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Strong solidity of q-Gaussian algebras

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Abstract

In 2011 Avsec showed strong solidity of the q-Gaussian algebras, building upon previous results of Houdayer and Shlyakhtenko, and Ozawa and Popa. In this work we study this result as well as the necessary literature and q-mathematics needed to replicate the proof. The literature is combined within this thesis, whilst additionally filling in gaps in Avsec's proofs. Overall, the thesis aims to present an improved and more accessible proof of the strong solidity of the q-Gaussian algebras.

Contents

1

Introduction

Introduced in the 20th century, von Neumann algebras are a special type of C^*-algebra with a widespread presence in mathematics. These algebras play a significant role in various fields of mathematics, such as functional analysis, quantum mechanics, probability, ergodic theory, and group theory.

A Von Neumann algebra is a strongly closed $*$-subalgebra of bounded operators on a Hilbert space that contains the identity operator. In this thesis we will be studying a very specific type of von Neumann algebra, namely the q-Gaussian algebra. These von Neumann algebras form an active area of research, see [2, 6, 7, 14] for instance.

The q-Gaussian algebras stem from creation and annihilation operators a^* and a on a Fock space that satisfy the q-relations, which are as follows:

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle 1,$$

where $-1 \leq q \leq 1$. An the appropriate Fock space is needed for the creation and annihilation operators to be each others adjoints, as demonstrated by Bożejko and Speicher in [3]. We reinvestigate these results and verify the construction, and make improvements where possible.

In the aforementioned paper Bożejko and Speicher laid the foundation for the q-Gaussian algebras. Subsequently, Bożejko, Kümmerer and Speicher. further investigated the subject in [4]. Given a real Hilbert space $H_{\mathbb{R}}$ with complexification H, $-1 < q < 1$, and the associated q-Fock space $\mathcal{F}_q(H)$, we define the q-Gaussians $\omega(f)$ for $f \in H_{\mathbb{R}}$ by

$$\omega_f := a^*(f) + a(f).$$

The von Neumann algebra generated by the q-Gaussians on $\mathcal{F}_q(H)$ is what we call the q-Gaussian algebra, denoted by $\Gamma_q(H_{\mathbb{R}})$. Taking inspiration from [4], we perform this construction and introduce the Wick words. Given a word (tensor), the Wick operator assigns an element of $\Gamma_q(H_{\mathbb{R}})$, namely the Wick word. The Wick operator will prove to be a powerful tool in working with the q-Gaussian algebra.

A von Neumann algebra whose center consists only of multiples of the identity operator is called a factor, which in forms an interesting class of von Neumann algebras. In Chapter 4 we show that this definition applies to the q-Gaussian algebra whenever $\dim H_{\mathbb{R}}$ is at least 2, utilising proofs from [16]. In the subsequent chapter we introduce the q-Gaussian functor, which assigns to a contraction on $H_{\mathbb{R}}$ a linear map on $\Gamma_q(H_{\mathbb{R}})$. We also present the notion of bimodules for von Neumann algebras, and in particular the coarse bimodule $L^2(\Gamma_q(H_{\mathbb{R}})) \otimes L^2(\Gamma_q(H_{\mathbb{R}}))$.

The notion of strong solidity for a von Neumann algebra was established by Ozawa and Popa in [15] for the free group factors. The application of strong solidity arises from the use of Cartan subalgebras, which are maximal abelian subalgebras whose normalizer generates the entire von Neumann algebra. The existence, of a Cartan subalgebra implies that the von Neumann algebra allows for a generalized crossed product decomposition [12]. On the other hand, if no Cartan subalgebra exists, there is no generalized crossed product decomposition of the von Neumann algebra. Provided the von Neumann algebra is not amenable, strong solidity offers a stronger property than the lack of a Cartan subalgebra. That is, it allows us to place the von Neumann algebra in the second category.

Not long after Ozawa and Popa's paper, Avsec proved that the q-Gaussian algebras satisfy the conditions for strong solidity in [1]. The final chapter is dedicated to studying Avsec's proof of the aforementioned. As such, the proofs are largely based on those of Avsec. We aim to present a more accessible version of said proof, and in doing so close some gaps and make further improvements.

2

Preliminaries

Familiarity with functional analysis, and particularly $*$-algebras is assumed. We refer to [8] and [13] for the basics not covered in the preliminaries. We shall state a number of important definitions and results, and establish notation.

Definition 2.1. *For $q \in \mathbb{R}$ the q-bracket for $n \in \mathbb{N}$ is given by*

$$[n]_q = \sum_{i=0}^{n-1} q^i,$$

and the q-factorial is defined by as

$$[n]_q! = \prod_{i=1}^{n} [i]_q.$$

Note that if $q \neq 1$ we have $[n]_q = \frac{1-q^n}{1-q}$, and for $q = 1$ we have $[n]_q = n$ and $[n]_q! = n!$.

Definition 2.2. *Let G be a group and $\phi : G \to \mathbb{C}$. We call ϕ is positive definite on G if the matrix $(\phi(\pi^{-1}\sigma))_{\pi,\sigma\in F}$ is positive definite, for any finite $F \subseteq G$.*

Remark 2.3. *Considering the definition of positive definite matrices, we see that the above definition is equivalent with requiring that*

$$\sum_{\pi,\sigma\in F} \phi(\pi^{-1}\sigma)r(\sigma)\overline{r(\pi)} > 0$$

for arbitrary non-zero $r : F \to \mathbb{C}$.

Proposition 2.4. *The product of two positive definite functions is positive definite.*

Proof. Let $\phi, \psi : X \to \mathbb{C}$ be positive definite functions, and let $F \subseteq X$ be finite. Then $(\phi(\pi^{-1}\sigma))_{\pi,\sigma\in F}$ and $(\psi(\pi^{-1}\sigma))_{\pi,\sigma\in F}$ are positive definite matrices. Ergo, $(\phi(\pi^{-1}\sigma))_{\pi,\sigma\in F} \otimes (\psi(\pi^{-1}\sigma))_{\pi,\sigma\in F}$ is positive definite. As ϕ and ψ are scalar-valued, we can conclude that the diagonal $((\phi\psi)(\pi^{-1}\sigma))_{\pi,\sigma\in F}$ is a positive definite matrix. □

Below are some classical theorems that we shall make use of later on.

Theorem 2.5 (Goldstine's Theorem)**.** *Let X be a Banach space. Then $B_1(X)$ lies weak* dense in $B_1(X^{**})$.*

We recall the definition of the Schatten class. Let $p \geq 1$ and H a Hilbert space. The Schatten class $S_p(H)$ is defined as

$$S_p(H) := \{T \in B(H) \mid \mathrm{tr}(|T|^p) < \infty\},$$

equipped with the norm $\|T\|_p = \mathrm{tr}(|T|^p)^{1/p}$. The Powers-Størmers inequality provides us with he following norm estimate for the difference of two operators in $S_2(H)$:

Theorem 2.6 ([5])**.** *Let H be a Hilbert space. Then for trace class operators $T, S \in S_2(H)$ we have*

$$\|T - S\|_{S_2}^2 \;\leq\; \|S^2 - T^2\|_{S_1}.$$

The Kaplanksy Density Theorem is stated as:

Theorem 2.7 ([17])**.** *Let $A \subset B(H)$ be a $*$-algebra represented on a Hilbert space H. Then the unit ball of A lies strongly dense in the unit ball of the weak closure of A.*

We recall the definition of a von Neumann algebra:

Definition 2.8. *A von Neumann algebra is a strongly closed $*$-subalgebra of bounded operators on a Hilbert space that contains the identity operator.*

The commutant of a subset S of an algebra A is defined as the set of all elements in S that commute with A, i.e. $S' := \{a \in A \mid as = sa \text{ for all } s \in S\}$. The double commutant theorem provides us an easy property to verify if a $*$-algebra is in fact a von Neumann algebra:

Theorem 2.9 (Double commutant theorem)**.** *Let H be a Hilbert space, and A be a $*$-algebra of operators acting on H such that $1_H \in A$. Then A is a von Neumann algebra if and only if $A'' = A$ holds.*

This gives rise to the following equality:

Theorem 2.10. *Let A be a subset of $B(H)$ and denote the von Neumann algebra generated by A on H by $vNA(A)$. We have:*

$$\mathrm{vNA}(A) = \mathrm{A}''.$$

For a discrete group we define the von Neumann group algebra $L(G)$ as the von Neumann algebra generated by the image of the left regular representation of G on $B(\ell^2(G))$.

A trace on a von Neumann algebra M is a function $\tau : M_+ \to [0, \infty]$ such that $\tau(x + y) = \tau(x) + \tau(y)$, $\tau(\lambda x) = \lambda\tau(x)$ and $\tau(x^*x) = \tau(xx^*)$ for any $x \in M_+$ and $\lambda \geq 0$. A trace can have multiple interesting properties. We call τ

- faithful if $\tau(x) > 0$ for any $x > 0$ in M.
- finite if $\tau(1) < \infty$.
- normal if $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ for any bounded increasing net (x_i) in M_+.

A tracial state is then a state that satisfies the property of a trace as well.

An interesting class of operators on von Neumann algebras are the conditional expectations. We define these as follows:

Definition 2.11 ([5])**.** *Let A, B with $B \subset A$ be C^*-algebras. A conditional expectation from A onto B is contractive completely positive map $E : A \to B$ such that $E\mid_B = id_B$, and $E(b_1xb_2) = b_1E(x)b_2$ for every $x \in A$ and $b_1, b_2 \in B$.*

We remark that E is a projection. In the case of von Neumann algebras, such a conditional expectation is provided by the following proposition:

Proposition 2.12 ([5])**.** *Let M be a von Neumann algebra with a faithful normal tracial state τ, and $N \subset M$ a von Neumann subalgebra. If $1_M \in N$ then there exists a unique, trace-preserving, normal conditional expectation E_N from M onto N.*

Let M be a von Neumann algebra and assume it has a faithful normal tracial state. In this scenario we can define the spaces $L^1(M)$ and $L^2(M)$. Let $M_\tau := \{xy \mid x, y \in M \text{ and } \tau(x^*x), \tau(y^*y) < \infty\}$. We can extend τ to a linear functional on M_τ such that it retains its properties, see [17] for the details. By the required properties imposed on the trace we can create a norm on M_τ through $\|x\|_1 := \tau(|x|)$. The completion of M_τ with respect to $\|\cdot\|_1$ gives us the definition of $L^1(M)$.

Likewise, we can perform a similar construction on $\{x \in M \mid \tau(x^*x) < \infty\}$ in the case that M is semi-finite, and construct the inner product $\langle x, y\rangle := \tau(y^*x)$. The completion of $\{x \in M \mid \tau(x^*x) < \infty\}$ then yields the definition for $L^2(M)$.

2.1. Factors

Whenever we talk of a projection in a von Neumann algebra, we assume it is orthogonal, unless explicitly stated otherwise. After we classify projections, we will introduce the concept of factors for von Neumann algebras.

Definition 2.13. *Let M be a von Neumann algebra acting on H, and $p, q \in M$ be projections. Define the following:*

- $p \sim q$ *if there exists* $u \in M$ *such that* $u^*u = p$ *and* $uu^* = q$.
- $p \preceq q$ *if there exists exists a projection* $p' \in M$ *such that* $p' \leq q$ *and* $p \sim p'$. *Strictness corresponds to case that* $p \not\sim q$.
- p *is minimal if there exists no non-zero projection* $q \in M$ *such that* $q < p$. *We also call* p *an atom.*
- *If for all projections* $q \in M$ *such that* $p \sim q$ *we have that* $q \leq p$ *implies* $p = q$, *we say* p *is finite. Otherwise,* p *is called infinite.*

In turn, these classifications will let us name some properties of von Neuman algebras. Let M be a von Neumann algebra acting on H.

- M is called injective if there exists a projection $p : B(H) \to M$ such that $\|p\| = 1$.
- M is said to be (in)finite if the identity is (in)finite.
- M is diffuse if it has no non-zero minimal projections.
- M is called atomic if for every non-zero projection $p \in M$ there exists a non-zero projection $q \in M$ such that $q \leq p$ and q is minimal.

If a von Neumann algebra's center is trivial, i.e. it contains only multiples of the identity operator, then we call it a *factor.* A factor can have different types, namely type I, type II and type III.

Definition 2.14. *Let M be a factor. Then M is of type*

- *I if there is a minimal projection.*
- *II if it does not contain a minimal projection, but does contain non-zero finite projections. We further classify it as type*
  - II_1 *if* M *is finite.*
  - II_∞ *otherwise.*
- *III if it contains no non-zero finite projections.*

Takesaki [17] lists a number of useful results for factors, and we refer to this book for further study on factors. The following two results will also be applied later on:

Proposition 2.15 ([17])**.** *An Abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint element.*

Theorem 2.16 ([17])**.** *Let A be a diffuse abelian von Neumann algebra on a separable Hilbert space. If A is diffuse then it is isomorphic with* $L^\infty(0, 1)$.

The proof of Theorem [17], which utilizes the preceding proposition, can be adjusted to work in the case that A is atomic instead. This in turn yields an isomorphism with $\ell^\infty(X)$ instead, where X is of the same cardinality as the set of atoms.

Central in this thesis will be the tensor product, which we use to construct the Fock space. It is defined as follows:

Definition 2.17. *Let H be a (complex) Hilbert space. For $n \geq 1$ define the n-fold tensor product*

$$H^{\otimes n} := H \underbrace{\otimes \cdots \otimes}_{n} H,$$

We take Ω to be an abstract vector that we designate as the unit of $\mathbb{C}$, such that for $n = 0$ we can set $H^{\otimes 0} = \mathbb{C}\Omega$. Now define the full Fock space $\mathcal{F}(H)$ as

$$\mathcal{F}(H) := \bigotimes_{n=0}^{\infty} H^{\otimes n}.$$

Let us equip $\mathcal{F}(H)$ with the inner product $\langle \cdot, \cdot \rangle_0$ given by the linear extension of

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_0 := \delta_{nm} \langle f_1, g_1 \rangle \cdots \langle f_n, g_n \rangle.$$

As we consider this the standard inner product on a Fock space, we may omit the subscript in subsequent use. Later on it will become apparent how the Fock space will be linked to von Neumann algebras.

There are numerous tensor product to consider. We start with the *spatial tensor product*. Let A and B be C^*-algebras and (H, ϕ) and (K, ψ) be universal representations of A and B respectively, where the universal representation of A is defined as the direct sum of all GNS representations associated to states on the respective space. Then by Theorem 6.3.3 from [13] there exists a unique injective $*$-homomorphism $\pi : A \otimes B \to B(H \hat{\otimes} K)$ such that $\pi(a \otimes b) = \phi(a) \hat{\otimes} \psi(b)$. We find the following C^*-norm on $A \otimes B$:

$$\|c\|_{min} := \|\pi(c)\|,$$

where we recall that a C^*-norm is a norm such that $\|c^*\| = \|c\|$ and $\|c^*c\| = \|c\|^2$. We call the above norm the spatial norm. $A \otimes_* B$ is defined as the completion of $H \otimes K$ with respect to the spatial norm. By Theorem 6.4.18 of [13] this equals the min-norm [13], hence the notation.

More generally, if γ is a C^*-norm, then with $A \otimes_\gamma B$ we denote the completion of $A \otimes B$ with respect to γ. One more norm to consider is

$$\|c\|_{max} := \sup_{\gamma \text{ is a } C^*\text{-norm on } A \otimes B} \gamma(c),$$

which defines a C^*-norm [13].

Yet another tensor product to consider is the binormal tensor product, or $A \otimes_{bin} B$, which is defined as

$$A \otimes_{bin} B := \{ f \in S(A \otimes B) \mid (a, b) \mapsto f(a \otimes b) \text{ is separately weak}* \text{ continuous} \},$$

where $S(A \otimes B)$ denotes the set of states on $A \otimes B$. We recall that a state is a positive linear functional with norm 1.

If in particular we are working with von Neumann algebras, there is a tensor product to consider such that it is a von Neumann algebra as well. Let M and N be von Neumann algebras on H_1 and H_2 respectively. The tensor product of M and N, denoted with $M \bar{\otimes} N$, is the von Neumann algebra on $H_1 \times H_2$ generated by $x \otimes y$ for $x \in M$, $y \in N$. However, Chapter 6 in [13] provides us with the fact that the representations ϕ and ψ used in the definition can be replaced with the GNS representation. Utilizing this fact we simply see $M \bar{\otimes} N$ as the closure of $M \otimes N$ represented on $H_1 \otimes H_2$.

Proposition 2.18. *Let M and N be von Neumann algebras. Then the weak* closure of $M \otimes_{min} N$ equals $M \bar{\otimes} N$.*

Proof. First let $x \in M \bar{\otimes} N$ and assume $\|x\| = 1$. Then by the double commutant theorem x lies in $\overline{M \otimes N}^{SOT}$, which by Kaplanksy's density theorem lies in $\overline{B_1(M \otimes N)}^{SOT}$. As the strong operator topology is finer than the weak operator topology, we find that $x \in \overline{B_1(M \otimes N)}^{WOT}$. Lastly, by Theorem 4.2.4 of [13] $\overline{B_1(M \otimes N)}^{WOT}$ coincides with the weak* closure of $B_1(M \otimes N)$ which in turn lies in weak* closure of $M \otimes_{min} N$.

Conversely, as the weak* topology is finer than the weak operator topology, the weak* closure of $M \otimes_{min} N$ is contained in $\overline{M \otimes_{min} N}^{WOT}$. Applying Theorem 4.2.5 from [13] the finishes the proof. □

3

The q-Gaussian Algebra

In this chapter we introduce the q-Gaussians, the Wick words, and most importantly the q-Gaussian algebra $\Gamma_q(H_\mathbb{R})$. In general q can range from -1 to 1, and although $\Gamma_{-1}(H_\mathbb{R})$ and $\Gamma_1(H_\mathbb{R})$ are in themselves interesting spaces, in this thesis we will concern ourselves with the case $-1 < q < 1$.

3.1. q-Fock Spaces

In this section we establish the q-Fock space. As the name implies, it is similar to the usual (full) Fock space, but we use an inner product dependent on q instead to create a Hilbert space.

Let $H_\mathbb{R}$ be a real Hilbert space, and define $H := H_\mathbb{R} \oplus iH_\mathbb{R}$ to be its complexification.

As we did for the Fock space, we designate an abstract vector Ω as the unit of $\mathbb{C}$. Henceforth, we shall refer to Ω as the *vacuum vector*. Let $H^{\otimes n}$ be as in the preliminaries. We define

$$\mathcal{F}^{finite}(H) := \operatorname{Span}\left\{f_1 \otimes \cdots \otimes f_n \in H^{\otimes n} \mid n \in \mathbb{N}_0\right\}.$$

Definition 3.1. *Let $n \in \mathbb{N}$ and denote the symmetric group of n elements with S_n. For a permutation $\pi \in S_n$, define the number of inversions $i(\pi)$ as*

$$i(\pi) := \#\{(i,j) \mid 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

Observe the following: for a permutation $\pi \in S_n$ we have that $i(\pi) = i(\pi^{-1})$. Indeed, any pair (i,j) such that $\pi(i) > \pi(j)$ yields $i' = \pi(j)$ and $j' = \pi(i)$ such that $i' < j'$ and $\pi^{-1}(i') > \pi^{-1}(j')$.

We will now make our first steps towards defining the q-inner product $\langle \cdot, \cdot \rangle_q$.

Definition 3.2. *Let $f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \in \mathcal{F}^{finite}$. For $q \in (-1,1)$, set*

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_q := \delta_{nm} \sum_{\pi \in S_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle.$$

Define $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}^{finite}$ as its sesquilinear extension.

Note that it agrees with the inner product in Definition 2.17 for $q = 0$.

Let $\|\cdot\|_q$ denote the norm induced on $\mathcal{F}^{finite}(H)$ by the q-inner product. For example, $\|f_1 \otimes \cdots \otimes f_n\|_q = \|f_1 \otimes \cdots \otimes f_n\|_0$ if $f_1, \ldots, f_n$ are pairwise orthogonal in H.

Theorem 3.3. *$\langle \cdot, \cdot \rangle_q$ is positive definite on $\mathcal{F}^{finite}$.*

The proof is structured as follows: First we define an operator P_q such that we can write $\langle \cdot, \cdot \rangle_q = \langle \cdot, P_q \cdot \rangle_0$. Then, by showing that $\pi \mapsto q^{i(\pi)}$ is a strictly positive definite function in the sense Definition 2.2, we can conclude that P_q is positive definite. From this we can conclude the theorem.

We start by defining P_q on $\mathcal{F}^{finite}$ through the linear extension of $P_q\Omega := \Omega$ and

$$P_q f_1 \otimes \cdots \otimes f_n := \sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}. \tag{3.1}$$

A simple substitution then shows that

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes n\rangle_q = \langle f_1 \otimes \cdots \otimes f_n, P_q g_1 \otimes \cdots \otimes g_n\rangle_0$$

as desired. By our definition of $\langle \cdot, \cdot \rangle_q$ we therefore have that $\langle \xi, \eta\rangle_q = \langle \xi, P_q \eta\rangle_0$ for any $\xi, \eta \in \mathcal{F}^{finite}(H)$.

In order to simplify the notation and help with the proofs, we use the unitary representation $\pi \mapsto U_\pi$ of S_n on $H^{\otimes n}$ defined through

$$U_\pi f_1 \otimes \cdots \otimes f_n := f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}.$$

That is, we can write

$$P_q f_1 \otimes \cdots \otimes f_n = \sum_{\pi \in S_n} q^{i(\pi)} U_\pi f_1 \otimes \cdots \otimes f_n. \tag{3.2}$$

We now arrive the main proof of this section:

Proof of theorem 3.3. The following proof it taken from [3]. We first show that the function

$$\phi_q : S_n \to \mathbb{C}, \phi_q : \pi \mapsto q^{i(\pi)}$$

is a positive definite function, and conclude from there.

For arbitrary non-zero $r : S_n \to \mathbb{C}$ we need to show that

$$\sum_{\pi,\sigma \in S_n} q^{i(\pi^{-1}\sigma)} r(\sigma)\overline{r(\pi)} > 0.$$

Define

$$\begin{aligned} \Phi &:= \{(i,j) \mid i \neq j, 1 \leq i,j \leq n\} \\ \Phi^+ &:= \{(i,j) \mid 1 \leq i < j \leq n\}. \end{aligned}$$

For any $\pi \in S_n$ and $A \subseteq \Phi$ we set

$$\pi(A) := \{(\pi(i), \pi(j)) \mid (i,j) \in A\}.$$

Note that $\pi(A) \subseteq \Phi$ and $|\pi(A)| = |A|$. From the definition of $i(\pi)$ as the number of inversions, we have that $i(\pi) = |\pi(\Phi^+)\backslash\Phi^+|$ for any $\pi \in S_n$. Combining the above facts, and using that $i(\pi) = i(\pi^{-1})$ we find that

$$\begin{aligned} 2i(\pi) &= i(\pi) + i(\pi^{-1}) \\ &= |\pi(\Phi^+)\backslash\Phi^+| + |\pi^{-1}(\Phi^+)\backslash\Phi^+| \\ &= |\pi(\Phi^+)\backslash\Phi^+| + |\Phi^+\backslash\pi(\Phi^+)| \\ &= |\pi(\Phi^+)\triangle\Phi^+|. \end{aligned}$$

Let $\sigma \in S_n$. Substituting $\pi^{-1}\sigma$ in the above and applying π to $\pi(\Phi^+)\triangle\Phi^+$ we deduce that $2i(\pi^{-1}\sigma) = |\sigma(\Phi^+)\triangle\pi(\Phi^+)|$.

Observe that for $q = 0$ we have $\phi_q(\pi)$ is 1 for the identity and otherwise 0, which is positive definite. For $q \neq 0$, we split the problem into the cases $q < 0$ and $q > 0$.

For the first case, let $0 < q < 1$. Take $\lambda \geq 0$ such that $q = e^{-\lambda}$, and let $\mathbf{1}_A$ be the indicator function for a set A. We find:

$$\begin{aligned} q^{i(\pi^{-1}\sigma)} &= \exp\left(-\frac{\lambda}{2}|\sigma(\Phi^+)\triangle\pi(\Phi^+)|\right) \\ &= \exp\left(-\frac{\lambda}{2}\sum_{x\in\Phi}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\right) \\ &= \prod_{x\in\Phi}\exp\left(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\right). \end{aligned}$$

By Proposition 2.4 it suffices to prove positive definiteness for a single term, i.e. for the function $x \mapsto \exp\left(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\right)$. Applying the aforementioned proposition then yields positive definiteness for the entire product as a function on S_n.

Let $x \in \Phi$, and $r : S_n \to \mathbb{C}$ be an arbitrary function. Moreover, define $s : \{0,1\} \to \mathbb{C}$ by

$$s(0) := \sum_{\substack{\sigma\in S_n \\ x\notin\sigma(\Phi^+)}} r(\sigma), \quad s(1) := \sum_{\substack{\sigma\in S_n \\ x\in\sigma(\Phi^+)}} r(\sigma)$$

We have:

$$\begin{aligned} &\sum_{\pi,\sigma\in S_n}\exp\left(-\frac{\lambda}{2}|\mathbf{1}_{\sigma(\Phi^+)}(x) - \mathbf{1}_{\pi(\Phi^+)}(x)|^2\right) r(\sigma)\overline{r(\pi)} \\ = &\sum_{\substack{\pi,\sigma\in S_n \\ x\notin\sigma(\Phi^+)\triangle\pi(\Phi^+)}} r(\sigma)\overline{r(\pi)} \;+\; e^{-\frac{\lambda}{2}} \sum_{\substack{\pi,\sigma\in S_n \\ x\in\sigma(\Phi^+)\triangle\pi(\Phi^+)}} r(\sigma)\overline{r(\pi)} \\ = &\sum_{\substack{\sigma\in S_n \\ x\notin\sigma(\Phi^+)}} r(\sigma) \sum_{\substack{\pi\in S_n \\ x\notin\pi(\Phi^+)}} \overline{r(\pi)} + \sum_{\substack{\sigma\in S_n \\ x\in\sigma(\Phi^+)}} r(\sigma) \sum_{\substack{\pi\in S_n \\ x\in\pi(\Phi^+)}} \overline{r(\pi)} \\ &+e^{-\frac{\lambda}{2}}\left[\sum_{\substack{\sigma\in S_n \\ x\in\sigma(\Phi^+)}} r(\sigma) \sum_{\substack{\pi\in S_n \\ x\notin\pi(\Phi^+)}} \overline{r(\pi)} + \sum_{\substack{\sigma\in S_n \\ x\in\sigma(\Phi^+)}} r(\sigma) \sum_{\substack{\pi\in S_n \\ x\notin\pi(\Phi^+)}} \overline{r(\pi)}\right] \\ = &\; s(0)\overline{s(0)} + s(1)\overline{s(1)} + e^{-\frac{\lambda}{2}}\left[s(0)\overline{s(1)} + s(1)\overline{s(0)}\right] \\ = &\sum_{i,j=0}^{1} e^{-\frac{\lambda}{2}|i-j|^2} s(i)\overline{s(j)}. \end{aligned}$$

As $x \mapsto e^{-\frac{\lambda}{2}|i-j|^2}$ is a positive definite function, we deduce that

$$\sum_{i,j=0}^{1} e^{-\frac{\lambda}{2}|i-j|^2} s(i)\overline{s(j)} \geq 0$$

and conclude that ϕ_q is positive definite for $0 < q < 1$.

The case of $-1 < q < 0$ remains. By definition of ϕ_q we obviously have that $\phi_q(\pi) = q^{i(\pi)} = (-1)^{i(\pi)}\phi_{-q}(\pi)$. We claim that $\pi \mapsto (-1)^{i(\pi)}$ is positive definite. Provided this is the case, we can call upon Proposition 2.4 again to conclude $\phi_q(\pi)$ is positive definite.

The first step is to show that $\pi, \sigma \in S_n$, $i(\pi\sigma)$ and $i(\pi) + i(\sigma)$ have the same parity. Indeed, if we have $(i,j) \in \Phi^+$ such that $(\pi\sigma)(i) > (\pi\sigma)(j)$, then either (i,j) yields an inversion for σ, or $(\sigma(i),\sigma(j))$ yields an inversion for π. If (i,j) does not yield an inversion, then it either yields one for both σ and π, or none for either. Ergo, we have that $i(\pi\sigma)$ and $i(\pi) + i(\sigma)$ have the same parity, and therefore

$(-1)^{i(\pi^{-1}\sigma)} = (-1)^{i(\pi^{-1})}(-1)^{i(\sigma)} = (-1)^{i(\pi)}(-1)^{i(\sigma)}$. That is, the sign is a representation of S_n.

Now let $r : S_n \to \mathbb{C}$ be an arbitrary function. Using the result above we conclude the claim as follows:

$$\begin{aligned}
\sum_{\pi,\sigma\in S_n} (-1)^{i(\pi^{-1}\sigma)} r(\sigma)\overline{r(\pi)} &= \sum_{\pi,\sigma\in S_n} (-1)^{i(\sigma)} r(\sigma)(-1)^{i(\pi)}\overline{r(\pi)} \\
&= \sum_{\sigma\in S_n} (-1)^{i(\sigma)} r(\sigma) \overline{\sum_{\sigma\in S_n} (-1)^{i(\sigma)} r(\sigma)} \\
&\geq 0.
\end{aligned}$$

We are now able to finish the proof. We need only show that $\langle \eta, P_q\eta\rangle_0 > 0$ for non-zero $\eta \in H^{\otimes n}$ for $n \in \mathbb{N}_0$. Let $\eta \in H^{\otimes n}$ be non-zero and $\{\xi_i\}_{i\in I}$ be a complete orthonormal system for $H^{\otimes n}$, i.e. $\eta = \sum_{i\in I} \langle \eta, \xi_i\rangle_0 \xi_i$. Using (3.2) and that ϕ_q is positive-definite, we have:

$$\begin{aligned}
\langle \eta, P_q\eta\rangle_0 &= \sum_{\pi\in S_n} q^{i(\pi)} \langle \eta, U_\pi \eta\rangle_0 \\
&= \frac{1}{n!} \sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)} \langle \eta, U_{\pi^{-1}\sigma}\eta\rangle_0 \\
&= \frac{1}{n!} \sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)} \langle U_\pi\eta, U_\sigma\eta\rangle_0 \\
&= \frac{1}{n!} \sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)} \sum_{i\in I} \langle U_\pi\eta, \xi_i\rangle_0 \langle \xi_i, U_\sigma\eta\rangle_0 \\
&= \frac{1}{n!} \sum_{i\in I} \left[\sum_{\pi,\sigma\in S_n} q^{i(\pi^{-1}\sigma)} \overline{\langle \xi_i, U_\pi\eta\rangle_0} \langle \xi_i, U_\sigma\eta\rangle_0 \right] \\
&> 0.
\end{aligned}$$

Now that we have proven $\langle\cdot,\cdot\rangle_q$ to be positive definite, and therefore an inner product, we can properly define the q-Fock space. □

Definition 3.4. *We define the q-Fock space $\mathcal{F}_q(H)$ as the completion of $\mathcal{F}^{finite}(H)$ with respect to the inner product $\langle\cdot,\cdot\rangle_q$.*

3.2. The creation and annihilation operators

Just as they exist on regular Fock spaces, we define creation and annihilation operators on the q-Fock space. Applied to a tensor in $H^{\otimes n}$, as hinted by the names, the creation operator creates an extra tensor leg to obtain a new tensor in $H^{\otimes(n+1)}$. On the other hand, the annihilation operator sends a tensor in $H^{\otimes n}$ to an element in $H^{\otimes(n-1)}$.

Definition 3.5. *Given $f \in H$, we define the creation operator a^* on $\mathcal{F}_q(H)$ as the linear extension of*

$$\begin{aligned}
a^*(f)\Omega &:= f, \\
a^*(f) g_1 \otimes \cdots \otimes g_n &:= f \otimes g_1 \otimes \cdots \otimes g_n.
\end{aligned}$$

The annihilation operator a on $\mathcal{F}_q^{finite}(H)$ is defined as the linear extension of

$$\begin{aligned}
a(f)\Omega &:= 0, \\
a(f) g_1 \otimes \cdots \otimes g_n &:= \sum_{i=1}^{n} q^{i-1} \langle f, g_i\rangle g_1 \otimes \cdots \otimes \check{g}_i \otimes \cdots \otimes g_n,
\end{aligned}$$

where $\check{g}_i$ indicates that it has to be deleted in the tensor.

We will also briefly consider the right variants of the creation and annihilation operator, denoted by a_r^* and a_r respectively. These are defined identically on the vacuum vector, and otherwise

$$\begin{aligned} a_r^*(f)g_1 \otimes \cdots \otimes g_n &:= g_1 \otimes \cdots \otimes g_n \otimes f \\ a_r(f)g_1 \otimes \cdots \otimes g_n &:= \sum_{i=1}^{n} q^{n-i}\langle f, g_i\rangle g_1 \otimes \cdots \otimes \check{g}_i \otimes \cdots \otimes g_n. \end{aligned}$$

Later on we will apply some of the results for the creation and annihilation operator to their right variants, the proof of which are analogous and will therefore be skipped. There are however some interesting relations between the left and right analogues that we would like to make explicit. Namely, the following remark allows us to switch between the two:

Remark 3.6. *Let S be the operator on $\mathcal{F}_q(H)$ that turns the order of tensors around. For $f \in H$ we have:*

$$\begin{aligned} a_r^*(f) &= Sa^*(f)S \\ a_r(f) &= Sa(f)S. \end{aligned}$$

We now arrive at two important properties that the creation and annihilation operators that we have defined satisfy.

Lemma 3.7. *For all $f, g \in H$ and $q \in (-1, 1)$, on $\mathcal{F}^{finite}(H)$ the operators a^* and a satisfy*

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle 1,$$

which we shall refer to as the q-relations. Moreover, $a^*(f)$ and $a(f)$ are adjoints with respect to the q-inner product.

Proof. By linearity it suffices to show both statements hold on $H^{\otimes n}$. We first show that the q-relations are indeed true.

Let $h_1 \otimes \cdots \otimes h_n \in H^{\otimes n}$. Then:

$$\begin{aligned} (a(f)a^*(g))h_1 \otimes \cdots \otimes h_n &= a(f)g \otimes h_1 \otimes \cdots \otimes h_n \\ &= \langle f, g\rangle h_1 \otimes \cdots \otimes h_n + q\sum_{i=1}^{n} q^{i-1}\langle f, g_i\rangle g \otimes h_1 \otimes \cdots \otimes \check{h}_i \otimes \cdots \otimes h_n \\ &= (\langle f, g\rangle + qa^*(g)a(f))h_1 \otimes \cdots \otimes h_n, \end{aligned}$$

which yield the q-relations. It remains to show that the creation and annihilator operators are each other's adjoint. For $f \in H$ and $g_1 \otimes \cdots \otimes g_n \in H^{\otimes n}$ and $h_1 \otimes \cdots \otimes h_{n+1} \in H^{\otimes(n+1)}$ we set out to prove that

$$\langle a^*(f)g_1 \otimes \cdots \otimes g_n, h_1 \otimes \cdots \otimes h_{n+1}\rangle_q = \langle g_1 \otimes \cdots \otimes g_n, a(f)h_1 \otimes \cdots \otimes h_{n+1}\rangle_q,$$

We proceed with a proof by induction on n.

The case $n = 0$ follow directly from the definition. Let $n \geq 1$ and assume the above statement holds for all $n' < n$. Consider the subset of permutations of S_n which map 1 to i. We denote:

$$S_n^{(i)} := \{\pi \in S_n \mid \pi(1) = i\}.$$

Recall that

$$\langle g_1 \otimes \cdots \otimes g_n, h_1 \otimes \cdots \otimes h_n\rangle_q = \sum_{\sigma \in S_n} q^{i(\sigma)}\langle g_1 \otimes \cdots \otimes g_n, U_\sigma h_1 \otimes \cdots \otimes h_n\rangle_0.$$

For $i = 1, \ldots, n+1$, we claim that we can rewrite $\sum_{\sigma \in S_n} q^{i(\sigma)}\langle g_1 \otimes \cdots \otimes g_n, U_\sigma h_1 \otimes \cdots \otimes \check{h}_i \otimes \cdots \otimes h_{n+1}\rangle_0$ using elements from $S_{n+1}^{(i)}$ instead. Identify $\sigma \in S_n$ with $\pi \in S_{n+1}^{(i)}$ where

$$\pi(l) = \begin{cases} i & \text{if } l = 1, \\ \sigma(l) & \text{if } \sigma(l) < i, \\ \sigma(l) + 1 & \text{if } \sigma(l) \geq i. \end{cases}$$

Note that this construction yields a bijection between $S_{n+1}^{(i)}$ and S_n. Moreover, we can see that pairs of the form $(1,j)$ for $j = 2, \cdots, n+1$ yield exactly $i-1$ inversions by taking $j' = \sigma^{-1}(j)$ and noting that $\sigma^{-1}(j) > 1$. All the other inversions of π have a one-to-one correspondence to the inversions of σ by the above construction, and hence

$$i(\pi) = i - 1 + i(\sigma),$$

Thus, for $\sigma \in S_n$ and $i = 1, \ldots, n+1$, if we pick the associated $\pi \in S_{n+1}^{(i)}$ we can then write

$$q^{i(\sigma)}\langle g_1 \otimes \cdots \otimes g_n, U_\sigma h_1 \otimes \cdots \otimes \check{h}_i \otimes \cdots \otimes h_{n+1}\rangle_0 \quad = \quad q^{i(\pi)-i+1}\langle g_1, h_{\pi(2)}\rangle \cdots \langle g_n, h_{\pi(n+1)}\rangle,$$

where we note that the i-th leg of $h_1 \otimes \cdots \otimes h_{n+1}$ is not accessed in the right hand side. For all $i = 1 \cdots n+1$ we have arrived at the conclusion of the claim:

$$\langle g_1 \otimes \cdots \otimes g_n, h_1 \otimes \cdots \otimes \check{h}_i \otimes \cdots \otimes h_{n+1}\rangle_q = \sum_{\pi \in S_{n+1}^{(i)}} q^{i(\pi)-i+1}\langle g_1, h_{\pi(2)}\rangle \cdots \langle g_n, h_{\pi(n+1)}\rangle$$

Using the fact that $\{S_{n+1}^{(i)} \mid i = 1, \cdots, n+1\}$ forms a partition of S_{n+1}, we complete the proof as follows:

$$\begin{aligned}
\langle g_1 \otimes \cdots \otimes g_n, a(f)h_1 \otimes \cdots \otimes h_{n+1}\rangle_q &= \sum_{i=1}^{n+1} q^{i-1}\langle f, h_i\rangle\langle g_1 \otimes \cdots \otimes g_n, h_1 \otimes \cdots \otimes \check{h}_i \otimes \cdots \otimes h_{n+1}\rangle_q \\
&= \sum_{i=1}^{n+1} q^{i-1}\langle f, h_i\rangle \sum_{\pi \in S_{n+1}^{(i)}} q^{i(\pi)-i+1}\langle g_1, h_{\pi(2)}\rangle \cdots \langle g_n, h_{\pi(n+1)}\rangle \\
&= \sum_{i=1}^{n+1} \sum_{\pi \in S_{n+1}^{(i)}} q^{i(\pi)}\langle f, h_{\pi(1)}\rangle\langle g_1, h_{\pi(2)}\rangle \cdots \langle g_n, h_{\pi(n+1)}\rangle \\
&= \sum_{\pi \in S_{n+1}} q^{i(\pi)}\langle f, h_{\pi(1)}\rangle\langle g_1, h_{\pi(2)}\rangle \cdots \langle g_n, h_{\pi(n+1)}\rangle \\
&= \langle a^*(f)g_1 \otimes \cdots \otimes g_n, h_1 \otimes \cdots \otimes h_{n+1}\rangle_q.
\end{aligned}$$

□

The next object of our attention will be the boundedness of a^* and a. Then by a density argument we can conclude that Lemma 3.7 holds on the entirety of the q-Fock space as well.

Remark 3.8. *The above proof allows us to compute the norm of $f^{\otimes n}$ explicitly for $f \in H$. The above proof shows that*

$$\|f^{\otimes(n+1)}\|_q^2 = \sum_{i=1}^{n+1} q^{i-1}\|f\|^2\|f^{\otimes n}\|_q^2 = [n+1]_q\|f\|^2\|f^{\otimes n}\|_q^2.$$

Hence, by a simple induction argument we can conclude that $\|f^{\otimes n}\|_q = \sqrt{[n]_q!}\|f\|^n$.

We can in fact compute the exact norms of a^* and a on $\mathcal{F}_q(H)$:

Lemma 3.9. *a^* and a are bounded on $\mathcal{F}_q(H)$, and*

$$\|a^*(f)\|_q = \|a(f)\|_q = \begin{cases} \frac{\|f\|}{\sqrt{1-q}} & \text{if } 0 \le q < 1, \\ \|f\| & \text{if } -1 < q < 0. \end{cases} \tag{3.3}$$

Proof. Part of the lemma in itself is that we can in fact extend a and a^* to $\mathcal{F}_q(H)$. That is, we show that the above results instead hold for $\mathcal{F}^{finite}(H)$ with the q-inner product, and then by a standard extension argument we find the result.

The first equality follows from a^* and a being adjoints, and thus we only need to compute the norm for a^*. Let us split the problem into two, the case of negative q and non-negative q. We start with the former, i.e. $-1 < q < 0$.

Let $\xi \in \mathcal{F}^{finite}(H)$ and $f \in H$. From the q-relations and the adjointness of a^* and a of we have:

$$\begin{aligned} \|a^* f\xi\|_q^2 &= \langle a^*(f)\xi, a^*(f)\xi\rangle_q \\ &= \langle a(f)a^*(f)\xi, \xi\rangle_q \\ &= \langle(\langle f, f\rangle + qa^*(f)a(f))\xi, \xi\rangle_q \\ &= \|f\|^2\|\xi\|_q^2 + q\|a(f)\xi\|_q^2 \\ &\leq \|f\|^2\|\xi\|_q^2. \end{aligned} \tag{3.4}$$

That is, a and a^* to $\mathcal{F}_q(H)$ are indeed bounded on $\mathcal{F}^{finite}(H)$ and $\|a(f)\| = \|a^*(f)\| \leq \|f\|$. As $a^*(f)\Omega = f$ we have equality as well, proving the statement for $-1 < q < 0$.

The case of $0 \leq q < 1$ is somewhat more involved. Let us first show that $\frac{1}{\sqrt{1-q}}$ is an upper bound for the norm of a^*.

Let $f \in H$ and $\xi \in H^{\otimes n}$. We shall define $\pi_{ij} \in S_n$ as the transposition of i and j, where $1 \leq i, j \leq n$ and $i \neq j$. For notational purposes we set π_{ii} as the identity. Moreover, let us embed S_n into S_{n+1} in the trivial sense. Recall that P_q was such that $\langle\xi, \xi\rangle_q = \langle\xi, P_q\xi\rangle_0$. Let $P_q^{(n)}$ denote the restriction of P_q to $H^{\otimes n}$, where we note that if maps to $H^{\otimes n}$. Moreover, as the definition involves a finite sum it is also bounded.

From the proof of Lemma 3.7 we can derive

$$\begin{aligned} P_q^{(n)} &= \sum_{\pi\in S_n} q^{i(\pi)}U_\pi\xi \\ &= \sum_{i=1}^{n} q^{i-1} \sum_{\sigma\in S_{n-1}} q^{i(\sigma)}(\mathbf{1}\otimes U_\sigma)U_{\pi_{1i}} \\ &= \sum_{i=1}^{n} q^{i-1}\left(\mathbf{1}\otimes \sum_{\sigma\in S_{n-1}} q^{i(\sigma)}U_\sigma\right)U_{\pi_{1i}} \\ &= \left[(\mathbf{1}\otimes P_q^{(n-1)})\sum_{i=1}^{n} q^{i-1}U_{\pi_{1i}}\right]. \end{aligned}$$

Since both $P_q^{(n)}$ and $U_{\pi_{1i}}$ are self-adjoint with respect to the standard inner product, we have that

$$\left((\mathbf{1}\otimes P_q^{(n-1)})\sum_{i=1}^{n} q^{i-1}U_{\pi_{1i}}\right)^* = \sum_{i=1}^{n} q^{i-1}U_{\pi_{1i}}(\mathbf{1}\otimes P_q^{(n-1)}).$$

This yields:

$$\begin{aligned} P_q^{(n)}\left(P_q^{(n)}\right)^* &= (\mathbf{1}\otimes P_q^{(n-1)})\sum_{i=1}^{n} q^{i-1}U_{\pi_{1i}}\sum_{i=1}^{n} q^{i-1}U_{\pi_{i1}}(\mathbf{1}\otimes P_q^{(n-1)}) \\ &\leq (\mathbf{1}\otimes P_q^{(n-1)})\left[\sum_{i=1}^{n} q^{i-1}\right]^2(\mathbf{1}\otimes P_q^{(n-1)}) \\ &\leq \frac{1}{(1-q)^2}(\mathbf{1}\otimes P_q^{(n-1)})(\mathbf{1}\otimes P_q^{(n-1)})^*. \end{aligned}$$

Hence $P_q^{(n)} \leq \frac{1}{1-q}\mathbf{1} \otimes P_q^{(n-1)}$ as $0 \leq q < 1$. This yields

$$\begin{aligned} \|a(f)\|_q^2 &= \langle a(f)\xi, a(f)\xi\rangle_q \\ &= \langle f \otimes \xi, f \otimes \xi\rangle_q \\ &= \langle f \otimes \xi, P_q^{(n+1)}(f \otimes \xi)\rangle_q \\ &\leq \frac{1}{1-q}\langle f \otimes \xi, \mathbf{1} \otimes P_q^{(n-1)}(f \otimes \xi)\rangle_0 \\ &= \frac{1}{1-q}\|f\|^2\langle \xi, \xi\rangle_q. \end{aligned} \tag{3.5}$$

By the same argument as before a and a^* are bounded operators on $\mathcal{F}_q(H)$ with $\|a(f)\| = \|a^*(f)\| \leq \|f\|/\sqrt{1-q}$ for $0 \leq q < 1$. Let $f^{\otimes n} = f \otimes \cdots \otimes f \in H^{\otimes n}$. Lastly, we have that

$$\begin{aligned} \|a^*(f)f^{\otimes n}\|_q^2 &= \langle f \otimes f^{\otimes n}, f \otimes f^{\otimes n}\rangle_q \\ &= \langle f \otimes f^{\otimes n}, \mathbf{1} \otimes P_q^{(n-1)} \sum_{i=1}^{n} q^{i-1} U_{\pi_{1i}}(f \otimes f^{\otimes n})\rangle_0 \\ &= \sum_{i=1}^{n} q^{i-1}\langle f, f\rangle\langle f^{\otimes n}, P_q^{(n)} f^{\otimes n}\rangle_0 \\ &= \frac{1-q^{n+1}}{1-q}\|f\|^2\|f^{\otimes n}\|_q^2, \end{aligned}$$

which gives us equality in norm as $n \to \infty$. □

Now that we have established our creation and annihilation operators on $\mathcal{F}_q(H)$, as well as the q-relations, are ready to introduce the q-Gaussians.

Definition 3.10. *For $f \in H_{\mathbb{R}}$, we define the q-Gaussians $\omega(f)$ as*

$$\omega(f) := a^*(f) + a(f).$$

With these, we define $\Gamma_q(H_{\mathbb{R}})$ as the von Neumann algebra generated by the q-Gaussians in $B(\mathcal{F}_q(H))$. That is,

$$\Gamma_q(H_{\mathbb{R}}) := \{a^*(f) + a(f) \mid f \in H_{\mathbb{R}}\}'' \subseteq B(\mathcal{F}_q(H)).$$

Lemma 3.11. *The vacuum vector is cyclic and separating for $\Gamma_q(H_{\mathbb{R}})$, and defines a trace through $\tau(x) := \langle \Omega, x\Omega\rangle_q$. Moreover, $\Gamma_q(H_{\mathbb{R}})$ is a finite von Neumann algebra.*

Proof. See [4]. □

Remark 3.12. *The right analogues of the above construction are defined by substituting the right creation and right annihilation operators. That is, $\omega_r(f) := a_r^*(f) + a_r(f)$, and $\Gamma_{q,r}(H_{\mathbb{R}}) := \{\omega_r(f) \mid f \in H_{\mathbb{R}}\}''$.*

3.3. WICK WORDS

Consider an element from $\Gamma_q(H_{\mathbb{R}})$ and apply it to the vacuum vector. Since the vacuum vector is separating, we obtain a unique element in $\mathcal{F}_q(H)$. Conversely, certain vectors from $\mathcal{F}_q(H)$ can be identified with elements in $\Gamma_q(H_{\mathbb{R}})$, which we will see as we introduce the Wick words. We first consider the results on $\mathcal{F}^{finite}(H)$, after which it can be extended to $\mathcal{F}_q(H)$.

Let us first define some notation. For $n \in \mathbb{N}$ and $k = 0, \ldots, n$ let us define

$$P_k^{(n)} := \{I, J \subseteq \{1, \ldots, n\} \mid |I| = k, |J| = n-k, I \cup J = \{1, \ldots, n\}\},$$

where we take each I and J as ordered sequences, i.e. for $(I, J) \in P_k^{(n)}$:

$$\begin{aligned} I &= (i_1, \ldots, i_k), \qquad i_1 \leq \cdots \leq i_n, \\ J &= (j_1, \ldots, j_{n-k}), \quad j_1 \leq \cdots \leq j_{n-k}. \end{aligned}$$

Note that I and J are disjoint. Moreover, for $(I,J) \in P_k^{(n)}$ we set

$$\iota(I,J) := \#\{(p,q) \mid i_q > j_q, 1 \leq p \leq k, 1 \leq q \leq n-k\}.$$

As the final step in defining the q-Wick product, we collect all of the $P_k^{(n)}$ in

$$P^{(n)} := \bigcup_{k=0}^{n} P_k^{(n)}.$$

Definition 3.13. *We define the Wick product of an element in $\mathcal{F}^{finite}(H)$ on $\mathcal{F}_q(H)$ through the linear extension of*

$$W(f_1 \otimes \cdots \otimes f_n) := \sum_{(I,J)\in P^{(n)}} q^{\iota(I,J)} a^*(f_{i_1}) \cdots a^*(f_{i_{|I|}}) a(f_{j_1}) \cdots a(f_{j_{|J|}}).$$

for $f_1 \otimes \cdots \otimes f_n \in H^{\otimes n}$.

Observe that

$$W(f_1 \otimes \cdots \otimes f_n)^* = W(f_n \otimes \cdots \otimes f_1).$$

There is the question of whether these operators are elements of $\Gamma_q(H_\mathbb{R})$. For a single $f \in H$ we have $W(f) = a^*(f) + a(f) = \omega(f) \in \Gamma_q(H_\mathbb{R})$. For higher order terms, we turn deduce a recursive formula for $W(f_1 \otimes \cdots \otimes f_{n+1})$. Clearly we have

$$\begin{aligned} W(f_1 \otimes \cdots \otimes f_{n+1}) &= a^*(f_1) \sum_{(I,J)\in P^{(n+1)}, 1\in I} q^{\iota(I,J)} a^*(f_{i_2}) \cdots a^*(f_{i_{|I|}}) a(f_{j_1}) \cdots a(f_{j_{|J|}}) \\ &\quad + \sum_{(I,J)\in P^{(n+1)}, 1\in J} q^{\iota(I,J)} a^*(f_{i_1}) \cdots a^*(f_{i_{|I|}}) a(f_1) a(f_{j_2}) \cdots a(f_{j_{|J|}}) \\ &=: A + B \end{aligned}$$

We can use the q-relations to move $a(f_1)$ forward in $a^*(f_{i_1}) \cdots a^*(f_{i_{|I|}}) a(f_1) a(f_{j_2}) \cdots a(f_{j_{|J|}})$ which make up the terms in B, to obtain that

$$\begin{aligned} B &= a(f_1) \sum_{(I,J)\in P^{(n)}} q^{\iota(I,J)} a^*(f_{i_1+1}) \cdots a^*(f_{i_{|I|}+1}) a(f_{j_1+1}) \cdots a(f_{j_{|J|}+1}) \\ &\quad - \sum_{i=1}^{n+1} q^{i-1} \langle f_1, f_{i+1} \rangle W(f_2 \otimes \cdots \otimes \check{f}_{i+1} \otimes \cdots \otimes f_{n+1}). \end{aligned}$$

through an exercise of expanding the terms. Combining these statements we get

$$W(f_1 \otimes \cdots \otimes f_{n+1}) = \omega(f_1) W(f_2 \otimes \cdots \otimes f_{n+1}) - \sum_{i=1}^{n} q^{i-1} \langle f_1, f_{i+1} \rangle W(f_2 \otimes \cdots \otimes \check{f}_{i+1} \otimes \cdots \otimes f_{n+1}).$$

Thus, by a simple induction argument with the above formula we conclude that $W(f_1 \otimes \cdots \otimes f_n) \in \Gamma_q(H_\mathbb{R})$.

We elaborate upon the aforementioned relation between $\Gamma_q(H_\mathbb{R})$ and $\mathcal{F}^{finite}(H)$. For any $\xi \in \mathcal{F}^{finite}(H)$ we have that

$$W(\xi)\Omega = \xi.$$

For any $f_1 \otimes \cdots \otimes f_n \in H^{\otimes n}$ this is easily seen, as applying any annihilation operator to the vacuum vector will yield zero, and so only the term $a^*(f_1) \cdots a^*(f_n)\Omega$ remains. Moreover, since Ω is separating, $W(\xi)$ is the only element that yields ξ. In this fashion, we can uniquely extend W to the entirety of $\Gamma_q(H_\mathbb{R})\Omega$ by requiring that $W(\xi)\Omega = \xi$ for $\xi \in \Gamma_q(H_\mathbb{R})\Omega$. Thus, if we have an $x \in \Gamma_q(H_\mathbb{R})$, we have $\xi := x\Omega \in \mathcal{F}_q(H)$ such that $x = W(\xi)$.

Remark 3.14. *For the Wick words we can also define the right-analogue W_r by using the right creation and annihilation operators. Let S be the operator that reverses the order of tensors once again. Using Remark 3.6 we can deduce the following:*

$$W_r(S\xi) = SW(\xi)S,$$

which shows us that $\Gamma_q(H_{\mathbb{R}})\Omega = \Gamma_{q,r}(H_{\mathbb{R}})\Omega$.

In particular, for $W(\xi), W(\eta) \in \Gamma_q(H_{\mathbb{R}})$ this gives us the following nice equality:

$$W(\xi)\eta = W(\xi)W_r(\eta)\Omega = W_r(\eta)W(\xi)\Omega = W_r(\eta)\xi.$$

3.4. The q-Gaussian functor

Suppose we have a contractive map $u : H_{\mathbb{R}} \to H_{\mathbb{R}}$. This section is dedicated to constructing a map $\Gamma_q(u)$ between $\Gamma_q(H_{\mathbb{R}})$ and $\Gamma_q(H_{\mathbb{R}})$ which is completely positive and trace preserving. We call this construction the q-Gaussian functor.

Lemma 3.15. *Let $T : \mathcal{F}^{finite}(H) \to \mathcal{F}^{finite}(H)$ be an operator such that it commutes with P_q, where P_q defines the inner product on $\mathcal{F}^{finite}(H)$ as in section 3.1. Then $\|T\|_0 = \|T\|_q$.*

Proof. Let $\xi \in \mathcal{F}^{finite}(H)$. By the functional calculus we have that we have that $P_q^{1/2}$ and T commute as well, and so:

$$\begin{aligned}
\|T\xi\|_q^2 &= \langle T\xi, T\xi\rangle_q \\
&= \langle T\xi, P_q T\xi\rangle_0 \\
&= \langle P_q^{1/2} T\xi, P_q^{1/2} T\xi\rangle_0 \\
&= \langle T P_q^{1/2}\xi, T P_q^{1/2}\xi\rangle_0 \\
&= \langle P_q^{1/2}\xi, T^*T P_q^{1/2}\xi\rangle_0 \\
&\leq \|T^*T\|_0 \langle P_q^{1/2}\xi, P_q^{1/2}\xi\rangle_0 \\
&= \|T^*T\|_0 \|\xi\|_q^2.
\end{aligned}$$

Therefore

$$\|T\|_q^2 \leq \|T^*T\|_0 \leq \|T\|_0^2.$$

Applying the above with P_q^{-1} instead we find the reverse inequality, and so $\|T\|_q = \|T\|_0$. □

Now suppose we have a contraction $T : H_{\mathbb{R}} \to H_{\mathbb{R}}$. We can define the map $\mathcal{F}(T)$ from $\mathcal{F}^{finite}(H)$ to $\mathcal{F}^{finite}(H)$ through the linear extension of

$$\begin{aligned}
\mathcal{F}_q(T)\Omega &= \Omega, \\
\mathcal{F}_q(T)(f_1 \otimes \cdots \otimes f_n) &= (Tf_1) \otimes \cdots \otimes (Tf_n).
\end{aligned}$$

for $f_1 \otimes \cdots \otimes f_n \in H^{\otimes n}$. Note that as T is a contraction we have that $\|\mathcal{F}_q(T)\|_0 < \infty$. We also note that for another contraction S on $H_{\mathbb{R}}$ we have $\mathcal{F}_q(TS) = \mathcal{F}_q(T)\mathcal{F}_q(S)$.

It is easily verified that $P_q\mathcal{F}_q(T) = \mathcal{F}_q(T)P_q$. Thus, we can apply Lemma 3.15 to find that $\mathcal{F}_q(T)$ is bounded with respect to the q-norm, and therefore we can extend it to a bounded operator on $\mathcal{F}_q$. With the operator established on the entirety of $\mathcal{F}_q$, we can move on to the main theorem of this section:

Theorem 3.16. *Let $T : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ be a contraction. Define $\Gamma_q(T) : \Gamma_q(H_{\mathbb{R}}) \to \Gamma_q(H_{\mathbb{R}})$ through*

$$(\Gamma_q(T)X)\Omega = \mathcal{F}_q(T)(X\Omega).$$

for $X \in \Gamma_q(H_{\mathbb{R}})$. Then $\Gamma_q(T)$ a unique linear, bounded, completely positive trace preserving map. Moreover, if T is orthogonal, then $\Gamma_q(T)$ is a $*$-automorphism. If T is an orthogonal projection then $\Gamma_q(T)$ is a conditional expectation.

Proof. Linearity is clear, and uniqueness follows from the separating property of the vacuum vector in $\Gamma_q(K_{\mathbb{R}})$. The unital property is also easily verified by the definition. Recall that for $W(\xi) \in \Gamma_q(H_{\mathbb{R}})$ we have that $W(\xi)^* = W(\xi^*)$. Utilizing this fact in combination with the definition of $\Gamma_q(T)$ yields that it respects the $*$ operation.

Let $S : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ be a contraction as well. Then

$$(\Gamma_q(TS)X)\Omega = \mathcal{F}_q(TS)(X\Omega) = \mathcal{F}_q(T)\mathcal{F}_q(S)(X\Omega) = (\Gamma_q(T)\Gamma_q(S)X)\Omega,$$

by the multiplicativity of $\mathcal{F}_q(\cdot)$. Thus, a factorization of T into contractions yields a factorization of $\Gamma_q(T)$. Given a factorization of T, tt suffices to prove the statements for each case separately as these translate to the result throughout the composition.

Set $K_{\mathbb{R}} = H_{\mathbb{R}} \oplus H_{\mathbb{R}}$. We claim that we can write $T = POI$, where

- $I : H_{\mathbb{R}} \to K_{\mathbb{R}}$ is an isometric embedding,
- $O : K_{\mathbb{R}} \to K_{\mathbb{R}}$ is orthogonal,
- $P : K_{\mathbb{R}} \to H_{\mathbb{R}}$ is an orthogonal projection.

We choose I to be the canonical embedding of $H_{\mathbb{R}}$ into $H_{\mathbb{R}} \oplus H_{\mathbb{R}}$ in the first coordinate, which clearly satisfies the required properties. Similarly, for P we choose the projection onto first coordinate.

It remains to find the appropriate operator for O. Since T is a contraction, $\sqrt{1 - T^*T}$ is well-defined. Now consider the operator O on $K_{\mathbb{R}} \to K_{\mathbb{R}}$ given by

$$O = \begin{pmatrix} T & \sqrt{1 - T^*T} \\ \sqrt{1 - TT^*} & -T^* \end{pmatrix}.$$

Note that $(1 - T^*T)T^* = T^*(1 - TT^*)$, and so by the continuous functional calculus we obtain that T^* and $\sqrt{1 - T^*T}$ commute. Consequently, it is easy to verify that O is orthogonal.

We now show that the statements hold for the individual factors. Let us start with the orthogonal map O. We claim that

$$\Gamma_q(O)X = \mathcal{F}_q(O)X\mathcal{F}_q(O)^*,$$

Obviously it holds for the unit. First we show that it holds for Wick words of length 1. Let $f \in H$. We observe that

$$\begin{aligned} \mathcal{F}_q(T)a^*(f) &= a^*(Tf)\mathcal{F}_q(T) \\ a(f)\mathcal{F}_q(T^*) &= \end{aligned}$$

Recall that $\langle Tf, Tg\rangle = \langle f, g\rangle$ for any $g \in H$. Utilizing this, and that $OO^* = 1_{K_{\mathbb{R}}}$, for $g_1 \otimes \cdots \otimes g_n \in H^{\otimes n}$ we find:

$$\begin{aligned} \mathcal{F}_q(O)W(f)\mathcal{F}_q(O)^* &= \mathcal{F}_q(O)(a^*(f) + a(f))\mathcal{F}_q(O)^* \\ &= \mathcal{F}_q(O)\mathcal{F}_q(O^*)a^*(Of) + \mathcal{F}_q(O)\mathcal{F}_q(O^*)a(Of) \\ &= \mathcal{F}_q(OO^*)(a^*(Of) + a(Of)) \\ &= W(Of). \end{aligned}$$

To obtain the result for higher order Wick words, we call upon the recursive formula we found in Section 3.3. Suppose the result holds for Wick words of length less than n. Now for $g_1 \otimes \cdots \otimes g_n \in H^{\otimes n}$ the fact that

$$\begin{aligned} \mathcal{F}_q(O)W(f_1)W(f_2 \otimes \cdots f_n)\mathcal{F}_q(O)^* &= \mathcal{F}_q(O)\mathcal{F}_q(O)^*W(Of_1)W(\mathcal{F}_q(O)f_2 \otimes \cdots f_n) \\ &= W(Of_1)W(\mathcal{F}_q(O)f_2 \otimes \cdots f_n) \end{aligned}$$

suffices, as the recursive formula then yields the result. With our explicit expression for $\Gamma_q(O)X$ we can deduce that it is bounded and completely positive.

It remains to check that it preserves the trace. Indeed,

$$\begin{aligned}\tau(\mathcal{F}_q(O)X\mathcal{F}_q(O^*)) &= \langle \Omega, \mathcal{F}_q(O)X\mathcal{F}_q(O^*)\Omega\rangle_q \\ &= \langle \mathcal{F}_q(O)^*\Omega, X\mathcal{F}_q(O^*)\Omega\rangle_q \\ &= \langle \mathcal{F}_q(O^*)\Omega, X\mathcal{F}_q(O^*)\Omega\rangle \\ &= \langle \Omega, X\Omega\rangle_q \\ &= \tau(X).\end{aligned}$$

Utilizing that $\mathcal{F}_q(O)^*\mathcal{F}_q(O) = \mathcal{F}_q(O)\mathcal{F}_q(O)^* = 1_K$ we can see that yields an automorphism. Thus, in the case that T is orthogonal, we could conclude the proof here as $\Gamma_q(T)$ is a $*$-automorphism.

Let us now consider the factor P. The above proof for boundedness, complete positivity, and the trace preserving property can be applied to identically to P, as $PP^* = 1_{H_\mathbb{R}}$ (however, the automorphism result fails). In a similar manner, if T is a projection, if we verify that it yields a conditional expectation to finish the proof. As for any $f \in H$ and $g \in \text{ran}P$ we have $\langle f, g\rangle = \langle Pf, g\rangle = \langle Pf, Pg\rangle$ we can repeat a similar argument with $\mathcal{F}_q(P)$ and the Wick operator to conclude this case.

In the case that T is neither a projection nor orthogonal, only the first factor remains. Let P_1 be the orthogonal projection of $K_\mathbb{R}$ into the first coordinate. Then through its definition we find $\mathcal{F}_q(P_1)$ is a projection in $\mathcal{F}_q(K)$ equal to projecting on the first coordinate. That is,

$$\mathcal{F}_q(H) \simeq \mathcal{F}_q(K)\mathcal{F}_q(P_1).$$

Let $\omega_K(f)$ be the q-Gaussians on $\mathcal{F}_q(K)$. We set:

$$\Gamma_q^K(H_\mathbb{R}) := \text{vNA}(\omega_K(f \oplus 0) \mid f \in H_\mathbb{R}).$$

Embedding $\mathcal{F}_q(H)$ in the first coordinate we have $\Gamma_q^K(H_\mathbb{R})\mathcal{F}_q(H) \subset \mathcal{F}_q(H)$. By the previous identifications we can deduce

$$\Gamma_q(H_\mathbb{R}) \cong \Gamma_q^K(H)\mathcal{F}_q(P_1).$$

But this homomophism is in fact equal to I. As a $*$-homomorphism $\Gamma_q(I)$ is therefore completely positive. Lastly, using that $\mathcal{F}_q(P_1)\Omega = \Omega$ we see that it preserves the trace. Having now shown the statements for all three factors, we conclude the proof.

$\square$

4

Factoriality of $\Gamma_q(H_{\mathbb{R}})$

Recall that a von Neumann algebra is a factor if its centre consists of multiples of the identity. The main result of this chapter is the following:

Theorem 4.1. *If* $\dim H \geq 2$ *then* $\Gamma_q(H_{\mathbb{R}})$ *is a factor.*

The proof of the theorem rests on the next proposition:

Proposition 4.2. *Suppose* $\dim H \geq 2$*, and* $e \in H_{\mathbb{R}}$ *is such that* $\|e\| = 1$*. Then* $\{W(e)\}''$ *is a maximal abelian subalgebra.*

First we fix some notation. Recall that for $\xi = \xi_1 \otimes \cdots \xi_a \in H^{\otimes a}$ and $\eta = \eta_1 \otimes \cdots \eta_b \in H^{\otimes b}$ then

$$\langle \xi_1 \otimes \cdots \xi_a, \eta_1 \otimes \cdots \eta_n \rangle_q = \delta_{a,b} \langle \xi_1 \otimes \cdots \xi_a, P_q^a \eta_1 \otimes \cdots \eta_b \rangle,$$

where P_q^a is P_q restricted to $H^{\otimes a}$. This is possible by Theorem 3.3. Let us denote the Hilbert space as H_q^a and let $\|\xi\|_{H_q^a}$ denote the norm of ξ with the above norm.

We now split $H^{\otimes n}$ into two, namely a part of size $n-k$ and a part of size k, for $k \leq n$. If we take $\xi \otimes \eta, \xi' \otimes \eta' \in H^{\otimes(n-k)} \otimes H^{\otimes k}$ we have the inner product

$$\langle \eta \otimes \zeta, \xi' \otimes \eta' \otimes \zeta' \rangle_{H_q^a} = \langle \xi \otimes \eta, P_q^{n-k} \xi' \otimes P_q^k \eta' \rangle.$$

Note that this does not coincide with the inner product on $H^{\otimes n}$. To link the two, we define the unique operator $R_{n,k} : H^{\otimes n} \to H^{\otimes n}$ through its adjoint by

$$P_q^n = (P_q^{n-k} \otimes P_q^k) R_{n,k}^*,$$

where we embed $H^{\otimes(n-k)} \otimes H^{\otimes k}$ in $H^{\otimes n}$ in the canonical way.

It can be verified (see [16]) that

$$R_{n,k} = \sum_{\pi \in S_n \backslash S_{n-k} \times S_k} q^{i(\pi)} U_\pi,$$

where the representative of each right coset is chosen such that the number of inversions is minimal. This provides us with an upper estimate for the norm, namely

$$\|R_{n,k}\| \leq \prod_{i=1}^{\infty} (1 - |q|^i)^{-i}.$$

Set $C_q := \prod_{i=1}^{\infty} (1 - |q|^i)^{-i}$.

Lemma 4.3. *The embedding of $H_q^{n-k} \otimes H_q^k$ in H_q^n has norm at most $\sqrt{C_q}$.*

Proof. We have:

$$\begin{aligned}(P_q^n)^2 &= P_q^n(P_q^n)^* \\ &= (P_q^{n-k} \otimes P_q^k)R_{n,k}^* R_{n,k}(P_q^{n-k} \otimes P_q^k)^* \\ &\leq \|R_{n,k}^* R_{n,k}\|(P_q^{n-k} \otimes P_q^k)^2 \\ &\leq C_q^2(P_q^{n-k} \otimes P_q^k)^2.\end{aligned}$$

Thus, $P_q^n \leq C_q P_q^{n-k} \otimes P_q^k$ and we conclude that for any $\nu \in H_q^n$ we have

$$\begin{aligned}\|\nu\|_{H_q^n}^2 &= \langle P_q^n \nu, \nu\rangle \\ &\leq C_q\langle P_q^{n-k} \otimes P_q^k \nu, \nu\rangle \\ &= C_q\|\nu\|^2_{H_q^{n-k}\otimes H_q^k}.\end{aligned}$$

□

Remark 4.4. *Applying the above result inductively on $e_1 \otimes \cdots \otimes e_n$ where $e_i \in H$ are unit vectors we have*

$$\|e_1 \otimes \cdots \otimes e_n \otimes e^{\otimes m}\|_q \leq C_q\|e_1 \otimes \cdots \otimes e_n\|_q\|e^{\otimes m}\|_q \leq C_q^{n/2}\sqrt{[m]_q!}.$$

One more definition and result before we start on the proof of the proposition:

Definition 4.5. *Let $e \in H$. Set $E_e \subset \mathcal{F}_q(H)$ as*

$$E_e := \overline{\operatorname{Span}}\{e^{\otimes n} \mid n \geq 0\}.$$

We note that $E_e = \mathcal{F}_q(\mathbb{R}e)$.

We construct an identification between $\{W(e)\}''$ and $E_e \cap \Gamma_q(H_{\mathbb{R}})\Omega$. On the one hand, suppose we have $W(\xi) \in \{W(e)\}''$. Then as $\{W(e)\}''$ is closed in the strong operator topology, we can find $\xi_i \in E_e$ such that $W(\xi_i)$ converges weakly to $W(\xi)$. But then applying both to Ω and using that $W(\xi_i)\Omega = \xi_i$ we directly find that $\xi \in E_e$.

On the other hand, suppose we have $\xi \in E_e \cap \Gamma_q(H_{\mathbb{R}})\Omega$. Let $P_{\mathbb{R}e} : H_{\mathbb{R}} \to \mathbb{R}e$ be the orthogonal projection onto $\mathbb{R}e$. Then

$$\begin{aligned}(\Gamma_q(P_{\mathbb{R}e})W(\xi))\Omega &= \mathcal{F}_q(P_{\mathbb{R}e})(W(\xi)\Omega) \\ &= \mathcal{F}_q(P_{\mathbb{R}e})\xi \\ &= P_{E_e}\xi \\ &= \xi \\ &= W(\xi)\Omega. \end{aligned} \tag{4.1}$$

As Ω is separating, this implies that $W(\xi) = \Gamma_q(P_{\mathbb{R}e})W(\xi)$, and therefore $W(\xi) \in \{W(e)\}''$.

Proof of Proposition 4.2. Let $W(\xi) \in \Gamma_q(H_{\mathbb{R}}) \cap \{W(e)\}'$. We need to show that $W(\xi) \in \{W(e)\}''$, or equivalently $\xi \in E_e$.

Let $\eta \in E_e$ and consider $W(\eta)$. By our assumption $W(\xi)$ and $W(\eta)$ commute. Applying $(W(\xi)W(\eta) - W(\eta)W(\omega))$ to the vacuum vector and utilising Remark 3.14 we have:

$$\begin{aligned}0 &= (W(\xi)W(\eta) - W(\eta)W(\xi))\Omega \\ &= W(\xi)\eta - W(\eta)\xi \\ &= W_r(\eta)\xi - W(\eta)\xi \\ &= (W_r(\eta) - W(\eta))\xi.\end{aligned} \tag{4.2}$$

Thus

$$\xi \in \bigcap_{W(\eta)\in\{W(e)\}''} \ker(W_r(\eta) - W(\eta)).$$

To show that $\xi \in E_e$, it suffices to show that

$$E_e^{\perp} \subset \overline{\text{Span}}\left(\bigcup_{W(\eta)\in\{W(e)\}''} \text{ran}(W_r(\eta) - W(\eta))\right).$$

Now extend $\{e\}$ to an orthonormal basis $(e_i)_{i\geq 0}$ of $H_{\mathbb{R}}$, where we take $e_0 = e$. Clearly we have

$$E_e^{\perp} = \overline{\text{Span}}\{e_{i_1} \otimes \cdots \otimes e_{i_n} \mid n > 1, e_{i_k} \neq 0 \text{ for some k}\}.$$

Take $z = e_{i_1} \otimes \cdots \otimes e_{i_n} \in E_e^{\perp}$. It suffices to show that z is the weak limit of elements in $\text{Span}\{\text{Ran}(W_r(\eta) - W(\eta)) \mid W(\eta) \in \{W(e)\}''\}$.

As $\{W(e)\}''$ is commutative and diffuse by we have $\{W(e)\}'' \simeq L^{\infty}([0,1])$ equipped with the Lebesgue measure. We can identify the Rademacher functions

$$r_i(x) := \text{sign}\sin(2^i \pi x), \quad i = 1, 2, \cdots$$

on $[0,1]$ with elements in $\{W(e)\}''$.

Let $(\eta_i)_i \subset E_e$ be the sequence such that $W(\eta_i) \simeq r_i$ for $i \geq 1$. Then $W(\eta_i)^2 \simeq r_i^2 = 1$, and similarly we see that $W(\eta_i)$ is self-adjoint. Moreover, $r_i \to 0$ weakly as $i \to \infty$ in $L^2([0,1])$, and so we find weak convergence for η_i in $\mathcal{F}_q(H)$ as well.

Let us now define the following sequence in $\mathcal{F}_q(H)$:

$$z_i := (W(\eta_i) - W_r(\eta_i))(W(\eta_i)z).$$

Expanding the brackets we note that

$$\begin{aligned} z_i &= W(\eta_i)^2 z - W_r(\eta_i)W(\eta_i)z \\ &= z - W_r(\eta_i)W(\eta_i)z. \end{aligned}$$

Thus, it suffices to show that $W_r(\eta_i)W(\eta_i)z$ converges weakly to zero. To simplify our notation, set

$$y_i := W_r(\eta_i)W(\eta_i)z.$$

Obviously the norm of $W(\eta_i)$ is at most 1, through the identification with r_i. Remark 3.14 then gives us the same estimate for $W_r(\eta_i)$. Hence, $\|y_i\| \leq \|z\|$, and thus it is sufficient to show that for any pure tensor $t = e_{j_1} \otimes \cdots \otimes e_{j_p} \in \mathcal{F}_q(H)$ that $\langle y_i, t\rangle_q \to 0$ as $i \to 0$. By Remark 3.14 we have:

$$\begin{aligned} \langle y_i, t\rangle_q &= \langle W_r(\eta_i)W(\eta_i)z, t\rangle_q \\ &= \langle W_r(z)\eta_i, W_r^*(\eta_i)t\rangle_q \\ &= \langle W_r(z)\eta_i, W(t)\eta_i\rangle_q. \end{aligned}$$

Recall the definition of the Wick word from Definition 3.13 (as well as the the right-analogue in this case). Substituting the sum in the above expression and moving the sums out of the inner product, we obtain

$$\langle W_r(z)\eta_i, W(t)\eta_i\rangle_q = \sum_{\substack{(I,J)\in P^{(n)} \\ (I',J')\in P^{(p)}}} q^{\iota(I,J)} q^{\iota(I',J')} \langle a_r^*(e_{i_{r_1}}) \cdots a_r^*(e_{i_{r_m}}) a_r(e_{i_{s_1}}) \cdots a_r(e_{i_{s_{n-m}}})\eta_i, \tag{4.3}$$

$$a^*(e_{j_{r'_1}}) \cdots a^*(e_{j_{r'_l}}) a(e_{j_{s'_1}}) \cdots a(e_{i_{s'_{p-l}}})\eta_i\rangle_q,$$

where

$$\begin{aligned}(I,J) &= ((r_1,\cdots,r_m),(s_1,\cdots,s_{n-m}))\\ (I',J') &= ((r'_1,\cdots,r'_l),(s'_1,\cdots,s'_{p-l})).\end{aligned}$$

As the number of terms in $P^{(n)}$ and $P^{(p)}$ is finite and depends only on n and p, ans $|q|<1$, it is sufficient to show that each inner product goes to 0 as $i\to\infty$. Since we intend to show the result for all pure tensors z and t, and can thus freely reorder them, we can drop the double indexing. That is, we may assume $I=(1,\ldots,m)$ and $J=(l+1,\ldots,n)$ since this will be the case for some (different) z, and similarly for I' and J'.

By way of this simplification, we can write the inner product terms in the right-hand side in (4.3) in the following manner:

$$I_i := \langle a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})a_r(e_{i_{m+1}})\cdots a_r(e_{i_n})\eta_i, a^*(e_{j_1})\cdots a^*(e_{j_l})a(e_{j_{l+1}})\cdots a(e_{i_p})\eta_i\rangle_q,$$

which gives us a more manageable expression. To aim is now to show that $I_i\to 0$ as $i\to\infty$.

We look at the terms within the inner product in more detail. As $\eta_i\in E_e$, it follows that $a_r(e_{i_k})\eta_i=0$ whenever $e_{i_k}\neq e$, as it lies in the orthogonal complement. As such, we can assume that $e_{i_n}=e$. Moreover, $a_r(e)\eta_i\in E_e$, and so we can repeat the argument to argue that $e_{i_{l+1}}=e_{i_{l+2}}=\cdots=e_{i_{s_n}}=e$. Thus, in order to be non-zero, our term must be of the form

$$a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})a_r(e)^{n-m}\eta_i.$$

For $k\geq 1$ we have that $a_r(e)e^{\otimes k}=[k]_q e^{\otimes(k-1)}$. Write $\eta_i=\sum_{k\geq 0}a_k^i e^{\otimes k}$. Note that as $W(\eta_i)$ is self-adjoint we necessarily have that all a_k are real. We can now write:

$$\begin{aligned}(a_r(e))^{n-m}\eta_i &= \sum_{k\geq 0} a_k^i (a_r(e))^{n-m}e^{\otimes k}\\ &= \sum_{k\geq n-m} a_k^i \frac{[k]_q!}{[k+m-n]_q!}e^{\otimes k+m-n}.\end{aligned}$$

Thus, overall this yields the result that for non-zero terms we have

$$a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})a_r(e)^{n-m}\eta_i = \sum_{k\geq n-m} a_k^i \frac{[k]_q!}{[k-(n-m)]_q!}a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})e^{\otimes k-(n-m)}.$$

Now consider the other term in the inner product, that is, $W(t)\eta_i$. Nearly identically we also derive:

$$a^*(e_{j_1})\cdots a^*(e_{j_l})a(e)^{p-l}\eta_i = \sum_{k\geq p-l} a_k^i \frac{[k]_q!}{[k+l-p]_q!}a^*(e_{j_1})\cdots a^*(e_{j_l})e^{\otimes k+l-p}.$$

We now substitute these terms to find that

$$\begin{aligned}& I_i\\ =\ & \langle \sum_{k\geq n-m} a_k^i \frac{[k]_q!}{[k+m-n]_q!}a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})e^{\otimes k+m-n}, \sum_{k'\geq p-l} a_{k'}^i \frac{[k']_q!}{[k'+l-p]_q!}a^*(e_{j_1})\cdots a^*(e_{j_l})e^{\otimes k'+l-p}\rangle_q\\ =\ & \sum_{\substack{k\geq m\\ k'\geq l}} a_{k-2m+n}^i a_{k'-2l+p}^i \frac{[k-2m+n]_q!}{[k-m]_q!}\frac{[k'-2l+p]_q!}{[k'-l]_q!}\langle a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})e^{\otimes k-m}, a^*(e_{j_1})\cdots a^*(e_{j_l})e^{\otimes k'-l}\rangle_q\\ =\ & \sum_{\substack{k\geq m\\ k\geq l}} a_{k-2m+n}^i a_{k-2l+p}^i \frac{[k-2m+n]_q!}{[k-m]_q!}\frac{[k-2l+p]_q!}{[k-l]_q!}\langle a_r^*(e_{i_1})\cdots a_r^*(e_{i_m})e^{\otimes k-m}, a^*(e_{j_1})\cdots a^*(e_{j_l})e^{\otimes k-l}\rangle_q.\end{aligned}$$

We now investigate the inner product terms appearing in the above sum. Let v be the largest index such that $e_{i_1} = e_{i_2} = \cdots = e_{i_{v-1}} = e$. By choice of z there must exist i_k with $k \leq m$ with $e_{i_k} \neq e$, so $v \leq m$. Taking the adjoint for $a_r^*(e_{i_1}) \cdots a_r^*(e_{i_v})$ we have

$$\begin{aligned} & \langle a_r^*(e_{i_1}) \cdots a_r^*(e_{i_m}) e^{\otimes k-m}, a^*(e_{j_1}) \cdots a^*(e_{j_l}) e^{\otimes k-l} \rangle_q \\ = \; & \langle a_r^*(e_{i_{v+1}}) \cdots a_r^*(e_{i_m}) e^{\otimes k-m}, a_r(e_{i_v}) \cdots a_r(e_{i_1})(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes k-l}) \rangle_q \\ = \; & \langle a_r^*(e_{i_{v+1}}) \cdots a_r^*(e_{i_m}) e^{\otimes k-m}, a_r(e_{i_v}) a_r(e)^{v-1}(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes k-l}) \rangle_q . \end{aligned}$$

We consider $a_r(e)^{v-1}(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes k-l})$. Each application of $a_r(e)$ cancels an occurrence of e multiplying by factor of q depending on n and the index of the cancelled tensor leg. Suppose $k \geq v$ so that we do not (necessarily) cancel all of the tensors.

We investigate how each term in the expanded result looks. As we are working with the right-annihilation, let us consider the indices as counted from the right. Let $(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes (k-l)})_{h,r}$ for $h = (h_1, \ldots, h_{v-1})$ denote the term such that the tensor leg at index h_1 as counted from the right has been cancelled, after which we repeat the procedure to the remaining tensor with $(h_2, \ldots, h_{v-1})$. Necessarily we must have $1 \leq h_i \leq k - i + 1$ for $i = 1, \ldots, v-1$. Whenever we cancel an $e_i \neq e$ the term yields zero. In this spirit, let δ_h be 0 if an $e_i \neq e$ is cancelled, and otherwise 1. Altogether, we can express

$$a_r(e)^{v-1}(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes k-l}) = \sum_{\substack{h \in \{1,\ldots,k\}^{v-1} \\ 1 \leq h_i \leq k-i+1}} \delta_h q^{-v+1+\sum_{i=1}^{v-1} h_i} (e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes (k-l)})_{h,r}.$$

In order to express the whole term, remains to apply $a_r(e_{i_v})$. Let us assume $k \geq n + p$, which ensures that $k \geq l + v$. By our choice of v we have $e_{i_v} \neq e$, so in order for it not to yield zero, we need to cancel an element from $e_{j_1} \otimes \cdots \otimes e_{j_l}$. As we have removed $v - 1$ tensor legs, the tensor now has a length of $k - v + 1$, of which at most $l - v + 1$ remain from $e_{j_1} \otimes \cdots \otimes e_{j_l}$.

The assumption $k \geq n + p$ guarantee that at least 1 tensor leg coming from $e^{\otimes (k-l)}$ remains in the term. In particular, there are at least $k - l - v + 1$ such legs. If we consider the definition of the right-annihilation operator, we find that the exponent in the q-term is at least $k - l - v + 1$ by virtue of the previous sentence. The remaining tensor now has length $k - l$, of which at most l of $e_{j_1} \otimes \cdots \otimes e_{j_l}$ remain and at most $k - l$ of $e^{\otimes (k-l)}$.

By Remark 4.4 and the fact that $C_q \geq 1$, $a \mapsto [a]_q!$ is increasing, we can conclude that

$$\|a_r(e_{i_v})(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes (k-l)})_{h,r}\|_q \leq C_q^{l/2} \sqrt{[k-l]_q} \leq C_q^p \sqrt{[k-l]_q!}.$$

Let us start absorbing constants, starting with $C_{q,p} = C_q^p$. Substituting the above in (4.4) together with the above result yields that

$$\|a_r(e_{i_v}) a_r(e)^{v-1}(e_{j_1} \otimes \cdots \otimes e_{j_l} \otimes e^{\otimes k-l})\|_q \leq \sum_{\substack{h \in \{1,\ldots,k\}^{v-1} \\ 1 \leq h_i \leq k-i+1}} |q|^{v-1+\sum_{i=1}^{v-1} h_i} C_{q,p} \sqrt{[k-l]_q!}.$$

We claim that we can bound $\sum_{\substack{h \in \{1,\ldots,k\}^{v-1} \\ 1 \leq h_i \leq k-i+1}} |q|^{v-1+\sum_{i=1}^{v-1} h_i}$ by a constant depending only on v and q. Considering the last index separately, we have that

$$\begin{aligned} \sum_{\substack{h \in \{1,\ldots,k\}^{v-1} \\ 1 \leq h_i \leq k-i+1}} |q|^{v-1+\sum_{i=1}^{v-1} h_i} &= |q|^{v-1} \sum_{\substack{h \in \{1,\ldots,k\}^{v-2} \\ 1 \leq h_i \leq k-i+1}} |q|^{\sum_{i=1}^{v-2} h_i} \sum_{h_{v-1}=1}^{k-v+2} |q|^{h_{v-1}} \\ &\leq |q|^{v-1} \sum_{\substack{h \in \{1,\ldots,k\}^{v-2} \\ 1 \leq h_i \leq k-i+1}} |q|^{\sum_{i=1}^{v-2} h_i} \frac{1}{1-|q|}. \end{aligned}$$

Applying an induction argument, we can argue that

$$\sum_{\substack{h\in\{1,\dots,k\}^{v-1}\\ 1\le h_i\le k-i+1}} |q|^{v-1+\sum_{i=1}^{v-1} h_i} \;\le\; \frac{|q|^{v-1}}{(1-|q|)^{v-2}} = C_{q,v}.$$

Since v itself depends on n, we have therefore found that

$$\|a_r(e_{i_v})a_r(e)^{v-1}(e_{j_1}\otimes\cdots\otimes e_{j_l}\otimes e^{\otimes k-l})\|_q \le C_{q,n,p}\sqrt{[k-l]_q!}.$$

We arrive at the end of the proof. Let $N > 2\max(n,p)$, and split the sum in I into the parts $A_i^{(N)}$ with $k < N$ and $B_i^{(N)}$ with $k \ge N$.

On the one hand we have

$$|A_i^{(N)}| \le \sum_{\substack{N>k\ge m\\ N>k\ge l}} |a^i_{k-2m+n}||a^i_{k-2l+p}|C_{N,q,n,p}.$$

by the fact that the sum is now finite. As η_i converges weakly to zero, we have that a^i_k converges to zero as $i\to\infty$. Thus, we find that $\lim_{i\to\infty} A_i^{(N)} = 0$.

We now consider $B_i^{(N)}$. As $\|e^{\otimes k}\| = \sqrt{[k]_q!}$, and $\|\eta\| = 1$, we necessarily have that $|a^i_k| \le \frac{1}{[k]_q!}$. We apply Cauchy-Schwarz and see that

$$\begin{aligned}
|B_i^{(N)}| &\le \sum_{k\ge N} |a^i_{k-2m+n}||a^i_{k-2l+p}|\frac{[k-2m+n]_q!}{[k-m]_q!}\frac{[k-2l+p]_q!}{[k-l]_q!}\|a_r^*(e_{i_{v+1}})\cdots a_r^*(e_{i_m})e^{\otimes k-m}\|_q\,\cdot\\
&\qquad \cdot\|a_r(e_{i_v})a_r(e)^{v-1}(e_{j_1}\otimes\cdots\otimes e_{j_l}\otimes e^{\otimes k-l})\|_q\\
&\le C_{q,n,p}\sum_{k\ge N}\frac{\sqrt{[k-2m+n]_q!}}{[k-m]_q!}\frac{\sqrt{[k-2l+p]_q!}}{[k-l]_q!}|q|^k\sqrt{[k-l]_q!}\|e_{i_{v+1}}\otimes\cdots\otimes e_{i_m}\otimes e^{\otimes k-m}\|_q\\
&\le C_{q,n,p}\sum_{k\ge N}\frac{\sqrt{[k-2m+n]_q!}}{[k-m]_q!}\frac{\sqrt{[k-2l+p]_q!}}{\sqrt{[k-l]_q!}}|q|^k\sqrt{[k-m]_q!}\\
&= C_{q,n,p}\sum_{k\ge N}\sqrt{\frac{[k-2m+n]_q!}{[k-m]_q!}}\sqrt{\frac{[k-2l+p]_q!}{[k-l]_q!}}|q|^k
\end{aligned}$$

Note that

$$\frac{[k-2m+n]_q!}{[k-m]_q!} = \prod_{i=k-m+1}^{k-2m+n}[i]_q \;\le\; \frac{1}{(1-|q|)^{n-m}} \;\le\; \frac{1}{(1-|q|)^n}.$$

Likewise, we find a similar expression for $[k-2l+p]_q!/[k-l]_q!$. We can conclude thus conclude

$$\begin{aligned}
|B_i^{(N)}| &\le C_{q,n,p}\sum_{k\ge N}\frac{1}{(1-|q|)^n(1-|q|)^p}\\
&\le C_{q,n,p}\frac{|q|^N}{(1-|q|)^{n+p}}\\
&= C_{q,n,p}|q|^N.
\end{aligned}$$

We emphasize that the constant $C_{q,n,p}$ does not depend on the choice of N. Consequently, we have

$$\limsup_{i\to\infty}|I_i| \;\le\; \lim_{i\to\infty}|A_i^{(N)}| + \limsup_{i\to\infty}|B_i^{(N)}| \;\le C_{q,n,p}|q|^N.$$

Since this must hold for arbitrary large N, we conclude that $I_i = 0$. □

Now that we have completed the proof of the proposition, we can show the proof of Theorem 4.1 and conclude the chapter.

Proof of Theorem 4.1. We now conclude with the theorem. Let $x = W(\xi) \in Z(\Gamma_q(H_{\mathbb{R}}))$. Then by Proposition 4.2 we have $W(\xi) \in \{W(e)\}''$ for any unit $e \in H_{\mathbb{R}}$. Hence, by our previous assertion we have $\xi \in E_e$ for any $e \in H_{\mathbb{R}}$ of norm one. But since $\dim H \geq 2$ and we can thus find two non-zero orthogonal elements, we necessarily have $\xi = 0$. That is, ξ is a multiple of Ω we conclude that x is a multiple of identity. $\square$

5

Deformations and bimodules

In this chapter we set up the deformation α_t that we will end up using to prove strong solidity of $\Gamma_q(H_\mathbb{R})$. We also introduce the coarse bimodule. The content of this chapter are largely based on [1].

5.1. The deformation α_t

Let us start by defining the necessary functions.

For $t \geq 0$ define the map $u_t : H_\mathbb{R} \to H_\mathbb{R}$ through

$$u_t h := e^{-t} h, \quad h \in H_\mathbb{R}.$$

Then u_t is obviously a contraction, so we satisfy the requirements for Theorem 3.16. Hence, said theorem provides us with a trace-preserving, completely positive, unital map $T_t = \Gamma_q(u_t)$. As we have that $u_s \circ u_t = u_{s+t}$, we can deduce that $T_s \circ T_t = T_{s+t}$ by multiplicativity of the functor. Moreover, $T_0 = \Gamma_q(1_{H_\mathbb{R}}) = 1_{\Gamma_q(H_\mathbb{R})}$. Therefore, $(T_t)_{t\geq 0}$ forms a semigroup of completely positive, unital maps on $\Gamma_q(H_\mathbb{R})$. From now on we refer to this function whenever we write T_t.

Definition 5.1. *Let us now consider $\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R})$, and set $R_t : H_\mathbb{R} \oplus H_\mathbb{R} \to H_\mathbb{R} \oplus H_\mathbb{R}$ as the rotation*

$$R_t = \begin{pmatrix} e^{-t} & -\sqrt{1-e^{-2t}} \\ \sqrt{1-e^{-2t}} & e^{-t} \end{pmatrix}.$$

By Theorem 3.16 this yields a group $(\Gamma_q(R_t))_{t\geq 0}$ of $*$-automorphisms of $\Gamma_q(H \oplus H)$. Let us denote $\alpha_t := \Gamma_q(R_t)$. We shall refer to α_t, sometimes referred to as an s-malleable deformation in the literature, as the deformation or the deformation of $\Gamma_q(H_\mathbb{R})$.

Now that we have introduced $\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R})$, we are interested in how it interacts with $\Gamma_q(H_\mathbb{R})$. By considering $\Gamma_q(\iota)$ of the canonical embedding $\iota : H_\mathbb{R} \to H_\mathbb{R} \oplus H_\mathbb{R}$ in the first coordinate we can embed $\Gamma_q(H_\mathbb{R})$ in the first coordinate in $\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R})$. Additionally, we observe the following relation between α_t and T_t:

Similar to the above we can project elements from $\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R})$ onto the first coordinate. That is, let $P_1 : H_\mathbb{R} \oplus H_\mathbb{R} \to H_\mathbb{R}$ be the projection onto the first coordinate, such that $\Gamma_q(P_1)$ gives us the desired projection. Then $T_t(x) = \Gamma_q(P_1) \circ \alpha_t(x)$, which one can verify using Wick words.

Definition 5.2. *Let $P \subset \mathcal{M}$ be a von Neumann subalgebra, and let $(\theta_t)_{t\geq 0}$ be a continuous family of completely positive maps $\theta_t : \mathcal{M} \to \mathcal{M}$. Then P is rigid with respect to $(\theta_t)_{t\in\mathbb{R}}$ if*

$$\lim_{t\downarrow 0} \sup_{\substack{x\in P \\ \|x\|\leq 1}} \|\theta_t(x) - x\|_{L^2(M)} = 0.$$

We will exclusively apply this definition with $\Gamma_q(H_\mathbb{R})$ and $(T_t)_{t\geq 0}$ or $\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R})$ and the deformations $(\alpha_t)_{t\geq 0}$. As it is now always easy to check the above definition, we formulate some equivalent conditions in the case that we are working with α_t and $H_\mathbb{R}$ is finite-dimensional.

Theorem 5.3. *Let $H_\mathbb{R}$ be finite-dimensional and $B \subset \Gamma_q(H_\mathbb{R})$ be a von Neumann subalgebra. Then the following are equivalent:*

(i) B is rigid with respect to α_t.

(ii) B is rigid with respect to T_t.

(iii) B is atomic.

Proof. We start by showing that *(ii)* $\Rightarrow$ *(iii)*. Suppose B is diffuse, and B is rigid with respect to T_t. Let $U(B)$ denote the set of unitaries in B. By definition of rigidity there exists t_0 such that

$$\|T_t x - x\|_2 \leq \frac{1}{2}$$

for all $t < t_0$ and $x \in U(B)$.

Let $A \subset B$ be a maximal abelian subalgebra of B. We claim that A must be diffuse as well, which we argue by contradiction. Suppose we can find a minimal projection $p \in A$. Then as p is not minimal in B by assumption, pBp must contain elements different from multiples of p. Indeed, if $pBp = \mathbb{C}p$, then we could find no smaller projection in B than p, contradicting that B is diffuse. Thus, let $x \in pBp$ be such that it is not a multiple of p.

We claim that x commutes with elements in A. Let $a \in A$. Then by orthogonality of p we can write $a = p^\perp a p^\perp + pap$. Moreover, since p is minimal we have $pAp = \mathbb{C}p$, so we can write $a = \lambda p$ for some scalar p. Using that $pxp = x$, we note:

$$\begin{aligned} ax &= apxp \\ &= (p^\perp a p^\perp + pap)pxp \\ &= 0 + papxp \\ &= \lambda pxp. \end{aligned}$$

Repeating the same computation with xa reveals that x commutes with elements from A, which contradicts the assumption that A is a maximal abelian subalgebra.

Hence, by Theorem 2.16 we have that $A \simeq L^\infty(0,1)$, where $L^\infty(0,1)$ is equipped with the Lebesgue measure. This identification allows us to find a sequence $(x_n)_{n=1}^\infty$ in A such that $\|x_n\| = 1$, $\tau(x_n) = 0$ and x_n converges weakly to zero in $L^2(\Gamma_q(H_\mathbb{R}))$. In $L_\infty(0,1)$, consider the function $t \mapsto e^{int}$, and define x_n to be the associated sequence element in A. Firstly, the norm is indeed 1. Secondly, as the trace in $L_\infty(0,1)$ being given by Lebesgue integration, the trace is zero. And lastly, the Riemann-Lebesgue Lemma yields weak convergence to zero.

We now argue that $\|T_t x_n\|_2 \to 0$ as $n \to \infty$. Note that all the eigenvalues of T_t are of the form e^{-kt}, for $k \geq 0$. The corresponding eigenvectors are direct sums of tensors with a total length of k, and result of H being finite-dimensional it follows that the corresponding eigenspaces are finite-dimensional. Thus, we can conclude that T_t is compact. But a compact operator maps weakly convergent sequences to norm-convergent sequences, and thus $\|T_t x_n\|_2 \to 0$ as $n \to \infty$ for any fixed t. We conclude this implication by noting that

$$\|T_t(x_n) - x_n\|_2 = \|x_n\|_2 = 1, \quad \text{as } n \to \infty.$$

which contradicts (5.1).

Secondly, we prove that *(iii)* $\Rightarrow$ *(ii)*. Suppose B is atomic. Using Chapter V.1 we can write

$$B \simeq \bigoplus_{i \in I} B(H_i)$$

where I is an index set. We can derive some extra properties here. Firstly, we necessarily have that $\dim H_i < \infty$ for all $i \in I$, as $\Gamma_q(H_\mathbb{R})$ is a finite von Neumann algebra. Secondly, as the predual of $\Gamma_q(H_\mathbb{R})$

is separable by virtue of $\dim(H_{\mathbb{R}}) < \infty$, we can conclude that I is countable. That is, we can take $I \subset \mathbb{N}$.

Let z_i be the projection on coordinate i in $\bigoplus_{i\in I} B(H_i)$. Then $\{z_i\}_{i\in I}$ forms an orthogonal family, $z_i B z_i = B(H_i)$, and $\sum_{i\in I} \tau(z_i) = 1$. For a set $F \subset I$ define $\iota_F : B \to B$ by

$$\iota_F(x) := \sum_{i\in F} z_i x z_i.$$

Now let $\varepsilon > 0$. By the fact that $\sum_{i\in I} \tau(z_i) = 1$ we can find a finite $F \subset I$ such that

$$\sum_{i\in F^C} \tau(z_i) < \varepsilon.$$

A crude estimation then shows that for $x \in P$ such that $\|x\| \leq 1$ we have

$$\|\iota_{F^C}(x)\|_2^2 < \sum_{i\in F^C} \tau(z_i) < \varepsilon.$$

Let us write $x = \iota_F(x) + \iota_{F^C}(x)$. Then we have:

$$\begin{aligned} \|T_t(x) - x\|_2 &= \|(T_t(\iota_F(x)) - \iota_F(x)) + (T_t(\iota_{F^C}(x)) - \iota_{F^C}(x))\|_2 \\ &\leq \|(T_t - id)(\iota_F(x))\|_2 + \|(T_t - id)(\iota_{F^C}(x))\|_2 \end{aligned}$$

Since the range of ι_F is finite-dimensional, it follows that $\|(T_t - id)(\iota_F(x))\|_2 \to 0$ uniformly in t, as T_t weakly converges to the identity. For the other term, by the properties of T_t we deduce that

$$\|T_t(\iota_{F^C}(x))\|_2 < \sqrt{\varepsilon}.$$

As T_t is a contraction, we have that $\|(T_t - id)(\iota_{F^C}(x))\|_2 \leq 2\|\iota_{F^C}(x)\|_2$. Since this holds for any $\varepsilon > 0$, we conclude that

$$\lim_{t\downarrow 0} \sup_{\substack{x\in B \\ \|x\|\leq 1}} \|T_t(x) - x\|_2 = 0.$$

That is, B is rigid with respect to T_t.

The computations to show that $(i) \Leftrightarrow (ii)$ are more straightforward. Assume B is rigid with respect to α_t. Let $x \in \Gamma_q(H_{\mathbb{R}})$. From the embedding it follows that

$$\langle x, \alpha_t(x)\rangle = \langle x, T_t(x)\rangle.$$

Applying the above and Cauchy Schwarz, we have:

$$\begin{aligned} \|\alpha_t(x) - x\|_2^2 &= 2\langle x, x\rangle - \langle \alpha_t(x), x\rangle - \langle x, \alpha_t(x)\rangle \\ &= 2(\langle x, x\rangle - \langle x, T_t(x)\rangle) \\ &= 2\langle x, x - T_t(x)\rangle \\ &\leq 2\|x\|\|T_t(x) - x\|_2. \end{aligned}$$

The assumption that $\|T_t(x) - x\|_2$ converges uniformly to 0, in combination with the above inequality, yields that the same holds for α_t.

We can in fact obtain the reverse inequality of the above. Let $E_{\Gamma_q(H_{\mathbb{R}})}$ be the conditional expectation of $\Gamma_q(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$ onto $\Gamma_q(H_{\mathbb{R}})$ as given by Proposition 2.12. Then for $x \in \Gamma_q(H)$ we have

$$\begin{aligned} \|T_t(x) - x\|_2 &= \|E_{\Gamma_q(H)}(\alpha_t(x) - x)\|_2 \\ &\leq \|\alpha_t(x) - x\|_2, \end{aligned}$$

and by the same argument α_t converges uniformly to the identity under the assumption B is rigid with respect to T_t. $\square$

5.2. Bimodules for von Neumann algebras

Let M and N be von Neumann algebras, and let N^{op} denote the opposite algebra. That is, in N^{op} the multiplication is reversed. Recall the binormal tensor product from Chapter 2. We define an M-N-bimodule as the $*$-representation of $M \otimes_{bin} N^{op}$ on a Hilbert space, where M acts on the first tensor leg, and N acts on the second tensor leg with reversed multiplication. One particular bimodule that will play a very important role is the coarse bimodule:

Definition 5.4. *Let M and N be von Neumann algebras. We call the M-N-bimodule $L^2(M) \otimes L^2(N)$ the coarse bimodule.*

For two M-N-modules we can define a notion of weak containment of one bimodule in the other. We define it as follows:

Definition 5.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be M-N-bimodules. If for all $\varepsilon > 0$, all $\xi \in \mathcal{H}$, and all finite subsets $F \subset M$ and $E \subset N$, there exist $\eta_1, \dots \eta_n \in \mathcal{K}$ such that*

$$|\langle \xi, x\xi y\rangle - \sum_{j=1}^{n} \langle \eta_j, x\eta_j y\rangle| < \varepsilon, \qquad \text{for all } x \in F \text{ and } y \in E,$$

we say that $\mathcal{H}$ is weakly contained in $\mathcal{K}$. Let us denote this as $\mathcal{H} \prec \mathcal{K}$

Remark 5.6. *If two M-N-bimodules $\mathcal{H}_1$ and $\mathcal{H}_2$ are weakly contained in $\mathcal{K}$, then we can see $\mathcal{H}_1 \oplus \mathcal{H}_2$ as an M-N-bimodule by the canonically extending the action. Then, for $\mathcal{H}_1 \oplus \mathcal{H}_2$ it is easily checked that it is weakly contained in $\mathcal{K}$.*

Recall that $S(A \otimes_{max} B)$ denotes the set of states on $A \otimes_{max} B$. Let $\mathcal{H}$ be an M-N-bimodule, and let $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$. We can define an element φ_ξ of $S(M \otimes_{max} N)$ by the linear extension of

$$\varphi_\xi(x \otimes y) := \langle \xi, x\xi y\rangle.$$

From here, we define a map $T_{\varphi_\xi} : M \to N$ through

$$T_{\varphi_\xi}(x)(y) = \varphi_\xi(x \otimes y), \quad x \in M,\ y \in N.$$

In particular, for $x \in M$ and $y \in N$ we therefore have $T_{\varphi_\xi}(x)(y) = \langle \xi, x\xi y\rangle$. As we will have great interest in if a bimodule is weakly contained in the coarse bimodule, we derive equivalent conditions to give us more options to verify weak containment, given some conditions on the von Neumann algebras.

Lemma 5.7. *Let M and N be finite von Neumann algebras with separable predual and $\mathcal{H}$ be an M-N-bimodule. Consider the following statements:*

(i) For $\xi \in \mathcal{H}$, $T_\xi = T_{\varphi_\xi}$ extends to an element $S_2(L^2(M), L^2(N))$.

(ii) For $\xi \in \mathcal{H}$ such that $\|\xi\|_{\mathcal{H}} = 1, \varphi_\xi \in S(M \otimes_{max} N)$ is continuous with respect to the minimal tensor norm.

(iii) $\mathcal{H} \prec L^2(M) \otimes L^2(N)$.

Then (i) implies the others, and (ii) and (iii) are equivalent.

Proof. First we show that $(i) \Rightarrow (ii)$. We set up an identification of the coarse bimodule $L^2(M) \otimes L^2(N)$ with $S_2(L^2(M), L^2(N))$. Indeed, for a pure tensor $\xi \otimes \eta \in Ł^2(M) \otimes L^2(N)$ let $\theta_{\xi,\eta} : L^2(M) \to L^2(N)$ be defined through the linear extension of

$$\theta_{\xi,\eta}(\zeta) = \langle \zeta, \overline{\xi}\rangle\eta, \quad \zeta \in L^2(M).$$

clearly $\theta_{\xi,\eta} \in S_2(L^2(M), L^2(N))$. Conversely, S_2 operators allow for an approximation in the form of a linear combination of $\theta_{\xi,\eta}$ for $\xi \in L^2(M)$, $\eta \in L^2(N)$ (see Chapter 2.4 of [13]).

By the assumption that T_{φ_ξ} can be extended to an element in $S_2(L^2(M), L^2(N))$. Using the identification $S_2(L^2(M), L^2(N)) \simeq L^2(M) \otimes L^2(N) \simeq L^2(M \otimes_{bin} N^{op})$, we can find $\zeta \in L^2(M \otimes_{bin} N^{op})$ that corresponds with T_{φ_ξ}. Moreover, as M and N are finite, we have that $\zeta \in L^1(M \otimes_{bin} N^{op})$. Lastly, by Proposition 2.18 we can identify $L^1(M\bar{\otimes}N)$ with $M \otimes_{\min} N$ and conclude that $\|\varphi_\xi\|_{min^*} \leq \|T_{\varphi_\xi}\|_{S_2}$.

We continue by showing that $(iii) \Rightarrow (ii)$. If $\mathcal{H} \prec L^2(M) \otimes L^2(N)$, using the definition of weak containment we can find $\eta_1, \cdots \eta_n \in L^2(M) \otimes L^2(N)$ such that

$$|\varphi_\xi(x \otimes y) - \sum_{j=1}^{n} \varphi_{\eta_j}(x \otimes y)| < \varepsilon$$

for all $x \in E$ finite, $y \in F$ and $\varepsilon > 0$. Let π be the $*$-representation as in the definition of the coarse bimodule. We have:

$$\varphi_{\eta_j}(x \otimes y) = \langle \eta_j, \pi(x \otimes y)\eta_j \rangle,$$

and hence φ_{η_j} lies in the predual of $B(L^2(M) \otimes L^2(N))$. As a limit of elements of the form $\sum_{j=1}^{n} \varphi_{\eta_j}$, Section 2 in [9] yields that φ_ξ is min-continuous.

Lastly, we prove that $(ii) \Rightarrow (iii)$. Assume φ_ξ is continuous with respect to the min-norm. Consequently, it can be extended to $M \otimes_{min} N$ and is therefore an element of $M \otimes_{min} N$. By Proposition 2.18 we have that $M \otimes_{min} N$ equals the closure of $M \otimes N$ with respect to the norm of $B(L^2(M) \otimes L^2(N))$. Using the Hahn-Banach extension theorem we can extend φ_ξ to the entirety of $B(L^2(M) \otimes L^2(N))$. That is, $\varphi_\xi \in B(L^2(M) \otimes L^2(N))^*$.

By Goldstine's Theorem (Theorem 2.5), we have that the predual of $B(L^2(M) \otimes L^2(N))$ lies dense in $B(L^2(M) \otimes L^2(N))^*$ with respect to the weak* topology. For $\eta \in L^2(M) \otimes L^2(N)$ let us define a continuous linear functionals θ_η on $B(L^2(M) \otimes L^2(N))$ through

$$\theta_\eta(\zeta) := \langle \eta, \zeta\eta \rangle, \quad \zeta \in B(L^2(M) \otimes L^2(N)).$$

Then

$$X := \mathrm{Span}\{\theta_\eta \mid \eta \in L^2(M) \otimes L^2(N)\}$$

lies dense in the predual of $B(L^2(M) \otimes L^2(N))$ with respect to the norm, and therefore it lies weak* dense in $B(L^2(M) \otimes L^2(N))^*$.

Thus, we can find $(\psi_i)_{i=1}^\infty$ in X in the form $\psi_i = \sum_{j=1}^{n_i} \theta_{\eta_{ij}}$ such that for any $x \in B(L^2(M) \otimes L^2(N))$ we have pointwise convergence:

$$\psi_i(x) \to \varphi_\xi(x), \text{ as } i \to \infty.$$

This holds for finite sets of $B(L^2(M) \otimes L^2(N))$, so in particular for any $\varepsilon > 0$ and $E \subset M$, $F \subset Y$ finite we can find i such that:

$$|\langle \xi, x\xi y \rangle - \sum_{j=1}^{n_i} \langle \eta_{ij}, x\eta_{ij} y \rangle| \ = \ |\varphi_\xi(x \otimes y) - \psi_i(x \otimes y)| \ < \ \varepsilon,$$

for all $x \in E, y \in F$, which yields us weak containment.

□

6

Strong solidity of $\Gamma_q(H_\mathbb{R})$

In this chapter we reach the core result of the thesis, which is to show that $\Gamma_q(H_\mathbb{R})$ is strongly solid. This result is based on Avsec's paper on the subject [1]. We replicate a number of the proofs, and make improvements where possible. In particular, we fill some gaps in the proofs and offer more expanded proofs.

The definition of strong solidity is in order now. For this we first need the concept of amenability:

Definition 6.1. *Let M be a von Neumann algebra with tracial state τ. Then M is is called amenable if there exists a state $\varphi \in B(L^2(M))$ such that $\varphi\mid_M = \tau$ and $\varphi(ax) = \varphi(xa)$ for all $a \in M$ and $x \in L^2(M)$.*

At last, we can define one of the central definitions in this thesis.

Definition 6.2. *Let M be a von Neumann algebra, and let $P \subset M$ be a subalgebra. The normalizer of P in M is defined as*

$$\mathcal{N}_M(P) := \{u \in U(M) \mid u^*Pu = P\}.$$

We call M strongly solid if for all diffuse, amenable subalgebras $P \subset M$, $\mathcal{N}_M(P)$ generates an amenable subalgebra.

Sadly, from here on forth we will be restricted to finite dimensional $H_\mathbb{R}$, as this will prove necessary in some of the proofs that are to follow.

6.1. Weak containment in the coarse bimodule

We reiterate that $H_\mathbb{R}$ is from now on taken to be finite-dimensional. Let $L^2(\Gamma_q(H_\mathbb{R} \oplus 0))^\perp$ denote the orthocomplement of $L^2(\Gamma_q(H_\mathbb{R} \oplus 0))$ in $L^2(\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R}))$.

Definition 6.3. *Let $m \geq 0$. Define the following subspaces of $L^2(\Gamma_q(H_\mathbb{R} \oplus 0))^\perp$:*

$$F_m := \mathrm{Span}\{W(f_1 \otimes \cdots \otimes f_n) \mid \text{ m legs of $f_1 \otimes \cdots \otimes f_n$ are from $0 \oplus H_\mathbb{R}$ and $n-m$ from $H_\mathbb{R} \oplus 0$}\}^{\|\cdot\|_2},$$

and

$$E_m := \bigoplus_{k=0}^{m} F_k.$$

We can view F_k and E_k as $\Gamma_q(H_\mathbb{R})$-$\Gamma_q(H_\mathbb{R})$-bimodules by restricting the action of $\Gamma_q(H_\mathbb{R} \oplus H_\mathbb{R})$. This section will be dedicated to showing that E_m is weakly contained in the coarse bimodule $L^2(\Gamma_q(H_\mathbb{R})) \oplus \Gamma_q(H_\mathbb{R})$ for appropriate m.

Theorem 6.4. *Let* $m > -\frac{\log(d)}{2\log|q|}$ *with* $d = \dim(H)$. *Then* $E_{m-1}^{\perp} \prec L^2(\Gamma_q(H_{\mathbb{R}}))\otimes L^2(\Gamma_q(H_{\mathbb{R}}))$.

Throughout this section we will identify elements from E_k and F_k with elements from $\mathcal{F}_q(H)$ through the vacuum vector. To simplify the notation, we shall set $M := \Gamma_q(H_{\mathbb{R}})$, $\widetilde{M} := \Gamma_q(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$. We shall need a number of lemmas before proving the proposition. The following lemma is dependent on $H_{\mathbb{R}}$ being finite-dimensional:

Lemma 6.5. *Let* $K = \oplus_{j=0}^{\infty} H_j$ *be a Hilbert space and* $A = [A_{ij}] : K \to K$ *be an operator such that*

(i) $A_{ij} = 0$ *if* $|i - j| \geq L$ *for a fixed* $L > 0$.

(ii) There exists j_0 *such that*

$$\|A_{ij}\| \leq Cr^{jk}$$

for all $j \geq j_0$ *and for constants* $0 < r < 1$, k, *and* C *independent of* i *and* j.

(iii) $\dim(H_j) = d^j$ *for a fixed* d.

Then $A \in S_p(K)$ *for* $p > -\frac{\log(d)}{k\log(r)}$.

Proof. Let $K_1 = \oplus_{j=0}^{j_0-1} H_j$ and $K_2 = \oplus_{j\geq j_0} H_j$. Then

$$\begin{aligned} \|A\|_{S_p} &\leq \|A : K_1 \to K_1\|_{S_p} + \|A : K_1 \to K_2\|_{S_p} + \|A : K_2 \to K_1\|_{S_p} + \|A : K_2 \to K_2\|_{S_p} \\ &= \|A : K_1 \to K_1\|_{S_p} + \|A : K_1 \to K_2\|_{S_p} + \|A^* : K_1 \to K_2\|_{S_p} + \|A : K_2 \to K_2\|_{S_p} \quad (6.1) \end{aligned}$$

As K_1 is finite-dimensional, we can estimate the first three terms using a constant, and so only the last term remains. Suppose $p > -\frac{\log(d)}{k\log(r)}$, or equivalently $dr^{pk} < 1$. Applying the min-max theorem for singular values we obtain

$$\begin{aligned} \|A : K_2 \to K_2\|_{S_p} &\leq C' + \sum_{l=-L}^{L} \left\| \sum_{j\geq L} A_{j,j+l} \right\|_{S_p} \\ &\leq C' + \sum_{l=-L}^{L} \left(\sum_{j=j_0}^{\infty} d^j \left(Cr^{(j+l)k}\right)^p \right)^{\frac{1}{p}} \\ &= C' + C'' \left(\sum_{j=j_0}^{\infty} C^p d^j r^{jkp} \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

□

Let E_M denote the conditional expectation on M, and define $\Phi_{\xi,\eta} : L^p(M) \to L^p(M)$ by $\Phi_{\xi,\eta}(x) := E_M(W(\xi)^* x W(\eta))$ for $\xi, \eta \in F_k$. We intend to show that we can apply the above lemma to $\Phi_{\xi,\eta}$ for $\xi, \eta \in F_k$ for $p = 2$ and for $k \geq -\frac{\log(d)}{2\log(|q|)}$.

The following proposition will yield us the result:

Proposition 6.6. *Let* $\xi \in (H\oplus H)^{\otimes n_1} \cap F_k$, $\eta \in (H\oplus H)^{\otimes n_2} \cap F_k$, *and* $\zeta_1 \in (H\oplus 0)^{\otimes m}$, *and* $\zeta_2 \in (H\oplus 0)^{\otimes l}$. *Then we have that*

$$|\langle \zeta_2, \Phi_{\xi,\eta}(\zeta_1)\rangle_q| \quad \leq \quad C_{q,\xi,\eta}|q|^{jk}\|\zeta_1\|_q\|\zeta_2\|_q,$$

for a constant $C_{q,\xi,\eta}$ *depending only on* q, ξ *and* η.

Recall how the splitting of $H^{\otimes n}$ into 2 parts in Chapter 4. We perform a similar construction, but this time we consider $H^{\otimes a} \otimes H^{\otimes b} \otimes H^{\otimes c}$ such that $a + b + c = n$. Recall that with H_q^n we denote $H^{\otimes n}$ equipped with the q-inner product.

Consider the space $H_q^a \otimes H_q^b \otimes H_q^c$. For $\xi \otimes \eta \otimes \zeta, \xi' \otimes \eta' \otimes \zeta' \in H_q^a \otimes H_q^b \otimes H_q^c$ we have the inner product

$$\langle \xi \otimes \eta \otimes \zeta, \xi' \otimes \eta' \otimes \zeta' \rangle_{H_q^n} = \langle \xi \otimes \eta \otimes \zeta, P_q^a \xi' \otimes P_q^b \eta' \otimes P_q^c \zeta' \rangle_0.$$

Note that this does not coincide with the inner product on $H_q^{\otimes(a+b+c)}$. To link the two, we define the unique operator $R_{a,b,c}$ through its adjoint by

$$P_q^{a+b+c} = (P_q^a \otimes P_q^b \otimes P_q^c) R_{a,b,c}^*.$$

similarly to how we defined $R_{n,k}$ in Chapter 4.

To aid us in proving Proposition 6.6 we need the following lemma:

Lemma 6.7. *For $\nu \in H^{\otimes(a+b+c)}$ we have*

$$\|\nu\|_{H_q^{a+b+c}} \leq C_q \|\nu\|_{H_q^a \otimes H_q^b \otimes H_q^c}$$

for a constant C_q depending only on q.

Proof. Recall the operator $R_{n,k}^*$ from Chapter 4 and the explicit form for $R_{n,k}$. On the hand we have that

$$P_q^{a+b+c} = R_{a+b+c,c}(R_{a+b,b} \otimes 1_{H_q^c})(P_q^a \otimes P_q^b \otimes P_q^c).$$

But then we must have

$$R_{a,b,c}^* = (R_{a+b,b}^* \otimes 1_{H_c^q}) R_{a+b+c,c}^*.$$

Applying Lemma 4.3 twice we obtain the result. □

The following operator will be needed to effectively use $R_{a,b,c}^*$:

Definition 6.8. *For $j \in \mathbb{N}$ we define*

$$m_n : H_q^{\otimes j} \otimes H_q^{\otimes j} \to \mathbb{C}$$

as the inner product pairing, where $m_j(v \otimes w) = \langle v, w \rangle$. Suppose now have a triple tensor product space $H_q^n \otimes H_q^m \otimes H_q^k$. For $1 \leq a < b \leq 3$ and appropriate j we define m_j^{ab} as the operator that applies the inner product pairing to the right end of the a-th space after splitting, and the left end of the b-th space after splitting. For example, applying m_j^{13} to an element from $H_q^{n-j} \otimes H_q^j \otimes H_q^m \otimes H_q^j \otimes H_q^{k-j}$ would pair elements from both H_q^j spaces, and applying the identity on the remainder, mapping to an element in $H_q^{n-j} \otimes H_q^m \otimes H_q^{k-j}$

We now set out to prove the proposition.

Proof of Proposition 6.6. Let $\xi \in (H \oplus H)^{\otimes n_1} \cap F_k$, $\eta \in (H \oplus H)^{\otimes n_2} \cap F_k$ and $\zeta_1 \in (H \oplus 0)^{\otimes m}$, and $\zeta_2 \in (H \oplus 0)^{\otimes l}$. Let E_M denotes the conditional expectation of $\widetilde{M}$ onto M. By that fact that $\zeta_1 \in (H \oplus 0)^m$ we have that

$$\begin{aligned}
\langle \zeta_2, \Phi_{\xi,\eta}(\zeta_1) \rangle_q &= \langle W(\zeta_2)\Omega, E_M(W(\xi)^* W(\zeta_1) W(\eta))\Omega \rangle_q \\
&= \langle E_M W(\zeta_2)\Omega, W_\xi^* W(\zeta_1) W(\eta)\Omega \rangle_q \\
&= \langle W(\zeta_2)\Omega, W(\xi)^* W(\zeta_1) W(\eta)\Omega \rangle_q. \\
&= \langle \zeta_2, W(\xi^*) W(\zeta_1) W(\eta)\Omega \rangle_q.
\end{aligned}$$

Proposition 4.9 in [6] states that

$$\begin{aligned}
&\langle \zeta_2, W(\xi)^* W(\zeta_1) W(\eta)\Omega \rangle_q \\
= \; &\langle \zeta_2, \sum_{\substack{j,r,s \geq 0 \\ r+s \leq n_1 \\ s+j \leq m \\ j+r \leq n_2}} q^{r(m-j-s)} m_r^{13} m_s^{12} m_j^{23} R_{n_1-r-s,r,s}^*(\xi^*) \otimes R_{s,m-s-j,j}^*(\zeta_1) \otimes R_{j,r,n_2-j-r}^*(\eta) \rangle_q.
\end{aligned}$$

Note that the terms in the above sum in which a tensor from $0 \oplus H$ occurs yield zero in the inner product by virtue of ζ_2. Thus, we only need consider the terms in which all of the tensors in $0 \oplus H$ from ξ get cancelled against all of the tensors in $0 \oplus H$ from η. As both ξ and η have exactly k such tensors, we can see that the non-zero terms must satisfy the condition that $r \geq k$. Thus, by applying Cauchy-Schwarz we find

$$\begin{aligned} & |\langle \zeta_2\Omega, W(\xi)^*W(\zeta_1)W(\eta)\Omega\rangle_q| \\ \leq \quad & \|\zeta_2\|_{H_q^l}\| \sum_{\substack{j,s\geq 0, r\geq k \\ r+s\leq n_1 \\ s+j\leq m \\ j+r\leq n_2}} q^{r(m-j-s)} m_r^{13} m_s^{12} m_j^{23} R^*_{n_1-r-s,r,s}(\xi^*) \otimes R^*_{s,m-s-j,j}(\zeta_1) \otimes R^*_{j,r,n_2-j-r}(\eta)\|_q \end{aligned}$$

Now that we have obtained $\|\zeta_2\|_{H_q^l}$, we continue by estimating the norm of the sum. Since we can at most remove $n+k$ elements through the pairing of ζ_1 with ξ and η, we have that $m-j-s \geq m-n_1-n_2$ and therefore $q^{r(m-j-s)} \leq q^{r(m-n_1-n_2)}$. In combination with 6.7 and the triangle inequality we see:

$$\begin{aligned} & \| \sum_{\substack{j,s\geq 0, r\geq k \\ r+s\leq n_1 \\ s+j\leq m \\ j+r\leq n_2}} q^{r(m-j-s)} m_r^{13} m_s^{12} m_j^{23} R^*_{n_1-r-s,r,s}(\xi^*) \otimes R^*_{s,m-s-j,j}(\zeta_1) \otimes R^*_{j,r,n_2-j-r}(\eta)\|_q \\ \leq \quad & C_q|q|^{r(m-n_1-n_2)} \sum_{\substack{j,s\geq 0, r\geq k \\ r+s\leq n_1 \\ s+j\leq m \\ j+r\leq n_2}} \|m_r^{13} m_s^{12} m_j^{23} R^*_{n_1-r-s,r,s}(\xi^*) \otimes R^*_{s,m-s-j,j}(\zeta_1) \otimes \\ & \qquad \otimes R^*_{j,r,n_2-j-r}(\eta)\|_{H_q^{n_1-r-s}\otimes H_q^{m-s-j}\otimes H_q^{n_2-j-r}} \end{aligned}$$

As we have previously established that $\|R^*_{n,k,l}\| \leq C_q$, all that remains is to find an estimate for m_i^{ab}. Lemma 4.11 in [6] offers an estimate for m_i, which is dependent on the dimension of the spaces involved. As j, r, s are bounded by either n_1 or n_2, this lemma immediately yields the result as H is finite-dimensional.
Thus far, we have that

$$\begin{aligned} & \|m_r^{13} m_s^{12} m_j^{23} R^*_{n_1-r-s,r,s}(\xi^*) \otimes R^*_{s,m-s-j,j}(\zeta_1) \otimes R^*_{j,r,n_2-j-r}(\eta)\|_{H_q^{n_1-r-s}\otimes H_q^{m-s-j}\otimes H_q^{n_2-j-r}} \\ \leq \quad & C_{\xi,\eta}\|R^*_{n_1-r-s,r,s}(\xi^*) \otimes R^*_{s,m-s-j,j}(\zeta_1) \otimes R^*_{j,r,n_2-j-r}(\eta)\|_{H_q^{n_1-r-s}\otimes H_q^{m-s-j}\otimes H_q^{n_2-j-r}} \\ \leq \quad & C_{q,\xi,\eta}\|\xi^* \otimes \zeta_1 \otimes \eta\|_{H_q^{n_1}\otimes H_q^m \otimes H_q^{n_2}} \\ = \quad & C_{q,\xi,\eta}\|\xi^*\|_{H_q^{n_1}}\|\zeta_1\|_{H_q^m}\|\eta\|_{H_q^{n_2}}. \end{aligned}$$

Lastly, summarising our results we see:

$$\begin{aligned} |\langle \zeta_2, W(\xi)^*W(\zeta_1)W(\eta)\Omega\rangle_q| \quad & \leq \quad C_{q,\xi,\eta}|q|^{r(m-n_1-n_2)}\|\zeta_2\|_{H_q^l} \sum_{\substack{j,s\geq 0, r\geq k \\ r+s\leq n_1 \\ s+j\leq m \\ j+r\leq n_2}} \|\xi^*\|_{H_q^{n_1}}\|\zeta_1\|_{H_q^{\otimes m}}\|\eta\|_{H_q^{\otimes n_2}}. \\ & \leq \quad C_{q,\xi,\eta}|q|^{km}\|\zeta_1\|_{H_q^m}\|\zeta_2\|_{H_q^l}, \end{aligned}$$

as we can bound the number of terms in the sum depending on only n and k. □

We are now equipped to prove the theorem.

Proof of Theorem 6.4. We first show that $\Phi_{\xi,\eta}$ is Hilbert-Schmidt by applying Lemma 6.5 to it. Let $\zeta_1 \in H^{\otimes m}$ and $\zeta_2 \in H^{\otimes l}$. From the fact that

$$\langle \zeta_2, \Phi_{\xi,\eta}(\zeta_1)\rangle_q = \langle W(\zeta_2)\Omega, W(\xi)^*W(\zeta_1)W(\eta)\Omega\rangle_q = \langle \zeta_2, W(\xi)^*W(\zeta_1)\eta\rangle_q$$

we can easily see that $\langle \zeta_2, \Phi_{\xi,\eta}(\zeta_1)\rangle_q = 0$ if $|m-l| \geq L$ for some L depending on n and k that depends only on ξ and η. From this we deduce that the first criterion of Lemma 6.5 is satisfied. The third criterion is satisfied trivially. The last statement follows directly from 6.6.

We will now argue that that F_k is weakly contained in $L^2(M) \otimes L^2(M)$. Let $\xi \in F_k$ for $k \geq -\frac{\log(d)}{2\log(|q|)}$. Take $x, y \in M$. With the M-M-bimodule structure on F_k, on the one hand we have

$$\begin{aligned}\langle \xi, x\xi y\rangle &= \tau(W(\xi)^* x W(\xi) y) \\ &= \tau(E_M(W(\xi)^* x W(\xi)) y) \\ &= \tau(\Phi_{\xi,\xi}(x) y).\end{aligned}$$

On the other hand, we recall φ_ξ and T_{φ_ξ} from Lemma 5.7, whereas $T_{\varphi_\xi}(x)(y) = \langle \xi, x\xi y\rangle$. That is, $\Phi_{\xi,\xi}$ coincides with T_{ϕ_ξ} as defined in Lemma 5.7. By choice of k and the previous lemma we have it lies in S_2 and as such we find that T_{ϕ_ξ} is Hilbert-Schmidt. Thus, Lemma 5.7 yields that $F_k \prec L^2(M) \otimes L^2(M)$. Lastly, by Remark 5.6, the direct sum of weakly contained bimodules is still weakly contained in another, and so we conclude:

$$E_{m-1}^{\perp} = \oplus_{k \geq m} F_k \prec L^2(M) \otimes L^2(M).$$

□

6.2. Strong solidity

Provided that k is large enough, by Proposition 6.4 we have that $E_{k-1}^{\perp}$ is weakly contained in the coarse bimodule. To make use of this, we need to relate α_t to $E_{k-1}^{\perp}$.

Let us use P_{F_k} and $P_{E_{k-1}}$ to denote the orthogonal projection with respect to the q-inner product onto their respective spaces. In particular, we explicitly construct the projection onto F_k, which, with some abuse of notation, we do through the identification given by the application of the vacuum vector.

Take $x = (f_{1,1} \oplus f_{1,2}) \otimes \cdots \otimes (f_{n,1} \oplus f_{n,2}) \in (H \oplus H)^{\otimes n}$ and let $A \subset \{1, \cdots, n\}$ of size k. We define $\Lambda_A x \in F_k$ as the tensor in which we project the i-th leg of x on $0 \oplus H$ if $i \in A$ and on $H \oplus 0$ otherwise. That is,

$$(\Lambda_A x)_i = \begin{cases} 0 \ \oplus f_{i,2} & \text{if } i \in A, \\ f_{i,1} \oplus \ 0 & \text{if } i \notin A. \end{cases}$$

We claim that $P_{F_k} : (H \oplus H)^{\otimes n} \to (H \oplus H)^{\otimes n}$ is given by

$$P_{F_k}(x) = \sum_{\substack{A \subset \{1, \cdots, n\} \\ |A| = k}} \Lambda_A x.$$

Since $\Lambda_B \Lambda_A(x) = \delta_{AB} \Lambda_A(x)$ it is obvious that if we extend by linearity the right-hand side of the above expression defines a projection, and has F_k as its range. To see that it is orthogonal, let $y = (g_{1,1} \oplus g_{1,2}) \otimes \cdots \otimes (g_{n,1} \oplus g_{n,2}) \in (H \oplus H)^{\otimes n}$. and we find that

$$\langle P_{F_k} x, y\rangle = \sum_{\substack{A \subset \{1, \cdots, n\} \\ |A| = k}} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{i \in A} \langle f_{i,2}, g_{\pi(i),2}\rangle \prod_{i \in A^C} \langle f_{i,1}, g_{\pi(i),1}\rangle = \langle x, P_{F_k} y\rangle.$$

As the F_k are orthogonal for different k, the above provides us with an explicit form for P_{E_m}, namely $P_{E_m} = \sum_{k=0}^{m} P_{F_k}$.

Remark 6.9. *For Wick words of length* 1 *we one can verify that* $P_{E_m}(W(f)W(h)W(g)) = W(f)P_{E_m}(W(h))W(g)$ *for* $W(f), W(g) \in E_{m-1}$ *and* $W(h) \in \Gamma_q(H_{\mathbb{R}\oplus\mathbb{R}})$. *Then, using the recursive formula determined in Section 3.3 for Wick words, we can extend the result to Wick words of arbitrary length.*

Proposition 6.10. *Let* $k \in \mathbb{N}$ *be given. Then there exists a constant* C_k *depending only on* k *such that*

$$\|(\alpha_{t_k} - id)(x)\|_2 \leq C_k \|P_{E_{k-1}^{\perp}} \alpha_t(x)\|_2$$

for $x \in \bigoplus_{m \geq k} H^{\otimes m} \subseteq \mathcal{F}_q(H)$ *and* $t < 2^{-k}$.

Proof. We first prove it for pure tensors. Note that all operators involved preserve tensor lengths, so it suffices to show the inequality on $H^{\otimes n}$ for $n \geq k$. Let $x = f_1 \otimes \cdots \otimes f_n$ and $y = g_1 \otimes \cdots \otimes g_n$.

Firstly, we have:

$$\begin{aligned} \langle P_{E_{k-1}^{\perp}}\alpha_t(x), P_{E_{k-1}^{\perp}}\alpha_t(y)\rangle_q &= \langle \sum_{m=k}^{n} P_{F_m}\alpha_t(x), \sum_{m=k}^{n} P_{F_m}\alpha_t(y)\rangle_q \\ &= \sum_{m=k}^{n} \langle P_{F_m}\alpha_t(x), P_{F_m}\alpha_t(y)\rangle_q. \end{aligned}$$

Applying our previously found formula for P_{F_m}, we can see that

$$P_{F_m}\alpha_t(x) = \sum_{\substack{A\subset\{1,\cdots,n\}\\ |A|=m}} e^{-t(n-m)}\sqrt{1-e^{-2t}}^{m}\Lambda_A x$$

by bringing the scalars out of the tensor product. Hence, we have:

$$\langle P_{F_m}\alpha_t(x), P_{F_m}\alpha_t(y)\rangle_q = \sum_{\substack{A,B\subset\{1,\cdots,n\}\\ |A|=|B|=m}} e^{-2t(n-m)}(1-e^{-2t})^m \langle \Lambda_A x, \Lambda_B y\rangle_q.$$

In computing $\langle \Lambda_A x, \Lambda_B y\rangle$ we note that the terms in the q-inner product can only be non-zero if A is matched exactly with B. But for a fixed A, if we iterate over all possible $B \subset \{1, \cdots, n\}$ and iterate over all permutations matching A to B, it is equivalent to iterating over S_n entirely. Thus, we obtain:

$$\begin{aligned} \sum_{m=k}^{n} \langle P_{F_m}\alpha_t(x), P_{F_m}\alpha_t(y)\rangle_q &= \sum_{m=k}^{n} \sum_{\substack{A\subset\{1,\cdots,n\}\\ |A|=m}} e^{-2t(n-m)}(1-e^{-2t})^m \langle x, y\rangle_q \\ &= \sum_{m=k}^{n} \binom{n}{m} e^{-2t(n-m)}(1-e^{-2t})^m \langle x, y\rangle_q. \end{aligned}$$

Let us now look at that which we wish to estimate using the above. Note that $\langle \alpha_{t^k}(x), \alpha_{t^k}(x)\rangle_q = \langle x, y\rangle_q$ as $(e^{-t^k})^2 + (\sqrt{1-e^{-2t^k}})^2 = 1$. Similarly, $\langle x, \alpha_{t^k}(y)\rangle_q = e^{-nt^k}\langle x, y\rangle_q = \langle \alpha_{t^k}(x), y\rangle_q$. Thus, we have:

$$\begin{aligned} \langle (\alpha_{t^k} - id)(x), (\alpha_{t^k} - id)(y)\rangle_q &= \langle \alpha_{t^k}(x), \alpha_{t^k}(y)\rangle_q + \langle x, y\rangle_q - (\langle x, \alpha_{t^k}(y)\rangle_q + \langle \alpha_{t^k}(x), y\rangle_q) \\ &= 2(1-e^{-nt^k})\langle x, y\rangle_q. \end{aligned}$$

Thus, it suffices to show that $2(1-e^{-nt^k}) < C_k \sum_{m=k}^{n} \binom{n}{m} e^{-2(n-m)t}(1-e^{-2t})^m$ for some C_k depending only on k. For the case $k = 0$ any $C > 2$ suffices, so let us assume $k \geq 2$.

First pick C_m for $m = 0, \ldots, k-1$ such that

$$\binom{n}{m} e^{2mt}(1-e^{-2t})^m \leq C_m n^m t^m.$$

for $m = 0, \ldots, k-1$ and all $t < 2^{-k}$ and $n \in \mathbb{N}$. This is clearly possible as for $n >> k$ we have $\binom{n}{k} \approx \frac{n^k}{k!}$, and the other two factors in the left-hand side are bounded from above by e^m. Now pick M_k such that $e^{-2nt}\sum_{m=0}^{k-1} C_m n^m t^m < \frac{1}{2}$ for $nt > M_k$, which is possible since e^{-nt} dominates the sum as nt becomes large. Suppose that $nt < M_k$. First we use that

$$\sum_{m=k}^{n} \binom{n}{m} e^{-2t(n-m)}(1-e^{-2t})^m > \binom{n}{k} e^{-2t(n-k)}(1-e^{-2t})^k.$$

For $t < 2^{-k}$ and $n \in \mathbb{N}$ we have $1 - e^{-nt^k} \leq nt^k \leq n^k t^k$ and $(1-e^{-2t})^k \geq t^k$. As such, it suffices to find C_k such that $\binom{n}{k} e^{-2t(n-k)} > C_k n^k > 0$. For one part we see that $e^{-2t(n-k)} > e^{-2nt} > e^{-2M_k}$. For the

other part, as before we use that if $n >> k$ then $\binom{n}{k} \approx \frac{n^k}{k!}$ and so $\binom{n}{k} > C_k n^k$ and thus we conclude this case.
Now suppose that $nt > M_k$. Then $t < 2^{-k}$ and $n > 2^k$ so applying the Binomial Theorem to $(1 + (1 - e^{-2t}))^n$ we have

$$\begin{aligned}
\sum_{m=k}^{n} \binom{n}{m} e^{-2t(n-m)}(1-e^{-2t})^m &= 1 - \sum_{m=0}^{k-1} \binom{n}{m} e^{-2t(n-m)}(1-e^{-2t})^m \\
&= 1 - e^{-2nt} \sum_{m=0}^{k-1} \binom{n}{m} e^{2tm}(1-e^{-2t})^m \\
&\geq 1 - e^{-2nt} \sum_{m=0}^{k-1} C_m n^m t^m \\
&\geq \frac{1}{2}.
\end{aligned}$$

As $2(1 - e^{-nt^k}) < 2$ we have proved the statement for all n and $t < 2^{-k}$. □

We have almost arrived at the main conclusion where we show $\Gamma_q(H_\mathbb{R})$ is strongly solid. Before we do so, we need to introduce the concept of weak compactness for a subalgebra $P \subset \Gamma_q(H_\mathbb{R})$ from Ozawa and Popa's paper [15].

Definition 6.11. *Let $P \subset \Gamma_q(H_\mathbb{R})$. We say P is weakly compact inside $\Gamma_q(H_\mathbb{R})$ if there exists a net $(\eta_i) \in L^2(P \oplus \bar{P})$ such that*

(i) $\lim_i \|\eta_i - (v \otimes \bar{v})\eta_i\|_2 = 0$ *for $v \in U(P)$,*

(ii) $\lim_i \|\eta_i - (u \otimes \bar{u})^* \eta_i (u \otimes \bar{u})\|_2 = 0$ *for $u \in \mathcal{N}_{\Gamma_q(H_\mathbb{R})}(P)$,*

(iii) $\langle (1 \otimes \bar{x})\eta_i, \eta_i \rangle = \tau(x) = \langle \eta_i, (x \otimes 1)\eta_i \rangle$ *for $x \in P$ and any i.*

With all set in place, we now enter the final stage of the thesis, as we have arrived at the main theorem of this work. In the proof of strong of the q-Gaussian algebras, Avsec borrows from Houdayer and Shlyakhtenko work [11], and as such the proof of the following theorem is an amalgamation of [1] and [11].

Theorem 6.12. *For $-1 < q < 1$ and finite-dimensional $H_\mathbb{R}$ we have that $\Gamma_q(H_\mathbb{R})$ is strongly solid.*

Proof. Recall the definition of strong solidity from Definition 6.2. Let $P \subset \Gamma_q(H_\mathbb{R}) = M$ be a diffuse, amenable subalgebra. We need to show that $\mathcal{N}_M(P)''$ is amenable.

By Lemma 2.2 in [10], if we can show that for any non-zero $z \in Z(\mathcal{N}_M(P)' \cap M)$ and any finite $F \subset U(\mathcal{N}_M(P)'')$ we have that

$$\left\| \sum_{u \in F} uz \otimes \overline{uz} \right\|_{M \bar{\otimes} \bar{M}} = |F|, \tag{6.2}$$

we find that $\mathcal{N}_M(P)$ is amenable. Thus, we we set out to show that the above it satisfied.

Note that $\left\|\sum_{u\in F} uz \otimes \bar{uz}\right\|_{M\bar{\otimes}\bar{M}} \leq |F|$ holds by the triangle inequality and the fact that $\|uz \otimes \bar{uz}\|_{M\otimes\bar{M}} = 1$ by definition of the spatial norm. The rest of the proof will be focussed on showing that the reverse inequality holds.

Let $z \in Z(\mathcal{N}_M(P)' \cap M)$ be a non-zero projection. Since P is diffuse, Theorem 5.3 yields us that P is not rigid with respect to α_t. Thus, α_t cannot converge uniformly to the identity on $B_1(P)$. A simple contradiction argument tells us that Pz is diffuse as well, so we can instead apply Theorem 5.3 to $B_1(Pz)$.

Since we can write any element as the sum of 4 unitary operators within the algebra, α_t cannot converge uniformly on $U(Pz)$ either. Thus, we can find a sequence $(u_k)_{k\in\mathbb{N}} \subset U(Pz)$ together with (t_k) such that $t_k \to 0$ such that $\|\alpha_t(u_k^m z) - u_k z\|_2 \geq c\|u_k z\|_2 = c\|z\|_2$ for some $c > 0$. We now apply 6.10 to obtain

$$\|P_{E_{m-1}^\perp}\alpha_{t_k}(u_k z)\| \;\geq\; C_m\|\alpha_{t_k^m}(u_k z) - u_k z\|_2 \;\geq\; C_m\|z\|_2,$$

whereas obviously $C_m < 1$. From now on, we fix the constant C_m as it is above. By Pythagoras' Theorem we find that

$$\|P_{E_{m-1}}\alpha_{t_k}(u_k z)\|_2 \leq \sqrt{1-C_m^2}\|\alpha_{t_k}(u_k z)\|_2 = \sqrt{1-C_m^2}\|(u_k z)\|_2 = \sqrt{1-C_m^2}\|z\|_2. \tag{6.3}$$

We shall show that this will lead to a contradiction. Set $\delta = \frac{1-\sqrt{1-C_m^2}}{6}$ and pick k_0 such that $\|\alpha_{t_k}(z) - z\|_2 \leq \delta$ for any $k \geq k_0$.

Combining the main result from [18] with Theorem 3.5 from [15] we have that P is weakly compact inside $\Gamma_q(H_{\mathbb{R}})$. Let (η_n) be as described in the definition of weakly compact. Let us now define the following sequences for $k \geq k_0$:

$$\begin{aligned} \eta_n^k &:= (\alpha_{t_k}\otimes 1)\eta_n, \\ \xi_n^k &:= ((P_{E_{m-1}})^\perp\alpha_{t_k}\otimes 1)\eta_n, \\ \zeta_n^k &:= (P_{E_{m-1}^\perp}\alpha_{t_k}\otimes 1)\eta_n. \end{aligned}$$

Lemma 6.13. *Fix $\delta = \frac{1-\sqrt{1-C_m^2}}{6}$ as before, and let $k \geq k_0$. Then*

$$\lim_n\|(z\otimes 1)\zeta_n^k\|_2 \geq \delta.$$

Proof. We give a proof by contradiction. Assume we can choose $k \geq k_0$ such that $\lim_n\|(z\otimes 1)\zeta_n^k\|_2 < \delta$. Using that $\eta_n^k = \xi_n^k + \zeta_n^k$ we note that

$$\begin{aligned} \lim_n\|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}\alpha_{t_k}(u_k)z\otimes \bar{u_k})\xi_n^k\|_2 \;&\leq\; \lim_n\|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}\alpha_{t_k}(u_k)z\otimes\bar{u_k})\eta_n^k\|_2 \\ &\quad + \lim_n\|(P_{E_{m-1}}\alpha_{t_k}(u_k)z\otimes\bar{u_k})\zeta_n^k\|_2. \end{aligned}$$

On the one hand we have

$$\lim_n\|(P_{E_{m-1}}\alpha_{t_k}(u_k))z\otimes\bar{u_k})\zeta_n^k\|_2 \;\leq\; \lim_n\|(z\otimes 1)\zeta_n^k\|.$$

Before we continue with the other term, we make a small detour. Let $x \in \widetilde{M}$. Recall that α_t is a $*$-automorphism. In combination with the third property of (η_n) for any k we have

$$\begin{aligned} \|(x\otimes 1)\eta_n^k\|_2^2 &= \langle(x\alpha_{t_k}\otimes 1)\eta_n, (x\alpha_{t_k}\otimes 1)\eta_n\rangle \\ &= \langle\eta_n, (\alpha_{t_k}^* x^* x\alpha_{t_k}\otimes 1)\eta_n\rangle \\ &= \langle\eta_n, (\alpha_{t_k}^*(x^*x)\alpha_{t_k}^*\alpha_{t_k}\otimes 1)\eta_n\rangle \\ &= \tau(\alpha_{t_k}^*(x^*x)) \\ &= \tau(E_M\alpha_{t_k}^*(x^*x)) \\ &= \tau(x^*x) \\ &= \|x\|_2^2. \end{aligned} \tag{6.4}$$

Now let $x \in M$ instead. Then by Remark 6.9 we find

$$\begin{aligned} \|(x\otimes 1)\zeta_n^k\|_2 &= \|(x\otimes 1)(P_{E_{m-1}^\perp}\otimes 1)\eta_n^k\|_2 \\ &= \|(P_{E_{m-1}^\perp}\otimes 1)(x\otimes 1)\eta_n^k\|_2 \\ &\leq \|(x\otimes 1)\eta_n^k\|_2 \\ &= \|x\|_2. \end{aligned}$$

We now continue with our estimation. We have:

$$\begin{aligned}
&\lim_n \|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}\alpha_{t_k}(u_k)z \otimes \bar{u_k})\eta_n^k\|_2 \\
\leq\ & \lim_n \|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 + \lim_n \|(P_{E_{m-1}}(\alpha_{t_k}(u_k)z - z\alpha_{t_k}(u_k)) \otimes \overline{u_k})\eta_n^k\|_2 \\
\leq\ & \lim_n \|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 + \lim_n \|((\alpha_{t_k}(u_k)z - z\alpha_{t_k}(u_k)) \otimes 1)\eta_n^k\|_2 \\
=\ & \lim_n \|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 + \lim_n \|\alpha_{t_k}(u_k)z - z\alpha_{t_k}(u_k)\|_2.
\end{aligned}$$

Recall that $z \in Z(\mathcal{N}_M(P)')$. Since obviously $U(P) \subset \mathcal{N}_M(P)$ we have that u_k commutes with z for any k. Hence, by reusing α_t is a $*$-homomorphism we deduce that

$$\begin{aligned}
\|\alpha_{t_k}(u_k)z - z\alpha_{t_k}(u_k)\|_2 &\leq \|\alpha_{t_k}(u_k)z - \alpha_{t_k}(z)\alpha_{t_k}(u_k)\|_2 + \|\alpha_{t_k}(z)\alpha_{t_k}(u_k) - z\alpha_{t_k}(u_k)\|_2 \\
&\leq \|\alpha_{t_k}(u_k)z - \alpha_{t_k}(u_k)\alpha_{t_k}(z)\|_2 + \|\alpha_{t_k}(z) - z\|_2 \\
&\leq 2\|\alpha_{t_k}(z) - z\|_2.
\end{aligned}$$

For the remaining term we first note that

$$\lim_n \|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k)\otimes \bar{u_k})\eta_n^k\|_2 \ \leq\ \lim_n \|(z\otimes 1)\zeta_n^k\|_2 + \lim_n \|(zP_{E_{m-1}}\otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k)\otimes \bar{u_k})\eta_n^k\|_2$$

We now claim that $\lim_n \|(zP_{E_{m-1}} \otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 = 0$. By the fact that $z \in M$, Remark 6.9, and property (ii) of η_i we obtain

$$\begin{aligned}
\lim_n \|(zP_{E_{m-1}} \otimes 1)\eta_n^k - (P_{E_{m-1}}z\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 &\leq \lim_n \|(zP_{E_{m-1}}x \otimes 1)\eta_n^k - (zP_{E_{m-1}}\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 \\
&\leq \lim_n \|\eta_n^k - (\alpha_{t_k}(u_k) \otimes \bar{u_k})\eta_n^k\|_2 \\
&\leq \lim_n \|(\alpha_{t_k} \otimes 1)(\eta_n - (u_k \otimes \bar{u_k})\eta_n)\|_2 \\
&\leq \lim_n \|\eta_n - (u_k \otimes \bar{u_k})\eta_n\|_2 \\
&= 0.
\end{aligned}$$

Combing the results we arrive at

$$\lim_n \|(z\otimes 1)\eta_n^k - (P_{E_{m-1}}\alpha_{t_k}(u_k z) \otimes \bar{u_k})\xi_n^k\|_2 \ \leq\ 4\lim_n \|(z\otimes 1)\zeta_n^k\|_2 \ < 4\delta. \tag{6.5}$$

We use this to create a contradiction with (6.3). Recall the assumption that $\|\alpha_{t_{k'}}(z) - z\|_2 \leq \delta$ for any $k' \geq k_0$. Using the reverse triangle inequality we obtain

$$\begin{aligned}
\|P_{E_{m-1}}\alpha_{t_k}(u_k z)\|_2 &\geq \|(P_{E_{m-1}}\alpha_{t_k}(u_k))z\| - \|P_{E_{m-1}}\alpha_{t_k}(u_k z) - (P_{E_{m-1}}\alpha_{t_k}(u_k))z\|_2 \\
&\geq \|(P_{E_{m-1}}\alpha_{t_k}(u_k))z\| - \|\alpha_{t_k}(u_k)\alpha_{t_k}(z) - \alpha_{t_k}(u_k)z\|_2 \\
&\geq \|(P_{E_{m-1}}\alpha_{t_k}(u_k))z\| - \|\alpha_{t_k}(z) - z\|_2 \\
&\geq \|(P_{E_{m-1}}\alpha_{t_k}(u_k))z\| - \delta.
\end{aligned}$$

Applying the formula from (6.4) and using the properties of the conditional expectation again we find

$$\begin{aligned}
\|(P_{E_{m-1}}\alpha_{t_k}(u_k))z\|_2 &\geq \|P_{E_{m-1}}((P_{E_{m-1}}\alpha_{t_k}(u_k))z)\|_2 \\
&= \|(P_{E_{m-1}} \otimes 1)((P_{E_{m-1}}\alpha_{t_k}(u_k))z \otimes 1)\eta_n^k\|_2 \\
&\geq \|(P_{E_{m-1}}(\alpha_{t_k}(u_k))z \otimes \bar{u_k})\xi_n^k\|_2
\end{aligned}$$

Now applying the inequality we found in (6.5) we can conclude our contradiction by

$$\begin{aligned}
\|P_{E_{m-1}}\alpha_{t_k}(u_k z)\|_2 &\geq \lim_n \|(P_{E_{m-1}}(\alpha_{t_k}(u_k))z \otimes \bar{u_k})\xi_n^k\|_2 - \delta \\
&\geq \lim_n \|(z\otimes 1)\eta_n^k\|_2 - 5\delta \\
&= \|z\|_2 - 5\delta \\
&> \sqrt{1 - C_m^2}\|z\|_2. \\
&= \delta.
\end{aligned}$$

□

We continue with proving strong solidity. Let $\rho(M^{op})$ denote the space of right actions of $\Gamma_q(H_{\mathbb{R}})$ on E_{m-1}. Let $\zeta_n^{z,k} := (z\otimes 1)\zeta_n^k$. We define the state $\varphi^{z,k}$ on $B(E_{m-1})\cap\rho(M^{op})'$ as follows:

$$\varphi^{z,k}(x) := \lim_n \frac{1}{\|\zeta_n^{z,k}\|_2^2}\langle (x\otimes 1)\zeta_n^{z,k}, \zeta_n^{n,k}\rangle.$$

Lemma 6.14. *Let $a\in\mathcal{N}_M(P)''$. Then $\varphi^{z,k}(ax - xa)$ converges uniformly to 0 on the unit ball of $B(E_{m-1})\cap\rho(M^{op})'$.*

Proof. Let $u\in\mathcal{N}_M(P)$. Since z and u commute, we have that

$$\begin{aligned}
\lim_n\|\zeta_n^{z,k} - (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*\|_2 &= \lim_n\|(z\otimes 1)(\zeta_n^{z,k} - (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*)\|_2\\
&\leq \lim_n\|\zeta_n^{k} - (u\otimes\bar{u})\zeta_n^{k}(u\otimes\bar{u})^*\|_2\\
&= \lim_n\|\zeta_n^{k}(u\otimes\bar{u}) - (u\otimes\bar{u})\zeta_n^{k}\|_2
\end{aligned}$$

By the Pythagorean Theorem we have that

$$\|\eta_n^k(u\otimes\bar{u}) - (u\otimes\bar{u})\xi_n^k\|_2^2 + \|\eta_n^k(u\otimes\bar{u}) - (u\otimes\bar{u})\zeta_n^k\|_2^2 = \|\eta_n^k(u\otimes\bar{u}) - (u\otimes\bar{u})\eta_n^k\|_2^2,$$

allow us to see that

$$\lim_n\|\zeta_n^{z,k} - (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*\|_2 \leq \lim_n\|\eta_n^k(u\otimes\bar{u}) - (u\otimes\bar{u})\eta_n^k\|_2.$$

By a simple triangle inequality we have

$$\begin{aligned}
\lim_n\|\eta_n^k(u\otimes\bar{u})^* - (u\otimes\bar{u})\eta_n^k\|_2 \leq\ & \lim_n\|\eta_n^k(u\otimes\bar{u}) - \eta_n^k(\alpha_{t_k}(u)\otimes\bar{u})\|_2\\
&+\lim_n\|\eta_n^k(\alpha_{t_k}(u)\otimes\bar{u}) - (\alpha_{t_k}(u)\otimes\bar{u})\eta_n^k\|_2\\
&+\lim_n\|(\alpha_{t_k}(u)\otimes\bar{u})\eta_n^k - (u\otimes\bar{u})\eta_n^k\|_2.
\end{aligned}$$

By (6.4) we can

$$\begin{aligned}
\lim_n\|(\alpha_{t_k}(u)\otimes\bar{u})\eta_n^k - (u\otimes\bar{u})\eta_n^k\|_2 &\leq \lim_n\|(\alpha_{t_k}(u)\otimes 1)\eta_n^k - (u\otimes 1)\eta_n^k\|_2\\
&= \|\alpha_{t_k}(u) - u\|_2.
\end{aligned}$$

We find the same estimate for $\lim_n\|\eta_n^k(u\otimes\bar{u}) - \eta_n^k(\alpha_{t_k}(u)\otimes\bar{u})\|_2$. Thus, we find

$$\begin{aligned}
\lim_n\|\eta_n^k(u\otimes\bar{u})^* - (u\otimes\bar{u})\eta_n^k\|_2 &\leq 2\|u-\alpha_{t_k}(u)\|_2 + \lim_n\|(\alpha_{t_k}\otimes 1)((u\otimes\bar{u})\eta_n - \eta_n(u\otimes\bar{u}))\|_2\\
&\leq 2\|u-\alpha_{t_k}(u)\|_2 + \lim_n\|(u\otimes\bar{u})\eta_n - \eta_n(u\otimes\bar{u})\|_2\\
&= 2\|u-\alpha_{t_k}(u)\|_2 + \lim_n\|\eta_n - (u\otimes\bar{u})^*\eta_n(u\otimes\bar{u})\|_2\\
&= 2\|u-\alpha_{t_k}(u)\|_2.
\end{aligned}$$

Moreover, for any $x\in B(E_{m-1})\cap\rho(M^{op})'$, using the fact that u is unitary, we have

$$\begin{aligned}
\varphi^{z,k}(u^*xu) &= \lim_n\frac{1}{\|\zeta_n^{z,k}\|_2^2}\langle (u^*xu\otimes 1)(1\otimes\bar{u})\zeta_n^{z,k}, (1\otimes\bar{u})\zeta_n^{n,k}\rangle\\
&= \lim_n\frac{1}{\|\zeta_n^{z,k}\|_2^2}\langle (x\otimes 1)(u\otimes\bar{u})\zeta_n^{z,k}, (u\otimes\bar{u})\zeta_n^{n,k}\rangle\\
&= \lim_n\frac{1}{\|\zeta_n^{z,k}\|_2^2}\langle (x\otimes 1)(u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*, (u\otimes\bar{u})\zeta_n^{n,k}(u\otimes\bar{u})^*\rangle
\end{aligned}$$

Combining these statements with Lemma 6.13 we obtain:

$$\begin{aligned}
|\varphi^{z,k}(u^*xu) - \varphi^{z,k}(x)| &= \lim_n \frac{1}{\|\zeta_n^{z,k}\|_2^2} |\langle (x\otimes 1)(u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*, (u\otimes\bar{u})\zeta_n^{n,k}(u\otimes\bar{u})^*\rangle \\
&\quad - \langle (x\otimes 1)\zeta_n^{z,k}, \zeta_n^{n,k}\rangle| \\
&\leq \lim_n \frac{\|x\|}{\|\zeta_n^{z,k}\|_2^2} \big(|\langle (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*, (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^* - \zeta_n^{z,k}\rangle| \\
&\quad + |\zeta_n^{z,k}, \zeta_n^{z,k} - (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*| \big) \\
&\leq \lim_n \frac{\|x\|}{\|\zeta_n^{z,k}\|_2^2} \big(\|(u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*\|_2 \|(u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^* - \zeta_n^{z,k}\|_2 \\
&\quad + \|\zeta_n^{z,k}\|_2 \|\zeta_n^{z,k} - (u\otimes\bar{u})\zeta_n^{z,k}(u\otimes\bar{u})^*\|_2 \big) \\
&\leq \lim_n \frac{\|x\|}{\|\zeta_n^{z,k}\|_2^2} \big(2\|\zeta_n^{z,k}\|_2 \cdot 2\|u - \alpha_{t_k}(u)\|_2 \big) \\
&\leq \frac{4}{\delta}\|u - \alpha_{t_k}(u)\|_2.
\end{aligned}$$

Letting $k \to \infty$ and multiplying by u this yields

$$\lim_{k\to\infty} |\varphi^{z,k}(ux - xu)| = 0$$

for any $u \in \mathcal{N}_M(P)$, with uniform convergence for $x \in B(E_{m-1}) \cap \rho(M)'$. Consequently, it holds for any $a \in \mathrm{Span}(\mathcal{N}_M(P))$ and as well.

We now intend to apply Kaplansky's density theorem (Theorem 2.7), but it remains to check that $\varphi^{z,k}$ is indeed bounded. Let $a \in E_{m-1}$. Then

$$\begin{aligned}
|\varphi^{z,k}(xa)| &\leq \frac{1}{\delta^2}\|x\| |(az\otimes 1)\zeta_n^k, \zeta_n^k| \\
&= \frac{1}{\delta^2}\|x\| \|za\|_2 \\
&\leq \frac{1}{\delta^2}\|x\| \|a\|_2. \qquad (6.6)
\end{aligned}$$

We can estimate $|\varphi^{z,k}(ax)|$ similarly, and using Kaplanky's density theorem yields the result. □

We continue to the final stage of the proof. Let μ be the representation of $M \otimes_{bin} M^{op}$ from the definition of E_{m-1} as an M-M-bimodule. Then through Lemma 5.7 we find continuity with respect to the minimal norm, and thus we can extend it to a unital completely positive map $\tilde{\mu} : B(L^2(M))\bar{\otimes} M^{op} \to B(E_{m-1})$. Now define Ψ on $B(E_{m-1})$ by $\Psi(x) := \tilde{\mu}(x \otimes 1)$. Note that Ψ extends the left action on E_{m-1}. Moreover, as M^{op} is in the multiplicative domain of $\tilde{\mu}$, elements in the range of Ψ commute with the right action. Set $\psi^{z,k} := \phi^{z,k} \circ \Psi$, which is a state on $B(L^2(M))$.

Let $u \in \mathcal{N}_M(P)$. Using Lemma 6.14 we can see that for any $x \in B(L^2(M))$ we have that

$$\begin{aligned}
\lim_k |\psi^{z,k}((uz)^*x(uz) - x| &= \lim_k |\phi^{z,k}(\Psi((uz)^*x(uz)) - \Psi(x))| \\
&= \lim_k |\phi^{z,k}((uz)^*\Psi(x(uz)) - \Psi(x))| \\
&= \lim_k |\phi^{z,k}(u^*\Psi(xu) - \Psi(x))| \\
&= 0,
\end{aligned}$$

where z disappears simply by applying the definition of $\phi^{z,k}$. Moreover, by Lemma 6.14 this convergence is uniform on the unit ball. By applying the Hahn-Banach separation theorem we can find a sequence $(\mu^{z,k}) \subset S_1(L^2(M))^+$ such that $\|\mu^{z,k}\|_1 = 1$ and

$$\lim_k \|\mu^{z,k} - (uz)\mu^{z,k}(uz)^*\|_1 = 0 \qquad (6.7)$$

for all $u \in U(\mathcal{N}_M(P))$. Since f is finite we may replace $\mu^{z,k}$ by $z\mu^{z,k}z$, which we note is positive, and scale it by $\|z\mu^{z,k}z\|_1$ to obtain elements a sequence of $(\mu^{z,k})$ of norm 1 elements such that $z\mu^{z,k}z = \mu^{z,k}$ and the limit holds for $u \in F$.

Let us define a sequence $(\nu^{z,k})_k \in S_2(L^2(M))$ through $\nu^{z,k} := (\mu^{z,k})^{1/2}$. The elements have norm 1 and satisfy $z\nu^{z,k}z = \nu^{z,k}$, as well as

$$\lim_k \|\nu^{z,k} - (uz)\nu^{z,k}(uz)^*\|_2 = 0 \tag{6.8}$$

for all $u \in F$ by Theorem 2.6. Let us identify $S_2(L^2(M))$ with $L^2(M) \otimes L^2(\bar{M})$ as bimodules through the identification of $(\xi \otimes \eta)$ with $\theta_{\xi,\eta}(v) := \langle v, \eta\rangle\xi$. By Theorem 6.4.19 in [13], this yields the spatial tensor norm on $L^2(M) \otimes L^2(\bar{M})$. Using the above results, we can conclude:

$$\begin{aligned}
|F| &= \|\sum_{u\in F} \nu^{z,k}\|_2 \\
&\leq \lim_k \left\|\sum_{u\in F}(uz)\nu^{z,k}(\nu^{z,k})^*\right\|_2 + \lim_k \left\|\sum_{u\in F} \nu^{z,k} - (uz)\nu^{z,k}(uz)^*\right\|_2 \\
&= \lim_k \left\|\sum_{u\in F}(uz)\nu^{z,k}(uz)^*\right\|_2 \\
&= \left\|\sum_{u\in F} uz \otimes \bar{u}\bar{z}\right\|_{M\overline{\otimes}\bar{M}}.
\end{aligned}$$

□

7

Conclusion

The main focus of this thesis was to show strong solidity of the q-Gaussian algebras for $-1 < q < 1$, in which we have succeeded in Chapter 6. Borrowing results from [1] and [11], we combined the coarse bimodule and the deformation α_t to show that the normalizer of diffuse amenable subalgebras are amenable. This result is based on Avsec's paper [1], on which we have made some adjustments and improvements.

In our journey to achieve the main result, we have been exposed to some of the existing literature on q-mathematics, and have seen the details of the construction of the q-Gaussian algebras. In Chapter 3 we introduced and constructed the q-analogue of the Fock space. Together with the creation and annihilation operator, which satisfy the q-relations, these form the basis of the q-Gaussian algebras. Moreover, this chapter also introduced the Wick operator, which together with the vacuum vector, allows us to study elements in $\Gamma_q(H_\mathbb{R})$. The Wick operator proved to be extremely useful in subsequent chapters.

In an intermezzo we showed that the q-Gaussian algebras are factors by an application of the Wick operator. In the subsequent chapter, we introduced the deformation α_t, which comes from a rotation of $H_\mathbb{R}$ into $H_\mathbb{R} \oplus H_\mathbb{R}$. This chapter also introduced the coarse bimodule, which together with the deformation formed a crucial element in the proof of strong solidity of $\Gamma_q(H_\mathbb{R})$.

Lastly, Chapter 6 was dedicated to the main focus of the thesis. Here we showed that E_{m-1} is contained in the coarse bimodule for sufficiently large m dependent on q. We borrow a result from the literature which gives us weak compactness for amenable subalgebras of $\Gamma_q(H_\mathbb{R})$. Unfortunately the methods used in the proof only allowed for finite dimensional $H_\mathbb{R}$. Nevertheless, the aforementioned results in combination with an application of the deformation α_t allowed us to conclude that $\Gamma_q(H_\mathbb{R})$ is in fact strongly solid.

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