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ABSTRACT

This work addresses the problem of pattern analysis in networks consisting of delay-coupled identical Lur'e systems. We study a class of nonlinear systems, which, being isolated, are globally asymptotically stable. Assembling such systems into a network via time-delayed coupling may result in the change of network equilibrium stability under parameter variation in the coupling. In this work, we focus on cases where a Hopf bifurcation causes the change of stability of the network equilibrium and leads to the occurrence of oscillatory modes (patterns). Moreover, some of these patterns can co-exist for the same set of coupling parameters, which makes the analysis by means of common methods, such as the Lyapunov–Krasovskii method or the analysis of Poincaré maps, cumbersome. A numerically efficient algorithm, aiming at the computation of the oscillatory patterns occurring in such networks, is presented. Moreover, we show that our approach is able to deal with co-existing patterns, and both stable and unstable regimes can be simultaneously computed, which gives deep insight into the network dynamics. In order to illustrate the efficiency of the method, we present two examples in which the instability of the network equilibria is caused by a subcritical and a supercritical Hopf bifurcation. In addition, a bifurcation analysis of the subcritical case is performed in order to further explain the occurrence of the detected coexisting modes.

We consider the dynamics of delay-coupled nonlinear systems of Lur'e type. Although each system is globally asymptotically stable in the absence of coupling, the collection of such systems may induce an oscillatory behavior. In our work, we focus on the case where oscillatory patterns arise from a Hopf bifurcation. The bifurcation can be either subcritical and there is an unstable limit cycle in the neighborhood of the bifurcation point, or supercritical, which results in a stable limit cycle. Two parameters in the coupling function, namely, coupling strength γ and time delay τ, take the role of bifurcation parameters. Networks with two (or more) bifurcation parameters show rich and complex dynamical behavior. This implies that basic bifurcation analysis is required in order to understand network behavior. Moreover, the co-existence of several modes (patterns) often occurs in such networks, which makes the usage of common methods of analysis, such as the Lyapunov–Krasovskii method and the analysis of Poincaré maps, cumbersome. In this work, a numerically efficient method for detecting (coexisting) oscillatory patterns (stable and unstable) in delay-coupled Lur'e systems is presented. We develop an approach based on the multivariable harmonic balance (MHB) method.[1] The idea of the method is to turn the problem of determination of an oscillatory profile (offset, amplitudes, phases, and frequency) into an eigenvalue problem. Solving the eigenvalue problem for chosen parameters γ and τ, by numerically efficient optimization routine, structurally possible patterns can be simultaneously found.

I. INTRODUCTION

The investigation of complex networks formed by coupled nonlinear dynamical systems has been an important subject in mathematical biology, control theory, applied physics, and interdisciplinary fields in recent years. Complex networks are prevalent in the world and can be used to describe the behavior of neuronal systems, heart cells synchronization, social networks, flocks of birds, and other kinds of collective behavior.[2,3] Complex networks exhibit rich dynamical behavior, with synchronized or partially synchronized motion being only a few out of many possibilities. To explore the complex behavior that can occur, bifurcation analysis is essential, as it allows to characterize the behavior that appears due to parameter variations.[4] As such, it brings an insight into complex behavior that may occur near well-understood dynamical regimes, such as the synchronized state.[5] A second motivation to investigate bifurcations in networked systems is to assess the robustness of a regime to parameter variations and to identify structurally possible regimes that are defined by a network topology. While a well-developed theory for bifurcations of nonlinear dynamical systems with delays exists,[6,7] these results are restricted to low-dimensional systems and do not exploit the network structure. About one decade later, theoretical bifurcation results were pursued that take into account the structure in the networks.[8] It is possible to exploit the theoretical results for the development of numerically efficient software to assess the emergent behavior of complex systems and explore different oscillatory regimes occurring in such systems.

Historically, the problem of the emergence of oscillations in coupled systems (systems that are globally asymptotically stable when the coupling is absent) arises from Turing's work on morphogenesis.[9] About two decades later, a nonlinear analysis of Turing instability or diffusion-driven instability was performed by Smale.[10] An example of two identical diffusively coupled fourth order systems is studied. Each individual system takes the role of a cell, which is inert or "dead" in the sense that it is globally asymptotically stable in isolation. However, when the cells are interconnected via diffusive coupling, "the cellular system pulses (or expressed perhaps overdramatically, becomes alive!) in the sense that the concentration of enzymes in each cell will oscillate infinitely." The author posed the question of determining conditions under which an oscillatory behavior in networks of initially globally asymptotically stable systems can be caused by diffusive coupling. A partial answer to this question was given in Ref. 11. The authors studied the dynamics of two diffusively coupled Lur'e systems using frequency domain methods and showed that diffusion-driven oscillations are possible with third-order systems. In Ref. 12, it was proven that the emergence of diffusion-driven oscillations from a unique equilibrium, as a result of the first bifurcation, is not possible in systems of an order lower than three. Moreover, in that same paper, conditions for the emergence of diffusion-driven oscillations were presented. Results for different synchronous regimes, such as synchronization,[13,14] partial synchronization,[15,16] and pattern formation including standing, traveling, and rotating waves,[1,17,18] are found in given references.

The above-mentioned studies all consider diffusive coupling,[19] which is (usually) symmetric and delay-free. In this work, we focus on pattern analysis of delay-coupled systems and study networks of Lur'e systems interconnected via linear time-delayed coupling functions. More precisely, the coupling function for a single system in a network is defined to be the weighted difference of the time-delayed output of its neighbors and its own non-delayed output. Physically, the adopted delay model represents the time that a signal takes to propagate from its source to its destination, and, therefore, the systems have access to their own outputs immediately. Such time-delay coupling functions appear, among others, in the network of neurons,[20] electrical circuits,[21] and networked control systems.[22] As a starting point for this research, the work of Ref. 23 is used. The authors derived conditions under which a network of nonlinear dynamical systems which are globally asymptotically stable by themselves, being interconnected via linear time-delay coupling functions, display oscillatory behavior and different patterns can emerge, such as partially synchronous oscillations and rotating waves. The (network) equilibrium loses its stability via a Hopf bifurcation, which leads to the pattern formation. The Hopf bifurcation changes the dynamics of the network from a stable equilibrium to oscillatory patterns. As a corollary to these results, the authors have shown that a network of inert systems with time-delay coupling can be globally oscillatory (considering the case of a subcritical Hopf bifurcation) only if the systems are at least of second order.

In this paper, we develop a numerically efficient method aiming at detecting co-existing oscillatory modes (patterns) even if some of them are unstable, in networks of nonlinear systems of Lur'e form interconnected via linear time-delayed coupling functions. Lur'e models are widely used in many fields (e.g., computational biology, control theory, etc.) and simplify the analysis by means of the harmonic linearization due to a scalar nonlinearity. The harmonic function approximation is characterized by offset components, amplitudes, phases, and a frequency of oscillations. We consider the case where a Hopf bifurcation of both types (subcritical and supercritical) causes the instability of the network dynamics. The coupling strength and time-delay term take the role of bifurcation parameters. A challenge is that there may be co-existence of different oscillatory patterns for the same parameters (coupling strength and time delay), e.g., stable relaxation oscillations and unstable harmonic oscillations. Due to the co-existence of different modes, the application of common methods for the analysis of delay-coupled system behavior, such as the Lyapunov–Krasovskii method and the analysis of Poincaré maps, becomes highly non-trivial.

To solve these problems, we present an extension of the multivariable harmonic balance (MHB) method for delay-coupled systems and elaborate it by a tool for bifurcation analysis called ddebiftool. The method allows us to analyze the behavior of the complex network and to determine an oscillatory profile that approximates the output of the studied network without simulating delay differential equations (DDEs), which is time-consuming for networks of many agents. The method presented in this paper extends the approach presented in Ref. 1 toward networks with a time-delay coupling and shows additional possibilities of the multivariable harmonic balance method, such as the simultaneous detection of stable and unstable coexisting periodic solutions. The MHB approach is numerically efficient and scalable due to the exploitation of a network structure and allows us to compute all co-existing regimes, even if some of them are unstable. The results of the MHB analysis can be also used as an input for continuation tools (e.g., DDE-BIFTOOL, AUTO-07P, MATHCONT), since they require accurate initial conditions.

It is worth mentioning that the conventional harmonic balance method is sometimes considered as a rather empirical approach. This issue is overcome at least for scalar nonlinearities by the mathematical analysis in Refs. 24–26. The extension of these results to the MHB method, which facilitates the study of the oscillatory networks, is an open and interesting problem, which lies outside the scope of this paper. This paper focuses on the numerical efficiency of the MHB method. In addition, we provide numerical evidence that the MHB method can accurately predict the oscillation profiles.

This paper is organized as follows. Preliminary results and the problem statement are given in Sec. II. More precisely, in Subsection II A, networks of Lur'e systems with time-delay coupling are introduced. Conditions for the occurrence of oscillations are presented in Subsection II B. We give some details about two types (subcritical and supercritical) of a Hopf bifurcation and show how to compute analytically critical values of the time-delay component in Subsection II C. The problem is stated in Subsection II D. In Sec. III, the harmonic balance method is introduced and the MHB equation for delay-coupled systems is derived. Numerical examples are provided in Sec. IV. The latter includes the results for both the supercritical Hopf bifurcation case and the subcritical Hopf bifurcation case. In Sec. V, the conclusions are given.

Throughout the paper, the following notations are used. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. j stands for the imaginary unit, $j^2 = -1$. Notation $\mathbb{C}_+$ corresponds to the open right half plane, $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. For a positive integer k, we let I_k denote the identity matrix of the size $k \times k$ and $\mathbf{1}_k$ denotes the column vector of length k with all entries equal to 1. The symbol $\otimes$ stands for the Kronecker product and $\oplus$ stands for the Kronecker sum (i.e., $A \oplus B = A \otimes I_B + I_A \otimes B$). Symbols $^\top$ and * stand for the transposition and conjugate transposition, respectively. Given positive integers p, q, for $\mathscr{X} \subset \mathbb{R}^p$ and $\mathscr{Y} \subset \mathbb{R}^q$, we denote the space of C^r r-times continuously differentiable functions from $\mathscr{X}$ into $\mathscr{Y}$ as $\mathscr{C}^r(\mathscr{X}, \mathscr{Y})$.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Networks of Lur'e systems with delays

We consider a network of identical delay-coupled systems. N single-input-single-output (SISO) nonlinear systems of Lur'e type are and described by following equations:

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \\ u_i &= u_{ci} + \psi(y_i), \end{aligned} \tag{1}$$

where $i \in 1, \ldots, N$ stands for the system index, $x_i \in \mathbb{R}^n$ is the state of ith system, $u_i \in \mathbb{R}$ is the combined input of the ith system, which consists of two components: $\psi(\cdot)$, which is a C^1 scalar nonlinear time-invariant memoryless function, and $u_{ci} \in \mathbb{R}$, which is the input receiving data from the coupling of multiple systems. $y_i \in \mathbb{R}$ is the output of the ith system used for the coupling and A, B, and C are constant matrices of appropriate dimension with CB being a positive constant. In addition, we make the following assumption.

Assumption 1. The isolated ($u_{ci} \equiv 0$) system (1), described by $\dot{x}_i := f(x_i) := Ax_i + B\psi(Cx_i)$, has a unique equilibrium point x_{i0}, which is globally asymptotically stable and locally exponentially stable.

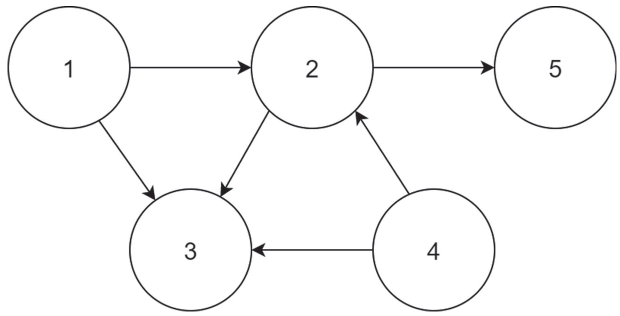

FIG. 1. An example of the directed network consisting of dynamical systems.

We remark that local exponential stability of the equilibrium is equivalent to the Jacobian matrix $J_0 = J(x_{i0}) = \frac{\partial f}{\partial x_i}(x_{i0})$ at x_{i0} being Hurwitz, i.e., all eigenvalues of J_0 have negative real parts.

The N systems (1) are interconnected via linear time-delay coupling functions of the form

$$u_{ci}(t) = \gamma \sum_{l} \gamma_{il}[y_l(t-\tau) - y_i(t)], \tag{2}$$

where positive constant γ stands for the coupling strength, positive constant τ stands for the time delay, and non-negative constants $\gamma_{il} \in [0, 1]$ are the interconnection weights. We remark that γ_{il} is positive if and only if there is a connection from system l to system i. Graphically, the connection between the nodes l and i is represented by an edge (arrow) between l and i (see Fig. 1). Positive constants γ and τ will be used as bifurcation parameters later on.

In order to specify the interaction structure of the network, we define the (weighted) adjacency matrix of the graph $\Gamma = (\gamma_{il}) \in \mathbb{R}^{N\times N}$ as

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} & \ldots & \gamma_{1k} \\ \gamma_{21} & 0 & \ldots & \gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k1} & \gamma_{k2} & \ldots & 0 \end{pmatrix}. \tag{3}$$

We allow the graph to be directed in sense that all the edges are directed from one node to another. The graph is assumed to be simple in the sense that self-connections and multi[al edges joining the same pair of nodes are forbidden. Moreover, we assume that every pair of systems can be joined by a sequence of directed edges, which is equivalent to strong connectivity of the graph. In addition, the following assumption is made.

Assumption 2. The sum of each row of the matrix Γ equals 1.

Previously, this assumption was used for the delay-coupled nonlinear system analysis in Refs. 27 and 28, and it was used to ensure that a synchronous (oscillatory) state exists in such networks. Moreover, according to the Gershgorin Disc Theorem,[29] Assumption 2 implies that all eigenvalues of Γ are located in the closed unit disc in $\mathbb{C}$.

B. Conditions for oscillations

The goal of this subsection is to present conditions that guarantee that the network (1) and (2) exhibits oscillatory behavior. Let $\sigma \in \mathscr{C}([0,\infty],\mathbb{R})$ be uniformly bounded. Such a function is oscillatory (in the sense of Yakubovich) if $\lim_{t\to+\infty}\sigma(t)$ does not exist.[30] Likewise, we say that a system is oscillatory if it admits the following properties: (1) all its solutions are uniformly (ultimately) bounded $[0,\infty)$ and (2) the system has a finite number of hyperbolically unstable equilibria. Note that nonlinear systems (1) connected via coupling (2) are described by Delay Differential Equations (DDEs). An equilibrium solution of a DDE is called hyperbolic if the roots of its characteristic equation have nonzero real part.[31]

Since the goal of this work is to analyze oscillatory patterns in the networks, following Ref. 23, we establish conditions for oscillatory behavior of networks (1) and (2). Suppose Assumptions 1 and 2 hold true and

- the solutions of the network (1) and (2) are uniformly bounded and uniformly ultimately bounded;[32]
- networks (1) and (2) have a unique but hyperbolically unstable equilibrium at $x_0 = \mathbf{1}_N \otimes x_{10}$.

Under these conditions, the coupled systems will exhibit oscillatory behavior. Uniqueness of the equilibrium is not necessary for the existence of oscillations. However, the stability analysis of additional equilibria could be cumbersome as the locations of these equilibria heavily depend on the overall coupling strength γ. In addition, for a unique equilibrium, the coupled systems can be oscillatory only if one of outputs is an oscillatory function of time. In case none of the outputs is an oscillatory function, the value of each coupling function converges to zero and the system cannot be oscillatory by Assumption 1.

We now discuss conditions under which the solutions are bounded. We consider a single system (1) and let $u_i(\cdot)$ be a piecewise continuous input function defined on $[0,T)$, $T\in\mathbb{R}_+$, and taking values in a compact set $\mathscr{U}\subset\mathbb{R}$. Let $x_i(\cdot)=x_i(\cdot,x_{i0},u_i[0,T))$ be the solution of system (1) corresponding to input $u_{ci}(\cdot)$ and initial conditions x_{i0} and $t=0$. Then, a (strictly) $\mathscr{C}^r$-semipassive system is defined as follows.

Definition 1. Assume that there is storage function $S\in\mathscr{C}^r(\mathbb{R}^n,\mathbb{R}_+\cup\{0\})$, such that

$$S(x_i(t)) - S(x_i(0)) \le \int_0^t [(y_i u_i(s)) - H(x_i(s))]\,\mathrm{d}s, \tag{4}$$

with $H\in\mathscr{C}(\mathbb{R}^n,\mathbb{R})$ and $t\in(0,T]$. If there is a constant $R>0$ and a non-negative nondecreasing function $h:\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+\cup\{0\}$ such that

$$H(s) \ge h(|s|),\ \forall |s|\ge R, \tag{5}$$

then system (1) is called $\mathscr{C}^r$-semipassive. If (5) holds for all $|s|\ge R$ and function h is strictly increasing and such that $h(s)\to\infty$ as $s\to\infty$, then system (1) is strictly $\mathscr{C}^r$-semipassive.

Lemma 1 (Boundedness). *Let w_0, $w_1:[0,\infty)\to[0,\infty)$ be strictly increasing functions, which satisfy $w_0(0)=w_1(0)=0$ and $w_0(s),w_1(s)\to\infty$ as $s\to\infty$. Suppose that each system (1) is strictly $\mathscr{C}^1$-semipassive with storage function S that satisfies*

$$w_0(|x_i(t)|) \le S(x_i(t)) \le w_1(|x_i(t)|).$$

Then, for each fixed γ and τ, the solutions of the coupled systems (1) and (2) are uniformly bounded and uniformly ultimately bounded.

The proof of this lemma can be found in Ref. 33. Note that, unlike in the Ordinary Differential Equation (ODE) (finite-dimensional) case, uniform ultimately boundedness does not imply uniform boundedness in the DDE case.[34]

In order to verify the second condition, we need to show that the equilibrium point x_0 is unique and unstable. Using Assumption 2, we can write the network dynamics as

$$\dot{x}(t) = F(x(t)) + \gamma[(\Gamma\otimes BC)x(t-\tau) - (I_N\otimes BC)x(t)]. \tag{6}$$

The linearization of (6) around the network equilibrium x_0 results in

$$\dot{\hat{x}}(t) = [I_N\otimes(J_0-\gamma BC)]\hat{x}(t) + (\gamma\Gamma\otimes BC)\hat{x}(t-\tau). \tag{7}$$

It is well known that the zero solution of the linear system (7) is unstable for some γ and τ if its associated characteristic equation

$$\Delta(\lambda;\gamma,\tau)=0, \tag{8}$$

with

$$\Delta(\lambda;\gamma,\tau) := \det(\lambda I_{Nn} - I_N\otimes(J_0-\gamma BC) - (\gamma\Gamma\otimes BC)\,e^{-\lambda\tau}),$$

which has a root in $\mathbb{C}_+$.[35] However, computing the roots of the characteristic equation in the (γ,τ)-parameter space is cumbersome for large N. To address this issue, we use sufficient conditions for instability of the network equilibrium presented in Ref. 23. In order to do so, we denote by

$$\mathscr{H}(s) := C(sI_n - J_0)^{-1}B = \frac{p(s)}{q(s)},$$

the transfer function from u_{ci} to y_i of system (1), linearized around its equilibrium. Here, p and q are polynomials of degree $n-1$ and n, respectively, because system (1) has relative degree one ($CB>0$). The following two lemmas are reported in Ref. 23:

Lemma 2 (Instability). *Suppose that Assumption 2 holds true. Let*

$$\nu := \inf_{\omega>0}\mathscr{R}(\mathscr{H}(j\omega)).$$

If $\nu<0$, then for each $\gamma\ge\frac{-1}{2\nu}$, there exists a $\tau>0$ such that the characteristic equation (8) has a root in $\mathbb{C}_+$.

Note that using Lemma 2, a value of $\gamma_{min} := \frac{-1}{2\nu}$ can be computed. It is important to mention that the condition for instability is delay-dependent. We continue with conditions for uniqueness of the network equilibrium.

Lemma 3 (Uniqueness of the network equilibrium). *Let Assumption 1 hold true and denote the eigenvalues of Γ by λ_i^Γ, $i=1,\ldots,N$. Let λ^* be the smallest real-valued eigenvalue of Γ. Choose $\bar{\gamma}\in(0,\infty]$ as the largest number for which matrix*

$$J(\chi) - \gamma(1-\lambda^*)BC$$

is nonsingular for all $\chi\in\mathbb{R}^n$ and all $\gamma\in[0,\bar{\gamma})$ with the Jacobian matrix $J(\chi)=\frac{\partial f}{\partial x_i}(\chi)$. Then, the network equilibrium solution $x_0=1_N\otimes x_{10}$ is the unique equilibrium solution of (6) for $\gamma\in[0,\bar{\gamma})$.

Using the conditions presented in Lemmas 2 and 3, one can easily determine a range of coupling strengths $\gamma \in [\gamma_{min}, \bar{\gamma})$ for which there exists delay values such that oscillations are present.

C. Critical values of time delay

In this subsection, we discuss two types of Hopf bifurcations that cause the instability of the network equilibria and may lead to pattern formation in oscillatory networks and show how to compute critical time delays. It will be shown how to detect Hopf bifurcation in such networks for the bifurcation parameter space (coupling strength γ and time delay τ). Networks (1) and (2) are described by DDEs of retarded type. We remark that for the considered class of delay systems, even though they are infinite-dimensional, the Hopf bifurcation theorem holds, and the center manifold corresponding to a Hopf bifurcation is finite-dimensional. Accordingly, when a linearized equilibrium becomes unstable, the number of unstable modes is always finite. This allows a direct extension of tools for finite-dimensional systems to detect/compute bifurcations and periodic solutions nearby.[31]

The Hopf bifurcation results in the birth of a limit cycle from an equilibrium in dynamical systems. For ODEs and DDEs, the Hopf bifurcation is characterized by eigenvalues crossing the imaginary axis. The bifurcation can be supercritical or subcritical, resulting in stable or unstable limit cycle, respectively. According to normal form theory (see Ref. 36, and references therein), oscillations are sinusoidal-like near the bifurcation point. It justifies using the describing function method to replace the nonlinearity by its gain approximation and then to analyze the system of nonlinear (die to nonlinear dependence of the gain on the offset and amplitudes) equations by means of the harmonic balance method.

In order to determine bifurcation points, we need to compute critical values of τ_c for γ_{min}. In our approach, we fix γ first and then compute critical values of the delay parameter, values for which the linearized system has imaginary axis eigenvalues. The critical delays of a DDE can be computed by solving a nonlinear two-parameter eigenvalue problem.[37] The solution of this two-parameter problem can be translated to solving a quadratic eigenvalue problem. Recall the characteristic equation (8). In order to compute critical delays needed to find bifurcation points, γ is fixed at $\gamma = \gamma_{min}$. One substitutes λ by $j\omega$, with $\omega \geq 0$.

Define the time-delay component as follows:

$$\mathscr{P} := \exp(-j\omega\tau_c). \tag{9}$$

Solve a quadratic eigenvalue problem for $\mathscr{P}$

$$\det\left[\left(\mathscr{A} + \mathscr{B}\mathscr{P}\right) \oplus \left(\mathscr{A}^\top + \mathscr{B}^\top\left(\frac{1}{\mathscr{P}}\right)\right)\right] = 0, \tag{10}$$

with $\mathscr{A} = I_n \otimes (J_0 - \gamma_{min}BC)$ and $\mathscr{B} = (\gamma_{min}\Gamma) \otimes (BC)$ and select the eigenvalues on the unit circle. For each such eigenvalue $\mathscr{P}$, we compute the imaginary axis eigenvalues of $\mathscr{A} + \mathscr{B}\mathscr{P}$. Next, the delay values τ_c are derived from (9) for each pair $(\mathscr{P}, \omega)$. Note that there are infinitely many values, separated by $2\pi/\omega$. The key of this approach is that we do not need to compute eigenvalues of a delay system, only of matrices.

Each pair (γ_{min}, τ_c) is a Hopf bifurcation point. Since we have a parameter space of dimension two (coupling strength γ and time delay τ), it is convenient to combine the presented above approach with the numerical computation of bifurcation curves. In order to compute bifurcations curves, it is necessary to employ a numerical bifurcation analysis tool for delay systems such as ddebiftool.[38]

D. Problem statement

Networks, whose individual dynamics and coupling functions are represented by (1) and (2), respectively, exhibit oscillatory behavior with a certain pattern and this pattern is a function of the parameters γ and τ. Such oscillatory patterns are characterized by a set of parameters (oscillatory profiles): offsets α_0 amplitudes α, phases ϕ, and frequency ω. These patterns, either stable or unstable, may co-exist for the same values of parameters γ and τ. Since we consider the case where oscillations appear via Hopf bifurcation, either subcritical, or supercritical, the use of an approach based on the harmonic balance method is justified. The harmonic balance method helps us to avoid the direct Lyapunov method, which is cumbersome when we deal with co-existing periodic solutions. Using the harmonic balance method, it is possible to compute the oscillatory profile and approximate oscillations by

$$y(t) \cong \alpha_0 + \alpha\sin(\omega t + \phi_0 + \phi). \tag{11}$$

Moreover, the harmonic balance method is numerically efficient, it can be used as additional tool in terms of bifurcation analysis, and it provides good oscillatory profiles for continuation procedures. In this paper, we consider the problem of deriving oscillatory profiles and computing approximations (11) of all patterns of oscillations in a given network.

III. MULTIVARIABLE HARMONIC BALANCE FOR DELAY COUPLED SYSTEMS

In this section, the harmonic balance method is introduced and extended to networks consisting of nonlinear dynamical systems of Lur'e type with linear time-delay coupling. The method allows us to compute an approximation to a periodic solution of ODEs and DDEs. We show how the Harmonic balance equation for a SISO nonlinear system of Lur'e type is derived. Consider the following system:

$$\begin{aligned} \dot{\xi}(t) &= A_L\xi(t) + B_L w(t), \\ w(t) &= \zeta(\eta(t)), \\ \eta(t) &= C_L\xi(t), \end{aligned} \tag{12}$$

where $\xi(t)$ is the state of the system, $w(t)$ is the input of the system, $\eta(t)$ is the output of the system, $\zeta(\cdot)$ is a continuously differentiable scalar nonlinear, time-invariant, memoryless function, and A_L, B_L, and C_L are constant matrices of appropriate dimension with $C_LB_L > 0$.

Suppose $\eta(t)$ is ω-periodic and let

$$\eta(t) = \sum_{k=-\infty}^{+\infty} a_k e^{kj\omega t}, \quad a_k = \bar{a}_{-k}. \tag{13}$$

Using Fourier series, the system's input is rewritten as follows:

$$w(t) = \zeta(\eta(t)) = \sum_{k=-\infty}^{+\infty} c_k\, e^{kj\omega t}, \quad c_k = \bar{c}_{-k}. \tag{14}$$

We seek for a set of Fourier coefficients a_k, c_k and frequency ω, which satisfy the system's equations. Using the orthogonality of the functions $\{\exp(kj\omega t)\}_k$, on an interval of length ω, we find that the Fourier coefficients a_k and c_k must satisfy

$$a_k = W(kj\omega)c_k \tag{15}$$

for all integers k and $W(s) = C_L(sI - A_L)^{-1}B_L$, which is the transfer function representing the linear time-invariant part of system (12). Because $W(kj\omega) = \bar{W}(-kj\omega)$, $a_k = \bar{a}_{-k}$, and $c_k = \bar{c}_{-k}$, we need only look at (15) for $k \geq 0$. Since $W(kj\omega)$ is strictly proper, most information of the system behavior is contained in the first harmonics ($k = 1$). Moreover, when the nonlinearity is not odd, an offset component related to $k = 0$ appears. According to this, we consider equations for $k = 0, 1$.[24] Relations between coefficients a_k and c_k can be established via the linear part of the system, which holds (due to the separation principle) frequency wise, and via the expression $w(t) = \zeta(\eta(t))$ which is approximated using the describing functions. Using these relations, we can rewrite Eq. (15) as follows:

$$\begin{aligned} [1 - W(0)K_0(a_0, a_1)]\, a_0 &= 0, \\ \left[1 - W(j\omega)K_1(a_0, a_1)\right] a_1 &= 0, \end{aligned} \tag{16}$$

where $K_0(a_0, a_1)$ and $K_1(a_0, a_1)$ stand for describing functions which are computed as follows:

$$\begin{aligned} K_0(a_0, a_1) &= \frac{c_0}{a_0} = \frac{1}{2a_0\pi}\int_0^{2\pi} \zeta(\eta(\omega t))\, \mathrm{d}(\omega t), \\ K_1(a_0, a_1) &= \frac{c_1}{a_1} = \frac{1}{a_1\pi}\int_0^{2\pi} \zeta(\eta(\omega t))\sin(\omega t)\, \mathrm{d}(\omega t). \end{aligned} \tag{17}$$

Note that describing functions K_0 and K_1 are real due to the fact that nonlinear function $\zeta(\cdot)$ is time-invariant and memoryless.[24] It is worth mentioning that the describing function method approximates a nonlinearity by a gain using the first harmonic, which means that the approximation is the most accurate when high-order terms are negligibly. In the neighborhood of the Hopf bifurcation point, oscillations are sinusoidal and can be accurately described by the harmonic balance method due to negligible small high-order nonlinear terms. A rigorous mathematical justification of this fact is outside the scope of our paper. For a classical harmonic balance method, its mathematical justification is derived in Ref. 25, where the accuracy of the method is estimated by the contraction mapping argument. The result in Ref. 25 provides a conservative estimate, which is derived under restrictive assumptions; however, these assumptions do not require the amplitude of the oscillations to be small. It would be very interesting but challenging to extend the result from Ref. 25 to the case of the multivariable harmonic balance method, studied in this paper.

Our goal is to study a network behavior. The harmonic balance method extended for networks is called Multivariable Harmonic Balance (MHB). We are now going to apply the MHB method to coupled systems (1) and (2). The idea of the method is to turn the problem of determining an approximate periodic solution in the form of a harmonic signal, characterized by offsets α_0, amplitudes α, phases ϕ and frequency ω, into an eigenvalue problem. By means of the Laplace transformation, the time-invariant linear part of coupled systems (1) and (2) is rewritten into the transfer function representation, leading to the following description of the closed loop system:

$$\begin{aligned} y &= W(s)I_N u, \\ u &= M(\Delta_\tau)y + \nu, \\ \nu &= \Psi(y), \end{aligned} \tag{18}$$

where y, u, and ν are of dimension $N \times 1$, $I_N \in \mathbb{R}^{N\times N}$ is an identity matrix, $W(s) = C(sI - A)^{-1}B$, $M(\Delta_\tau) = \gamma\Delta_\tau\Gamma - \gamma I_N$ represents the coupling structure, Δ_τ is the delay operator ($\Delta_\tau f(t) = f(t-\tau)$) and $\Psi(y) \in \mathbb{R}^N$ is a column vector of nonlinearities. We remark that s stands for the differentiation operator.

Recall (18) and rewrite the coupled system's input u as

$$u = M(\Delta_\tau)W(s)I_N u + \nu, \tag{19}$$

$$W(s)\left[\frac{1}{W(s)}I_N - M(\Delta_\tau)\right]u = \nu. \tag{20}$$

Defining $R(s, \Delta_\tau) = \left[\frac{1}{W(s)}I_N - M(\Delta_\tau)\right]^{-1}$, the input u takes the form

$$u = \frac{1}{W(s)}R(s, \Delta_\tau)\nu. \tag{21}$$

Substituting (21) in (18), we obtain

$$y - R(s, \Delta_\tau)\Psi(y) = 0. \tag{22}$$

Premultiplying (22) by $R(s, \Delta_\tau)^{-1}$, we derive

$$\left[\frac{1}{W(s)}I_N - M(\Delta_\tau)\right]y - \Psi(y) = 0. \tag{23}$$

The nonlinearities $\Psi(y)$ are approximated by their describing functions, whose components can be computed using (17). The network output signal y is approximated by a sum of the offsets and signal's first harmonic

$$y \cong q_0 + q_1,\ q_0 = \alpha_0 = \text{const},\ q_1 = \alpha\sin(\omega t + \phi_0 + \phi), \tag{24}$$

where $\alpha_0 = [\alpha_{01}, \alpha_{02}, \ldots, \alpha_{0N}]^\top$, $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^\top$, and $\phi = [\phi_1, \phi_2, \ldots, \phi_N]^\top$. The objective is to determine an oscillatory profile consisting of the following parameters: α_0 is a constant offset vector, α is a vector of amplitudes, ϕ is a vector of phases, and ω stands for the frequency of oscillations.

Recalling the Fourier expansion and substituting s by $kj\omega$, and Δ_τ by $\exp(-kj\omega\tau)$, we obtain

$$\left[\frac{1}{W(kj\omega)}I_N - M(\exp(-kj\omega\tau)) - \mathrm{diag}\left[K_k(\alpha_{0i}, \alpha_i), \ldots, K_k(\alpha_{0N}, \alpha_N)\right]\right] q_k = 0. \tag{25}$$

We now have two multivariable harmonic balance equations for the networks with delay coupling from (25) for $k = 0, 1$, respectively.

These can be expressed as

$$J_0(\alpha_0,\alpha)q_0 := \left[\frac{1}{W(0)}I_N - M(1) - \text{diag}\left[K_0(\alpha_{0i},\alpha_i),\ldots,K_0(\alpha_{0N},\alpha_N)\right]\right]q_0 = 0, \quad (26)$$

and

$$J_1(\alpha_0,\alpha,\omega)q_1 := \left[\frac{1}{W(j\omega)}I_N - M(\exp(-j\omega\tau)) - \text{diag}\left[K_1(\alpha_{0i},\alpha_i),\ldots,K_1(\alpha_{0N},\alpha_N)\right]\right]q_1 = 0. \quad (27)$$

All parameters of the oscillatory profile are encoded in the relations (26) and (27). These equations are nonlinear due to the describing functions (17). In order to determine these parameters, we formulate an optimization problem. Since we are not interested in the trivial solution $q_0 = 0$ and $q_1 = 0$, we look for a set of phases ϕ_i, amplitudes α_i, offset components α_{0i} and frequency ω, which solve nonlinear equations (26) and (27). We define the objective function to minimize as

$$F(g) := Q_0^* Q_0 + Q_1^* Q_1 + p^\top p, \quad (28)$$

where

$$g := [\phi;\alpha;\alpha_0;\omega] \in \mathbb{R}^{3N+1}, \quad p := \alpha - |q_1| \in \mathbb{R}^N,$$
$$Q_0 := J_0(\alpha_0,\alpha)q_0 \in \mathbb{C}^N, \quad Q_1 := J_1(\alpha_0,\alpha,\omega)q_1 \in \mathbb{C}^N,$$

where $|q_1| = [|q_{11}|,|q_{12}|,\ldots,|q_{1N}|]^\top$. Vector g consists of phases, amplitudes, offset components, and frequency, the component p represents the second equation of (24). To simplify the problem, the phase of the first node is fixed at $\phi_1 = 0$ without loss of generality, and boundaries are chosen as: $\phi_i \in [0, 2\pi]\ \forall i \neq 1$, $\alpha_i \in [0,\beta]\ \forall i$, $\alpha_{0i} \in [-\beta,\beta]\ \forall i$, $\omega \in [0, 2\omega_{bif}]$, where β is a user defined number that bounds the region of interest in the search space.

It is worth mentioning that the relative phases in ring networks can be extracted from the symmetry argument.[39,40] An advantage of the MHB approach is that it is not limited only to the networks with a ring structure and aiming at computing the whole oscillatory profile (i.e., the offset, amplitudes, phases, and frequency) in one shot. Another large advantage of the multivariable harmonic balance method is the small number of parameters of the harmonic approximation looked for (at the price of a possible over-approximation of the solution's profile). This makes the method particularly suitable for the efficient detection of coexisting patters, given the reduced small space.

Solutions to nonlinear equations (26) and (27) are found by minimizing the squared residual norm of the objective function (28). The local solver fmincon included in MATLAB is used.[41] The solver uses the *interior-point* algorithm[42] in order to find a minimum of a constrained nonlinear multivariable function satisfying the imposed constraints. Since Eqs. (26) and (27) are nonlinear and multiple solutions exist, these solutions depend on the initial conditions. Multiple values of the initial conditions are generated by the MultiStart algorithm within the defined region of interest.[43]

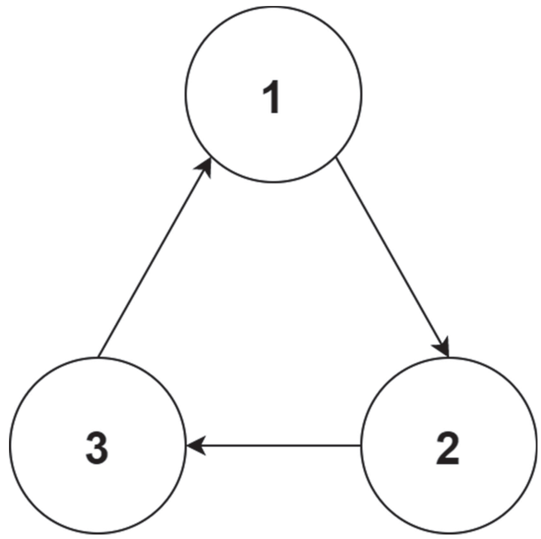

FIG. 2. Network structure for numerical examples.

A big advantage of the MultiStart method is that starting points for the minimization procedure are not needed to be specified. A set of random points choosing from the feasible region is used as initial conditions. The key point in the procedure is to choose the correct boundaries for the starting points and the unknown variables. Moreover, the approach can deal with co-existed periodic solution. Stable and unstable solutions can be simultaneously found. Having a set of parameters g, oscillatory profiles can be reconstructed using (24).

IV. NUMERICAL EXAMPLES

In this section, it is shown how the multivariable harmonic balance method is applicable to networks consisting of dynamical systems of Lur'e form interacting via linear time-delay coupling functions. We present two different networks in terms of individual system dynamics to show that both cases result in instability of the network equilibrium: supercritical and subcritical Hopf bifurcations. Illustrative examples are given with the same network structure which is a directed ring (Fig. 2) in order to show different possibilities of the MHB approach. Note that the MHB method can be easily applied for different networks in terms of the structure.

The graph topology is described by the following adjacency matrix:

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \quad (29)$$

The software package with demo examples, which are presented below, can be downloaded by the following link: https://github.com/RogovKO/DMHB.git.

A. Network with a supercritical Hopf bifurcation

In this subsection, we present the example of coupled nonlinear systems of Lur'e type that we studied in Ref. 1 with delay-free coupling. In this work, the coupling has a time-delay component and is given by (2). The goal is to show that the MHB method for delayed systems gives a very accurate approximation when oscillations emerge as a result of a supercritical Hopf bifurcation. The individual dynamics of the nodes, as in Ref. 15, are given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ u_i &= u_{ci} - \psi(z_i), \\ z_i &= Zx_i, \\ y_i &= Cx_i, \end{aligned} \tag{30}$$

with

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & -3 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\top,\ C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},\ Z = B^\top P,$$

where P is the solution to the Lyapunov equation

$$A^\top P + PA = -I_3.$$

The nonlinear function $\psi(\cdot)$ is given as follows:

$$\psi(z_i) = z_i^3;$$

and computing the describing functions using (17), we obtain

$$K_0(\alpha_{0i}, \alpha_i) = \alpha_{0i} + \frac{3}{2}\alpha_i^2, \quad K_1(\alpha_{0i}, \alpha_i) = 3\alpha_{0i} + \frac{3}{4}\alpha_i^2.$$

Note that nonlinearity $\psi(z_i)$ is odd and, due to this, the offset component satisfies $\alpha_{0i} = 0$.

For coupled systems (30) and (2), MHB equations (26) and (27) become as follows:

$$J_0(\alpha_0, \alpha)q_0 := \left[\frac{1}{W_y(0)}I_N - M(0) + \frac{W_z(0)}{W_y(0)}K_0(\alpha_0, \alpha)I_N\right] q_0 = 0, \tag{31}$$

and

$$J_1(\alpha_0, \alpha, \omega)q_1 := \left[\frac{1}{W_y(j\omega)}I_N - M(j\omega) + \frac{W_z(j\omega)}{W_y(j\omega)}K_1(\alpha_0, \alpha)I_N\right] q_1 = 0, \tag{32}$$

with vectors $K_0(\alpha_0, \alpha) = [K_0(\alpha_{01}, \alpha_1), \ldots, K_0(\alpha_{0N}, \alpha_N)]^\top$, $K_1(\alpha_0, \alpha) = [K_1(\alpha_{01}, \alpha_1), \ldots, K_1(\alpha_{0N}, \alpha_N)]^\top$, and frequency response functions $W_y(j\omega)$ and $W_z(j\omega)$ defined as

$$\begin{aligned} W_y(j\omega) &= C(j\omega I_3 - A)^{-1}B, \\ W_z(j\omega) &= Z(j\omega I_3 - A)^{-1}B. \end{aligned}$$

As the first step of the MHB procedure for delay systems, the critical values of coupling strength γ and time delay τ are computed by means of Lemma 2 and (9) in order to detect a Hopf bifurcation point. It is found that $\nu = \inf_{\omega>0} \mathbb{R}(\mathscr{H}(i\omega)) = -0.771$, which is attained at $\omega = \omega^* = 0.8146$. Thus, the minimal coupling strength to have characteristic roots on the imaginary axis for some $\tau > 0$ is $\gamma_{min} = 0.6481$. The critical value of time delay $\tau_c = 1.5304$ is computed by solving quadratic eigenvalue problem (10). The Hopf bifurcation point is detected for the parameter values $\gamma_{min} = 0.6481$ and $\tau_c = 1.5304$.

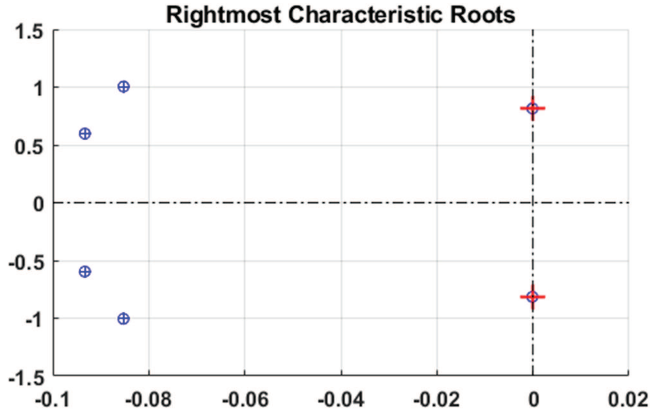

FIG. 3. Rightmost characteristic roots computed for fixed $\gamma_{min} = 0.6481$ and $\tau_c = 1.5304$.

In order to verify our calculations, we compute rightmost characteristics roots of our coupled systems described by DDEs using the method described in Refs. 44 and 45. Figure 3 shows that the rightmost characteristic roots are located on the imaginary axes. They correspond to a Hopf bifurcation of the nonlinear system (note that the bifurcation leading to instability of the network equilibrium must be a Hopf bifurcation because otherwise the condition of Lemma 3 would be violated).

Now we fix the time-delay parameter at $\tau = 1.5304$ and increase the coupling strength to make sure that there is a pair of eigenvalues with positive real part. Figure 4 shows that there is one pair of unstable eigenvalues with the magnitude of imaginary part 0.8162, which is used in the MHB analysis to set up the search space for the frequency of oscillations. For this pair of parameters, there is one pair of eigenvalues with $\text{Re}(\lambda) > 0$ while all others are confined to the open left half plane $\text{Re}(\lambda) < 0$.

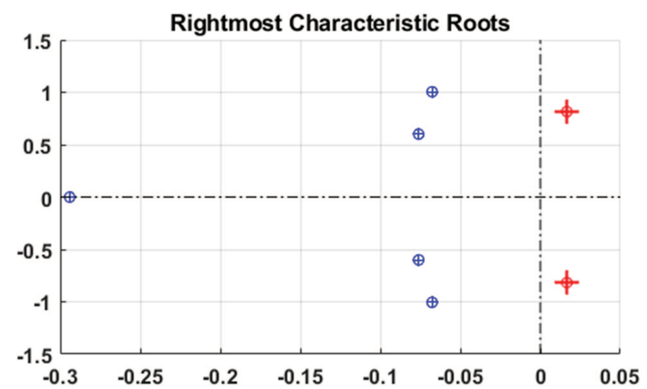

FIG. 4. Rightmost characteristic roots computed for fixed $\gamma = 0.7129$ and $\tau_c = 1.5304$.

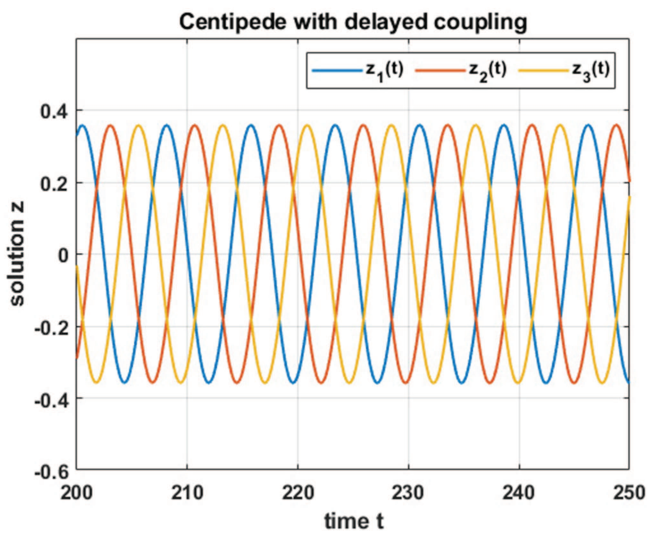

FIG. 5. Network behavior obtained by simulating DDEs.

The oscillatory profile obtained by simulating the system of DDEs is shown in Fig. 5. The results are obtained by simulating DDEs using the dde23 solver in MATLAB environment. The standard set of options is used and the initial data are chosen as a constant vector function defined on $[-\tau, 0]$ whose values are (uniformly distributed) random numbers.

The oscillatory pattern emerges from the supercritical Hopf bifurcation and has the sinusoidal-like form close to the bifurcation point. It implies that the MHB method gives a very accurate approximation of the pattern. In Table I, one can see the oscillatory profiles derived from the DDE simulation and computed by the MHB approach for delay-coupled systems.

In order to measure the accuracy of the MHB method, we provide the value of objective function (28) in computed minimum and compare the offset components, amplitudes, and frequency with the ones that are extracted from the DDE simulation as follows:

$$\Delta\alpha_0 = 100||\alpha_0 - \alpha_{0s}||/||\alpha_{0s}||,$$

$$\Delta\alpha = 100||\alpha - \alpha_s||/||\alpha_s||,$$

$$\Delta\omega = 100(\omega - \omega_s)/\omega_s,$$

where α_{0s}, α_s, and ω_s are obtained from the numerical simulation.

The result presented in Table II shows that the MHB approach for delay-coupled Lur'e systems accurately predicts oscillatory patterns in the neighborhood of a Hopf bifurcation point.

TABLE I. Oscillatory profiles of the network extracted from the DDE simulation and computed by the MHB method.

Method	ω	α_0	α_1	α_2	α_3	ϕ_1	ϕ_2	ϕ_3
DDE Sim	0.8239	0	0.3992	0.3991	0.3992	0	240	120
MHB	0.8242	0	0.3624	0.3627	0.3625	0	240	120

TABLE II. Comparison of the DDE simulation and the MHB method outputs.

Residual	$\Delta\omega$ in %	$\Delta\alpha_0$ in %	$\Delta\alpha$ in %
2.87×10^{-8}	0.02	0	0.94

In order to show that the MHB approach for delay-coupled Lur'e systems is numerically efficient, we perform the pattern analysis of 3 nodes, 10 nodes, 20 nodes, and 50 nodes ring networks (the structure is given in Fig. 6) with node's individual dynamics (30) and coupling (2). Bifurcation parameter values are chosen as $\gamma = 0.7129$ and $\tau = 1.5304$ for all networks. Results are given in Table III. The direct simulation results depend on initial conditions and contain only one pattern. Since there are co-existing patterns, it is nontrivial to find the initial conditions for each mode. The MHB approach deals with multistability. In our example, we have ten starting points distributed in the region of interest and this allows us to compute all structurally possible patterns in the network for selected parameters. Note that MHB computational time is given for all starting points together.

Data in Table III show that the MHB approach is more numerically efficient than the direct simulation by dde23 solver when we need to analyze the behavior of complex networks. The value of $\Delta\alpha$ increases as we take larger network. It can be explained by the fact that amplitudes become smaller when the amount of nodes increases. It is necessary to mention that frequency and phases are accurately computed for all patterns.

B. Network with a subcritical Hopf bifurcation

In this subsection, we consider a collection of three inert FitzHugh–Nagumo (FHN) model neurons interconnected by a linear time-delay coupling functions.[46] The network structure is

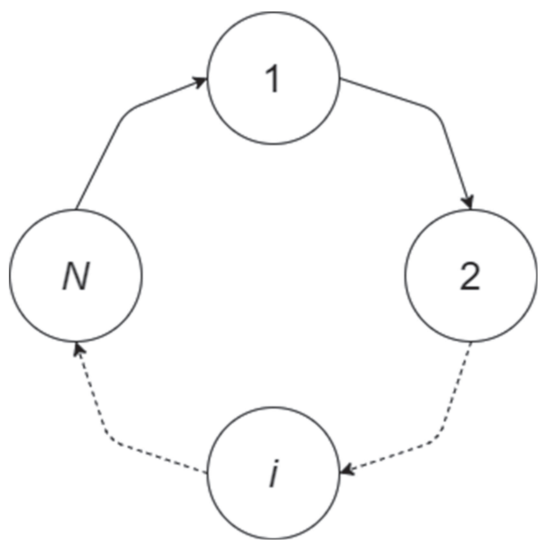

FIG. 6. Ring network structure with N nodes.

depicted in Fig. 2 and the adjacency matrix is given by (29). The bifurcation analysis is performed using the ddebiftool software package and shows that relaxation oscillations, observed in simulations, are emerged from a subcritical Hopf bifurcation. The MHB analysis is used to show that both stable and unstable oscillatory patterns exist for the same set of parameters and can be simultaneously computed by our approach.

The dynamics of this model neuron are given by the following equations:

$$\begin{aligned} \dot{x}_{i1} &= 0.08(x_{i2} - 0.8x_{i1}), \\ \dot{x}_{i2} &= x_{i2} - (x_{i2})^3/3 - x_{i1} - 0.559 + u_i, \\ y_i &= x_{i2}. \end{aligned} \tag{33}$$

One can easily verify that the FHN model has a locally exponentially stable equilibrium at $\bar{x}_i = (-1.225 - 0.980)^\top$. Moreover, the FHN model neuron is strictly C^∞-semipassive with a quadratic storage function (see Ref. 47). Hence, the solutions of any network consisting of FHN neurons are uniformly bounded for any non-negative γ and τ.

In order to simplify the MHB analysis of the network of FHN neurons, we perform a change of coordinates. For that, we define new variables $z_{i1} = x_{i1} - \bar{x}_{i1}$ and $z_{i2} = x_{i2} - \bar{x}_{i2}$. System (33) is equivalent to the system

$$\begin{aligned} \dot{z}_{i1} &= -0.064z_{i1} + 0.08z_{i2}, \\ \dot{z}_{i2} &= -z_{i1} + (1 - \bar{x}_{i2}^2)z_{i2} - (z_{i2}^3 + 3\bar{x}_{i2}z_{i2}^2)/3 + u_i, \\ y_i &= z_{i2}. \end{aligned} \tag{34}$$

It is easy to see that this system has an equilibrium at the origin and is of form (1), with

$$A = \begin{bmatrix} -0.064 & 0.08 \\ -1 & 0.0391 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$\psi(y_i) = -y_i^3/3 + 0.98y_i^2,$$

and $i = 1, 2, 3$. The network topology is defined by (29).

The nonlinear function $\psi(y_i)$ is not odd, which leads to a nonzero offset component of signal y_i. This implies that $y_0 \neq 0$ and both Eqs. (26) and (27) have to be solved.

The describing functions for nonlinearity $\psi(y_i)$ are given as

$$K_{0i}(\alpha_{0i}, \alpha_i) = 0.98\alpha_{0i} - 0.5\alpha_i^2 - 0.33\alpha_{0i}^2 + 49\alpha_i^2/(100\alpha_{0i}),$$

$$K_{1i}(\alpha_{0i}, \alpha_i) = -0.25\alpha_i^2 - \alpha_{0i}^2 + 1.96\alpha_{0i}.$$

TABLE III. Computational time and comparison of the MHB approach and the direct simulation.

Size of the ring	3	10	20	50
MHB (s)	2.14	9.09	35.07	629.5
dde23 (s)	2.45	160.5	355.7	4455
$\Delta\omega$ in %	0.02	0.009	0.04	0.89
$\Delta\alpha$ in %	0.94	6.97	14.35	20.82
Multistability	No	No	Yes	Yes

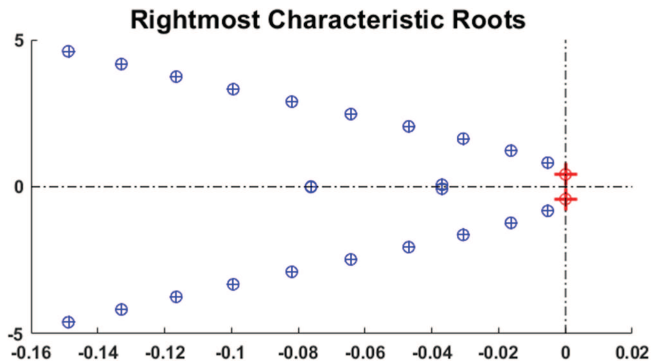

FIG. 7. Rightmost characteristic roots computed for fixed $\gamma_{min} = 2.5376$ and $\tau_c = 4.7643$.

In order to determine the bifurcation point, we recall the condition from Lemma 2. It is found that $\nu = \inf_{\omega>0} \mathbb{R}(\mathscr{H}(i\omega)) = -0.197$, which is attained at $\omega = \omega^* = 0.4202$. Thus, the minimal coupling strength to have characteristic roots on the imaginary axis for some $\tau > 0$ is $\gamma_{min} = 2.5376$. The critical value of time delay $\tau_c = 4.7643$ is computed by solving quadratic eigenvalue problem (10). We verify the calculations by computing rightmost characteristic roots by means of the method described in Ref. 44. Figure 7 shows that a couple of characteristic roots is located on the imaginary axis, which corresponds to the Hopf Bifurcation.

The basic MHB analysis of the given network was performed in Ref. 48. It was found that the simulation output consists of oscillations of the relaxation type. According to normal form theory,[36] oscillations in the neighborhood of the Hopf bifurcation must be sinusoidal-like with zero offset. One may think that the Hopf bifurcation of the subcritical type occurs in the example under consideration. In order to verify this, a bifurcation analysis of the FHN neuron network is performed by means of the software package `ddebiftool`.[38]

As a first step, we compute Hopf bifurcation curves in the parameter space (γ, τ) which are shown in Fig. 8.

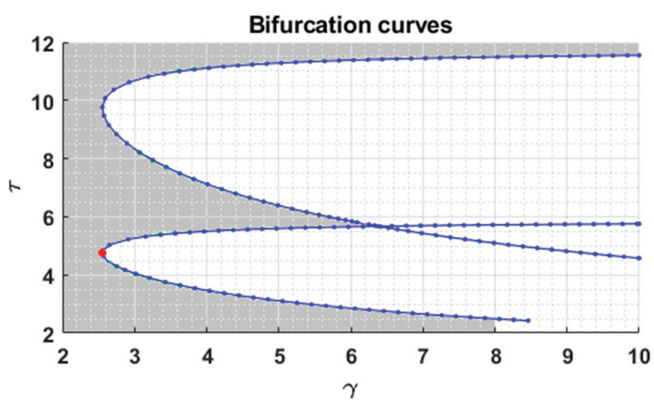

FIG. 8. Bifurcation curves in (τ, γ) parameter space.

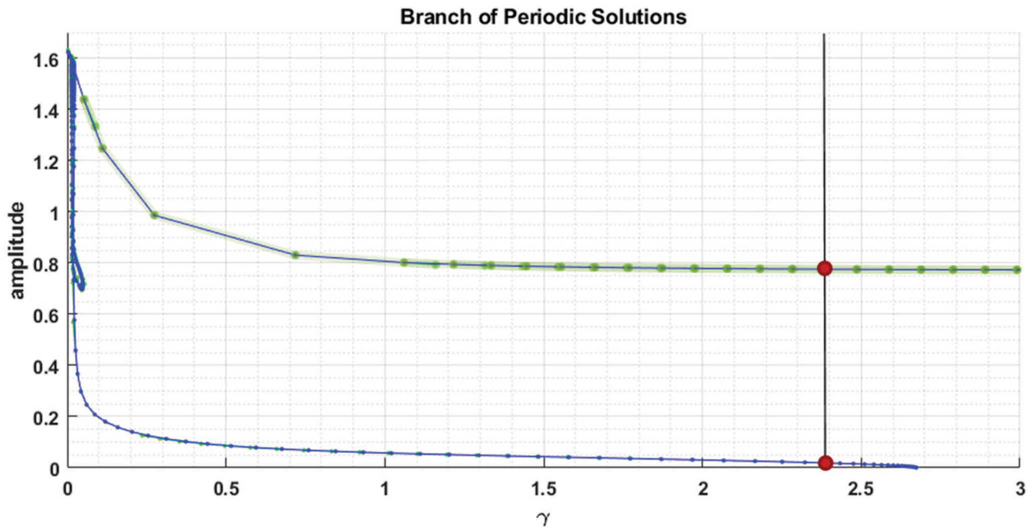

FIG. 9. Branch of periodic solutions that emerges from the subcritical Hopf bifurcation.

These curves computed for the (τ, γ) bifurcation parameter space for networks (34) and (2). The gray area stands for the region in which the network equilibrium is stable. When a bifurcation curve is crossed, the Hopf bifurcation occurs and the network equilibrium loses stability. The red dot represents a previously computed pair of parameters (τ_c, γ_{min}) and lies on a bifurcation curve, which also verifies our computations of γ_{min} and τ_c.

The second step of our analysis is the continuation procedure. We start from bifurcation point $(\tau = 4.4063, \gamma = 2.6733)$, compute the first point of a branch of periodic solutions, and then we fix

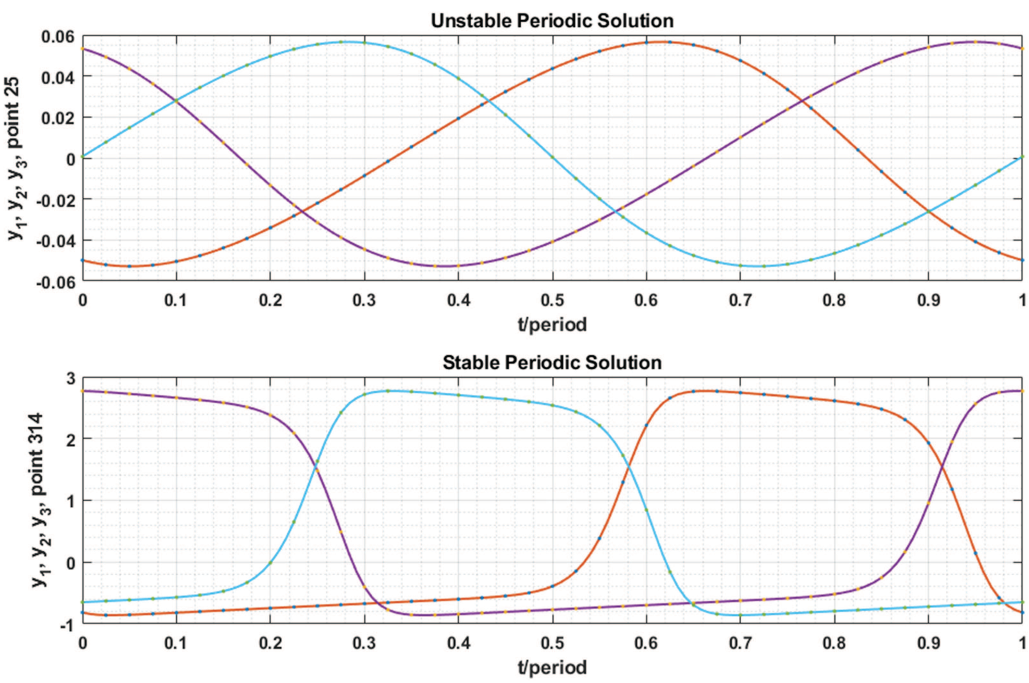

FIG. 10. Oscillatory profiles of the periodic solutions represented by red dots in Fig. 9.

TABLE IV. Stable and unstable co-existing oscillatory patterns, which are computed by the MHB method.

Periodic solution	ω	α_0	α_e	ϕ_1	ϕ_2	ϕ_3
Unstable	0.4491	0.0011	0.0529	0	120	240
Stable	0.4491	0.7063	1.8803	0	120	240

bifurcation parameter τ and change parameter γ with an adaptive step in order to compute the next point of the branch. The computed branch of periodic solutions is shown in Fig. 9. Each dot (320 in total) on this plot is a periodic solution computed by means of ddebiftool. The branch is represented in the (γ, amplitude) space. The blue line and dots stand for the unstable periodic solutions that emerge from the subcritical Hopf bifurcation. The brushed green line and dots stand for the stable periodic solutions. This confirms the hypothesis given in Ref. 48 about the subcritical type of the Hopf bifurcation.

In order to show that the MHB for delay-coupled systems deals with coexisting patterns, we choose two periodic solutions (red dots in Fig. 9). These solutions coexist for the same set of parameters ($\tau = 4.4063$, $\gamma = 2.383$). Oscillatory profiles, obtained by bifurcation analysis software, are shown in Fig. 10.

One can see that the unstable periodic solution located in a neighborhood of the bifurcation point has a sinusoidal-like shape. The stable periodic solution that co-exists with the unstable one for the same pair of parameters is represented by relaxation oscillations. We employ the MHB method to simultaneously compute the oscillatory profiles of these two regimes. The oscillatory patterns are given in Table IV.

Using data from Table IV and (24), the oscillations can be reconstructed. The MHB approach gives good approximation of the harmonic-like oscillation, which are in the neighborhood of the bifurcation point.

It is clear that the accuracy of the MHB approach cannot be good when it is applied for oscillations of relaxation type (see Table V), but it gives an accurate prediction in terms of phases of the oscillations and good starting points for the tools to accurately compute periodic solutions such as collocation based approaches. The MHB method for delay coupled systems deals with co-existing patterns even if some of them are unstable. In addition, one may say that since oscillatory profiles presented in Table IV share the phases and the frequency and belong to one branch of periodic solutions, the MHB approach can be used for basic bifurcation analysis.

TABLE V. Comparison of the DDE simulation and the MHB method outputs.

Periodic solution	Residual	$\Delta\omega$ in %	$\Delta\alpha_0$ in %	$\Delta\alpha$ in %
Unstable	3.67×10^{-10}	1.60	1.85	2.02
Stable	$2.94 \times 10 - 13$	2.25	26.04	4.95

V. CONCLUSIONS

This work addresses the problem of detecting (coexisting) oscillatory patterns in networks of delay-coupled nonlinear systems of Lur'e form. We focused on the cases where oscillatory patterns appear in the networks via both subcritical and supercritical Hopf bifurcations. We developed fast and efficient numerical tools, which help us to avoid time inefficient simulations of DDEs and can handle co-existing periodic solutions. The multivariable harmonic balance method was modified and extended for systems described by DDEs, and applied to two examples with subcritical and supercritical Hopf bifurcations in order to show that the method can successfully compute a good approximation of periodic solutions, which appear in such networks, even if the periodic solution does not occur in the neighborhood of the bifurcation point. We showed that the method is numerically efficient in comparison with the direct simulation approach. Moreover, the MHB method for delay coupled systems can simultaneously compute co-existing periodic solutions even if some of them are unstable and perform basic bifurcation analysis. In addition, we performed a bifurcation analysis of the delay coupled systems with the subcritical Hopf bifurcation by means of ddebiftool, using MHB results as starting points.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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