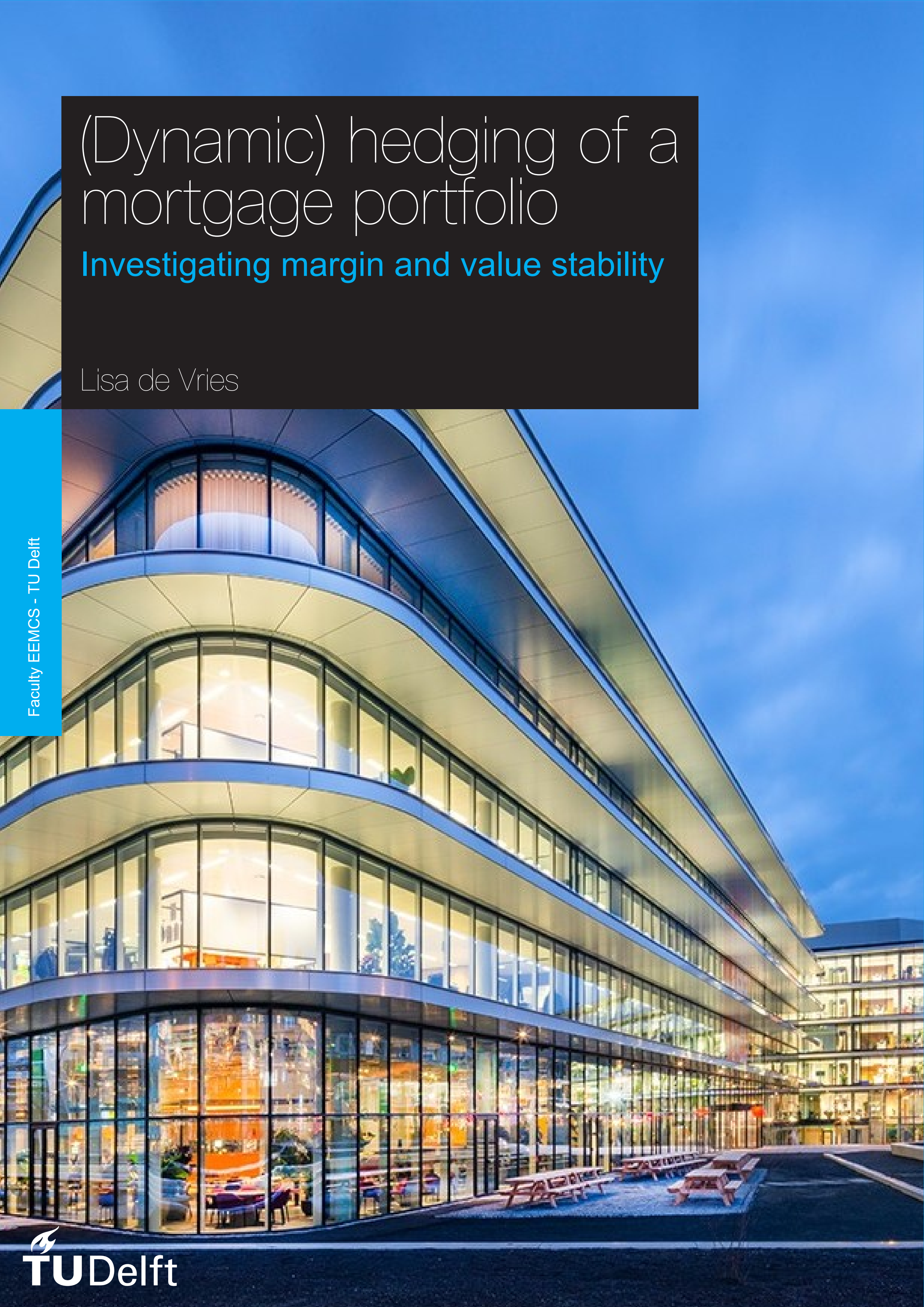
(Dynamic) hedging of a mortgage portfolio

Investigating margin and value stability

Lisa de Vries

Faculty EEMCS - TU Delft



(Dynamic) hedging of a mortgage portfolio

Investigating margin and value stability

by

Lisa de Vries

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Thesis committee:	Prof. dr. A. Papapantoleon,	TU Delft, responsible supervisor
	Dr. F. Barsotti,	TU Delft, daily supervisor
	Dr. N. Parolya,	TU Delft
	N. van Pelt MSc,	ING

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Delft University of Technology

Abstract

Banks issue mortgages with an embedded option for borrowers to prepay a part of the loan. However, this behaviour poses a risk to banks as it disrupts the level and timing of mortgage cash flows. From an earning perspective, when interest rates decrease, customers are financially incentivised to prepay their mortgages, resulting in a decrease in the bank's income when the cash proceeds are reinvested at a lower rate. Conversely, from a value perspective, with an increase in interest rates, reducing the financial incentive to prepay, cash flows are moved further ahead in time, thereby increasing the duration of the mortgage. These two scenarios highlight the instability in the bank's margin and value caused by prepayments. To address this risk, banks employ hedging strategies to mitigate the prepayment risk and achieve margin and value stability. This research aims to identify an effective hedging strategy that can accomplish both.

The research utilised the one-factor Hull-White model to simulate various interest rate scenarios, while an interest rate-dependent logistic prepayment model provided monthly prepayment rates based on the mortgagors' refinancing incentives. Ten different hedging techniques were explored, including the internal funding, a static and dynamic notional hedge, and a static and dynamic value hedge. Additionally, a calibrated receiver swaption was included in each of these five hedging approaches. Subsequently, each of these hedging approaches was assessed for its margin stability, measured by the variance of the net interest margin, and its value stability, evaluated through the variance of the net present value, the average basis point value, and the NPV-at-Risk in ± 200 basis point shocked interest rate scenarios.

The analysis indicated that relying solely on internal funding performs poorly in terms of both margin and value stability. Dynamic hedges were found to generally outperform their static counterparts, due to their ability to respond to market changes. Furthermore, the notional hedge demonstrates superior margin stability, while the value hedge exhibits the best value stability. Additionally, the analysis revealed that the incorporation of a receiver swaption significantly improves the NPV-at-Risk but has limited impact on the other risk metrics.

Based on the conducted research, it is concluded that for a bank aiming for both value and margin stability, the most effective hedge strategy is the dynamic value hedge without the utilisation of a swaption. However, it should be noted that the ultimate choice for a hedging strategy depends heavily on the risk appetite of each bank. If a bank prioritises attaining margin stability, the recommended choice would be the dynamic notional hedge without the incorporation of the receiver swaption. On the other hand, for a bank that prefers value stability over margin stability, the dynamic value hedge without the inclusion of a swaption should be considered. Moreover, the final decision may also be influenced by mandatory requirements imposed by financial regulators, such as the European Central Bank.

Keywords: Prepayment risk, hedging, value stability, margin stability, swaptions

Preface

This thesis has been submitted in partial fulfilment of the requirements for the degree of the Master of Science in Applied Mathematics at Delft University of Technology. From January 2023 to July 2023, I have been working on the construction of various hedging portfolios to offset the prepayment risk arising from mortgages, focusing on maintaining margin and value stability. I conducted this project in collaboration with ING, a large Dutch commercial bank, as part of the Model Validation team that focuses on interest rate risks in the banking book models.

My supervisor within this team has been N. van Pelt. I would like to thank him for our (bi-)weekly meetings in which we discussed the progress of my research as well as my personal well-being. His relevant insights and guidance were useful throughout the entire process, and his enthusiasm made it seem like this research project would never reach its conclusion. However, he also taught me a crucial lesson that I will cherish: never be afraid to ask questions.

My academic supervisor of this project has been Dr. F. Barsotti, of the Applied Probability group at the Delft Institute of Applied Mathematics. Throughout this project, she provided valuable guidance, offering advice on enhancing my report. Additionally, she consistently ensured that I maintained a clear overview of the research. Despite facing personal challenges, she remained dedicated and checked in on my progress. I am truly thankful for her support and would like to express my appreciation to her.

Furthermore, I would like to thank Prof. Dr. A. Papapantoleon for being my responsible supervisor and Dr. N. Parolya for completing the committee. I also want to thank ING for the provided input, but mostly my colleagues who made me feel like I was part of the team from day one, and who graciously shared their expertise and provided assistance when needed. I could not have wished for a better environment to carry out my thesis project. Finally, I would like to thank my friends and family, who supported me unconditionally during these six months.

Lisa de Vries
Amsterdam/Delft, July 2023

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1

Introduction

This chapter serves as an introduction to the research topic, covering several key aspects. It begins by providing background information, offering an understanding of the context in which the research is situated. Subsequently, it elaborates on the research objective, presenting the main research question. It then outlines the methodology employed to achieve the research objective, along with an overview of the paper's structure.

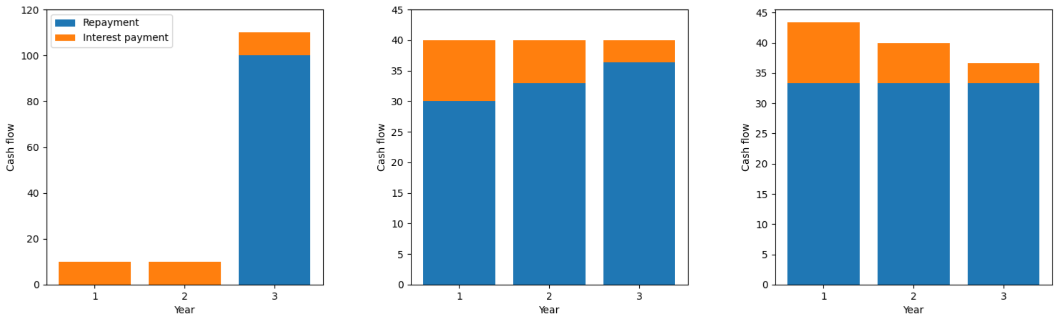
1.1. Background

Fluctuations in interest rates can have a significant impact on a bank's financial position and earnings. Interest rate risk in the banking book (hereafter IRRBB) specifically refers to the potential impact on its banking book activities, which include assets and liabilities held for longer-term investment purposes, such as loans, deposits, and investments. The risk arises because changes in interest rates can affect the bank's net interest margin, net income, and economic value. Effective management of IRRBB is crucial for banks to maintain a stable and profitable financial position while minimising potential losses. Moreover, regulators such as the European Banking Authority (EBA) set guidelines for banks on the management of interest rate risk in the banking book.

In the Netherlands, one area where this is particularly relevant is in the management of mortgages. Mortgages are long-term loans that are typically used to finance the purchase of a home, land, or other type of real estate. They are a key component of a bank's banking book, not only as they form a large part of the balance sheet for most big Dutch banks but also as they typically have a longer-term duration than other loans and are often held to maturity. As such, changes in interest rates can have a significant impact on a bank's mortgage portfolio.

Whenever a mortgage is settled, the borrower, or the *mortgagor*, receives a payment schedule he is obliged to follow. These *repayments* ensure that the initial amount, or *notional*, together with interest, is fully paid back to the bank. Mortgages come in different types, distinguished by the repayment amounts and the compounding methods used. The three main types are the bullet mortgage, the annuity mortgage, and the linear mortgage. The bullet mortgage is the simplest, with only interest payments required, and the notional fully redeemed at maturity in one single payment (bullet). This type is therefore also known as an interest-only mortgage. The annuity mortgage requires a fixed payment amount (the annuity) each period, including both interest and notional repayment. Note that as the outstanding notional decreases over time, the portion of the annuity allocated to interest payment decreases. Finally, customers with a linear mortgage repay the same amount of notional each period, resulting in a decrease in their periodically costs. Figure 1.1 summarises the cash flow schemes for each of these mortgage types.

While these mortgages differ in their repayment schedules, they have one key feature in common: the embedded prepayment option. Clients with an outstanding mortgage loan have the right to prepay (part of) the mortgage notional in addition to the contractually agreed-upon repayments. In the Netherlands, this option can be exercised penalty-free up to a certain percentage of the notional, typically 10% on a yearly basis. If borrowers act rationally, there is a growing incentive to repay when interest rates are declining. That is because they can take out a loan at a lower rate and use it to pay off



(a) (b) (c)

Figure 1.1: An overview of the cash flows of a (a) bullet mortgage (b) annuity mortgage (c) linear mortgage, divided into repayments and interest payments.

their existing mortgage. On the contrary, in a rising interest rate environment, prepayments generally decrease because the holders wait longer before they reinvest at the higher level.

This behaviour poses a risk for Dutch banks as they disrupt the level and timing of mortgage cash flows. From an earning perspective, when interest rates decrease (and therewith the mortgage rates), customers will become financially incentivised to prepay their mortgages. Cash proceeds arising from such prepayments are reinvested at a lower rate, resulting in an income decrease for the bank. From a value perspective, an increase in interest rates and thus a decreasing financial incentive to prepay, leads to reduced prepayments. Consequently, cash flows are moved further ahead in time, thereby increasing the duration of the mortgage, indicating a greater interest rate sensitivity. Therefore, rising rates result in more exposure to interest rates variations.

It is thus for Dutch banks of great interest to manage the risk arising from the embedded option and, therefore, they need models to gain insight into prepayment behaviour. The main driver of prepayment is the refinancing incentive, which refers to the potential cost savings that a borrower may achieve by refinancing their existing mortgage loan, typically by obtaining a lower interest rate. However, research shows that prepayment rates in the Netherlands may also be influenced by, among others, the time of the year and the age of the mortgagor (e.g. Charlier and van Bussel, 2003; Jacobs et al., 2005). Indeed, borrowers are more likely to prepay in December than in August due to tax benefits, and younger clients generally prepay more as they tend to move more often than elderly people. Models that take into account such different risk drivers can be used to understand the evolution of prepayment rates. This is highly relevant for Dutch banks as accurate predictions of prepayment rates are essential for estimating future cash flows and managing the associated risk. Given the current trend of increasing interest rates after a long period of historically low interest rates, it is important to adjust existing models to reflect this upward-moving environment.

A prepayment model allows the bank to understand client behaviour and how to fund and hedge a portfolio of mortgages. In essence, banks aim to manage and, if possible, offset the arising prepayment risk, from both their own risk management perspective as that of the regulators. One way is by means of a hedge portfolio, which is widely used by financial institutions. The concept is simple: one needs to find a portfolio of financial instruments (e.g. bonds, swaps, swaptions, etc.) that matches some characteristics of the underlying mortgage portfolio. However, due to the convexity arising from the embedded prepayment option, a linear hedge may not suffice, and consequently non-linear derivatives such as swaptions must be considered. However, even with a well-designed hedge portfolio, there may still be residual risk that cannot be perfectly offset. To address this, dynamic hedging can be used, which involves rebalancing the hedge position to account for changing market conditions and prepayment behaviour. This approach can improve the hedge and reduce the residual risk.

In practice, there are two common approaches to hedge the embedded option risk: notional hedging and value hedging. The first aims to match the outstanding notional of the expected mortgage amortisation profile with the notional of the replicating portfolio, ensuring margin stability. In contrast, the second method tries to maintain value stability by achieving a net basis point value of zero. Note that a basis point value (BPV) is a risk metric that estimates the interest rate risk for the bank, as it represents the gain or loss in the value for a parallel movement of 0.01% in the yield curve. A net BPV of zero implies that the net value of the total portfolio, including mortgages and the selected financial instruments, remains approximately constant. Both methods correspond to a different objective of the bank, margin and value stability, respectively. Ideally, both should be maintained, but in reality, a pure focus on one of the two would destabilise the other (Seidel, 2018). Banks should be aware of this trade-off and hedge accordingly.

1.2. Research objective

The objective of this thesis is to construct a (dynamic) portfolio of financial instruments that captures not only the linear behaviour of a hypothetical Dutch fixed-rate bullet mortgage portfolio, but also the convexity arising from the embedded prepayment option. Specifically, the resulting hedge portfolio should be as stable as possible in terms of both net interest margin (NIM) and net present value (NPV), thereby aligning with the two main objectives of a bank.

In order to perform the analysis, three models are required: an interest rate model, an interest rate-dependent prepayment model, and a hedging model. Given the research focus on constructing a hedge portfolio, the first two models will be presented rather than developed from scratch. Specifically, the interest rate simulations will be based on the Hull-White model, and a logistic prepayment model will be utilised. These models have been chosen in collaboration with the Dutch bank involved in this research.

Combining all above, the following main research question is established:

”Which portfolio of financial instruments provides the most effective hedge for a given mortgage portfolio, ensuring stability in terms of net interest margin and net present value under various interest rate simulations?”

1.3. Thesis outline and methodology

The research question will be answered step-by-step throughout this report. First, a literature review will be conducted in Chapter 2 to provide an overview of existing models regarding interest rates, prepayments, and hedging approaches. While the functional form of the first two models has already been determined by the bank, exploring alternative models from the literature remains crucial. This not only enhances our understanding of existing knowledge but also identifies avenues for future research extensions. Following this, the Hull-White interest rate model will be presented in Chapter 3. This chapter will delve into some useful properties of the Hull-White model, examining its strengths and limitations in capturing interest rate dynamics. Additionally, a calibration process will be executed to fit the model parameters to current data, ensuring its accurate representation of the observed interest rate behaviour. The simulated rates will then be utilised as input for an interest rate-dependent prepayment model, which will be discussed in the first part of Chapter 4. This part includes both a theoretical examination of the properties of the logistic function and an empirical analysis that involves investigating the provided historical data and calibrating the model’s parameters accordingly. In the second part of Chapter 4, the calibrated prepayment model is augmented with the contractual agreed-upon repayment cash flows of a given mortgage portfolio. This integration results in a model capable of generating the total mortgage cash flows given an interest rate scenario. This is followed by Chapter 5, in which various hedge portfolios will be constructed, encompassing static and dynamic hedges, as well as incorporating swaptions as hedging instruments. The performance of these different hedges will be evaluated, considering the stability of the net interest margin, as well as the net present value. Finally, Chapter 6 will conclude the research and provide a discussion addressing its limitations, along with recommendations and suggestions for future research.

2

Literature review

This chapter describes previous work and scientific publication relevant to the construction of a hedge portfolio for mortgages. It is subdivided into three parts. The first section examines the evolution of interest rate models throughout the years, starting from the one-factor Merton model introduced in 1973 and progressing to the two-factor model proposed by Brigo and Mercurio (2001). The second section focuses on the vast body of literature on prepayment models. We explore the various factors that drive the prepayment behaviour observed in the market and different approaches for modelling prepayment rates. Lastly, the third section covers diverse hedging approaches. We discuss different financial instruments and techniques employed in hedging.

These three models form the basis of this research and are related as shown in Figure 2.1. At the core of the framework is the interest rate model that generates simulations of future interest rates. These simulated rates serve as inputs for the subsequent models and analyses. Building upon the interest rate model, a prepayment model is employed to simulate prepayment rates. With these rates determined, a cash flow model generates cash flows associated with a mortgage portfolio. These cash flows consist of fixed contractual interest rate payments and repayments, as well as prepayments that depend on the interest rate. Similarly, the cash flows of interest rate-dependent financial instruments, such as swaptions, are generated in different interest rate scenarios. Finally, by examining the cash flows of both the mortgage portfolio and the portfolio of market-tradable instruments, a hedging model is developed. This model takes into account the interplay between interest rate movements, prepayment rates, and cash flows to identify suitable hedging strategies.

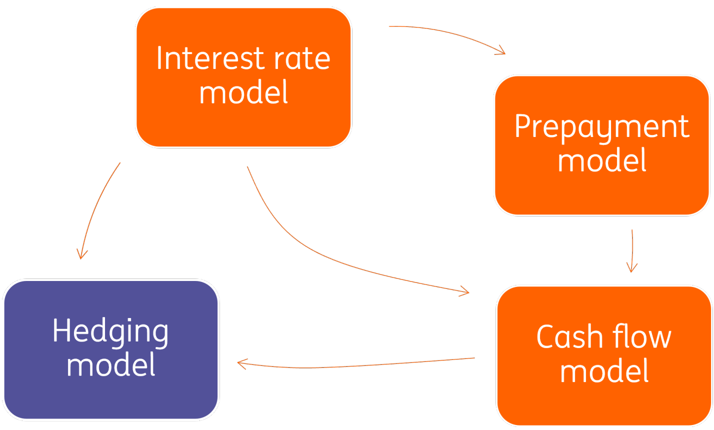


Figure 2.1: An overview of the interdependencies between the four models used in this research.

2.1. Interest rate models

Interest rate models are widely used in the financial world for pricing and risk management purposes. They provide insight in the evolution of interest rates of all different maturities over time. The simplest type is the one-factor model that assumes that the short-term rates, or short rates, are the only drivers in place. This implies that a single short rate model suffices to characterise the complete term structure of interest rates.

Merton (1973) pioneered the use of a short rate model in finance by proposing a simple Brownian motion with a constant drift term. The process implies a positive probability of negative interest rates, which was seen as an undesirable feature at the time of its inception. After the crisis of 2008, however, this stance was being reconsidered. A few years later, Vasicek (1977) introduced an improved model, not in the sense of the unwanted negativity, but by being able to capture mean reversion. This essential characteristic describes the converging behaviour of interest rates to a long-run equilibrium over time, a widely observed phenomenon in the market. In 1985, J. C. Cox et al. succeeded with their CIR model to fix the pre-crisis drawback of rates below zero by including a square root component while preserving the mean-reverting property. Until recently, the CIR model was preferred over the Merton and Vasicek models. Today, however, with many central banks experimenting with negative interest rates (Haksar & Kopp, 2020), this preference may not hold anymore. A disadvantage of the CIR model is that it may be inadequate to reproduce the initial interest rate curve since the model only uses time-independent parameters. As response, Hull and White (1990) introduced an extension of the Vasicek and CIR models that allows the choice of parameters to match the term structure of interest rates. This mean-reverting model allows negative rates and is currently the most widely used model in interest rate modelling due to its simple yet accurate approach.

The aforementioned models are all examples of one-factor models. However, it is commonly criticised that it is unrealistic that the behaviour of interest rates is solely induced by short rates. As a result, multi-factor models were introduced, in which two or more rates are assumed to drive the term structure. In fact, principal component analyses by Jamshidian and Zhu (1996) and Rebonato (1998) conclude that two- or three-factor models are adequate to simulate a realistic zero-coupon curve, as two and three components can explain approximately 90% and 95% of the total variance, respectively. Some examples of multi-factor models are those of Hull and White (1994), Longstaff and Schwartz (1992), and an extension of the latter by Brigo and Mercurio (2001) (two-factor), and the model proposed by Chen (1996) (three-factor).

Even though interest rates with multiple drivers seem to capture the behaviour more accurately, one-factor models are preferred in practice. The first reason is related to the difficulty of applying even a two-factor model, as the extra factor may add unnecessary complexity without significantly improving its accuracy. Secondly, the single-factor approach assumes that all rate changes are solely driven by short rates, making it possible to capture parallel movements in the term structure. This means that a shock to the yield curve is transmitted equally through all maturities, since all interest rates are perfectly correlated. While this assumption clearly does not hold in reality, as non-parallel shifts in the yield curve can be observed, such movements occur significantly less often than parallel movements. Therefore, there is no need to disregard the use of one-factor models altogether. In fact, for this thesis, the one-factor Hull-White model is selected for the simulation of interest rates as it is able to capture the parallel movements in the yield curve and provide accurate results. Moreover, as the main objective of this research is to construct a hedging portfolio, this simple yet accurate interest rate model will suffice.

2.2. Prepayment models

Prepayment models are used to forecast the number of prepayments that borrowers will make on their loans, particularly mortgages. These models are of great importance for banks as prepayments can have a significant impact on the level and timing of mortgage cash flows, resulting in uncertainty. Over the last 40 years, numerous research have been conducted to gain insight into the option embedded in mortgages and its driving factors. This section aims to provide an overview of the developed models. To establish a clear context for the discussion, let us first consider an illustrative example.

Imagine a homeowner named Bob, who recently obtained a fixed-rate bullet mortgage amounting to $€100,000$ at an annual rate of 6%. Under this arrangement, Bob is obligated to make monthly payments of $€500$, with an additional $€100,000$ due at the end of the contract term. Now, suppose that after one year, a significant decline in interest rates occurs, causing mortgage rates to drop to

4%. In this situation, Bob finds himself presented with an opportunity to refinance his existing loan. Through refinancing, Bob can secure a new loan at the reduced rate, leading to significantly lower interest payments compared to the original 6%. As a result, Bob can save up to €166, 67 on interest payments each month over the remaining term of the mortgage. However, it is important to note that if the market moves in the opposite direction, with mortgage rates increasing, Bob will have little or no incentive to pursue refinancing.

The example illustrates one of the main drivers of prepayment behaviour: the refinancing incentive. The majority of conducted studies support this notion, arriving at a shared conclusion: prepayment rates are dependent on the fluctuations in interest rates. Mathematically, the refinancing incentive is defined as a measure of the difference between the current contractually agreed-upon coupon rate and a reference rate, which usually corresponds to an alternative way to invest cash or refund the loan. An example of a reference rate is the current coupon rate or the swap rate that matches the remaining maturity of the mortgage, such as the 10Y-Euribor swap rate for a mortgage that matures in 10 years. The refinancing incentive can be interpreted as the absolute difference between the two rates, as Casamassima (2018) and Green and Shoven (1986) did, or as their ratio (Kang and Zenios, 1992; Richard and Roll, 1989). Another alternative proposed in the literature is the net present value gained by refinancing the mortgage, i.e., the difference in the net present value of the mortgage left uncalled and refinanced, as suggested by Charlier and van Bussel (2003) and Jacobs et al. (2005).

If people were to fully act in a rational way, they would repay each time the refinancing incentive exceeds a certain level ϵ, for example, when the absolute difference or net gain is positive, or the ratio is strictly larger than 1. As a consequence, the functional form of the prepayment rates, depending on the refinancing incentive, would resemble a step function with a jump at ϵ, as represented by the blue line in Figure 2.2. The type of models based on this rationality assumption are called optimal prepayment models. Examples of this approach are given by Kuijpers and Schotman (2007) and Stanton (1995).

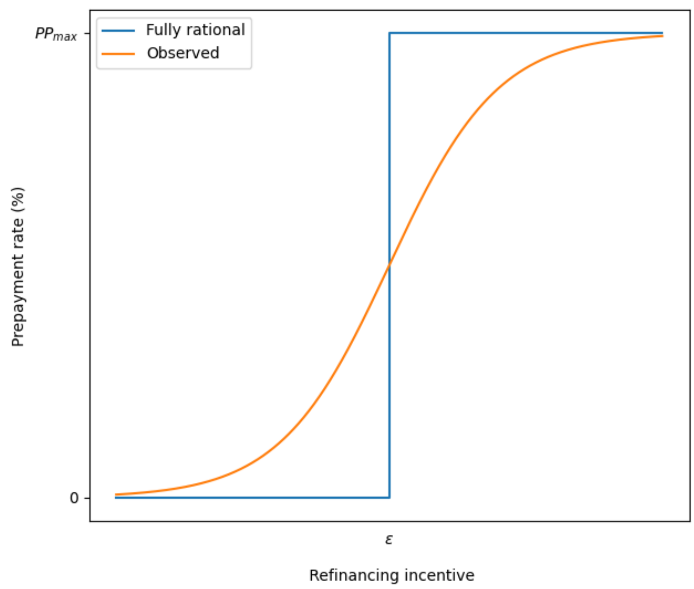


Figure 2.2: Rational (blue) and observed (orange) prepayment behaviour of mortgagors against the refinancing incentive.

In reality, however, this behaviour is not always observed, as people are simply not fully rational. Clients holding back from prepaying in the lowest interest rate environments or exercising their prepayment option even though the current rates are higher are both observed in practice. Models that acknowledge this non-optimal behaviour are called exogenous models and can be further divided in two categories. On the one hand there are the endogenous models, which are closely related to the optimal prepayment models. The only difference is that variables insensitive to interest rates are added to capture the irrationality. For example, Dunn and McConnell (1981) modelled the suboptimal prepayments by including a Poisson-driven process. On the other hand, strictly empirical models try to explain the prepayments through a set of variables. The literature on this category is extensive, as there are numerous different approaches. Firstly, one could decide to stick to the interest rate and thus the refinancing incentive as single driver. Casamassima (2018) substantiates this decision by arguing that this choice for the functional form of the prepayment rates allows for both the irrationality of the mortgagors and their reaction time, thus capturing all prepayment behaviour. Specifically, he proposed a logistic function as plotted in orange in Figure 2.2. Also, other sigmoid (s-shaped) functions can be applied to model the influence of the refinancing incentive. Kolbe (2008), for example, used an arc-tangent function.

An alternative but common approach to modelling prepayments is the proportional hazard technique (D. Cox and Reid, 1984), which is widely used not only in economics but also in demography and medical research. It assumes that the probability of prepayment can be split into two multiplicative factors: a baseline hazard that represents the proportion of people that would prepay in the *base* situation and a factor that describes how the hazard varies in response to some exogenous factors. Green and Shoven (1986) applied this technique to estimate the percentage reduction in prepayment probability associated with interest rate changes. This method can also incorporate additional drivers, such as mortgagor and loan characteristics or macroeconomic factors. Charlier and van Bussel (2003) developed a model in this way for Dutch bullet mortgages and specified four main determining factors. The first factor is the refinancing incentive, which is related to the term structure of interest rates. The second is called burnout, which encompasses various behaviours of mortgagors. Some clients may react instantly to a positive incentive to prepay, while others may miss the opportunity and act the next time the same scenario occurs. Following this trend, one could argue that the older the portfolio of mortgages, the lower the prepayment rate for a certain refinancing incentive. Another effect related to the age of the contract is called seasoning. Customers are not very likely to prepay just after the loan contract has been signed. This likelihood gradually increases with time until it reaches a maximum level. Finally, the month of the year plays a role in the prepayment behaviour. Charlier and van Bussel observed that the Dutch prepayment rates peak in the month of December. This can be explained by the tax effect, as clients aim to reduce their taxable savings sum by the end of the year, or by the additional salary payments. Similar conclusions were made by Richard and Roll (1989), while using a different type of prepayment model, known as the Goldman Sachs mortgage prepayment model. Instead of the proportional hazard approach, they used functions for the aforementioned effects and multiplying all four. Although this method has a high explanatory power in estimating prepayments, the final functional form is complex and nonlinear due to the multiplication. Kang and Zenios (1992) addressed this drawback by combining basis functions to achieve an arbitrary level of accuracy while maintaining the explanatory power.

The main drivers of prepayment rates are typically the four factors discussed earlier. However, there is a broad spectrum of research dedicated to alternative drivers. For instance, Yang et al. (1998) introduced a model that incorporates stochastic processes for house prices and household income, as well as mortgage underwriting constraints. This approach narrowed the differences between theoretical and observed results, but at the expense of departing from rigorous theories in the standard mortgage option pricing literature. Research by Perry et al. (2001) showed that both house price inflation and prepayment charges have significant impact on prepayment behaviour in the UK. In the same research it was concluded that the influence of the age of the borrower and the employment levels were relatively low. In contrast, Jacobs et al. (2005) found evidence for a relation between prepayment rates and the age of the mortgagor in the Netherlands. They discovered that older clients usually repay less than the younger ones. This finding is supported by the fact that younger people tend to move more often, whereas the presence of children usually forces adults to settle down. Additionally, they found that the size of the loan and the prepayment rates have a negative relationship, where larger loans tend to have lower prepayment rates.

Besides the characteristics of the contract and the mortgagor, external (macroeconomic) factors can also be included in a prepayment model, as demonstrated by Kolbe (2008). He incorporated a factor based on quarterly GDP growth rate, as he found strong empirical evidence for its impact on prepayment behaviour, particularly the turnover (non-refinancing) component. Recent research has also examined the relationship between race and client behaviour. Gerardi et al. (2023) noticed that black and Hispanic homeowners refinance at rates 0.75% and 1.19% lower than white and Asian borrowers on average, respectively. While observable differences, such as lower credit scores and higher leverage, explain about 80% of this refinancing gap, a significant portion remains unexplained.

It is clear that there is no perfect prepayment model that can fully explain the behaviour of mortgagors. Different types of factors can be included, related to either the contract itself, the mortgagor, or external macroeconomic factors. However, as the primary objective of this project is to construct a hedge portfolio, a simplified prepayment model will meet the requirements. Therefore, a logistic function, with the refinancing incentive as the only driver, is selected to model the prepayment rates.

2.3. Hedging approaches

Hedging models are vital from a risk management perspective as they allow financial institutions to (partly) offset potential losses. To illustrate the significance of hedging models, let us continue with the example of Bob, but now consider the perspective of a bank. As a bank, offering mortgages exposes the institution to risks arising from interest rate fluctuations and prepayment behaviour of borrowers like Bob. As the mortgage rates drop to 4%, Bob may choose to refinance his existing mortgage, securing a new loan at the lower rate. Consequently, the interest payments he makes would be substantially lower compared to the original 6% rate. This, in turn, results in a decreased interest income for the bank, amounting to approximately €$166,67$ each month over the remaining term of Bob's contract. Conversely, if the mortgage rates increase rather than decrease, Bob is less inclined to refinance his existing mortgage. As a result, the cash flows from the mortgage are postponed, extending the duration of the loan. This implies that the bank remains exposed to interest rate changes for a longer period.

These situations highlight the importance of managing interest rate fluctuations and prepayment behaviour for the bank. By utilising hedging models and strategies, the bank can actively mitigate the effects of both decreasing and increasing mortgage rates. Hedging involves taking an opposite position in a portfolio that has the same value and risk profile as the one requiring hedging. In the context of this project, which focuses on the embedded prepayment option in mortgages, the objective is to construct a portfolio of financial instruments that aligns with the risk characteristics associated with this optionality. Within this approach, two key questions arise:

1. Which instruments should be included in the portfolio?
2. How do we assess the quality of the hedge?

Addressing the first question involves selecting appropriate financial instruments to include in the hedging portfolio. These instruments should mirror the risk factors associated with the mortgage optionality and provide a suitable hedge against interest rate fluctuations and prepayment behaviour. The second question involves evaluating how well the portfolio's performance aligns with the dynamics of the mortgage option and the associated risks. Various metrics and evaluation methods are employed to determine the quality of the hedge. In this section, we will delve into these questions and explore different hedging approaches and models, providing insights into how banks and other financial institutions manage interest rate and prepayment risks.

Literature to address the first question is relatively scarce due to limited access to the required data. This is mainly because most banks maintain strict confidentiality regarding their customers' prepayment behaviour and other related information, making it challenging for researchers to build models based on real-world data. Thus, the few publicly available researches on this subject have been carried out in cooperation with financial institutions. One such example is given by Casamassima (2018), who had access to monthly observations over a period from February 2011 to July 2017 of an average of more than 400,000 mortgages. The data consisted of more than 20 explanatory variables that covered various contract and client characteristics. Based on this data, an Index Amortising Swap (IAS) was constructed to represent the mortgage portfolio. Typically, an IAS is an interest rate swap of which the notional is gradually reduced over the life of the agreement, and, for the purpose of his research, it was adjusted to follow the notional of the mortgage portfolio exactly. In particular, the amortisation schedule of the swap was predetermined as function of the prepayment rates, which were assumed to be fully dependent on a chosen interest rate. Theoretically, this instrument would provide the perfect hedge should it actually be traded, but unfortunately, it can only be used as a representation and a starting point for a practical hedge. Indeed, Casamassima replicated the IAS through a portfolio composed of market-tradable swaps and swaptions. His choice for these types of instruments was substantiated by the fact that the IAS, and thus the mortgage portfolio, can be seen as a swap, and the embedded prepayment optionality as a swaption. Specifically, from a bank perspective, this is translated as a long position in a receiver swap and a short position in a receiver swaption, respectively. He tested different compositions and concluded that a portfolio with swaps and nine swaptions seems to be the best compromise in terms of good hedging quality and a limited number of instruments. On the other hand, Jochems and van Haastrecht (2019) proposed a portfolio of zero-coupon bonds and receiver swaptions with a wide variety of tenors to replicate the prepayment option. This was based on a hypothetical portfolio of 1000 mortgages, constructed to capture the same characteristics, in particular, the same interest rate sensitivity as a typical mortgage portfolio of a bank. They ended up with a good fit and noted that the obtained delta was close to that of the mortgage portfolio.

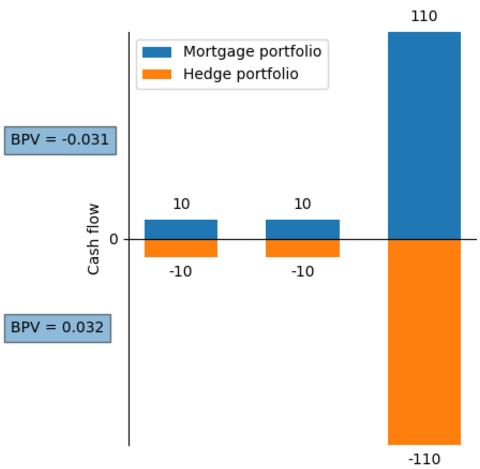
To get a broader picture of hedging approaches, let us consider a different type of financial company that is exposed to prepayment risk: mortgage investors. These institutions purchase mortgages from banks, which they may either hold in their portfolio or repackage and sell as mortgage-backed securities (MBS). In case of the latter, the embedded risk is transferred to the buyers, and the institution is no longer exposed to it. However, if they retain the mortgages, they may need to implement a hedging strategy. Jaffee (2003) evaluated the exposure of the retained mortgages of two of such mortgage investors, Fannie Mae and Freddie Mac. He shows that they used interest rate swaps and option-based derivatives (in particular, swaptions) for their hedging strategy. The swaps were intended to improve the maturity match of long-term mortgages to short-term debt, while the swaptions were used to offset the risks associated with the prepayment option. This approach aligns with the practices of most banks. Since mortgage-backed securities are publicly traded, data on their characteristics and prices are readily available, allowing researchers to construct real data-based models. So, the few articles available on mortgage portfolios, the many studies discuss hedging techniques for MBSs.

Although an MBS differs from a typical bank mortgage portfolio, their hedging results may still be valuable, because both are exposed to prepayment risk. Treasury note futures, particularly those with a lifespan of 10 years, are often used to hedge MBSs. Boudoukh et al. (1994) argues that these instruments are very liquid, and their prices are related to the underlying yield curve, directly creating a link to the value of MBSs. As well as G. Koutmos and Pericli (1999), he proposes a model that captures the relation between the returns of MBSs and the T-note futures. Both papers focus heavily on the choice of an optimal hedge ratio, which is the number of short future contracts needed for one unit of MBS. This is logical because this value immediately gives the related hedge portfolio. Batlin (1987) used a similar approach but with the 30-year treasury bond futures. Moreover, all three articles conclude independently that this strategy only works when implemented dynamically because the derived hedge ratio may become suboptimal when interest rates change during the lifespan of the MBS. In contrast to a static hedge, where purchases at a single moment should cover a wide range of interest rate scenarios, this variant allows rebalancing. That is, when the interest rate market changes sufficiently and extreme events may become more likely, you are allowed to adjust your position in the market accordingly.

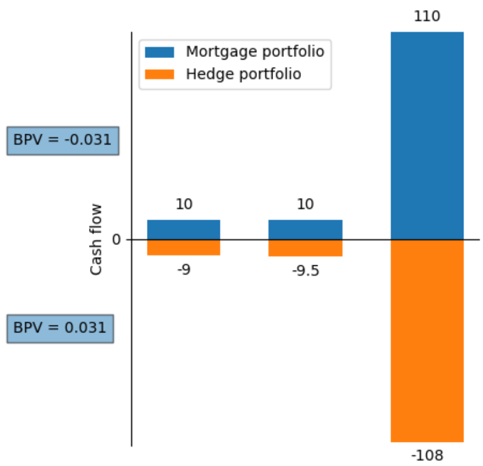
At first sight, it may seem that dynamically hedging is highly favourable to its static equivalent in general because of its ability to capture the interest rate sensitivity more accurately. Moreover, the relatively high costs for instruments with long maturities are avoided, as the hedge portfolio is only purchased when the relevant markets have become more efficient. However, rebalancing incurs transaction costs, which can amount to a large sum of money. In fact, the more frequently you wish to rebalance, the more money is lost to these charges. Nevertheless, research by G. G. Koutmos (2005) conclude that the economic gains by choosing the dynamic technique are substantial. Empirical evidence from the mortgage-backed security market suggests that dynamic hedging is superior in terms of total variance reduction and expected utility maximisation. This conclusion holds even when realistic transaction costs are incorporated into the analysis. However, implementing dynamic hedging may impact the long-term yield curve, as excessively rebalancing can alter the evolution of interest rates, according to Fernald et al. (1994) and Perli and Sack (2003). Therefore, the dynamic strategy should be considered with caution. However, this aspect is outside the scope of this research.

To answer the second question stated at the start of this section, it is important to understand the different types of risk appetite of a bank. Clearly, banks are exposed to interest rate risk, and if we zoom in on the prepayment option, this can be expressed in twofold. On the one hand, when interest rates are low, clients are incentivised to prepay, thus forcing banks to reinvest at a lower rate resulting in a decrease in income. For some banks, this may be undesirable, and instead, they may prefer stable earnings. For those institutions, the notional hedge is the best approach. This technique focuses on optimising the hedge portfolio such that its notional matches that of the mortgage portfolio, ensuring the most stable margin. Examples are given by Casamassima (2018) and Jaffee (2003). On the other hand, when interest rates increase, the prepayment rates tend to be lower. In this scenario, customers most likely will strictly stick to their amortisation schedule, and the cash flows (which include prepayments) may turn out to be lower than expected. This moves the payments up in time, increasing the duration, and thus the bank is exposed to interest rate changes for a longer period. Again, this may be a scenario the bank wishes to prevent. In this case, the objective of the hedge should be to stabilise the value, which can be achieved by either net present value (NPV) matching or basis point value (BPV) matching. As the names suggest, these approaches aim to find a hedge portfolio with the same NPV and BPV as the mortgage portfolio, respectively. For example, Jochems and van Haastrecht (2019) used this approach.

Optimally, a bank would prefer to employ a hedging strategy that can guarantee both a stable margin and value. However, achieving this goal is not feasible due to the disparity in used discount factors. Typically, cash flows from the mortgage portfolio are discounted using the coupon rate, while the external contracts are discounted using the appropriate swap rate plus an internal funding rate known as the FTP. In general, the former rate exceeds the latter, so that smaller notionals in the hedge portfolio are needed to match the value of the mortgage portfolio. As a result, a value hedge is generally cheaper, but comes at the cost of less stable income. To illustrate this, let us consider two banks, Bank X and Bank Y, both offering a simple mortgage of €100 with an annual rate of 10% and annual payments maturing in three years. Bank X aims for a stable margin and constructs a hedge portfolio with cash outflows of 10, 10 and 110, over the contract term, as depicted in Figure 2.3(a). This strategy ensures a net interest income of €0 after each year, thus achieving the desired stable margin. However, the total basis point value (BPV) is nonzero, indicating a mismatch between the BPV of the hedging portfolio and the mortgage portfolio. While Bank X may tolerate this discrepancy, Bank Y, which seeks value stability, may not be satisfied. Therefore, Bank Y may need to adjust the portfolio to match the BPV. One potential outcome is presented in Figure 2.3(b). Note that with this value hedge portfolio, the cash flows of the two portfolios no longer align. This example illustrates that the choice between margin and value stability poses a trade-off for banks. The decision on which approach to adopt depends on several factors, including the bank's specific objectives and risk appetite. However, it is important to note that banks are also obligated to follow the guidelines and regulations set by financial regulators.



(a)



(b)

Figure 2.3: An example of (a) a cash flow hedge and (b) a value hedge with discount factors of 3% and 4% for the hedging portfolio and mortgage portfolio, respectively.

Seidel (2018) explored the differences and the trade-off between hedging from earnings and (economic) value perspective. He noticed that the former merely focuses on the movements of the net interest income when interest rates change, and thus captures short-term effects. In contrast, the latter is more concerned with the long-term effects of interest rate movements, that is, its impact on the value of the bank's assets and liabilities. In the paper, a hedging approach for both perspectives was proposed and tested based on a self-constructed portfolio consisting of bullet loans and mortgages across six interest rate scenarios. Seidel achieved a successful hedge twice, whereby the final hedging portfolio became largely insensitive to either short-term or long-term interest rate movements, depending on the approach. It is important to note that the analyses did not consider prepayment. The author extended the research to incorporate a constant prepayment rate of 1% and concluded that the original hedge portfolio became ineffective under this scenario. Thus, it is crucial to include prepayments in the cash flow analysis before implementing the hedging strategy. Moreover, no clear evidence was provided on which approach was better, arguing that it all comes down to a trade-off between earnings and economic value volatility. The optimal hedging strategy is, therefore, completely based on the bank's preference.

Tang (2020) analysed the effectiveness of the notional hedge and two different approaches within the BPV hedge, namely the pure interest and risk adjusted hedge, in achieving a most stable NII-at-Risk. The distinction between the latter two lies in the composition of mortgage contract rates, which can be split into the prevailing base rate, or the pure interest component, and the margin. The pure interest hedge aims to capture the underlying base rate only, while the risk adjusted hedge includes the margin. After applying the three different strategies to a portfolio of nearly 380,000 mortgages, the author concluded that the risk-adjusted hedge poses the highest repricing risk under interest rate dependent prepayment behaviour. Additionally, there is a lengthening risk (increase of duration) in case of an interest rate up shock because of a reduction in prepayments. However, this risk is insignificant for the considered mortgage portfolio due to a low degree of sensitivity in the prepayments. Therefore, the research concludes that the notional and pure interest hedge are superior to the risk-adjusted hedge from an NII-at-Risk stability perspective.

Until now, all the hedging portfolios mentioned have been based on either value or margin stability. However, other factors could also be considered. For example, Jochems and van Haastrecht (2019) added a constant transaction cost for each instrument to penalise for a large portfolio. Moreover, they included an error term related to the Greek delta in the optimisation problem. Delta measures the change of the value to interest rate changes. Along with the transaction penalty, this approach led to a hedge portfolio of reasonable size that matches the mortgage portfolio both in value and delta as closely as possible. In a different example, Casamassima (2018) investigated the consequences of hedging the gamma sensitivity. This Greek measures the change of delta with respect to changes in the yield curve and is thus considered the second-order sensitivity. It is evident that other Greeks, such as vega and theta (sensitivities with respect to the volatility and time, respectively) can also be incorporated.

It is clear that there are numerous approaches to hedge the prepayment risk embedded in mortgages, both in terms of instruments and quality assessment. In this project, our aim is to maintain both margin and value stability. To achieve this, different combinations of instruments, such as bonds and swaptions, will be evaluated for their ability to ensure a stable margin and value.

3

Interest rate model

This chapter discusses the Hull-White model, which is used to simulate interest rates. Firstly, the dynamics of this model together with some of its key features are presented. Next, the calibration process is explained and finally, the data required for calibrating are described and the resulting outcomes are given.

3.1. Hull-White model

The Hull-White model is an interest rate model designed to eliminate arbitrage opportunities by ensuring its drift term $\theta(t)$ is a deterministic function of time, selected to precisely match the current yield curve. To achieve this, the model fits theoretical bond prices to the observed yield curve in the market. The model assumes a normal distribution of the short-term interest rate, which results in a positive probability of negative interest rates, which is desirable given the current environment of low or negative interest rates. The one-factor version of the Hull-White model (HW1F) assumes that the short-term interest rate, $r(t)$, follows a mean-reversion process and, under the risk-neutral measure $\mathbb{Q}$, has the dynamics given by

$$\mathrm{d}r(t) = (\theta(t) - \alpha r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}}(t), \tag{3.1}$$

where

- $\theta(t)$ is the time-dependent drift term,
- α is the mean-reversion coefficient,
- $\sigma > 0$ is the short rate volatility, and
- $W^{\mathbb{Q}}(t)$ represents a Brownian motion under the measure $\mathbb{Q}$.

This stochastic differential equation results in a normally distributed short rate. In Appendix B.1.1 we obtained the following expressions for its mean and variance:

$$r(t) \sim \mathcal{N}(\mu_{r,\mathbb{Q}}(t), \sigma^2_{r,\mathbb{Q}}(t))$$

with

$$\mu_{r,\mathbb{Q}}(t) = e^{-\alpha(t-t_0)} r(t_0) + \int_{t_0}^{t} e^{-\alpha(t-s)}\theta(s)\,\mathrm{d}s,$$
$$\sigma^2_{r,\mathbb{Q}}(t) = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-t_0)}\right).$$

Furthermore, the Hull-White model belongs to the class of so-called affine term structure models (see Appendix B.1.2). This means that its drift $\bar{\mu}$, volatility $\bar{\sigma}^2$ and interest rate component r can be expressed in an affine form:

$$\bar{\mu}(r(t)) = \theta(t) - \alpha r(t),$$
$$\bar{\sigma}(r(t))^2 = \sigma^2 + 0 \cdot r(t),$$
$$\bar{r}(r(t)) = 0 + 1 \cdot r(t).$$

This characteristic is extremely convenient as it enables us to find closed-form solutions for the value of various financial instruments. In Appendix B.2, this will be discussed in more detail. Particularly, the price of a zero-coupon bond, denoted by $P(t,T)$, holds significant value as it allows us to simulate a wide range of interest rates. An illustrative example is the Euribor rate, which represents the average rate at which banks within the Eurozone borrow from each other. The most widely used rate is the 6M-Euribor rate, which resets every 6 months. At time t, this rate is defined as

$$E(t;t,t+6/12) = \frac{1 - P(t,t+6/12)}{\frac{6}{12}P(t,t+6/12)}. \tag{3.2}$$

For further insights into the details and derivation of this forward rate, please refer to Appendix A.1.2.

It evident that the prices of financial instruments and the values of the simulated interest rates are dependent on the parameters of the Hull-White model, namely α and σ. The α parameter determines the speed at which the interest rate reverts to its mean level. So, a larger α corresponds to a faster mean reversion and leads to a flatter yield curve, as it reduces the differences in interest rates for bonds with different maturities. This can impact the pricing of financial instruments. On the other hand, the σ parameter represents the volatility of interest rates. A higher σ value results in more significant fluctuations in interest rates over time. This volatility affects the values of bonds and derivatives, leading to larger changes in their prices.

To ensure accurate simulations and pricing that align with current market conditions, it is necessary to calibrate the model's parameters based on observed market data. The calibration process involves selecting α and σ that best fit the market dynamics and prevailing conditions. In the upcoming section, the calibration method will be discussed in detail.

3.2. Calibration method of the Hull-White model

Calibration involves adjusting the values of the Hull-White model's parameters to match the observed behaviour of interest rates in the market. This is important for two reasons: firstly, a properly calibrated Hull-White model can generate realistic interest rate scenarios. Secondly, it can be used to accurately value financial products such as bonds, options, and swaps, which is essential for this research. Typically, the calibration is based on swaption volatility data, ensuring that the modelled volatility is consistent with market observations. While this paper does not cover the full calibration process, only a pseudocode will be provided, as presented in Algorithm 1.

Algorithm 1: Pseudocode calibration Hull-White

Input: Set of at-the-money swaption volatilities with different option expiries and swaption maturities, current yield curve for the underlying currency

Output: Calibrated Hull-White model parameters α and σ

1. Build the initial term structure by an interpolation method such as linear or cubic.
2. Calculate the swaption price for each volatility using the created yield curve.
3. Calculate the price of each swaption using the Hull-White model based on the created yield curve.
4. Define a pricing fit error function that calculates the difference between the market quotes and the model prices.
5. Use an optimisation algorithm such as Levenberg-Marquardt to find the calibrated Hull-White model parameters that minimise the error.

return *Calibrated α and σ.*

Executing this pseudocode allows for the calibration of α and σ in the Hull-White model. However, a drawback of this approach is the need for regular updates of these values due to ever-changing market conditions. As market dynamics evolve, the previously calibrated values may become less accurate, requiring recalibration. Another drawback is the limited market data coverage. The calibration process as described in Algorithm 1 relies on current yield curve data and available swaption data, which may not capture the full range of market instruments. The calibration accuracy may be affected by this limited market coverage, potentially leading to pricing errors. Lastly, it is worth noting that the Hull-White model is a relatively simple model that assumes a single driver of the yield curve, enabling it to capture parallel movements in interest rate. However, it cannot simulate non-parallel movements such as flattening or steepening of the yield curve. Despite this limitation, since parallel movements are frequently observed in the market, the Hull-White model is widely employed to simulate interest rates.

Having examined the calibration method and its potential shortcomings, the focus now shifts towards the data used in the calibration process and the corresponding results. In the following section, we will discuss the data sources and variables employed for calibration, as well as analyse and interpret the calibrated results.

3.3. Calibration data and results of the Hull-White model

The required data for calibrating the Hull-White model is obtained on February 17, 2023, using the generic market data feed of the bank. This data feed is a reliable source and contains up-to-date market data.

The first dataset we acquired is the yield curve data. It includes spot rates for maturities from 1 day to 6 months and par rates for maturities from 1 year to 50 years. These rates are based on the 6M-deposit rates. To provide a visual representation, the yield curve based on this data is plotted, as illustrated in Figure 3.1. This curve serves as a crucial component for setting up the initial term structure in the calibration process.

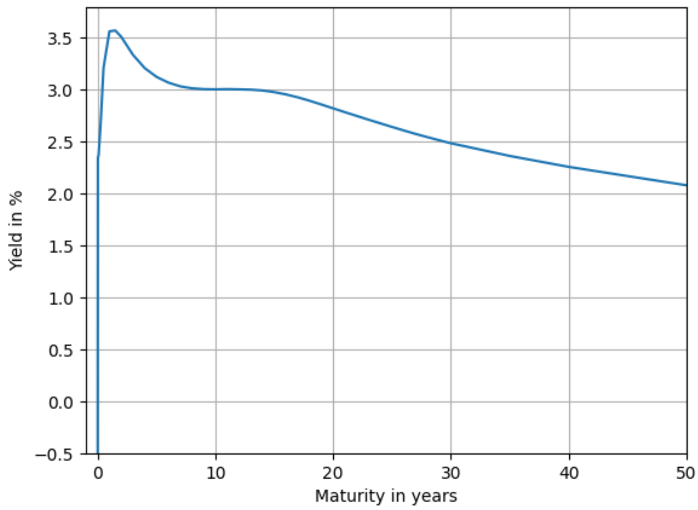


Figure 3.1: The yield curve based on the 6M-deposit rates as of February 17, 2023. *Note.* The yield curve data is obtained from a generic market data feed of the bank.

The second dataset we obtained consists of swaption volatilities. Specifically, our focus was on at-the-money swaptions with the 6M-Euribor as the underlying asset. We considered combinations of underlying swap maturities (5, 7, 10, and 20 years) and swaption maturity dates (1, 5, and 10 years) to derive an overview of the normal swaption volatilities. The corresponding values are presented in Table 3.1.

Swaption tenor vs swap tenor	5 years	7 years	10 years	20 years
1 year	103.453	103.169	100.967	92.552
5 years	91.005	89.229	86.878	77.948
10 years	79.391	77.232	74.347	64.633

Table 3.1: The normal swaption volatilities in basis points (1 basis point = 0.01%) for different combinations of swap and swaption maturities as of February 17, 2023. *Note.* The swaption volatility data is obtained from a generic market data feed of the bank.

The two datasets, the yield curve and the swaption volatilities, are utilised as input in Algorithm 1 to calibrate the parameters α and σ of the Hull-White model. The calibration process aims to align the model's simulated swaption prices with the observed market quotes, taking into account the information provided by these datasets. The resulting calibrated values of α and σ, which can be found in Table 3.2, reflect the market conditions as of February 17, 2023.

Parameter	Outcome calibration
α	0.0458
σ	0.0116

Table 3.2: The calibrated parameters of the Hull-White model as of February 17, 2023, using Algorithm 1.

From these results we can conclude that the model exhibits a moderate speed of mean reversion and captures relatively small interest rate fluctuations. In other words, interest rates tend to revert towards their long-term average at a moderate pace, while the volatility of interest rates in the model is relatively low. As we proceed with the thesis, these calibrated parameters will be considered as fixed.

The fixed parameter values allow for the simulation of interest rates using the Hull-White model. As an example, Figure 3.2 illustrates the simulation of 1000 future 6M-Euribor rates for the next 10 years, accompanied by historical values. The simulation values are generated using Equation (3.2), which incorporates the dynamics of the Hull-White model.

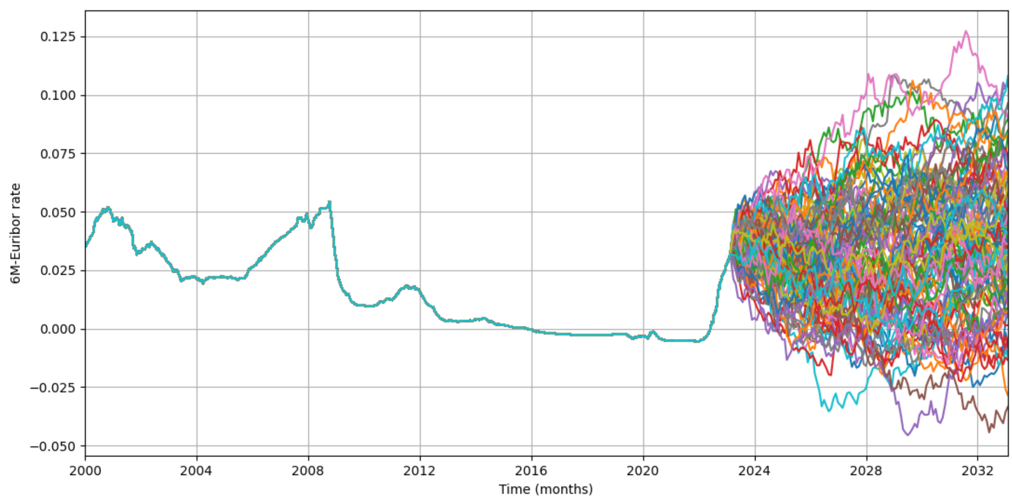


Figure 3.2: The historical 6M-Euribor rates together with 100 simulations using the calibrated Hull-White parameters $\alpha = 0.0458$ and $\sigma = 0.0116$.

The simulations in Figure 3.2 provide valuable insights into the potential behaviour of the 6M-Euribor rates under the assumptions of the Hull-White model. However, when it comes to managing a mortgage portfolio, it is essential to consider not only interest rate movements but rather their impact on prepayments. Understanding and modelling prepayment behaviour is crucial for effectively hedging a mortgage portfolio and mitigating the associated risks. Therefore, in the upcoming chapter, we will focus on prepayment modelling. Specifically, we will present a prepayment model and analyse its effect on cash flows of a given mortgage portfolio.

4

Cash flow model

This chapter presents a cash flow model that includes both the monthly contractual interest payments and the prepayments of the mortgagors. It starts with describing an approach to model prepayment rates and continues by discussing how this model can be used to understand the behaviour of mortgagors in different interest rate scenarios. Subsequently, these prepayments are combined with the contractual repayments, resulting in a total cash flow model, which is applied to the relevant mortgage portfolio.

4.1. Prepayment model

Prepayment behaviour refers to the act of borrowers paying off their mortgages earlier than the scheduled term. It is influenced by a multitude of factors, each with its own impact on the decision-making process. Mortgagor characteristics, such as age, creditworthiness, and income stability, can influence the likelihood of prepayment. Economic conditions, such as interest rates and unemployment rates, also play a role in shaping borrower behaviour.

However, among these various drivers, the refinancing incentive has consistently proven to be the most influential factor behind prepayment behaviour. The refinancing incentive refers to the potential savings that borrowers can achieve by refinancing their existing mortgages at a lower interest rate. When interest rates decline, borrowers have a stronger incentive to refinance their mortgages, as it allows them to reduce their monthly payments or shorten their loan term. Consequently, refinancing activities tend to increase during periods of declining interest rates, leading to higher prepayment rates. To quantify the refinancing incentive, we can define it mathematically as the difference between the mortgage rate and a relevant swap rate plus a fixed margin. Specifically, for a mortgage with a coupon rate of c and a maturity of T, the incentive at a given time t, denoted by $Inc(t)$, can be expressed as follows:

$$Inc(t) = c - S(t, t, T) - 1.5\%, \tag{4.1}$$

where $S(t, t, T)$ represents the 6M-Euribor swap rate at time t with tenor T. The 6M-Euribor swap rate refers to the interest rate associated with a 6M-Euribor swap and serves as a benchmark for short-term euro interest rates. So, by incorporating this rate in the refinancing incentive formula, we account for the prevailing market interest rates and their impact on the attractiveness of refinancing. The fixed margin of 1.5% included in the formula accounts for additional costs involved in refinancing. It acknowledges that there may be expenses or considerations beyond the pure interest rate differential that borrowers need to consider when contemplating refinancing their mortgages. Therefore, the complete formula provides a reliable measure of the financial advantage borrowers may gain by refinancing their mortgages.

With the definition of the refinancing incentive, we can now explore its relationship with prepayment rates. From a rational perspective, one might expect prepayment behaviour to follow a step function, where borrowers only choose to prepay when the incentive is positive. However, observations in the market reveal a more nuanced reality. People prepay even when the refinancing incentive is negative, and in some cases, they may not prepay significantly even when the incentive is high. This discrepancy suggests the need for a more flexible modelling approach. To capture these non-linear dynamics, we turn to the logistic function. By employing a logistic function, we can effectively describe the relationship between the refinancing incentive and prepayment rates, accounting for the observed behaviour in the market. In the upcoming subsection, we will delve into the details of the logistic function and its application in prepayment behaviour.

4.1.1. Logistic function

The logistic or double-asymptotic function is a mathematical function that is commonly used to model S-shaped curves. In the context of prepayment rates, denoted as $PP(t)$ in month t, we can express it as follows:

$$PP(Inc) := f(Inc) = a + \frac{b}{1 + e^{-c(Inc-d)}}.$$

This formulation involves four parameters that determine the shape of the function and carry specific interpretations. Parameter a represents the minimum prepayment rate, which is typically slightly above zero as mortgagors tend to prepay to some extent consistently. Parameter b defines the range of prepayment rates, with $a + b$ representing the maximum prepayment rate. The sensitivity of prepayments to the refinancing incentive is determined by parameter c. A positive value of c indicates that as the incentive increases, the prepayment rates grow. Conversely, when c has a negative value, an increase in incentive is followed by a decline in prepayments. Given that mortgagors generally tend to prepay when the market rates are low, indicating a high refinancing incentive, the value of c is typically positive. Lastly, parameter d represents the incentive value at which borrowers are the most sensitive to changes in the refinancing incentive. To visualise the impact of these parameters, consider Figure 4.1 which presents two examples of the logistic function with distinct parameter settings.

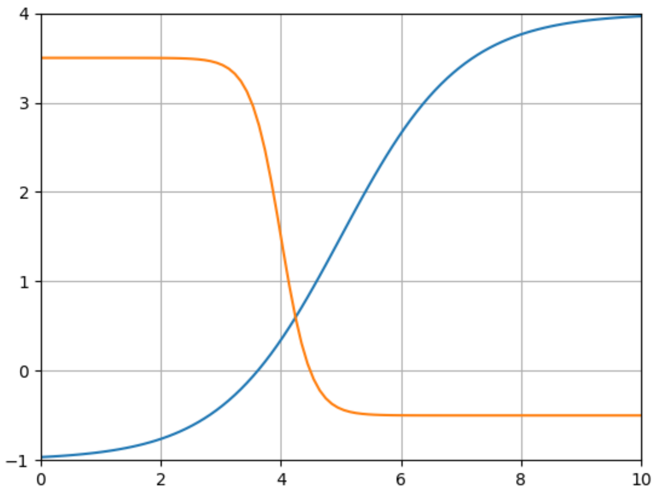


Figure 4.1: The logistic function $f(x) = a + \frac{b}{1+e^{-c(x-d)}}$ with 1) $a = -1$, $b = 5$, $c = 1$ and $d = 5$ (blue) and 2) $a = -0.5$, $b = 4$, $c = -4$ and $d = 4$ (orange).

By analysing the example curves in Figure 4.1, we can observe how different parameter combinations alter the shape and characteristics of the logistic function. Consequently, the four parameters allow for flexibility in tailoring the function to a specific data set. This flexibility is crucial when aiming to develop a prepayment model that accurately captures observed prepayment behaviour. To achieve this, a calibration process is required. In the upcoming subsection, we will discuss the calibration of these parameters using monthly prepayment data and analyse the resulting outcomes.

4.1.2. Calibration method of the prepayment model

To model the relationship between prepayment rates PP and refinancing incentives Inc using a logistic function, the parameters must be calibrated based on historical prepayment data. This calibration involves determining the values of a, b, c, and d that best fit the data within the logistic function

$$PP(Inc) = f(Inc) := a + \frac{b}{1 + e^{-c(Inc-d)}}.$$

To achieve an optimal fit, the mean-squared error (MSE) between the observed monthly prepayment rates and the corresponding modelled rates is minimised. The MSE is a widely used metric that quantifies the level of deviation in the model's prediction. By squaring the differences, the MSE places more weight on large errors, thereby encouraging accurate predictions. For the minimisation of the mean-squared error, the SLSQP algorithm is employed. This algorithm is specifically designed to minimise a scalar function of one or more variables using Sequential Least Squares Programming. It requires only an initial guess for the parameter values. The SLSQP algorithm is chosen for its ability to handle optimisation problems with constraints efficiently. The complete calibration process is outlined in Algorithm 2.

Algorithm 2: Mean-squared error minimisation using the SLSQP algorithm.

Input: Real data $\mathbf{y} \in \mathbb{R}^N$ function $f(\cdot)$ that computes $\hat{\mathbf{y}} \in \mathbb{R}^N$ for a given parameter vector $\theta = (a, b, c, d)$.
Output: Calibrated parameters $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$

1. Initialise parameters $\theta_0 := (a_0, b_0, c_0, d_0)$.
2. Define objective function $MSE(a, b, c, d) = \frac{1}{N}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2$.
3. Call the SLSQP optimisation algorithm with the objective function $MSE(\cdot)$, and initial guess θ_0.

return $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$

To successfully calibrate the logistic function, it is essential to utilise historical prepayment. By analysing the historical data, we gain insights in how the mortgagors' behaviour has been influenced by changing market conditions. In the following subsection, we will discuss the calibration data, assess its characteristics, and analyse the resulting calibration outcomes.

4.1.3. Calibration data and results of the prepayment model

The data used for the calibration of the prepayment model is from an internal project of the bank, specifically focusing on prepayment behaviour in Italy. The dataset comprises monthly observations of the historical refinancing incentives and corresponding prepayment rates. The data covers a time period from January 1, 2010, to February 1, 2022, allowing us to capture a significant span of prepayment behaviour over several years. To facilitate the calibration process, the data is grouped by incentive level, providing aggregated information on the notional amount and the associated monthly prepayment rate. Table 4.1 presents an extract from the historical Italian prepayment data used for calibrating the logistic function.

Date	Incentive	Notional	Monthly prepayment rate
⋮	⋮	⋮	⋮
01/01/2010	1.25%	420220000	1.62%
01/01/2010	1.50%	40640000	1.34%
01/01/2010	1.75%	410000	0.00%
01/02/2010	-2.25%	520000	0.00%
01/02/2010	-2.00%	700000	0.00%
⋮	⋮	⋮	⋮

Table 4.1: An extract of the historical Italian monthly prepayment data used for the calibration of the logistic function. *Note.* The historical data is obtained from an internal project of the bank.

It is important to note that the use of Italian data rather than Dutch data does not pose any issues for this project. Since we are only considering the refinancing incentive as the driver of prepayments and not incorporating any country-specific factors, the calibration can effectively capture the relationship between prepayment rates and refinancing incentives.

To provide an overview of the complete historical dataset, Figure 4.2 shows the distribution of monthly prepayment rates against the refinancing incentive.

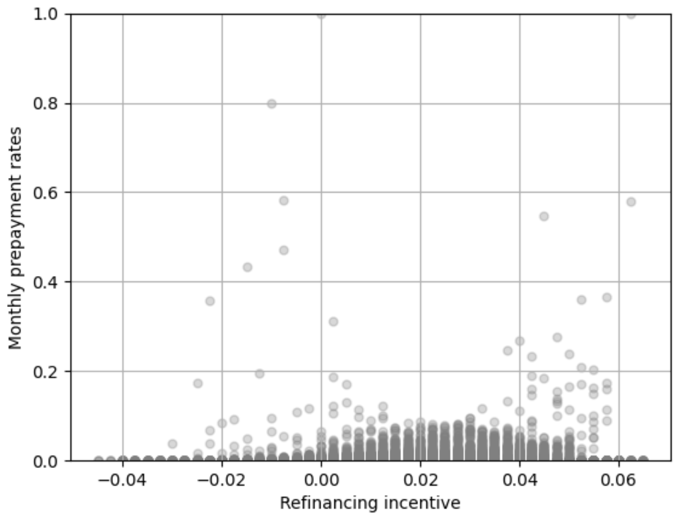


Figure 4.2: A scatter plot of historical Italian monthly prepayment rates against the refinancing incentive. *Note.* The historical data is obtained from an internal project of the bank.

From this figure several observations can be made regarding the relationship between prepayment rates and the refinancing incentive. First of all, it is evident that as the incentive rate increases, the monthly prepayment rates generally exhibit an upward trend. This implies that mortgagors are more inclined to prepay their mortgage when the incentive is higher, indicating lower interest rates. Secondly, historical incentive rates span from -4.50% to 6.5%, while prepayment rates vary from 0% to 100%. However, a significant proportion of the data, approximately 99%, falls within the range of 0% and 10%. It is worth noting that the remaining 1% of the data, and in specific the full prepayments (100%), should not be considered as outliers. In fact, it is not uncommon for clients to (almost) fully prepay their mortgage, especially when they are moving to a new house. Thirdly, the figure reveals a substantial proportion of the observations corresponding to zero prepayments, particularly when there is a negative incentive. Further analysis indicates that these zero prepayment instances represent approximately 40% of the data. This high portion suggests that a significant number of borrowers do not take advantage of refinancing opportunities. Consequently, we expect that the calibration process will pull the fitted logistic function towards the x-axis to account for this prevalent behaviour.

These observations provide valuable insights into the characteristics of historical prepayment data and allows us to set up the calibration process. For this, only an initial guess for the parameters is required. Based on Figure 4.2, the following parameter values are selected:

$$(a_0, b_0, c_0, d_0) = (0, 0.1, 200, 0.01).$$

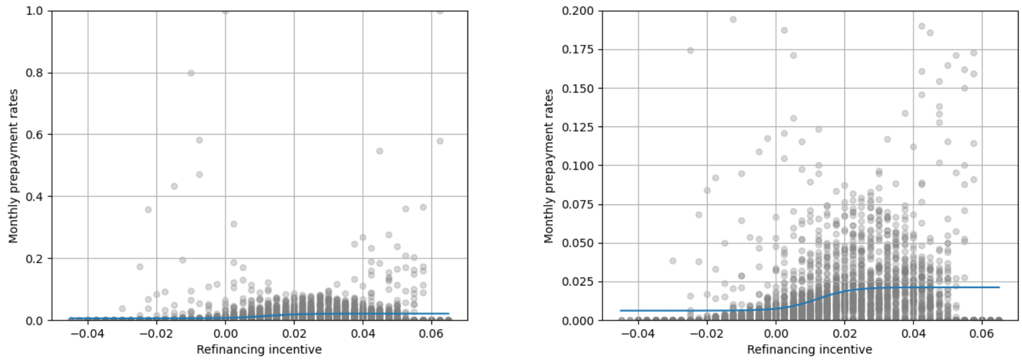
This parameter vector, along with the historical prepayment data, serves as input in Algorithm 2 to calibrate the prepayment model. The resulting calibrated parameters are presented in Table 4.2.

These calibrated parameters can be interpreted as follows: The fitted monthly prepayment rates range from 0.62% to 2.12% and demonstrate a positive relationship with the refinancing incentive. This implies that, since c has a positive value, there is more prepayment behaviour when the incentive increases. Additionally, the clients are the most sensitive when the refinancing incentive is around 1.22%. A small deviation from this value leads to a relatively large change in prepayment rates. The goodness of fit can be read from the mean-squared error. An MSE of 0.0017 means that on average the squared difference between the predicted values and the actual values is 0.0017. Since a significant portion of the data falls within the 0% to 10% range, this value can be considered a reasonably good

Parameter	Outcome calibration	Mean squared error
$\tilde{a}$	0.0062	
$\tilde{b}$	0.0150	
$\tilde{c}$	200.00	
$\tilde{d}$	0.0122	$1.7160 \cdot 10^{-3}$

Table 4.2: The calibrated parameters of the logistic prepayment model $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$.

fit. To further assess the quality of the fit, it can be insightful to visualise the results through a figure. Figure 4.3 illustrates the calibration data together with the fitted logistic function, presenting both the entire dataset and a zoom-in view.



(a) The entire dataset.

(b) Zoom-in view of the dataset.

Figure 4.3: Two scatter plots of historical monthly prepayment against the refinancing incentive together with the fitted logistic function, using the parameters from Table 4.2.

At first sight, the fitted logistic function seems to effectively capture the prepayment behaviour. However, a closer examination of the figure reveals that some significant deviations from the fitted curve occur in cases of the highest prepayment rates. In particular, there are instances of (almost) full prepayments, whereas the fitted maximum prepayment rate is limited to 2.12%. As a results, these deviations will contribute substantially to the overall MSE as the squared error are amplified. Our previous data analysis revealed that approximately 1% of the data exceeds the 10% prepayment threshold. While this percentage may not seem significant, the deviation of these data points greatly influences the size of the MSE. However, when considering the notionals associated with these mortgages, it becomes apparent that they account for only 0.04% of the total notional. This suggests that these mortgages may not be as important as the remaining 99% in terms of their impact on the total prepayment behaviour. To address this issue, one approach is to introduce weighting to the data points based on their respective notionals. By assigning greater importance to mortgages with larger notional values, the impact of these mortgages on the analysis can be appropriately reflected. This weighted approach is particularly relevant for banks as it acknowledges the higher exposure to prepayment risk that banks face with such mortgages in absolute terms.

Figure 4.4 presents the scatter plot in which the data is weighted by their respective notionals, demonstrating a more compact set of data points with the fading of the more extreme values. This is due to the fact that these extremes correspond to mortgages with small notionals.

To incorporate the notionals in the calibration process, Algorithm 2 must be adjusted accordingly. This can be accomplished by introducing a weight factor based on the relative notional value into the objective function. By doing so, more emphasis is placed on minimising the error term for the larger mortgages. The modified pseudocode is presented in Algorithm 3 and the results of the new calibration process are given in Table 4.3.

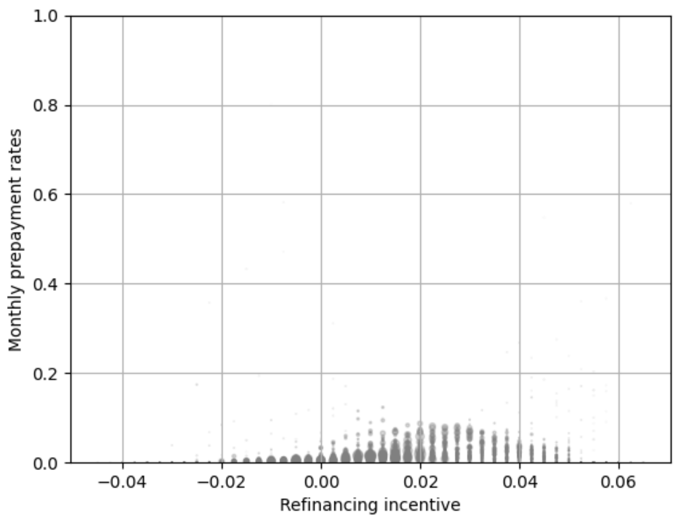


Figure 4.4: Scatter plots of historical monthly prepayment against the refinancing incentive, weighted by their respective notional.

Algorithm 3: Weighted mean squared error minimisation using the SLSQP algorithm

Input: Real data $\mathbf{y} \in \mathbb{R}^{3528}$ with corresponding notional $\mathbf{N} \in \mathbb{R}^{3528}$, the total notional N_{tot}, and the function $f(\cdot)$ that computes $\hat{\mathbf{y}} \in \mathbb{R}^{3528}$ for a given parameter vector $\theta_w = (a_w, b_w, c_w, d_w)$

Output: Calibrated parameters $(\tilde{a}_w, \tilde{b}_w, \tilde{c}_w, \tilde{d}_w)$

1. Initialise parameters $\theta_{w,0} := (a_{w,0}, b_{w,0}, c_{w,0}, d_{w,0}) = (0, 0.02, 200, 0.01)$.
2. Define objective function $MSE(a_w, b_w, c_w, d_w) = \frac{1}{3528} \sum_{i=1}^{3528} \frac{N_i}{N_{tot}} \cdot (\hat{y}_i - y_i)^2$.
3. Call the SLSQP optimisation algorithm with the objective function $MSE(\cdot)$, and initial guess $\theta_{w,0}$.

return $(\tilde{a}_w, \tilde{b}_w, \tilde{c}_w, \tilde{d}_w)$

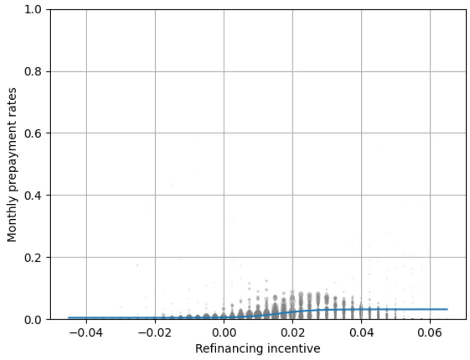
Parameter	Outcome calibration	Mean squared error
$\tilde{a}_w$	0.0046	
$\tilde{b}_w$	0.0272	
$\tilde{c}_w$	200.00	
$\tilde{d}_w$	0.0162	$4.9673 \cdot 10^{-8}$

Table 4.3: The calibrated parameters of the logistic prepayment model for the weighted case $(\tilde{a}_w, \tilde{b}_w, \tilde{c}_w, \tilde{d}_w)$.

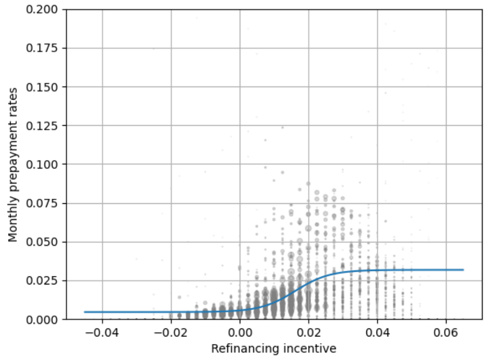
Although the newly calibrated parameters show little variation in comparison to the unweighted case in Table 4.2, the mean-squared error is notably less in the weighted scenario. This can be explained by the fact that the extreme prepayment rates correspond to mortgages with a relatively small notional. Their significant deviation from the fitted curve heavily influences the total error in the unweighted case, but this effect is scaled down when we include the relative weight of their notionals. This effect is also visible in Figure 4.5, which shows the calibration data together with the new fitted logistic function, presenting both the entire dataset and a zoom-in view.

We conclude that the results obtained in the weighted scenario are more reasonable as it places more emphasis on mortgages with a larger notional. Hence, for the remainder of the thesis, the final prepayment function will be

$$PP(Inc) = a + \frac{b}{1 + e^{-c(Inc-d)}}, \tag{4.2}$$



(a) The entire dataset.



(b) Zoom-in view of the dataset.

Figure 4.5: Two scatter plots of historical monthly prepayment against the refinancing incentive together with the fitted logistic function, using the parameters from Table 4.3.

with

$$a = 0.0046, \quad b = 0.0272, \quad c = 200, \quad d = 0.0162.$$

This function provides us with a valuable tool for understanding the potential impact of changing interest rates on mortgage prepayments. In the next section, we will consider a specific mortgage portfolio and analyse its cash flows, considering both scenarios with and without prepayments. Through this analysis, we aim to uncover valuable insights into the effects of interest rate fluctuations on the timing and magnitude of mortgage prepayments.

4.2. Mortgage cash flows

In this thesis, we consider a hypothetical portfolio (hereafter, the mortgage portfolio) comprising six bullet mortgages, with varying notionals, maturities and coupon rates. The portfolio, displayed in Table 4.4, has been carefully constructed to mirror the exposure and sensitivity to interest rate movements found within a mortgages portfolio of a Dutch bank. Working with this mortgage portfolio allows us to conduct insightful analyses while ensuring the confidentiality of the bank's disclosed data.

Mortgage	1	2	3	4	5	6
Notional	217,594	217,791	776,889	233,714	210,013	144,000
Remaining FIRP (years)	2	3	5	7	10	8
Coupon rate (annual)	6.84%	6.00%	5.13%	4.34%	3.59%	2.83%

Table 4.4: A representative mortgage portfolio of six bullet mortgages with various notionals, remaining fixed interest rate periods (FIRPs) and coupon rates.

Upon examination of this table, several observations can be made. Firstly, it is evident that five out of the six mortgages have a notional value of approximately €$200,000$, signifying a consistent amount across the majority of the portfolio. However, the third mortgage stands out with a substantially larger notional value of around €$800,000$. The second row of the table indicates the remaining fixed interest rate period (FIRP) for each of the mortgages. It is assumed that the entire notional will be fully repaid at the end of this period. On average, the remaining fixed interest rate period amounts to 5.8 years, which aligns with a realistic value of 5 years. Finally, we observe that the coupon rates are generally higher for older mortgages. This pattern can be explained by the fact that these older mortgages are usually deeper in the money, leading to higher coupon rates being applied.

Without considering prepayments, the cash flows of the mortgage portfolio primarily consist of monthly interest payments and the final repayments at maturity of each individual mortgages. For illustration, we consider a simplified mortgage with a notional value of €1000, a monthly coupon rate of 1%, and a maturity of one year. The monthly cash flows for this mortgage are presented in Figure 4.6.

Time	1	2	…	11	12
Interest payment	10	10	…	10	10
Repayment	-	-	…	-	1000

Figure 4.6: The monthly cash flows of a mortgage with a monthly coupon rate of 1%, a notional value of €1000 and a maturity of one year, excluding prepayments.

This example shows the stable interest income each month, with the full redemption of the notional amount upon maturity. In case of the mortgage portfolio from Table 4.4, the cash flows without prepayments can be obtained in a similar manner. Figure 4.7 provides an overview of the aggregated cash flows over the entire 10-year period. Note that the spikes in the figure represent the full repayments occurring at the end of the fixed interest rate periods, matching the level and timing indicated in the first and second rows of Table 4.4, respectively.

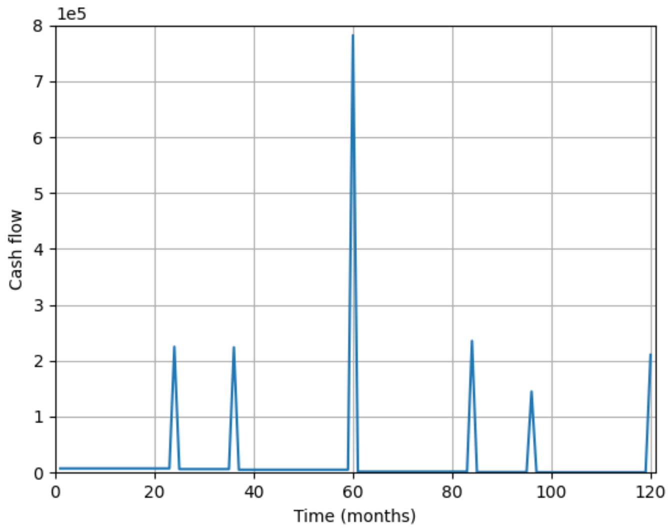


Figure 4.7: The cash flows of the total mortgage portfolio without prepayments.

The cash flows depicted in Figure 4.7 are purely theoretical, as they do not account for the potential impact of prepayments by customers. How incorporating the prepayments affect the actual cash flows of the mortgage portfolio will be analysed in the next subsection.

4.2.1. Mortgage cash flows with prepayments

Mortgagors have the option to prepay (part of) one's notional. For this thesis, we consider the refinancing incentive as sole driver for this behaviour, which is defined as the difference between the coupon rate and a reference rate that represents an alternative way to invest or to refinance the loan. Using Equation (4.1), the refinancing incentive for the first mortgage is thus given by

$$Inc_1(t) = 6.84\% - S(t,t,2) - 1.5\%,$$

with $S(t,t,2)$ the 6M-Euribor swap rate with a two-year maturity, as defined in Equation (B.6). Similarly, we can define the refinancing incentive for each mortgage of the portfolio. The relation between these incentives and the prepayment rates was derived in the previous section, and is repeated here for the sake of completeness:

$$PP(Inc) = a + \frac{b}{1+e^{-c(Inc-d)}}, \tag{4.3}$$

with

$$a = 0.0046, \quad b = 0.0272, \quad c = 200, \quad d = 0.0162.$$

With the use of this formula, the prepayment rates for each of the mortgages under different interest scenarios can be simulated. These prepayments may impact not only the timing but also the magnitude of the cash flows within the mortgage portfolio. To illustrate this phenomenon, we will consider an example.

Let us revisit the mortgage with a notional amount of €1000, a monthly coupon rate of 1%, and a maturity of one year. However, this time we will allow for prepayments. Suppose the customer decides to prepay 2% of its notional in the first month, which equates to €20. As a result, the outstanding notional in the second month reduces to €980, and the interest payment in this month decreases correspondingly to €9.80. These changes can be observed in Figure 4.8.

Time	1	2	…	11	12
Prepayment	20	-	…	-	-
Interest payment	10	9.8	…	9.8	9.8
Repayment	-	-	…	-	980

Figure 4.8: The monthly cash flows of a mortgage with a monthly coupon rate of 1%, a notional value of €1000 and a maturity of one year, including a prepayment of 2% in the first month.

The table highlights the impact of prepayments on both the timing and the level of the subsequent cash flows of the mortgage. While this example only considers a single prepayment, it is important to note that prepayments can occur monthly. In general, if we represent the prepayment made in month t by Λ_t, the cash flows of a bullet mortgage with notional N, annual coupon rate c, and maturity T can be expressed as illustrated in Figure 4.9. This representation shows the dynamic nature of cash flows that arise when prepayments are incorporated into the analysis.

Time	1	2	…	$T-1$	T
Prepayment	Λ_1	Λ_2	…	Λ_{T-1}	-
Interest payment	$\frac{c}{12} \cdot N$	$\frac{c}{12} \cdot (N - \Lambda_1)$	…	$\frac{c}{12} \cdot \left(N - \sum_{i=1}^{T-2} \Lambda_i\right)$	$\frac{c}{12} \cdot \left(N - \sum_{i=1}^{T-1} \Lambda_i\right)$
Repayment	-	-	…	-	$N - \sum_{i=1}^{T-1} \Lambda_i$

Figure 4.9: The monthly cash flows of a mortgage with annual coupon rate c, notional value N and a maturity of T months, including monthly prepayments $\Lambda_1, \dots \Lambda_{T-1}$.

Using this scheme, it is possible to simulate the cash flows for the mortgage portfolios under various interest rate scenarios. The initial step involves generating Euribor swap rates using the Hull-White model. These swap rates are subsequently employed to calculate the refinancing incentive for each mortgage, applying Equation (4.1). The obtained incentives serve as inputs to the prepayment function, represented in Equation (4.3), to determine the corresponding monthly prepayment rates denoted as $\lambda_{m,t}$ for mortgage m and time t. With the prepayment rates identified, the prepayment amounts can be computed using the following formula:

$$\Lambda_{m,t} = \lambda_{m,t} \prod_{i=1}^{t-1} (1 - \lambda_{m,t}).$$

Note that the scenario where $\lambda_{m,t} = 0 \; \forall t$ corresponds to the absence of prepayments.

The effect of allowing prepayments to the mortgage cash flows can be seen in Figure 4.10, which considers the first mortgage from Table 4.4. Without prepayments, the cash flows consist of the steady monthly contractual interest payments and a repayment of the full notional after two years. In contrast, if prepayments are allowed, the cash flows become more variable and are substantially higher over the course of the mortgage, except for the final payment. This is because each prepayment reduces the outstanding notional, resulting in a final payment that is less than what it would be without prepayments.

It is clear that allowing mortgagors to prepay part of their mortgage can significantly affect the level and timing of the mortgage cash flows. For banks, particularly those with large mortgage portfolios, these effects can result in major mismatches and potential challenges. Therefore, understanding and effectively managing prepayment risk is of high importance to banks. To address this concern, one approach is the implementation of a hedging portfolio. By adopting a hedging strategy, banks seek to

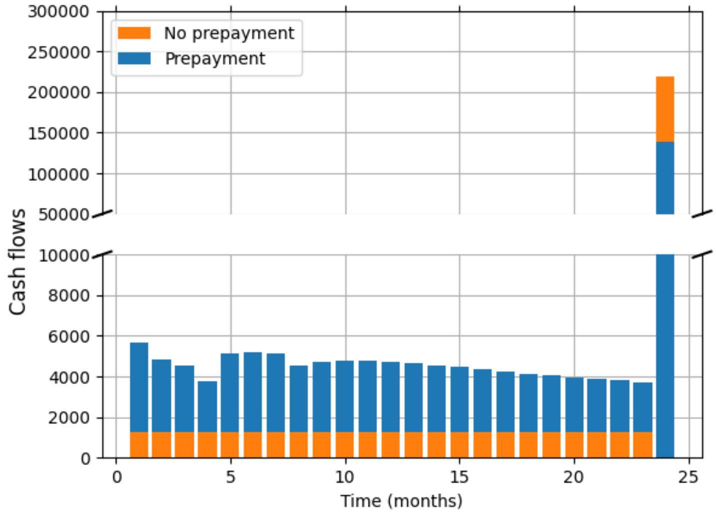


Figure 4.10: One simulation of the cash flows of the first mortgage of the portfolio with prepayments (blue) and without prepayments (orange).

mitigate the effects of prepayment risk on their earnings and value. The forthcoming chapter will delve into an investigation of such hedging methods. The primary objective will be to achieve stability from an earnings and value perspective.

5
Hedging approaches

This chapter contains the core research of the thesis. It starts with an introduction which explains the general mortgage issuance process within a bank, introduces the corresponding baseline hedge, and outlines the bank's objectives for a hedge portfolio. Subsequently, hedge approaches aiming for margin stability are examined and analysed using introduced risk metrics, followed by a similar analysis for hedging portfolios focusing on value stability. Finally, all proposed hedging portfolios are assembled and evaluated based on all risk metrics, with the aim of determining the most efficient hedge portfolio.

5.1. Introduction

Banks typically fund their mortgages by internal contracts. In our analysis, these internal contracts are assumed to be simple bullet loans or deposits, traded at par at inception (swap + FTP) and held till maturity. By strategically constructing a portfolio of such contracts, the bank can effectively transfer the interest rate risk associated with its mortgage portfolio to the Group Treasury. This department then aggregates all received risks and enters the financial markets to hedge the total balance sheet (assets and liabilities) of the bank. An overview of this risk transfer process is given in Figure 5.1.

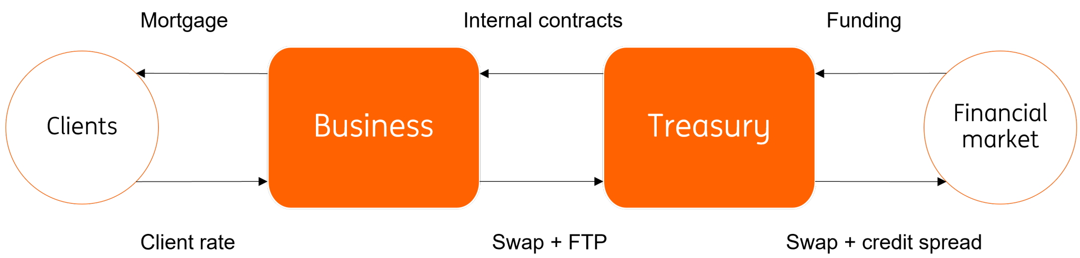


Figure 5.1: A schematic overview of risk transfer process within a bank.

In this figure, the 'Business' represents the retail part of the bank, which interacts with the clients, issuing mortgages and receiving interest payments. The business division then finances these transactions with 'Treasury' using internal contracts and paying the prevailing internal funding rate. This rate consists of a relevant swap rate with an additional Funds Transfer Pricing (FTP) spread. Typically, this swapFTP rate is smaller than the client rate, so that the business receives a positive margin. Finally, Treasury uses the financial market as well as the internal deposits to fund the internal loans.

Without any additional hedging strategies, the baseline hedge for one mortgage consists of a single internal contract in which the business receives the notional of the mortgage and pays interest, based on the swapFTP rate with a tenor equal to the maturity of the mortgage. The cash flows of this internal funding are illustrated in Figure 5.2.

Time	0	1	2	3	...	T
Internal funding	N	$-\frac{SFTP(0)}{12} \cdot N$	$-\frac{SFTP(0)}{12} \cdot N$	$-\frac{SFTP(0)}{12} \cdot N$	...	$-\left(\frac{SFTP(0)}{12} + 1\right) \cdot N$

Figure 5.2: The monthly cash flows of the internal funding of a mortgage with notional value N and a maturity of T months, with interest based on the relevant swapFTP rate at time 0.

It is clear that this baseline hedging model does not account for the prepayment behaviour and the associated interest rate risk. Therefore, this portfolio should be adjusted to effectively transfer the risks to Group Treasury. In particular, the aim of this research is to construct the most effective hedge portfolio for the given hypothetical mortgage portfolio in terms of value and margin stability. These factors are crucial for managing interest rate risk within a bank. Margin stability ensures consistent profitability by minimising the impact of interest rate fluctuations on the net interest margin, while value stability helps mitigate potential value losses in response to interest rate changes. In addition to these bank's own motivations, financial regulators also play a crucial role. Regulators, such as the European Banking Authority (EBA), impose restrictions on banks to ensure prudent risk management practices, making margin and value stability essential for maintaining overall stability of the banking system (EBA, 2022b).

In the subsequent subsections, we will refine the baseline hedge portfolio, consisting of only an internal funding, by prioritising margin stability and value stability. This will involve examining relevant risk metrics and exploring various hedging approaches.

5.2. Margin stability

The net interest margin (NIM) is a financial metric that quantifies the profitability of a bank or financial institution. It is defined as the ratio of net interest income to the average interest-earning assets. Net interest income represents the difference between interest rate revenues and interest rate expenses.

In our analysis, we consider monthly cash flows arising from the mortgage portfolio and the hedging portfolio. This implies that the denominator simplifies to the outstanding notional of the mortgage portfolio since it represents the average interest-earning assets over a given month. Consequently, if we denote the mortgage and hedging cash flows in month j and simulation i by $M_{i,j}$ and $H_{i,j}$, respectively, then the corresponding monthly NIM can be expressed as follows:

$$NIM_{i,j} = \frac{M_{i,j} + H_{i,j}}{\text{Outstanding notional}_{i,j}}.$$

5.2.1. Risk metrics margin stability

Based on the evolution of interest rates, the NIM may change over time. Clients may decide to prepay, disrupting the cash flows arising from the mortgage portfolio and resulting in an unstable margin. Our aim is to achieve a stable net interest margin, ensuring its consistency over time and across various interest rate scenarios. A common way to attain stability is by minimising the variance of the NIM across all simulations and time periods. Since the true distribution and variance of the NIM are unknown, we will employ the unbiased sample variance as an alternative estimation method. With $Y_1, \ldots, Y_N$ the relevant samples, the unbiased sample variance, denoted by S^2, is defined as follows:

$$S^2 = \frac{1}{N-1} \sum_{k=1}^{N} (Y_k - \bar{Y})^2,$$

where $\bar{Y}$ represents the sample mean, given by

$$\bar{Y} = \frac{1}{N} \sum_{k=1}^{N} Y_k.$$

Consequently, considering the hypothetical mortgage portfolio used in this research, with 100 simulations conducted over a period of 10 years, the risk metric for margin stability will be the following quantity:

$$VAR_{NIM} := \frac{1}{100 \cdot 120 - 1} \sum_{i=1}^{100} \sum_{j=1}^{120} (NIM_{i,j} - \overline{NIM})^2,$$

where $\overline{NIM}$ is the sample mean of the NIM values, defined as:

$$\overline{NIM} = \frac{1}{100 \cdot 120} \sum_{i=1}^{100} \sum_{j=1}^{120} NIM_{i,j}.$$

It is worth noting that for any two hedging portfolios, the one with the lowest variance will be considered as the better portfolio in terms of margin stability, as a low variance suggests low variability in the net interest margin.

5.2.2. Hedging approaches margin stability

We start our hedge with the baseline hedge as described at the start of this chapter. For a single mortgage without prepayments, this method leads to perfect NIM stability. This can be seen in Figure 5.3, which shows the cash flows of one mortgage and its internal funding.

Time	0	1	2	...	T
Mortgage	$-N$	$\frac{c}{12} \cdot N$	$\frac{c}{12} \cdot N$	...	$\left(\frac{c}{12} + 1\right) \cdot N$
Internal funding	N	$-\frac{SFTP(0)}{12} \cdot N$	$-\frac{SFTP(0)}{12} \cdot N$	...	$-\left(\frac{SFTP(0)}{12} + 1\right) \cdot N$
NIM	0	$\frac{c-SFTP(0)}{12}$	$\frac{c-SFTP(0)}{12}$	...	$\frac{c-SFTP(0)}{12}$

Figure 5.3: The monthly cash flows of a mortgage with annual coupon rate c, notional value N, and a maturity of T months, together with its internal funding with interest based on the relevant swapFTP rate at time 0, denoted by $SFTP(0)$.

However, considering a portfolio of two mortgage, the internal funding already becomes inefficient in terms of margin stability. This becomes apparent in Figure 5.4.

Time	0	1	2	...	T
Mortgage 1	$-N_1$	$\frac{c_1}{12} \cdot N_1$	$\frac{c_1}{12} \cdot N_1$	...	$\left(\frac{c_1}{12} + 1\right) \cdot N_1$
Mortgage 2	$-N_2$	$\frac{c_2}{12} \cdot N_2$	$\frac{c_2}{12} \cdot N_2$	...	$\left(\frac{c_2}{12} + 1\right) \cdot N_2$
Internal funding 1	N_1	$-\frac{SFTP_1(0)}{12} \cdot N_1$	$-\frac{SFTP_1(0)}{12} \cdot N_1$	...	$-\left(\frac{SFTP_1(0)}{12} + 1\right) \cdot N_1$
Internal funding 2	N_2	$-\frac{SFTP_2(0)}{12} \cdot N_2$	$-\frac{SFTP_2(0)}{12} \cdot N_2$	...	$-\left(\frac{SFTP_2(0)}{12} + 1\right) \cdot N_2$
NIM	0	$\frac{\frac{c_1-SFTP_1(0)}{12} \cdot N_1 + \frac{c_2-SFTP_2(0)}{12} \cdot N_2}{N_1+N_2}$	$\frac{\frac{c_1-SFTP_1(0)}{12} \cdot N_1 + \frac{c_2-SFTP_2(0)}{12} \cdot N_2}{N_1+N_2}$	...	$\frac{\frac{c_1-SFTP_1(0)}{12} \cdot N_1 + \frac{c_2-SFTP_2(0)}{12} \cdot N_2}{N_1+N_2}$

Figure 5.4: The monthly cash flows of two mortgages with annual coupon rates c_1, c_2, notional values N_1, N_2, and a maturity of T months, together with their internal funding with interest based on the relevant swapFTP rates at time 0, denoted by $SFTP_1(0)$ and $SFTP_1(0)$.

From this figure, we conclude that only if the spread between the coupon rate and the relevant internal funding rate is fixed, complete margin stability is attained. In other words, if

$$c_1 - SFTP_1(0) = c_2 - SFTP_2(0),$$

the net interest margin will have a value of $\frac{c_1-SFTP_1(0)}{12}$ throughout the complete lifespan of the mortgage portfolio. A similar conclusion can be drawn if we consider more than two mortgages.

For the mortgage portfolio considered in this research, the differences between the coupon rate and the relevant swapFTP rate are not fixed. However, due to only small variations across the six different mortgages, we still achieve a low variance of the NIM, as depicted in Table 5.1. This indicates that internal funding provides good margin stability when prepayments are excluded.

	Variance NIM
Without prepayments	2.205×10^{-7}
With prepayments	1.097×10^{-1}

Table 5.1: The variance of the net interest margin for the baseline model, with and without allowing prepayments.

However, if we allow the customers to prepay, Table 5.1 shows that the variance of the NIM substantially increases and as a result, the baseline model no longer suffices to maintain margin stability. The main reason for this large difference in variance, is the fact that prepayments change the outstanding notional amount of the mortgage. If a mortgage starts with a notional amount of N, and the customer makes a prepayment of Λ_1 in the first month, the outstanding notional in the second month becomes $N - \Lambda_1$. The interest paid on the mortgage now decreases, where the interest on the internal loan remains the same, resulting in a change in net interest margin.

A common method to address this, and attain margin stability, is by applying a notional hedge. This approach tries to match the expected outstanding notional of a mortgage with that of the hedge. For this, the baseline hedge should be adjusted so that its notional decreases by the exact prepayment amount. In the next two subsections, a static and a dynamic application of this hedge are discussed.

Static notional hedge

A static notional hedge is a type of notional hedge that is constructed at time $t = 0$ and fixed throughout the timeline of the mortgage portfolio. As the real prepayment amounts are unknown at this time, and therefore the outstanding is not known, their expected values are utilised, which can be derived from the expected swap rates. For instance, let us consider the first mortgage of our portfolio, which has a coupon rate of 6.84% and a remaining fixed interest rate period of 2 years. For this mortgage, the expected prepayment rate at time t can be expressed as

$$\widetilde{\lambda_{1,t}} = \widetilde{PP}_1(t) = a + \frac{b}{1 + e^{-c*(\widetilde{Inc}_1(t)-d)}},$$

with the parameter a, b, c and d as given in Table 4.3 and

$$\widetilde{Inc}_1(t) = 6.84\% - S(0, t, 2) - 1.5\%.$$

Here, $S(0, t, 2)$ denotes the 6M-Euribor swap rate at time t with a maturity of two years, as expected at time $t = 0$. It is calculated by applying Equation (B.6) using the current prices of zero-coupon bonds, which can be obtained from a generic market data feed of the bank. With the use of the expected prepayment rates, the monthly expected prepayment amounts can be calculated, which we will denote by $\widetilde{\Lambda}_t$ for month t.

Now in order to apply a notional hedge, zero-coupon bonds will be used, one for each month, as illustrated in Figure 5.5. The notional of each bond is specifically chosen, so that if the expected notionals match the observed notionals, a stable net interest margin is attained. This can easily be checked by setting $\widetilde{\Lambda}_t = \Lambda_t \; \forall t$.

Time	0	1	2	...	T
Mortgage	$-N$	$\frac{c}{12}\cdot N$	$\frac{c}{12}\cdot(N-\Lambda_1)$	...	$\left(\frac{c}{12}+1\right)\cdot\left(N-\sum_{t=1}^{T-1}\Lambda_t\right)$
Prepayment	-	Λ_1	Λ_2	...	-
Internal funding	N	$-\frac{SFTP(0)}{12}\cdot N$	$-\frac{SFTP(0)}{12}\cdot N$	...	$-\left(\frac{SFTP(0)}{12}+1\right)\cdot N$
Zero-coupon bonds	$P_1(0)$	$-\widetilde{\Lambda}_1$	-	...	-
	$P_2(0)$	-	$-\widetilde{\Lambda}_2+\frac{SFTP(0)}{12}\widetilde{\Lambda}_1$	...	-
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$P_T(0)$	-	-	...	$\left(\frac{SFTP(0)}{12}+1\right)\sum_{t=1}^{T-1}\widetilde{\Lambda}_t$
NIM	$\sum_{i=1}^{T}\frac{P_i(0)}{N}$	$\frac{c-SFTP(0)}{12}$ $+\frac{\Lambda_1-\widetilde{\Lambda}_1}{N}$	$\frac{\frac{c}{12}(N-\Lambda_1)-\frac{SFTP(0)}{12}(N-\widetilde{\Lambda}_1)}{N-\Lambda_1}$ $+\frac{\Lambda_2-\widetilde{\Lambda}_2}{N-\Lambda_1}$	...	$\frac{\left(\frac{c}{12}+1\right)\left(N-\sum_{t=1}^{T-1}\Lambda_t\right)}{N-\sum_{t=1}^{T-1}\Lambda_t}$ $-\frac{\left(\frac{SFTP(0)}{12}+1\right)\left(N-\sum_{t=1}^{T-1}\widetilde{\Lambda}_t\right)}{N-\sum_{t=1}^{T-1}\Lambda_t}$

Figure 5.5: The monthly cash flows of a static notional hedge with zero-coupon bonds with notionals based on the expected cash flows from a mortgage with annual coupon rate c, notional value N, and a maturity of T months.

It is clear that the realised prepayment rates may deviate from the expected rates, resulting in a discrepancy in the notional matching and changes in the margin. This is demonstrated through the simulations of interest rate scenarios and the corresponding NIM depicted in Figure 5.6, where ten simulations are shown.

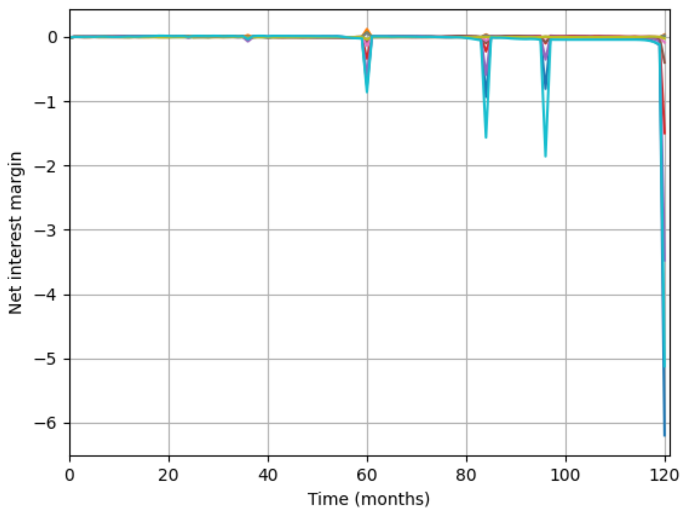


Figure 5.6: Ten simulations of the net interest margin for the static notional hedge with zero-coupon bonds.

From this figure it is evident that the current portfolio is inadequate to ensure margin stability. In particular, at the maturities of the mortgages portfolio large deviations in the net interest margin are visible, demonstrating the interest rate sensitivity of the hedging portfolio. To gain more insight into these significant variations in the NIM across different interest rate scenarios, a closer examination of the relation between the NIM and the market behaviour is required.

Considering the bottom row of Figure 5.5, which indicates the NIM, we see that any disparity in prepayment amount leads to a fluctuation in the margin. In particular, if we write $\widetilde{\Lambda}_t = \Lambda_t + \epsilon_t$, then the

NIM can be rewritten to

$$NIM_t = \begin{cases} \frac{c - SFTP(0)}{12} + \frac{\frac{SFTP(0)}{12}\sum_{i=1}^{t-1}\epsilon_i - \epsilon_t}{N - \sum_{i=1}^{t-1}\Lambda_i} & t < T; \\ \frac{c - SFTP(0)}{12} + \left(\frac{SFTP(0)}{12} + 1\right)\frac{\sum_{t=i}^{T-1}\epsilon_i}{N - \sum_{i=1}^{T-1}\Lambda_i} & t = T. \end{cases} \tag{5.1}$$

This notation indicates that whenever ϵ_t becomes significantly less than zero at a particular time t, the NIM at maturity can become substantially negative. Therefore, the primary reason behind the prominent negative spikes observed in Figure 5.6 may be an interest rate environment that is lower than anticipated. Figure 5.7, which displays the evolution of a 6M-Euribor swap rate in the same ten simulations, supports this notion by showing that the extreme negative scenarios align with the lowest interest rates. In contrast, relatively minor positive spikes are visible in Figure 5.6. These spikes arise from interest rates surpassing their expected values, thereby resulting in fewer prepayments than expected. However, as the denominator of the error term in Equation (5.1) is the real outstanding notional, which remained substantial, the resulting NIM remains modest. Consequently, the impact of this disparity is less apparent in the figure.

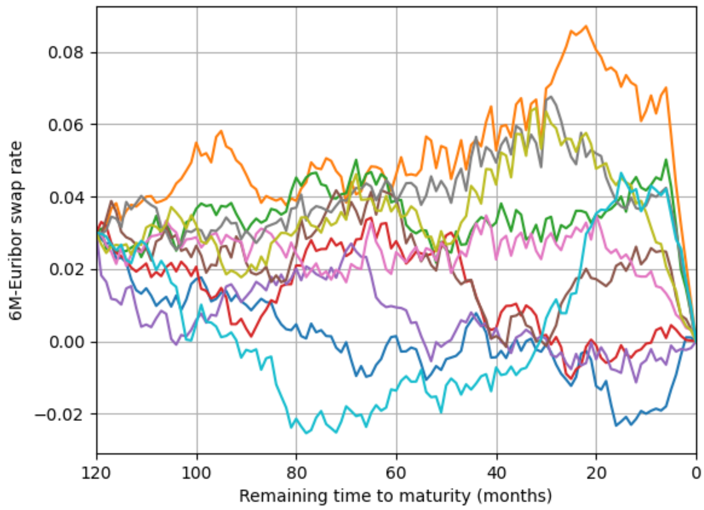


Figure 5.7: Ten simulations of the 6M-Euribor swap rates over time for a swap that matures in ten years.

Hedging approach	Variance NIM
Internal funding	1.097×10^{-1}
Static notional hedge	2.466×10^{-2}

Table 5.2: The variance of the net interest margin for the baseline model and the static notional hedge with zero-coupon bonds.

From this analysis, it can be concluded that whenever the market behaves differently than anticipated, the static hedge using zero-coupon bonds does not effectively hedge the margin stability. However, there is an improvement in comparison to the baseline model. The variance of the net interest margin for this static hedge, as displayed in Table 5.2, exhibits a significant reduction of 75% compared to the variance associated with the internal funding. Although this reduction is noteworthy, it is important to note that the effectiveness of this approach heavily relies on the assumption that the current yield curve accurately reflects the market's expectation of future interest rates. Any discrepancies between the yield curve and the actual future interest rates could result in differences between the implied prepayment rates and the actual prepayment rates. Consequently, the notional match of the hedge is highly sensitive to interest rate changes, as the zero-coupon bond cash flows are predetermined. To address this issue, the subsequent step would be to consider a dynamic model that incorporates realised prepayment amounts. The following section will be dedicated to exploring this dynamic approach.

Dynamic notional hedge

A dynamic notional hedge is similar to a static notional hedge but distinguishes itself by allowing rebalancing the hedge portfolio periodically. This implies that each month, the hedge can be adjusted according to the observed market conditions, incorporating realised prepayment behaviour rather than relying solely on expected prepayments.

Consider a mortgage with an initial notional amount N. If a prepayment of Λ_1 is made in the first month, the outstanding notional reduces to $N - \Lambda_1$ in the second month. To account for this change, the hedge portfolio can be adjusted by issuing an internal deposit with an amount equal to Λ_1. then in the subsequent months, we earn interest on this deposit with respect to the prevailing swapFTP curve. The total notional of the hedge portfolio thus decreases to $N - \Lambda_1$, matching the notional of the mortgage portfolio as required. This process can be repeated on a monthly basis to ensure that the notionals of the two portfolios are aligned through the entire lifespan of the mortgage portfolio. For a detailed understanding of this dynamic approach, refer to Figure 5.8, which provides a comprehensive overview of the relevant cash flows and the corresponding net interest margin per month.

Time	0	1	2	…	T
Mortgage	$-N$	$\frac{c}{12} \cdot N$	$\frac{c}{12} \cdot (N - \Lambda_1)$	...	$\left(\frac{c}{12} + 1\right) \cdot \left(N - \sum_{t=1}^{T-1} \Lambda_t\right)$
Prepayment	-	Λ_1	Λ_2	…	0
Internal contracts	N	$-\frac{SFTP(0)}{12} \cdot N$	$-\frac{SFTP(0)}{12} \cdot N$	...	$-\left(\frac{SFTP(0)}{12} + 1\right) \cdot N$
	-	$-\Lambda_1$	$\frac{SFTP(1)}{12} \cdot \Lambda_1$	…	$\left(\frac{SFTP(1)}{12} + 1\right) \cdot \Lambda_1$
	-	-	$-\Lambda_2$	…	$\left(\frac{SFTP(2)}{12} + 1\right) \cdot \Lambda_2$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	-	-	-	...	$\left(\frac{SFTP(T-1)}{12} + 1\right) \cdot \Lambda_{T-1}$
NIM	0	$\frac{c-SFTP(0)}{12}$	$\frac{\frac{c-SFTP(0)}{12} \cdot N - \frac{c-SFTP(1)}{12} \Lambda_1}{N-\Lambda_1}$	...	$\frac{\frac{c-SFTP(0)}{12} \cdot N - \sum_{t=1}^{T-1} \frac{c-SFTP(t)}{12} \Lambda_t}{N - \sum_{t=1}^{T-1} \Lambda_t}$

Figure 5.8: The monthly cash flows of a dynamic notional hedge with internal contracts with notionals based on the observed cash flows from a mortgage with annual coupon rate c, notional value N, and a maturity of T months.

Focusing on the final row of Figure 5.8, it becomes apparent that the monthly NIM would be stabilised if the swapFTP curve remains flat. Indeed, if we express the swapFTP curve as

$$SFTP(t) = SFTP(0) + \delta_t,$$

then the NIM can be expressed as

$$NIM_t = \frac{c - SFTP(0)}{12} + \frac{1}{12} \frac{\sum_{i=1}^{t-1} \delta_i \Lambda_i}{N - \sum_{i=1}^{t-1} \Lambda_i}. \tag{5.2}$$

This equation demonstrates that a perfectly stable net interest margin can be attained if $\delta_t = 0$ for all t. It is however important to note that this ideal scenario may not hold in reality. In practice, the swapFTP curve is subject to market dynamics, making it unlikely to remain completely flat over time. Nevertheless, these fluctuations in the swapFTP curve are generally expected to be less significant than the discrepancies between expected and realised prepayments. As a result, the dynamic notional hedge will likely yield better margin stability compared to the static approach.

To assess this hypothesis, we examine the stability of the net interest margin for both the dynamic and static notional hedge using the same interest rate scenarios. Figure 5.9 displays the evolution of the net interest margin over a ten-year period for the dynamic notional hedge. Comparing this figure to Figure 5.6 reveals notable differences in both the size of the NIM and the shape of the curve. The dynamic approach yields values ranging approximately between -0.03 and 0.002, whereas the static approach exhibits a wider variation between -6 and 0.03. This significant difference can be attributed to the different error terms in Equation (5.1) and Equation (5.2). The first equation, corresponding to the static notional hedge, shows that the error term primarily depends on the discrepancies between the expected and realised prepayments, while the error term in Equation (5.2), for the dynamic approach, relies on the fluctuations in the swapFTP curve. As the fluctuations in the swapFTP curve are in general less significant than the discrepancies between expected and realised prepayments, this leads to a smaller error in the dynamic approach and thus a smaller range of NIM values.

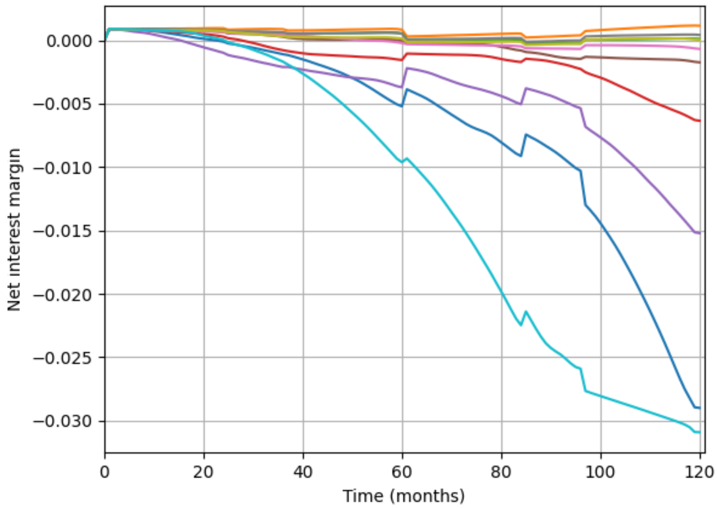


Figure 5.9: Ten simulations of the net interest margin for the dynamic notional hedge with internal contracts.

Additionally, despite sharing the same underlying interest rate scenarios, the two NIM curves exhibit distinct shapes. Figure 5.9 displays a mostly monotonic decreasing curve, whereas Figure 5.6 shows fluctuations with spikes aligning with the mortgage maturities in the considered mortgage portfolio. The latter behaviour was explained in the previous section through Equation (5.1), where the error term could be both positive and negative depending on the differences in expected and observed prepayments, with relatively larger negative errors. In the case of the dynamic notional hedge, we observe a similar pattern in Equation (5.2). Positive values of δ_t correspond to an increased swapFTP curve in comparison to the start date, suggesting fewer prepayments and a smaller decrease in the outstanding notional. As a result, the error term is a small positive number. Conversely, negative values of δ correspond to a decreased swapFTP rate, indicating more prepayments and a larger decrease in the outstanding notional. Combined, this results in a relatively large negative error. Consequently, this suggest that the two blue lines in Figure 5.9, which correspond to the largest deviations in the net interest margin, should correspond to the lowest interest rate environment. This is in line with Figure 5.7, which shows the corresponding simulated 6M-Euribor swap rates over time.

Beside the significant positive change in the behaviour of the net interest margin for the dynamic approach, as there are visibly less fluctuations, we also see this back in Table 5.3 which displays the variance of the NIM for the three hedging approaches discussed so far. From this table, it becomes clear that the dynamic notional hedge ensures substantially more margin stability than the baseline model and the static notional hedge.

In the next section we will move from earnings perspective to value perspective and explore various hedging approaches to attain value stability. Moreover, we will discuss different metrics to measure this stability.

Hedging approach	Variance NIM
Internal funding	1.097×10^{-1}
Static notional hedge	2.466×10^{-2}
Dynamic notional hedge	1.080×10^{-5}

Table 5.3: The variance of the net interest margin for the baseline model, the static notional hedge with zero-coupon bonds, and the dynamic notional hedge with internal contracts.

5.3. Value stability

The net present value (NPV) is a financial measure used to assess the value of a portfolio by calculating the present value of expected future cash flows, taking into account the time value of money. It compares the present value of incoming cash flows to the present value of outflows, using a specified discount rate. Let us consider a portfolio with net cash flows $CF_1, \ldots, CF_N$ and their corresponding discount factors $DF_1, \ldots, DF_N$. Then the NPV can be calculated using the following formula:

$$NPV = \sum_{k=1}^{N} DF_k CF_k.$$

In the context of the mortgage portfolio, the cash flows include monthly interest payments, prepayments, and the final payment of the outstanding notional. It is worth noting that there are no principal repayments included since we are considering bullet mortgages. As for the hedging portfolio, the relevant cash flows are those arising from the selected financial instruments. Since these payments are all made in the future, they are worth less than the same payments made today due to the opportunity to earn interest through investments or deposits. Therefore, discounting the cash flows using an appropriate rate is required.

The foundation of the discounting curve is the reference curve, which serves as the risk-free component. In the context of this study, spot rates are considered the relevant rates as they reflect the current rates for immediate contract settlements. These spot rates are derived using forward rates, which are rates used for contracts starting in the future. The relationship between spot rates (SR_t) and forward rates ($FR_{s-1,s}$) is defined by Equation (5.3).

$$(1 + SR_t)^t = (1 + FR_{0,1}) \cdot (1 + FR_{1,2}) \cdot \ldots \cdot (1 + FR_{t-1,t}). \tag{5.3}$$

Since the mortgage portfolio under consideration involves monthly cash flows, monthly forward rates are required. In this case, the 1M-Euribor rate is chosen and can be simulated using the formula provided in Appendix A.1.2.

It is important to note that prior to the 2008 financial crisis, it was common practice to use a single curve, such as the 1M-Euribor, for the calculation of the NPV. This was because the Euribor rate was considered close to the risk-free rate and using it for both forecasting and discounting simplified valuation. However, during the crisis, the spread between Euribor rates and the 'real risk-free' overnight index swap (OIS) rates widened significantly, making the single-curve framework no longer sufficient (Ametrano and Bianchetti, 2013; Bianchetti and Carlicchi, 2011). Therefore, in the present market, a multi-curve approach is typically employed to account for this spread. Nevertheless, for the purpose of this research, we will utilise the single-curve framework, as the multi-curve approach introduces unnecessary complexity that is not required for this specific analysis.

In order to calculate the NPV accurately, spot rates alone are insufficient, as they only account for the risk-free component. To incorporate a market value perspective, a risky component needs to be added. A common approach to address this is by including instrument-specific spreads to the reference curve. For the mortgage portfolio, this entails adding a commercial margin to compensate for the risk such as defaulting clients. A margin of 1.5% is chosen, which aligns with the margin included in the mortgage rate. On the other hand, the hedging portfolio requires an internal Funds Transfer Pricing (FTP) spread, that approximates the liquidity spreads and is set at 50% of the margin, equivalent to 0.75%. It is worth noting that the commercial margin is typically larger than the FTP spread.

Taking all of this in account, we define the discount factor DF_t for time t as

$$DF_t = \frac{1}{(1 + SR_t + \beta)^t}.$$

In this formula, SR_t represents the spot rate for time t, and β is the additional spread. The value of β depends on the type of portfolio:

$$\beta = \begin{cases} 0.015 & \text{for the mortgage portfolio;} \\ 0.0075 & \text{for the hedging portfolio.} \end{cases}$$

By incorporating the spread into the discount factor formula, the cash flows from the mortgage and hedging portfolio can be appropriately discounted. Note that the cash flows from the mortgage portfolio are discounted with a smaller value. As a results, it will have a smaller NPV compared to a hedge portfolio with identical cash flows.

5.3.1. Risk metrics value stability

Our objective is to achieve a stable net present value by ensuring its consistency across different interest rate scenarios. One approach to ensure this is by minimising the variance of the NPV, similar to the approach used for margin stability. However, in this case, only one value is obtained for each interest rate scenario, as opposed to obtaining values for each month and simulation. Hence, we aim to minimise the following objective over 100 conducted simulations:

$$VAR_{NPV} := \frac{1}{100-1}\sum_{i=1}^{100}(NPV_i - \overline{NPV})^2,$$

with $\overline{NPV}$ the average NPV, defined by

$$\overline{NPV} = \frac{1}{100}\sum_{i=1}^{100} NPV_i.$$

Additionally, we will evaluate the stability of the value by considering the net basis point value (BPV) of the mortgage and hedging portfolio. The BPV is a metric that reflects the sensitivity of the value to interest rate changes. Specifically, it gives an approximation of the change in value when the rates shift up by 1 basis point (0.01%), and it is defined as

$$BPV = \frac{NPV_{up} - NPV_{down}}{20bps},$$

where NPV_{up} represents the NPV value of a portfolio when the yield curve is shifted up by 10 basis points, while NPV_{down} does for the yield curve shifted down by 10 basis points. A net BPV of zero thus implies, on average, no change in value for parallel movements in the yield curve. Therefore, minimising the net BPV should result in increased value stability.

5.3.2. Hedging approaches value stability

Similar to margin stability, we initiate our hedge with the baseline hedge described earlier in this chapter. Figure 5.10 presents the cash flows and corresponding discount factors for a single mortgage including prepayments and its internal funding.

Based on this figure, several observations can be made. First of all, it is clear that the discount factor for the mortgage is consistently smaller than that of the internal funding. This is a general pattern that remains unchanged regardless of the hedge portfolio. Therefore, in order to achieve a value match, the cash flows of the hedge should be smaller than the mortgage cash flows. Secondly, we note that the total NPV for the baseline hedge can be expressed as

$$NPV = \sum_{t=0}^{T} \frac{CF_{mort,t}}{(1+SR_{t/12}+0.015)^{\frac{t}{12}}} + \sum_{t=0}^{T} \frac{CF_{int\,fund,t}}{(1+SR_{t/12}+0.0075)^{\frac{t}{12}}},$$

with $CF_{mort,t}$ and $CF_{int\,fund,t}$ representing the cash flows of the mortgage and the internal funding, respectively, as shown in the first and third row of Table 5.10. This expression shows that changes in the NPV arise from changes in spot rates, prepayment amounts, or both, as the other parameters

Time	0	1	2	...	T
Mortgage	$-N$	$\frac{c}{12}\cdot N+\Lambda_1$	$\frac{c}{12}\cdot(N-\Lambda_1)$	...	$\left(\frac{c}{12}+1\right)\cdot\left(N-\sum_{t=1}^{T-1}\Lambda_t\right)$
Discount factor	1	$\frac{1}{(1+SR_{1/12}+0.015)^{\frac{1}{12}}}$	$\frac{1}{(1+SR_{2/12}+0.015)^{\frac{2}{12}}}$	...	$\frac{1}{(1+SR_{T/12}+0.015)^{\frac{T}{12}}}$
Internal funding	N	$-\frac{SFTP(0)}{12}\cdot N$	$-\frac{SFTP(0)}{12}\cdot N$	...	$-\left(\frac{SFTP(0)}{12}+1\right)\cdot N$
Discount factor	1	$\frac{1}{(1+SR_{1/12}+0.0075)^{\frac{1}{12}}}$	$\frac{1}{(1+SR_{2/12}+0.0075)^{\frac{2}{12}}}$	...	$\frac{1}{(1+SR_{T/12}+0.0075)^{\frac{T}{12}}}$

Figure 5.10: The monthly cash flows and discount factors of a mortgage with annual coupon rate c, notional value N, a maturity of T months, and prepayment amounts $\Lambda_1, \dots \Lambda_{T-1}$, together with its internal funding with interest based on the relevant swapFTP rate at time 0.

remain constant. Therefore, we expect different values of this risk metric across various interest rate scenarios. As a consequence, a hedge that solely relies on internal funding, which does not take into account fluctuations in spot rates and prepayment rates, will not provide effective value stability.

A common approach to address this, is by neutralising the net basis point value (BPV). This metric approximates the change in value in response to a one basis point parallel shift of the relevant yield curve. This implies that a net BPV of zero ensures that the impact of interest rate changes on the portfolio's value is minimal. To maintain a net BPV of zero, a BPV (or value) hedge can be applied, which tries to find a hedge portfolio of which the BPV matches the BPV of the mortgage portfolio. In the upcoming two subsection, we will delve into two applications of this value hedge, subsequently a static and a dynamic approach.

Static value hedge

The static value hedge is a type of value hedge that is established at time $t = 0$ and remains unchanged until the mortgage portfolio reaches maturity. Like the static notional hedge, it relies on zero-coupon bonds constructed using projected prepayment behaviour, as this is the only information accessible at initiation.

The objective of the value hedge is to achieve value stability by neutralising the net basis point value. One approach to accomplish this, is by focusing on the net BPV over the entire duration of the mortgage portfolio. This approach provides a comprehensive view of interest rate sensitivity over the complete time horizon. However, it may overlook short-term market fluctuations and their impact on the portfolio, thereby failing on capturing potential risks that arise within shorter time periods. Given that the mortgage portfolio in this research involves monthly cash flows, this method may not be sufficient. An effective alternative is to assess the net BPV on a month-to-month basis. By achieving a monthly net BPV of zero, this approach will ultimately result in a total net BPV of zero, as the total net BPV can be obtained by summing the individual monthly net BPVs. This method is preferred as it maintains long-term risk management while allowing for a more granular analysis of interest rate sensitivity.

Executing the month-to-month value hedge requires several steps. First, the monthly BPV of the expected cash flows of the mortgage is calculated, discounting with the expected spot rates plus the additional fixed margin of 1.5%. Next, the monthly BPV of the internal funding, discounted with the sum of the expected spot rates and the FTP spread of 0.75%, is calculated. Finally, for each month, the net BPV, the sum of the BPV of the mortgage and the internal funding, is determined, and a zero-coupon bond is constructed to offset its value, leading to an expected net BPV of zero.

Mathematically, if we let $BPV_{hedge,t}$ and $BPV_{mort,t}$ be the BPV of the hedge portfolio and the mortgage portfolio at time t, respectively, then BPV matching translates to attaining the following equation:

$$net\ BPV_t := BPV_{hedge,t} + BPV_{mort,t} = 0 \quad \forall t.$$

Note that at first, we have $BPV_{hedge,t} = BPV_{int\ fund,t}$ and we assume that the corresponding net BPV is nonzero, since otherwise the BPV matching is already obtained. Now let $BPV_{zcb,t}$ be the BPV of a zero-coupon bond that pays out $CF_{zcb,t}$ at time t. Then this zero-coupon bond should be constructed such that

$$BPV_{zcb,t} = -BPV_{int\ fund,t} - BPV_{mort,t},$$

so that

$$\begin{aligned} net\ BPV_t &= BPV_{hedge\ new,t} + BPV_{mort,t} \\ &= (BPV_{int\ fund,t} + BPV_{zcb,t}) + BPV_{mort,t} \\ &= (BPV_{int\ fund,t} - BPV_{int\ fund,t} - BPV_{mort,t}) + BPV_{mort,t} \\ &= 0. \end{aligned}$$

This implies that the value of $CF_{zcb,t}$ should satisfy

$$\begin{aligned} -BPV_{int\ fund,t} - BPV_{mort,t} = BPV_{zcb,t} & \\ &= \frac{NPV_{zcb,up,t} - NPV_{zcb,down,t}}{20} \\ &= \frac{CF_{zcb,t} \cdot DF^{+}_{hedge,t} - CF_{zcb,t} \cdot DF^{-}_{hedge,t}}{20} \\ &= CF_{zcb,t} \frac{DF^{+}_{hedge,t} - DF^{-}_{hedge,t}}{20}, \end{aligned}$$

with the discount factors for a 10 basis points shock up (+) and down (−) scenario defined as:

$$DF^{+}_{hedge,t} = \frac{1}{(1 + SR_{t/12} + 0.0075 + 0.001)^{\frac{t}{12}}} \quad \text{and} \quad DF^{-}_{hedge,t} = \frac{1}{(1 + SR_{t/12} + 0.0075 - 0.001)^{\frac{t}{12}}} \tag{5.4}$$

This expression can be rewritten, resulting in the following expression for the required cash flow in month t:

$$CF_{zcb,t} = -\frac{20 \cdot (BPV_{hedge,t} + BPV_{mort,t})}{DF^{+}_{hedge,t} - DF^{-}_{hedge,t}} \tag{5.5}$$

However, as we consider a static approach of the value hedge, the future monthly cash flows of the mortgage and therefore, its corresponding basis point values are unknown. Therefore, Equation (5.5) becomes impractical to use. To address this, we can rely on the expected prepayment rates and the expected spot rates, which allow us to predict the future BPV values. Based on this, we issue zero-coupon bonds that pay out an amount equal to the expected required cash flows, denoted by $\widetilde{CF}_{zcb,t}$. Using Equation (5.5), we define this value as

$$\widetilde{CF}_{zcb,t} = -\frac{20 \cdot (\widetilde{BPV}_{int\ fund,t} + \widetilde{BPV}_{mort,t})}{\widetilde{DF}^{+}_{hedge,t} - \widetilde{DF}^{-}_{hedge,t}}$$

with $\widetilde{BPV}_{int\ fund,t}$ and $\widetilde{BPV}_{mort,t}$ representing the expected BPV of the internal funding and the mortgage portfolio, respectively, and $\widetilde{DF}^{+}_{hedge,t}$ and $\widetilde{DF}^{-}_{hedge,t}$ representing the expected discount factors of the hedge in a 10 basis points shock up- and down interest rate scenario, respectively. Figure 5.11 provides an overview of the cash flows that arise within this hedging approach together with the mortgage cash flows.

From this figure, it is not immediately clear how efficient this hedge will be in terms of value stability. In particular, it does not provide a clear formula for the NPV, similar to Equation (5.1) and (5.2) for the NIM, to analyse the potential error term. Therefore, in order to assess the efficiency of the static value hedge, we simulate 100 interest rate scenarios and calculate the variance of the NPV and the average net BPV of the mortgage and hedge portfolio combined. The results are displayed in Table 5.4, which also includes the values of the risk metrics in case of the internal funding.

Hedging approach	Variance NPV	Average net BPV
Internal funding	2.767×10^{9}	2.474×10^{2}
Static value hedge	3.668×10^{8}	4.830×10^{1}

Table 5.4: The variance of the net present value and the average net basis point value for the baseline model and the static value hedge with zero-coupon bonds.

Time	0	1	2	...	T
Mortgage	$-N$	$\frac{c}{12} \cdot N$	$\frac{c}{12} \cdot (N - \Lambda_1)$	...	$\left(\frac{c}{12} + 1\right) \cdot \left(N - \sum_{t=1}^{T-1} \Lambda_t\right)$
Prepayment	-	Λ_1	Λ_2	...	0
Internal contracts	N	$-\frac{SFTP(0)}{12} \cdot N$	$-\frac{SFTP(0)}{12} \cdot N$	...	$-\left(\frac{SFTP(0)}{12} + 1\right) \cdot N$
	$-P_1(0)$	$\widetilde{CF}_{zcb,1}$	-	...	-
	$-P_2(0)$	-	$\widetilde{CF}_{zcb,2}$	...	-
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$-P_T(0)$	-	-	...	$\widetilde{CF}_{zcb,T}$

Figure 5.11: The monthly cash flows of a static value hedge with zero-coupon bonds with notionals based on the expected cash flows from a mortgage with annual coupon rate c, notional value N, and a maturity of T months.

This table shows a decrease in both risk metrics, indicating an improvement in value stability. Nevertheless, it is important to note that a perfectly stable value entails a zero variance of the NPV and a net BPV of zero across all simulation. Thus, these results still fall short of being satisfactory, particularly considering the observed variance. One of the key factors contributing to this discrepancy is the fact that the static value hedge completely depends on expectations of the market. Therefore, any deviation may lead to a mismatch in BPV and potentially to an unstable NPV. In the remaining of this section, we will take a closer look at the simulated scenarios to understand this impact.

Firstly, let us consider the discount factors. In the static value hedge, the expected discount factors are utilised to calculate the present value of the future cash flows. These expected discount factors are derived from the expected 1M-Euribor rates, which are defined as

$$\widetilde{E}(0; t, t + 1/12) = \frac{1 - P(0, t + 1/12)}{\frac{1}{12} P(0, t + 1/12)},$$

where $P(0, x)$ represents the price of a zero-coupon bond with maturity x at time $t = 0$. However, in reality, interest rates may deviate from their expected values, leading to disparities in spot rates and, consequently, in the discount factors. Figure 5.12 illustrates this behaviour of the spot rates for ten simulations, highlighting these variations.

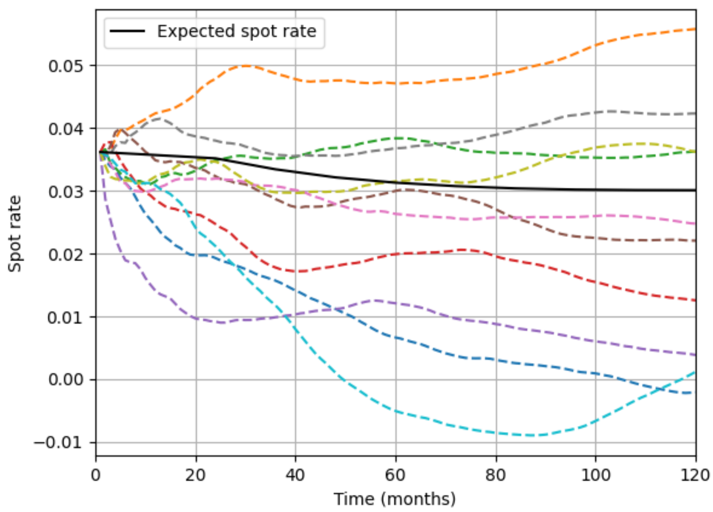
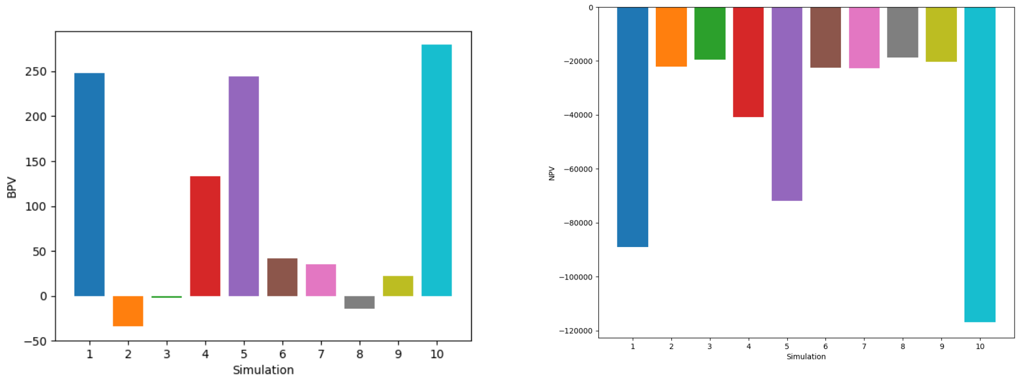


Figure 5.12: Ten simulations of the evolution of spot rates together with their expected evolution.

In this figure, the black solid line represents the expected curve of the spot rates, and the dotted lines are the simulated spot rates. We observe that the dark green and the light green curves are



(a) The basis point value (BPV)

(b) The net present value (NPV)

Figure 5.13: Ten simulations of the (a) basis point value and (b) net present value in the static value hedge with zero-coupon bonds.

the closest to the expected curve, whereas the two blue lines as well as the orange line correspond to the largest deviations. Consequently, as the hedge was constructed to ensure a net BPV of zero, we expect that the static value hedge performs best, in terms of having a zero net BPV, in the green scenarios and worst in the blue and orange scenarios.

Figure 5.13(a), which displays the total net BPV for the same ten interest rate scenarios, however, does not fully support this. We do observe large BPV values for the light and dark blue scenario, but the orange scenario does not lead to the expected large deviation of BPV compared to zero. One explanation for this may be the fact that the in case of a high interest rate environment, the discount factors are small, resulting in small NPVs. Consequently, the basis point value may be small as well. This becomes clearer if we consider a simplified definition of the BPV. For this, let x be the spot rate including the additional spread and δ be the value of the relevant shock. Then for a cash flow with a notional amount of N, the BPV can be simplified to

$$\begin{aligned} BPV_{simp} = \frac{1}{2\delta}\left(\frac{N}{1+x+\delta} - \frac{N}{1+x-\delta}\right) &= \frac{N}{2\delta}\frac{1+x-\delta-(1+x+\delta)}{(1+x)^2-\delta^2} \\ &= \frac{N}{2\delta}\frac{-2\delta}{(1+x)^2-\delta^2}. \end{aligned}$$

This expression shows that for a large x the BPV will be smaller than for a small x, assuming all other parameters being equal. This explains why the orange curve corresponds to a relatively small BPV, despite deviating heavily from the expected spot rates. So, we conclude that a shock in a high interest rate environment has less effect on the NPV, than the same shock in a low interest rate environment.

The NPV shows a similar response to the interest rate scenarios, as displayed in Figure 5.13(b). The two blue scenarios correspond to the most negative values, but most of all, the most divergent net present values. This is in line with Figure 5.13(a), as a large positive BPV implies that a 1 basis point shock up leads to a large increase in value. Conversely, a downward shock of 1 basis point results in a significant value decrease.

The two figures, Figure 5.13(b) and 5.13(a), show that the static value hedge is not efficient to ensure value stability. In particular, when interest rates are lower than expected, the net BPV becomes relatively large, resulting in a substantial decrease in the net present value. These scenarios contribute significantly to the observed NPV variance of 10^8, as indicated in Table 5.4. Therefore, it is evident that by improving the hedging of these specific scenarios, there is a strong likelihood of improving the overall value stability.

It is important to recognise that the static value hedge provides a framework for aligning the hedge portfolio with the mortgage portfolio based on expected market conditions, but it cannot eliminate the uncertainties and variability within. Therefore, in the subsequent subsection the dynamic value hedge will be explored, which should address this issue by considering realised rather than expected market behaviour.

Dynamic value hedge

The dynamic value hedge is similar to the static value hedge but utilises internal contracts instead of zero-coupon bonds to obtain a net BPV of zero. Moreover, the hedge portfolio can be adjusted on a monthly basis, based on the observed market behaviour. In specific, for each month, the BPV of the cash flows of the mortgage is calculated, discounting with the sum of the simulated spot rates and the fixed margin of 1.5%. Additionally, the monthly BPV of the internal funding, discounted with the simulated spot rates and an additional FTP spread of 0.75% is calculated. Finally, the net BPV is determined, and an internal contract is issued to offset this value, leading to a monthly net BPV of zero. Note that the notional of the internal contract depends on the BPV of the mortgage and hedge in the same way as the cash flow of the zero-coupon bond in the static value hedge. The only difference is that in the dynamic hedge, each simulation may lead to a different strategy.

Following the same procedure as in the previous section, we find that the notional of the internal contract that is issued that time t, denoted by $N_{IC,t}$, should equal

$$N_{IC,t} = -\frac{20 \cdot (BPV_{hedge,t} + BPV_{mort,t})}{DF^{+}_{hedge,t} - DF^{-}_{hedge,t}},$$

with DF^{+} and DF^{-} as defined in Equation (5.4). A comprehensive overview of the cash flows of the corresponding dynamic value hedge can be found in Figure 5.14.

Time	0	1	2	…	T
Mortgage	$-N$	$\frac{c}{12} \cdot N$	$\frac{c}{12} \cdot (N - \Lambda_1)$	…	$\left(\frac{c}{12} + 1\right) \cdot \left(N - \sum_{t=1}^{T-1} \Lambda_t\right)$
Prepayment	-	Λ_1	Λ_2	…	0
Internal contracts	N	$-\frac{SFTP(0)}{12} \cdot N$	$-\frac{SFTP(0)}{12} \cdot N$	…	$-\left(\frac{SFTP(0)}{12} + 1\right) \cdot N$
	-	$N_{IC,1}$	$-\frac{SFTP(1)}{12} \cdot N_{IC,1}$	…	$-\left(\frac{SFTP(1)}{12} + 1\right) \cdot N_{IC,1}$
	-	-	$N_{IC,2}$	…	$-\left(\frac{SFTP(2)}{12} + 1\right) \cdot N_{IC,2}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	-	-	-	…	$-\left(\frac{SFTP(T-1)}{12} + 1\right) \cdot N_{IC,T-1}$
	-	-	-	…	$N_{IC,T}$

Figure 5.14: The monthly cash flows of a dynamic value hedge with internal contracts with notionals based on the observed cash flows from a mortgage with annual coupon rate c, notional value N, and a maturity of T months.

This figure illustrates the additional complexity in comparison to the static hedge, as the issued internal contracts at times $1, \ldots, t-1$ influence the BPV of the hedge at time t, and therefore the required notional $N_{IC,t}$. However, this complexity brings an advantage as all available market information is used in the calculation. As a consequence, the efficiency of the hedge, in terms of achieving a zero net BPV, is independent of the market conditions, because it incorporates all information of the market. This also becomes apparent if we simulate 100 interest rate scenarios and apply the dynamic value hedge to each of these, as shown in Table 5.5, where a significantly lower average net BPV is observed.

Hedging approach	Variance NPV	Average net BPV
Internal funding	2.767×10^{9}	2.474×10^{2}
Static value hedge	3.668×10^{8}	4.830×10^{1}
Dynamic value hedge	4.308×10^{5}	6.480×10^{-12}

Table 5.5: The variance of the net present value and the average net basis point value for the baseline model, the static value hedge with zero-coupon bonds, and the dynamic value hedge with internal contracts.

Additionally, the results show a substantial reduction in the variance of the NPV in comparison with the static value hedge. This may be attributed to the fact that a small BPV implies a smaller dependency on the interest rates, so that various interest rate scenarios result in more similar values. Despite this significant improvement in variance of approximately 99.98% compared to the internal funding, the variance still attains a large value, indicating significant fluctuations in the value. Figure 5.15 provides a visual representation of the NPV values of ten simulations in the dynamic value hedge. By analysing the underlying interest rate simulations displayed in Figure 5.12, we observe that the most diverse NPV outcomes correspond to the most extreme interest rate scenarios. Hence, the variability in NPV can largely be attributed to the relatively high and low interest rate scenarios. Therefore, these scenarios require extra attention, particularly because of the additional presence of convexity risk.

As observed, the prepayment option embedded in the mortgage portfolio gives rise to non-linear behaviour in response to interest rates fluctuations, resulting in increased portfolio volatility during extreme scenarios. Therefore, we are interested in understanding and managing potential losses resulting in these scenarios. In this regard, considering the NPV-at-Risk can provide additional insights. This risk metric offers a quantitative estimate of the potential loss that could occur due to unfavourable interest rate scenarios.

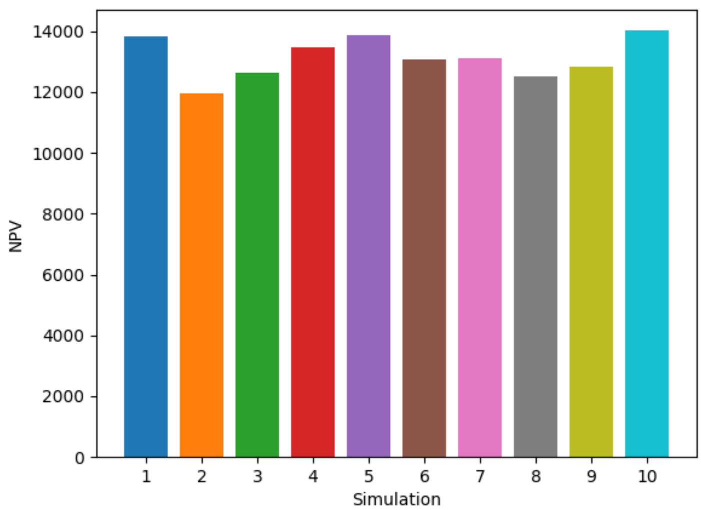


Figure 5.15: Ten simulations of the net present value (NPV) for the dynamic value hedge with internal contracts.

In the next subsection, we will explore NPV-at-Risk in greater detail, examining how this risk metric can be employed to offset some of the interest rate risk leading to value loss. By identifying the interest rate scenarios contributing most to this risk, we can develop strategies to manage and minimise the potential negative impact.

5.4. NPV-at-Risk analysis

The NPV-at-Risk (NPVaR) is a financial risk metric used to assess the potential changes in the net present value in a particular scenario compared to a baseline scenario. The NPVaR for a given scenario s, denoted as $NPVaR_s$, is determined by calculating the difference between the NPV of that scenario and the NPV of the baseline scenario b, $NPVaR_b$. Mathematically, it can be expressed as

$$NPVaR_s = NPV_s - NPV_b. \tag{5.6}$$

The baseline scenario is typical built upon expected market conditions. It serves as a benchmark against which other scenarios are compared to evaluate their impact on the NPV. Comparing scenario specific NPVs with that of the baseline scenario allows us to gain insight in the potential gains or losses arising from different interest rate scenarios. To effectively capture the effect of the prepayment option and the convexity within, it is crucial to choose interest rate shock scenarios of significant magnitude.

Following the guidelines offered by the European Banking Authority (EBA), parallel shocks of ± 200 basis points are selected (EBA, 2022a). These two shock scenarios, that set out the change in interest rates, are specified to assess the impact on the value and provide a standardised framework for risk evaluation.

For each of the hedging approaches discussed up to now, we have determined the value in the baseline scenario, assuming the expected market behaviour, and the value under the two shocked scenarios. Thereafter, the NPVaRs for the 200 basis points up- and down scenario are calculated, applying Equation (5.6), which are displayed in the first and second column of Table 5.6, respectively.

We observe that the value of the NPV-at-Risk can either be positive or negative, indicating whether the scenario leads to an increase or decrease in value, respectively. When the NPVaR is positive, it signifies a favourable change in value in scenario s, whereas a negative NPVaR corresponds to a decrease in value, which is a disadvantageous scenario for the bank. Therefore, the NPVaR with the most negative value is the most interested quantity for a bank. This value is given in the third column in Table 5.6.

		Value stability		
	Hedging approach	NPV-at-Risk +200bps	NPV-at-Risk -200bps	NPV-at-Risk
Static	Internal funding	3.196×10^4	-9.240×10^4	-9.240×10^4
	Notional hedge with zero-coupon bonds	-3.154×10^2	-5.382×10^4	-5.382×10^4
	Value hedge with zero-coupon bonds	-2.083×10^3	-2.394×10^4	-2.394×10^4
Dynamic	Notional hedge with internal contracts	3.302×10^4	-9.260×10^4	-9.260×10^4
	Value hedge with internal contracts	-9.396×10^3	-3.230×10^4	-3.230×10^4

Table 5.6: An overview of the NPV-at-Risks for all proposed hedging approaches.

From this table it becomes apparent that for all hedging approaches, the most significant potential value loss occurs when there is a change in interest rates of -200 basis points. This finding is consistent with the analysis of the static value hedge, where we observed smaller NPV values when the realised interest rates were lower than expected.

For a bank, a large negative NPVaR is considered unfavourable as it indicates potential losses and may compromise financial stability. Therefore, banks aim to minimise and neutralise the NPVaR to eliminate the associated risks. One common strategy to achieve this is by incorporating interest rate derivatives into their existing hedging portfolios. By entering into an interest rate option contract, banks have the right, but not the obligation, to take a position based on future interest rate movements, offering protection against unfavourable markets. In specific, we are interested in the impact of including a swaption, which gives its owner the right to enter an interest rate swap at a predetermined rate in the future. To thoroughly analyse the effect of incorporating a swaption, it is important to understand the definition and characteristics of both an interest rate swap and a swaption. In the upcoming two subsections, we will delve into these concepts, establishing a solid foundation for the subsequent analysis.

Swaps

An interest rate swap is an agreement involving one party agreeing to pay fixed cash flows equal to a predetermined fixed rate r_{fixed} on a notional amount, while the other party pays floating interest $r_{floating}$ on the same notional amount, typically based on an Euribor rate E. The payments are made at predetermined future times $T_{m+1}, \dots T_s$, with fixed intervals corresponding to the tenor of the Euribor rate. Note that the Euribor rate is a forward rate, so in the payment at T_k, the floating rate equals $E(T_{k-1})$ rather than $E(T_k)$. The analytical value of a swap, V_{swap} can be found in Appendix B.2.2.

One can distinguish between two types of interest rate swaps: the *payer swap*, in which the buyer receives the floating leg and pays the fixed leg, and the *receiver swap*, in which the buyer receives the fixed cash flows and pays the floating cash flows. As we focus on increasing the value in the scenario with a -200 basis points shock, we are interested in a swap that generates the largest cash flows in a low interest rate environment. A receiver swap satisfies this requirement, as can be seen in Figure 5.16, which illustrates the cash flows of a receiver swap.

Time	0	$\dots$	T_{m+1}	$\dots$	T_s
Receiver swap	$-V_{swap,rec}(0)$	$\dots$	$\tau_{m+1} \cdot N \cdot \left[r_{fixed} - E(T_m)\right]$	$\dots$	$\tau_s \cdot N \cdot \left[r_{fixed} - E(T_{s-1})\right]$

Figure 5.16: The monthly cash flows of a receiver interest rate swap with notional N and payments at $T_{m+1}, \dots, T_s$, where $\tau_k = T_k - T_{k-1}$.

Typically, a swap is considered to be "at-the-money" which means that the total value of the fixed interest rate cash flows is exactly equal to the expected value of the floating interest rate cash flows. This implies that the fixed rate is chosen to be at a level where the value of the swap at initial time equals zero, thereby allowing one to enter the contract for free. The rate at which this hold is referred to as the *swap rate*, and it is defined as

$$S(t, T_m, T_s) := \frac{P(t, T_m) - P(t, T_s)}{\sum_{k=m+1}^{s} \tau_k P(t, T_k)}. \tag{5.7}$$

For the derivation of the swap rate, refer to Appendix B.2.2. It should be noted that this expression does not depend on either the notional amount or the type of swap.

Swaption

A swaption is a derivative that gives its holder the right, but not the obligation, to enter into a swap contract in the future. The swaption can be a *receiver* or *payer* swaption, depending on the underlying type of swap. Upon entering, the counterparties must agree on the notional N, the fixed rate r_{fixed}, the underlying floating rate $r_{floating}$ (typically an Euribor rate), the expiration date of the swaption T_m, and the maturity of the swap T_s. At $t = T_m$, the swaption holder needs to decide whether to enter the swap or not based on the prevailing market conditions. If favourable, cash flows will be exchanged from T_{m+1} to T_s. If not, the option expires, and no further cash flows are exchanged. These two scenarios are summarised in Figure 5.17, in which we consider a receiver swaption with value $V_{swaption,rec}$ (see Appendix B.2.3).

Time	0	$\dots$	T_{m+1}	$\dots$	T_s
Exercise	$-V_{swaption,rec}(0)$	$\dots$	$\tau_{m+1} \cdot N \cdot \left[r_{fixed} - E(T_m)\right]$	$\dots$	$\tau_s \cdot N \cdot \left[r_{fixed} - E(T_{s-1})\right]$
No exercise	$-V_{swaption,rec}(0)$	$\dots$	0	$\dots$	0

Figure 5.17: The monthly cash flows of a receiver interest rate swaption with notional N, expiry date T_m and swap maturity T_s.

Given that the receiver swap generates the largest cash flows in the -200 basis points interest scenario, a receiver swaption will be selected for inclusion in the hedge portfolios. In particular, we have opted for a single receiver swaption rather than a diverse selection of different swaptions. This decision is based on the fact that the price of a swaption is typically significant and incorporating multiple swaptions would substantially increase the overall cost. By focusing on a single receiver swaption, we may effectively manage the risk exposure while maintaining the expenses within acceptable bounds.

To facilitate this implementation, the characteristics of the swaption need to be determined. Among these characteristics, the starting time, the swap maturity and the underlying floating rate will be fixed for all hedge approaches, whereas the option maturity, notional and strike rate will be left to a calibration. The starting time and swap maturity are set to align with the initiation and maturity of the mortgage portfolio being hedged. Additionally, the 6M-Euribor rate is selected as the underlying floating rate, as the prepayment behaviour is directly related to the corresponding swap rate. These choices are summarised in Table 5.7.

On the other hand, the remaining characteristics of the swaption, the option maturity, notional, and strike rate, will be subject to a calibration for each hedge approach separately. By calibrating these parameters, banks can tailor their hedging portfolios to neutralise the associated NPVaR. The calibration process will be elaborated upon in the next subsection, where the optimal values of option maturity,

Swaption characteristics	
Starting time	0
Type	Receiver
Swap maturity	10 years
Underlying floating rate	6M-Euribor rate

Table 5.7: An overview of the fixed characteristics of the receiver swaption to hedge the -200 basis points interest rate scenario.

notional, and strike rate for each hedge portfolio will be determined to achieve the desired NPVaR neutralisation.

5.4.1. Calibration of the swaptions

For the calibration of the notional N, option maturity T_m and strike rate K of the swaption, we select an objective function that aims to offset the NPV-at-Risk in an interest rate scenario with -200 basis point. That is, our goal is to minimise the following expression:

$$|NPVaR_{-200bps}(T_m, N, K)| := |NPV_{-200bps}(T_m, N, K) - NPV_b(T_m, N, K)|. \tag{5.8}$$

It is important to note the use of the absolute value in the objection function. This ensures that the swaption is constructed solely to offset the NPVaR, without generating a profit. Excluding the absolute value and formulating it as a maximisation problem could result in disproportionately large notional amounts and strike rates. Increasing the strike rate, for example, would amplify the cash flows of the swaption and therewith the value. The same applies to the notional amount. However, such inflated values are unrealistic, and if they were feasible, banks would be reluctant to purchase them due to their high costs.

To minimise the objective function described in (5.8), the Nelder-Mead algorithm will be employed. This algorithm is suitable for optimising scalar functions of one or more variables and is particularly robust for nonlinear optimisation problem with unknown derivatives. Given the non-linearity of our objective function, the Nelder-Mead algorithm is deemed suitable.

When utilising the Nelder-Mead algorithm, we are required to provide an initial guess for the parameters and are able to specify boundary conditions. For the option maturity T_m, we set the minimum value at 1 month and the maximum 114 months, which align with the first and last instances where the underlying swap contributes to a value change. This choice is based on the understanding that a 6M-Euribor swap initiates six months after T_m. Our initial guess is chosen as the midpoint of the lifespan of the mortgage portfolio, which is 60 months or 5 years. As for the notional, it should be a positive value since a negative notional has no meaning. We initialise it at a value of $10,000$, considering that the NPV-at-Risk for all hedging approaches were of the order 10^4 (refer to Table 5.6). Finally, the strike rate should be chosen such that the swaption will be exercised in the scenario with a -200 basis points shock, but not in the baseline scenario. Therefore, as we consider a receiver swaption, the strike rate K should satisfy the following inequality:

$$S(t, T_m, T_s) - 0.02 < K < S(t, T_m, T_s), \tag{5.9}$$

where $S(t, T_m, T_s)$ denotes the expected 6M-Euribor swap rate at time T_m for a swap that matures at time T_s. It is clear that the relevant expected swap rate is unknown at the start of the calibration, as it depends on the option maturity T_m being calibrated. However, we can deduce that the expected swap rates range between 0% and 3%, so these values will serve as the minimum and maximum bounds for the strike rate, respectively. Additionally, we set an initial value of 2.8% for the strike rate, to increase the likelihood of satisfying Inequality (5.9) and, consequently, achieving a successful calibration. In Algorithm 4, a complete overview of the calibration can be found.

Algorithm 4: Neutralisation of the NPV-at-Risk in a -200 basis points shock using the Nelder-Mead algorithm.

Input: The swaption characteristics in Table 5.7, NPV_b and $NPV_{-200bps}$ of the hedge and the mortgage excluding the swaption.
Output: Calibrated parameters $(\tilde{T}_m, \tilde{N}, \tilde{K})$

1. Initialise parameters $\theta_0 := (T_{m,0}, N_0, K_0) = (60, 10000, 0.028)$.
2. Define objective function $|NPVaR_{-200bps}(T_m, N, K)| = |NPV_{-200bps}(T_m, N, K) - NPV_b(T_m, N, K)|$.
3. Call the Nelder-Mead optimisation algorithm with the objective function $|NPVaR_{-200bps}(\cdot)|$, and initial guess θ_0.

return $(\tilde{T}_m, \tilde{N}, \tilde{K})$

For each hedging approach covered thus far, starting with the internal funding and ending with the dynamic value hedge, Algorithm 4 is executed. The algorithm used the corresponding NPV_b and $NPV_{-200bps}$ of the hedge and the mortgage (excluding the swaption) as input and calibrated the required parameters. The resulting values, along with the corresponding swaption price and NPVaR, are presented in Table 5.8. The data in the final column of the table shows that for each hedge approach, the NPVaR for a -200 basis points interest rate scenario is increased to nearly zero, indicating that incorporating a single receiver swaption effectively mitigated the value exposure.

		Swaption characteristics				**NPV-at-Risk**
	Hedging approach	Option maturity	Notional	Strike (%)	Price	$NPVaR_{-200bps}$
Static	Internal funding	43 months	9.657×10^5	2.673	6.851×10^4	1.234×10^{-6}
	Notional hedge with zero-coupon bonds	45 months	5.339×10^5	2.781	4.106×10^4	-1.943×10^{-6}
	Value hedge with zero-coupon bonds	48 months	2.366×10^5	2.805	1.766×10^4	1.929×10^{-6}
Dynamic	Notional hedge with internal contracts	42 months	8.700×10^5	2.735	6.450×10^4	-1.679×10^{-5}
	Value hedge with internal contracts	46 months	3.060×10^5	2.877	2.516×10^4	6.681×10^{-6}

Table 5.8: An overview of the calibrated characteristics of the receiver swaption to hedge the -200 basis points interest rate scenario for each proposed hedging approaches.

Upon examining the characteristics of the calibrated swaptions, the first notable observation is that the optimal option maturities for the different hedging approaches vary between 42 and 48 months, equivalent to 3.5 to 4 years. Given that the mortgage portfolio we consider has a lifespan of 10 years, this indicates that the swaptions, if exercised, influence the cash flows of the hedge for the majority of the portfolio's duration. Secondly, the notional amounts are consistently in tens of thousands, which is a factor of ten higher than our initial guess. This implies that a relatively large notional is required to offset a smaller loss in value. In other words, the notional amount should exceed the value loss. Examining the third column of Table 5.8, we find that the calibrated strike rates are clustered around the initially chosen 2.8%. It is important to recall that the 2.8% was chosen to increase the likelihood of satisfying Inequality (5.9). To verify if this inequality holds for the calibrated strike rates, we need the expected swap rates $S(t, \widetilde{T}_m, T_s)$, where $\widetilde{T}_m$ represents the corresponding calibrated option maturity. These rates are displayed in Table 5.9, and upon verification, we find that the inequality indeed holds for all hedging approaches.

Considering the prices of the swaption, it is important to note that their values are of the same size as

Option maturity ($\tilde{T}_m$)	Expected swap rate (%)
43	2.878
45	2.881
48	2.886
42	2.877
46	2.882

Table 5.9: The expected 6M-Euribor swap rates $S(t, \tilde{T}_m, T_s)$ for each calibrated option maturity $\tilde{T}_m$ in Table 5.8.

the value exposures we want to hedge, as given in Table 5.6. This means that no matter the evolution of the interest rates, you will always lose tens of thousands on the purchase of the swaption, which you otherwise will only lose if the interest rates exhibit a large decline in value. This raises concerns about the effectiveness of including a swaption in the hedging portfolios. However, it is worth investigating whether the incorporation of this financial instrument can have a positive effect on the other risk metrics, making it valuable addition. This aspect will be further investigated in the next section of this chapter.

Finally, comparing the swaptions associated with the hedging approaches having the largest and smallest NPV-at-Risk, prior to the incorporation of the swaptions, provides valuable insights into the relationship between the risk metric and the characteristics of the financial instrument. It becomes evident that the hedges with greater exposure require a larger notional, a shorter option maturity, and a lower strike rate. The larger notional can be justified by the resulting larger cash flows, consequently leading to an overall increase in value. The shorter maturity can be attributed to the fact that an earlier exercised swaption has more influence on the value, as the swap may generate more cash flows. The lower strike rate however goes against initial expectations. In a receiver swaption, a higher strike rate typically results in larger cash flows, as the swaption holder receives the fixed strike rate and pays the floating rate. Therefore, a higher swap rate was initially anticipated. However, upon further analysis, an explanation was found. Rather than considering the strike rate on its own, we focus on the fixed cash flows of the swap, or the product of the notional and the strike rate, as displayed in Table 5.10. These values align with our expectations, as a higher fixed cash flow corresponds to a more exposed hedge approach.

	Hedging approach	Notional × Strike
Static	Internal funding	2.581×10^4
	Notional hedge with zero-coupon bonds	1.485×10^4
	Value hedge with zero-coupon bonds	6.637×10^3
Dynamic	Notional hedge with internal contracts	2.379×10^4
	Value hedge with internal contracts	8.804×10^3

Table 5.10: The fixed cash flows of the calibrated receiver swaption to hedge the -200bps interest rate scenario for each proposed hedging approaches.

Up to now, we have examined two risk appetites of a bank, margin stability and value stability. For margin stability, we analysed both a static and dynamic notional hedge, and concluded that the dynamic approach resulted in the lowest variance in the net interest margin. Subsequently, we shifted our focus to value stability, aiming to achieve a net basis point value of zero and minimise the variance in net present value. To accomplish this, we employed a static and dynamic value hedge, and once again found that the dynamic approach was superior. Furthermore, we delved deeper into value stability by considering the NPV-at-Risk when subject to a ± 200 basis points shock in the expected interest rate curve. In particular, the large negative shock could lead to a significant value decrease. However, by including a calibrated receiver swaption, we were able to mitigate this exposure. While we evaluated these different aspects individually, the optimal goal of a bank is to have a comprehensive risk management. This entails achieving both margin stability and value stability, while also maintaining a low NPV-at-Risk. Therefore, in the next section, we will assess the various hedging approaches using all discussed risk metrics simultaneously. This evaluation will enable us to provide a final recommendation for the most effective hedge approach.

5.5. Assessment hedge approaches

For each of the ten hedging approaches introduced in the previous sections, all introduced risk metrics are calculated. An overview for the approaches, with and without the incorporation of the calibrated receiver swaption, are given in Table 5.11 and 5.12, respectively.

		Margin stability	**Value stability**		
	Hedging approach	NIM variance	NPV variance	Average net BPV	NPV-at-Risk
Static	Internal funding	1.097×10^{-1}	2.767×10^{9}	247.4	-9.240×10^{4}
	Notional hedge with zero-coupon bonds	2.466×10^{-2}	5.021×10^{8}	71.33	-5.382×10^{4}
	Value hedge with zero-coupon bonds	2.022×10^{-2}	3.668×10^{8}	48.30	-2.394×10^{4}
Dynamic	Notional hedge with internal contracts	1.080×10^{-5}	1.480×10^{9}	28.57	-9.260×10^{4}
	Value hedge with internal contracts	5.862×10^{-5}	4.308×10^{5}	6.480×10^{-12}	-3.230×10^{4}

Table 5.11: An overview of the risk metrics for all proposed hedging approaches.

From Table 5.11 we observe some expected and surprising results. First of all, we note that the dynamic notional has the lowest net interest margin variance, whereas the dynamic value hedge exhibits the lowest net present value variance and the best net basis point value. Both can be justified as they were constructed with the main focus on margin and value stability, respectively. However, we also observe that the NIM variance of the dynamic value hedge is comparable to that of the dynamic notional hedge, which may seem surprising at first sight. However, upon closer examination, we find that the dynamic notional and value hedge exhibit similar behaviour. In both hedge approaches, the notional of the internal contracts increases when the interest rates are low, and vice versa. In the case of the notional hedge, this is due to the higher number of repayments in a low interest rate environment. While this relation for the value hedge may not immediately be apparent, our earlier analysis in this chapter led us to the conclusion that a shock in a high interest rate environment has a lesser effect on the NPV compared to the same shock in a low interest rate environment. Consequently, we will observe higher BPV values when the rates are low, resulting in larger notional amounts of the internal contracts.

For our final observations we note that the hedging approach with the best NPV-at-Risk is the static value hedge, but the difference with all other hedges is minimal. Additionally, we see that the internal funding performs the worst in almost all aspects. This emphasises again the need of a hedging portfolio.

Table 5.12 shows the risk metrics for all hedging approaches, but with the inclusion of the receiver swaption as calibrated in the previous section. With the colouring from light green to red indicating a large positive until a large negative effect of the incorporation the swaption. Clearly, the swaptions have a full positive effect on the NPV-at-Risk as it is constructed to minimise the NPVaR in a -200 basis points interest rate scenario. Consequently, the new value exposure, which is derived in Appendix C, is automatically lower. As for the other metrics, we observe diverse results. For all static approaches the inclusion of a swaption does not significantly affect their risk metrics. We only see little improvements and deteriorations. The final two rows show however a different picture. In particular, for the dynamic value hedge, all risk metrics, except for the NPVaR worsened. This can be attributed to the fact that the notionals of the internal contracts were constructed based on the observed cash flows of the mortgage

		Margin stability	Value stability		
	Hedging approach	NIM variance	NPV variance	Average net BPV	NPV-at-Risk
Static	Internal funding with a swaption	1.117×10^{-1}	1.637×10^{9}	229.7	1.235×10^{-6}
	Notional hedge with zero-coupon bonds and a swaption	2.531×10^{-2}	4.540×10^{8}	58.26	-315.4
	Value hedge with zero-coupon bonds and a swaption	1.874×10^{-2}	1.351×10^{8}	42.50	-2083
Dynamic	Notional hedge with internal contracts and a swaption	1.77×10^{-3}	2.395×10^{9}	9.674	-1.679×10^{-5}
	Value hedge with internal contracts and a swaption	2.817×10^{-4}	4.856×10^{8}	-7.816	-9396

Table 5.12: An overview of the risk metrics for all proposed hedging approaches including the calibrated receiver swaption. The colours red, orange, dark green and light green represent a significant deterioration, a slight deterioration, a slight improvement, and a significant improvement, respectively, in comparison to the hedging approaches excluding the swaption.

and hedge *excluding* the swaption. This implies that if the swaption is exercised, the BPV is no longer matched. On the contrary, in the case of no exercise the hedge would perform as required.

From this analysis, we can conclude that the incorporation of the swaption has the most effect on the NPV-at-Risk but does not significantly affect the other risks metrics in a positive matter. Therefore, taking into account our earlier observation of the swaption price being of the same order of magnitude as the value exposure we intend to hedge, it raises questions about its value. However, further research taking into account other risk factors is necessary in order to make conclusive statements on this matter.

Combining all results, we can construct a separate ranking of the hedging approaches for each risk metric. This is displayed in Table 5.13, in which 1 refers to the best performing hedge for that specific risk metric, and 10 refers to the worst performing. Based on this ranking, a final conclusion about the most effective hedging portfolio can be drawn.

The table shows that the dynamic notional hedge has the best margin stability, the dynamic value hedge has the best NPV variance and the best net BPV, and the internal funding with the swaption has the best NPV-at-Risk. A bank is however interested in a comprehensive risk management rather than the mitigation of one single risk metric. Therefore, from all hedging approaches discussed within this research, the dynamic value hedge without swaptions can be seen as the best choice for a bank without a strong risk appetite. This is due to the fact that even though it is constructed to maintain a stable value, it simultaneously results in a stable margin, as can be seen by its second place in the corresponding ranking. On the other hand, a bank which prioritises margin stability over value stability should consider a dynamic notional hedge without swaptions. This approach simply leads to the lowest variance in the net interest margin. For a bank that prefers value stability, the dynamic value hedge without a swaption exhibits the best results. However, it should be noted that the ultimate choice will not only depend on the bank's preference but may also be subject to the mandatory requirements of financial regulators, such as the European Central Bank.

	Hedging approach	Margin stability NIM variance	Value stability NPV variance	Value stability Average net BPV	Value stability NPV-at-Risk
Static	Internal funding	9	10	10	9
	Internal funding with a swaption	10	8	9	1
	Notional hedge with zero-coupon bonds	7	6	8	8
	Notional hedge with zero-coupon bonds and a swaption	8	4	7	3
	Value hedge with zero-coupon bonds	6	3	6	6
	Value hedge with zero-coupon bonds and a swaption	5	2	5	5
Dynamic	Notional hedge with internal contracts	1	7	4	10
	Notional hedge with internal contracts and a swaption	4	9	3	2
	Value hedge with internal contracts	2	1	1	7
	Value hedge with internal contracts and a swaption	3	5	2	4

Table 5.13: The final rankings of all proposed hedging approaches for each risk metric. The number 1 denotes the best performing approach, while 10 represents the worst performing approach.

6

Conclusion and discussion

6.1. Conclusion

Throughout this thesis we have investigated different methodologies to hedge the prepayment risk arising from a given mortgage portfolio, evaluating their effectiveness in maintaining value and margin stability under various interest rate scenarios. The one-factor Hull-White model was employed to simulate interest rate scenarios, while an interest rate-dependent logistic prepayment model provided monthly prepayment rates based on the mortgagors' refinancing incentive. We explained that during period of low interest rates, customers are more inclined to prepay their mortgages since they can refinance their existing loan at a lower rate. This impacts the size and timing of incoming mortgage cash flows for a bank, posing possible challenges to the stability of the net interest margin and value of the mortgage. Such instability is undesirable for banks and also regulatory bodies, such as the European Central Bank, acknowledge the interest rate risk and impose restrictions on banks to ensure its effective management. This highlights the importance of implementing effective hedging strategies.

In total, ten different hedging techniques are discussed within this research. We started with the internal funding, explored the static and dynamic notional hedge, which aim to maintain margin stability, and proceeded to a static and dynamic value hedge, focusing on value stability. Additionally, we included a receiver swaption into each of these five hedging approaches. Upon analysing all hedge portfolios, several noteworthy findings arise. Firstly, the sole utilisation of internal funding performs the worst in terms of both margin and value stability. Secondly, the dynamic hedges generally outperformed their static counterparts, which can be attributed to their flexibility to respond to market changes. Thirdly, as expected, the notional hedge demonstrates the best margin stability, while the value hedge exhibits the best value stability. Moreover, analysis reveals that incorporating a swaption into the hedging strategies leads to a significant improvement in NPV-at-Risk. However, the other risk metrics remain relatively stable or may even worsen. Furthermore, considering that the price of a swaption is comparable in magnitude to the potential exposure in value in a -200 basis points shocked scenario, the inclusion of a swaption does not contribute significant value to the initially constructed hedge portfolios.

Taking into account all observations, we can address the research question posed at the beginning of this report:

"Which portfolio of financial instruments provides the most effective hedge for a given mortgage portfolio, ensuring stability in terms of net interest margin and net present value under various interest rate simulations?"

Based on the research conducted in this thesis, there is no definitive answer, as the choice for a hedging approach depends heavily on the risk appetite of a bank. Therefore, we provide recommendations for three different types of banks. If a bank prioritises attaining margin stability, the recommended choice would be the dynamic notional hedge without the incorporation of the receiver swaption. On the other hand, for a bank that prefers value stability over margin stability, the dynamic value hedge without the inclusion of a swaption should be considered. Finally, for a bank that is indifferent and aims to ensure both value and margin stability to the greatest extent possible, the optimal choice would also be the dynamic value hedge without the utilisation of a swaption. It should however be noted that the final decision for a hedge portfolio should not solely rely on the bank's risk appetite but may also be influenced by mandatory requirements imposed by financial regulators, such as the European Central Bank.

6.2. Discussion

The research conducted in this thesis has shed light on effective hedging strategies to mitigate the prepayment risk associated with a mortgage portfolio. However, certain limitations should be acknowledged, highlighting potential areas for future research.

Firstly, the interest rate model and prepayment model employed in this study were relatively simplistic. The one-factor Hull-White model assumes a single driver for the complete yield curve, enabling it to only capture parallel movement in the interest rate curve. In practice, we also observe non-parallel movements such as the flattening of the yield. To capture these more sophisticated interest rate dynamics, further research could explore advanced interest rate models, such as the two-factor Hull-White model or the CIR model. Similarly, enhancing the prepayment model by incorporating additional drivers, such as the age of the contract or the time of the year, would provide a more comprehensive understanding of prepayment risk. Together, these improvements could lead to more realistic market scenarios, and therefore more realistic risk metrics. However, we do not expect this to change the final conclusions drawn in this research, as all hedging approaches used assumed the same underlying interest rate scenario.

Another limitation lies in the calculations of the different risk metrics. In this research, the net interest margin was determined under the assumption of a run-off balance sheet, where no new business was included. However, in practice, banks are more concerned with maintaining a stable NIM on a going-concern basis, considering the continuous issuance of new mortgages. Therefore, future research could consider a going-concern approach for the net interest margin, offering a more realistic perspective. Additionally, the discount factors used in the net present value calculations were based on the single-curve framework. We justified this choice for the potential unnecessary complexity introduced by any alternative approach, which is not required for the analyses conducted in this research. However, in the present market, the single-curve framework is seen as obsolete, and the multi-curve approach is typically employed instead. An interesting and valuable extension of this thesis would thus be to investigate the impact of incorporating the multi-curve framework on the hedging effectiveness. Finally, for the calculation of the basis point value and NPV-at-Risk, a simplified approach was used for the implementation of interest rate shocks. This involved applying a uniform shock to all relevant rates, including the spot rate and the swap rate. However, in practice, it is common to shock the swap rates and subsequently construct spot rates using bootstrapping techniques. Future studies could consider incorporating these nuances and assess their influence on the risk metrics and the conclusions drawn. Moreover, alternative scenarios for interest rate shocks, such as shifted curves or varying shock sizes, could be explored when evaluating the NPV-at-Risk. This would provide a more comprehensive understanding of the effectiveness of the proposed hedging approaches.

In this thesis we briefly touched upon the hedging costs when comparing the NPV-at-Risks with the price of the corresponding calibrated swaptions. We noted that their sizes were comparable, raising questions about the value of including swaptions in the hedging portfolio. However, it is important to note that hedging costs should not be limited to swaptions alone. By considering the hedging costs associated with each of the hedging approaches, future studies can achieve a more accurate evaluation of their overall effectiveness. Furthermore, in a dynamic hedge approach, there are additional costs associated with rebalancing that were not taken into account in this research. Future studies should incorporate the impact of rebalancing costs into the analysis, as this factor could potentially influence the preference for a dynamic hedging approach. An important aspect to consider within this research would be the frequency of rebalancing. Evaluating the optimal frequency becomes crucial as it may reveal that banks should not rebalance their hedge portfolios on a monthly basis but instead consider (semi-)annually rebalancing. This adjustment in frequency could potentially impact the value and margin stability of the corresponding hedge approach and result in the revision of the conclusions drawn in this thesis.

The final potential extension we propose is related to the risk metrics. This research focused on margin and value stability and assessed the various hedging approaches by considering the variance of the net interest margin, the variance of the net present value, the average basis point value, and the NPV-at-Risk for a parallel shock of 200 basis points. In addition to these metrics, alternative metrics could be explored to evaluate the effectiveness of hedging strategies. For example, one could incorporate the economic value of equity or the net interest income (NII) and the NII-at-Risk. This could provide a more comprehensive assessment of the risk exposure of the mortgage portfolio and the effectiveness of the proposed hedge portfolios.

Addressing these limitations and exploring the suggested future research topics would further enrich our understanding of effective hedging strategies for mortgage portfolios and contribute to the advancement of risk management practices in the financial industry.

A

Preliminary financial mathematics

This appendix provides a brief overview of preliminary financial mathematics concepts that are relevant to the research conducted in this thesis. It covers key definitions, as well as important theorems, aiming to ensure a comprehensive understanding of the underlying mathematical principles necessary for the analysis presented in this thesis.

A.1. Financial definitions

A.1.1. Money-market account

The money market account $M(t)$ represents the value of a bank account at time t and is defined by the following system:

$$\begin{cases} \mathrm{d}M(t) = M(t)r(t)\,\mathrm{d}t; \\ M(0) = 1, \end{cases}$$

where $r(t)$ denotes the bank deposit interest rate. By solving this system, we obtain the equivalent definition:

$$M(t) = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right).$$

This notation plays a crucial role in pricing financial products since it enables the discounting of future cash flows.

A.1.2. Euribor rate

The Euribor rate is an interest rate based on the average interest rates at which banks within the Eurozone borrow from one another. It is widely used as a reference rate in the European money market for various financial instruments, including mortgages, swaps and swaptions. The Euribor rate comes in different maturities, ranging from 1 week to 1 year, where the 6M-Euribor rate is used most often.

The Euribor rate is denoted by $E(t; T_{k-1}, T_k)$, which can be read as the interest rate at time t for settlement date T_{k-1} and maturity date T_k. This means that $E(t; T_{k-1}, T_k)$ is a forward rate over the period $[T_{k-1}, T_k]$ which resets at t. To derive the simply compounded forward rate we look at two counterparties: A and B. Suppose A wants to borrow 1 euro from B at future time T_{k-1} and will pay this back at time T_k, including some interest E. The value of such a contract for B at time t will be

$$\begin{aligned} V(t) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{-1}{M(T_{k-1})} + \frac{1 + E \cdot (T_k - T_{k-1})}{M(T_k)} \middle| \mathcal{F}(t)\right] \\ &= -P(t, T_{k-1}) + P(t, T_k) \cdot (1 + E \cdot (T_k - T_{k-1})), \end{aligned} \tag{A.1}$$

where $P(t, T)$ denotes the price of zero-coupon bond at time t that pays out €1 at time T. Setting this equation equal to zero and rewriting gives the fair value for the interest rate E:

$$E = \left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right) \cdot \frac{1}{T_k - T_{k-1}} = \frac{P(t, T_{k-1}) - P(t, T_k)}{(T_k - T_{k-1}) \cdot P(t, T_k)}$$

Typically, we write

$$E(t;T_{k-1},T_k)=\frac{P(t,T_{k-1})-P(t,T_k)}{\tau_k P(t,T_k)},$$

with $\tau_k = T_k - T_{k-1}$.

A.2. Financial theorems

In the probability space $(\Omega,\mathcal{F},\mathbb{P})$, where Ω is the set of all possible outcomes and $\mathcal{F}$ the σ-algebra encompassing all feasible events, the real-world measure $\mathbb{P}$ reflects the true probability of events that could occur. Specifically, the probability measure $\mathbb{P}$ assigns probabilities to each set $A\in\mathcal{F}$, so that $\mathbb{P}:\mathcal{F}\to[0,1]$. In finance, however, pricing is done under the so-called risk-neutral measure $\mathbb{Q}$. This measure is equivalent to $\mathbb{P}$, meaning they agree on which sets are assigned probability zero to, but they may disagree on the probability of the feasible scenarios. The risk-neutral measure is artificially constructed so that the expected return on any asset equals the risk-free rate of return. This characteristic enables universal pricing of any derivative, regardless of the underlying asset's actual probabilities, by stating that its price is precisely the present value of its expected payoff. This feature will be used to price the zero-coupon bonds, swaps and swaptions used in this thesis.

A.2.1. Change of measure

The risk-neutral measure is essential in pricing financial products. However, the pricing of exotic derivatives can be problematic due to the complexity of their payoff functions. One way to address this is through the change of measure approach, which can simplify the instrument's payoff significantly and may even lead to a closed form solution. Bayes' theorem is a well-known theorem that relates two expectations under different measures and can be used in this context.

Theorem 1 (Bayes)**.** *Let $\mathbb{N}\sim\mathbb{M}$ be two absolutely continuous probability measures on probability space $(\Omega,\mathcal{F})$ and $\mathcal{G}\subseteq\mathcal{F}$ a σ-algebra. Let $X\in L^1(\Omega,\mathcal{F},\mathbb{N})$, then*

$$\mathbb{E}_{\mathbb{N}}[X|\mathcal{G}]=\frac{\mathbb{E}_{\mathbb{M}}\left[\Lambda_{\mathbb{N},\mathbb{M}}X|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{M}}\left[\Lambda_{\mathbb{N},\mathbb{M}}|\mathcal{G}\right]},$$

with Radon-Nikodym derivative

$$\Lambda_{\mathbb{N},\mathbb{M}}:=\left.\frac{\mathrm{d}\mathbb{N}}{\mathrm{d}\mathbb{M}}\right|_{\mathcal{G}}.$$

Once the Radon-Nikodym derivative is known, the Bayes' theorem enables us to calculate expectations under different measures. The following theorem provides a formula for this derivative.

Theorem 2 (Change of measure)**.** *Let X be the price of any traded asset and M a numéraire. Suppose that $\frac{X}{M}$ is a martingale under $\mathbb{M}\sim\mathbb{Q}$, i.e.*

$$\mathbb{E}_{\mathbb{M}}\left[\frac{X(T)}{M(T)}\middle|\mathcal{F}(t_0)\right]=\frac{X(t_0)}{M(t_0)}.$$

Furthermore, let V be the price of any self-financing derivative, N a quantity and suppose that $\frac{V}{N}$ is a martingale under $\mathbb{N}\sim\mathbb{Q}$, i.e.

$$\mathbb{E}_{\mathbb{N}}\left[\frac{V(T)}{N(T)}\middle|\mathcal{F}(t_0)\right]=\frac{V(t_0)}{N(t_0)}.$$

Then the Radon-Nikodym derivatives is given by

$$\Lambda_{\mathbb{N},\mathbb{M}}(T):=\left.\frac{\mathrm{d}\mathbb{N}}{\mathrm{d}\mathbb{M}}\right|_{\mathcal{F}(T)}=\frac{N(T)M(t)}{N(t)M(T)}.$$

The Radon-Nikodym derivative is most often used in combination with the Girsanov theorem. This theorem provides insights into how a stochastic process changes when changing from one measure to another. Specifically, we will present the theorem in the context of a special case when the underlying process is a Brownian motion.

Theorem 3 (Girsanov). *Let $\mathbb{Q} \sim \mathbb{P}$ with corresponding Radon-Nikodym derivative $\Lambda_{\mathbb{Q},\mathbb{P}}$. Let X be a martingale such that $\Lambda_{\mathbb{Q},\mathbb{P}}$ is the solution of the system*

$$\begin{cases} \mathrm{d}\Lambda_{\mathbb{Q},\mathbb{P}}(t) = \Lambda_{\mathbb{Q},\mathbb{P}}(t)\,\mathrm{d}X(t); \\ \Lambda_{\mathbb{Q},\mathbb{P}}(t_0) = 1. \end{cases} \tag{A.2}$$

Then $W^{\mathbb{P}}$ is a martingale under $\mathbb{P}$ if and only if $W^{\mathbb{Q}} = W^{\mathbb{P}} - [W^{\mathbb{P}}, X]$ is a martingale under $\mathbb{Q}$. Consequently, using system A.2, the dynamics of $W^{\mathbb{Q}}$ become

$$\begin{aligned} \mathrm{d}W^{\mathbb{Q}}(t) &= \mathrm{d}W^{\mathbb{P}}(t) - \mathrm{d}W^{\mathbb{P}}(t)\,\mathrm{d}X(t) \\ &= \mathrm{d}W^{\mathbb{P}}(t) - \frac{\mathrm{d}\Lambda_{\mathbb{Q},\mathbb{P}}(t)}{\Lambda_{\mathbb{Q},\mathbb{P}}(t)}\,\mathrm{d}W^{\mathbb{P}}(t). \end{aligned}$$

Change of measure from $\mathbb{Q}$ to $\mathbb{Q}^T$

Any tradable asset can act as a numéraire, but as the scope of this thesis is limited to a handful financial products, we will only consider two assets - the money-market account and the zero-coupon bond - which will be discussed below.

1. The risk-neutral measure $\mathbb{Q}$ corresponds to the money-market account $M(t)$, so that

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{X(t)}{M(t)}\middle|\mathcal{F}(t_0)\right] = \frac{X(t_0)}{M(t_0)}.$$

2. The forward measure $\mathbb{Q}^T$ is associated with the zero-coupon bond $P(t,T)$ as numéraire, so that

$$\mathbb{E}_{\mathbb{Q}^T}\left[\frac{X(t)}{P(t,T)}\middle|\mathcal{F}(t_0)\right] = \frac{X(t_0)}{P(t_0,T)}.$$

Consequently, the relevant change of measure is that of $\mathbb{Q}$ to $\mathbb{Q}^T$ since the opposite is simply the inverse. From Theorem 2 we know that for $t_0 < t < T$ the corresponding Radon-Nikodym derivative is given by

$$\Lambda_{\mathbb{Q},\mathbb{Q}^T}(t) := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^T}\right|_{\mathcal{F}(t)} = \frac{M(t)P(t_0,T)}{M(t_0)P(t,T)}.$$

A.2.2. Itô's lemma

A different essential tool in mathematical finance is Itô's lemma. It provides a formula for the stochastic differential of a function of a stochastic process and reads as follows:

Lemma 1 (Itô's lemma). *Let $f(t,x)$ be a function with well-defined and continuous partial derivatives f_t, f_x and f_{xx}, and let $X(t)$ a stochastic variable with dynamics*

$$\mathrm{d}X(t) = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W(t).$$

Then $f(t,X(t))$ has the dynamics

$$\mathrm{d}f(t,X(t)) = \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t + \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}X(t) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\,\mathrm{d}X(t)\,\mathrm{d}X(t). \tag{A.3}$$

To illustrate the use of Itô's lemma, let us consider the function $f(x) = \ln(x)$. Suppose $X(t)$ follows the stochastic differential equation

$$\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t),$$

then the dynamics of $\ln(X(t))$ are as follows:

$$
\begin{aligned}
\mathrm{d}\ln(X(t)) &= \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t + \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}X(t) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\,\mathrm{d}X(t)\,\mathrm{d}X(t) \\
&= 0\,\mathrm{d}t + \frac{1}{X(t)}\,\mathrm{d}X(t) + \frac{1}{2}\cdot -\frac{1}{X(t)^2}\,\mathrm{d}X(t)\,\mathrm{d}X(t) \\
&= \frac{1}{X(t)}\,[\mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t)] - \frac{1}{2}\frac{1}{X(t)^2}\sigma^2 X(t)^2\,\mathrm{d}t \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t),
\end{aligned}
$$

where we used Itô's multiplication table, Table A.1, to determine the second order term.

$\times$	$\mathrm{d}W(t)$	$\mathrm{d}t$
$\mathrm{d}W(t)$	$\mathrm{d}t$	0
$\mathrm{d}t$	0	0

Table A.1: Itô's multiplication table for the multiplication of stochastic/deterministic infinitesimal increments.

B

Additional insights about the Hull-White model

The one-factor Hull-White model is a valuable tool for simulating interest rate scenarios. Moreover, its simplicity enables us to derive analytical formulas for pricing various financial instruments. In this appendix, we will explore some of its useful characteristics that will be subsequently employed in pricing a zero-coupon bond, an interest rate swap, and a swaption

B.1. Characteristics of the Hull-White model

This section discusses two important characteristics of the Hull-White model: it generates normally distributed short rates, and it belongs to class of affine term structure models.

B.1.1. Normality assumption

The dynamics of the Hull-White model are given by

$$\mathrm{d}r(t) = (\theta(t) - \alpha r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}}(t). \tag{B.1}$$

One can see that this stochastic differential equation corresponds to a normally distributed short rate by applying Itô's lemma (Lemma 1) to $x(t) = e^{\alpha t}\,r(t)$:

$$\begin{aligned}
\mathrm{d}x(t) &= \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,\mathrm{d}t + \frac{\mathrm{d}x(t)}{\mathrm{d}r(t)}\,\mathrm{d}r(t) + \frac{1}{2}\frac{\mathrm{d}^2x(t)}{\mathrm{d}r(t)^2}\,\mathrm{d}r(t)\,\mathrm{d}r(t) \\
&= \alpha e^{\alpha t}r(t)\,\mathrm{d}t + e^{\alpha t}\,\mathrm{d}r(t) + 0 \\
&= \alpha e^{\alpha t}r(t)\,\mathrm{d}t + e^{\alpha t}\left[(\theta(t) - \alpha r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}}(t)\right] && \text{(Using (B.1))} \\
&= e^{\alpha t}\theta(t)\,\mathrm{d}t + \sigma e^{\alpha t}\,\mathrm{d}W^{\mathbb{Q}}(t).
\end{aligned}$$

Integrating on both sides gives

$$x(t) - x(t_0) = \int_{t_0}^{t} e^{\alpha s}\theta(s)\,\mathrm{d}s + \sigma\int_{t_0}^{t} e^{\alpha s}\,\mathrm{d}W^{\mathbb{Q}}(s),$$

which can be rewritten by using the definition of $x(t)$ to

$$r(t) = e^{-\alpha(t-t_0)}r(t_0) + \int_{t_0}^{t} e^{-\alpha(t-s)}\theta(s)\,\mathrm{d}s + \sigma\int_{t_0}^{t} e^{-\alpha(t-s)}\,\mathrm{d}W^{\mathbb{Q}}(s).$$

Taking into account that an Itô integral is normally distributed, we can easily find that under the $\mathbb{Q}$−measure

$$r(t) \sim \mathcal{N}\left(e^{-\alpha(t-t_0)}r(t_0) + \int_{t_0}^{t} e^{-\alpha(t-s)}\theta(s)\,\mathrm{d}s,\ \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-t_0)}\right)\right),$$

where the variance is obtained by applying Itô's isometry:

$$\mathbb{E}\left[\left(\int_t^T X(s)\,\mathrm{d}W(s)\right)^2\right] = \mathbb{E}\left[\int_t^T X(s)^2\,\mathrm{d}s\right].$$

B.1.2. Affine term structure model

Suppose we have the following system of stochastic differential equations:

$$\mathrm{d}\mathbf{X}(t) = \bar{\mu}(\mathbf{X}(t))\,\mathrm{d}t + \bar{\sigma}(\mathbf{X}(t))\,\mathrm{d}\widetilde{\mathbf{W}}(t),$$

where $\widetilde{\mathbf{W}}(t)$ are independent Brownian motions. This process is in the affine diffusion class, if we can write the drift, volatility, and interest rate components in affine form. That is,

$$\begin{aligned}
\bar{\mu}(\mathbf{X}(t)) &= a_0 + a_1\mathbf{X}(t) && \text{for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n},\\
\bar{\sigma}(\mathbf{X}(t))\bar{\sigma}(\mathbf{X}(t))^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}(t) && \text{for } (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n},\\
\bar{r}(\mathbf{X}(t)) &= r_0 + r_1^T\mathbf{X}(t) && \text{for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

From the stochastic differential equation (B.1) it is clear to see that the Hull-White model belongs to the affine term structure class, with

$$\begin{aligned}
a_0 &= \theta(t), & a_1 &= -\alpha,\\
c_0 &= \sigma^2, & c_1 &= 0,\\
r_0 &= 0, & r_1 &= 1.
\end{aligned}$$

With this class comes a powerful theorem that shows formula for the corresponding discounted characteristic function (Duffie et al., 2000) and therefore is vital for pricing financial instruments under the Hull-White model.

Theorem 4. *Consider the discounted characteristic function for an affine term structure model, defined as*

$$\phi(\boldsymbol{X}(t), t, T, \boldsymbol{u}) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(\boldsymbol{X}(s))\,\mathrm{d}s} e^{i\boldsymbol{u}\,\boldsymbol{X}(T)} \middle| \mathcal{F}(t)\right] \quad \textit{for } \boldsymbol{u} \in \mathbb{C}^n,$$

with boundary condition

$$\phi(\boldsymbol{X}(T), T, T, \boldsymbol{u}) = e^{i\boldsymbol{u}^T\boldsymbol{X}(T)}.$$

Then its solution is of the following form:

$$\phi(\boldsymbol{X}(t), t, T, \boldsymbol{u}) = e^{A(\boldsymbol{u},t,T) + \boldsymbol{B}(\boldsymbol{u},t,T)^T\boldsymbol{X}(t)}.$$

If we define $\tau = T - t$, we can write $A(\boldsymbol{u}, t, T) = A(\boldsymbol{u}, \tau)$ and $\boldsymbol{B}(\boldsymbol{u}, t, T) = \boldsymbol{B}(\boldsymbol{u}, \tau)$. These coefficients have to satisfy the following system of Ricatti-type of ordinary differential equations:

$$\begin{aligned}
\frac{\mathrm{d}A(\boldsymbol{u},\tau)}{\mathrm{d}\tau} &= -r_0 + \boldsymbol{B}(\boldsymbol{u},\tau)a_0 - \frac{1}{2}\boldsymbol{B}(\boldsymbol{u},\tau)^T c_0 \boldsymbol{B}(\boldsymbol{u},\tau),\\
\frac{\mathrm{d}\boldsymbol{B}(\boldsymbol{u},\tau)}{\mathrm{d}\tau} &= -r_1 + a_1^T\boldsymbol{B}(\boldsymbol{u},\tau) + \frac{1}{2}\boldsymbol{B}(\boldsymbol{u},\tau)^T c_1 \boldsymbol{B}(\boldsymbol{u},\tau).
\end{aligned}$$

B.2. Pricing financial instruments under the Hull-White model

In this thesis, it is important to determine the prices of a zero-coupon bond, interest rate swap, and swaption under the Hull-White model. In particular, understanding the cost associated with a receiver swaption is vital for evaluating its incorporation in the hedging approaches, and this calculation relies on the prices of the other instruments. The valuation of all the aforementioned instruments will be consecutively discussed in the following sections.

B.2.1. Zero-coupon bond

The cash flows of a standard zero-coupon bond consist of a single payment of 1 at maturity. So, under the risk neutral measure, the price at time t of a zero-coupon bond maturing at time T is given by

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,\mathrm{d}s}\middle|\mathcal{F}_t\right].$$

From the fact that the Hull-White model is an affine term structure model and Theorem 4, we know that the price of the zero-coupon bond has the form

$$P(t,T) = e^{A(t,T)+B(t,T)r(t)}, \tag{B.2}$$

where

$$\begin{cases} \frac{\mathrm{d}A(\tau)}{\mathrm{d}\tau} & = -r_0 - a_0 B(\tau) + \frac{1}{2}c_0 B(\tau)^2, \\ & = 0 + \theta(T-\tau)B(\tau) + \frac{1}{2}\sigma^2 B(\tau)^2 \\ \frac{\mathrm{d}B(\tau)}{\mathrm{d}\tau} & = -r_1 + a_1 B(\tau) + \frac{1}{2}c_1 B(\tau)^2 \\ & = -1 - \alpha B(\tau) + 0. \end{cases} \tag{B.3}$$

Moreover, we know that

$$P(T,T) = \phi(r(T),T,T,0) = 1,$$

so that

$$A(0) = 0 \quad \text{and} \quad B(0) = 0$$

or equivalently,

$$A(T,T) = 0 \quad \text{and} \quad B(T,T) = 0.$$

Now we can solve system (B.3) and find that

$$\begin{aligned} A(t,T) &= -\int_t^T \theta(s)B(s,T)\,\mathrm{d}s + \frac{1}{2}\sigma^2\int_t^T B(s,T)^2\,\mathrm{d}s \\ B(t,T) &= \frac{1}{\alpha}\left(e^{-\alpha(T-t)} - 1\right). \end{aligned}$$

However, this is not yet sufficient since the coefficient $A(t,T)$ depends on the drift $\theta(t)$, and this term is still unknown. This can be solved since we know that it was chosen by fitting the theoretical bond prices to the yield curve observed in the market. For this we can use the instantaneous forward rate $F(t,T)$, defined as

$$F(t,T) = -\frac{\mathrm{d}}{\mathrm{d}T}\log P(t,T).$$

Substituting Equation (B.2) and the derived equations for the coefficients $A(t,T)$ and $B(t,T)$, we obtain

$$\begin{aligned} F(t,T) &= -\frac{\mathrm{d}}{\mathrm{d}T}\log P(t,T) \\ &= -\frac{\mathrm{d}}{\mathrm{d}T}A(t,T) - \frac{\mathrm{d}}{\mathrm{d}T}B(t,T)r(t) \\ &= -\frac{\mathrm{d}}{\mathrm{d}T}\left[-\int_t^T \theta(s)B(s,T)\,\mathrm{d}s + \frac{1}{2}\sigma^2\int_t^T B(s,T)^2\,\mathrm{d}s\right] - \frac{\mathrm{d}}{\mathrm{d}T}B(t,T)r(t) \\ &= -\left[-\int_t^T \theta(s)\frac{\mathrm{d}}{\mathrm{d}T}B(s,T)\,\mathrm{d}s + \frac{1}{2}\sigma^2\int_t^T 2B(s,T)\frac{\mathrm{d}}{\mathrm{d}T}B(s,T)\right] - \frac{\mathrm{d}}{\mathrm{d}T}B(t,T)r(t). \end{aligned} \tag{B.4}$$

The final equation gives a relation between the forward rate and the drift term, which can be used to derive the analytical form of the coefficient $A(t,T)$, which will be done in the remaining of this subsection.

Using Equation (B.2), we can write

$$
\begin{aligned}
\log\left(\frac{P(0,T)}{P(0,t)}\right) &= A(0,T) - A(0,t) + (B(0,T) - B(0,t))r(0) \\
&= -\int_0^T \theta(s)B(s,T)\,\mathrm{d}s + \frac{1}{2}\sigma^2\int_0^T B(s,T)^2\,\mathrm{d}s \\
&\quad + \int_0^t \theta(s)B(s,t)\,\mathrm{d}s - \frac{1}{2}\sigma^2\int_0^t B(s,t)^2\,\mathrm{d}s + (B(0,T) - B(0,t))r(0) \\
&= -\int_t^T \theta(s)B(s,T)\,\mathrm{d}s - \int_0^t \theta(s)(B(s,T) - B(s,t))\,\mathrm{d}s \\
&\quad + \frac{1}{2}\sigma^2\int_t^T B(s,T)^2\,\mathrm{d}s + \frac{1}{2}\sigma^2\int_0^t (B(s,T)^2 - B(s,t)^2)\,\mathrm{d}s + (B(0,T) - B(0,t))r(0).
\end{aligned}
$$

Now since $B(s,T) - B(s,t) = -B(t,T)\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)$, we can rewrite this to

$$
\begin{aligned}
\log\left(\frac{P(0,T)}{P(0,t)}\right) &= -\int_t^T \theta(s)B(s,T)\,\mathrm{d}s + \frac{1}{2}\sigma^2\int_t^T B(s,T)^2\,\mathrm{d}s + B(t,T)\int_0^t \theta(s)\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)\,\mathrm{d}s \\
&\quad - \frac{1}{2}\sigma^2 B(t,T)\int_0^t (B(s,T) + B(s,t))\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)\,\mathrm{d}s - B(t,T)\frac{\mathrm{d}}{\mathrm{d}t}B(0,t)r(0) \\
&= A(t,T) + B(t,T)\left[\int_0^t \theta(s)\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)\,\mathrm{d}s - \frac{1}{2}\sigma^2\int_0^t 2B(s,t)\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)\,\mathrm{d}s - \frac{\mathrm{d}}{\mathrm{d}t}B(0,t)r(0)\right] \\
&\quad - \frac{1}{2}\sigma^2 B(t,T)\int_0^t (B(s,T) - B(s,t))\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)\,\mathrm{d}s \\
&= A(t,T) + B(t,T)F(0,t) + \frac{1}{2}\sigma^2 B(t,T)^2\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}t}B(s,t)\right)^2\mathrm{d}s \qquad \text{(By (B.4))} \\
&= A(t,T) + B(t,T)F(0,t) + \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha t}\right)B(t,T)^2.
\end{aligned}
$$

Rewriting gives

$$
A(t,T) = \log\left(\frac{P(0,T)}{P(0,t)}\right) - B(t,T)F(0,t) - \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha t}\right)B(t,T)^2.
$$

Finally, we conclude that the price for a zero-coupon bond under the Hull-White model is given by

$$
P(t,T) = e^{A(t,T)+B(t,T)r(t)},
$$

where

$$
\begin{aligned}
A(t,T) &= \log\left(\frac{P(0,T)}{P(0,t)}\right) - B(t,T)F(0,t) - \frac{\sigma^2}{4\alpha}\left(1 - e^{-2\alpha t}\right)B(t,T)^2 \\
B(t,T) &= \frac{1}{\alpha}\left(e^{-\alpha(T-t)} - 1\right).
\end{aligned}
$$

Note that this price is independent of the drift term $\theta(t)$.

Dynamics zero-coupon bonds under the risk-neutral Hull-White model

With the use of the price derived in the previous section, we can determine the dynamics of the zero-coupon bond under the risk neutral measure:

$$\begin{aligned}
\mathrm{d}P(t,T) &= \frac{\mathrm{d}P(t,T)}{\mathrm{d}t}\,\mathrm{d}t + \frac{\mathrm{d}P(t,T)}{\mathrm{d}r(t)}\,\mathrm{d}r(t) + \frac{1}{2}\frac{\mathrm{d}^2P(t,T)}{\mathrm{d}r(t)\,\mathrm{d}r(t)}\,\mathrm{d}r(t)\,\mathrm{d}r(t) \\
&= P(t,T)\left(\frac{\mathrm{d}A(t,T)}{\mathrm{d}t} + \frac{\mathrm{d}B(t,T)}{\mathrm{d}t}r(t)\right)\mathrm{d}t + P(t,T)B(t,T)\,\mathrm{d}r(t) + \frac{1}{2}P(t,T)B(t,T)^2\,\mathrm{d}r(t)\,\mathrm{d}r(t) \\
&= P(t,T)\left(-\theta(t)B(t,T) - \frac{1}{2}\sigma^2B(t,T)^2 + (1+\alpha B(t,T))r(t)\right)\mathrm{d}t \\
&\quad + P(t,T)B(t,T)\left((\theta(t)-\alpha r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}}\right) + \frac{1}{2}P(t,T)B(t,T)^2\sigma^2\,\mathrm{d}t \\
&= r(t)P(t,T)\,\mathrm{d}t + \sigma B(t,T)P(t,T)\,\mathrm{d}W^{\mathbb{Q}}
\end{aligned}$$

B.2.2. Interest rate swap

Consider a swap with payments at $T_{m+1},\dots,T_s$ with a notional amount of N. Then the fixed leg at T_k equals $\tau_k N r_{fixed}$ and the floating leg is equal to $\tau_k N r_{floating}(T_k)$, where we define $\tau_k = T_k - T_{k-1}$. Typically, the floating rate is an Euribor rate E such that

$$r_{floating}(T_k) = E(T_{k-1};T_{k-1},T_k),$$

with the Euribor rate over the period $[T_{k-1},T_k]$ which resets at t is defined as

$$E(t;T_{k-1},T_k) = \frac{P(t,T_{k-1}) - P(t,T_k)}{\tau_k P(t,T_k)}.$$

Note that the lengths in between the payment dates correspond to the tenor of the Euribor rate. Then the payoff of a payer and receiver swap is given by

$$V_{swap}(T_m,\dots,T_s) = \sum_{k=m+1}^{s} \bar{\alpha}\tau_k N(E(T_{k-1};T_{k-1},T_k) - r_{fixed}),$$

with

$$\bar{\alpha} = \begin{cases} 1 & \text{for a payer swap;} \\ -1 & \text{for a receiver swap.} \end{cases}$$

In order to find the value of an interest rate swap at time t, the future cash flows at $T_{m+1},\dots,T_s$ should be discounted. So, under the risk neutral measure, the price at time t of a payer ($\bar{\alpha} = 1$) and receiver ($\bar{\alpha} = -1$) interest rate swap that starts with payments at $T_{m+1},\dots,T_s$ is given by

$$\begin{aligned}
V_{swap}(t) &= \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=m+1}^{s}\frac{M(t)}{M(T_k)}\bar{\alpha}\tau_k N(E(T_{k-1};T_{k-1},T_k) - r_{fixed})\middle|\mathcal{F}(t)\right] \\
&= \bar{\alpha}M(t)N\sum_{k=m+1}^{s}\tau_k\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{M(T_k)}(E(T_{k-1};T_{k-1},T_k) - r_{fixed})\middle|\mathcal{F}(t)\right] \\
&= \bar{\alpha}N\sum_{k=m+1}^{s}\tau_k P(t,T_k)(\mathbb{E}_{\mathbb{Q}^{T_k}}[E(T_{k-1};T_{k-1},T_k)|\mathcal{F}(t)] - r_{fixed}) && \text{(Change of measure: } \mathbb{Q} \text{ to } \mathbb{Q}^{T_k}\text{)} \\
&= \bar{\alpha}N\sum_{k=m+1}^{s}\tau_k P(t,T_k)(E(t;T_{k-1},T_k) - r_{fixed}) && (E(T_{k-1};T_{k-1},T_k) \text{ martingale under } \mathbb{Q}^{T_k})
\end{aligned}$$

Now we can substitute the definition of the Euribor rate,

$$\begin{aligned} V_{swap}(t) &= \bar{\alpha}N \sum_{k=m+1}^{s} \tau_k P(t,T_k)\left(\frac{P(t,T_{k-1}) - P(t,T_k)}{\tau_k P(t,T_k)} - r_{fixed}\right) \\ &= \bar{\alpha}N \sum_{k=m+1}^{s} (P(t,T_{k-1}) - P(t,T_k)) - \bar{\alpha}N r_{fixed} \sum_{k=i+1}^{m} \tau_k P(t,T_k) \\ &= \bar{\alpha}N(P(t,T_m) - P(t,T_s)) - \bar{\alpha}N r_{fixed} \sum_{k=m+1}^{s} \tau_k P(t,T_k) \qquad \text{(Telescopic sum)} \end{aligned} \tag{B.5}$$

Equation (B.5) gives the final price of a payer ($\bar{\alpha} = 1$) and receiver ($\bar{\alpha} = -1$) interest rate swap. This expression was achieved without any assumptions about the underlying interest rate model. Therefore, this formula holds in general.

In practice, a swap is usually considered to be "at-the-money", which refers to a scenario where the value of the fixed cash flows matches that of the floating cash flows, resulting in zero value of the swap. This allows for entering into the swap contract without any initial costs. The fixed rate required to achieve this, known as the swap rate, can be derived by setting Equation (B.5) equal to zero, and it is given by

$$S(t,T_m,T_s) := \frac{P(t,T_m) - P(t,T_s)}{\sum_{k=m+1}^{s} \tau_k P(t,T_k)}. \tag{B.6}$$

B.2.3. Swaption

The price of a swaption at time t, denoted by $V_{swaption}(t,T_m,T_s)$ can be expressed as the expected sum of discounted cash flows. These cash flows are either zero, or equal to those of the underlying swap, depending on whether the option is exercised. In practice, one would compare the swap rate at time T_m to the contractually agreed-upon fixed rate. Then in case of a receiver swaption, the holder would exercise if the observed rate were below the fixed rate, and the converse holds in case of a payer swaption. Mathematically, we can write

$$x_{pay}(t) = \begin{cases} 1 & \text{if } r_{fixed} < S(t,T_m,T_s); \\ 0 & \text{otherwise,} \end{cases} \tag{B.7}$$

and

$$x_{rec}(t) = \begin{cases} 1 & \text{if } r_{fixed} > S(t,T_m,T_s); \\ 0 & \text{otherwise,} \end{cases} \tag{B.8}$$

where $S(t,T_m,T_s)$ denotes the swap rate at time t of a swap with exchanges at future times $T_{m+1}, \dots T_s$ and is defined as in (B.6). The reasoning behind this is that the market swap rate is derived such that the present value of the swap equals zero. So, whenever the cash flows of the agreed swap exceed those expected by the market, you may expect a profit. Equivalently, we could say that one should only exercise if the value of the swap at T_m is positive. Therefore, the value of a receiver swaption can be written either by (1) using Equations (B.7) and (B.8), or by (2) the value of a swap as derived in (B.5):

$$(1)\;\; V_{swaption,rec}(t,T_m,T_s) = \mathbb{E}_{\mathbb{Q}}\left[\frac{M(t)}{M(T_m)} x_{rec}(T_m) V_{swap,rec}(T_m) \middle| \mathcal{F}(t)\right]$$

$$(2)\;\; V_{swaption,rec}(t,T_m,T_s) = \mathbb{E}_{\mathbb{Q}}\left[\frac{M(t)}{M(T_m)} \max\left(V_{swap,rec}(T_m), 0\right) \middle| \mathcal{F}(t)\right]$$

A similar equation can be derived for a payer swap. In practice, method (1) is used more often, as the swap rate can directly be observed in the market, and thus the comparison can be made without any computations. Theoretically, however, method (2) is preferable because there is a wide range of theorems that can be applied while using this notation, which will be discussed hereafter.

So, using notation (2), the price at time t of a receiver swaption with expiry T_m and maturity of the underlying swap T_s under the risk neutral measure is given by

$$\begin{aligned} V_{swaption,rec}(t,T_m,T_s) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{M(t)}{M(T_m)}\max\left(V_{swap,rec}(T_m),0\right)\middle|\mathcal{F}(t)\right] \\ &= P(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\max\left(V_{swap,rec}(T_m),0\right)|\mathcal{F}(t)\right] \quad \text{(Change of measure } \mathbb{Q} \text{ to } \mathbb{Q}^{T_m}) \end{aligned}$$

Using the expression obtained in the previous section, we can write

$$\begin{aligned} V_{swaption,rec}(t,T_m,T_s) &= P(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\max\left(N(P(T_m,T_s)-P(T_m,T_m))+Nr_{fixed}\sum_{k=m+1}^{s}\tau_k P(T_m,T_k),0\right)\middle|\mathcal{F}(t)\right] \\ &= NP(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\max\left(\sum_{k=m+1}^{s}c_k P(T_m,T_k)-1,0\right)\middle|\mathcal{F}(t)\right], \end{aligned}$$

where $c_k = r_{fixed}\tau_k$ for $k = m+1,\dots,s-1$ and $c_s = 1 + r_{fixed}\tau_k$. Now by using "Jamshidian's trick" (Jamshidian, 1989), we obtain

$$\max\left(\sum_{k=m+1}^{s}c_k P(T_m,T_k)-1,0\right) = \sum_{k=m+1}^{s}c_k\max\left(P(T_m,T_k)-P_k^*,0\right),$$

where $P_k^* := e^{A(T_m,T_k)+B(T_m,T_k)r^*}$ and r^* chosen such that

$$\sum_{k=m+1}^{s}c_k e^{A(T_m,T_k)+B(T_m,T_k)r^*} = 1.$$

Substituting this expression gives

$$\begin{aligned} V_{swaption,rec}(t,T_m,T_s) &= NP(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\sum_{k=m+1}^{s}c_k\max\left(P(T_m,T_k)-P_k^*,0\right)\middle|\mathcal{F}(t)\right] \\ &= NP(t,T_m)\sum_{k=m+1}^{s}c_k\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\max\left(P(T_m,T_k)-P_k^*,0\right)\middle|\mathcal{F}(t)\right]. \end{aligned}$$

Notice that this is no more than a scaled sum of call options on a zero-coupon bond. Indeed, the value of this derivative at time t with strike P_k^*, expiry T_m and bond maturity T_k reads

$$\begin{aligned} V_{ZCB,call}(t,T_m;T_k,P_k^*) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{M(t)}{M(T_m)}\max\left(P(T_m,T_k)-P_k^*,0\right)\middle|\mathcal{F}(t)\right] \\ &= P(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\max\left(P(T_m,T_k)-P_k^*,0\right)\middle|\mathcal{F}(t)\right] \quad \text{(Change of measure } \mathbb{Q} \text{ to } \mathbb{Q}^{T_m}). \end{aligned}$$

Finally, we obtain

$$V_{swaption,rec}(t,T_m,T_s) = N\sum_{k=m+1}^{s}c_k V_{ZCB,call}(t,T_m;T_k,P_k^*)$$

as required.

From this expression it is a logical next step to find a closed-form formula for a call option on a zero-coupon bond under the Hull-White model. This is given by the following theorem:

Theorem 5. *The value of a call option on a zero-coupon bond at time t with strike K, expiry T_m and bond maturity T_n is given by*

$$V_{ZCB,call}(t,T_m;T_n,K) = P(t,T_n)\Phi(h) - KP(t,T_m)\Phi(h-\sigma_p),$$

with $\Phi(\cdot)$ the cumulative distribution function of a standard normal distribution and

$$h = \frac{\log\left(\frac{P(t,T_n)}{K\cdot P(t,T_m)}\right) + \frac{1}{2}\sigma_p^2}{\sigma_p}$$

$$\sigma_p = -\sigma\cdot B(T_m,T_n)\sqrt{\frac{1-e^{-2\alpha(T_m-t)}}{2\alpha}}.$$

Proof. Under the forward measure Q^{T_m}, the value of the call option on a zero-coupon bond can be expressed as

$$V_{ZCB,call}(t,T_m;T_n,K) = P(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[\max\left(P(T_m,T_n)-K,0\right)\middle|\mathcal{F}(t)\right].$$

To compute this expression, the dynamics of the short rate r under the T_m-forward measure Q^{T_m} is required. Remember that under the risk-neutral measure $\mathbb{Q}$, we have

$$\mathrm{d}r(t) = (\theta(t)-\alpha r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}}(t).$$

Then by Girsanov's theorem (Theorem 3), we obtain

$$\mathrm{d}W^{\mathbb{Q}^{T_m}}(t) = \mathrm{d}W^{\mathbb{Q}}(t) - \frac{\mathrm{d}\Lambda_{Q^{T_m},\mathbb{Q}}(t)}{\Lambda_{Q^{T_m},\mathbb{Q}}(t)}\,\mathrm{d}W^{\mathbb{Q}}(t),$$

with

$$\Lambda_{Q^{T_m},\mathbb{Q}}(t) = \frac{P(t,T_m)M(t_0)}{P(t_0,T_m)M(t)} \quad \text{and}$$

$$\begin{aligned}
\mathrm{d}\Lambda_{Q^{T_m},\mathbb{Q}}(t) &= \frac{M(t_0)}{P(t_0,T_m)}\left(\frac{1}{M(t)}\,\mathrm{d}P(t,T_m) - \frac{P(t,T_m)}{M(t)^2}\,\mathrm{d}M(t)\right) && \text{(Itô's lemma)}\\
&= \frac{M(t_0)}{P(t_0,T_m)}\left(\frac{1}{M(t)}\left(r(t)P(t,T_m)\,\mathrm{d}t + \sigma B(t,T_m)P(t,T_m)\,\mathrm{d}W^{\mathbb{Q}}(t)\right) - \frac{P(t,T_m)}{M(t)^2}r(t)M(t)\,\mathrm{d}t\right)\\
&= \frac{M(t_0)}{P(t_0,T_m)}\frac{\sigma B(t,T_m)P(t,T_m)\,\mathrm{d}W^{\mathbb{Q}}(t)}{M(t)},
\end{aligned}$$

where we used the dynamics of the zero-coupon bond and the money market account as given in B.2.1 and A.1.1, respectively. Substituting leads to

$$\begin{aligned}
\mathrm{d}W^{\mathbb{Q}^{T_m}}(t) &= \mathrm{d}W^{\mathbb{Q}}(t) - \sigma B(t,T_m)\,\mathrm{d}W^{\mathbb{Q}}(t)\,\mathrm{d}W^{\mathbb{Q}}(t)\\
&= \mathrm{d}W^{\mathbb{Q}}(t) - \sigma B(t,T_m)\,\mathrm{d}t && \text{(Itô's multiplication)}
\end{aligned}$$

Then finally,

$$\mathrm{d}r(t) = (\theta(t) + \sigma^2 B(t,T_m) - \alpha r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}^{T_m}}(t).$$

Consequently, we can write

$$r(t) \sim \mathcal{N}(\mu_{r,T_m}(t), \sigma^2_{r,T_m}(t))$$

with

$$\begin{aligned}
\mu_{r,T_m}(t) &= e^{-\alpha(t-t_0)}r(t_0) + \int_{t_0}^{t} e^{-\alpha(t-s)}(\theta(s) + \sigma^2 B(s,T_m))\,\mathrm{d}s\\
\sigma^2_{r,T_m}(t) &= \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(t-t_0)}\right).
\end{aligned}$$

As a consequence, applying the general Black's formula (Black, 1976), the call option on a zero-coupon bond becomes

$$V_{ZCB,call}(t, T_m; T_n, K) = P(t, T_n)\Phi(h) - KP(t, T_m)\Phi(h - \sigma_p),$$

with $\Phi(\cdot)$ the cumulative distribution function of a standard normal distribution and

$$h = \frac{\log\left(\frac{P(t,T_n)}{K \cdot P(t,T_m)}\right) + \frac{1}{2}\sigma_p^2}{\sigma_p}$$

$$\sigma_p = -\sigma \cdot B(T_m, T_n)\sqrt{\frac{1 - e^{-2\alpha(T_m - t)}}{2\alpha}}.$$

□

C

Derivation NPV-at-Risk after incorporating swaptions

In Section 5.4.1 we have calibrated receiver swaptions to offset the value exposure in a parallel -200 basis points shock in the interest rates. The reason for this specific scenario is twofold. On the one hand, the European Banking Authority (EBA) suggests using a parallel shock with a magnitude of 200 basis points for outlier testing. Moreover, we observed in Table 5.6 that the downward shock resulted in larger losses than the upward shock. To derive the new NPVaR in the 200 basis points parallel shock, we should again consider the exposure in an upward and downward scenario, as given in the first and second column of Table C.1, respectively. This table shows that the NPV-at-Risk in the upward scenario does not change when the calibrated swaption is included in the hedge portfolio. This is due to the fact that a receiver swaption is only exercised if the prevailing swap rate at the option maturity is below the strike rate. In an interest rate environment with a 200 basis points upward shock, it is highly unlikely this would occur. On the other hand, the NPV-at-Risk for the downward scenario improved significantly since the exposure for all hedging approaches is reduced to approximately zero. Consequently, the total NPV-at-Risk, which is defined as the most negative of the two, is improved as well, as visible in the final column of Table C.1.

		Value stability		
	Hedging approach	NPV-at-Risk +200bps	NPV-at-Risk -200bps	NPV-at-Risk
Static	Internal funding	3.196×10^{4}	1.235×10^{-6}	1.235×10^{-6}
	Notional hedge with zero-coupon bonds	-315.4	-1.943×10^{-6}	-315.4
	Value hedge with zero-coupon bonds	-2083	1.929×10^{-6}	-2083
Dynamic	Notional hedge with internal contracts	3.302×10^{4}	-1.679×10^{-5}	-1.679×10^{-5}
	Value hedge with internal contracts	-9396	6.681×10^{-6}	-9396

Table C.1: An overview of the NPV-at-Risks for all proposed hedging approaches including the swaption. The colours green and grey represent a positive and no change in comparison to the hedging approaches excluding the swaption.

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