lambda^{\text{CIR}}\theta^{\text{CIR}}}{(\eta^{\text{CIR}})^2}}, \\ B(t,T) &= \frac{2\left(e^{(T-t)h}-1\right)}{2h+(\lambda^{\text{CIR}}+h)\left(e^{(T-t)h}-1\right)}. \end{aligned}$$

The price at time $t_0$ of a European call option on a zero-coupon bond is also given in a closed form. In the following formulas, the maturity of the option is $T$, the strike price is $K$, the underlying bond matures at time $T_S$ and $\chi^2(\cdot;n,s)$ denotes the noncentral chi–squared cumulative distribution function with $n$ degrees of freedom and non-centrality parameter $s$.

**Proposition 6** (European option on a zero-coupon bond)**.**

$$\begin{aligned} V_{\text{ZCB,Call}}(t_0,T;T_S) = P(t_0,T_S)\chi^2&\left(2\hat{r}[\rho+\psi+B(T,T_S)];\frac{4\lambda^{\text{CIR}}\theta^{\text{CIR}}}{(\eta^{\text{CIR}})^2},\frac{2\rho^2[r(t)-\varphi^{\text{CIR}}(t,\Upsilon)]e^{h(T-t)}}{\rho+\psi+B(T,T_S)}\right)+ \\ -KP(t_0,T)\chi^2&\left(2\hat{r}[\rho+\psi];\frac{4\lambda^{\text{CIR}}\theta^{\text{CIR}}}{(\eta^{\text{CIR}})^2},\frac{2\rho^2[r(t)-\varphi^{\text{CIR}}(t,\Upsilon)]e^{h(T-t)}}{\rho+\psi}\right), \end{aligned} \tag{2.24}$$

with

$$\begin{aligned} \rho &= \frac{2h}{(\eta^{\text{CIR}})^2\left(e^{h(T-t)}-1\right)}, \\ \psi &= \frac{\lambda^{\text{CIR}}+h}{(\eta^{\text{CIR}})^2}, \\ \hat{r} &= \frac{1}{B(T,T_S)}\left[\log\left(\frac{A(T,T_S)}{K}\right)-\log\left(\frac{P^M(0,T)A(0,t)e^{-B(0,t)x_0}}{P^M(0,t)A(0,T)e^{-B(0,T)x_0}}\right)\right]. \end{aligned} \tag{2.25}$$

From the put-call parity also the price of a put option on a ZCB is obtained

$$V_{\text{ZCB,Put}}(t_0,T;T_S) = V_{\text{ZCB,Call}}(t_0,T;T_S) - P(t_0,T_S) + KP(t_0,T).$$

Finally, similarly to the Hull-White model, the swaption price is obtain using the Jamshidian's technique, leading the swaption prices to be determined from a series of bond options.

**Proposition 7** (Payer swaption price)**.**

$$V_{\text{Swpt,Pay}}(t_0;T_m,T_n) = NP(t_0,T_m)\sum_{k=m+1}^{n} c_k V_{\text{ZCB,Put}}(t_0,T_m;T_k,\widetilde{K}) \tag{2.26}$$

where $\widetilde{K} = \frac{P^M(0,T_k)A(0,T_m)e^{-B(0,T_m)x_0}}{P^M(0,T_m)A(0,T_k)e^{-B(0,T_k)x_0}} A(T_m,T_k)e^{-B(T_m,T_k)\left(r^*-\varphi^{\text{CIR}}(T_m;\Upsilon)\right)}$ and $r^*$ such that

$$1-\sum_{k=m+1}^{n} c_k \frac{P^M(0,T_k)A(0,T_m)e^{-B(0,T_m)x_0}}{P^M(0,T_m)A(0,T_k)e^{-B(0,T_k)x_0}} A(T_m,T_k)e^{-B(T_m,T_k)\left(r^*-\varphi^{\text{CIR}}(T_m;\Upsilon)\right)} = 0.$$

## 2.5. CALIBRATION OF THE INTEREST-RATE MODELS

Calibrating the interest-rate models consists in finding the correct set of parameter that is able to reproduce the price (or the implied volatilities) observed in the market. In this section we will give details about the procedure followed to calibrate the Hull-White and the CIR++ models.

### 2.5.1. RELATION WITH THE CURRENT BOND STRUCTURE

The first step in the calibration of the considered interest-rate models regards the relationship with the market bond structure. In the HW model, this is governed by the time-dependent long-term average $\theta^{\text{HW}}(t)$, while in CIR++ the quantity that permits a perfect fitting of the bonds is the deterministic shift $\varphi^{\text{CIR}}(t;\Upsilon)$. Note that the market structure consists of a set of discount factors which, after interpolation, gives rise to a function $P^M(0,t)$, as shown in Figure 2.1. Both functions $\theta^{\text{HW}}(t)$ and $\varphi^{\text{CIR}}(t;\Upsilon)$ rely on the definition of the forward-rate,

$$f^r(0,T) = -\frac{\partial}{\partial T} P^M(0,T),$$

and their aim is essentially to shift the forward-rate returned by the model in order to impose the match with the market. For HW, this is translated to defining

$$\theta^{\text{HW}}(t) = \frac{1}{\lambda}\frac{\partial}{\partial t} f^r(0,t) + f^r(0,t) + \frac{\eta^2}{2\lambda^2}\left(1 - e^{-2\lambda t}\right),$$

while, for CIR++

$$\begin{aligned}
\varphi^{\text{CIR}}(t;\Upsilon) &= f^r(0,t) - f^{\text{CIR}}(0,t;\Upsilon),\\
f^{\text{CIR}}(0,t;\Upsilon) &= \frac{\lambda^{\text{CIR}}\theta^{\text{CIR}}\left(e^{th}-1\right)}{2h + (\lambda^{\text{CIR}}+h)\left(e^{th}-1\right)} + x_0 \frac{4h^2 e^{th}}{\left[2h + (\lambda^{\text{CIR}}+h)\left(e^{th}-1\right)\right]^2},
\end{aligned}$$

with $h = \sqrt{(\lambda^{\text{CIR}})^2 + 2(\eta^{\text{CIR}})^2}$. Since the bonds typically show a decaying behaviour similar to an exponential, the interpolation of the initial bond structure is accomplished by finding a set of parameters $\alpha$ such that $P^M(0,t) = e^{\alpha_0+\alpha_1 t+\alpha_2 t^2+\ldots+\alpha_n t^n}$ fits exactly the market discount factors. Later in this thesis there is the will to show the sensitivity of the price of the mortgage portfolio with respect to the instruments used to calibrate the interest rate model, thus, we point out that the discount factors that we used are actually obtained constructing the yield curve from a set of swap rates $S_{t_0,T_i}(t_0)$ (with $i = 1,\ldots,M$) that are quoted in the market. The construction of the yield curve is a well-known topic and its methodology does not influence the construction of the pricing model nor the hedging strategy, therefore we do not introduce it formally.

### 2.5.2. CALIBRATION TO SWAPTIONS

After determining the correct shift of the model, in order to perform a proper calibration there are three main ingredients that are necessary. First, a correct choice of the instruments. Second, a formula or a procedure that returns the price of such instruments given a set of parameters for the model. Third, a correct loss function. The answers to these questions may differ according to the reason for which the interest rate models were required in the first place, but can also depend on the efficiency of the pricing formulas or, more importantly, to the current market structure and the ability of the model to replicate it. In our case, the prepayment model and the hedging strategy will require the use of swaptions, thus, even if the calibration of the mean-reverting speed and of the volatility could be also done using caplets, we choose to focus on prices and volatilities of swaptions. However, the choice of the correct set of swaptions is also a delicate aspect. In principle, one would like to calibrate on all the market data available, in order to have more information and obtain a

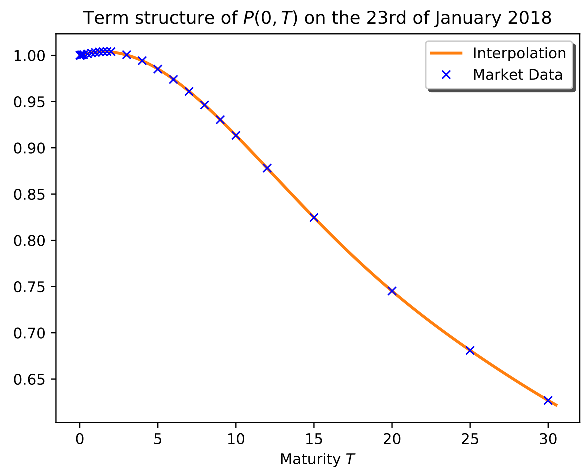



Figure 2.1: ZCB structure observed on the 23rd of January 2018. Data from Bloomberg.

more precise calibration, nevertheless, the fact that the volatility is a constant parameter in both HW and CIR++ makes it impossible to recover smiles and skews. More details and motivation about the final set of swaptions selected are presented in the chapter dedicated to the hedging strategies. The second ingredient, the formulas to price payer and receiver swaptions under HW or CIR++, have been already derived, thus only the choice of a proper loss function is lacking. The main alternatives consist in comparing the (relative) error between the prices or between the implied volatilities of the market and of the model. We choose to calibrate on the prices, so, for HW

$$(\eta,\lambda)_{\mathrm{HW}} = \underset{\lambda,\eta}{\arg\min} \sum_{i=1}^{N_{Swpt}} \left[ V^{\mathrm{HW}}_{\mathrm{Swpt}}\left(t_0, T_m^{(i)}, T_n^{(i)}; \lambda, \eta\right) - V^{\mathrm{Market}}_{\mathrm{Swpt}}\left(t_0, T_m^{(i)}, T_n^{(i)}\right)\right]^2,$$

while, for CIR++,

$$(\eta,\lambda,\theta,x_0)_{\mathrm{CIR++}} = \underset{\lambda,\eta}{\arg\min} \sum_{i=1}^{N_{Swpt}} \left[ V^{\mathrm{CIR++}}_{\mathrm{Swpt}}\left(t_0, T_m^{(i)}, T_n^{(i)}; \lambda, \eta, \theta, x_0\right) - V^{\mathrm{Market}}_{\mathrm{Swpt}}\left(t_0, T_m^{(i)}, T_n^{(i)}\right)\right]^2.$$

However, the market prices are not directly quoted, whereas implied volatilites are. Moreover, two different quotes are present for the implied volatility of the swaptions, depending on the assumed distribution of the underlying swap. We analyse this aspect separately.

### 2.5.3. NORMAL AND LOGNORMAL IMPLIED VOLATILITIES

The market quotes the swaptions in terms of implied volatility. This means that, in order to get the correct price, the volatility has to be plugged into a pricing formula in which all the other parameters (maturity, tenor, frequency of payment...) are known. A pricing formula can be obtained giving information about the distribution of the underlying swaps, and, typically, this leads to choose a lognormal distribution. However, negative interest rates cause issues also regarding this aspect, because the lognormal distribution cannot recover negative values and, nowadays, this problem is avoided in two ways: either the distribution is shifted, or the distribution is changed. A natural choice for a distribution which allows negative values is the Normal distribution, that is why, today, there are two quotes of the swaption implied volatilities: one for Lognormal, the other for the Normal assumption.

Either cases, it is possible to find an analytical formula for the price of a swaption, and the missing parameter to obtain the price is indeed the volatility quoted in the market. Using the Annuity

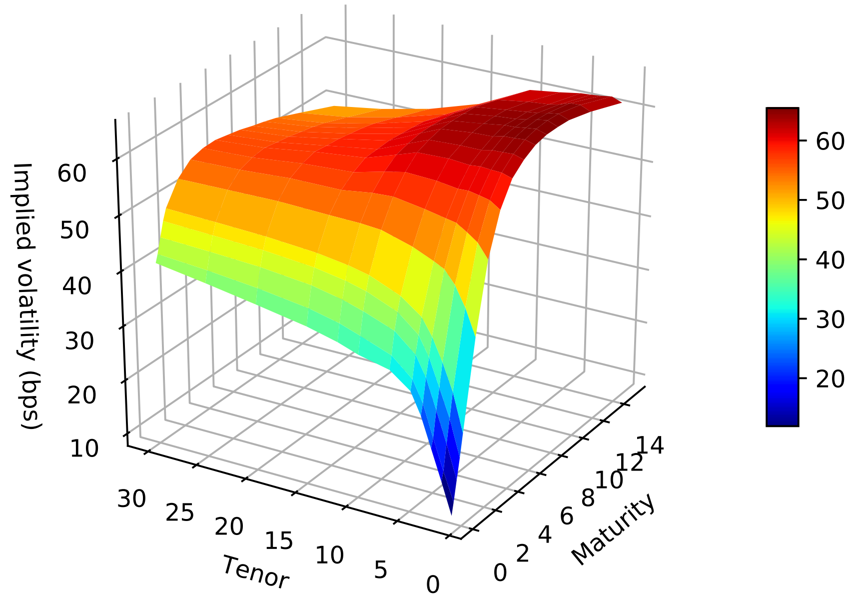



Figure 2.2: Implied volatility surface from swaptions with Normal underlying swap, 23rd of January 2018.

$A_{m,n}(t) = \sum_{i=m+1}^{n} \tau_i P(t, T_i)$ as numéraire, we can express the payoff of a swaption as

$$A_{m,n}(T_m)\big(S_{m,n}(T_m) - K\big)^+,$$

therefore, the swaption becomes a call option on the swap rate $S_{m,n}(T_m)$. Now, assuming a lognormal distribution for the swap rate, we end up using the well known Black-76 formula [25]

$$V^{\mathrm{L}}_{\mathrm{Swpt}}(t_0, T_m, T_n) = N A_{m,n}(t_0)\left(S_{m,n}(t_0)\omega\Phi(\omega d_1) - K\omega\Phi(\omega d_2)\right),$$

$$d_1 = \frac{\log\left(\frac{S_{m,n}(t_0)}{K}\right) - \frac{(\sigma^{\mathrm{L}}_{m,n})^2(T_m - t_0)}{2}}{\sigma^{\mathrm{L}}_{m,n}\sqrt{T_m - t_0}}, \quad d_2 = d_1 + \sigma^{\mathrm{L}}_{m,n}\sqrt{T_m - t_0},$$

where $\omega = 1$ (payer swaption) or $\omega = -1$ (receiver). Alternatively, we can reach a similar result assuming a Normal distribution, which leads to

$$V^{\mathrm{N}}_{\mathrm{Swpt}}(t_0, T_m, T_n) = N A_{m,n}(t_0)\left(\omega(S_{m,n}(t_0) - K)\Phi(\omega d) + \frac{\sigma^{\mathrm{N}}_{m,n}\sqrt{T_m - t_0}}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}\right),$$

$$d = \frac{S_{m,n}(t_0) - K}{\sigma^{\mathrm{N}}_{m,n}\sqrt{T_m - t_0}}.$$

Notice that we distinguished between the two quoted volatilities $\sigma^{\mathrm{L}}_{m,n}$ and $\sigma^{\mathrm{N}}_{m,n}$, which refer to the quotes with the Lognormal or Normal assumption.

### MARKET DATA

The market volatilities used to calibrate the interest rate models are taken from Table 2.1, in which we find the values of $\sigma^{\mathrm{N}}_{m,n}$. In a column, we read the quotes for swaptions with different maturities

| | 1Yr | 2Yr | 3Yr | 4Yr | 5Yr | 7Yr | 10Yr | 12Yr | 15Yr | 20Yr | 25Yr | 30Yr |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1Mo | 8.86 | 14.61 | 20.09 | 25.55 | 29.49 | 32.30 | 33.58 | 34.88 | 36.28 | 37.92 | 39.46 | 40.86 |
| 3Mo | 11.05 | 16.25 | 21.74 | 27.52 | 31.34 | 34.65 | 36.71 | 38.00 | 39.30 | 40.60 | 41.54 | 42.25 |
| 6Mo | 14.75 | 20.81 | 26.38 | 31.74 | 35.37 | 38.63 | 40.71 | 41.84 | 43.01 | 43.99 | 44.51 | 44.90 |
| 9Mo | 18.37 | 24.96 | 30.00 | 34.98 | 38.46 | 41.59 | 43.86 | 44.82 | 45.79 | 46.63 | 47.07 | 47.48 |
| 1Yr | 22.40 | 29.34 | 34.17 | 38.25 | 41.26 | 44.01 | 46.31 | 47.15 | 47.97 | 48.68 | 49.07 | 49.39 |
| 2Yr | 37.99 | 42.76 | 46.29 | 48.58 | 49.53 | 51.14 | 52.78 | 53.08 | 53.21 | 53.52 | 53.41 | 53.29 |
| 3Yr | 50.12 | 52.65 | 53.99 | 55.17 | 55.53 | 56.28 | 57.18 | 56.98 | 56.22 | 56.18 | 55.60 | 55.35 |
| 4Yr | 57.30 | 58.56 | 58.98 | 59.35 | 59.27 | 59.80 | 60.07 | 59.31 | 57.96 | 57.42 | 56.43 | 56.04 |
| 5Yr | 61.34 | 61.93 | 62.02 | 62.17 | 61.98 | 62.30 | 62.13 | 60.91 | 59.14 | 58.14 | 56.82 | 56.17 |
| 6Yr | 63.21 | 63.79 | 63.69 | 63.63 | 63.24 | 63.24 | 62.98 | 61.53 | 59.30 | 57.89 | 56.46 | 55.77 |
| 7Yr | 64.41 | 64.95 | 64.79 | 64.72 | 64.26 | 64.13 | 63.56 | 61.99 | 59.49 | 57.76 | 56.15 | 55.36 |
| 8Yr | 64.88 | 65.46 | 65.24 | 64.98 | 64.91 | 64.37 | 63.66 | 62.03 | 59.34 | 57.38 | 55.68 | 54.73 |
| 9Yr | 64.89 | 65.62 | 65.37 | 64.99 | 64.68 | 64.38 | 63.61 | 61.86 | 59.17 | 57.05 | 55.26 | 54.21 |
| 10Yr | 64.75 | 65.46 | 65.20 | 64.81 | 64.60 | 64.18 | 63.49 | 61.62 | 58.91 | 56.54 | 54.64 | 53.58 |
| 12Yr | 63.62 | 64.43 | 64.33 | 63.71 | 63.34 | 63.01 | 62.17 | 60.37 | 57.63 | 55.04 | 53.09 | 51.90 |
| 15Yr | 61.33 | 62.05 | 61.77 | 61.58 | 61.45 | 60.84 | 60.09 | 58.26 | 55.59 | 52.70 | 50.70 | 49.30 |

Table 2.1: Swaption matrix obtained from Bloomberg on the 23rd of January 2018. Index tenor: 3 months Euribor. Values expressed in basis points for volatilities of ATM swaption, assuming a Normal distribution of the underlying.

but with same length for the underlying swaps. In a row, we read the quotes for swaptions with a fixed maturity but with an increasing length of the underlying swap. The payments for the fixed and floating lengths are supposed to happen with same frequency, every semester. In Figure 2.2 we can see the volatility surface obtained from the data. If $\sigma^{\mathrm{L}}_{m,n}$ was chosen, Table 2.1 would show some empty entries in the top left-corner.

# 3

# Pricing model

This chapter is dedicated to the pricing of a portfolio of mortgages. The aim is establishing a clear link between the value of the portfolio and the level of the interest rates in the market, leaving aside other possible drivers for the prepayments. This assumption lacks in precision of the forecast of the CPR, but improves the evaluation of the interest rate risk to which a bank is exposed, clearing the way to set up a more accurate hedging strategy. In fact, the other prepayment drivers explained in Chapter 1 can not be hedged in a standard way, because the reason of their existence can not be linked to any instrument in the market. However, their effect could be hedged buying some derivatives that help to mitigate their impact on the notional, but this would result in a hedging strategy in which the non-linear instruments are exercised irrationally from a financial perspective. We consider this as a possible extension of the model, but it is out of the scope of this thesis.

The central idea behind the pricing model is replicating the mortgage portfolio with an Index Amortizing Swap (IAS), whose notional depends on the characteristics of the contract, on the prepayment rate and, of course, on the level of the interest rates in the market. Even if our model will consider only the Refinancing Incentive as main driver of the prepayments, this will not lead to a full-rational model, because a smoother functional form for $\Lambda$ will be preferred to a step-function, which is typical of the optimal prepayment models. The analysis of more than thirty millions of rows of prepayment data will lead to the historical calibration of the Refinancing Incentive, providing the correct bridge between the risk-neutral world of the interest rates and the prepayment rate, for which a risk-neutral evaluation does not exist. This framework makes the extension to a stochastic environment natural, because the paths of the short-rate can be linked directly to the paths of the notional of the IAS. This approach will have a great impact on the hedging strategy, since the possibility of evaluating the whole distribution of the notional unveils the necessity to include non-linear instruments.

The chapter is organized as follows. First, the prepayment literature is reviewed. Secondly, the reasons behind the choice of the IAS will be justified, and all the dependencies that its notional has will be explained and motivated according to the literature. Then, examples of the outcomes of the pricing procedure will be provided for two important examples, showing the differences between a full-rational model and a more realistic one. Finally, the results will be discussed.

## 3.1. Prepayment literature

The greatest challenge in all prepayment models, that also makes the problem interesting, is the impact of mortgagors' behaviour. People not always act in a rational way, meaning that it often happens that even if there is a strong incentive in prepaying people do not prepay, or they prepay when it would not be suggested to do so. This fact influences considerably the type of model that could be chosen to forecast prepayments. There are two main branches of prepayment models: *optimal prepayment models* and *exogenous models*. Optimal prepayment models rely on the assumption that people behave purely in a rational way, and they will prepay when the value of their mortgage is greater than the outstanding debt (including penalties for refinancing). A great example of this typology could be found in [14], where an optimal or suboptimal prepayment rule is used to forecast the prepayment behaviour, and the mortgages are seen as callable bonds. Alternatively, in [26] a binomial tree is proposed to evaluate the cost of the prepayment option exactly or with some boundaries, depending on the type of mortgage. A similar approach, but with more implementation aspects, is given in [27]. In these cases, the evaluation is performed backward, comparing at each time the present value of the future cash flows of the mortgage with the outstanding debt. The problem with these models is that they only depend on the interest rate level in the market, leaving aside all the endogenous variables (for instance, client age, mortgage age, period of the year). On the other hand, exogenous models could be further divided in two categories: they could either be *based on endogenous models*, or be *strictly empirical*. The first ones have the same approach as optimal prepayment models, but they add prepayments that are not directly caused by the level of interest rates. There are many examples of this kind of models in a large time horizon, from [28] to [29]. A Poisson process is sometimes used to model the unrational behaviour, but this increases the price of the mortgage excessively, because in the reality pure irrational decisions rarely (maybe never) happen, they are attributable to the borrower's personal situation. Finally, strictly empirical models try to attribute the prepayments to a set of variables which are believed to be correlated to the prepayments. The authors in [1] analyse the pros and cons of the two main possibilities, which are survival analysis and logistic regression.

The following subsections are meant to retrace the main alternatives for the functional forms of the CPR and of the interest rate incentive. This will help making correct choices to link the prepayments to the interest rates' market behaviour.

### 3.1.1. FUNCTIONAL FORMS FOR THE CONDITIONAL PREPAYMENT RATE

Equation (1.1) defines the CPR as the ratio between the unscheduled prepayments and the scheduled notional at time $t$. However, this definition only helps to define the CPR and evaluate it for past data, in which the prepaid amount is known. When the focus is on the future, people's behaviour is unknown, therefore the future unscheduled notional payments are ignored. Thus, what it is required is finding a reasonable functional form that links the CPR to the main drivers of prepayments, so that, given the data about future interest rate level and mortgage and client characteristics, it becomes possible to estimate the future trend of the CPR. Note that the exogenous variables (like the future interest rate level or the housing market) need to be estimated or simulated, while the values of the endogenous variables are known a priori.

In the literature review the different typologies of model have been introduced, from the most empirical to the most theoretical. At this stage, the idea is to go through the literature again, analysing the most common functional forms for the CPR in order to identify the one that suits the best the intent of this thesis. Considering that one of the main purposes is to obtain a stochastic evaluation of a portfolio of mortgages, *strictly empirical* models are left aside, because (logistic) regressions do not leave that much of space for this kind of extension.

In the literature, a common functional form used to estimate the monthly prepayment is the model developed by [16] and firstly adopted by Goldman Sachs. In this framework, the CPR is estimated starting from the Single Monthly Mortality (SMM), a quantity already encountered in Definition 1.1,

$$\mathrm{SMM}(t) = \mathrm{RI}(t) \cdot \mathrm{SM}(t) \cdot \mathrm{MM}(t) \cdot \mathrm{BM}(t), \tag{3.1}$$

where

RI($t$) is the Refinancing Incentive, which takes into account the incentive given by a (possibly) reduced level of the interest rate market at time $t$.

SM($t$) is the Seasoning Multiplier, representing the age of the mortgage.

MM($t$) is the Month Multiplier, which considers the peak of prepayments happening during certain periods of the year.

BM($t$) is the Burnout Multiplier, used to model the empirical phenomenon of burnout. Essentially, people who received a strong incentive in the past, and they did not refinance, are less likely to refinance.

Examples of models which used this form are [14], [13] and [12]. However, in [12], the model of Equation (3.1) is only used to verify the results of a variation of the models itself, which is

$$\mathrm{SMM}(t) = \mathrm{RI}(t) \cdot \mathrm{BM}(t) + \mathrm{MM}(t) \cdot \mathrm{SM}(t).$$

Another category of models is named *intensity-based prepayment models*. Here, besides the (possibly stochastic) drivers of prepayment, the baseline hazard rate itself is a stochastic process, allowing randomness for the base of prepayment. In this framework, the hazard rate is none other than the probability of prepayment in the next interval of time, and the baseline hazard rate is the amount of prepayment that is assumed to always happen, and that it can be altered by the effects of all the considered drivers. An example of this approach is given in [6]

$$p(t) = e^{f(\mathbf{x}(t),\beta)+p_0(t)},$$

where the continuously compounded annual prepayment speed $p(t)$ can be converted to a (discrete) prepayment rate using

$$\mathrm{CPR}(t) = e^{p(t)} - 1.$$

In this model $p_0(t)$, the baseline hazard rate, is a stochastic quantity that is assumed to follow a mean reverting process, while $\mathbf{x}(t)$ and $\beta$ are the vectors of the considered significant variables with the respective weights. Moreover, a correlation with the GDP growth is added, since strong empirical evidence was found. Alternatively, the authors in [30] proposed a CIR process for the prepayment rate, flanked by another process modeling the possibility of default of the borrower.

In the framework of the hazard-rate based model, [31] proposes an estimation of the CPR via the following formulation

$$\mathrm{CPR}(t) = \Phi(\alpha \cdot f(\mathbf{x}(\mathbf{t}), \beta)),$$

where, again, $\mathbf{x}(t)$ and $\beta$ are the vectors of the considered significant variables with the respective weights, $\alpha$ is another parameter vector and $\Phi$ indicates the standard normal cumulative distribution function.

Finally, a more modern approach is provided by [19]. Here, the borrower's wealth is assumed to follow a process of the form

$$X(t) = X(0) + \mu t + \sigma\rho W^M(t) + \sigma\sqrt{1-\rho^2}\,W(t),$$

where $\mu$ and $\sigma$ are the drift and the volatility of the process, while $\rho$ is the correlation between the Brownian motion $W^M$ supposed to catch the market effects, and the Brownian motion $W$ representing idiosyncratic risk factors. Depending on how the borrower's wealth evolves in time, four different scenarios could be reached (from "best case" to "worst case", borrower's perspective): prepayment, normal evolution, refinancing and default.

### 3.1.2. FUNCTIONAL FORMS FOR THE REFINANCING INCENTIVE

The interest rate incentive is the reason to prepay that mortgagors receive from the market. To reach a correct definition,answering the following questions is needed:

1. What represents properly the rate offered by the market at time $t$?
2. How do people compare rationally this rate with the one they have?
3. How do they react to such incentive?

The answer to the first question is obtained giving the assumption that a reasonable benchmark for the price of a mortgage is a swap rate which matches the maturity and the frequency of payments of the mortgage, where a spread is added to cover liquidity risks. In fact, banks themselves derive the at-the-money mortgage rate for new clients starting from the current level of swap rates. Studies about which swap rate should be taken as ultimate determinant of mortgage rates have been performed, often leading to the choice of the 10 year LIBOR swap rate [32]. However, the choice of linking the mortgage rate to a swap rate with the same characteristics is more consistent and it avoids the further assumption of converging all the dependency on the 10 year swap. Thus, to finally give an answer, the initial mortgage rate is indicated as $K$ while the new mortgage rate that could be found in the market at time $t$ for a mortgage with maturity $T$ is

$$\kappa(t;T,\zeta) \approx S_{t,T}(t) + \zeta, \tag{3.2}$$

where $\zeta$ is a deterministic spread which covers liquidity risk and profit of the bank. We assume it to be constant over time and independent on the level of interest rates. We also point out that the spread $\zeta$ will only be useful for the pricing model, but all the evaluations regarding the hedging will assume $\zeta = 0$, since a bank is not willing to hedge completely the fixed coupon received by the mortgagors, but only the amount that corresponds to the funding cost.

The second question raises the problem of choosing the functional form for the incentive. The question could be also interpreted as: given the mortgage rate $K$ and a refinancing rate $\kappa$, how should they be combined to reflect the incentive? As it is understandable, the literature is thriving about this topic as well. Mainly, there are two possibilities. The first one is the difference between the rates

$$\epsilon(t) = K - \kappa(t). \tag{3.3}$$

In this case, the smaller the market rate, the greater the difference, with an at-the-money mortgage showing $\epsilon(t) = \epsilon^* = 0$. This has been the choice of [6, 8, 12]. An alternative could be using the ratio

$$\epsilon(t) = \frac{K}{\kappa(t)}. \tag{3.4}$$

This ratio implies that a par mortgage has $\epsilon(t) = \epsilon^* = 1$, and that a positive incentive is represented by $\epsilon > 1$, while a negative incentive (sometimes called disincentive) happens when $\epsilon < 1$. The reasons behind a possible choice of the ratio come from the evaluation of the present value of an annuity per Euro of monthly payment, as defined by Equation (1.6). The ratio $\frac{\text{An}(t;\kappa(t))}{\text{An}(t_0;K)}$ provides a decent

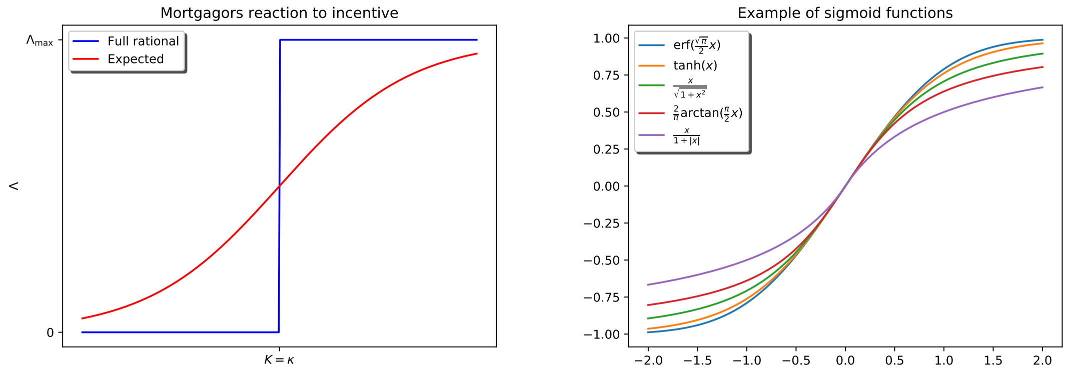



Figure 3.1: Left: real versus expected reaction of the people to the Refinancing Incentive. Right: some examples of sigmoid functions, rescaled to have steepness one in the origin.

benchmark for the Refinancing Incentive, but this ratio is well approximated by Equation (3.4), that is why the latter is usually chosen, since it has a more immediate meaning. The ratio incentive has been introduced by [16] and it is used, among the others, by [13, 14]. Moreover, in the literature, also the net present value gained by refinancing sometimes found its use, for instance in [7, 31].

Finally, the answer to the third question. At first sight, modeling how borrowers react to incentives could appear problematic, and this is one of the crucial steps of the whole thesis. People do not act in a rational way, meaning that they do not prepay whenever there is the incentive do to so, and they sometimes prepay when it would not be suggested. This is usually one of the main drawbacks of full-rational models, where it is assumed that the exercise of the prepayment option is always rational. If this were true, the “reaction function” would be represented by the blue line of the left graph of Figure 3.1, where the maximum level of prepayments is reached instantly whenever $\epsilon(t) > \epsilon^*$. Nevertheless, an “S-shape” graph like the one sketched by the red line is well-known in the literature, and it seems more realistic than a step function which models a full rational exercise. The S-shape is actually a feature which defines a whole class of functions, called “sigmoid” functions, and some examples are plotted in the right graph. This smoother function mitigates the rationality and allows for non-rational behaviour, possibly including a reaction time to the incentive. To capture this non-linear behaviour, the arctangent function has been widely used [6, 12, 13, 18]

$$\mathrm{RI}(t) = \alpha_1 + \alpha_2 \arctan\left(\alpha_3 \epsilon(t) + \alpha_4\right).$$

Alternatively, the authors [12, 31] used the Cumulative Distribution Function of a Normal with shift and scale parameters

$$\mathrm{RI}(t) = \Phi\left(\beta_2 \cdot \left(\epsilon(t) - \beta_1\right)\right).$$

## 3.2. PRICING A MORTGAGE PORTFOLIO

The goal of this section is setting up a method to evaluate a portfolio of mortgages which naturally takes into account the prepayment option embedded in such contracts. The solution adopted consists in pricing the portfolio using instruments that replicate *perfectly* the value of the mortgage portfolio, therefore we will introduce a variation of an Index Amortizing Swap (IAS) and specific reasons behind this choice will be given. Once the IAS is established, the literature about prepayments modeling will be retraced briefly in order to create the correct link between the (stochastic) amortizing plan of the swap and the probability of prepayments. Finally, using the mathematics developed in the previous chapter, a numerical procedure to obtain a price of this highly non-linear instrument will be set, also counterchecked by an analytical price in the special case of a deterministic prepayment rate.

### 3.2.1. THE INDEX AMORTIZING SWAP

An Index Amortizing Swap (IAS) is an over-the-counter interest rate swap which combines the characteristics of a plain vanilla interest rate swap and, partially, of a swaption. Amortizing swaps with a deterministic scheme of amortization are commonly traded instruments, and they are obtained by means of linear combinations of payer and receiver swaps, as we will show in the hedging section. However, the peculiarity of the IAS which makes it "hybrid" is that its amortization scheme is predetermined only as a function of a specific interest rate. This feature creates uncertainty about the price of the IAS, since the amount of fixed-for-floating rate exchanged in the future will depend on the current level of the selected interest rate. Therefore, the IAS effectively gives an option on that rate. Typically, the amortization function has a lower and upper bound which avoids extreme scenarios, and it could also be linked to some seasonal effects.

As it may be already noticed, the dependency of the IAS on the interest rate level is the key point in the whole prepayment replication. In fact, also in reality, an IAS is already used to mitigate the risk of changes in the interest rates, with a greater (smaller) amortization linked to a drop (raise) of the interest rates level. As it is mentioned, the amortization of an IAS could be also connected to seasonal effects, which, in principle, would improve the replication of prepayments. Nevertheless, these risks cannot be hedged, since there are no traded instruments that manifest a similar response to prepayment drivers like specific months of the year or specific ages of the mortgage. Their impact could be hedged trying to replicate the effect with some tradeable instruments, however, this would lead to an exercise that is not rational from a financial perspective, and therefore we decide to not consider it. The reaction to interest rate movements of the IAS is exactly the one expected from prepayments because, as it has been explained in the previous chapters, a positive incentive of prepaying arises from a decrease of interest rates. A typical amortization plan $a(t; l(t))$ which depends on a reference rate $l(t)$ is [33]:

$$a(t; l(t)) \begin{cases} 0 & l(t) \geq l_0 \\ a_1 & l_0 \geq l(t) > l_1 \\ a_2 & l_1 \geq l(t) > l_2 \\ a_3 & l_2 \geq l(t) > l_3 \\ 1 & l_3 \geq l(t). \end{cases} \tag{3.5}$$

It is worthy to notice that it is possible to augment the number of levels arbitrarily, spanning from a step-function (when there is only one level) to a quasi-linear amortization schedule. More importantly, there are clear analogies with Figure 3.1, where the functions supposed to capture mortgagor's reaction to the incentive are nothing more than particular cases of the possible amortization plans of an IAS represented by (3.5).

At this point, considered that an IAS and a mortgage portfolio share the same embedded optionality, it should be understandable why a taylor-made construction of the former provides an optimal representation of the latter. In fact, particular choices of reference rate and amortization plan can fully replicate the observed decaying trend of the notional of a portfolio of mortgages[1]. The final step before proceeding to the pricing of this instrument consists in giving insight in how a

[1] Since the focus of the thesis is on the prepayment risk, we are only considering the prepayment option in this thesis, and the full model is based on this assumption. Other options which bring default risk or pipeline risk could be modeled similarly and should be included in a comprehensive evaluation of a mortgage portfolio.

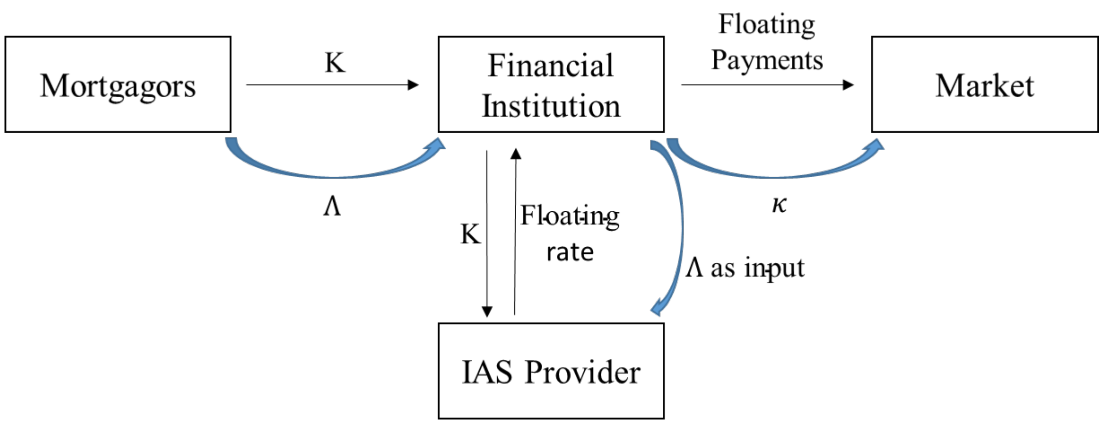



Figure 3.2: Schematic representation of how a bank would use an IAS to hedge the prepayment risk, using the prepayments as input to exchange the correct amount of fixed rate for a floating one.

bank would use this derivative to hedge the prepayment risk. Remember that, at this stage, there is only the intent of motivating why the IAS is the perfect candidate on which to build the evaluation procedure of a mortgage portfolio and, more importantly, of the hedging strategy of the next chapter.

From a pure interest-rate risk perspective, only fixed rate loans contain prepayment risk, because loans with a variable rate pay a coupon that is adjusted to the market rates. This way, the mismatch between the pre-agreed mortgage rate and the at-the-market rate is zero, because effectively $K \equiv \kappa(t)$ for every time $t$. Therefore, the fixed rate loans are usually hedged to reduce the pure interest rate risk, and this is achieved by coupling mortgages with interest rate swaps [9]. However, the swaps are not contractually linked to the mortgage loan, thus, prepayments cause misalignments of the cash flow of the hedge. In Figure 3.2 the mechanism is represented through the black lines. Prepayments are indicated through the blue lines, and the idea behind an IAS is to use prepayments as input, which corrects the mismatch between the plain-vanilla swaps and the mortgage portfolio, and makes it possible to exchange the exact quantity of the fixed rate for a floating one. Once the floating rate is received, it is reused to either fulfill some floating rate commitments of the bank, or fund the new production of mortgages with an at-the-money market rate $\kappa$.

The are few papers which consider the link between an Index Amortizing Swaps and portfolios of mortgages, for example [34–36] or the textbook [33], so the idea itself of using an IAS to represent the behaviour of a mortgage portfolio does not consists a novelty. However, the specific construction that we will provide in the following section represents innovation. At this time, there is no trace of any paper which clearly establishes a link from the observed market rates to the amortizing scheme of the IAS, taking into account historical behaviour of people as well as the different repayments schemes that distinguishes mortgages in bullets or annuities. Even more important, there is no specific information in the literature about the performance of different hedging strategies that attempt to replicate the non-linear payoff caused by prepayments.

### 3.2.2. PRICING MODEL CONSTRUCTION

This subsection explains the precise construction of the Index Amortizing Swap which replicates the mortgage portfolio. The central challenge is modeling the notional of the IAS, which embodies the (possibly stochastic) amortization via a complex function of the typology of the mortgage, the history of prepayments, the prepayments drivers and the prepayments functional form. All the dependencies are stated and motivated according to the literature.

The starting point for the evaluation of an IAS is the following expectation

$$V_{\mathrm{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M} \tau_i \frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)} \cdot \left(K - L(T_{i-1};T_{i-1},T_i)\right)\middle|\mathcal{F}(t_0)\right], \tag{3.6}$$

where the notation is the same as introduced in the chapter dedicated to the mathematical framework. At this stage, the assumption is that the payment dates $T_1,...,T_M$ of the IAS are the same as of the mortgage portfolio, and for simplicity we assume yearly payments so that $\tau_i \equiv 1 \forall i$. Moreover, the notional of the IAS at time $T_i$ is the notional of the mortgage at time $T_{i-1}$, so we could write $N_{\mathrm{IAS}}(T_i) = N_{\mathrm{Mortgage}}(T_{i-1})$, but we preferred to simplify the notation writing $N(T_{i-1})$, supposing that the notional specified in Equation (3.6) is directly the one of the mortgage. Note also that the prepayment rate $\Lambda(T_i)$ is written directly as one of the dependencies of the notional. Considering what has been stated so far about the notional of the IAS, it is worhty to point out that the first payment is deterministic because $L(t_0;t_0,T_1)$ is observed in the market, moreover, at time $t_0$ no prepayments are possible and, by definition, $N(t_0) = N_0$. Finally, $K$ is exactly the mortgage rate that at time $t_0$ was at-the-money. The intention now is to dissect all the dependencies that the notional has in order to choose step-by-step the optimal choices that can help to replicate the behaviour of the notional of a mortgage portfolio. Choices are motivated with respect to the literature analysis and the mathematics developed in the previous chapter.

The first level of dependency is given by the typology of mortgage that the portfolio consists of. In Chapter 1 the different kinds of mortgages have been introduced, and the focus has been pointed on annuities and bullets. These two contracts present different amortization plans, therefore the impact of prepayments varies depending on how the contractual repayments are. We already found a closed form expression to evaluate how a constant prepayment rate $\Lambda \equiv \Lambda_0$ affects the value of annuities and bullets with Equations (1.4) and (1.12). In the general case, where $\Lambda \equiv \Lambda(T_i)$, it is necessary to explicitly state the dependency on the repayment scheme at each time. So,

$$N(T_i) = N(T_{i-1}) \cdot \Psi(\Lambda(T_i)), \tag{3.7}$$

where the function $\Psi(\cdot)$ is a discrete version of the formulas derived in Chapter 1 such that

$$\Psi = \begin{cases} 1 - \Lambda(T_i) & \text{Bullet,} \\ 1 + \frac{K(\Lambda(T_i)-1)}{1-(1+K)^{-(T_M-T_{i-1})}} + K - \Lambda(T_i)\cdot(K+1) & \text{Annuity.} \end{cases} \tag{3.8}$$

Considering that $i = 1,...,M$ and that $N(T_0) = N_0$, Equation (3.7) is well defined.

The second level of dependence involves the modeling of the prepayment rate $\Lambda$. In the literature review many possibilities have been presented, all including one or more explanatory variables. The main feature required by our scope is the possibility to introduce stochasticity in the evaluation of the portfolio. This is essential because it gives the possibility of taking into account a whole cloud of paths of the notional, rather than only the simple average path represented by the case in which $\Lambda \equiv \Lambda_0 \forall t$. In practice, it is common for banks to adopt a constant forecast of $\Lambda$ for a whole year, which is empirically not correct. This incorrect expectation is reflected in the hedging strategy, which not only is not precise, but also avoids completely the modeling of the non-linear part of the prepayment risk. We will see that the assumption of a constant prepayment rate implies that the hedge is only made of swaps, and that this linear hedge performs poorly in more realistic scenarios. The literature is meagre about stochastic models for a notional of a mortgage portfolio. One of the few attempts of using a stochastic process to forecast the prepayment rate is [19], which provides a rigorous modeling; however, estimating drift and volatility of each borrower's wealth process for a portfolio of mortgages seems to have feasibility drawbacks and their method lacks applicability. On the other hand, strictly empirical models do not offer an easy way to distinguish the interest rate incentive from the other variables, making a stochastic extension difficult to implement. At this level, the assumption we make is that the prepayment rate is only determined by the Refinancing Incentive (RI),

$$\Lambda = \Lambda(\mathrm{RI}(T_i)).$$

This assumption finds its motivations in the fact that the interest rate incentive is universally considered as the main driver of prepayments, and it is also a quantity that can be linked to the market and estimated through a stochastic process. Other exogenous variables like the housing market could also be modelled through a stochastic process, although they are not attributable to traded instruments that can be used to hedge the changes. The same limitations apply to endogenous variables like the age of the mortgage or the age of the mortgagor. Moreover, even if using only one variable to approximate the expected prepayments may sound limiting, the scope of this thesis is not modeling the prepayment rate, but use it to price a mortgage portfolio and, more importantly, to model and test hedging strategies. In a real case, the prepayment rate would be probably estimated with more advanced techniques which include more variables and, also considering the recent developments, maybe the use of Artificial Intelligence. However, the added value of this thesis is that the prepayment rate has a direct relation with the interest rate level and this can help any financial institution to hedge the prepayment risk with respect to sudden market movements.

The analysis goes now into the third layer, where we are required to choose a functional form for the refinancing incentive. In Figure 3.1 the difference between a "full-rational" exercise and a more realistic model is shown. The former case corresponds to a step function, while the latter is modelled by a sigmoid function[2] which, with the characteristic s-shape, can take into account the reaction time of mortgagors as well as non rational behaviour. Choosing a smoother function for RI helps to capture the empirical behaviour of mortgagors, and it is our way to compromise with the fact that prepayments are guided only by one variable which, in theory, would lead to a full-rational model. These two functional forms will be both used during the hedging experiments to validate the numerical procedure (assuming full-rational prepayments) and to analyse real scenarios. For this reason, the Refinancing Incentive is written in the following double form

$$\mathrm{RI}(T_i) = \begin{cases} \Lambda_{\max} \mathbb{1}_{\{\epsilon(T_i) > \epsilon^*\}} & \text{Optimal,} \\ \alpha_1 + \alpha_2 \left(1 + e^{\alpha_3 \epsilon(T_i) + \alpha_4}\right)^{-1} & \text{Realistic.} \end{cases} \tag{3.9}$$

As it is noticeable, among all the sigmoid functions, the logistic functional has been chosen. There is not any rigorous reason to prefer it over other "s-shaped" functions, however logit functions are common in the prepayment literature, and this can make appear our assumption more consistent with the standards. In fact, this functional form is employed when prepayments are modelled via a logistic regression, as well as in special cases of survival analysis in which the hazard rate is a logit function. Thus, we consider that the logistic functional form can reach a broader audience and can be more welcomed over arctangent or hyperbolic tangent functions, which would fit the problem anyway. In any case, we postpone further considerations about this choice to the next subsection, where prepayments data are analysed and the historical calibration of (3.9) is performed.

The final step consists in choosing the form of $\epsilon(K, \kappa(t))$ which we recall being either the difference or the ratio between the mortgage rate $K$ and the market rate $\kappa(t)$. Our reasoning begins with Equation (1.6), which introduced a formula for the present value at time $t$ of an annuity An(t;K), assuming a discount rate equal to the mortgage rate $K$. Now, the option of prepaying can be seen

[2] https://en.wikipedia.org/wiki/Sigmoid_function.

as an American (Bermudan) call option on the annuity with an actual value $\mathrm{An}(t;\kappa(t))$ and with the strike price equal to the initial value $\mathrm{An}(t;K)$ plus refinancing costs [16]. If these penalties costs $\Xi$ were proportional[3] to the actual value $\mathrm{An}(t;\kappa(t))$, then the payoff of the option would be

$$\max\left[\mathrm{An}(t;\kappa(t)) - \mathrm{An}(t;K)(1+\Xi), 0\right],$$

and the option would be in-the-money if and only if $\frac{\mathrm{An}(t_0;K)}{\mathrm{An}(t;\kappa(t))}$ crosses some critical value. Since the last fraction is well approximated by $\frac{K}{\kappa(t)}$, this should imply that this ratio is more suited to model $\epsilon(K,\kappa(t))$ than the simple difference. Nevertheless, even if there could be reasons to choose the ratio, the only incentive available in the data we have at our disposition is the difference, therefore this is the final choice. Moreover, [1] tested different possibilities on real prepayment data of Dutch mortgages, and concluded that the difference returned better results. Thus, we are confident in using

$$\epsilon(T_i) = K - \kappa(T_i).$$

All the dependencies of the notional of the IAS have been investigated and motivated, so it is possible to give a final formulation for the evaluation of a Dutch mortgage portfolio. Equation (3.10) summarises all the arguments in compact form:

$$\begin{cases} V_{\mathrm{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M} \tau_i \frac{N(T_{i-1})}{M(T_i)}\left(K - L(T_{i-1};T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right] & \text{Index Amortizing Swap,} \\ N(T_i) = N(T_{i-1})\cdot\Psi(\Lambda(T_i)) & \text{Mortgage type (3.8),} \\ \Lambda(T_i) = \mathrm{RI}(T_i;\epsilon(T_i)) & \text{Prepayment driver (3.9),} \\ \epsilon(T_i) = K - \kappa(T_i) & \text{Market conditions.} \end{cases} \tag{3.10}$$

### 3.2.3. HISTORICAL CALIBRATION OF THE REFINANCING INCENTIVE

The last subsection concluded with Equation (3.10), which embodies the pricing model of the IAS. However, after the construction of the model, one important step is still missing, and this is the calibration of the Refinancing Incentive (3.9). This subsection is dedicated to motivate the use of a non-linear function that links prepayment to the interest rate incentive, as well as to the data analysis of a prepayment data set where the previous considerations find their verification.

One of the main drawbacks of any prepayment model is that prepayments need to be calibrated historically, because there is not any traded instrument which can give information about the risk-neutral price of the option of prepaying. Our approach to obtain a risk-neutral evaluation of a mortgage portfolio consists of two blocks. First of all, we calibrate the interest rate model prices and volatililites observed in the market. Secondly, the prepayments are linked to the market level of interest rates through a historical calibration of mortgagor's reaction to the interest rate incentive. Thus, the functional form of the Refinancing Incentive is the fundamental bridge that connects a risk-free simulation of the level of interest rates to the forecast of the prepayment rate. As it is understandable, this has an incredible importance not only for the mere pricing of the portfolio, but also for the hedging strategy adopted to protect the institution from the prepayment risk. In fact, also according to the literature, it is common practice to estimate the prepayment rate as a constant percentage over a time unit, which leads to a constant decaying of the notional of the portfolio, resulting in a situation in which plain vanilla swaps would provide a perfect hedge. Nevertheless, this result is artificial and misleads the conception of being hedged when, in reality, the non-linear risk

[3] This motivation has more theoretical value. In practice, even when the penalties are proportional to the actual value of the mortgage, they are capped to a certain amount.

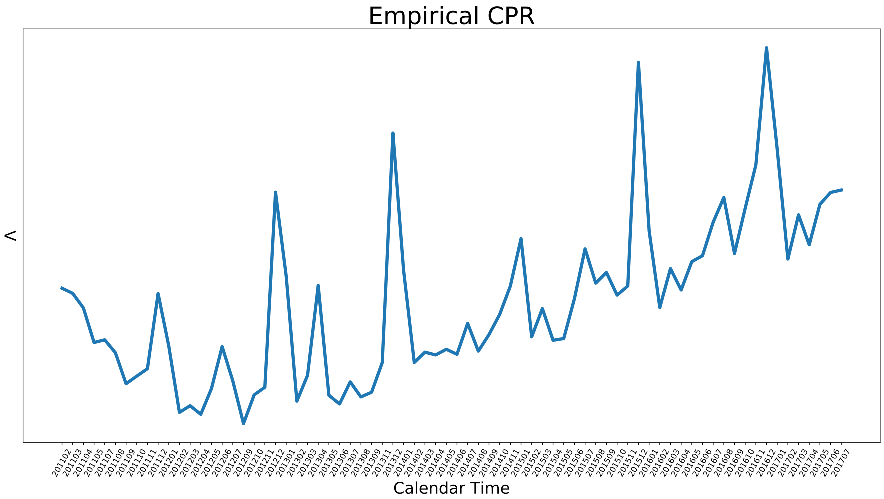



Figure 3.3: Empirical CPR obtained from more than 31 millions observations collected in a timespan of 74 months.

is not even modelled. It will be possible to appreciate this effect when the Greeks will be introduced, because the linear and non-linear risk are represented by the Delta and the Gamma that an instrument shows. For the moment, there will just be a qualitative description of the fact that a constant prepayment rate does not produce sufficient precision in the appraisal of the risks included in the mortgage portfolio.

### DATA ANALYSIS

We decided to calibrate the Refinancing Incentive historically, therefore prepayments data are required. For this purpose, we use the data collected over the past years and that are currently used to calibrate a dynamic prepayment model. However, often a constant conditional prepayment rate (constant CPR) is used to measure and manage the interest rate risk in the banking book related to prepayments and, when a bank has a large quantity of mortgages and loans, the assumptions with regard to prepayments have important implications for interest rate risk measurement and steering. Therefore, it is important to monitor changes in the prepayment behavior over time that is ignored by the static CPR model and so is having a hedging strategy that can take into account sudden market movements. The focus is on the fixed rate personal mortgages for the moment, as that is by far the largest segment, presenting a notional of more than 150 billions Euro. Other segments, like rollover/variable mortgages are available in the dataset, but are excluded from the analysis since they represent different asset classes and hence require most likely different risk drivers than fixed rate personal retail mortgages. The mortgage savings account (liability) is also out-of-scope, as it was anticipated in Chapter 1.

The dataset in our possession accounts for monthly observations of an average of more than 400 000 mortgages, over a period of 74 months from February 2011 to July 2017, resulting in a total of 31 752 378 observations (some months are missing). Because of the great dimension of the dataset, the analysis has been performed using Python[4] as programming language, since it supports an open-source library[5] for data analysis. More than 20 explanatory variables are available which cover various client, contract and mortgage rate characteristics, without any major variable

[4] https://www.python.org/.
[5] https://pandas.pydata.org/.

missing according to the literature. Among all the accessible variables, the ones to extrapolate from each row are:

- "StartingBalance" is the notional of the mortgage at the end of the month, which does not take into account the prepayment but only the contractual repayments.

- "PrepaidAmount" indicates the magnitude of the prepayment. In theory, there would be four different variables, linking the prepayment to its cause ("EarlyRedemptionVoluntary", "EarlyRedemptionMovement", "EarlyRepricingVoluntary", "EarlyRepricingMovement"). The reason is that, when mortgagors move to another house, they have the option to prepay fully without any penalty applied, while usually the amount that is possible to prepay without extra costs is capped at 20% a year. So, the idea is that voluntary prepayments might display different behaviour, which could be useful in an in-depth data analysis. However, four our analysis, this distinction is not necessary, so we collapsed the four causes in one variable.

- "Period" records the month and the year of the observation.

- "InterestRateIncentive". In the dataset there are four possibilities for the interest rate incentive, depending on which rate is used to create the incentive. Imagine a mortgage of ten years, observed after three years from its initiation. Then, one can argue that the incentive is better represented by a market rate for a ten years mortgage, or for a seven years mortgage. Consistently with the pricing model developed in this thesis, the second alternative is selected.

Once the significant variable are defined, using the definition of prepayment rate (1.1) we extrapolate the empirical CPR observed in the data set evaluating

$$\begin{aligned} \text{SMM}(\text{Period}_i) &= \frac{\sum_{j=1}^{N_i} \text{PrepaidAmount}_{i,j}}{\sum_{j=1}^{N_i} \text{StartingBalance}_{i,j}}, \\ \text{CPR}(\text{Period}_i) &= 1 - (1 - \text{SMM}(\text{Period}_i))^{12}, \end{aligned} \tag{3.11}$$

where $N_i$ is the number of mortgages observed during the "Period" (month) $i$. Once again, it is important to underline the fact that the CPR (indicated as $\Lambda$) is an annual measure, that is why the monthly observations are annualised. Figure 3.3 shows the empirical prepayment rate for each of the 74 months analysed, obtained using Equation (3.11). The first feature that has to be noticed is that $\Lambda$ is everything but constant, stressing out once again that assuming a constant prepayment rate is a poor assumption. Secondly, the other characteristic that should be noticed is the fact that $\Lambda$ increases over time on average. This effect can be attributed to the low (or even negative) level of interest rates that is characterizing the market over the last years, producing high incentive of prepaying. Finally, the peaks that seem to appear every year form actually a well-known pattern. As it has been already mentioned when the causes of prepayment have been discussed, during certain months (January and June) mortgagors tend to prepay more thanks to a larger availability of money that salary bonuses offer.

The focus now moves to the Refinancing Incentive, with the goal of finding the correct pattern between prepayments and level of interest rates. The following analysis ignores the timing of prepayment, requiring the data to be clustered no longer according to the calendar month. Instead, they need to be batched in bundles of mortgages sharing all the same Refinancing Incentive. This distinction is essential, because the mortgages observed at a certain period $i$ do not share the same characteristics of maturity, rate, incentive and can even be different contracts (annuities or bullet). Thus, collecting the mortgages of a month in an "average" mortgage with an average rate, average

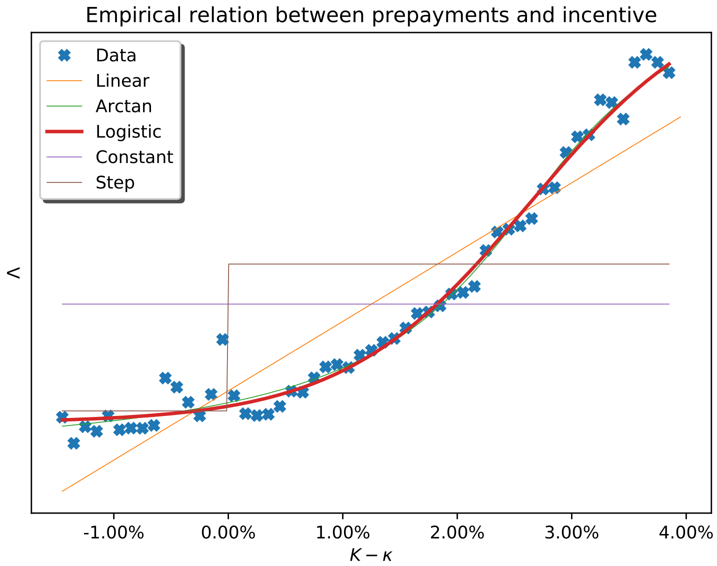



Figure 3.4: Real prepayment data show an empirical non-linear relation between the level of prepayment and the incentive $K-\kappa$, where $K$ indicates the old mortgage rate and $\kappa$ is the at-the-money market rate when the prepayment happened.

maturity and average incentive would result in an incredible loss of data. We decide to circumvent this problem performing a mortgage-wise analysis. Thus, we define

$$\text{SMM}_m = \frac{\text{PrepaidAmount}_m}{\text{StartingBalance}_m},$$

which represents the monthly prepayment rate for each observation $m$, with $m = 1, ..., 31752378$. Then, recalling that the mortgage rate is $K$ while the market rate is $\kappa$, we decide to concentrate on the range of incentive $K-\kappa = [-1.5\%; 4\%]$ because there are not many observations outside this interval and this would result in noise and low reliability of the data. The range is divided in 56 bins with equal size and, for each of them, we would like to extrapolate the corresponding prepayment rate. Thus, we group the millions of rows $\text{SMM}_m$ using the bins as criterium of separation. Basically, the prepayment rate $\text{SMM}_m$ ends up in the bin $b$ if the incentive read on the same row $m$ belongs the the interval covered by $b$. So far, the result is a quantity $\text{SMM}_{m,b}$ where the subscripts indicate the index of the observation and the bin to which it belongs. Finally,

$$\text{SMM}_b = \frac{1}{N_b}\sum_{m=1}^{N_b} \text{SMM}_{m,b},$$
$$\text{CPR}_b = 1-(1-\text{SMM}_b))^{12},$$

where $N_b$ is the number of observations that ended in the bins $b$. The results are shown in Figure 3.4, and they manifest a clear pattern. Different functions have been “calibrated” carrying out a least-squares regression with the intent of finding the one fitting the data the best. Both the step function and the constant function perform poorly, while the linear regression manages to follow at least the upward trend. However, the functions that clearly succeed to capture the trend of the prepayments are both the logistic and arctangent functions. They both belong to the class of s-shaped functions introduced in the previous subsection, and they confirm the expectation that there is a non-linear relation between prepayments and interest rate incentive. This outcome states once again that the assumption of constant prepayment rate is highly inadequate and can lead to a misinterpretation of the relation that prepayments have with the market movements. On the other hand, models assuming a full-rational behaviour of the mortgagors also lack precision, since the step function does not return reliable results. Between the two sigmoid functions plotted, none is visibly better, thus, as explained already, the choice goes in advantage of the logisitc function because it is more often found in the literature. Finally, a couple of remarks about the quality of the data collected. We already explained why only the range $K-\kappa = [-1.5\%; 4\%]$ is observed. However, what we have not specified is that the function selected will also provide information outside of this range, therefore assuming that the pattern will continue the s-shape. This is an assumption required by the lack of data available, and it could be proved wrong in the future. The second remark is about the outlier value observable in Figure 3.4 and corresponding to an incentive of 0%. This has a clear explanation, and it is due to the fact that, when prepayment data were collected, if the incentive was somehow not available, an incentive of 0% was added by default, resulting in a peak of observations for that value.

All the details of Equation (3.10) have been provided and motivated. At this point, we move to the pricing of the derivative.

### 3.2.4. IAS WITH DETERMINISTIC CPR

The initial point for the evaluation of the IAS is Equation (3.6). Even though all the dependencies of the notional have been specified in (3.10), we decide to still postpone the general case after the analysis of a simpler one. Thus, in this subsection, it is assumed that the prepayment rate is a deterministic function of time

$$\Lambda \equiv \Lambda(t).$$

This simplification implies that the system of Equation (3.10) reduces to only two "levels"

$$\begin{cases} V_{\text{AS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M} \tau_i \frac{N(T_{i-1})}{M(T_i)} (K - L(T_{i-1}; T_{i-1}, T_i)) \middle| \mathcal{F}(t_0)\right] & \text{Index Amortizing Swap,} \\ N(T_i) = N(T_{i-1}) \cdot \Psi(\Lambda(T_i)) & \text{Mortgage type (3.8),} \end{cases}$$

because there is no need to further define the prepayment rate. Note that the name of this derivative has been changed to Amortizing Swap (AS) since there is not any index according to which the amortization is determined. The difference between IAS and AS will become more relevant during the hedging analysis. In these circumstances, we are able to reach an analytical solution for the price of the AS, which is useful for two reasons. First of all, since the solution is obtained without the help of any interest-rate model, it provides an easy countercheck for the stochastic extension we are going to implement later in this thesis. This is possible because, when the short-rate will be simulated, if $\Lambda$ will still be considered deterministic and independent on the market, the price of the IAS provided by the simulations of bonds and Libor rate should still return the same price of the AS. Thus, we will have a tool to review the implementation. Furthermore, a deterministic case could be interpreted as a situation in which a forecast of the prepayment rate is performed separately, for example with an in-depth data analysis and/or with the use of Artificial Intelligence. Having the possibility to price analytically a case in which a function $\Lambda(t)$ is given is then beneficial also for scenario-analysis in which different frameworks are tested. However, this does not deliver information about the distribution of the notional of the IAS, since only one path is considered, that is why there will be the necessity to extend the pricing to a stochastic environment.

The procedure followed to price the IAS with a time-dependent notional is just a small variation of the one used to price a standard swap. In the previous chapter the forward Libor rate has been defined (4), while the useful toolkit of the change of numèraire has been explained in Section 2.2.1. The central result in the whole pricing of a swap is that the Libor rate is a martingale under the forward measure $\mathbb{Q}^T$, which is easily provable using its definition and the fact that the zero-coupon bond price $P(t,T)$ is the numèraire of the measure

$$\mathbb{E}^{T_i}\left[L(T_{i-1};T_{i-1},T_i)\middle|\mathcal{F}(t)\right] = \mathbb{E}^{T_i}\left[\frac{P(T_{i-1},T_{i-1})-P(T_{i-1},T_i)}{\tau_i P(T_{i-1},T_i)}\middle|\mathcal{F}(t)\right] = \frac{P(t,T_{i-1})-P(t,T_i)}{\tau_i P(t,T_i)} = L(t;T_{i-1},T_i).$$

Thus, the price of the AS with a time-dependent notional is

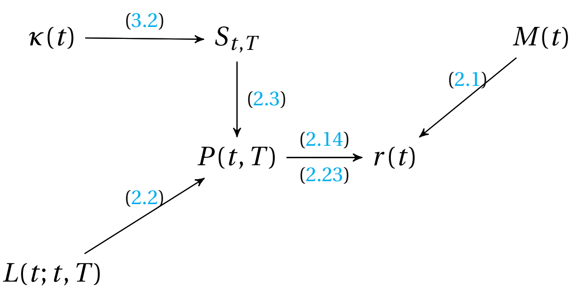
$$\begin{aligned} V_{\text{AS}}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M}\tau_i\frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)}\cdot\left(K-L(T_{i-1};T_{i-1},T_i)\right)\middle|\mathcal{F}(t_0)\right] \\ &= \sum_{i=1}^{M}\tau_i P(t_0,T_i)N(T_{i-1})\left(K-\mathbb{E}^{T_i}\left[L(T_{i-1};T_{i-1},T_i)\middle|\mathcal{F}(t_0)\right]\right) \\ &= \sum_{i=1}^{M}N(T_{i-1})\left[\tau_i P(t_0,T_i)K-\tau_i P(t_0,T_i)\frac{P(t_0,T_{i-1})-P(t_0,T_i)}{\tau_i P(t_0,T_i)}\right] \\ &= \sum_{i=1}^{M}N(T_{i-1})\left[P(t_0,T_i)(\tau_i K+1)-P(t_0,T_{i-1})\right]. \end{aligned} \tag{3.12}$$

### 3.2.5. IAS WITH STOCHASTIC CPR

"If prepayments are forecast under the unrealistic assumption that the refinancing rate is not random, then we cannot expect to obtain very realistic forecasts" [16].

The stochastic prepayment rate and the resulting IAS' stochastic notional are one of the core extensions that this thesis provides to the literature. A clear link from a market simulation to the impact of prepayments on a mortgage portfolio evaluation is something that is difficult to find in the literature, as well as specifics about the typology and quantity of the instruments that should be used to hedge the prepayment risk. Most of the articles cover specific aspects of the whole procedure discussed here, however the big picture is often confused. We conclude the pricing section setting up the algorithm that makes possible to simulate the pricing model (3.10).

The system is already structured in a way that it makes possible to start from $\kappa(T_i)$ and reach $N(T_i)$. Consequently, the quantities that need to be simulated are the discount factor $M(T_i)$, the Libor rate $L(T_{i-1};T_{i-1},T_i)$ and the market mortgage rate $\kappa(T_i)$. In the incipit of Chapter 2 we motivated the choice of simulating the short-rate $r(t)$ over using a Market Model. Among the short-rates model, Hull-White and CIR++ have been selected because of their capacity of recovering analytically the price of zero coupon bonds as well as of European options on bonds and swaptions. This advantages now become crucial, because, looking at the definitions of Libor rate, swap rate and money-savings account it is possible to establish the following connections




where the arrows indicate the quantity and the equations from which an element can be recovered. Two clarifications are required. First, the arrow going from the bond price to the short rate is combined with two formulas because it depends whether HW or CIR++ is considered. Second, the value of the money-savings account $M(T_i)$ results from numerical integration[6]. Thus, once the paths of the short rate $r(t)$ have been simulated, all the information necessary to price the IAS is already provided. Before concluding, a final remark about the simulation of the short rate. The timestep $\mathrm{d}t$ of the simulation has been set way smaller than the one of the payment dates, in order to get more precision at the points in time where the values of $r(t)$ are required. Therefore, even if we assumed yearly payments of the IAS for simplicity, $r(t)$ follows a much finer mesh, resulting in a better accuracy at the payment dates $T_1, ..., T_M$. All the steps required to obtain the price of the IAS have been condensed in Algorithm 2, which can give a better feeling about how to implement the whole procedure.

**Algorithm 2** Simulation of the Index Amortizing Swap

**procedure** IAS
  Calibrate the interest-rate model as described in Section 2.5.
  Initialize the short rate and simulate $N_{Sim}$ paths at times $t = \Delta t, ..., T - \Delta t, T$.
  Extrapolate the payment dates $T_1, ..., T_M$ from the more fine mesh of simulations of the short-rate.
  **for** $k \in \{1, ..., N_{Sim}\}$ **do**
    **for** $i \in \{1, ..., M\}$ **do**
      Obtain all the bond prices $P(T_i, T_j)$ with $j = i, i+1, ..., M$ from (2.14) or (2.23).
      Compute $L(T_{i-1}; T_{i-1}, T_i)$ with (2.2).
      Calculate the new rate $S_{T_i, T_M}$ using (2.3).
      Evaluate the difference $K - \kappa(T_i)$ .
      Estimate the prepayment via (3.9).
      Get the correct value of the notional $N(T_i)$ considering (3.7).
      Integrate numerically on the fine mesh to obtain $M(T_i)$.
    Acquire the price of the IAS for the simulation $k$ using (3.6)
  Average on the simulations to get $V_{\text{IAS}}$

[6]During all the numerical simulation, the trapezoidal rule has been selected as the technique used for approximating definite integrals. Thus, $\int_a^b f(x)\mathrm{d}x \approx \sum_{k=1}^{N} \frac{f(x_{k-1})+f(x_k)}{2}\Delta_k$, where $\{x_k\}$ is a partition of the interval $[a, b]$ such that $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$ and $\Delta_k = x_k - x_{k-1}$.

## 3.3. RESULTS

In this section we discuss the results returned by the pricing procedure presented previously and which we can explicitly find in Algorithm 2. All the simulations here are performed using the Hull-White model as interest rate model, since CIR++ will be used to test the robustness of the hedging algorithm in the next chapter. The calibration of the Hull-White model is a topic that is deeply related to the hedging chapter, mostly for the numerical approximation of the Greeks but also for the choice of the instruments on which to calibrate, so, we postpone all the details and discussions to the next chapter, where everything will find a natural place in the development of the hedging strategy. Therefore, for the moment we assume that the Hull-White model has been calibrated and the parameters are the following

$$\lambda = 0.264, \quad \eta = 0.017, \tag{3.13}$$

while the zero-coupon bond curve is obtained interpolating the discount factors observed in the market as it has already been shown in Figure 2.1. The section is organised as follow: first, the case with a deterministic prepayment rate is used to countercheck the implementation of the numerical procedure; then, the evaluation of the portfolio is extended to the case with a stochastic prepayment rate and the two refinancing incentive functional forms (step function or sigmoid) are employed to perform experiments on both bullets and annuities; finally, the results are discussed.

### 3.3.1. DETERMINISTIC PREPAYMENT RATE AND NUMERICAL COUNTERCHECK

The main goal of this subsection is showing that the price returned by the numerical procedure we implemented converges to the one returned by Equation (3.12), which is the formula to price the IAS in case the prepayment rate is assumed to be deterministic and known (so the IAS becomes an AS). This experiment will validate the numerical scheme and will give confidence about the reliability of the next results. Obtaining a perfect match of the prices, however, is not something easily achievable, because even when the implementation is executed properly, Monte Carlo simulations might not return perfect results. Specifically, the Libor rate under the forward measure $\mathbb{Q}^T$ is a martingale as proved in Equation (3.2.4). Nevertheless, this is not always the case with Monte Carlo simulations, and the two values (numerical and analytical) could not coincide. That is why techniques for the reduction of the variance have been developed, for instance the Antithetic Sampling or Moment Matching methods. This premise was necessary as some basis points of difference between the analytical results and the simulated one are accepted, and every financial institution has its own thresholds with which a model is classified as "validated" or not. A simple but still reasonable way to try to overcome the aforementioned problems is keeping the same seed for the random number generator and augment the number of paths in order to observe, hopefully, some sort of convergence of the numerical price to the expected one. The latter is exactly the approach we follow, and to evaluate the results the following quantity is defined

$$\delta_{N_{Sim}} = \left| V_{AS}(t_0) - \widetilde{V}_{AS}(t_0; N_{Sim}) \right|,$$

where $N_{Sim}$ refers to the number of paths, $V_{AS}(t_0)$ indicates the price resulting from (3.12) while $\widetilde{V}_{AS}(t_0; N_{Sim})$ comes from the numerical simulations. The mortgage rate has been chosen at-the-money, and since bullets and annuities have different plans of amortization this resulted in different rates for mortgages with the same maturities.

Different maturities, constant prepayment rates and mortgage types have been combined and the results are presented in Table 3.1. As expected, the error reduces when the number of paths increases, but obtaining an accuracy better than 1bps seems challenging. However, up to five basis points of mismatch is usually allowed, therefore we are confident that the numerical experiment can be extended to include the stochastic prepayment rate.

| Mortgage Type | $T_M$ (Years) | $K$ (bps) | $\Lambda$ (%) | Analytical Price (bps) | $\delta_{1000}$ (bps) | $\delta_{10000}$ (bps) | $\delta_{50000}$ (bps) |
|---|---|---|---|---|---|---|---|
| Bullet | 10 | 88.83 | 4 | 67.82 | 26.21 | 1.56 | 1.81 |
| Bullet | 30 | 151.18 | 12 | 43.49 | 11.02 | 2.77 | 2.53 |
| Annuity | 10 | 52.93 | 12 | 73.56 | 7.56 | 1.24 | 0.68 |
| Annuity | 30 | 131.25 | 4 | 18.80 | 10.06 | 3.86 | 2.92 |

Table 3.1: Numerical results of the simulations of an Amortizing Swap starting from the short-rate. Since it is possible to get an error smaller than five basis points the implementation is supposed to be validated and will be used for the extended case in which also $\Lambda$ is stochastic.

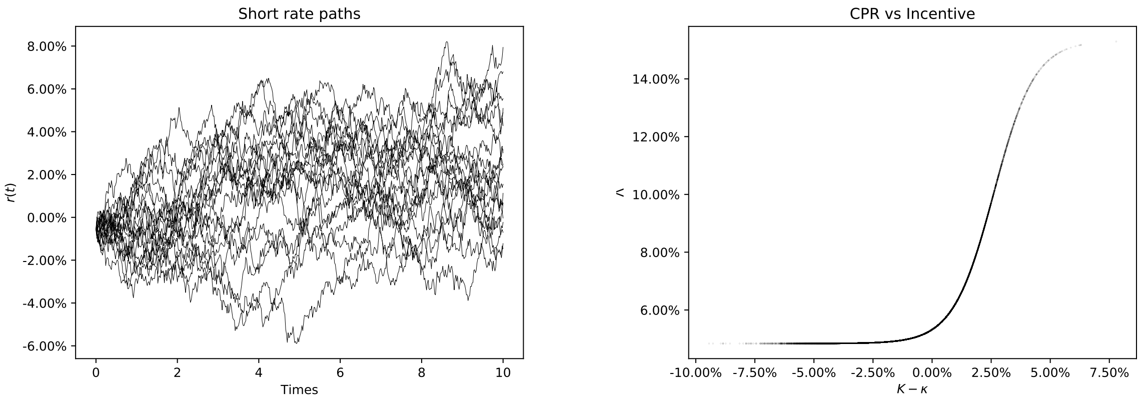



Figure 3.5: Left: examples oh paths of the short rate. Right: the sigmoid function which models the prepayment rate $\Lambda$.

### 3.3.2. STOCHASTIC PREPAYMENT RATE

The extension that allows $\Lambda$ to be a stochastic quantity is straightforward since all the steps required to incorporate the evolution of the short-rate $r(t)$ in the prepayments have been already discussed in Section 3.2.5. Thus, here the outcomes of the simulations are presented.

Figure 3.5 shows some paths of the short rate obtained using the parameters (3.13), and the Refinancing Incentive modelled by the logistic functional. Figure 3.6 illustrates various examples of the notional $N(t)$ assuming different contract types and alternating a Refinancing Incentive in the form of a step function or sigmoid. For completeness, the figure is matched with Table 3.2, which presents the details of the four cases considered.

| Mortgage Type | $T_M$ (Years) | $K$ (bps) | $\Lambda$ | Numerical Price (bps) |
|---|---|---|---|---|
| Bullet | 10 | 88.83 | Full-rational | −63.02 |
| Bullet | 10 | 88.83 | Sigmoid | 74.93 |
| Annuity | 10 | 52.93 | Full-rational | −24.09 |
| Annuity | 10 | 52.93 | Sigmoid | 36.28 |

Table 3.2: Values corresponding to the graphs of Figure 3.6.

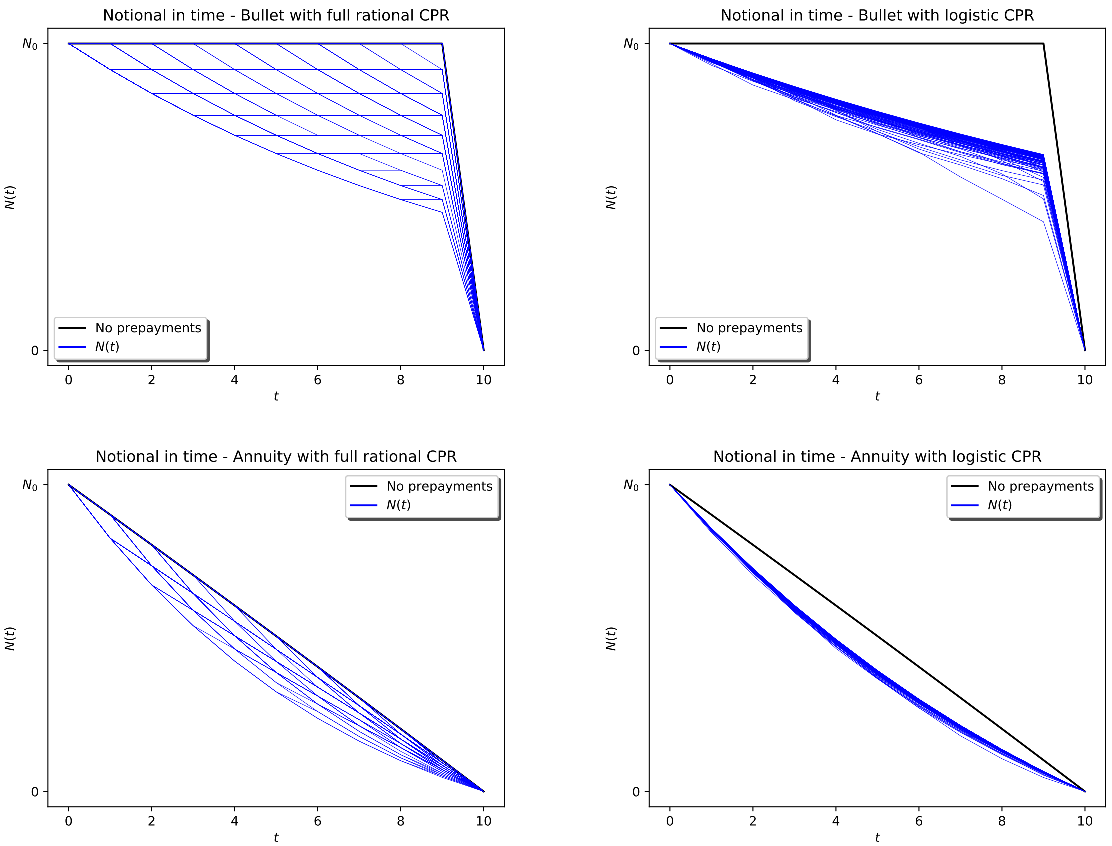



Figure 3.6: Examples of stochastic notionals for different contracts and different assumptions. First row: bullet. Second row: annuity. First column: full-rational prepayment rule. Second column: prepayments follow a logistic function calibrated on the historical data.

### 3.3.3. DISCUSSION

We implemented the pricing model of a mortgage portfolio. It has been possible to take into account the contract specifics and the evolution of a calibrated interest-rate model, so that a more realistic, and stochastic, amortization of the notional produced a more reliable evaluation of the price. Looking at Figure 3.6, many features appear and need to be discussed.

The first consideration concern the simulations with a full-rational prepayment rule, the one that assumes that a maximum level of prepayment $\Lambda_{max}$ is reached as soon as there is a positive incentive. In this case, the notional ended up being represented by a grid, which is consistent with the insight that, at each time, a mortgagor chooses to either continue with the scheduled plan of amortization, or prepay a fixed portion of the outstanding notional. However, since $\Lambda$ is by definition (1.1) proportional to the outstanding notional, the resulting grid has no fixed steps, because a bigger amortization is reached when $N(t)$ is larger. Thus, in the top and left regions of the grid we observe bigger jumps, while in the bottom and right sides the grid seems to become more fine. This effect implies that, regardless of the magnitude of $\Lambda_{max}$, a prior termination of the loan is not possible in this formulation. A way to include it would be to redefine the prepayment rate as a quantity proportional to the initial notional $N_0$, however the assumption that an early conclusion is impossible makes sense when the focus is on a whole portfolio of mortgages. At a mortgage level, prior terminations can be observed, but modelling a scenario in which a whole portfolio would prepaid fully sounds unrealistic. This assumption is strictly related to the definition of prepayment rate also because, already when it was introduced, we stressed that at a mortgage level there is no prepayment rate observable, because a single mortgagor usually prepays no more than once or twice during the whole lifetime of the contract, nevertheless, as also Figure 3.3 showed, at a portfolio level the prepayment rate is actually a clear effect.

The second analysis focuses on the experiments with the s-shaped Refinancing Incentive, so we are looking at the images of the second column of Figure 3.6. First of all, we should notice that at the top an artificial black path is added to indicate how $N(t)$ should have behaved in case no prepayment occurred. Then, it is also possible to see that the cloud of blue paths separates clearly the whole graphs in three regions. Below the black line and above $N(t)$ there is a white area which represents the prepayments supposed to always happen, regardless of the rationality of the action. This effect is attributable to the calibrated logistic function, which is always positive and thus implies a minimum amount of prepayments regardless of the market conditions. On the other hand, below the paths of the notional there is the territory of the prepayments that are supposed to never take place, despite any possible incentive. As it is understandable, this effect is caused by the fact that the s-shaped function also presents a maximum, presupposing that prepayments over a certain level are not reachable.

Table 3.2 shows the results of the simulations, expressing the price of the IAS in basis points, and there are two comments that need to be done. First, we notice that, in case the Refinancing Incentive is the step function, a negative price is returned. This is consistent with the idea of full-rational exercise, because if a mortgagor exercised the prepayment option only driven by the financial rationality, all the opportunities to save money would be caught, and the prepayments would certainly lead to a loss for the bank. However, in case the Refinancing Incentive allowed for non-rational exercise as well as for a slow reaction to the incentive, a positive price can be returned. This does not mean that hedging the prepayment risk is not necessary, because it is not assured that the non-rational prepayments are greater than the rational ones in every state of the world. The second effect that is worthy of note is that the annuity displayed less deviation in both experiments from the at-the-money price, which is zero. This is attributable to the fact that, as it will be explained in the next paragraphs, annuities have much less contractual freedom, therefore they have intrinsically less power to react to incentives. We postpone further considerations about the prices of both the IAS and of the prepaying option to Section 4.2.3, where the reasoning necessary to construct the hedging portfolio will shed light on both topics.

We start moving toward the hedging, therefore we discuss already two features of the graphs that we can observe now and that will be useful during the future reasoning. Consider again the case with the full-rational prepayment rule. The instruments that will be used to hedge the prepayment risk will be derivatives with an embedded optionality supposed to replicate the prepayment option of mortgages. Usually, these products are priced assuming an optimal exercise, therefore we expect that the case of the notional with the form of a grid is simpler to hedge. In fact, the same way we used the deterministic $\Lambda$ to countercheck the stochastic implementation, we will use the grid case to first test and validate the hedge procedure. Remember that, even if the grid does not seem stochastic, the actual distribution of $N(t)$ at each time is not known, because the number of times that a certain path is covered is not clear from the figure.

Finally, a last consideration about the difference between bullets and annuities. We already specified that the contractual difference resides in the fact that bullets do not suppose any repayments, while annuities do. However, only now it is possible to ascertain that this has consequences also in the way they include prepayments. In fact, it seems that the paths of an annuity are much closer to each other than the ones of a bullet. This could be caused by the fact that annuities already include contractual repayments, therefore there is less space for prepayments, while bullets leave more freedom because there are less obligations in the contract. Since there seems to be less optionality in an annuity, we expect that the non-linear hedge of the latter will be cheaper than the one of the bullet.

# 4

# HEDGING STRATEGIES

This chapter explains the hedging procedure used to hedge the prepayment risk, starting from the pricing model of the previous chapter. The type of hedge we aim to build is "static", which means that the risk is perfectly addressed to a portfolio of tradeable instruments that does not need to be recalibrated in the future. In reality, most of the times, reaching a perfect match of the characteristics of the derivative with the ones of the hedging portfolio is either very complicated or very expensive. Therefore, a subset of instrument is often selected and the position is dynamically adjusted in order to maintain the value of the hedge and of the derivative close enough. However, our analysis aims to a static replication which can provide more insight in the risks embedded in the dynamics of a mortgage portfolio. For this reason, a specific investigation of the non-linear risk produced by the prepayment option is provided and the Greeks are calculated. A common way to hedge the prepayment risk is using swaps, however this sometimes can lead to a poor approximation of the Greeks and of the price of the IAS under all the possible scenarios, therefore, throughout this chapter, the difference between a linear and non-linear hedging strategy will be particularly stressed. In the literature, finding specifics about the hedge of the non-linear risk is almost impossible, and at this time, to our knowledge there are no articles that price a mortgage portfolio and consistently hedge it. Thus, even if the choice of static hedging can be seen as a limitation, it actually helps creating a framework that can be seen as optimal starting point for a dynamic hedge. In the context of prepayments and mortgage portfolio evaluation, understanding the risks as well as choosing the hedging instruments, calibrating the portfolio and assessing the performances of different hedges are all operations that can extensively help any financial institution. Moreover, since the value of the derivative that has to be hedged is not the only quantity that banks aim to replicate, a calibration on the Greeks is also provided. Finally, the choice of a particular interest-rate model is questioned through the use of an alternative model with a different distribution. Essentially, the hedge calibrated under the Hull-White model will be tested on the paths of the CIR++, to see how much the distribution of the former impacts in a more realistic scenario in which the real distribution of the short-rate is not known.

## 4.1. LINEAR HEDGE

Linear hedge means that the payoff of the hedging instrument is linear. So, a movement in the price of the underlying asset will affect linearly the present value of the instrument itself. The most common example of a linear instrument is a plain-vanilla swap (or simply "swap"), which has been introduced with Definition 6. The reason behind the choice of swaps as linear hedge is that the IAS actually resembles swaps, as it has been explained in Section 3.2.1. However, the similarities with swaptions have also been pointed out and, by definition of linear instrument, no optionality can

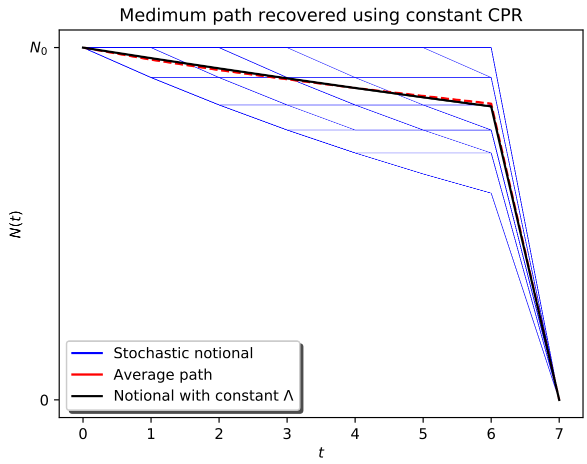



Figure 4.1: Example of medium path recovered using a constant prepayment rate. This result can be achieved regardless of the functional form of $\Lambda$. However, for this picture the full-rational one has been chosen just to let the graph be more clear and do not let the paths of $N(t)$ overlap.

be included, thus it is already possible to understand that swaps alone will not provide a good approximation of the risks included in the IAS. Nonetheless, this simple step will be useful to stress out more that the non-linear risk is important and should be considered. Moreover, an Amortizing Swap (AS) like the one constructed here will be also used in more complicated portfolios, therefore it is worth a subsection.

At this stage, the goal is executing the procedure used to price the IAS and then look for a portfolio of instruments that replicates the best way the paths of the notional $N(t)$. Since swaps do not involve any optionality, only one path is recoverable, therefore we decide to match the average one. Offsetting the average is straightforward since we can just use a combination of payer/receiver swaps to obtain the correct amortization schedule. In fact, let's use the following definition of a swap with a delayed start at time $T_l$ or/and anticipated end at time $T_m$

$$V_{\text{S}}(t_0; l, m) = \sum_{i=l+1}^{m} P(t_0, T_i)\tau_i \big(K - L(t_0; T_{i-1}, T_i)\big),$$

then, an AS is nothing more than a linear combination of co-terminal[1] standard swaps

$$V_{\text{AS}}(t_0) = \sum_{i=1}^{M} \Delta \bar{N}(T_{i-1}) V(t_0; i, M),$$

where we defined $\Delta \bar{N}(T_i) = \bar{N}(T_i) - \bar{N}(T_{i-1})$, and $\bar{N}(T_{-1}) = 0$. This should also clarify why swaps are linear instruments. Once the general construction of the AS is achieved, to create the correct amortization we define the average notional of the IAS at each time as

$$\bar{N}_{\text{IAS}}(T_i) = \frac{\sum_{j=1}^{N_{\text{Sim}}} N_{\text{IAS}}^{(j)}(T_i)}{N_{\text{Sim}}}.$$

Thus, the value of the linear hedge becomes

$$V_{\text{AS}}(t_0) = \sum_{i=1}^{M} \Delta \bar{N}_{\text{IAS}}(T_{i-1}) V(t_0; i, M). \tag{4.1}$$

[1] The decomposition in co-initial standard swaps is also possible.

As final remark, we notice that the average path of $N_{\text{IAS}}(t)$ can be even attributed to a constant prepayment rate. In fact, we are able to find a constant $\Lambda$ that recovers the average path of $N_{\text{IAS}}$, as Figure 4.1 reveals. However, we point out that this does not mean that the price of the IAS is the same as the price of a swap with deterministic amortization schedule resembling the average path of its stochastic notional. This happens because in general we cannot assume that the distribution of the notional is independent of the one of the Libor rate, meaning that

$$\mathbb{E}^{T_i}\left[N_{\text{IAS}}(T_{i-1})\big(K-L(T_{i-1};T_{i-1},T_i)\big)\Big|\mathcal{F}(t_0)\right]\neq\mathbb{E}^{T_i}\left[N_{\text{IAS}}(T_{i-1})\Big|\mathcal{F}(t_0)\right]\cdot\mathbb{E}^{T_i}\left[\big(K-L(T_{i-1};T_{i-1},T_i)\big)\Big|\mathcal{F}(t_0)\right].$$

## 4.2. NON-LINEAR HEDGE

This section is dedicated to analysing the non-linear risk generated by the prepayment option. Initially, the presence of non-linear risk in a mortgage contract will be proved obtaining an analytical price for a particular case of IAS. Then, the focus will move to the choice of the non-linear instruments and to the construction of a hedging portfolio. Then, an alternative calibration will be presented, in order to let the hedge replicate not only the value but also the Greeks of the IAS.

### 4.2.1. NON-LINEAR JUSTIFICATION

The existence of non-linear risk is an aspect that needs to be acknowledged in the pricing and hedging of a mortgage portfolio. The necessity of using non-linear instruments to achieve a better hedge will be shown here via two arguments.

Consider an annuity with initial notional $N(0)=N_0$, maturity $T_M=2$ and fixed rate $K$. Moreover, the "optimal" functional form for the Refinancing Incentive is assumed, as in (3.9). This toy model can help us reach a closed-form solution for the price of the IAS and therefore get insight in which instruments to select and which quantity to buy to hedge the prepayment risk. The price of such IAS is

$$V_{\text{IAS}}(t_0)=\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{2}\tau_i\frac{N(T_{i-1})}{M(T_i)}\left(K-L(T_{i-1};T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right].$$

Since the time horizon is limited, the only payment date at which prepaying is possible is $T_2$, therefore all the randomness is concentrated on the estimation of $N(T_1)$. In fact, as always, the first payment is deterministic and, changing measures, we find its present value

$$\begin{aligned}C_1&=M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{N(t_0)}{M(T_1)}\tau_1\big(K-L(t_0;t_0,T_1)\big)\Big|\mathcal{F}(t_0)\right]\\&=P(t_0,T_1)\mathbb{E}^{T_1}\left[N(t_0)\tau_1\big(K-L(t_0;t_0,T_1)\big)\Big|\mathcal{F}(t_0)\right]\\&=P(t_0,T_1)N(t_0)\big(K-L(t_0;t_0,T_1)\big).\end{aligned}$$

Now, since the prepayment rate is supposed to be a step-function, we can define the possible outcomes of the notional at time $T_1$ as $N^{\text{Up}}$ and $N^{\text{Low}}$, where the former embodies the normal amortization of the notional while the second includes prepayments. The left graph of Figure 4.2 should clarify the framework. Furthermore, in this special framework we also notice that

$$S_{T_1,T_2}(T_1)=L(T_1;T_1,T_2),$$

thus, the notional at time $T_1$ can be written as

$$\begin{aligned}N(T_1)&=N^{\text{Up}}\mathbb{1}_{\{K<L(T_1;T_1,T_2)\}}+N^{\text{Low}}\mathbb{1}_{\{K>L(T_1;T_1,T_2)\}}\\&=N^{\text{Up}}-\left(N^{\text{Up}}-N^{\text{Low}}\right)\mathbb{1}_{\{K>L(T_1;T_1,T_2)\}}.\end{aligned}$$

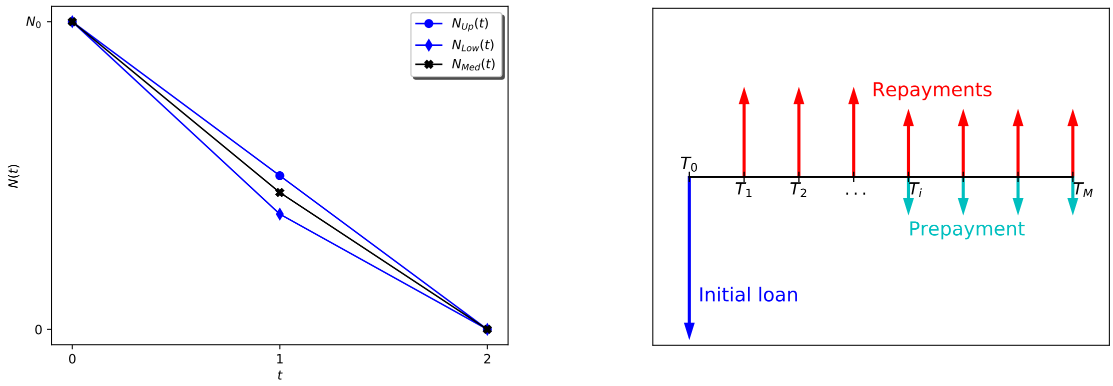



Figure 4.2: Left: the toy model which leads to an analytical price of the IAS. Right: Explanation of the effect of prepayments, which effectively reduce the notional of the mortgage giving the possibility of selling a swap (implying that the prepayment option is modelled via a short position in a swaption).

implying that the second payment $C_2$ equals

$$\begin{aligned}C_2 &= M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{N(T_1)}{M(T_2)}\tau_2\big(K-L(T_1;T_1,T_2)\big)\Big|\mathcal{F}(t_0)\right] = \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_2)}N^{\mathrm{Up}}\big(K-L(T_1;T_1,T_2)\big)\Big|\mathcal{F}(t_0)\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_2)}\left(N^{\mathrm{Up}}-N^{\mathrm{Low}}\right)\big(K-L(T_1;T_1,T_2)\big)^+\Big|\mathcal{F}(t_0)\right] \\ &= N^{\mathrm{Up}}\big((K+1)P(t_0,T_2)-P(t_0,T_1)\big) - \left(N^{\mathrm{Up}}-N^{\mathrm{Low}}\right)V_{\mathrm{Floorlet}}(t_0;T_1,T_2).\end{aligned}$$

Since we managed to write down all the payments explicitly, the final price of the instrument is

$$\begin{aligned}V_{\mathrm{IAS}}(t_0) &= C_1 + C_2 = \\ &= \sum_{i=1}^{2} N(T_i)\big((K+1)P(t_0,T_i)-P(t_0,T_{i-1})\big) - \left(N^{\mathrm{Up}}-N^{\mathrm{Low}}\right)V_{\mathrm{Floorlet}}(t_0;T_1,T_2) = \\ &= V_{\mathrm{AS}}(t_0) - \left(N^{\mathrm{Up}}-N^{\mathrm{Low}}\right)V_{\mathrm{Floorlet}}(t_0;T_1,T_2).\end{aligned} \tag{4.2}$$

The price we just derived[2] redefines the IAS as a combination of an Amortizing Swap and a Floorlet. This result is important because it starts distinguishing the linear component from the non-linear one, suggesting that a possible way to replicate an IAS is entering a long position in swaps and a short position in an option on the refinancing driver.

The second line of thought we present to model the non-linear risk is more qualitative. The main idea comes from the fact that mortgages are seen by financial institutions as swaps. This has already been justified pointing out that a bank raises funds for mortgages entering in swap positions, implying that the mortgage rate offered can be approximated by a swap rate plus a spread that covers liquidity costs and profits. Therefore, buying a mortgage can be seen as entering a long position in a payer swap, while the counterparty is long a receiver swap. Now, when a mortgagor prepays, what effectively happens is that the future installments of its mortgage decrease, and this action can be translated in a reduction of the notional of its swap for all the next payments. However, reducing the notional of a payer swap of an amount $C$ is equivalent to selling a payer swap whose notional is exactly $C$. This concept is illustrated in the right graph of Figure 4.2. On the other hand, from the bank perspective, being subjected to a diminution of the receiver swap corresponds

[2] All the formulas in this thesis, included this one, have been implemented and tested via Monte Carlo simulations.

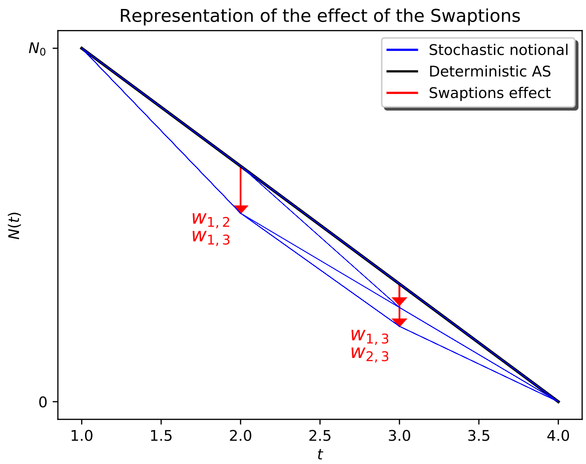



Figure 4.3: Explanation of the idea behind the calibration of the hedging portfolio. The black line represents an Amortizing Swap and it overlaps the upper-bound of the blue paths of the IAS. The red arrows indicates the notional $w_{i,j}$ of the swaptions starting in $T_i$ and ending in $T_j$, whose role is to reduce the notional of the hedging portfolio whenever a prepayment is triggered.

to being short another receiver swap. Therefore, since the prepayment is an optionality embedded in the swap representing the mortgage, giving the option of prepaying to a mortgagor is equal to give the possibility to enter an opposite contract. The right of entering a swap is nothing else than a swaption, thus, from the bank perspective, the option to prepay is translated in a short position in a receiver swaption. This conclusion makes sense also because, looking at Equation (4.2), the price for the special case introduced before was made up of a long position in a linear instrument and a short position in an option on the quantity determining the prepayment that, in that case, was the Libor rate. However, when the time horizon is extended, the market's mortgage rate is modelled with a swap rate, therefore we expect that the short position will be on an option on the swap rate itself. Since we managed to identify the instruments that should be used to hedge the IAS, we can move to the construction and calibration of the hedging portfolio.

### 4.2.2. PORTFOLIO CONSTRUCTION

Our goal is to replicate the IAS through a static hedge that addresses the different causes of risk, linear and non-linear, to specific tradeable instruments. Our reasoning led us to choose a portfolio $\Pi$ composed of a long position in receiver swaps and a short position in receiver swaptions. The combination of swaps creates an Amortizing Swap and the main idea behind the calibration of such portfolio is using the swaptions to reduce the notional of the AS at the same way prepayments reduce the notional of the IAS. One could argue that matching the notional $N_{\text{IAS}}$ of the IAS with the notional of the replicating portfolio $N_{\Pi}$ is not enough. However, an underlying assumption is that the swaps and the swaptions have the same frequency of payments of the IAS, moreover, the fixed and floating rates considered in the swaptions are the same as in the IAS. Therefore, the mismatch between the cash flow of the IAS and the one of the portfolio of swaps and swaptions is only caused by the differences in their notionals. This leads to claim that it is sufficient to calibrate the hedging portfolio choosing a combination of swaps and swaptions that recovers as precisely as possible the paths of $N_{\text{IAS}}$. The major consequence is that this static non-linear hedging aims to offset the behaviour of the IAS *path-wise*, while the linear hedge could only reproduce the average path of the notional. Before proceeding into the details, we point out that the paths of the IAS are required for the calibration of the portfolio, thus this will be some sort of "posterior calibration". The most important reference in this area is the article [36], where the authors reached a model-free calibration

| Swaptions | | | | |
|---|---|---|---|---|
| $w_{1,2}$ | $w_{1,3}$ | $w_{1,4}$ | $\cdots$ | $w_{1,M}$ |
| – | $w_{2,3}$ | $w_{2,4}$ | $\cdots$ | $w_{2,M}$ |
| $\vdots$ | – | $\ddots$ | | $\vdots$ |
| | $\vdots$ | | | |
| – | – | $\cdots$ | | $w_{M-1,M}$ |

Table 4.1: All the possible swaptions available to hedge the prepayment risk. The element $w_{i,j}$ of the table indicates the notional of the swaption whose value is $V_{\text{Swp}}(t_0; T_i, T_j)$.

of a portfolio of Bermudan swaptions to replicate a flexi-swap. However, the amortization of their IAS involves bounds on the amortization, and they reached their results assuming that the notional either continues without amortizaion or jumps from the upper bound to the lower bound. The impossibility of modelling any "half-redemption" makes their model difficult to apply to our framework, since we showed that historically people do not act in a full-rational way. A similar approach can be followed to calibrate our portfolio of European swaptions, as Appendix B illustrates. Other important references that regard this topic are [37, 38].

The swaps should be chosen in order to overlap the upper bound of the paths of the IAS, which means that they replicate the minimum amount of prepayments that is always supposed to happen. For the calibration of the swaptions, we first recall that the value of a swaption is indicated as $V_{\text{Swpt}}(t_0; T_m, T_n)$, meaning that a swaption needs two parameters to be uniquely identified. Then, we also remember that a mortgage has payment dates $T_1, ..., T_M$ and that the prepayments can occur at times $T_1, ..., T_{M-1}$, therefore the notional of the IAS is affected at times $T_2, ..., T_M$. As a result, all the swaptions of Table 4.1 are available, where $w_{i,j}$ indicates the notional of a swaption starting at time $T_i$ and ending at time $T_j$. Figure 4.3 summarizes all the concepts expressed so far.

Mathematically, we define the portfolio as

$$\Pi(t_0, \mathbf{w}) = \underbrace{V_{\text{AS}}(t_0, K)}_{\text{Long swaps}} - \underbrace{\sum_{i=1}^{M-1} \sum_{l=i+1}^{M} w_{i,l} V_{\text{Swp}}(t_0; T_i, T_l)}_{\text{Short receiver swaptions}}, \tag{4.3}$$

and the problem consists in finding a set of weights representing the notionals of the swaptions such that

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} F(\mathbf{w}), \tag{4.4}$$

where $F(\mathbf{w})$ is a function that measures the "distance" between $N_{\text{IAS}}$ and $N_\Pi$. Specifically, the value of $N_\Pi(T_k; \mathbf{w})$ under the simulation $j$ is

$$\begin{aligned} N_\Pi^{(j)}(T_k; \mathbf{w}) &= N_{\text{AS}}(T_k) - \sum_{i=1}^{k-1} \sum_{l=i+1}^{M} w_{i,l} \mathbb{1}^{(j)}_{\{K > \kappa(T_i)\}}, \\ N_{\text{AS}}(T_k) &= \max_j N_{\text{IAS}}^{(j)}(T_k), \end{aligned} \tag{4.5}$$

where the indicator function takes into account whether the prepayment is triggered or not. Notice that $N_{\text{AS}}(t)$ does not depend on $j$ because it is deterministic. Depending on the form of $F(\mathbf{w})$, different quantities are minimized and the optimal solution varies. A reasonable choice for $F(\mathbf{w})$ is

$$F(\mathbf{w}) = \sum_{\text{Time } t} \frac{1}{N_{Sim}} \sum_{j}^{N_{Sim}} \left[ \left| N_{\text{IAS}}^{(j)}(t) - N_\Pi^{(j)}(t; \mathbf{w}) \right|^2 \right], \tag{4.6}$$

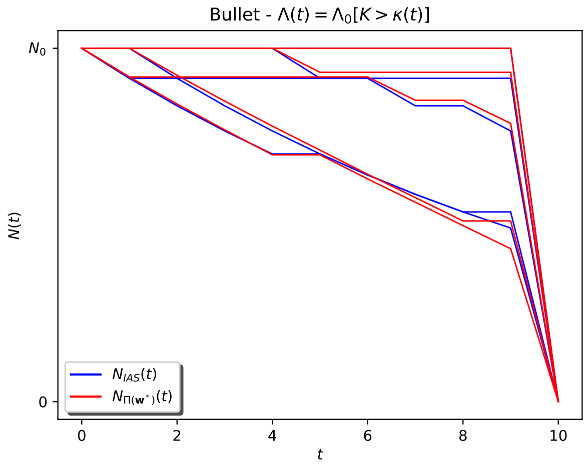



Figure 4.4: Example of how the calibrated portfolio is able to replicate the paths of the notional of the IAS. The case with a full-rational prepayment rule has been chosen because the paths are more spread and it is possible to appreciate the replication more.

because this way the mismatch between the notionals of the IAS and of $\Pi(\mathbf{w})$ is taken into account under every scenario $j$, and the squared errors are averaged on the simulations and integrated in time. The idea was to set up a "minimum-variance" hedging. Another result we accomplished is an analytical minimization for a special set of swaptions, as illustrated in the following proposition.

**Proposition 8** (Calibration of the diagonal swaptions on the paths of the IAS)**.** Consider the swaptions of the diagonal of Table 4.1, therefore the swaptions whose starting date and tenor sum up to the maturity of the mortgage. For example, for a mortgage with maturity $T_M = 10$, take the swaptions with notionals $w_{1,10}, w_{2,10}, \dots, w_{9,10}$. Define for simplicity

$$\mathbb{1}_{i,M}^{(j)} = \mathbb{1}_{\{K>\kappa^{(j)}(T_i)\}}.$$

Then, the solution to the problem (4.4) using the penalty function (4.6) corresponds to solving

$$\nabla F(\mathbf{w}) = \left[\frac{\partial F}{\partial w_{1,M}}, \dots, \frac{\partial F}{\partial w_{M-1,M}}\right] = \mathbf{0},$$

which results in a linear system $A\mathbf{w} = \mathbf{b}$. In particular,

$$A = \begin{bmatrix} (M-1)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{1,M}^{(j)}\mathbb{1}_{1,M}^{(j)} & (M-2)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{1,M}^{(j)}\mathbb{1}_{2,M}^{(j)} & \dots & (M-M+1)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{1,M}^{(j)}\mathbb{1}_{M-1,M}^{(j)} \\ (M-2)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{2,M}^{(j)}\mathbb{1}_{1,M}^{(j)} & (M-2)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{2,M}^{(j)}\mathbb{1}_{2,M}^{(j)} & \dots & (M-M+1)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{2,M}^{(j)}\mathbb{1}_{M-1,M}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ (M-M+1)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{M-1,M}^{(j)}\mathbb{1}_{1,M}^{(j)} & (M-M+1)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{M-1,M}^{(j)}\mathbb{1}_{2,M}^{(j)} & \dots & (M-M+1)\sum_{j=1}^{N_{Sim}} \mathbb{1}_{M-1,M}^{(j)}\mathbb{1}_{M-1,M}^{(j)} \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} w_{1,M} & w_{2,M} & \dots & w_{M-1,M} \end{bmatrix}^T,$$

$$\mathbf{b} = \begin{bmatrix} \sum_{k=2}^{M}\sum_{j=1}^{N_{Sim}} \mathbb{1}_{1,M}^{(j)}\left(\bar{N}(T_k) - N_{\text{IAS}}^{(j)}(T_k)\right) \\ \sum_{k=3}^{M}\sum_{j=1}^{N_{Sim}} \mathbb{1}_{2,M}^{(j)}\left(\bar{N}(T_k) - N_{\text{IAS}}^{(j)}(T_k)\right) \\ \vdots \\ \sum_{k=M-1}^{M} \mathbb{1}_{M-1,M}^{(j)}\left(\sum_{j=1}^{N_{Sim}} \bar{N}(T_k) - N_{\text{IAS}}^{(j)}(T_k)\right) \end{bmatrix}. \tag{4.7}$$

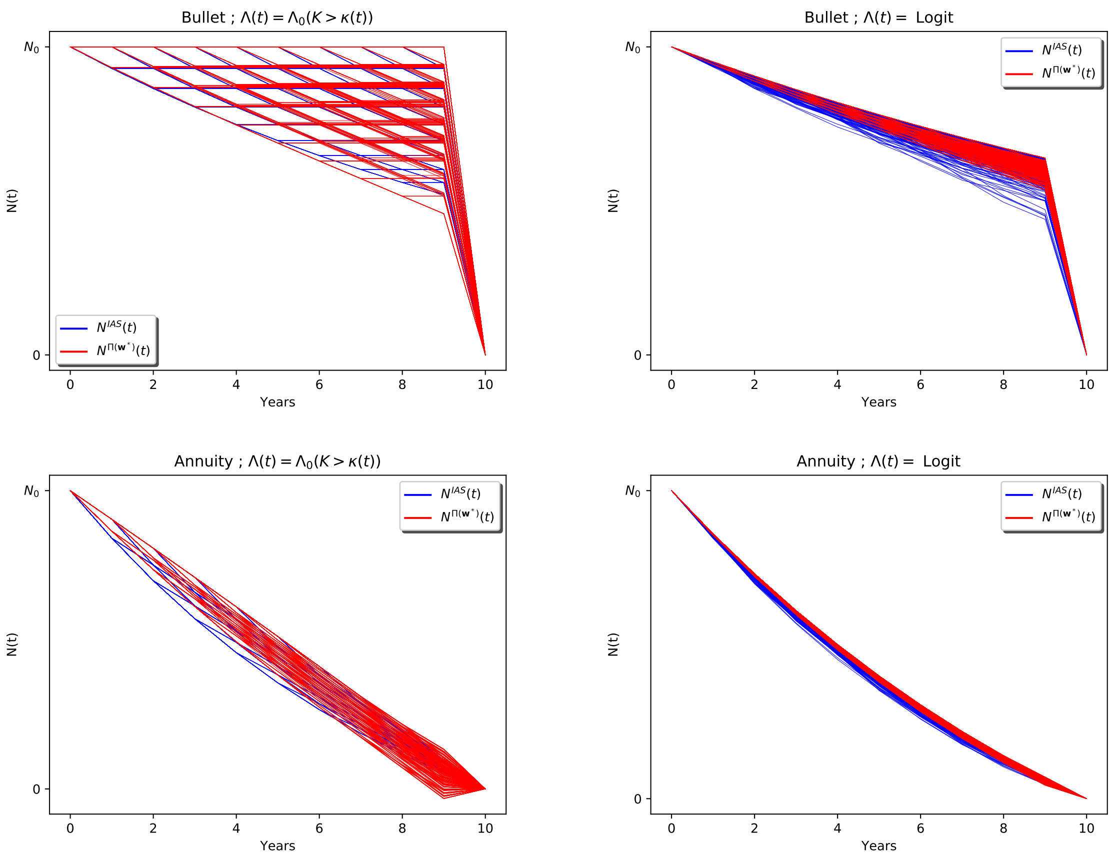



Figure 4.5: Examples of how the calibrated hedging portfolio composed of an amortizing swap and nine swaptions (red) is able to replicate the notional of the IAS (blue). Left: full-rational Refinancing Incentive. Right: s-shaped function. Top: bullets. Bottom: annuities.

| 1Y-9Y (bps) | 2Y-8Y (bps) | 3Y-7Y (bps) | 4Y-6Y (bps) | 5Y-5Y (bps) | 6Y-4Y (bps) | 7Y-3Y (bps) | 8Y-2Y (bps) | 9Y-1Y (bps) | Total (bps) |
|---|---|---|---|---|---|---|---|---|---|
| 1.16 | 1.34 | 1.41 | 1.43 | 1.40 | 1.34 | 1.23 | 0.99 | 0.61 | 10.95 |
| 0.61 | 0.66 | 0.58 | 0.48 | 0.35 | 0.25 | 0.12 | 0.01 | -0.07 | 3.01 |

Table 4.2: Composition of the calibrated portfolio of swaptions for a bullet and an annuity. The weights multiplied the prices of the swaptions and have been divided by the notional of the mortgage, in order to give an idea of the basis points that should be invested to apply the hedging strategy. Top: bullet. Bottom: annuity.

This result guarantees that the calibration of the hedging portfolio using the swaptions of the diagonal of Table 4.1 is performed without the use of any numerical procedure. For other sets of swaptions similar results could be reached, however we did not pursue this goal. Note that these swaptions correspond to the counter-diagonal of Table 2.1. Table 4.2 reports the composition of the portfolio of swaptions for bullets and annuities expressing their price in basis points, while Figure 4.4 shows an example of the replication power of the calibrated portfolio, where the red paths of $N_{\Pi(\mathbf{w}^*)}$ follow the behaviour of the blue paths of $N_{\text{IAS}}$. To appreciate the effect of the replication we can also look at the whole cloud of paths and, in this case, we can refer to Figure 4.5, which illustrates the result of the calibration of the 9 swaptions with which a mortgage of ten years can be hedged. The four graphs are similar to the ones of Figure 3.6, however this time, besides the blue paths of the notional of bullets and annuities, the red paths of the the notional of the portfolio appear. In theory, we would like to observe a perfect overlapping, which would mean that the IAS is perfectly replicated in every scenario. In reality, this could be only possible choosing a higher number of swaptions, as it will be shown soon. Looking carefully at the graphs, four effects need to be discussed, one for each of the plots. We start from the top-left corner, so the case of a bullet with a full-rational prepayment rule. We notice that there is a mismatch between $N_{\text{IAS}}$ and $N_{\Pi}$ next to maturity, and we would like to point out that, as it was already discussed, since the prepayment rate is proportional to the outstanding notional, the blue grid shrinks slightly close to maturity, making the replication more difficult in that area. Now this issue is reflected in the hedging because the swaptions can not be "partly" exercised. However, from this graph it is not possible to appreciate whether the paths that are missed are many or not, because there is no information about the distribution of $N_{\text{IAS}}$. Now, we move the focus to the second graph of the first line, which is the case of a bullet with an s-shaped function as RI. Here, we notice that the red paths do not touch the bottom-right portion of blue paths at all. This is worrying from a certain perspective, because it means that we are completely missing the replication of some scenarios, nevertheless it should not be that scary because we decided to minimize the sum of the squared differences path-wise, so it means that those paths did not have much weight. In fact, it seems that the core of the distribution is more on the top-right portion of paths. The last two thoughts regard the case of the annuity, so the graphs in the second line. In the left graph, next to maturity, an awkward effect appears, caused by the excessive amortization of the swaptions. Basically, the swaptions bought to help the replication of the first five years affects negatively the performance close to maturity, that is why negative notionals for the last swaptions are necessary. However, this effect is largely mitigated in reality (last graph) from the fact the paths of the annuity with an s-shaped RI are quite close one to each other, implying that the replication via swaps only should be already performing quite well and that not so many swaptions are needed. In the next subsections we will give insights about the performances of the linear hedge and different non-linear portfolios.

### 4.2.3. PRICE OF THE PREPAYMENT OPTION

Before proceeding in the evaluation of the different hedging strategies, some specific considerations about the results obtained so far are worthy. Equation (4.3) defines the replicating portfolio as a long positions in swaps and a short position in receiver swaptions. This has a strong impact on the analysis of the prepayment option, because it separates the effects of rational and non-rational prepayments, giving the possibility to obtain a price for the optionality embedded in the mortgage contracts.

In principle, one would obtain the price of the option as a difference between the price of a mortgage which does not allow prepayments, and the price returned by our simulation. This way, the prepayment option would have a price that is simply the difference between the case with or without prepayments. Nevertheless, this idea is misleading. We saw that there is a minimum level of prepayments that are always supposed to happen, this means that those prepayments do not constitute risk, therefore they should be excluded from the pricing of the prepayment option. In fact, these prepayments are correctly hedged via the linear part of the hedging portfolio, which, indeed, does not present any optionality. Therefore, the distinction we made between linear and non-linear risk is not only useful for hedging purposes, where it helps to address the risk to tradeable instruments, but also for the pricing itself. In practice, the mortgage rate that banks offer already includes the possibility that prepayments will happen, and the swap rate we start from is, indeed, only the starting point. On top of that, some basis points are added to cover the linear risk, and some others to cover the non-linear risk. In Equation (3.2) we mentioned this procedure simply assuming a constant spread. However, now, we are able to do the previous distinction, and to isolate the real prepayment risk.

At the end of Chapter 3, we postponed the discussion of the results obtained from the numerical simulations. The truth is that the price itself of the IAS does not give the correct information about how a financial institution should charge the mortgagor for the prepayment option embedded, because it is not able to express whether the linear part or the non-linear part is dominant, but only one of them represents the option's price. In fact, as Equation (4.3) states, those two effects go in different directions. This is why, in Table 3.2 we obtained both positive and negative prices. In the case of the full-rational prepayment rule, a negative price has been returned. Looking at the paths of the notional (Figure 3.6) we notice that the upper bound corresponds to the case without prepayments and, since the mortgage at time $t_0$ was at-the-money, the cost of the linear part of the hedge is zero. Thus, only the swaptions would intervene, leading the price to be negative. However, consider now the case with the sigmoid function as driver of the prepayments. In this case, the upper-bound of the notional was way lower than the line representing the case without prepayments, and the linear hedge kicked in. The price of the IAS happened to be positive, and now we can explain it saying that this happened because the value of the AS in (4.3) overcame the price of the swaptions.

The point of the whole reasoning is specifying that the prepayment option is always a cost for the financial institution providing it, therefore it should be priced correctly in order to charge the correct amount of basis points to the clients. However, the price of the IAS depends on many factors, such as the steepness and shape of the yield curve as well as the functional form of the Refinancing Incentive. Thus, the price has not a great explaining power, but our methodology does, because it considers only the real part of the price attributed to the non-linear risk. Therefore, the price of the prepayment option corresponds to the value of the non-linear part of the hedge, which is the price of the swaptions necessary to replicate the IAS. In case of an annuity, this is limited, and we motivated it saying that the contractual duties that it presents leave less freedom to react to the incentive. However, in the case of a bullet the price of the swaptions goes up to more than ten basis points (Table 4.2), or even more, as it we will explain in the next sections.

A final remark goes to the way of representing the price. Rather than doing it through the price

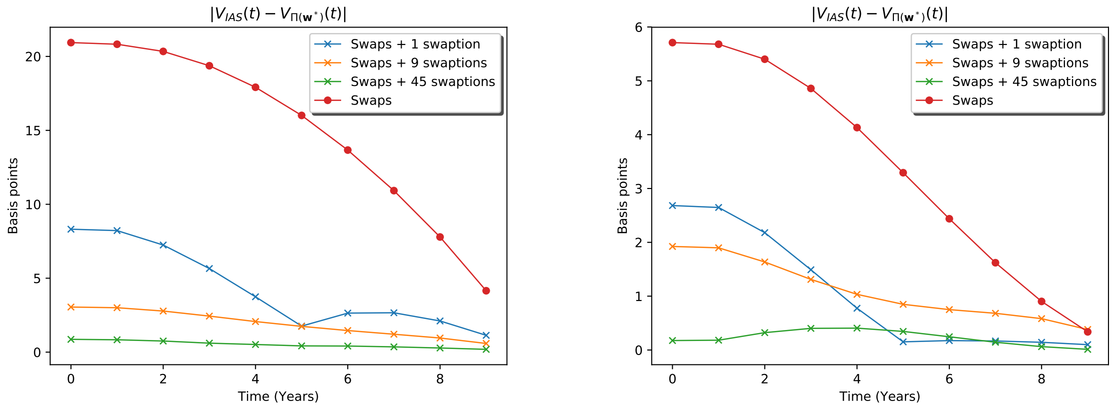



Figure 4.6: Comparison of the performances of different static hedging strategies. The completete replication is achieved using all the swaptions available, while already half of the risk is covered with only one swaption. The nine swaptions of the counter-diagonal seem to be the best compromise which allow for a decent hedging without using an excessive number of instruments. Left: bullet. Right: annuity.

of the swaptions, in practice people prefer to express it using a par swap rate, obtained imposing the value of the non-linear hedge equal to the value of a swap and solving for the swap rate. This procedure is also followed to describe the cost of the linear hedge, which, being already composed of swaps, finds in this idea a natural extension. However, when it is the turn of the non-linear part, there are problems in the computation of the par swap, because the notional is not well defined. Thus, in our framework, we just limit to express the price of the prepayment option through the price of the non-linear part of the hedging portfolio.

### 4.2.4. COMPARISON OF DIFFERENT HEDGING PORTFOLIOS

In this subsection we present another of the achievements we reached. We already showed how to calibrate the portfolio of swaps and swaptions on the paths of the IAS, so, at this point, we would like to assess the performance of different hedging strategies, in order to give insight about which swaptions are most useful in the replication of the IAS. Since we only have an analytical calibration for the swaptions of the counter-diagonal, for other portfolios a numerical minimization will be performed[3]. We define the value at time $T_k$ of the IAS as

$$\begin{aligned} V_{\text{IAS}}(T_k) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=k+1}^{M} \tau_i \frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)} \cdot \big(K - L(T_{i-1};T_{i-1},T_i)\big) \Big| \mathcal{F}(T_k)\right] \\ &= \frac{1}{N_{Sim}} \sum_{j=1}^{N_{Sim}} \sum_{i=k+1}^{M} \tau_i N_{\text{IAS}^{(j)}} \frac{M^{(j)}(T_k)}{M^{(j)}(T_i)} \cdot \big(K - L^{(j)}(T_{i-1};T_{i-1},T_i)\big), \end{aligned}$$

thus, we evaluate the accuracy of the hedge computing

$$\big|V_{\text{IAS}}(T_k) - \Pi(T_k, \mathbf{w}^*)\big|,$$

for each time $T_k$, $k = 0, \ldots, M-1$. In Figure 4.6 it is possible to appreciate the performances of the linear hedge and of non-linear hedges made of swaps and a varying number of swaptions. On the left, there are the results for a bullet, while the annuity is shown on the right. The values are expressed

[3] From the numerical experiments, it seemed that function (4.6) presented many local minima, meaning that the initial guess of the minimization affected largely the solution returned. For this reason, the model-free calibration reported in Appendix B has been chosen as reasonable initial guess. However, we can not guarantee that the portfolios calibrated numerically are actually the optimal. Only for the diagonal swaptions the result is true, thanks to Proposition 8.

in basis points and, even at a first glance, it should be already clear how the non-linear portfolios outperform the linear replication. First of all, it is essential to notice that a portfolio which uses the full matrix of swaptions replicates almost perfectly the IAS, as expected, keeping the error below one basis point. Then, another important phenomenon is represented by the portfolio with the swaption $5Y - 5Y$, which in case of a ten-years mortgage largely helps in replication also alone, with its peak of performance, as expected, happening at the fifth year. Finally, the portfolio with the nine swaptions from the diagonal seems to reach the best compromise between the full-replication and the linear hedging. We mentioned "best compromise" because, even if buying 45 swaptions would appear to be beneficial from this graph, in practice only the $85 \sim 90\%$ of the risk is hedged because fully-replicating a model does not mean that the position is hedged fully in reality, and, in case the model were not precise, handling 45 swaptions would be much more challenging. Thus, the goal of replicating "well-enough" the IAS using a reasonable number of instruments is achieved with a portfolio with 9 swaptions, especially in the case of a bullet. When we look at the annuity, instead, we recognize that the errors of the replication are all much smaller, and that the linear hedge is still worse than the non-linear hedge, but these may be acceptable. However, looking at the performance of the portfolio with one swaption and the benefit that comes with it, probably this would be the preferred solution of many, since the purchase of one swaption considerably improves the hedging. Before concluding, there is still one remark. In reality, the financial institutions mostly hedge the prepayment risk using a dynamic linear hedge, which means that the position in the swaps is updated in time typically every month or quarter of year. Nevertheless, these results give important insight in the benefits that a small number of swaptions could provide, not only to a static hedge but also to a dynamic one. To clarify this point, one should probably look at the Greeks of both the IAS and $\Pi(\mathbf{w}^*)$, which is what we will do in the next section. An alternative form for the function $F(\mathbf{w})$, which aims to minimize the errors at each time, is provided in Appendix C together with a closed-form solution for the diagonal swaptions and its numerical results.

## 4.3. RISKS FROM THE INTEREST RATE MODEL

In this section the details about the calibration of the interest rate models are provided. There are three topics that require special attention and that correspond to the three subsections. First of all, the choice of the instruments on which calibrate HW and CIR++ will be finally discussed, together with some drawbacks of the models. Then, the focus will move to the sensitivity of the price of the IAS respect to the aforementioned instruments. For this reason, the Greeks will be introduced and an alternative calibration of the hedging portfolio on the Greeks will be proposed. Finally, we will try to evaluate the risk behind the choice of the Hull-White model for a specific example, suggesting the comparison of the results under the different HW or CIR++ frameworks as methodology to evaluate the model risk.

### 4.3.1. CHOICE OF THE INSTRUMENTS

In theory, one would like to use all the information that the market provides in order to get the most accurate calibration of the parameters of a model. In practice, this is challenging (if not impossible) to achieve, especially with a single-factor model like the Hull-White or CIR++. A comprehensive calibration is unreachable because only the long-term average is time-dependent, while the mean-reverting speed and volatility are assumed to be constant parameters. In case of the CIR++, also the long-term average is actually constant, but the shift introduced to recover negative rates effectively acts like the time-dependent average for Hull-White. Once the necessity to select a subset of instruments has been acknowledged, the most natural choice to calibrate a short-rate model is selecting the market quotes of the instruments that the simulations will use the most. Basically, it means selecting an "area" of the volatility matrix showed in Table 2.1 which we would like the models to recover. In our case, since most of the results will gravitate around the swaptions of the counter-

| $\lambda^{HW}$ | $\eta^{HW}$ | $\lambda^{CIR}$ | $\eta^{CIR}$ | $\theta^{CIR}$ | $x_0$ |
|---|---|---|---|---|---|
| 0.264 | 0.017 | 0.185 | 0.039 | 0.184 | 0.079 |

Table 4.3: Calibration of the Hull-White and CIR++ models.

| Swaption (Maturity-Tenor) | $\sigma_{Market}$ (bps) | $\sigma_{HW}$ (bps) | $\sigma_{CIR}$ (bps) | $V_{Market}$ (bps) | $V_{HW}$ (bps) | $V_{CIR}$ (bps) |
|---|---|---|---|---|---|---|
| 1Y-10Y | 46.31 | 53.90 | 48.00 | 177.96 | 207.13 | 184.45 |
| 3Y-7Y | 56.28 | 56.22 | 55.40 | 261.61 | 261.34 | 257.53 |
| 5Y-5Y | 61.98 | 57.37 | 60.28 | 262.28 | 242.75 | 255.11 |
| 7Y-3Y | 64.79 | 61.75 | 66.36 | 191.63 | 182.62 | 196.27 |
| 9Y-1Y | 64.89 | 69.79 | 74.10 | 71.27 | 76.66 | 81.39 |

Table 4.4: Results of the calibrated swaptions.

diagonal (or diagonal, if we look at Table 4.1), if we intend to simulate a mortgage with maturity $T_M = 10$, then the swaptions $1Y-9Y, 2Y-8Y, ..., 9Y-1Y$ are the ones that should be selected to calibrate the interest rate models. This is also justified by the fact that for this mortgage the Refinancing Incentive depends on the swap rates with those maturities and tenors. Nevertheless, this choice is still too restrictive to accomplish satisfying results, because the smile/skew that the market presents for those swaptions cannot be displayed by the two interest rate models. One way to overcome this issue would consist in letting the volatility be time-dependent, but this goes out of the scope of this thesis. Thus, we opted to select an even smaller set of swaptions and, for a mortgage of ten years, the parameters returned by the calibration procedure explained in Section 2.5 are reported in Table 4.3, while the calibrated prices and implied volatilities are shown in Table 4.4.

A consistent choice of the instruments used to construct the yield curve would imply that the swap rates selected are the underlying instruments of the swaptions. However, we are about to introduce the Greeks and the concept of sensitivity respect to quotes of the market instruments. Since it is common practice to show the sensitivities with respect to spot instruments, we opt for constructing the curve from the swap rates $S_{t_0,T_i}(t_0)$ with $i = 1, ..., M$. The forward instruments would have been the swap rates $S_{T_m,T_n}(T_m)$ with $m + n = M$, however the derived "forward Delta-profile" would be less intuitive to interpret.

### 4.3.2. GREEKS APPROXIMATION

The value of the IAS obviously depends on the parameters of the interest rate model used to price it. This means in turn that the price depends on the market quotes involved in the calibration of the interest rate model. The sensitivities of the price of a derivative respect to the underlying instruments are called "Greeks", and we can distinguish them according to the order of derivative they represent, but also depending on the underlying quantity they refer to. The most common Greeks are Delta ($\Delta$), Gamma ($\Gamma$) and Vega ($\mathcal{V}$), and having insight in their value can extensively help in the hedge of a derivative. In fact, most traders are more concerned about the sensitivity that their portfolios display rather than the price itself, and a portfolio that is fully hedged must have all the Greeks counterbalanced. In this subsection we show the Greeks that both the IAS and the hedging portfolio present, and we conclude discussing an alternative calibration of the swaptions with which the Gamma of the IAS is replicated.

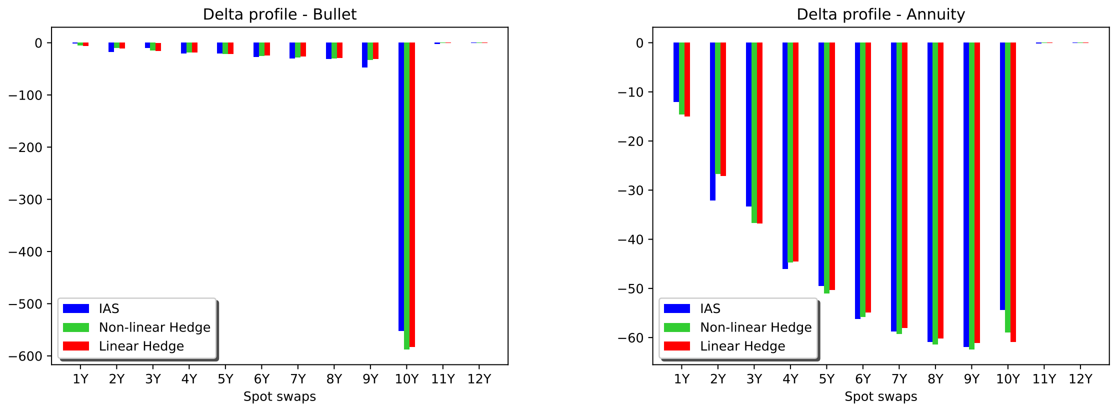



Figure 4.7: Delta profile of a bullet (left) and an annuity (right). On the x-axis there are the spot swap rates used to construct the yield curve. The Delta profile of the IAS is well-approximated by both the linear and non-linear hedge. This result was expected since the swaptions have marginal importance in the total hedging portfolio, accounting for only 10 basis point in case of a bullet and less than 5 for an annuity. Thus, the Delta is mostly influenced by the linear instruments. The notional of the mortgage was $N_0 = 1$ million.

We compute the Greeks of the IAS numerically, approximating them using finite differences methods. The instruments to which we can show the sensitivity are the spine swap rates used to construct the yield curve, which affect $\theta^{\text{HW}}$, and the implied volatilities on which the parameters $(\lambda^{\text{HW}}, \eta^{\text{HW}})$ have been calibrated. Obtaining the variation of the price of a derivative with respect to the latter quantities means considering the price of the IAS as a function of them, thus

$$V_{\text{IAS}}(t_0) = V_{\text{IAS}}(t_0; S_{m,n}, \sigma_{i,j}^{\text{Market}}).$$

The approach we set then is the following:

1. Pricing the IAS.
2. Shock of the market quote of one basis point ($h$). Typically, the implied volatility would be shocked of 100 basis points. However, we are using the Nognormal implied volatility, as explained in Section 2.5, and, since it is much smaller than the implied Logormal volatility, the shock of one basis point is still reasonable.
3. Recalibration of the interest rate model.
4. Pricing of the IAS using the same seed for the random number generator.
5. Evaluation of the difference with the old price.

Once we established the general procedure, we introduce the Greeks:

- Delta represents a first-order sensitivity and it is the Greek that is hedged primarily. In the case of the IAS, it measures the rate of change of $V_{\text{IAS}}$ with respect to changes in the spine swap rates. Since it is a first derivative, the hedge of Delta usually involves linear instruments. We approximate it by a numerical differentiation with a two-point scheme:

$$\Delta_{\text{IAS}}(S_{m,n}) = \frac{\partial V_{\text{IAS}}(t_0; S_{m,n})}{\partial S_{m,n}} \approx \frac{V_{\text{IAS}}(t_0; S_{m,n} + h) - V_{\text{IAS}}(t_0; S_{m,n} - h)}{2h}.$$

- Gamma measures the rate of change of Delta and it represents the second derivative of the IAS. An effective hedge also considers Gamma, since having a Delta-Gamma neutral position

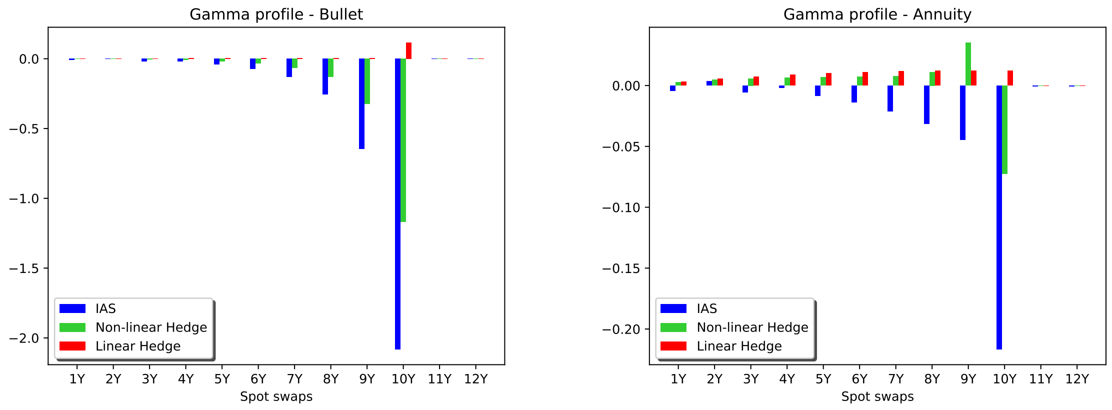



Figure 4.8: Gamma profile of a bullet (left) and an annuity (right). On the x-axis there are the spot swap rates used to construct the yield curve. The Gamma profile of the bullet shows higher values compared to the one of the annuity, and the advantages of the non-linear hedge are clear compared to the linear hedge. The Gamma profile of the annuity is awkward, however the values are negligable. The notional of the mortgage was $N_0 = 1$ million.

means being neutral over a wider range of market movements. We obtain Gamma from:

$$\Gamma_{\text{IAS}}(S_{m,n}) = \frac{\partial^2 V_{\text{IAS}}(t_0; S_{m,n})}{\partial S_{m,n}^2} \approx \frac{V_{\text{IAS}}(t_0; S_{m,n}+h) - 2V_{\text{IAS}}(t_0; S_{m,n}) + V_{\text{IAS}}(t_0; S_{m,n}-h)}{h^2}.$$

- Vega measures the sensitivity to the volatility, thus it is the first derivative of the price of the IAS with respect to the quotes of the swaptions' implied volatilities. Since these quotes only affect the calibratiton of the parameters of volatility and mean-reverting speed, we do not expect any Vega for the linear hedge. In this case, we opted to retrieve Vega from a one-point forward scheme:

$$\mathcal{V}_{\text{IAS}}(\sigma_{i,j}) = \frac{\partial V_{\text{IAS}}(t_0; \sigma_{i,j})}{\partial \sigma_{i,j}} \approx \frac{V_{\text{IAS}}(t_0; \sigma_{i,j}+h) - V_{\text{IAS}}(t_0; \sigma_{i,j})}{h}.$$

Following a similar approach, also the Greeks of the linear and non-linear hedging portfolios can be approximated. However, after recalibrating the interest rate model, the pricing of swaps and swaptions can be also done using the formulas presented in Chapter 2, without recurring to numerical simulations. Since we decided to focus on a hedging portfolio constituted of swaptions taken from the counter-diagonal, we show the results for this kind of non-linear hedge. Figure 4.7 and 4.8 report the Delta and Gamma profiles of bullets and annuities with maturity $T_M = 10$, while Figure 4.9 illustrates the Vega. The delta profiles of both the bullet and the annuity seem to be replicated, while the performance on the Gamma profiles is weaker. As expected, most of the Delta of a bullet with maturity of ten years is on the swap rate $S_{0,10}$ because, since a bullet does not involve any repayments, it is closer to a plain-vanilla swap than an annuity is. On the other hand, the Delta of an annuity is spread over the whole set of instruments because its amortization plan takes all of them into account. Mostly for the annuity, the Gamma of the hedge seems to be quite far from the Gamma of the IAS. However, one should notice that the magnitude of the Gamma embedded in an annuity seems to be pretty small, especially if compared to the one of the bullet. This behaviour was expected because, looking at the paths of the notional of an annuity (for example, Figure 4.5), we can observe that the cloud of paths is not as large as the one of the bullet, thus there is no need to involve many swaptions. In fact, the linear hedge was already a good replication of an annuity as Figure 4.6 pointed out. On the other hand, looking at the results of the bullet, the Gamma displayed by a bullet is much higher and therefore concerns could arise. Fortunately, in this case the hedge manages to

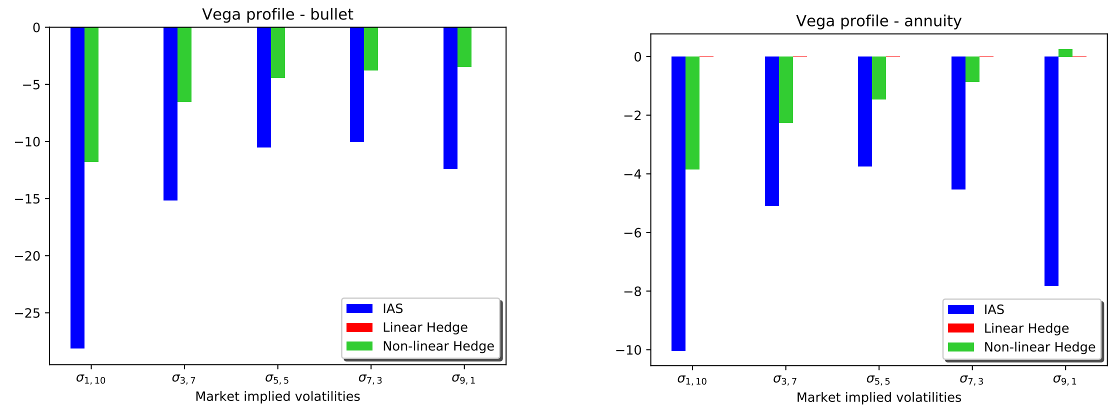



Figure 4.9: Vega profile of a bullet (left) and an annuity (right). On the x-axis there are the implied volatilities used to calibrate the Hull-White model. As expected, the non-linear hedge is able to approximate the Vega profile of the IAS, while the linear hedge does not present any sensitivity to changes in the volatility of the market. Similarly to what happened with Gamma, also in this case the bullet shows more sensitivity than the annuity. The notional of the mortgage was $N_0 = 1$ million.

offset part of the risk. Finally, the Vega profiles are partially replicated for both the bullet and the annuity by the non-linear hedge, while, of course, the swaps did not return any Vega, stressing out once again the necessity to include the non-linear hedge in the protection from the prepayment risk.

### 4.3.3. Calibrating the hedging portfolio on Gamma

After computing the Greeks of the IAS and of the replicating portfolio, we concluded that the bullets required special attention because of the high values of Gamma. Since the portfolio of nine swaptions only managed to offset the Gamma of the IAS partly, we now see how the composition of the portfolio changes when the calibration is based on the replication of the Gamma of the IAS. We recall that the weights of the swaptions are only marginal compared to the one of the amortizing swap, thus a variation/redistribution of the basis points invested in the swaptions does not change the Delta profile of the hedge significantly. Moreover, the swaptions are the instruments which affect the Gamma the most, so changing the linear part of the hedging would be useless for the offset of the Gamma. We calibrate the swaptions obtaining the weights from (4.4) using

$$F(\mathbf{w}) = \left\| \Gamma_{\text{IAS}} - \Gamma_{\Pi(\mathbf{w})} \right\|_2, \tag{4.8}$$

where $\Gamma_{\text{IAS}}$ and $\Gamma_{\Pi(\mathbf{w})}$ are the vectors whose entries are the Gamma values of the IAS and of the non-linear hedge. The minimization routine returned a vector of weights $\mathbf{w}^*_\Gamma$ which determined a new composition of the portfolio, as showed in Table 4.5. We notice that this solution costs almost the double of the original $\mathbf{w}^*$, which must result in a worse replication of the price of the IAS. Thus, we aimed to reach a compromise in a naive way, "averaging" the weights of the calibrations on the price ($\mathbf{w}^*$) and on Gamma ($\mathbf{w}^*_\Gamma$) and defining

$$\widetilde{\mathbf{w}} = \frac{\mathbf{w}^* + \mathbf{w}^*_\Gamma}{2}.$$

As Figure 4.10 shows, the full replication of the Gamma leads to a great mismatch in the prices, however the average portfolio $\widetilde{\mathbf{w}}$ manages to compromising, keeping the error on price below five basis points, and providing a better replication of the Gamma profile of the IAS.

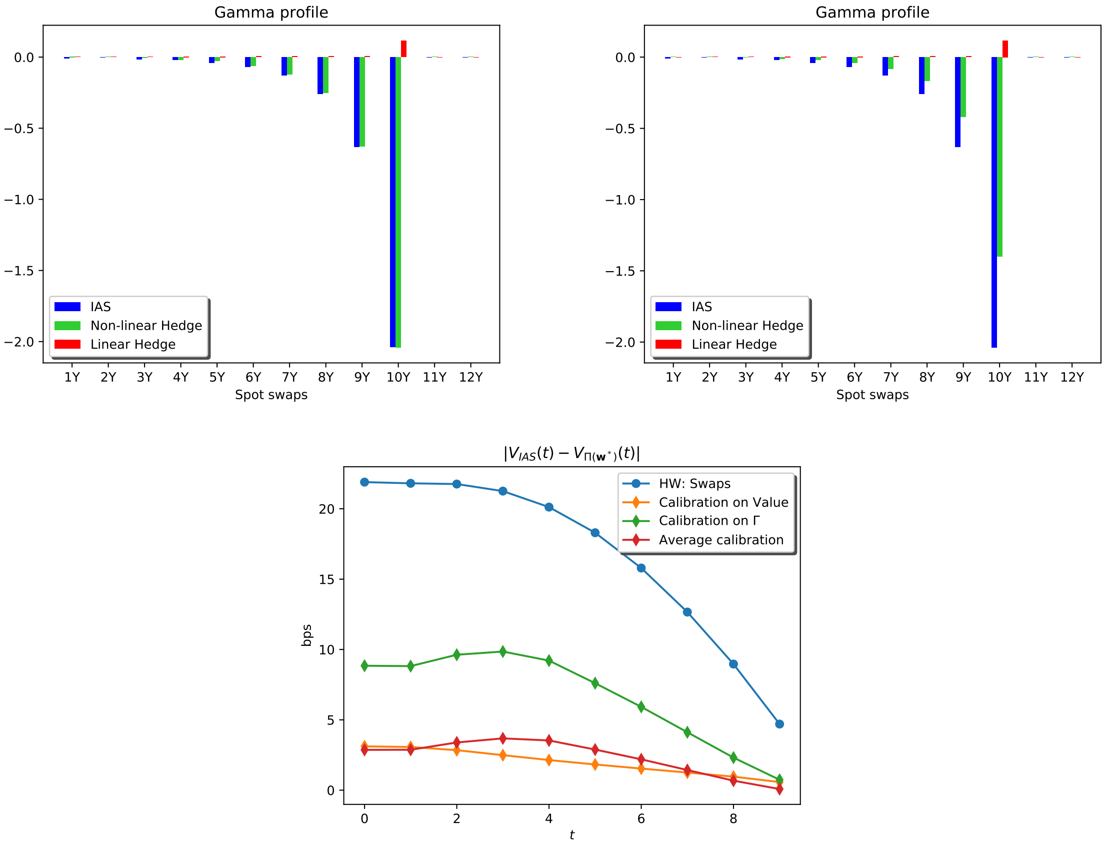



Figure 4.10: Top left: the Gamma profiles using $\Pi(\mathbf{w}^*_\Gamma)$. Top right: the gamma profiles using $\Pi(\widetilde{\mathbf{w}})$. Bottom: comparison of the approximation of the price of $\Pi(\mathbf{w}^*)$ (Calibration on value), $\Pi(\mathbf{w}^*_\Gamma)$ (calibration on Gamma) and $\Pi(\widetilde{\mathbf{w}})$ (average calibration).

| 1Y-9Y (bps) | 2Y-8Y (bps) | 3Y-7Y (bps) | 4Y-6Y (bps) | 5Y-5Y (bps) | 6Y-4Y (bps) | 7Y-3Y (bps) | 8Y-2Y (bps) | 9Y-1Y (bps) | Total (bps) |
|---|---|---|---|---|---|---|---|---|---|
| 1.16 | 1.34 | 1.41 | 1.43 | 1.40 | 1.34 | 1.23 | 0.99 | 0.61 | 10.95 |
| 1.76 | 0.00 | 3.36 | 2.65 | 3.53 | 3.51 | 3.27 | 2.85 | 1.80 | 22.78 |

Table 4.5: Composition of the calibrated portfolio of swaptions for a bullet with two different calibrations. The weights multiplied the prices of the swaptions and have been divided by the notional of the mortgage, in order to give the basis points that should be invested to apply the hedging strategy. Top: calibration performed minimizing (4.6) that returned $\mathbf{w}^*$. Bottom: calibration performed minimizing (4.8) that returned $\mathbf{w}^*_\Gamma$.

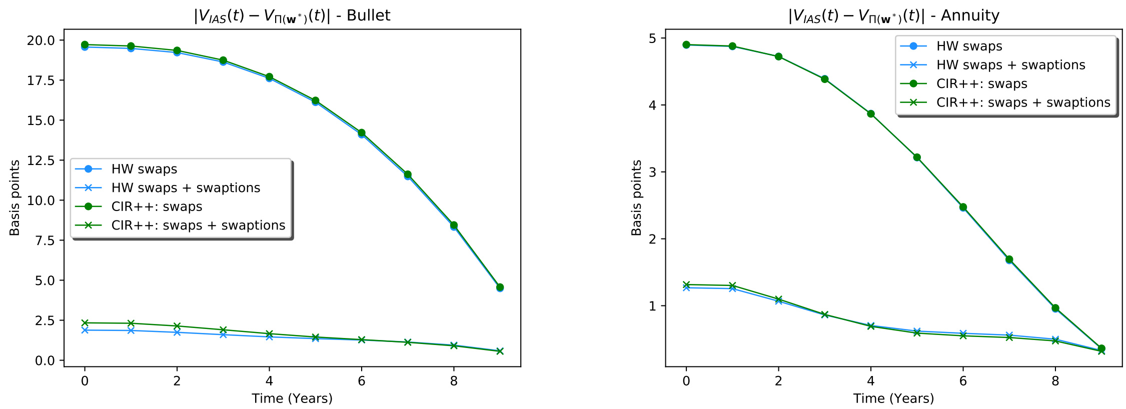



Figure 4.11: Assessment of the impact of a different realized distribution of the interest rate. The portfolio calibrated on the paths of the IAS under HW has been tested on the paths of the CIR++. Left: bullet. Rght: annuity.

### 4.3.4. RISK BEHIND THE CHOICE OF HULL-WHITE

The last analysis we perform investigates the risk behind the choice of the Hull-White model. The idea arises from the fact that we calibrated the hedging strategy on the realised paths of the IAS, thus one could argue that this procedure is artificial and does not provide useful results for a practical use. First of all, we already discussed that our results can give significant insight in the non-linear risk embedded in the prepayment option, and that any financial institution can profit from our research obtaining valuable knowledge about the replication power of different hedging portfolios. Secondly, we now show that, for the specific example we consider, the model-risk we introduce with the choice of Hull-White is limited. This is not a real stress-test, because only one alternative model and one maturity have been tried. However, it can help to answer some of the typical questions that are usually raised to verify the robustness of a model. To pursue this scope, in Chapter 2 also the CIR++ has been described. The CIR++ model provides another formulation for an SDE governing the short-rate and, most importantly, this comes with a different distribution of $r(t)$. In fact, the idea is to look at our model critically and verify what is the impact of a incorrect expectation of the reality. So, what if the reality did not behave like we predicted? Or, more properly, *what happens if the realized distribution is not the one we expect?*

The approach we followed to answer this question consists in calibrating a portfolio $\Pi(\mathbf{w}^*)$ on the path of the IAS simulated with HW, and then assess its performance using the paths generated by the CIR++. Of course, both models are calibrated on the same set of instruments. Nevertheless, the distinct distributions of the two models should provide a decent perception of the impact that a wrong choice of the model has in the evaluation and hedging of the mortgage portfolio. Algorithm 3 recapitulates the steps followed, while Figure 4.11 illustrates the results for a bullet (left) and for an annuity (right). The two hedging strategies selected for this experiment have been the linear hedge and the non-linear hedge with the swaptions from the diagonal. The surprising result is that the portfolio $\Pi(\mathbf{w}^*_{\text{HW}})$ seemed to perform better than $\Pi(\mathbf{w}^*_{\text{CIR}})$, which is suspicious since both of them have been calibrated on the paths of the CIR++. Nevertheless, this effect has an explanation. The reason behind this outcome is due to the choice of the minimization function $F(\mathbf{w})$, which, in this case, we recall being the sum in time of the average on the simulations of the path-wise squared error, as in Equation (4.6). So, we do not have to expect that the $\Pi(\mathbf{w}^*_{\text{CIR}})$ returns better results *at each time*, but that gives better accuracy in the replication *over the whole time horizon*, which is exactly what happens since, considering the paths of the CIR++ model, we obtain that

$$F(\mathbf{w}^*_{\text{CIR}}) < F(\mathbf{w}^*_{\text{HW}}).$$

We also point out that the graphs in Figure 4.11 do not show the value of the minimization function itself, because even if we decided to minimize the integrated variance, we preferred to show the results of the replication of the price. The reader might be interested to see also the graphs representing the trend of the variance in order to judge the replication path-wise. This analysis can be found in Appendix C together with the test of another minimization function $F(\mathbf{w})$ which minimizes the difference between the notionals *at each time*.

**Algorithm 3** Testing the Hull-White assumption.

1: **procedure** COMPARING HW AND CIR++
2: Calibrate the Hull-White and the CIR++ as described in Section 2.5.
3: Perform the simulation of the IAS following Algorithm 2.
4: Use the paths of the IAS to calibrate the non-linear hedge as explained in 4.2.2.
5: Assess the performance of the portfolio as in 4.2.4 testing how both $\Pi(\mathbf{w}^*_{\text{HW}})$ and $\Pi(\mathbf{w}^*_{\text{CIR}})$ performed on the paths of the IAS generated by the CIR++.

# 5

# Conclusion and outlook

## 5.1. Summary and conclusion

Throughout this thesis we have investigated the methods to price and hedge a portfolio of mortgages, with focus on the prepayment risk. After introducing the prepayment option and the various kinds of mortgage contracts, we have explained the importance of predicting the prepayment rate creating a direct link between the refinancing incentive and instruments observable in the market. This framework enables the pricing in a stochastic environment, where the paths of an interest rate model generate the paths of the notional of the mortgage. The advantages are twofold. First of all, a risk-neutral evaluation is possible. Secondly, the volatility implied by the non-linear instruments used for the calibration of the interest rate model arises the necessity to extend the hedging strategy to non-linear instruments. Currently, it is common to hedge the prepayment risk with linear instruments, however this is not in line with the real nature of the prepayment option which, being an option, provides non-linear risk. By implementing the pricing model and calibrating it on real market data, one is able to test different hedging portfolios on the simulated states of the market.

In Chapter 3 the pricing model is constructed. An Index Amortizing Swap has been selected as the instrument which replicates perfectly the mortgage portfolio, and specific justifications have been provided for all the dependencies that its notional must have in order to encapsulate the characteristics of the notional of a mortgage. As the Refinancing Incentive has been shown to be the main driver of prepayments in all the literature, it has been chosen as the only variable of a prepayment model which aims to return a risk-neutral price of a mortgage portfolio. This limitation has been mitigated by the use of an s-shaped function as functional form for the incentive, which, through a historical calibration on a data set of more than thirty millions of rows, is able to capture the non-rational behaviour of the mortgagors as well as the reaction time necessary to perform a rational prepayment. This function is the essential bridge between the risk-neutral world in which the Refinancing Incentive is located, and the real-world prepayment rate, for which a risk-neutral measure does not exists. Thanks to all the relations established, the market mortgage rate is obtained analytically from the paths of the short-rate, thus no further information is lost. To countercheck the implementation, the results of the numerical simulations have been compared with an analytical formula assuming a deterministic prepayment rate.

The function determining the notional of the IAS is tailor-made to incorporate the characteristics of a mortgage portfolio, however, such derivative is only useful for pricing purposes because it does not exist in the market. Thus, Chapter 4 aims to replicate the optionality embedded in the notional of the IAS addressing the risks to a set of tradeable instruments. We showed that improving the pricing from a deterministic to a stochastic framework unveils the limitations of a linear hedge, thus we justified the choice of building the replicating portfolio from a set of swaps and swaptions. The role of the swaptions is offsetting the prepayments reacting at the same way to market movements, in order to provide the correct convexity adjustment. At first, the swaptions have been calibrated to approximate the notional of the IAS under all the scenarios simulated, achieving decent results in terms of price replication. In fact, it has been showed that the full set of swaptions available replicates fully the IAS. Then, we selected the swaptions which explained most of the risk, so that a limited number of derivatives could be already effective. However, the replication of the Greeks seemed to have space for improvements, so we tested also a calibration based on the Gamma of the IAS, concluding that a decent compromise is provided by an average calibration. Finally, for a specific framework, we assessed the model-risk embedded in the choice of the Hull-White model, testing the hedging strategy that resulted optimal under Hull-White on the paths of the IAS resulting from the simulation of the CIR++. The impact of the different distributions changed according to the penalty function that leads to the calibration of the swaptions, and we concluded that, for this specific example, the strategy we propose in the main body of the thesis is robust with respect to changes in the distribution of the short-rate.

## 5.2. MODEL LIMITATIONS AND FUTURE RESEARCH

The goal of this thesis was creating a comprehensive framework which included a pricing model dependent on the market level and a consistent hedging strategy. Considering the lack of articles providing details about the non-linear hedging of the prepayment risk, this thesis constitutes a novelty. However, many assumptions are the basis of the construction of the pricing and hedging models, and while some of them are marginal, others are more restrictive. In what follows, we present a list of possible extensions that could improve the accuracy of some of the results, and we also suggest the topics for the future research.

The biggest assumption is the single-curve framework in which all the evaluations inhabit. The biggest drawback is that a pricing model inserted in this frame can not consider the different areas in which the market of interest rates is segmented. Pricing and hedging an interest rate derivative like the IAS mixes different rate tenors which might belong to divisions of the market for which different dynamics would be required. The single-curve thus might lead to inconsistencies or results that are not robust. Switching to the multi-curve environment would make our models way more valuable.

Assuming the Hull-White model as driver of the short-rate leads creates model-risk that has only been partially assessed. The limitations of models with constant volatility, like the Hull-White or CIR++, have been pointed out when we dealt with the problems of retrieving the smile/skew of the implied volatility surface. Even if short-rate models still present advantages, as explained in Chapter 2, implementing a Market-Model seems like a natural extension for the kind of questions we investigated.

Regarding the modelling of the IAS, the assumptions of yearly payments as well as the fact that the new production of mortgages is not taken into account seem to be limitations that could be easily overcome. However, to improve the hedging we suggest to cluster the mortgages not only according to the typology (bullet or annuities) but also depending on the maturity or the coupon they have. A proper data analysis should provide the correct tool to reduce the number of instruments required to hedge the whole portfolio of a bank.

In Equation (3.2) we approximate the at-the-money mortgage rate as a swap plus a spread which covers liquidity risks and profits. Including this spread in pricing results is immediate, however, it has not been taken into account in the hedging chapter. This finds the reasons in the fact that the spread is typically not hedged by the financial institution, which is willing to only transfer part of the fixed rate received from the mortgagor. Nevertheless, even in the pricing procedure, assuming a constant spread for any level of rates is inconsistent with the market practice. Finding a relation between this spread and the market movements would help largely.

We based all the non-linear hedging on a portfolio constituted of swaps and European swaptions. However, using Bermudan swaptions might return more accuracy in the replication of the IAS and should be considered.

Finally, we recommend extending the static-hedging to a dynamic strategy, in which the benefits of the swaptions would stand out even more. Replicating a swaption with linear instruments is practically impossible because it would require a continuous recalibration of the position in the underlying swaps. Thus, in case of sudden changes in the market, the use of swaptions would show the best of its power. Testing the performance of the non-linear hedge we propose with real market data is the feature that is really missing to the whole thesis. An interesting idea would be setting the experiment in the framework of the crisis of 2008, when interest rates dropped unexpectedly and jumps in the market caused immediate mismatches in the hedging positions of most of the financial institutions.

# A

# MORE INSIGHTS ABOUT THE PRICING UNDER THE HULL-WHITE MODEL

In this appendix, more details about the pricing of bonds, floorlets and swaptions are given. The main question that this appendix is supposed to answer is: considering that the functions $A(t,T)$ and $B(t,T)$ have been derived for a process under the $\mathbb{Q}$ measure, do they change when they refer to a process under another measure (for instance, $\mathbb{Q}^T$)? To clarify this aspect, an alternative derivation of the functions is proposed, and it is performed under both the risk-neutral measure and the T-forward measure. Then, a brief clarification about the floorlet pricing is also given, explaining with more details a passage of the proof in which, still under the $\mathbb{Q}$ measure, the discount factors change from the bank-account to the bond price.

## A.1. ALTERNATIVE DERIVATION OF THE FUNCTIONS A(T,T) AND B(T,T)

The motivation behind the following derivation is that during the pricing of the bond option

$$\begin{aligned} V_{\text{ZCB}}(t_0,T;T_S) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}(K-P(T,T_S))^+\middle|\mathcal{F}(t_0)\right] \\ &= P(t_0,T)e^{A_r(T,T_S)}\mathbb{E}^T\left[\left(\hat{K}-e^{B_r(T,T_S)r(T)}\right)^+\middle|\mathcal{F}(t_0)\right]. \end{aligned}$$

the mean and volatility of the Hull-White process under $\mathbb{Q}^T$ have been used, while the expressions for the functions $A(t,T)$ and $B(t,T)$ have been obtained under the $\mathbb{Q}$ measure. So the question is: shouldn't we be consistent and use some expression for $A(t,T)$ and $B(t,T)$ under the T-forward measure? To answer this question, we recall the following dynamics of the ZCB under the Hull-White model

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\mathrm{d}t + \eta B(t,T)\mathrm{d}W^{\mathbb{Q}}(t),$$

where $B(t,T)$ comes from to Equation (2.15). We first derive the price of ZCB under $\mathbb{Q}$ and then under $\mathbb{Q}^T$, and show that the functions $A(t,T)$ and $B(t,T)$ are the same.

### BOND PRICE UNDER $\mathbb{Q}$

In the previous SDE we recognise a Geometric Brownian Motion, common practice to find the solution is to define a new process $X(t) := g(t,P(t,T)) = \log(P(t,T))$, and use Itô's lemma to be able to find the solution to $X(t)$ and thus to $P(t,T)$. From now on since the whole problem concerns the process $P(t,T)$ either under $\mathbb{Q}$ or under $\mathbb{Q}^T$, we will stress out whether every process is considered under one or the other measure. Thus,

$$\mathrm{d}X(t) = \left(r(t)_{\mathbb{Q}} - \frac{1}{2}\eta^2 B^2(t,T)\right)\mathrm{d}t + \eta B(t,T)\mathrm{d}W^{\mathbb{Q}}(t),$$
$$X(T) - X(t) = \int_t^T r(u)_{\mathbb{Q}} - \frac{1}{2}\eta^2 B^2(u,T)\mathrm{d}u + \int_t^T \eta B(u,T)\mathrm{d}W^{\mathbb{Q}}(u),$$

so, imposing the final condition $P(T,T)=1$, which implies $X(T)=\log(1)=0$, we obtain,

$$P(t,T)_{\mathbb{Q}} = \exp\left\{-\int_t^T r(u)_{\mathbb{Q}} - \frac{1}{2}\eta^2 B^2(u,T)\mathrm{d}u - \int_t^T \eta B(u,T)\mathrm{d}W^{\mathbb{Q}}(u)\right\}.$$

Now, we already know from Equation (2.5) that

$$r(u)_{\mathbb{Q}} = e^{-\lambda(u-t)}r(t)_{\mathbb{Q}} + \lambda\int_t^u \theta(z)e^{-\lambda(u-z)}\mathrm{d}z + \eta\int_t^u e^{-\lambda(u-z)}\mathrm{d}W^{\mathbb{Q}}(z),$$

therefore we can compute the integral $\int_{t_0}^t r(u)_{\mathbb{Q}}\mathrm{d}u$. In particular,

$$\begin{aligned}\int_t^T r(u)_{\mathbb{Q}}\mathrm{d}u &= r(t)_{\mathbb{Q}}\int_t^T e^{-\lambda(u-t)}\mathrm{d}u + \lambda\int_t^T\int_t^u \theta(z)e^{-\lambda(u-z)}\mathrm{d}z\mathrm{d}u + \eta\int_t^T\int_t^u e^{-\lambda(u-z)}\mathrm{d}W^{\mathbb{Q}}(z)\mathrm{d}u\\ &= r(t)_{\mathbb{Q}}\int_t^T e^{-\lambda(u-t)}\mathrm{d}u + \lambda\int_t^T\int_z^T \theta(z)e^{-\lambda(u-z)}\mathrm{d}u\mathrm{d}z + \eta\int_t^T\int_z^T e^{-\lambda(u-z)}\mathrm{d}u\mathrm{d}W^{\mathbb{Q}}(z)\\ &= -r(t)_{\mathbb{Q}}B(t,T) - \lambda\int_t^T \theta(z)B(z,T)\mathrm{d}z - \eta\int_t^T B(z,T)\mathrm{d}u\mathrm{d}W^{\mathbb{Q}}(z),\end{aligned}$$

where the first passage follows from the fact that

$$\begin{cases} t<z<u \\ t<u<T\end{cases} \iff \begin{cases} t<z<T \\ z<u<T\end{cases}.$$

So, finally, we get that

$$\begin{aligned}P(t,T)_{\mathbb{Q}} &= \exp\left\{r(t)_{\mathbb{Q}}B(t,T) + \lambda\int_t^T \theta(z)B(z,T)\mathrm{d}z + \int_t^T \frac{1}{2}\eta^2 B^2(z,T)\mathrm{d}z\right\}\\ &= e^{r(t)_{\mathbb{Q}}B(t,T)+A(t,T)},\end{aligned}$$

where, defining

$$A(t,T) = \lambda\int_t^T \theta(z)B(z,T)\mathrm{d}z + \int_t^T \frac{1}{2}\eta^2 B^2(z,T)\mathrm{d}z,$$

we get the same formula of Equation (2.15).

### BOND PRICE UNDER UNDER T-FORWARD MEASURE

We consider the case in which the ZCB price is under the $T$-forward measure, namely $\mathbb{Q}^T$. Our starting point is the dynamics of the ZCB under $\mathbb{Q}$. After performing a change of measure in agreement with Equation (2.8), we get

$$\begin{aligned}\frac{\mathrm{d}P(t,T)}{P(t,T)} &= r(t)_{\mathbb{Q}}\mathrm{d}t + \eta B_r(t,T)\mathrm{d}W^{\mathbb{Q}}(t)\\ &= \left(r(t)_T + \eta^2 B_r^2(t,T)\right)\mathrm{d}t + \eta B_r(t,T)\mathrm{d}W^T(t).\end{aligned}$$

Under $T$, the Hull-White SDE has this solution

$$r(u)_T = e^{-\lambda(u-t)}r(t)_T + \lambda\int_t^u \hat{\theta}(z,T)e^{-\lambda(u-z)}\mathrm{d}z + \eta\int_t^u e^{-\lambda(u-z)}\mathrm{d}W^T(z),$$

where $\hat{\theta}(t,T)$ has been defined in Equation (2.11). Therefore, following the same reasoning as before, we get that

$$\begin{aligned}P(t,T)_T &= \exp\left\{r(t)_T B(t,T) - \int_t^T \eta^2 B^2(z,T)\mathrm{d}z + \lambda\int_t^T \hat{\theta}(z,T)B(z,T)\mathrm{d}z + \int_t^T \frac{1}{2}\eta^2 B^2(z,T)\mathrm{d}z\right\}\\ &= \exp\left\{r(t)_T B(t,T) + \lambda\int_t^T \theta(z)B(z,T)\mathrm{d}z + \int_t^T \frac{1}{2}\eta^2 B^2(z,T)\mathrm{d}z\right\}\\ &= e^{r(t)_T B(t,T)+A(t,T)}.\end{aligned}$$

Thus, we showed that the functions $A_r(t,T)$ and $B_r(t,T)$ are the same regardless of the measure under which we are considering the ZCB price $P(t,T)$.

## A.2. MORE EVIDENCE ABOUT THE FLOORLET PRICE

Looking at the derivations, we basically stated that

$$V_{\text{Floorlet}}(t_0,T_1,T_2) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_2)}\big(K-L(T_1;T_1,T_2)\big)^+\bigg|\mathcal{F}(t_0)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_1)}P(T_1,T_2)\big(K-L(T_1;T_1,T_2)\big)^+\bigg|\mathcal{F}(t_0)\right].$$

This passage could appear awkward since the bond price is an *expectation*, while here it seems that part of the randomness from $T_1$ to $T_2$ collapses into the the bond price $P(T_1,T_2)$ without using any expected value operator. However, the motivation comes from the Tower Property[1], which states that, considering $Z$ a $\mathcal{F}$-measurable random variable with $\mathbb{E}[|Z|<\infty]$ and $\mathcal{F}\subset\mathcal{G}$, then

$$\mathbb{E}\Big[\mathbb{E}\left[Z|\mathcal{G}\right]\Big|\mathcal{F}\Big]\,\mathbb{E}\left[Z|\mathcal{F}\right] = \mathbb{E}\Big[\mathbb{E}\left[Z|\mathcal{F}\right]\Big|\mathcal{G}\Big],$$

thus, considering that $\mathcal{F}(t_0)\subset\mathcal{F}(T_1)$, it is possible to write

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_2)}\bigg|\mathcal{F}(t_0)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_1)}\frac{M(T_1)}{M(T_2)}\bigg|\mathcal{F}(t_0)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_1)}\frac{M(T_1)}{M(T_2)}\bigg|\mathcal{F}(T_1)\right]\bigg|\mathcal{F}(t_0)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_1)}P(T_1,T_2)\bigg|\mathcal{F}(t_0)\right].\end{aligned}$$

[1] https://en.wikipedia.org/wiki/Law_of_total_expectation

# B

# Model-free calibration of the hedging portfolio

In this Appendix we test a calibration of the swaption portfolio similar to the one proposed in [36]. In the article, the authors managed to replicate a flexi-swap calibrating a set of Bermudan swaptions, without recurring to any model for the interest rates. The derivative they considered only presented an upper and lower bound, and they achieved their result assuming that a partial redemption was not allowed, which means that the notional could either continue with the normal amortization plan, that in their case corresponded to no-amortizaiton, or jump to the lower bound implying a maximum amortization. This assumption reminds the case of the full-rational prepayment rule which, however, we showed being quite unrealistic.

Following a similar approach, we assume that we only know the upper and lower bound of the notional of the IAS, which we choose to make correspond to the minimum and maximum level of prepayment possible according to the Logistic function (3.9) calibrated on the historical data. Thus we have a situation like the one in the left graph of Figure B.1. Now, the idea is to attribute *equally* the maximum decay of the Notional at time $T_k$ to all the possible swaptions that can influence the notional at that time, i.e. we want to impose

$$N_{\text{IAS}}^{\text{Diff}}(T_k) = \sum_{i=1}^{k-1} \sum_{l=k}^{M} w_{T_i,T_l}, \tag{B.1}$$

where

$$\begin{aligned} N_{\text{IAS}}^{\text{Max}}(T_k) &:= \max_j N_{\text{IAS}}^{(j)}(T_k), \\ N_{\text{IAS}}^{\text{Min}}(T_k) &:= \min_j N_{\text{IAS}}^{(j)}(T_k), \\ N_{\text{IAS}}^{\text{Diff}}(T_k) &:= N_{\text{IAS}}^{\text{Max}}(T_k) - N_{\text{IAS}}^{\text{Min}}(T_k). \end{aligned}$$

This results in an underdetermined linear system for which, however, we can find a least square solution.

For example, consider a mortgage with maturity $T_M = 5$. All the possible swaptions that we could use to create the hedging portfolio are shown in Table B.1. Applying Equation (B.1) we end up solving the following underdetermined linear system $\mathbf{A}\hat{\mathbf{w}} = \mathbf{n}$ of Equation (B.2). Since we wanted to compare the results with the other calibrations we proposed, we repeated the procedure in the

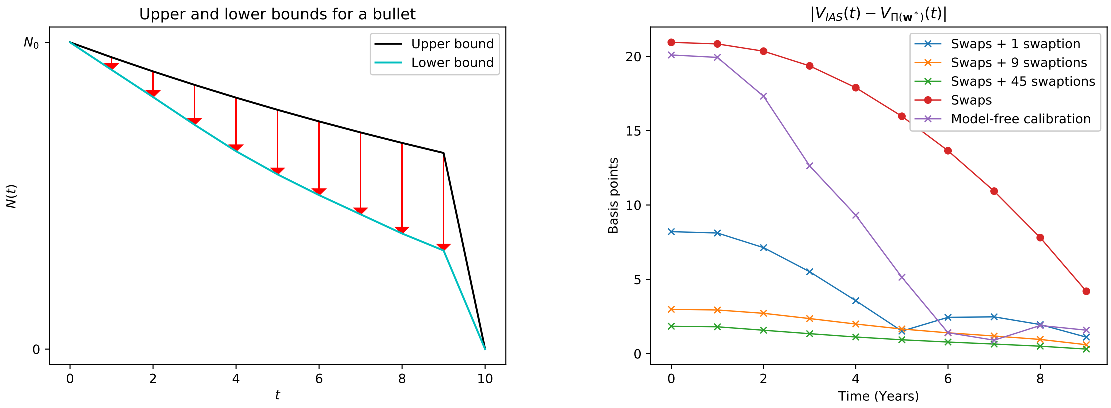



Figure B.1: Left: historical upper and lower bounds of a bullet with the red arrows indicating the gap that the swaptions have to recover. Right: The performance of the

| $T_2$ | $T_3$ | $T_4$ | $T_5$ |
|---|---|---|---|
| $w_{1,2}$ | $w_{1,3}$ | $w_{1,4}$ | $w_{1,5}$ |
| - | $w_{2,3}$ | $w_{2,4}$ | $w_{2,5}$ |
| - | - | $w_{3,4}$ | $w_{3,5}$ |
| - | - | - | $w_{4,5}$ |

Table B.1: Swaptions' notionals to hedge the prepayments of an Annuity with maturity $T = 5$.

case of a bullet with maturity $T_M = 10$ and in the right graph of Figure B.1 the performance of $\Pi(\hat{\mathbf{w}})$ is compared to the one of the other calibrations. Clearly, the other calibrations lead to a better replication and thus to a better addressing of the risks to the correct instruments.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_{1,2} \\ w_{1,3} \\ w_{1,4} \\ w_{1,5} \\ w_{2,3} \\ w_{2,4} \\ w_{2,5} \\ w_{3,4} \\ w_{3,5} \\ w_{4,5} \end{bmatrix} = \begin{bmatrix} N_{\text{IAS}}^{\text{Diff}}(T_2) \\ N_{\text{IAS}}^{\text{Diff}}(T_3) \\ N_{\text{IAS}}^{\text{Diff}}(T_4) \\ N_{\text{IAS}}^{\text{Diff}}(T_5) \end{bmatrix}. \tag{B.2}$$

# C

# Alternative calibration of the hedging portfolio

In this Appendix another minimization function $F(\mathbf{w})$ for the problem (4.4) is tested. We provided an analytical calibration on the paths of the IAS in Proposition 8 for a function of the form 4.6, however this time we will try to minimize the squared errors *between the discounted cash flow of each time*, so also the values of the money-saving account and of the Libor rate are involved in the minimization. Thus, $F(\mathbf{w})$ becomes a vectorial function

$$F(\mathbf{w}) = \begin{bmatrix} \left|\left(N^{(j)}_{\text{IAS}}(T_1) - N^{(j)}_{\Pi(\mathbf{w})}(T_1)\right)(K - L(t_0;t_0,T_1))M(T_1)\right|^2 \\ \dots \\ \left|\left(N^{(j)}_{\text{IAS}}(T_M) - N^{(j)}_{\Pi(\mathbf{w})}(T_M)\right)(K - L(T_{M-1};T_{M-1},T_M))M(T_M)\right|^2, \end{bmatrix} \tag{C.1}$$

and, after simple differentiation respect to the weights $w_{1,M},\dots,w_{M-1,M}$, imposing $\nabla F(\mathbf{w}) = 0$ leads to the following linear system:

$$A = \begin{bmatrix} \sum_{j=1}^{N_S} \frac{\left(K-L^{(j)}_{T_1}(T_2)\right)^2}{M^{(j)2}(T_2)} \mathbb{1}^{(j)}_{1,M} & 0 & 0 & \dots & 0 \\ \sum_{j=1}^{N_S} \frac{\left(K-L^{(j)}_{T_2}(T_3)\right)^2}{M^{(j)2}(T_3)} \mathbb{1}^{(j)}_{1,M}\mathbb{1}^{(j)}_{2,M} & \sum_{j=1}^{N_S} \frac{\left(K-L^{(j)}_{T_2}(T_3)\right)^2}{M^{(j)2}(T_3)} \mathbb{1}^{(j)}_{2,M} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \sum_{j=1}^{N_S} \frac{\left(K-L^{(j)}_{T_{M-1}}(T_M)\right)^2}{M^{(j)2}(T_M)} \mathbb{1}^{(j)}_{1,M}\mathbb{1}^{(j)}_{M-1,M} & \sum_{j=1}^{N_S} \frac{\left(K-L^{(j)}_{T_{M-1}}(T_M)\right)^2}{M^{(j)2}(T_M)} \mathbb{1}^{(j)}_{2,M}\mathbb{1}^{(j)}_{M-1,M} & \dots & & \sum_{j=1}^{N_S} \frac{\left(K-L^{(j)}_{T_{M-1}}(T_M)\right)^2}{M^{(j)2}(T_M)} \mathbb{1}^{(j)}_{M-1,M} \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} w_{1,M} & w_{2,M} & \dots & w_{M-1,M} \end{bmatrix}^T,$$

$$\mathbf{b} = \begin{bmatrix} \sum_{j=1}^{N_S} \left(\bar{N}(T_1) - N^{(j)}_{\text{IAS}}(T_1)\right)\left(K - L^{(j)}_{T_1}(T_2)\right)^2 M^{(j)2}(T_2)\mathbb{1}^{(j)}_{1,M} \\ \sum_{j=1}^{N_S} \left(\bar{N}(T_2) - N^{(j)}_{\text{IAS}}(T_2)\right)\left(K - L^{(j)}_{T_2}(T_3)\right)^2 M^{(j)2}(T_3)\mathbb{1}^{(j)}_{2,M} \\ \vdots \\ \sum_{j=1}^{N_S} \left(\bar{N}(T_{M-1}) - N^{(j)}_{\text{IAS}}(T_{M-1})\right)\left(K - L^{(j)}_{T_{M-1}}(T_M)\right)^2 M^{(j)2}(T_M)\mathbb{1}^{(j)}_{M-1,M} \end{bmatrix},$$

where, for simplicity,

$$\mathbb{1}^{(j)}_{i,M} = \mathbb{1}_{\{K>\kappa^{(j)}(T_i)\}} \quad \text{and} \quad L^{(j)}(T_{i-1};T_{i-1},T_i) = L^{(j)}_{T_i}.$$

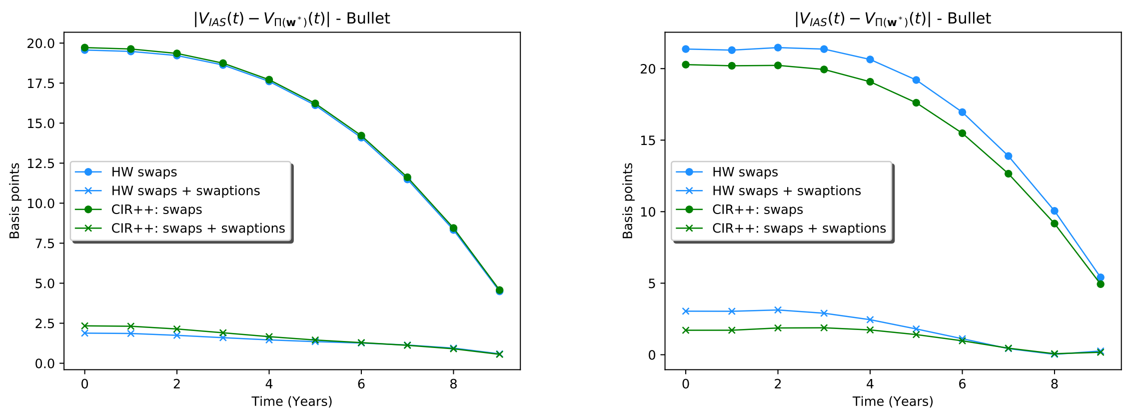



Figure C.1: Comparison of the hedging performances in the replication of the price of the IAS with the bullet characteristics. The calibration is performed according to (4.6) on the left or to (C.1) on the right.

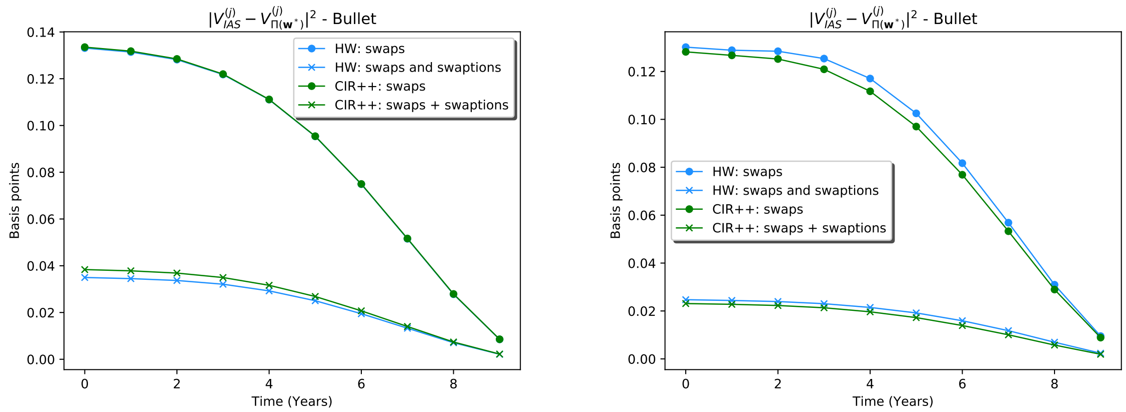



Figure C.2: Comparison of variances returned by the two different calibration for a IAS with the bullet characteristics. Left: (4.6). Right: (C.1). It is important to notice that on the right graph $\Pi(\mathbf{w}^*_{\text{CIR}})$ outperforms $\Pi(\mathbf{w}^*_{\text{HW}})$ *at each time.*

Analogously, also the linear hedge can be improved to consider also $M(t)$ and $L(t;t,T)$ and also comply with (C.1):

$$N_{\text{AS}}(T_k) = \frac{\sum_{j=1}^{N_{Sim}} N_{\text{IAS}}^{(j)}(T_k) \cdot \left(K - L_{T_k}^{(j)}\right)^2 M^{(j)2}(T_k)}{\sum_{j=1}^{N_{Sim}} \left(K - L_{T_k}^{(j)}\right)^2 M^{(j)2}(T_k)},$$

for $k = 1, ..., M-1$. The results are for the values are shown in Figure C.1 and, more important, the performance on the squared errors path-wise are reported in Figure C.2. It should be clear that the minimization (C.1) (in the right graph) effectively minimizes the squared errors each time, in fact, the squared errors of $\Pi(\mathbf{w}^*_{\text{HW}})$ are greater than the ones for $\Pi(\mathbf{w}^*_{\text{CIR}})$ at each time-step, validating our reasoning. Note that the variance has been divided by the squared notional and multiplied by 10000 in order to have a basis-points representation.

# BIBLIOGRAPHY


[1] B. J. Alink, *Mortgage prepayments in the Netherlands*, Ph.D. thesis, PhD-Thesis. Twente University (2002).

[2] NVB, *The dutch mortgage market,* Nederlandse Vereniging van Banken (2014).

[3] Fitch, *Fitch residential mortgage briefing,* London: Fitch Ratings (2013).

[4] Moody's, *Flexible mortgage lending to self-employed borrowers is credit positive for dutch rmbs,* London: Moody's (2014).

[5] CBS, *Trends in the netherlands,* Statistics Netherlands (2017).

[6] A. Kolbe, *Valuation of mortgage products with stochastic prepayment-intensity models*, Ph.D. thesis, PhD thesis, Technical University of Munchen, Centre for Mathematics (2008).

[7] J. Jacobs, R. Koning, and E. Sterken, *Modelling prepayment risk,* Dept. Economics, University of Groningen (2005).

[8] S. Perry, S. Robinson, and J. Rowland, *A study of mortgage prepayment risk,* Housing Finance International **16**, 36 (2001).

[9] M. Consalvi and G. S. di Freca, *Measuring prepayment risk: an application to UniCredit Family Financing*, Tech. Rep. (Working Paper Series, UniCredit&Universities, 2010).

[10] F. J. Fabozzi, *Mortgage and mortgage-backed securities markets* (Harvard Business School Press, 1992).

[11] E. Charlier and A. Van Bussel, *Prepayment behavior of dutch mortgagors: an empirical analysis,* Real estate economics **31**, 165 (2003).

[12] S. Z. Hoda and R. X. Kee, *Implementation of a mortgage backed security (mbs) pricing model,* Princeton University (2007).

[13] I. Bandic, *Pricing mortgage-backed securities and collateralized mortgage obligations,* M. sc. Essay (2004).

[14] P. J. M. van Bussel, *Valuation and interest rate risk of mortgages in the netherlands,* Maastricht University (1998).

[15] M. Bissiri and R. Cogo, *Modeling behavioral risk,* Social Science Research Network (2014).

[16] S. F. Richard and R. Roll, *Prepayments on fixed-rate mortgage-backed securities,* The Journal of Portfolio Management **15**, 73 (1989).

[17] A. Kalotay, D. Yang, and F. J. Fabozzi, *An option-theoretic prepayment model for mortgages and mortgage-backed securities,* International Journal of Theoretical and Applied Finance **7**, 949 (2004).

[18] A. Castagna and F. Fede, *Measuring and managing liquidity risk* (John Wiley & Sons, 2013).

[19] F. Ahmad, B. Hambly, and S. Ledger, *A stochastic partial differential equation model for the pricing of mortgage-backed securities,* Stochastic Processes and their Applications (2018).

[20] F. Fabozzi, *Fixed Income Mathematics, 4E* (McGraw-hill, 2005).

[21] D. Brigo and F. Mercurio, *Interest rate models-theory and practice: with smile, inflation and credit* (Springer Science & Business Media, 2007).

[22] J. Hull and A. White, *Pricing interest-rate-derivative securities,* The Review of Financial Studies **3**, 573 (1990).

[23] D. Duffie, J. Pan, and K. Singleton, *Transform analysis and asset pricing for affine jump-diffusions,* Econometrica **68**, 1343 (2000).

[24] F. Jamshidian, *An exact bond option formula,* The journal of Finance **44**, 205 (1989).

[25] F. Black, *The pricing of commodity contracts,* Journal of financial economics **3**, 167 (1976).

[26] B. H. Kuijpers and P. C. Schotman, *Optimal prepayment of dutch mortgages,* Statistica Neerlandica **61**, 137 (2007).

[27] M. Sherris, *Pricing and hedging loan prepayment risk*, Tech. Rep. (International Actuarial Association, 1993).

[28] K. B. Dunn and J. McConnell, *A comparison of alternative models for pricing gnma mortgage-backed securities,* The Journal of Finance **36**, 471 (1981).

[29] R. Stanton, *Rational prepayment and the valuation of mortgage-backed securities,* The Review of financial studies **8**, 677 (1995).

[30] J. B. Kau, D. C. Keenan, and A. A. Smurov, *Reduced-form mortgage valuation,* University of Georgia, Georgia (2004).

[31] M. Svenstrup, *Interest Rate Derivatives-Valuation and Applications*, Ph.D. thesis, PhD thesis, The Aarhus Business School (2002).

[32] C. S. Sirmans, S. D. Smith, and G. S. Sirmans, *Determinants of mortgage interest rates: Treasuries versus swaps,* The Journal of Real Estate Finance and Economics **50**, 34 (2015).

[33] D. Rich, *Modelling fixed income securities and interest rate options,* (1996).

[34] L. N. Galaif *et al.*, *Index amortizing rate swaps,* Notes **100**, 30 (1993).

[35] J. D. Fernald *et al.*, *The pricing and hedging of index amortizing rate swaps,* FRBNY Quarterly Review , 71 (1993).

[36] F. Jamshidian and I. Evers, *Replication of flexi-swaps,* Journal of risk and uncertainty , 67 (2005).

[37] M. Joshi, *Pricing discretely sampled path-dependent exotic options using replication methods,* Quantitative Research Centre, Royal Bank of Scotland Group Risk, Working paper (2001).

[38] P. Carr, K. Ellis, and V. Gupta, *Static hedging of exotic options,* The Journal of Finance **53**, 1165 (1998).