Regularity theory for a new class of fractional parabolic stochastic evolution equations

Kirchner, Kristin; Willems, Joshua

Important note
To cite this publication, please use the final published version (if applicable).
Please check the document version above.

Regularity theory for a new class of fractional parabolic stochastic evolution equations

Kristin Kirchner[1] · **Joshua Willems[1]**

Abstract

A new class of fractional-order parabolic stochastic evolution equations of the form $(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t)$, $t \in [0, T]$, $\gamma \in (0, \infty)$, is introduced, where $-A$ generates a C_0-semigroup on a separable Hilbert space H and the spatiotemporal driving noise $\dot{W}^Q$ is the formal time derivative of an H-valued cylindrical Q-Wiener process. Mild and weak solutions are defined; these concepts are shown to be equivalent and to lead to well-posed problems. Temporal and spatial regularity of the solution process X are investigated, the former being measured by mean-square or pathwise smoothness and the latter by using domains of fractional powers of A. In addition, the covariance of X and its long-time behavior are analyzed. These abstract results are applied to the cases when $A := L^\beta$ and $Q := \widetilde{L}^{-\alpha}$ are fractional powers of symmetric, strongly elliptic second-order differential operators defined on (i) bounded Euclidean domains or (ii) smooth, compact surfaces. In these cases, the Gaussian solution processes can be seen as generalizations of merely spatial (Whittle–)Matérn fields to space–time.

Keywords Spatiotemporal Gaussian processes · Matérn covariance · Nonlocal space–time differential operators · Mild solution · Mean-square differentiability · Strongly continuous semigroups

Mathematics Subject Classification Primary 60G15 · 60H15; Secondary 47D06 · 35R11

✉ Joshua Willems
  j.willems@tudelft.nl

  Kristin Kirchner
  k.kirchner@tudelft.nl

[1] Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

🖄 Springer

1 Introduction

1.1 Motivation and background

Gaussian processes play an important role for modeling in spatial statistics. Typical applications arise in the environmental sciences, where geographically indexed data is collected, including climatology [3, 66], oceanography [8], meteorology [38], and forestry [7, 43, 54]. More generally, hierarchical models based on Gaussian processes have been used in various disciplines, where spatially dependent (or spatiotemporal) data is recorded, such as demography [28, 62], epidemiology [48], finance [33], and neuroimaging [56].

Since a Gaussian process $(X(j))_{j \in \mathcal{I}}$ is fully characterized by its mean and its covariance function, *second-order-based approaches* focus on the construction of appropriate covariance classes. In the case that the index set $\mathcal{I}$ is given by a spatial domain in the Euclidean space $\mathcal{I} = \mathcal{D} \subseteq \mathbb{R}^d$, the *Matérn covariance class* [54] is an important and widely used model. The Matérn covariance function is given by

$$\varrho(x, y) = 2^{1-\nu} \sigma^2 [\Gamma(\nu)]^{-1} (\kappa \|x - y\|_{\mathbb{R}^d})^\nu K_\nu(\kappa \|x - y\|_{\mathbb{R}^d}), \quad x, y \in \mathcal{D}, \quad (1.1)$$

where K_ν denotes the modified Bessel function of the second kind. It is indexed by the three interpretable parameters $\nu, \kappa, \sigma^2 \in (0, \infty)$, which determine *smoothness*, *correlation length* and *variance* of the process. It is this feature that renders the Matérn class particularly suitable for making inference about spatial data [69].

When considering *spatiotemporal* phenomena, the following difficulties occur:

1. It is desirable to control the properties of the stochastic process named above (in particular, smoothness and correlation lengths) separately in space and time. For this reason, considering (1.1) in $d + 1$ dimensions is not expedient and it is a difficult task to construct appropriate spatiotemporal covariance models, see e.g. [22, 34, 36, 64, 65, 70].
2. Second-order-based approaches require the factorization of, in general, dense covariance matrices, causing computational costs which are cubic in the number of observations. The two common assumptions imposed on spatiotemporal covariance models to reduce the computational costs—separability (factorization into merely spatial and temporal covariance functions) and stationarity (invariance under translations)—have proven unrealistic in many situations, see [23, 55, 70]. In particular, Stein [70] criticized the behavior of separable covariance functions with respect to their differentiability.

Owing to these problems, the class of *dynamical models* has gained popularity. The name originates from focusing on the dynamics of the stochastic process which are described either by means of conditional probability distributions or by representing the process as a solution of a stochastic partial differential equation (SPDE). The latter approach was originally proposed in the merely spatial case, motivated by the following observation made by Whittle [73]: A stationary process $(X(x))_{x \in \mathcal{D}}$ indexed

by the entire Euclidean space $\mathcal{D} = \mathbb{R}^d$ which solves the SPDE

$$\left(\kappa^2 - \Delta\right)^\beta X(x) = \mathcal{W}(x), \quad x \in \mathcal{D}, \tag{1.2}$$

has a covariance function of Matérn type (1.1) with $\nu = 2\beta - d/2$. Here, Δ denotes the Laplacian and $\mathcal{W}$ is Gaussian white noise. This relation gave rise to the SPDE approach proposed by Lindgren, Rue, and Lindström [50], where the SPDE (1.2) is considered on a bounded domain $\mathcal{D} \subsetneq \mathbb{R}^d$ and augmented with Dirichlet or Neumann boundary conditions. Besides enabling the applicability of efficient numerical methods available for (S)PDEs, such as finite element methods [11, 13, 14, 21, 40, 50] or wavelets [16, 39], this approach has the advantage of allowing for

(a) nonstationary or anisotropic generalizations by replacing the operator $\kappa^2 - \Delta$ in (1.2) with more general strongly elliptic second-order differential operators,

$$(Lv)(x) = \kappa^2(x)v(x) - \nabla \cdot (a(x)\nabla v(x)), \quad x \in \mathcal{D}, \tag{1.3}$$

where $\kappa \colon \mathcal{D} \to \mathbb{R}$ and $a \colon \mathcal{D} \to \mathbb{R}^{d \times d}_{\text{sym}}$ are functions [6, 11, 13, 14, 21, 35, 40, 50];
(b) more general domains, such as surfaces [15, 40] or manifolds [39].

In the SPDE (1.2) the fractional exponent β defines the (spatial) differentiability of its solution, see e.g. [21]. A realistic description of spatiotemporal phenomena necessitates controllable differentiability in space and time. This motivates to consider the space–time fractional SPDE model

$$\begin{cases} \left(\partial_t + L^\beta\right)^\gamma X(t, x) = \dot{\mathcal{W}}(t, x), & t \in [0, T], \quad x \in \mathcal{D}, \\ \qquad\qquad X(0, x) = X_0(x), & x \in \mathcal{D}, \end{cases} \tag{1.4}$$

where L in (1.3) is augmented with boundary conditions on $\partial \mathcal{D}$, $(X_0(x))_{x \in \mathcal{D}}$ is the initial random field, $\dot{\mathcal{W}}$ denotes space–time Gaussian white noise, and $T \in (0, \infty)$ is the time horizon. Whenever $\beta = \gamma = 1$, the SPDE (1.4) simplifies to the stochastic heat equation and this spatiotemporal model had already been mentioned in [50] and it was used for statistical inference in [18, 67]. The novelty and sophistication of the SPDE model (1.4) lies in the fractional power $\gamma \in (0, \infty)$ of the parabolic operator. Notably, it is the interplay of the parameters β and γ that will facilitate controlling spatial and temporal smoothness of the solution process. For $\mathcal{D} = \mathbb{R}^d$, this has recently been investigated via Fourier techniques in [49], see also [4, 19, 44].

Besides the aforementioned benefits of the SPDE approach and in contrast to the SPDE $\left(\partial_t^\gamma + L^\beta\right) X = \dot{\mathcal{W}}$, considered for instance in [17, 30], the SPDE model (1.4) furthermore exhibits a long-time behavior resembling the spatial model (1.2).

1.2 Contributions

We introduce a novel interpretation of (1.4) with $X_0 = 0$ as a fractional parabolic stochastic evolution equation, and correspondingly define mild and weak solutions for it. To this end, we first give a meaning to fractional powers of an operator of

the form $\partial_t + A$, where $-A$ generates a C_0-semigroup. Generalizing the approach taken for $\gamma = 1$ in [25, Chapter 5], we prove that mild and weak solutions are equivalent under natural assumptions, and we investigate their existence, uniqueness, regularity, and covariance. Our main findings are that the problem (1.4) is well-posed, and the properties of its solution X with respect to smoothness and covariance structure generalize those of the spatial Whittle–Matérn SPDE model (1.2) and relate to the parameters $\beta, \gamma \in (0, \infty)$ in the desired way. Restricting the analysis to a zero initial field is justified by our primary interest in regularity related to the dynamics of (1.4) and the long-time behavior of solutions.

In comparison with [9, 10, 51, 57, 71]—the only previous works on an equation of the form $(\partial_t + L)^\gamma u = f$ known to the authors—the main contributions of this work, besides considering a stochastic right-hand side, are the fractional power β in (1.4) and the method of proving regularity using semigroups. As opposed to the extension approach in [9, 10, 51, 57, 71], this setting does not require a Euclidean structure.

1.3 Outline

Preliminary notation and theory will be introduced in Sect. 2. In Sect. 3 we give a meaning to the parabolic operator $\partial_t + A$ and its fractional powers in order to introduce well-defined mild and weak solutions of (1.4) with $X_0 = 0$. Subsequently, we analyze these in terms of spatiotemporal regularity. Section 4 is concerned with the covariance structure of solutions. Finally, in Sect. 5 we apply our results to the space–time Whittle–Matérn SPDE (1.4) considered on a bounded Euclidean domain or on a surface. This article is supplemented by two appendices: "Appendix A" contains several technical auxiliary results used in the proofs of Sect. 3. "Appendix B" collects necessary definitions and results from functional calculus.

2 Preliminaries

2.1 Notation

The sets $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the positive and non-negative integers, respectively. We write $s \wedge t$ (or $s \vee t$) for the minimum (or maximum) of two real numbers $s, t \in \mathbb{R}$. The real and imaginary parts of a complex number $z \in \mathbb{C}$ are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively; its argument, denoted $\arg z$, takes its values in $(-\pi, \pi]$. We write $\mathbf{1}_D$ for the indicator function of a set D. The restriction of a function $f \colon D \to E$ to a subset $D_0 \subseteq D$ is denoted by $f|_{D_0} \colon D_0 \to E$; the image of D_0 under a linear mapping T is written as $T D_0$. Given two parameter sets $\mathscr{P}, \mathscr{Q}$ and two mappings $\mathscr{F}, \mathscr{G} \colon \mathscr{P} \times \mathscr{Q} \to \mathbb{R}$, we use the expression $\mathscr{F}(p, q) \lesssim_q \mathscr{G}(p, q)$ to indicate that for each $q \in \mathscr{Q}$ there exists a constant $C_q \in (0, \infty)$ such that $\mathscr{F}(p, q) \leq C_q \mathscr{G}(p, q)$ for all $p \in \mathscr{P}$. We write $\mathscr{F}(p, q) \eqsim_q \mathscr{G}(p, q)$ if both relations, $\mathscr{F}(p, q) \lesssim_q \mathscr{G}(p, q)$ and $\mathscr{G}(p, q) \lesssim_q \mathscr{F}(p, q)$, hold simultaneously.

2.2 Banach spaces and operators

If not specified otherwise, E or F denote separable Banach spaces. We instead write H or U if we work with separable Hilbert spaces and wish to emphasize this. The scalar field $\mathbb{K}$ is either given by the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. A norm on E will be denoted by $\|\cdot\|_E$ and an inner product on H by $(\cdot, \cdot)_H$. We write I for the identity operator. The notation $E \hookrightarrow F$ indicates that E is continuously embedded in F, i.e., there exists a bounded injective map from E to F. The dual space of E is denoted by E^*. We write $\overline{E_0}^E$ for the closure of a subset $E_0 \subseteq E$ with respect to the norm on E; the superscript may be omitted when there is no risk of confusion. The Borel σ-algebra of E is denoted by $\mathcal{B}(E)$.

We write $T \in \mathscr{L}(E; F)$ if the linear operator $T: E \to F$ is bounded. Whenever $E = F$, we abbreviate $\mathscr{L}(E) := \mathscr{L}(E; E)$, and this convention holds also for all other spaces of operators to be introduced. The space $\mathscr{L}(E; F)$ is rendered a Banach space when equipped with the usual operator norm; the space of Hilbert–Schmidt operators $\mathscr{L}_2(U; H) \subseteq \mathscr{L}(U; H)$ is a Hilbert space with respect to the inner product $(T, S)_{\mathscr{L}_2(U;H)} := \sum_{j \in \mathbb{N}} (T e_j, S e_j)_H$, where $(e_j)_{j \in \mathbb{N}}$ is any orthonormal basis for U. We write $T^* \in \mathscr{L}(F^*; E^*)$ for the adjoint operator of $T \in \mathscr{L}(E; F)$. In the case that $T \in \mathscr{L}(U; H)$, we identify $U^* = U$ and $H^* = H$ via the Riesz maps, so that $T^* \in \mathscr{L}(H; U)$. An operator $T \in \mathscr{L}(H)$ is said to be self-adjoint if $T^* = T$, non-negative if $(Tx, x)_H \geq 0$ holds for all $x \in H$, and strictly positive if there exists a constant $\theta \in (0, \infty)$ such that $(Tx, x)_H \geq \theta \|x\|_H^2$ holds for all $x \in H$.

A linear operator A on E with domain $\mathsf{D}(A)$ is denoted by $A: \mathsf{D}(A) \subseteq E \to E$ and its range by $\mathsf{R}(A)$. We call A closed if its graph $\mathsf{G}(A) := \{(x, Ax) : x \in \mathsf{D}(A)\}$ is closed with respect to the graph norm $\|(x, Ax)\|_{\mathsf{G}(A)} := \|x\|_E + \|Ax\|_E$, and densely defined if $\mathsf{D}(A)$ is dense in E. The definition $\|x\|_{\mathsf{D}(A)} := \|(x, Ax)\|_{\mathsf{G}(A)}$ yields a norm on $\mathsf{D}(A)$. If $\mathsf{G}(A) \subseteq \mathsf{G}(\widetilde{A})$ for another linear operator $\widetilde{A}$ on E, then $\widetilde{A}$ is called an extension of A and we write $A \subseteq \widetilde{A}$. If $\overline{\mathsf{G}(A)}$ is the graph of a linear operator, then we call this operator the closure of A, denoted $\overline{A}$.

2.3 Function spaces

Let a measure space $(S, \mathcal{S}, \mu)$ be given. We abbreviate the phrases "almost everywhere" and "almost all" by "a.e." and "a.a.", respectively.

We say that a function $f: S \to E$ is strongly measurable if it is the μ-a.e. limit of measurable simple functions. For $p \in [1, \infty]$, the Bochner space of (equivalence classes of) strongly measurable, p-integrable functions is denoted by $L^p(S; E)$. It is equipped with the norm

$$\|f\|_{L^p(S;E)} := \begin{cases} \left(\int_S \|f(t)\|_E^p \, d\mu(t)\right)^{1/p} & \text{if } p \in [1, \infty), \\ \operatorname{ess\,sup}_{t \in S} \|f(t)\|_E & \text{if } p = \infty, \end{cases}$$

where ess sup denotes the essential supremum. The norm on $L^2(S; H)$ is induced by the inner product $(f, g)_{L^2(S;H)} := \int_S (f(t), g(t))_H \, d\mu(t)$.

Now let S be an interval $S := J \subseteq \mathbb{R}$, equipped with the Borel σ-algebra and the Lebesgue measure. The space of continuous functions from J to E will be denoted by $C(J; E)$ or $C^{0,0}(J; E)$ and be endowed with the supremum norm if J is compact. For $\alpha \in (0, 1]$ and compact J, we consider the space $C^{0,\alpha}(J; E)$ of α-Hölder continuous functions with norm

$$\|f\|_{C^{0,\alpha}(J;E)} := |f|_{C^{0,\alpha}(J;E)} + \|f\|_{C(J;E)}, \text{ where}$$

$$|f|_{C^{0,\alpha}(J;E)} := \sup_{t,s \in J, \, t \neq s} \frac{\|f(t) - f(s)\|_E}{|t - s|^\alpha}$$

is the α-Hölder seminorm. For $n \in \mathbb{N}_0$ and $0 \leq \alpha \leq 1$, the space $C^{n,\alpha}(J; E)$ consists of functions whose nth derivative exists and belongs to $C^{0,\alpha}(J; E)$. If J is compact we use the norm $\|f\|_{C^{n,\alpha}(J;E)} := \|f^{(n)}\|_{C^{0,\alpha}(J;E)} + \sum_{k=0}^{n-1} \|f^{(k)}\|_{C(J;E)}$, where $f^{(k)}$ denotes the kth derivative of f. Moreover, we define $C^\infty(J; E) := \bigcap_{n \in \mathbb{N}} C^{n,0}(J; E)$. We say that $f \in C^{n,\alpha}(J; E)$ is compactly supported if the support of f, defined by

$$\operatorname{supp} f := \overline{\{t \in J : f(t) \neq 0\}}^J,$$

is compact. The space consisting of such functions is denoted by $C_c^{n,\alpha}(J; E)$. If f vanishes at a point $t \in J$, then we use the notation $f \in C_{0,\{t\}}^{n,\alpha}(J; E)$. The spaces $C_c^\infty(J; E)$ and $C_{0,\{t\}}^\infty(J; E)$ are defined analogously.

For an open interval J, we say that $u \in L^2(J; E)$ belongs to $H^1(J; E)$ if there exists a function $v \in L^2(J; E)$ such that $\int_J v(t)\phi(t) \, \mathrm{d}t = -\int_J u(t)\phi'(t) \, \mathrm{d}t$ holds for all $\phi \in C_c^\infty(J; \mathbb{R})$. The function $\partial_t u := v$ is called the weak derivative of u and the norm on $H^1(J; E)$ is $\|u\|_{H^1(J;E)} := \left(\|u\|_{L^2(J;E)}^2 + \|\partial_t u\|_{L^2(J;E)}^2 \right)^{1/2}$. The completion of $C_c^\infty((0, \infty); E)$ with respect to the norm $\| \cdot \|_{H^1(0,\infty;E)}$ defines the space $H_{0,\{0\}}^1(0, \infty; E)$. Elements of $H_{0,\{0\}}^1(J; E)$ are restrictions of functions in $H_{0,\{0\}}^1(0, \infty; E)$ to $J \subseteq (0, \infty)$.

Whenever the function space contains functions mapping to $E = \mathbb{R}$, we omit the codomain, e.g., we write $L^p(S) := L^p(S; \mathbb{R})$ for the Lebesgue spaces.

2.4 Vector-valued stochastic processes

Throughout this article, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space equipped with a normal filtration $(\mathcal{F}_t)_{t \geq 0}$, i.e., $\mathcal{F}_0$ contains all elements $B \in \mathcal{F}$ with $\mathbb{P}(B) = 0$ and $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ for all $t \geq 0$. Statements which hold $\mathbb{P}$-almost surely are marked with "$\mathbb{P}$-a.s.".

We call every strongly measurable function $Z \colon \Omega \to E$ a (vector-valued) random variable, and the expectation of $Z \in L^1(\Omega; E)$ is defined as the Bochner integral $\mathbb{E}[Z] := \int_\Omega Z(\omega) \, \mathrm{d}\mathbb{P}(\omega)$. An E-valued stochastic process $X = (X(t))_{t \in [0,T]}$ indexed by the interval $[0, T]$, $T \in (0, \infty)$, is called integrable if $(X(t))_{t \in [0,T]} \subseteq L^p(\Omega; E)$ holds for $p = 1$, and square-integrable if this inclusion is true for $p = 2$. It is said to be predictable if it is strongly measurable as a mapping from $[0, T] \times \Omega$ to E, where

the former set is equipped with the σ-algebra generated by the family

$$\{(s, t] \times F_s : 0 \leq s < t \leq T, \; F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 : F_0 \in \mathcal{F}_0\}.$$

Given another E-valued process $\widetilde{X} := (\widetilde{X}(t))_{t \in [0,T]}$, we call $\widetilde{X}$ a modification of X, provided that $\mathbb{P}(X(t) = \widetilde{X}(t)) = 1$ holds for all $t \in [0, T]$. Moreover, X and $\widetilde{X}$ are said to be indistinguishable if $\mathbb{P}(\forall t \in [0, T] : X(t) = \widetilde{X}(t)) = 1$.

For a self-adjoint strictly positive operator $Q \in \mathscr{L}(H)$, $(W^Q(t))_{t \geq 0}$ denotes a cylindrical Q-Wiener process with respect to $(\mathcal{F}_t)_{t \geq 0}$ which takes its values in H, cf. [52, Proposition 2.5.2]; if $Q = I$, we omit the superscript and call $(W(t))_{t \geq 0}$ a cylindrical Wiener process.

3 Analysis of the fractional stochastic evolution equation

The aim of this section is to define and analyze solutions to the following stochastic evolution equation of the general fractional order $\gamma \in (0, \infty)$:

$$(\partial_t + A)^\gamma X(t) = \dot{W}^Q(t), \quad t \in [0, T], \qquad X(0) = 0. \tag{3.1}$$

We interpret this as an abstraction of (1.4) with $X_0 = 0$. As noted in the introduction, we restrict the discussion to a zero initial field, since we are primarily interested in properties resulting from the dynamics of the SPDE (1.4), respectively (3.1), and the long-time behavior for $0 \ll T < \infty$ of its solution. We also note that imposing non-zero boundary data for fractional problems is, in general, highly non-trivial, see e.g. the recent works [1, 5] on the fractional Laplacian.

In Sect. 3.1 we investigate the *parabolic operator* $\mathcal{B}$, which is defined as the closure of the sum operator $\partial_t + A$ on an appropriate domain. In particular, we consider the C_0-semigroup generated by $-\mathcal{B}$, which is used to define fractional powers $\mathcal{B}^\gamma$ for $\gamma \in \mathbb{R}$. Interpreting the expression $(\partial_t + A)^\gamma$ appearing in (3.1) as $\mathcal{B}^\gamma$, we use this result to define mild solutions in Sect. 3.2. In this part, we furthermore introduce a weak solution concept for (3.1), and prove equivalence of the two solution concepts as well as existence and uniqueness of mild and weak solutions. Spatiotemporal regularity of solutions is the subject of Sect. 3.3.

3.1 The parabolic operator and its fractional powers

In this subsection we define the parabolic operator $\mathcal{B}$ and fractional powers $\mathcal{B}^\gamma$. We start by formulating several assumptions on the linear operator A, to which we shall refer throughout the remainder of this work. For an overview of the theory of C_0-semigroups, we refer the reader to [31] or [61]. The complexification of a normed space or operator is indicated by the subscript $\mathbb{C}$; see Sect. B.2.1 in "Appendix B" for details.

Assumption 3.1 Let H be a separable Hilbert space over the real scalar field $\mathbb{R}$. We assume that the linear operator $A : \mathsf{D}(A) \subseteq H \to H$ satisfies

(i) $-A$ generates a C_0-semigroup $(S(t))_{t\geq 0}$.

Sometimes we additionally require one or more of the following conditions:

(ii) $(S(t))_{t\geq 0}$ is (uniformly) bounded analytic, i.e., the mapping $t \mapsto S_{\mathbb{C}}(t)$, where $S_{\mathbb{C}}(t) := [S(t)]_{\mathbb{C}}$, extends to a bounded holomorphic function on an open sector $\Sigma_\omega \subseteq \mathbb{C}$ for some angle $\omega \in (0, \pi)$ (see Definition B.1 in "Appendix B");
(iii) $A_{\mathbb{C}}$ admits a bounded H^∞-calculus with $\omega_{H^\infty}(A_{\mathbb{C}}) < \frac{\pi}{2}$, see Definition B.3;
(iv) A has a bounded inverse.

Under Assumption 3.1(i), Lemma B.6 allows us to use several results from [31, 37, 61] for C_0-semigroups and their generators on complex spaces also for $(S(t))_{t\geq 0}$ and $-A$. For instance, by [31, Theorem II.1.4] and [61, Chapter 1, Theorem 2.2] the operator A is closed and densely defined, and the C_0-semigroup $(S(t))_{t\geq 0}$ satisfies

$$\exists M \in [1, \infty),\ w \in \mathbb{R}: \quad \|S(t)\|_{\mathscr{L}(H)} = \|S_{\mathbb{C}}(t)\|_{\mathscr{L}(H_{\mathbb{C}})} \leq M e^{-wt} \quad \forall t \geq 0.$$
(3.2)

If the conditions (i), (ii) and (iv) are satisfied, then (3.2) holds for some $w \in (0, \infty)$, see e.g. [61, p. 70]. In this case, $(S(t))_{t\geq 0}$ is said to be *exponentially stable*. Moreover, we note that Assumption 3.1(ii) is equivalent to the operator $A_{\mathbb{C}}$ being sectorial with $\omega(A_{\mathbb{C}}) < \frac{\pi}{2}$ by Theorem B.2, and that consequently condition (iii) implies (ii) since $\omega(A_{\mathbb{C}}) \leq \omega_{H^\infty}(A_{\mathbb{C}})$ by Remark B.5. Whenever the conditions (i) and (ii) are satisfied, we have the following useful estimate (see [37, Proposition 3.4.3]):

$$\forall c \in [0, \infty): \quad \|A^c S(t)\|_{\mathscr{L}(H)} = \|A_{\mathbb{C}}^c S_{\mathbb{C}}(t)\|_{\mathscr{L}(H_{\mathbb{C}})} \lesssim_c t^{-c} \quad \forall t \in (0, \infty). \ (3.3)$$

As a first step towards defining the parabolic operator $\mathcal{B}$, we define the Bochner space counterpart $\mathcal{A}: \mathsf{D}(\mathcal{A}) \subseteq L^2(0, T; H) \to L^2(0, T; H)$ of A by

$$[\mathcal{A}v](\vartheta) := Av(\vartheta), \quad v \in \mathsf{D}(\mathcal{A}),\ \text{a.a. } \vartheta \in (0, T),$$
$$\mathsf{D}(\mathcal{A}) = L^2(0, T; \mathsf{D}(A)) := \left\{ v \in L^2(0, T; H) : \|Av(\cdot)\|_{L^2(0,T;H)} < \infty \right\}.$$
(3.4)

The C_0-semigroup $(S(t))_{t\geq 0}$ on H, generated by $-A$, can be associated to a family of operators $(\mathcal{S}(t))_{t\geq 0}$ on $L^2(0, T; H)$ in a similar way:

$$[\mathcal{S}(t)v](\vartheta) := S(t)v(\vartheta), \quad t \geq 0,\ v \in L^2(0, T; H),\ \text{a.a. } \vartheta \in (0, T). \quad (3.5)$$

It turns out that $(\mathcal{S}(t))_{t\geq 0} \subseteq \mathscr{L}(L^2(0, T; H))$ is again a C_0-semigroup, with infinitesimal generator $-\mathcal{A}$, see Proposition A.3 in "Appendix A".

In addition, we consider the family of zero-padded right-translation operators $(\mathcal{T}(t))_{t\geq 0}$ on $L^2(0, T; H)$, defined by

$$[\mathcal{T}(t)v](\vartheta) := \widetilde{v}(\vartheta - t), \quad t \geq 0,\ v \in L^2(0, T; H),\ \text{a.a. } \vartheta \in (0, T), \quad (3.6)$$

where $\widetilde{v} \in L^2(-\infty, T; H)$ denotes the extension of v by zero to $(-\infty, T)$. As shown in Proposition A.5 in "Appendix A", also $(\mathcal{T}(t))_{t \geq 0} \subseteq \mathscr{L}(L^2(0, T; H))$ is a C_0-semigroup and its infinitesimal generator is given by $-\partial_t$, where

$$\partial_t : \mathsf{D}(\partial_t) \subseteq L^2(0, T; H) \to L^2(0, T; H), \qquad \mathsf{D}(\partial_t) = H^1_{0,\{0\}}(0, T; H), \quad (3.7)$$

denotes the Bochner–Sobolev vector-valued weak derivative. We point out that the domain $\mathsf{D}(\partial_t) = H^1_{0,\{0\}}(0, T; H)$ encodes the zero initial condition of the SPDE (3.1). Furthermore, note that it readily follows from the definitions in (3.5) and (3.6) that, for all $t \geq 0$, every $v \in L^2(0, T; H)$, and a.a. $\vartheta \in (0, T)$,

$$[\mathcal{S}(t)\mathcal{T}(t)v](\vartheta) = [\mathcal{T}(t)\mathcal{S}(t)v](\vartheta) = S(t)\widetilde{v}(\vartheta - t),$$

i.e., the semigroups $(\mathcal{S}(t))_{t \geq 0}$ and $(\mathcal{T}(t))_{t \geq 0}$ commute.

We now define the sum operator $\partial_t + \mathcal{A} \colon \mathsf{D}(\partial_t + \mathcal{A}) \subseteq L^2(0, T; H) \to L^2(0, T; H)$ on its natural domain, that is

$$\begin{aligned} (\partial_t + \mathcal{A})v &:= \partial_t v + \mathcal{A}v, \\ v &\in \mathsf{D}(\partial_t + \mathcal{A}) = H^1_{0,\{0\}}(0, T; H) \cap L^2(0, T; \mathsf{D}(A)), \end{aligned} \tag{3.8}$$

with $\mathcal{A}$ and ∂_t as given in (3.4) and (3.7), respectively. The next proposition shows that the closure of $-(\partial_t + \mathcal{A})$ again generates a C_0-semigroup, namely the product semigroup of $(\mathcal{S}(t))_{t \geq 0}$ and $(\mathcal{T}(t))_{t \geq 0}$.

Proposition 3.2 *Let Assumption 3.1(i) be satisfied. The closure $\mathcal{B} := \overline{\partial_t + \mathcal{A}}$ of the sum operator $\partial_t + \mathcal{A}$ defined in (3.8) exists and $-\mathcal{B}$ generates the C_0-semigroup $(\mathcal{S}(t)\mathcal{T}(t))_{t \geq 0}$ on $L^2(0, T; H)$, which satisfies*

$$\|\mathcal{S}(t)\mathcal{T}(t)\|_{\mathscr{L}(L^2(0,T;H))} = \|\mathcal{T}(t)\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} = \begin{cases} \|S(t)\|_{\mathscr{L}(H)} & \text{if } 0 \leq t < T, \\ 0 & \text{if } t \geq T, \end{cases}$$

where $(\mathcal{S}(t))_{t \geq 0}$ and $(\mathcal{T}(t))_{t \geq 0}$ are defined as in (3.5) and (3.6), respectively.

Proof By the commutativity of the semigroups $(\mathcal{S}(t))_{t \geq 0}$ and $(\mathcal{T}(t))_{t \geq 0}$, we may conclude that $(\mathcal{T}(t)\mathcal{S}(t))_{t \geq 0}$ is a C_0-semigroup whose generator is an extension of $-(\partial_t + \mathcal{A})$, and the domain of the generator contains $H^1_{0,\{0\}}(0, T; H) \cap L^2(0, T; \mathsf{D}(A))$ as a subspace that is dense with respect to the graph norm, see [31, Example II.2.7]. Subsequently, Lemma A.2 shows that the generator is the closure of $-(\partial_t + \mathcal{A})$.

Fix $t \in [0, T)$. The inequality $\|\mathcal{T}(t)\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} \leq \|S(t)\|_{\mathscr{L}(H)}$ follows by the contractivity of $\mathcal{T}(t)$ and the operator norm isometry from Lemma A.1(a). Now we turn to the reverse inequality. By definition of the operator norm on $\mathscr{L}(H)$, there exists a normalized sequence $(x_n)_{n \in \mathbb{N}}$ in H such that $\|S(t)x_n\|_H \geq \|S(t)\|_{\mathscr{L}(H)} - \frac{1}{n}$ holds for all $n \in \mathbb{N}$. Correspondingly, define the sequence $(v_n)_{n \in \mathbb{N}}$ in $L^2(0, T; H)$ by $v_n(\vartheta) := (T - t)^{-1/2} \mathbf{1}_{(0, T-t)}(\vartheta) x_n$ for every $\vartheta \in (0, T)$ and all $n \in \mathbb{N}$. Note that

$\|v_n\|_{L^2(0,T;H)} = 1$ for every $n \in \mathbb{N}$, and

$$\|\mathcal{T}(t)\mathcal{S}(t)v_n\|_{L^2(0,T;H)} = \|(T-t)^{-1/2}\mathbf{1}_{(t,T)}\|_{L^2(0,T)}\|S(t)x_n\|_H \geq \|S(t)\|_{\mathscr{L}(H)} - \frac{1}{n}.$$

As this holds for all $n \in \mathbb{N}$, we conclude that $\|\mathcal{T}(t)\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} \geq \|S(t)\|_{\mathscr{L}(H)}$. The final assertion for $t \geq T$ follows from the fact that $\mathcal{T}(t) = 0$ for $t \geq T$. $\square$

Remark 3.3 The closure $\mathcal{B} = \overline{\partial_t + \mathcal{A}}$ appearing in Proposition 3.2 raises the question of when the sum operator itself is closed. The answer is intimately related to the subject of maximal L^p-regularity; we refer the reader to [29] or [47] for detailed accounts of this theory. In the Hilbert space setting, the sum turns out to be closed under Assumptions 3.1(i),(ii). Indeed, $[\partial_t]_{\mathbb{C}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}([\partial_t]_{\mathbb{C}}) \leq \frac{\pi}{2}$ since $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{T}_{\mathbb{C}}(t))_{t\geq 0}$ are contractive, see Definition B.3 in "Appendix B" and [42, Theorem 10.2.24]. By Assumption 3.1(ii) and Theorem B.2, we have $\omega(A_{\mathbb{C}}) < \frac{\pi}{2}$, and the same follows for $\mathcal{A}_{\mathbb{C}}$ by applying Lemma A.1(a) to its resolvent operators. Thus, we may conclude with [47, Theorem 12.13] that $[\partial_t + \mathcal{A}]_{\mathbb{C}}$ is closed, so that the same holds for $\partial_t + \mathcal{A}$.

We are now in the position to define fractional powers of the parabolic operator. For $\gamma \in (0, \infty)$ we work with the following representation (see "Appendix B.2.2"):

$$\mathcal{B}^{-\gamma} := \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma-1} \mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s = \frac{1}{\Gamma(\gamma)} \int_0^T s^{\gamma-1} \mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s. \quad (3.9)$$

Note that, for any $\gamma \in (0, \infty)$, this definition yields a well-defined bounded linear operator on $L^2(0, T; H)$, since the product semigroup $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$ was seen to be exponentially stable (in fact, eventually zero) in Proposition 3.2.

The next result shows that the pointwise evaluation of $\mathcal{B}^{-\gamma}f$ at $t \in [0, T]$ is meaningful, provided that $\gamma > \frac{1}{2}$.

Proposition 3.4 *Suppose Assumption 3.1(i) and let $p \in (1, \infty)$, $\gamma \in (1/p, \infty)$. Then*

$$f \mapsto \mathcal{B}_{\gamma,p}f, \quad [\mathcal{B}_{\gamma,p}f](t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s)f(s)\,\mathrm{d}s \quad \forall t \in [0, T],$$
$$(3.10)$$

defines a bounded linear operator, mapping $f \in L^p(0, T; H)$ into $C_{0,\{0\}}([0, T]; H)$.

In particular, if $\gamma \in (1/2, \infty)$, we have for the negative fractional parabolic operator $\mathcal{B}^{-\gamma}$ defined by (3.9) when acting on $f \in L^2(0, T; H)$ the pointwise formula

$$[\mathcal{B}^{-\gamma}f](t) = [\mathcal{B}_{\gamma,2}f](t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s)f(s)\,\mathrm{d}s \quad \forall t \in [0, T].$$
$$(3.11)$$

Proof By [25, Proposition 5.9], for $p \in (1, \infty)$ and $\gamma \in (1/p, \infty)$, the operator $\mathcal{B}_{\gamma,p}$ defined by (3.10) maps continuously from $L^p(0, T; H)$ to $C_{0,\{0\}}([0, T]; H)$.

Next, note that for all $f \in L^2(0, T; H)$ and a.a. $t \in [0, T]$, we obtain by (3.9)

$$[\mathcal{B}^{-\gamma}f](t) = \frac{1}{\Gamma(\gamma)}\int_0^\infty s^{\gamma-1}[\mathcal{S}(s)\mathcal{T}(s)f](t)\,\mathrm{d}s = \frac{1}{\Gamma(\gamma)}\int_0^t s^{\gamma-1}S(s)f(t-s)\,\mathrm{d}s$$

$$= \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}S(t-s)f(s)\,\mathrm{d}s = [\mathcal{B}_{\gamma,2}f](t).$$

Thus, by the first part of this proposition, for every $\gamma \in (1/2, \infty)$, we obtain that $R(\mathcal{B}^{-\gamma}) \subseteq C_{0,\{0\}}([0, T]; H)$ and the above identities hold pointwise in $t \in [0, T]$. $\square$

Remark 3.5 Propositions 3.2 and 3.4 require only Assumption 3.1(i), i.e., that $-A$ generates the C_0-semigroup $(S(t))_{t\geq 0}$. Exponential stability or uniform boundedness of $(S(t))_{t\geq 0}$ are not needed, since we consider linear operators on $L^2(0, T; H)$ (instead of $L^2(0, \infty; H)$), allowing us to use uniform boundedness of $(S(t))_{t\geq 0}$ on the compact interval $[0, T]$ to derive exponential stability of $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$.

In what follows, we may also consider the adjoint operator $\mathcal{B}^{-\gamma*} := (\mathcal{B}^{-\gamma})^*$. More specifically, we will use it in the next section to define a weak solution to the fractional parabolic SPDE (3.1). The following lemma provides useful results for the adjoint $\mathcal{B}^{-\gamma*}$ which are analogous to those for $\mathcal{B}^{-\gamma}$ in Proposition 3.4. For ease of presentation, the proof has been moved to Sect. A.3 of "Appendix A".

Lemma 3.6 *Suppose Assumption* 3.1(i) *and let* $\gamma \in (1/2, \infty)$. *The adjoint negative fractional parabolic operator* $\mathcal{B}^{-\gamma*}$ *maps* $g \in L^2(0, T; H)$ *into* $C_{0,\{T\}}([0, T]; H)$, *and*

$$[\mathcal{B}^{-\gamma*}g](s) = \frac{1}{\Gamma(\gamma)}\int_s^T (t-s)^{\gamma-1}[S(t-s)]^*g(t)\,\mathrm{d}t \quad \forall s \in [0, T]. \quad (3.12)$$

Finally, we note that $\mathcal{B}^{-\gamma*} = (\mathcal{B}^*)^{-\gamma}$. To see that the fractional power on the right-hand side is indeed well-defined, we use [61, Chapter 1, Corollary 10.6] and conclude that $-\mathcal{B}^*$ is the generator of the C_0-semigroup $([\mathcal{S}(t)\mathcal{T}(t)]^*)_{t\geq 0}$, which clearly inherits the exponential stability from $(\mathcal{S}(t)\mathcal{T}(t))_{t\geq 0}$ since their norms are equal. The identity is then obtained as follows,

$$\mathcal{B}^{-\gamma*} = \left(\frac{1}{\Gamma(\gamma)}\int_0^\infty s^{\gamma-1}\mathcal{S}(s)\mathcal{T}(s)\,\mathrm{d}s\right)^* = \frac{1}{\Gamma(\gamma)}\int_0^\infty s^{\gamma-1}[\mathcal{S}(s)\mathcal{T}(s)]^*\,\mathrm{d}s = (\mathcal{B}^*)^{-\gamma},$$

where the first and last identities are due to (3.9) and the second is a consequence of the general ability to interchange Bochner integrals and duality pairings.

3.2 Solution concepts, existence and uniqueness

We now turn towards defining solutions to (3.1) for fractional powers $\gamma \in (0, \infty)$. Recall from Sect. 2 that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$, and that $(W^Q(t))_{t\geq 0}$ is a cylindrical Q-Wiener process on H with respect to $(\mathcal{F}_t)_{t\geq 0}$, where $Q \in \mathcal{L}(H)$ is self-adjoint and strictly positive.

Having defined and investigated the parabolic operator $\mathcal{B}$, its domain and its fractional powers, we are now in particular able to invert the fractional operator $\mathcal{B}^\gamma$.

Equation (3.11) suggests the following definition of a fractional stochastic convolution as a *mild solution* to (3.1).

Definition 3.7 Let Assumption 3.1(i) hold and define, for $\gamma \in (0, \infty)$, the stochastic convolution

$$\widetilde{Z}_\gamma(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) \, dW^Q(s), \quad t \in [0, T]. \qquad (3.13)$$

A predictable H-valued stochastic process $Z_\gamma := (Z_\gamma(t))_{t \in [0,T]}$ is called a *mild solution* to (3.1) if, for all $t \in [0, T]$, it satisfies $Z_\gamma(t) = \widetilde{Z}_\gamma(t)$, $\mathbb{P}$-a.s.

We first address existence and mean-square continuity of mild solutions. Furthermore, we adapt the Da Prato–Kwapień–Zabczyk factorization method (see [24], [25, Section 5.3]) to establish the existence of a pathwise continuous modification.

Theorem 3.8 *Let Assumption 3.1(i) be satisfied and let $\gamma \in (0, \infty)$ be such that*

$$\exists \delta \in [0, \gamma): \quad \int_0^T \left\| t^{\gamma-1-\delta} S(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 dt < \infty. \qquad (3.14)$$

The stochastic convolution $\widetilde{Z}_\gamma(t)$ in (3.13) belongs to $L^2(\Omega; H)$ for all $t \in [0, T]$ if and only if (3.14) holds with $\delta = 0$. In this case, the mapping $t \mapsto \widetilde{Z}_\gamma(t)$ is an element of $C([0, T]; L^p(\Omega; H))$ for all $p \in [1, \infty)$; in particular, there exists a mild solution in the sense of Definition 3.7, and it is mean-square continuous.

Whenever (3.14) holds for some $\delta \in (0, \gamma)$, then for every $p \in [1, \infty)$ there exists a modification of $\widetilde{Z}_\gamma$ with continuous sample paths belonging to $L^p(\Omega; C([0, T]; H))$. In particular, the mild solution has a modification with continuous sample paths.

Proof We first consider the case $\delta = 0$ in (3.14). By the Itô isometry (see e.g. [52, Proposition 2.3.5 and p. 32]), we obtain the identity

$$\sup_{t \in [0,T]} \|\widetilde{Z}_\gamma(t)\|_{L^2(\Omega;H)}^2 = \frac{1}{|\Gamma(\gamma)|^2} \int_0^T \left\| t^{\gamma-1} S(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 dt.$$

Therefore, $\widetilde{Z}_\gamma(t) \in L^2(\Omega; H)$ holds for all $t \in [0, T]$ if and only if (3.14) is satisfied with $\delta = 0$. The fact that $t \mapsto \widetilde{Z}_\gamma(t)$ belongs to $C([0, T]; L^p(\Omega; H))$ for every $p \in [1, \infty)$ will be shown in greater generality in Proposition 3.18, see Sect. 3.3.3.

Moreover, note that $\widetilde{Z}_\gamma : [0, T] \times \Omega \to H$ is measurable and $(\mathcal{F}_t)_{t \in [0,T]}$-adapted, and that mean-square continuity implies continuity in probability, so that we may apply [63, Proposition 3.21] to conclude that there exists a predictable modification Z_γ of $\widetilde{Z}_\gamma$. Then, Z_γ is a mild solution to (3.1) in the sense of Definition 3.7.

Now suppose that (3.14) holds for some $\delta \in (0, \gamma)$ and let $p \in (1/\delta \vee 1, \infty)$. By the above considerations, $\widetilde{Z}_{\gamma-\delta}$ and $\widetilde{Z}_\gamma$ exist as elements of $C([0, T]; L^p(\Omega; H))$. In particular, $\widetilde{Z}_{\gamma-\delta}$ belongs to $L^p(0, T; L^p(\Omega; H))$, hence to $L^p(\Omega; L^p(0, T; H))$ by Fubini's theorem. For this reason, there exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 0$ such that $\widetilde{Z}_{\gamma-\delta}(\cdot, \omega) \in L^p(0, T; H)$ for all $\omega \in \Omega_0^c = \Omega \setminus \Omega_0$. We recall the linear operator

$\mathscr{B}_{\delta,p}\colon L^p(0,T;H) \to C_{0,\{0\}}([0,T];H)$ from (3.10) and claim that the process $\widehat{Z}_\gamma$ defined for $t \in [0,T]$ and $\omega \in \Omega$ by

$$\widehat{Z}_\gamma(t,\omega) := \begin{cases} [\mathscr{B}_{\delta,p}\widetilde{Z}_{\gamma-\delta}](t,\omega) & \text{if } (t,\omega) \in [0,T] \times \Omega_0^c, \\ 0 & \text{if } (t,\omega) \in [0,T] \times \Omega_0, \end{cases}$$

is the desired continuous modification of $\widetilde{Z}_\gamma$. To this end, firstly note that for all $\omega \in \Omega$ the mapping $t \mapsto \widehat{Z}_\gamma(t,\omega)$ indeed is continuous and $\widehat{Z}_\gamma$ belongs to $L^p(\Omega; C([0,T];H))$; this follows from Proposition 3.4 since $\delta \in (1/p, \infty)$. In order to show that $\widehat{Z}_\gamma$ is a modification of $\widetilde{Z}_\gamma$, we fix $t \in [0,T]$ and employ formulas (3.10) and (3.13) along with the semigroup property to obtain

$$
\begin{aligned}
\widehat{Z}_\gamma(t) &= [\mathscr{B}_{\delta,p}\widetilde{Z}_{\gamma-\delta}](t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} S(t-s) \widetilde{Z}_{\gamma-\delta}(s)\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)\Gamma(\gamma-\delta)} \int_0^t (t-s)^{\delta-1} S(t-s) \left[\int_0^s (s-r)^{\gamma-\delta-1} S(s-r)\,\mathrm{d}W^Q(r) \right] \mathrm{d}s \\
&= \frac{1}{\Gamma(\delta)\Gamma(\gamma-\delta)} \int_0^t \int_0^s (t-s)^{\delta-1}(s-r)^{\gamma-\delta-1} S(t-r)\,\mathrm{d}W^Q(r)\,\mathrm{d}s, \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

$$(3.15)$$

We set $\widetilde{M}_T := \sup_{t \in [0,T]} \|S(t)\|_{\mathscr{L}(H)}$, $K_T := \int_0^T \|t^{\gamma-1-\delta} S(t) Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\,\mathrm{d}t$ and find

$$
\begin{aligned}
&\int_0^t \left[\int_0^s \|(t-s)^{\delta-1}(s-r)^{\gamma-\delta-1} S(t-r) Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\,\mathrm{d}r \right]^{1/2} \mathrm{d}s \\
&\leq \widetilde{M}_T \int_0^t (t-s)^{\delta-1} \left[\int_0^s \|(s-r)^{\gamma-\delta-1} S(s-r) Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\,\mathrm{d}r \right]^{1/2} \mathrm{d}s \\
&= \widetilde{M}_T \int_0^t (t-s)^{\delta-1} \left[\int_0^s \|r^{\gamma-1-\delta} S(r) Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)}^2\,\mathrm{d}r \right]^{1/2} \mathrm{d}s \leq \frac{\widetilde{M}_T T^\delta \sqrt{K_T}}{\delta} < \infty.
\end{aligned}
$$

This estimate shows that

$$
\begin{aligned}
s &\mapsto \mathbf{1}_{(0,t)}(s)\mathbf{1}_{(0,s)}(\cdot)(t-s)^{\delta-1}(s-\cdot)^{\gamma-\delta-1} S(t-\cdot) Q^{\frac{1}{2}} \\
&\in L^1(0,T; L^2(0,T; \mathscr{L}_2(H))),
\end{aligned}
$$

and the stochastic Fubini theorem [63, Theorem 8.14] may be used in (3.15), yielding

$$\widehat{Z}_\gamma(t) = \frac{1}{\Gamma(\delta)\Gamma(\gamma-\delta)} \int_0^t \left[\int_r^t (t-s)^{\delta-1}(s-r)^{\gamma-\delta-1}\,\mathrm{d}s \right] S(t-r)\,\mathrm{d}W^Q(r), \quad \mathbb{P}\text{-a.s.}$$

Using the change of variables $u(s) := \frac{s-r}{t-r}$ and [59, Formula 5.12.1], we derive that

$$(t-r)^{1-\gamma} \int_r^t (t-s)^{\delta-1}(s-r)^{\gamma-\delta-1}\,\mathrm{d}s = \int_0^1 (1-u)^{\delta-1} u^{\gamma-\delta-1}\,\mathrm{d}u = \frac{\Gamma(\gamma-\delta)\Gamma(\delta)}{\Gamma(\gamma)},$$

which shows that $\widehat{Z}_\gamma(t) = \widetilde{Z}_\gamma(t)$ holds $\mathbb{P}$-a.s. Since $t \in [0, T]$ was arbitrary this implies that $\widehat{Z}_\gamma$ is a modification of $\widetilde{Z}_\gamma$ and completes the proof for $p \in (1/\delta \vee 1, \infty)$. The case $p \in [1, 1/\delta \vee 1]$ follows from the nestedness of the $L^p(\Omega; C([0, T]; H))$ spaces. $\qquad\square$

In order to provide a more rigorous justification for the Definition 3.7 of a mild solution to (3.1), we proceed as follows: We seek a further suitable solution concept of a *weak solution*, which follows "naturally" from (3.1) using $L^2(0, T; H)$ inner products, and show that weak and mild solutions are equivalent.

For this, we first define the *weak stochastic Itô integral* for $f : (0, T) \to \mathscr{L}(H)$ and $g : (0, T) \to H$ by

$$\int_0^t \big(f(s)\, \mathrm{d}W^Q(s), g(s)\big)_H := \int_0^t \widetilde{f_g}(s)\, \mathrm{d}W^Q(s), \quad t \in [0, T],$$

where $\int_0^T \big\| Q^{\frac{1}{2}}[f(s)]^* g(s) \big\|_H^2 \, \mathrm{d}s < \infty$ and $\widetilde{f_g} : (0, T) \to \mathscr{L}(H; \mathbb{R})$ is defined by

$$\widetilde{f_g}(s)x := (f(s)x, g(s))_H \quad \forall x \in H, \quad \forall s \in (0, T),$$

cf. [52, Lemma 2.4.2].

Definition 3.9 Let Assumption 3.1(i) hold and let $\gamma \in (0, \infty)$. A predictable H-valued stochastic process $Y_\gamma := (Y_\gamma(t))_{t\in[0,T]}$ is called a *weak solution* to (3.1) if it is mean-square continuous and, in addition,

$$\forall \psi \in \mathsf{D}(\mathcal{B}^{\gamma*}) : \quad (Y_\gamma, \mathcal{B}^{\gamma*}\psi)_{L^2(0,T;H)} = \int_0^T \big(\mathrm{d}W^Q(t), \psi(t)\big)_H, \quad \mathbb{P}\text{-a.s.} \,(3.16)$$

Remark 3.10 For $\gamma = 1$, a natural weak solution concept is the formulation given in [63, Definition 9.11]: A predictable H-valued process $(Y_1(t))_{t\in[0,T]}$ is a weak solution to (3.1) if $\sup_{t\in[0,T]} \|Y_1(t)\|_{L^2(\Omega;H)} < \infty$ and, for all $t \in [0, T]$ and $y \in \mathsf{D}(A^*)$,

$$(Y_1(t), y)_H = -\int_0^t (Y_1(s), A^*y)_H \, \mathrm{d}s + \big(W^Q(t), y\big)_H, \quad \mathbb{P}\text{-a.s.}$$

Provided that Assumption 3.1(i) and (3.14) are satisfied, by [63, Theorem 9.15] an H-valued stochastic process is a weak solution in this sense if and only if it is a mild solution in the sense of Definition 3.7 with $\gamma = 1$.

In the next proposition we generalize this result to an arbitrary fractional power γ and show that, under the same conditions, the mild solution in the sense of Definition 3.7 is equivalent to the weak solution in the sense of Definition 3.9.

Proposition 3.11 *Suppose that Assumption 3.1(i) holds and let $\gamma \in (0, \infty)$ be such that (3.14) is satisfied. Then, a stochastic process is a mild solution in the sense of Definition 3.7 if and only if it is a weak solution in the sense of Definition 3.9. Moreover, mild and weak solutions are unique up to modification. If one requires continuity of the sample paths, mild and weak solutions are unique up to indistinguishability.*

Proof First, we show that a mild solution Z_γ is a weak solution. Note that mean-square continuity follows from Theorem 3.8. Fix an arbitrary $\psi \in D(\mathcal{B}^{\gamma*})$. Then,

$$(Z_\gamma, \mathcal{B}^{\gamma*}\psi)_{L^2(0,T;H)} = \frac{1}{\Gamma(\gamma)} \int_0^T \left(\int_0^t (t-s)^{\gamma-1} S(t-s) \, dW^Q(s), [\mathcal{B}^{\gamma*}\psi](t) \right)_H dt$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \left(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s) \, dW^Q(s), [\mathcal{B}^{\gamma*}\psi](t) \right)_H dt \qquad (3.17)$$

holds $\mathbb{P}$-a.s. Here, we used that $\left(\int_0^T f(s) \, dW^Q(s), x \right)_H = \int_0^T (f(s) \, dW^Q(s), x)_H$ for all $f : (0,T) \to \mathscr{L}(H)$ and $x \in H$, which readily is derived from the definition of the weak stochastic integral and the continuity of inner products. We now would like to apply the stochastic Fubini theorem, see e.g. [63, Theorem 8.14], in order to interchange the inner weak stochastic integral and the outer deterministic integral. Again by the definition of the weak stochastic integral we have, for a.a. $t \in (0,T)$,

$$\int_0^T \left(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s) \, dW^Q(s), [\mathcal{B}^{\gamma*}\psi](t) \right)_H = \int_0^T \Psi(s,t) \, dW^Q(s), \quad \mathbb{P}\text{-a.s.},$$

where the integrand $\Psi(s,t) : H \to \mathbb{R}$ is deterministic and, for $s, t \in (0,T)$, defined by

$$\Psi(s,t)x := \left(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)x, [\mathcal{B}^{\gamma*}\psi](t) \right)_H \quad \forall x \in H. \qquad (3.18)$$

Thus, the usage of the stochastic Fubini theorem is justified if $t \mapsto \Psi(\cdot, t)Q^{\frac{1}{2}}$ is in $L^1(0,T; L^2(0,T; \mathscr{L}_2(H; \mathbb{R})))$. Given an orthonormal basis $(g_j)_{j \in \mathbb{N}}$ for H, we obtain

$$\left\| \Psi(s,t)Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\mathbb{R})}^2 = \sum_{j=1}^\infty \left| \left(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)Q^{\frac{1}{2}} g_j, [\mathcal{B}^{\gamma*}\psi](t) \right)_H \right|^2$$

$$\leq \left\| \mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s)Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \left\| [\mathcal{B}^{\gamma*}\psi](t) \right\|_H^2$$

by the Cauchy–Schwarz inequality on H. From this, it follows that

$$\left\| t \mapsto \Psi(\cdot, t)Q^{\frac{1}{2}} \right\|_{L^1(0,T;L^2(0,T;\mathscr{L}_2(H;\mathbb{R})))} = \int_0^T \left(\int_0^T \left\| \Psi(s,t)Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\mathbb{R})}^2 \, ds \right)^{1/2} dt$$

$$\leq \int_0^T \left(\int_0^t \left\| (t-s)^{\gamma-1} S(t-s)Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \left\| [\mathcal{B}^{\gamma*}\psi](t) \right\|_H^2 \, ds \right)^{1/2} dt$$

$$= \int_0^T \left(\int_0^t \left\| s^{\gamma-1} S(s)Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \, ds \right)^{1/2} \left\| [\mathcal{B}^{\gamma*}\psi](t) \right\|_H dt$$

$$\leq T^{1/2} \|\mathcal{B}^{\gamma*}\psi\|_{L^2(0,T;H)} \left(\int_0^T \left\| s^{\gamma-1} S(s)Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \, ds \right)^{1/2} < \infty,$$

where we used the Cauchy–Schwarz inequality on $L^2(0,T)$ in the last step. Owing to (3.14), the integral in the final expression is finite. Applying the stochastic Fubini

theorem to (3.17), taking adjoints in (3.18) and using the continuity of $(\cdot, \cdot)_H$ gives

$$
\begin{aligned}
(Z_\gamma, \mathcal{B}^{\gamma*}\psi)_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \Psi(s,t) \, dt \, dW^Q(s) \\
&= \int_0^T \left(dW^Q(s), \frac{1}{\Gamma(\gamma)} \int_s^T (t-s)^{\gamma-1} [S(t-s)]^* [\mathcal{B}^{\gamma*}\psi](t) \, dt \right)_H \\
&= \int_0^T \left(dW^Q(s), [\mathcal{B}^{-\gamma*}\mathcal{B}^{\gamma*}\psi](s) \right)_H = \int_0^T \left(dW^Q(s), \psi(s) \right)_H, \qquad \mathbb{P}\text{-a.s.,}
\end{aligned}
$$

where we used (3.12) in the third line. Therefore, Z_γ is a weak solution.

Conversely, suppose that Y_γ is a weak solution, let an arbitrary $\phi \in L^2(0, T; H)$ be given and set $\psi := \mathcal{B}^{-\gamma*}\phi \in \mathsf{D}(\mathcal{B}^{\gamma*})$. Substituting this into (3.16) gives

$$
(Y_\gamma, \phi)_{L^2(0,T;H)} = \int_0^T \left(dW^Q(t), [\mathcal{B}^{-\gamma*}\phi](t) \right)_H, \quad \mathbb{P}\text{-a.s.}
$$

Let $(\widetilde{Z}_\gamma(t))_{t\in[0,T]}$ be the stochastic convolution in (3.13). Since the condition for the stochastic Fubini theorem still holds after replacing $\mathcal{B}^{\gamma*}\psi$ by ϕ in (3.18), the proof of the previous implication can be read backwards to see that

$$
\forall \phi \in L^2(0, T; H): \quad \mathbb{P}\left((Y_\gamma, \phi)_{L^2(0,T;H)} = (\widetilde{Z}_\gamma, \phi)_{L^2(0,T;H)} \right) = 1.
$$

By separability of H, also $\mathbb{P}(Y_\gamma = \widetilde{Z}_\gamma$ in $L^2(0, T; H)) = 1$ holds so that by Fubini's theorem $Y_\gamma = \widetilde{Z}_\gamma$ in $L^2(0, T; L^2(\Omega; H))$ follows. Since both Y_γ and $\widetilde{Z}_\gamma$ are mean-square continuous, this shows that, for all $t \in [0, T]$, $Y_\gamma(t) = \widetilde{Z}_\gamma(t)$ in $L^2(\Omega; H)$. Therefore, for all $t \in [0, T]$, we have that $Y_\gamma(t) = \widetilde{Z}_\gamma(t)$, $\mathbb{P}$-a.s., i.e., Y_γ is a mild solution.

It thus suffices to prove uniqueness only for mild solutions. By Definition 3.7, mild solutions are modifications of the stochastic convolution $\widetilde{Z}_\gamma$ in (3.13), hence of each other. If two mild solutions are moreover known to have continuous sample paths, then they are indistinguishable by [63, Proposition 3.17]. □

3.3 Spatiotemporal regularity of solutions

We now investigate spatiotemporal regularity of the mild solution Z_γ in Definition 3.7. We start by stating our main results, Theorem 3.12 and Corollary 3.13, in Sect. 3.3.1. In Sect. 3.3.2 we derive a simplified condition for spatiotemporal regularity, which is easier to check in applications and sufficient whenever A satisfies Assumptions 3.1(i),(iii),(iv), see Proposition 3.14. In addition, we explicitly discuss the setting of a Gelfand triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ in the case that the operator A is induced by a (not necessarily symmetric) bilinear form $\mathfrak{a} \colon V \times V \to \mathbb{R}$ which is continuous and satisfies a Gårding inequality. Section 3.3.3 is devoted to the proof of Theorem 3.12.

3.3.1 Main results

In Theorem 3.12 below, the temporal regularity of the (weak or mild) solution is measured by the differentiability $n \in \mathbb{N}_0$ as well as the Hölder exponent $\tau \in [0, 1)$. Spatial regularity is expressed by means of vector spaces which are defined in terms of fractional powers of A (see Sect. B.2.2 in "Appendix B") as follows:

$$\dot{H}_A^\sigma := \mathsf{D}(A^{\sigma/2}), \qquad (x, y)_{\dot{H}_A^\sigma} := (A^{\sigma/2}x, A^{\sigma/2}y)_H, \quad \sigma \in [0, \infty).$$

For $\sigma \in (0, \infty)$, $\dot{H}_A^\sigma$ is a Hilbert space provided that Assumptions 3.1(i),(ii),(iv) are satisfied. In this case, we have the embeddings $\dot{H}_A^{\sigma'} \hookrightarrow \dot{H}_A^\sigma \hookrightarrow H$ for all $\sigma' \geq \sigma \geq 0$. Note, in particular, that we do not need to assume that A is self-adjoint.

Theorem 3.12 *Suppose that Assumptions 3.1(i),(ii) are satisfied and let* $n \in \mathbb{N}_0$, $\sigma \in [0, \infty)$ *and* $\gamma \in \left(\frac{\sigma - r}{2} + n, \infty\right)$, *where* $r \in [0, \sigma]$ *is such that* $Q^{\frac{1}{2}} \in \mathscr{L}(H; \dot{H}_A^r)$. *In the case that* $\sigma \in (0, \infty)$, *suppose furthermore that Assumption 3.1(iv) is fulfilled. Under the condition*

$$\int_0^T \left\| t^{\gamma - 1 - n} S(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}^2 \, \mathrm{d}t < \infty, \tag{3.19}$$

the mild solution Z_γ *(or, equivalently, the weak solution* Y_γ*) in the sense of Definition 3.7 (or 3.9) belongs to* $C^{n,0}([0, T]; L^p(\Omega; \dot{H}_A^\sigma))$ *for every* $p \in [1, \infty)$.

If additionally $\gamma \geq n + \tau + \frac{1}{2}$ *and* $A^{n + \tau + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \in \mathscr{L}_2(H; \dot{H}_A^\sigma)$ *are satisfied for some* $\tau \in (0, 1)$, *then we have* $Z_\gamma \in C^{n,\tau}([0, T]; L^p(\Omega; \dot{H}_A^\sigma))$ *for every* $p \in [1, \infty)$.

An application of the Kolmogorov–Chentsov continuity theorem, see e.g. [20, Theorem 3.9], allows us to (partially) transport the temporal regularity result of Theorem 3.12 to the pathwise setting, as seen in the next corollary.

Corollary 3.13 *Suppose that Assumptions 3.1(i),(ii) are satisfied. Let* $\sigma \in [0, \infty)$, $r \in [0, \sigma]$, $\gamma \in \left(\frac{\sigma - r}{2}, \infty\right)$ *and* $\tau \in (0, 1)$ *be such that* $Q^{\frac{1}{2}} \in \mathscr{L}(H; \dot{H}_A^r)$ *and* $\gamma \geq \tau + \frac{1}{2}$. *If* $\sigma \in (0, \infty)$, *suppose also that Assumption 3.1(iv) holds. If the condition*

$$\left\| A^{\tau + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)} + \int_0^T \left\| t^{\gamma - 1} S(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}^2 \, \mathrm{d}t < \infty$$

is satisfied, then for all $p \in [1, \infty)$ *and every* $\tau' \in [0, \tau)$ *there exists a modification* $\widehat{Z}_\gamma$ *of the mild solution* Z_γ *(or, equivalently, of the weak solution* Y_γ*) in the sense of Definition 3.7 (or 3.9) such that* $\widehat{Z}_\gamma$ *has* τ'-*Hölder continuous sample paths and belongs to* $L^p\left(\Omega; C^{0,\tau'}([0, T]; \dot{H}_A^\sigma)\right)$.

Proof We first invoke Theorem 3.12 with $n = 0$ and $\tau \in (0, 1)$ to establish that Z_γ belongs to $C^{0,\tau}([0, T]; L^q(\Omega; \dot{H}_A^\sigma))$ for every $q \in [1, \infty)$. The result then follows by choosing $q \geq 1$ sufficiently large, applying the Kolmogorov–Chentsov continuity theorem (see e.g. [20, Theorem 3.9]), and using nestedness of the L^p spaces. □

3.3.2 A simplified condition and its application to the Gårding inequality case

Whenever also Assumption 3.1(iii) holds, it is possible to replace the condition (3.19) by one which is simpler to check in practice. In this case, the operator A satisfies *square function estimates* (see Sect. B.2.3 in "Appendix B"), one of which is used to prove the next result.

Proposition 3.14 *Let Assumptions 3.1(i),(iii),(iv) be satisfied. Suppose that the constants $\sigma, \delta \in [0, \infty)$ and $\gamma \in \left(\frac{1}{2} + \delta, \infty\right) \cap \left[\frac{1}{2} + \delta + \frac{\sigma - r}{2}, \infty\right)$ are given, where $r \in [0, \sigma]$ is taken such that $Q^{\frac{1}{2}} \in \mathscr{L}(H; \dot{H}_A^r)$. Then,*

$$\int_0^\infty \left\| t^{\gamma - 1 - \delta} S(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}^2 \mathrm{d}t \eqsim_{(\gamma, \delta)} \left\| A^{\delta + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}^2.$$

Proof Applying Lemma B.7, see "Appendix B", with $a := \gamma - \delta - \frac{1}{2} \in (0, \infty)$ and $x := A^{\frac{\sigma}{2} + \delta + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} y \in H$ for $y \in H$ shows that

$$\int_0^\infty \left\| t^{\gamma - 1 - \delta} A^{\gamma - \delta - \frac{1}{2}} S(t) A^{\delta + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} y \right\|_{\dot{H}_A^\sigma}^2 \mathrm{d}t \eqsim_{(\gamma, \delta)} \left\| A^{\delta + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} y \right\|_{\dot{H}_A^\sigma}^2 \quad \forall y \in H.$$

Summing both sides over an orthonormal basis for H and using the Fubini–Tonelli theorem to interchange integration and summation on the left-hand side yields the desired conclusion. □

Remark 3.15 Proposition 3.14 shows that under the additional assumption that $A_\mathbb{C}$ admits a bounded H^∞-calculus with $\omega_{H^\infty}(A_\mathbb{C}) < \frac{\pi}{2}$, which e.g. is satisfied whenever A is self-adjoint and strictly positive, it suffices to check that $\gamma > n + \frac{(\sigma - r) \vee 1}{2}$ and $\gamma \geq n + \frac{1 + (\sigma - r) \vee (2\tau)}{2}$ and that the Hilbert–Schmidt norm $\| A^{n + \tau + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}$ is bounded to conclude the regularity results of Theorem 3.12. This condition coincides with the one imposed in [46, Section 4, Theorem 6] to derive regularity in the non-fractional case $\gamma = 1$ for $p = 2$, $\sigma = 0$, $n = 0$ and $\tau \in [0, 1/2]$.

Corollary 3.16 *Let $\delta \in [0, \infty)$ and $\gamma \in \left(\frac{1}{2} + \delta, \infty\right)$. Suppose that A satisfies Assumption 3.1(i) and that there exists a constant $\eta \in [0, \infty)$ such that $\widehat{A} := A + \eta I$ satisfies Assumptions 3.1(i),(iii),(iv) and $\widehat{A}^{\delta + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \in \mathscr{L}_2(H)$. Then, the mild solution Z_γ in the sense of Definition 3.7 exists and belongs to $C([0, T]; L^p(\Omega; H))$ for every $p \in [1, \infty)$. If $\delta > 0$, then for every $p \in [1, \infty)$ there exists a modification of Z_γ in $L^p(\Omega; C([0, T]; H))$ which has continuous sample paths.*

Proof Note that $S(t) = e^{\eta t} \widehat{S}(t)$ holds for every $t \geq 0$, where $(\widehat{S}(t))_{t \geq 0}$ denotes the C_0-semigroup generated by $-\widehat{A}$. Hence, by Proposition 3.14 we find that

$$\int_0^T \left\| t^{\gamma - 1 - \delta} S(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}t \leq e^{2\eta T} \int_0^T \left\| t^{\gamma - 1 - \delta} \widehat{S}(t) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \mathrm{d}t$$
$$\lesssim_{(\gamma, \delta)} e^{2\eta T} \left\| \widehat{A}^{\delta + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 < \infty.$$

The claim now follows from Theorem 3.8. □

We illustrate the utility of Corollary 3.16 in the following example. It is concerned with the case that the operator A is induced by a bounded bilinear form $\mathfrak{a}\colon V \times V \to \mathbb{R}$, where $V \hookrightarrow H$ is dense in H, and $\mathfrak{a}$ is not necessarily coercive on V; see also [37, Section 7.3.2]. We note that this setting applies to a variety of important applications, including symmetric and non-symmetric differential operators of even orders.

Example 3.17 Let $(V, (\,\cdot\,,\,\cdot\,)_V)$ be a Hilbert space which is densely and continuously embedded in H. Suppose that $A\colon \mathsf{D}(A) \subseteq H \to H$ is induced by a bounded bilinear form $\mathfrak{a}\colon V \times V \to \mathbb{R}$ which satisfies a Gårding inequality, i.e., there exist constants $\alpha_0, \alpha_1 \in (0, \infty)$ and $\eta \in [0, \infty)$ such that

$$|\mathfrak{a}(u, v)| \leq \alpha_1 \|u\|_V \|v\|_V \qquad \forall u, v \in V, \qquad (3.20)$$

$$\mathfrak{a}(u, u) \geq \alpha_0 \|u\|_V^2 - \eta \|u\|_H^2 \qquad \forall u \in V. \qquad (3.21)$$

The Gårding inequality (3.21) can be interpreted as coercivity of the bilinear form $\hat{\mathfrak{a}}(u, v) := \mathfrak{a}(u, v) + \eta(u, v)_H$ on V, associated with $\widehat{A} = A + \eta I$, while (3.20) implies that $\hat{\mathfrak{a}}$ is bounded. The complexified sesquilinear form $\hat{\mathfrak{a}}_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$, which is defined analogously to (B.2) and induces the operator $\widehat{A}_{\mathbb{C}}$, inherits the boundedness and coercivity from $\hat{\mathfrak{a}}$. Thus, there exist $\widehat{\alpha}_0, \widehat{\alpha}_1 \in (0, \infty)$ such that

$$|\hat{\mathfrak{a}}_{\mathbb{C}}(u, v)| \leq \widehat{\alpha}_1 \|u\|_{V_{\mathbb{C}}} \|v\|_{V_{\mathbb{C}}} \qquad \forall u, v \in V_{\mathbb{C}},$$

$$\operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u, u) \geq \widehat{\alpha}_0 \|u\|_{V_{\mathbb{C}}}^2 \qquad \forall u \in V_{\mathbb{C}}.$$

Therefore, $\widehat{\alpha}_0 \|u\|_{V_{\mathbb{C}}}^2 \leq \operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u, u) \leq |\hat{\mathfrak{a}}_{\mathbb{C}}(u, u)| \leq \widehat{\alpha}_1 \|u\|_{V_{\mathbb{C}}}^2 \leq \frac{\widehat{\alpha}_1}{\widehat{\alpha}_0} \operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u, u)$ follows for every $u \in V_{\mathbb{C}}$. If $V_{\mathbb{C}} \neq \{0\}$, these estimates imply that $\widehat{\alpha}_0 \leq \widehat{\alpha}_1$ and

$$|\operatorname{Im} \hat{\mathfrak{a}}_{\mathbb{C}}(u, u)|$$
$$= \sqrt{|\hat{\mathfrak{a}}_{\mathbb{C}}(u, u)|^2 - |\operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u, u)|^2} \leq \left(\frac{\widehat{\alpha}_1^2}{\widehat{\alpha}_0^2} - 1\right)^{1/2} \operatorname{Re} \hat{\mathfrak{a}}_{\mathbb{C}}(u, u) \quad \forall u \in V_{\mathbb{C}}.$$

This shows that $-\widehat{A}_{\mathbb{C}}$ generates a bounded analytic C_0-semigroup $(\widehat{S}_{\mathbb{C}}(t))_{t \geq 0}$ of contractions on $H_{\mathbb{C}}$, cf. [60, Theorem 1.54], where we used that $(-\infty, 0) \subseteq \rho(\widehat{A}_{\mathbb{C}})$ by [60, Proposition 1.22]. Applying [42, Theorems 10.2.24 and 10.4.21] and using that $\omega(\widehat{A}_{\mathbb{C}}) \in [0, \frac{\pi}{2})$ because $(\widehat{S}_{\mathbb{C}}(t))_{t \geq 0}$ is bounded analytic (see Theorem B.2), we find that $\widehat{A}_{\mathbb{C}}$ admits a bounded H^∞-calculus of angle $\omega_{H^\infty}(\widehat{A}_{\mathbb{C}}) = \omega(\widehat{A}_{\mathbb{C}}) \in [0, \frac{\pi}{2})$. Thus, we are in the setting of Corollary 3.16. In particular, the existence of a mean-square continuous mild solution to (3.1) for $\gamma > \frac{1}{2}$ follows if $\|\widehat{A}^{\frac{1}{2} - \gamma} Q^{\frac{1}{2}}\|_{\mathscr{L}_2(H)} < \infty$.

3.3.3 The proof of Theorem 3.12

We split the proof of Theorem 3.12 into several intermediate results. Before stating and proving these, we introduce the following function, which generalizes the integrand in (3.13) used to define mild solutions. Given $a \in \mathbb{R}$, $b \in [0, \infty)$ and $\sigma \in [0, \infty)$,

define $\Phi_{a,b} \colon (0, \infty) \to \mathscr{L}(H; \dot{H}_A^\sigma)$ by

$$\Phi_{a,b}(t) := t^a A^b S(t) Q^{\frac{1}{2}}, \qquad t \in (0, \infty). \tag{3.22}$$

Note that a mild solution Z_γ in the sense of Definition 3.7 satisfies the relation

$$\forall t \in [0, T]: \quad Z_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \Phi_{\gamma-1,0}(t - s) \, \mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.,}$$

where $\widehat{W}(t) := Q^{-\frac{1}{2}} W^Q(t)$, $t \geq 0$, is a cylindrical Wiener process.

The first result quantifies spatial regularity of the continuous-in-time stochastic convolution with $\Phi_{a,b}$ in $L^p(\Omega; \dot{H}_A^\sigma)$-sense. Recall from Sect. 2 that $(W(t))_{t \geq 0}$ denotes an (arbitrary) H-valued cylindrical Wiener process with respect to $(\mathcal{F}_t)_{t \geq 0}$.

Proposition 3.18 *Let Assumption* 3.1(i) *hold. Suppose that the constants* $a \in \mathbb{R}$, $b, \sigma \in [0, \infty)$ *and* $T \in (0, \infty)$ *are given. If* $\sigma \neq 0$, *then suppose moreover that Assumptions* 3.1(ii),(iv) *are satisfied. If the function* $\Phi_{a,b}$ *defined in* (3.22) *belongs to* $L^2(0, T; \mathscr{L}_2(H; \dot{H}_A^\sigma))$, *i.e.,*

$$\int_0^T \|\Phi_{a,b}(t)\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}^2 \, \mathrm{d}t < \infty,$$

then $t \mapsto \int_0^t \Phi_{a,b}(t - s) \, \mathrm{d}W(s)$ *belongs to* $C([0, T]; L^p(\Omega; \dot{H}_A^\sigma))$ *for all* $p \in [1, \infty)$.

Proof We first note that the assumption $\Phi_{a,b} \in L^2(0, T; \mathscr{L}_2(H; \dot{H}_A^\sigma))$, combined with the Burkholder–Davis–Gundy inequality (see [52, Theorem 6.1.2]) and the continuous embedding

$$L^2(\Omega; \dot{H}_A^\sigma) \hookrightarrow L^p(\Omega; \dot{H}_A^\sigma), \qquad p \in [1, 2), \ \sigma \in [0, \infty), \tag{3.23}$$

imply that $\int_0^t \Phi_{a,b}(t - s) \, \mathrm{d}W(s)$ indeed is a well-defined element of $L^p(\Omega; \dot{H}_A^\sigma)$ for all $t \in [0, T]$ and every $p \in [1, \infty)$.

It remains to check the $L^p(\Omega; \dot{H}_A^\sigma)$-continuity of $t \mapsto \int_0^t \Phi_{a,b}(t - s) \, \mathrm{d}W(s)$. For fixed $t \in [0, T)$ and $h \in (0, T - t]$, we split the stochastic integrals as follows:

$$\int_0^{t+h} \Phi_{a,b}(t + h - s) \, \mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t - s) \, \mathrm{d}W(s)$$

$$= \int_t^{t+h} \Phi_{a,b}(t + h - s) \, \mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t + h - s) - \Phi_{a,b}(t - s)] \, \mathrm{d}W(s).$$

For $p \in [2, \infty)$, the Burkholder–Davis–Gundy inequality yields

$$\left\| \int_t^{t+h} \Phi_{a,b}(t + h - s) \, \mathrm{d}W(s) + \int_0^t [\Phi_{a,b}(t + h - s) - \Phi_{a,b}(t - s)] \, \mathrm{d}W(s) \right\|_{L^p(\Omega; \dot{H}_A^\sigma)}$$

$$\lesssim_p \left[\int_t^{t+h} \|\Phi_{a,b}(t+h-s)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}s\right]^{1/2}$$

$$+ \left[\int_0^t \|\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}s\right]^{1/2}$$

$$= \left[\int_0^h \|\Phi_{a,b}(u)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}u\right]^{1/2} + \left[\int_0^t \|\Phi_{a,b}(r+h) - \Phi_{a,b}(r)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}r\right]^{1/2},$$

where $u := t + h - s$ and $r := t - s$. Since $\Phi_{a,b} \in L^2(0, T; \mathscr{L}_2(H; \dot{H}_A^\sigma))$ the first integral tends to zero as $h \downarrow 0$ by dominated convergence. The second term tends to zero by Lemma A.4, see "Appendix A".

For $t \in (0, T]$ and $h \in [-t, 0)$, the difference of stochastic integrals can be rewritten using $\int_0^t = \int_0^{t+h} + \int_{t+h}^t$. Thus, we obtain, for every $p \in [2, \infty)$, the bound

$$\left\|\int_0^{t+h} \Phi_{a,b}(t+h-s) \, \mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s) \, \mathrm{d}W(s)\right\|_{L^p(\Omega;\dot{H}_A^\sigma)}$$

$$\lesssim_p \left[\int_0^{-h} \|\Phi_{a,b}(r)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}r\right]^{1/2} + \left[\int_{-h}^t \|\Phi_{a,b}(r+h) - \Phi_{a,b}(r)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}r\right]^{1/2},$$

where we again used the change of variables $r := t - s$. Both terms on the last line tend to zero, again by dominated convergence and Lemma A.4, respectively.

Finally, we note that the result for $p = 2$ implies that for $p \in [1, 2)$ by (3.23). $\quad\square$

Furthermore, we obtain the following result regarding the temporal Hölder continuity of the stochastic convolution with the function $\Phi_{a,b}$ in (3.22).

Proposition 3.19 *Suppose that Assumptions* 3.1(i),(ii) *are fulfilled, let* $T \in (0, \infty)$, $a \in \left(-\frac{1}{2}, \infty\right)$, $b, \sigma \in [0, \infty)$ *and* $\tau \in \left(0, a + \frac{1}{2}\right] \cap (0, 1)$. *If* $\sigma \neq 0$, *then suppose also that Assumption* 3.1(iv) *holds. If* $A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \in \mathscr{L}_2(H; \dot{H}_A^\sigma)$ *and* $\Phi_{a,b}$ *is defined by* (3.22), *then* $t \mapsto \int_0^t \Phi_{a,b}(t-s) \, \mathrm{d}W(s)$ *belongs to* $C^{0,\tau}([0, T]; L^p(\Omega; \dot{H}_A^\sigma))$ *for all* $p \in [1, \infty)$.

Proof For $t \in [0, T)$ and $h \in (0, T - t]$, we obtain

$$\left\|\int_0^{t+h} \Phi_{a,b}(t+h-s) \, \mathrm{d}W(s) - \int_0^t \Phi_{a,b}(t-s) \, \mathrm{d}W(s)\right\|_{L^p(\Omega;\dot{H}_A^\sigma)}$$

$$\leq \left\|\int_0^t \left[\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\right] \mathrm{d}W(s)\right\|_{L^p(\Omega;\dot{H}_A^\sigma)}$$

$$+ \left\|\int_t^{t+h} \Phi_{a,b}(t+h-s) \, \mathrm{d}W(s)\right\|_{L^p(\Omega;\dot{H}_A^\sigma)}$$

$$\lesssim_{(p,a,\tau)} h^\tau \left\|A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}}\right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}$$

by Lemmas A.6 and A.7, see "Appendix A". The analogous result for the case that $t \in (0, T]$ and $h \in [-t, 0)$ follows upon splitting $\int_0^t = \int_0^{t+h} + \int_{t+h}^t$ and applying the lemmas with $\bar{t} := t + h \in [0, T)$ and $\bar{h} := -h \in (0, T - \bar{t}\,]$. $\quad\square$

We now investigate temporal mean-square differentiability. To this end, we need the following estimate which is implied by (3.3): For all $a \in \mathbb{R}$, $b \in [0, \infty)$, we have

$$\forall c \in [0, \infty) : \quad \|\Phi_{a,b}(t)x\|_H \lesssim_c t^{a-c} \|A^{b-c} Q^{\frac{1}{2}} x\|_H \quad \forall x \in \mathsf{D}(A^{b-c} Q^{\frac{1}{2}}). \quad (3.24)$$

The next lemma records some information about the derivatives of $\Phi_{a,b}$ in (3.22).

Lemma 3.20 *Let Assumptions* 3.1(i),(ii) *be satisfied, and let* $a \in \mathbb{R}$, $b, \sigma \in [0, \infty)$. *If* $\sigma \in (0, \infty)$, *suppose furthermore that Assumption* 3.1(iv) *holds. Then, the function* $\Phi_{a,b}$ *defined by* (3.22) *belongs to* $C^\infty((0, \infty); \mathscr{L}(H; \dot{H}_A^\sigma))$ *with kth derivative*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} \Phi_{a,b}(t) = \sum_{j=0}^k C_{a,j,k} t^{a-(k-j)} A^{b+j} S(t) Q^{\frac{1}{2}} = \sum_{j=0}^k C_{a,j,k} \Phi_{a-(k-j),b+j}(t),$$
$$(3.25)$$

where $C_{a,j,k} := (-1)^j \binom{k}{j} \prod_{i=1}^{k-j} (a - (k - j) + i)$ *for* $a \in \mathbb{R}$, $j, k \in \mathbb{N}_0$, $j \le k$.

Moreover, if $r \in [0, 2b + \sigma]$ *is such that* $Q^{\frac{1}{2}} \in \mathscr{L}(H; \dot{H}_A^r)$ *and* $n \in \mathbb{N}_0$ *satisfies* $n < a - b - \frac{\sigma - r}{2}$, *then* $\Phi_{a,b}$ *has a continuous extension in* $C^n([0, \infty); \mathscr{L}(H; \dot{H}_A^\sigma))$ *with all n derivatives vanishing at zero.*

Proof Since $(S(t))_{t \ge 0}$ is assumed to be analytic, $S(\cdot)$ is infinitely differentiable from $(0, \infty)$ to $\mathscr{L}(H)$, with jth derivative $(-A)^j S(\cdot)$ and, for $t \in (0, \infty)$, $\varepsilon := \frac{t}{2}$,

$$\begin{aligned}
\left[A^{b+\frac{\sigma}{2}} S(\cdot) \right]^{(j)}(t) &= \left[S(\cdot - \varepsilon) A^{b+\frac{\sigma}{2}} S(\varepsilon) \right]^{(j)}(t) \\
&= (-A)^j S(t - \varepsilon) A^{b+\frac{\sigma}{2}} S(\varepsilon) = (-1)^j A^{j+b+\frac{\sigma}{2}} S(t).
\end{aligned}$$

Here, the limits for the derivatives are taken in the $\mathscr{L}(H)$ norm. This is equivalent to $[A^b S(\cdot)]^{(j)}(t) = (-1)^j A^{j+b} S(t)$ with respect to the $\mathscr{L}(H; \dot{H}_A^\sigma)$ norm. The expression for the kth derivative of $\Phi_{a,b}$ thus follows from the Leibniz rule.

Now let $r \in [0, 2b+\sigma]$, $n \in \mathbb{N}_0$ be such that $n < a - b - \frac{\sigma - r}{2}$ and $Q^{\frac{1}{2}} \in \mathscr{L}(H; \dot{H}_A^r)$. To prove the second claim, we derive that for all $k \in \{0, 1, \ldots, n\}$ and $t \in (0, \infty)$

$$\begin{aligned}
\left\| \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Phi_{a,b}(t) \right\|_{\mathscr{L}(H; \dot{H}_A^\sigma)} &= \left\| \sum_{j=0}^k C_{a,j,k} t^{a-(k-j)} A^{b+j+\frac{\sigma-r}{2}} S(t) A^{\frac{r}{2}} Q^{\frac{1}{2}} \right\|_{\mathscr{L}(H)} \\
&\lesssim_{(a,b,k,r,\sigma)} t^{a-k-b-\frac{\sigma-r}{2}} \left\| Q^{\frac{1}{2}} \right\|_{\mathscr{L}(H; \dot{H}_A^r)}
\end{aligned}$$

by applying (3.24) to each summand with $c := b + j + \frac{\sigma-r}{2} \ge 0$. Furthermore, since $a - k - b - \frac{\sigma-r}{2} \ge a - n - b - \frac{\sigma-r}{2} > 0$, the above quantity tends to zero as $t \downarrow 0$. Hence, extending $t \mapsto \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Phi_{a,b}(t)$ by zero at $t = 0$ gives a function in $C([0, \infty); \mathscr{L}(H; \dot{H}_A^\sigma))$ for all $k \in \{0, 1, \ldots, n\}$. Inductively it follows then that the kth derivative of the zero extension is the zero extension of the original kth derivative. □

Proposition 3.21 *Let $\sigma \in [0, \infty)$, and additionally require Assumptions* 3.1*(i),(ii),(iv) whenever $\sigma \in (0, \infty)$. Suppose that $\Psi \in H^1_{0,\{0\}}(0, T; \mathscr{L}_2(H; \dot{H}^\sigma_A))$ and let Ψ' denote its weak derivative. Then, for every $p \in [1, \infty)$, the stochastic convolution*

$$t \mapsto \int_0^t \Psi(t - s) \, \mathrm{d}W(s)$$

is differentiable from $[0, T]$ to $L^p(\Omega; \dot{H}^\sigma_A)$, with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \Psi(t - s) \, \mathrm{d}W(s) = \int_0^t \Psi'(t - s) \, \mathrm{d}W(s) \quad \forall t \in [0, T]. \tag{3.26}$$

Proof For $t \in [0, T)$ and $h \in (0, T - t]$, we can write

$$\frac{1}{h} \left[\int_0^{t+h} \Psi(t + h - s) \, \mathrm{d}W(s) - \int_0^t \Psi(t - s) \, \mathrm{d}W(s) \right] - \int_0^t \Psi'(t - s) \, \mathrm{d}W(s)$$

$$= \int_0^t \left[\frac{\Psi(t + h - s) - \Psi(t - s)}{h} - \Psi'(t - s) \right] \mathrm{d}W(s)$$

$$+ \frac{1}{h} \int_t^{t+h} \Psi(t + h - s) \, \mathrm{d}W(s)$$

$$=: I_1^{h^+} + I_2^{h^+}.$$

For $t \in (0, T]$ and $h \in [-t, 0)$, we instead have

$$\frac{1}{h} \left[\int_0^{t+h} \Psi(t + h - s) \, \mathrm{d}W(s) - \int_0^t \Psi(t - s) \, \mathrm{d}W(s) \right] - \int_0^t \Psi'(t - s) \, \mathrm{d}W(s)$$

$$= \int_0^{t+h} \left[\frac{\Psi(t + h - s) - \Psi(t - s)}{h} - \Psi'(t - s) \right] \mathrm{d}W(s)$$

$$- \frac{1}{h} \int_{t+h}^t \Psi(t - s) \, \mathrm{d}W(s) - \int_{t+h}^t \Psi'(t - s) \, \mathrm{d}W(s) =: I_1^{h^-} + I_2^{h^-} + I_3^{h^-}.$$

We first deal with the terms $I_2^{h^\pm}$. Note that $\Psi \in H^1_{0,\{0\}}(0, T; \mathscr{L}_2(H; \dot{H}^\sigma_A))$ implies $\Psi(u) = \int_0^u \Psi'(r) \, \mathrm{d}r$ for all $u \in (0, |h|)$, see [32, §5.9.2, Theorem 2]. In conjunction with the Burkholder–Davis–Gundy inequality (combined with the embedding (3.23) if $p \in [1, 2)$) and the Cauchy–Schwarz inequality, this leads to

$$\left\| I_2^{h^\pm} \right\|_{L^p(\Omega; \dot{H}^\sigma_A)} \lesssim_p \frac{1}{|h|} \left[\int_0^{|h|} \| \Psi(u) \|^2_{\mathscr{L}_2(H; \dot{H}^\sigma_A)} \, \mathrm{d}u \right]^{1/2}$$

$$\leq \frac{1}{|h|} \left[\int_0^{|h|} \left(\int_0^u \| \Psi'(r) \|_{\mathscr{L}_2(H; \dot{H}^\sigma_A)} \, \mathrm{d}r \right)^2 \mathrm{d}u \right]^{1/2} \leq \| \Psi' \|_{L^2(0, |h|; \mathscr{L}_2(H; \dot{H}^\sigma_A))}.$$

Moreover, we find that

$$\|I_3^{h^-}\|_{L^P(\Omega;\dot{H}_A^\sigma)} \lesssim_P \left[\int_{t+h}^t \|\Psi'(t-s)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, ds\right]^{1/2}$$
$$= \left[\int_0^{|h|} \|\Psi'(u)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, du\right]^{1/2} = \|\Psi'\|_{L^2(0,|h|;\mathscr{L}_2(H;\dot{H}_A^\sigma))}.$$

Since $\Psi' \in L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))$, we have that $\|\Psi'\|_{L^2(0,|h|;\mathscr{L}_2(H;\dot{H}_A^\sigma))} \to 0$ as $h \to 0$ by dominated convergence. Thus, it remains to deal with the $I_1^{h^\pm}$ terms. For the case of positive h, we find using the definition of the difference quotient D_h (see Eq. (A.6) in Sect. A.4 of "Appendix A") that

$$\|I_1^{h^+}\|_{L^P(\Omega;\dot{H}_A^\sigma)} \lesssim_P \left[\int_0^t \left\|\frac{\Psi(t+h-s)-\Psi(t-s)}{h} - \Psi'(t-s)\right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, ds\right]^{1/2}$$
$$= \left[\int_0^t \left\|\frac{\Psi(u+h)-\Psi(u)}{h} - \Psi'(u)\right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, du\right]^{1/2}$$
$$= \|D_h\Psi - \Psi'\|_{L^2(0,t;\mathscr{L}_2(H;\dot{H}_A^\sigma))}.$$

For the case of negative h, we arrive at

$$\|I_1^{h^-}\|_{L^P(\Omega;\dot{H}_A^\sigma)} \lesssim_P \left[\int_0^{t+h} \left\|\frac{\Psi(t+h-s)-\Psi(t-s)}{h} - \Psi'(t-s)\right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, ds\right]^{1/2}$$
$$= \left[\int_{-h}^t \left\|\frac{\Psi(u+h)-\Psi(u)}{h} - \Psi'(u)\right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, du\right]^{1/2}$$
$$= \|D_h\Psi - \Psi'\|_{L^2(-h,t;\mathscr{L}_2(H;\dot{H}_A^\sigma))}.$$

The convergence $\lim_{h\to 0} \|I_1^{h^\pm}\|_{L^P(\Omega;\dot{H}_A^\sigma)} = 0$ follows then from Proposition A.8. ∎

We are now ready to prove Theorem 3.12.

Proof of Theorem 3.12 We first claim that the mild solution, viewed as a mapping $Z_\gamma \colon [0,T] \to L^P(\Omega;\dot{H}_A^\sigma)$, is n times differentiable and, for every $k \in \{0,1,\ldots,n\}$ and all $t \in [0,T]$, its kth derivative satisfies

$$Z_\gamma^{(k)}(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \Phi_{\gamma-1,0}^{(k)}(t-s) \, d\widehat{W}(s), \quad \mathbb{P}\text{-a.s.}, \tag{3.27}$$

where $\Phi_{\gamma-1,0}^{(k)}$ is the kth derivative of $\Phi_{\gamma-1,0}$ given by (3.25), and $\widehat{W}$ is the cylindrical Wiener process $\widehat{W}(t) := Q^{-\frac{1}{2}} W^Q(t)$, $t \geq 0$. We prove this by induction with respect to k. For $k = 0$, the identity (3.27) follows from Definition 3.7 and (3.22). Now let

$k \in \{0, 1, \ldots, n-1\}$ and suppose that Z_γ is k times differentiable and (3.27) holds. Then, the induction hypothesis and Lemma 3.20 show that, for all $t \in [0, T]$,

$$\frac{d^{k+1}}{dt^{k+1}} Z_\gamma(t) = \frac{d}{dt} Z_\gamma^{(k)}(t) = \frac{d}{dt}\left[\frac{1}{\Gamma(\gamma)} \int_0^t \Phi_{\gamma-1,0}^{(k)}(t-s)\, d\widehat{W}(s) \right]$$

$$= \frac{1}{\Gamma(\gamma)} \frac{d}{dt} \int_0^t \sum_{j=0}^k C_{\gamma-1,j,k} \Phi_{\gamma-1-(k-j),j}(t-s)\, d\widehat{W}(s), \quad \mathbb{P}\text{-a.s.}$$

Fixing an arbitrary $j \in \{0, 1, \ldots, k\}$, it suffices to verify that $\Psi := \Phi_{\gamma-1-(k-j),j}$ satisfies the conditions of Proposition 3.21, so that (3.26) holds for the cylindrical Wiener process $\widehat{W}$. Indeed, having proved this for an arbitrary j, by linearity

$$\frac{d^{k+1}}{dt^{k+1}} Z_\gamma(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \sum_{j=0}^k C_{\gamma-1,j,k} \Phi'_{\gamma-1-(k-j),j}(t-s)\, d\widehat{W}(s)$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^t \Phi_{\gamma-1,0}^{(k+1)}(t-s)\, d\widehat{W}(s), \quad \mathbb{P}\text{-a.s.},$$

follows, where the latter identity is an equality of the operator-valued integrands.

Using (3.3) with $c := b$, the identity $A^{\frac{\sigma}{2}} \Phi_{a,b}(t) = 2^a (t/2)^a A^b S(t/2) A^{\frac{\sigma}{2}} S(t/2) Q^{\frac{1}{2}}$ and a change of variables $u := t/2$, we observe that

$$\|\Phi_{a,b}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))} \lesssim_{(a,b)} \left[\int_0^T (t/2)^{2(a-b)} \left\| A^{\frac{\sigma}{2}} S(t/2) Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \frac{dt}{2} \right]^{1/2}$$

$$= \left[\int_0^T \|\Phi_{a-b,0}(t/2)\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \frac{dt}{2} \right]^{1/2} \leq \|\Phi_{a-b,0}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))} \tag{3.28}$$

holds for all $a \in \mathbb{R}$ and $b \in [0, \infty)$. For $\Psi = \Phi_{\gamma-1-(k-j),j}$ we use (3.28) to obtain

$$\|\Psi\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))} \lesssim_{(\gamma,k,j)} \|\Phi_{\gamma-1-k,0}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))}.$$

The norm on the right-hand side is finite by (3.19), since $k \leq n-1 < n$. Next, noting that $\gamma - 1 - k - \frac{\sigma-r}{2} \geq \gamma - n - \frac{\sigma-r}{2} > 0$, the second assertion of Lemma 3.20 implies that $t \mapsto \Psi(t)$ has a continuous extension in $C_{0,\{0\}}([0, T]; \mathscr{L}(H; \dot{H}_A^\sigma))$. Furthermore, also by Lemma 3.20, Ψ is differentiable from $(0, T)$ to $\mathscr{L}(H; \dot{H}_A^\sigma)$, with derivative

$$\Psi' = (\gamma - 1 - (k-j)) \Phi_{\gamma-1-(k-j)-1,j} - \Phi_{\gamma-1-(k-j),j+1}.$$

Applying the triangle inequality and (3.28) then shows that

$$\|\Psi'\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))} \lesssim_{(\gamma,k,j)} \|\Phi_{\gamma-1-(k+1),0}\|_{L^2(0,T;\mathscr{L}_2(H;\dot{H}_A^\sigma))},$$

where the norm on the right-hand side is finite by (3.19), as $k + 1 \leq n$. Since also $\mathscr{L}_2(H; \dot{H}_A^\sigma) \hookrightarrow \mathscr{L}(H; \dot{H}_A^\sigma)$, Lemma A.9 implies that Ψ is an element of

$H^1_{0,\{0\}}(0, T; \mathscr{L}_2(H; \dot{H}^\sigma_A))$. Thus, we may indeed use Proposition 3.21, and the differentiability follows.

It remains to show that the nth derivative $Z^{(n)}_\gamma$ is (Hölder) continuous, i.e., that $Z^{(n)}_\gamma \in C^{0,\tau}([0, T]; L^p(\Omega; \dot{H}^\sigma_A))$. To this end, we use (3.27) and (3.25), and write

$$\forall t \in [0, T]: \quad Z^{(n)}_\gamma(t) = \frac{1}{\Gamma(\gamma)} \sum_{j=0}^n C_{\gamma-1,j,n} \int_0^t \Phi_{\gamma-1-(n-j),j}(t-s) \, \mathrm{d}\widehat{W}(s), \quad \mathbb{P}\text{-a.s.}$$

The case $\tau = 0$ (i.e., continuity) follows after applying, for all $j \in \{0, 1, \ldots, n\}$, Proposition 3.18 with $a = \gamma - 1 - (n - j)$ and $b = j$. Note that $\Phi_{\gamma-1-(n-j),j}$ indeed is an element of $L^2(0, T; \mathscr{L}_2(H; \dot{H}^\sigma_A))$ for all $j \in \{0, \ldots, n\}$ by (3.19) and (3.28). For $\tau \in (0, \gamma - n - \frac{1}{2}] \cap (0, 1)$, the Hölder continuity of $Z^{(n)}_\gamma$ follows from Proposition 3.19 which we may apply, for all $j \in \{0, 1, \ldots, n\}$, with $a = \gamma - 1 - (n - j)$ and $b = j$, since $A^{n+\tau+\frac{1}{2}-\gamma} Q^{\frac{1}{2}} \in \mathscr{L}_2(H; \dot{H}^\sigma_A)$ is assumed. $\qquad\square$

4 Covariance structure

In this section, we study the covariance structure of solutions to (3.1). More specifically, we consider the mild solution process $(Z_\gamma(t))_{t\in[0,T]}$ from Definition 3.7. The covariance structure of Z_γ will be expressed in terms of the family of *covariance operators* $(Q_{Z_\gamma}(s, t))_{s,t\in[0,T]} \subseteq \mathscr{L}(H)$ which satisfies, for all $s, t \in [0, T]$, that

$$(Q_{Z_\gamma}(s, t)x, y)_H = \mathbb{E}[(Z_\gamma(s) - \mathbb{E}[Z_\gamma(s)], x)_H (Z_\gamma(t) - \mathbb{E}[Z_\gamma(t)], y)_H] \quad \forall x, y \in H.$$

Note that this family is well-defined whenever Z_γ is square-integrable, e.g., under the assumptions made in Theorem 3.8. Note also that $\mathbb{E}[Z_\gamma(t)] = 0$ for all $t \in [0, T]$.

We present three results on the covariance operators of the mild solution Z_γ. The most general result is Proposition 4.1, which provides an explicit integral representation of $Q_{Z_\gamma}(s, t)$. Corollary 4.2 is concerned with the asymptotic behavior of the covariance operator $Q_{Z_\gamma}(t, t)$ as $t \to \infty$. Subsequently, in Corollary 4.3 we consider a situation in which the covariance is separable in time and space, and prove that the temporal part is asymptotically of Matérn type.

Proposition 4.1 *Let Assumption* 3.1(i) *be satisfied and* $\gamma \in (0, \infty)$ *be such that* (3.14) *holds. The covariance operators* $(Q_{Z_\gamma}(s, t))_{s,t\in[0,T]}$ *of* Z_γ *admit the representation*

$$Q_{Z_\gamma}(s, t) = \frac{1}{\Gamma(\gamma)^2} \int_0^{s \wedge t} [(s - r)(t - r)]^{\gamma-1} S(t - r) Q[S(s - r)]^* \, \mathrm{d}r. \quad (4.1)$$

Proof Square-integrability of Z_γ is a consequence of Theorem 3.8 and (3.14). In order to prove the integral representation (4.1), for $s \in [0, T]$, $r \in (0, s)$ and $x \in H$, we define $f(s, r; x) \in \mathscr{L}(H; \mathbb{R})$ by

$$f(s, r; x)z := [\Gamma(\gamma)]^{-1}(z, (s - r)^{\gamma-1}[S(s - r)]^*x)_H, \quad z \in H.$$

We proceed similarly as in [45, Lemma 3.10] and obtain (4.1) from the Itô isometry combined with the polarization identity:

$$
\begin{aligned}
\mathbb{E}[(Z_\gamma(s), x)_H (Z_\gamma(t), y)_H] &= \mathbb{E}\left[\int_0^s f(s, r; x)\, \mathrm{d}W^Q(r) \int_0^t f(t, \tau; y)\, \mathrm{d}W^Q(\tau)\right] \\
&= \int_0^{s \wedge t} \left(f(s, r; x)Q^{\frac{1}{2}}, f(t, r; y)Q^{\frac{1}{2}}\right)_{\mathscr{L}_2(H;\mathbb{R})} \mathrm{d}r \\
&= \frac{1}{\Gamma(\gamma)^2} \int_0^{s \wedge t} [(s-r)(t-r)]^{\gamma-1} (S(t-r)Q[S(s-r)]^* x, y)_H\, \mathrm{d}r.
\end{aligned}
$$

Then, (4.1) follows from exchanging the order of integration and taking the inner product, which is justified since $(0, s \wedge t) \ni r \mapsto [(s-r)(t-r)]^{\gamma-1} S(t-r)Q[S(s-r)]^* x$ is integrable by (3.14). □

By imposing more assumptions on the operator A, one can obtain explicit representations of the asymptotic covariance structure of Z_γ as $t \to \infty$, as the next two corollaries show. Note that, if (3.14) holds for $\delta = 0$ and $T = \infty$, in Definition 3.7 the stochastic convolution $\widetilde{Z}_\gamma$ and the mild solution Z_γ are well-defined on the infinite time interval $[0, \infty)$. It is thus meaningful to consider the asymptotic behavior.

Corollary 4.2 *Let Assumptions* 3.1(i),(ii),(iv) *be satisfied and let* $\gamma \in (1/2, \infty)$. *Suppose that* (3.14) *holds for* $\delta = 0$ *and* $T = \infty$. *If for every* $t \in [0, \infty)$ *the operator* $S(t)$ *is self-adjoint and commutes with the covariance operator* Q *of* W^Q, *we have*

$$
\lim_{t \to \infty} Q_{Z_\gamma}(t, t) = \Gamma(\gamma - 1/2)\left[2\sqrt{\pi}\,\Gamma(\gamma)\right]^{-1} A^{1-2\gamma} Q \quad \text{in } \mathscr{L}(H).
$$

Proof Starting from the identity (4.1) for a fixed $t = s \in [0, \infty)$, we recall self-adjointness of the operators $(S(t))_{t \geq 0}$ and the commutativity with Q to obtain that

$$
\begin{aligned}
Q_{Z_\gamma}(t, t) &= \frac{1}{\Gamma(\gamma)^2} \int_0^t (t-r)^{2(\gamma-1)} S(t-r)Q S(t-r)\, \mathrm{d}r \\
&= \frac{1}{\Gamma(\gamma)^2} \int_0^t (t-r)^{2\gamma-2} Q S(2t-2r)\, \mathrm{d}r = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2} \int_0^{2t} u^{2\gamma-2} Q S(u)\, \mathrm{d}u,
\end{aligned}
$$

where we also used the semigroup property and the change of variables $u := 2(t - r)$. Now we interchange the bounded linear operator Q with the integral, and pass to the limit $t \to \infty$ in $\mathscr{L}(H)$, which by (B.3) with $\alpha := 2\gamma - 1 \in (0, \infty)$ gives

$$
\begin{aligned}
\lim_{t \to \infty} Q_{Z_\gamma}(t, t) &= 2^{1-2\gamma} \Gamma(2\gamma - 1)[\Gamma(\gamma)]^{-2} A^{1-2\gamma} Q \\
&= \Gamma(\gamma - 1/2)\left[2\sqrt{\pi}\,\Gamma(\gamma)\right]^{-1} A^{1-2\gamma} Q.
\end{aligned}
$$

The last equality is a consequence of the Legendre duplication formula for the gamma function (see e.g. [59, Formula 5.5.5]) applied to $\Gamma(2\gamma - 1) = \Gamma(2[\gamma - 1/2])$. □

Corollary 4.3 *Suppose the setting of Corollary 4.2 and let $A := \kappa I$ for $\kappa \in (0, \infty)$. Then the covariance function of Z_γ is separable and its temporal part is asymptotically of Matérn type, i.e., there is a function $\varrho_{Z_\gamma} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that*

$$\forall s, t \in [0, \infty) : \quad Q_{Z_\gamma}(s, t) = \varrho_{Z_\gamma}(s, t)\, Q,$$

$$\forall h \in \mathbb{R} \backslash \{0\} : \quad \lim_{t \to \infty} \varrho_{Z_\gamma}(t, t+h) = \frac{2^{\frac{1}{2} - \gamma} \kappa^{1-2\gamma}}{\sqrt{\pi}\, \Gamma(\gamma)} (\kappa |h|)^{\gamma - \frac{1}{2}} K_{\gamma - \frac{1}{2}}(\kappa |h|). \quad (4.2)$$

Remark 4.4 On the right-hand side of (4.2), one recognizes the Matérn covariance function (1.1) with smoothness parameter $\nu = \gamma - 1/2$, correlation length parameter κ and variance $\sigma^2 = \kappa^{1-2\gamma} \Gamma(\gamma - 1/2) [2\sqrt{\pi}\, \Gamma(\gamma)]^{-1}$.

Proof of Corollary 4.3 For $s, t \geq 0$, the integral representation (4.1) yields

$$Q_{Z_\gamma}(s, t) = \frac{1}{\Gamma(\gamma)^2} \int_0^{s \wedge t} [(s-r)(t-r)]^{\gamma - 1} e^{-\kappa(s+t-2r)}\, \mathrm{d}r\, Q = \varrho_{Z_\gamma}(s, t)\, Q,$$

where we moved the bounded operator $Q \in \mathscr{L}(H)$ out of the integral. Next, we fix $h \in (0, \infty)$, let $t \in [0, \infty)$ and perform the change of variables $u := h + 2(t - r)$,

$$\varrho_{Z_\gamma}(t, t+h) = \varrho_{Z_\gamma}(t+h, t) = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2} \int_h^{2t+h} [(u+h)(u-h)]^{\gamma - 1} e^{-\kappa u}\, \mathrm{d}u.$$

Thus, by passing to the limit $t \to \infty$, we obtain

$$\lim_{t \to \infty} \varrho_{Z_\gamma}(t, t+h) = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2} \int_h^\infty (u^2 - h^2)^{\gamma - 1} e^{-\kappa u}\, \mathrm{d}u$$

$$= \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2} \mathcal{L}\big[u \mapsto (u^2 - h^2)^{\gamma - 1} \mathbf{1}_{(h, \infty)}(u)\big](\kappa) = \frac{2^{1-2\gamma}}{\Gamma(\gamma)^2} \frac{(2h)^{\gamma - \frac{1}{2}} \Gamma(\gamma)}{\sqrt{\pi} \kappa^{\gamma - \frac{1}{2}}} K_{\gamma - \frac{1}{2}}(\kappa h),$$

where $\mathcal{L}[f](\kappa)$ denotes the Laplace transform of the function $f : [0, \infty) \to \mathbb{R}$ evaluated at κ, and the last identity follows from [58, Chapter I, Formula 3.13]. $\quad\square$

5 Spatiotemporal Whittle–Matérn fields

In this section, we demonstrate how the results of the previous Sects. 3 and 4 can be related to the widely used statistical models involving generalized Whittle–Matérn operators (1.3) on $H = L^2(\mathcal{X})$, where $\mathcal{X} = \mathcal{D} \subsetneq \mathbb{R}^d$ is a bounded domain in the Euclidean space (see Sect. 5.1) or a surface $\mathcal{X} = \mathcal{M}$ (see Sect. 5.2).

5.1 Bounded Euclidean domains

Throughout this subsection, let $\emptyset \neq \mathcal{D} \subsetneq \mathbb{R}^d$ be a bounded, connected and open domain. In order to rigorously define the symmetric, strongly elliptic second-order

differential operator L, formally given by (1.3), as a linear operator on $L^2(\mathcal{D})$, we make the following assumptions on its coefficients $\kappa \colon \mathcal{D} \to \mathbb{R}$ and $a \colon \mathcal{D} \to \mathbb{R}^{d \times d}_{\text{sym}}$, as well as on the spatial domain $\mathcal{D} \subsetneq \mathbb{R}^d$.

Assumption 5.1 (Euclidean domain—minimal conditions)

(i) $\mathcal{D}$ has a Lipschitz continuous boundary $\partial \mathcal{D}$;

(ii) $a \in L^\infty\big(\mathcal{D}; \mathbb{R}^{d \times d}_{\text{sym}}\big)$ is strongly elliptic, i.e.,

$$\exists \theta > 0 : \quad \underset{x \in \mathcal{D}}{\operatorname{ess\,inf}} \ \xi^\top a(x) \xi \geq \theta \|\xi\|^2_{\mathbb{R}^d} \quad \forall \xi \in \mathbb{R}^d;$$

(iii) $\kappa \in L^\infty(\mathcal{D})$.

Under these assumptions, we introduce the bilinear form

$$\mathfrak{a}_L \colon H^1_0(\mathcal{D}) \times H^1_0(\mathcal{D}) \to \mathbb{R}, \qquad \mathfrak{a}_L(u, v) := (a \nabla u, \nabla v)_{L^2(\mathcal{D})} + (\kappa^2 u, v)_{L^2(\mathcal{D})},$$

which is symmetric, continuous and coercive. We say that $u \in H^1_0(\mathcal{D})$ belongs to the domain $\mathsf{D}(L)$ of the differential operator L if and only if $|\mathfrak{a}_L(u, v)| \lesssim_u \|v\|_{L^2(\mathcal{D})}$ holds for all $v \in H^1_0(\mathcal{D})$. In this case, we define Lu as the unique element of $L^2(\mathcal{D})$ which satisfies the relation $\mathfrak{a}_L(u, v) = (Lu, v)_{L^2(\mathcal{D})}$ for all $v \in H^1_0(\mathcal{D})$.

By the Lax–Milgram theorem the inverse $L^{-1} \in \mathscr{L}(L^2(\mathcal{D}); H^1_0(\mathcal{D}))$ exists and can be extended to $L^{-1} \in \mathscr{L}(H^1_0(\mathcal{D})^*; H^1_0(\mathcal{D}))$. Moreover, it is a consequence of the Rellich–Kondrachov theorem (see [2, Theorem 6.3]) that L^{-1} is compact on $L^2(\mathcal{D})$. For this reason, the spectral theorem for self-adjoint compact operators is applicable and shows that there exist an orthonormal basis $(e_j)_{j \in \mathbb{N}}$ for $L^2(\mathcal{D})$ and a non-decreasing sequence $(\lambda_j)_{j \in \mathbb{N}}$ of positive real numbers accumulating only at infinity such that $Le_j = \lambda_j e_j$ holds for all $j \in \mathbb{N}$. Furthermore, the eigenvalues of L satisfy the following asymptotic behavior, known as Weyl's law [27, Theorem 6.3.1]:

$$\lambda_j \eqsim j^{2/d} \quad \forall j \in \mathbb{N}. \tag{5.1}$$

In this setting, for two differential operators L and $\widetilde{L}$ on $L^2(\mathcal{D})$ with coefficients a, κ and $\widetilde{a}, \widetilde{\kappa}$, respectively, we obtain the following corollary from the regularity results in Sect. 3 for spatiotemporal Whittle–Matérn fields, where $A := L^\beta$ and $Q := \widetilde{L}^{-\alpha}$.

Corollary 5.2 *Let* $\alpha, \beta, \sigma \in [0, \infty)$, *set* $r := \frac{\alpha}{\beta} \wedge \sigma$ *if* $\beta > 0$ *and* $r := \sigma$ *if* $\beta = 0$, *and suppose that* $n \in \mathbb{N}_0$, $\tau \in [0, 1)$ *and* $\gamma \in \big(n + \frac{(\sigma - r) \vee 1}{2}, \infty\big)$ *are such that*

$$\gamma \geq n + \frac{1 + (\sigma - r) \vee (2\tau)}{2} \quad \text{and} \quad \beta\gamma > \frac{d}{4} - \frac{\alpha}{2} + \beta\big(n + \tau + \tfrac{1+\sigma}{2}\big). \tag{5.2}$$

Let $L \colon \mathsf{D}(L) \subseteq H^1_0(\mathcal{D}) \to L^2(\mathcal{D})$ *and* $\widetilde{L} \colon \mathsf{D}(\widetilde{L}) \subseteq H^1_0(\mathcal{D}) \to L^2(\mathcal{D})$ *be symmetric, strongly elliptic second-order differential operators as defined above, cf. (1.3). Suppose that Assumption 5.1(i) holds for* $\mathcal{D} \subsetneq \mathbb{R}^d$, *and that the coefficients* a, κ *of* L *and* $\widetilde{a}, \widetilde{\kappa}$ *of* $\widetilde{L}$ *satisfy Assumptions 5.1(ii),(iii). Assume further that* L *and* $\widetilde{L}$ *diagonalize with respect to the same orthonormal basis* $(e_j)_{j \in \mathbb{N}}$ *for* $L^2(\mathcal{D})$, *i.e., there*

exist non-decreasing sequences $(\lambda_j)_{j \in \mathbb{N}}$, $(\widetilde{\lambda}_j)_{j \in \mathbb{N}}$ *of positive real numbers such that* $L e_j = \lambda_j e_j$ *and* $\widetilde{L} e_j = \widetilde{\lambda}_j e_j$ *for all* $j \in \mathbb{N}$.

Then, setting $A := L^\beta$ *and* $Q := \widetilde{L}^{-\alpha}$, *the mild solution* Z_γ *to (3.1) in the sense of Definition 3.7, see also (1.4), belongs to* $C^{n,\tau}([0,T]; L^p(\Omega; \dot{H}_A^\sigma))$ *for all* $p \in [1, \infty)$. *If the above conditions hold with* $n = 0$ *and* $\tau \in (0,1)$, *then for every* $p \in [1, \infty)$ *and all* $\tau' \in [0, \tau)$ *the mild solution* Z_γ *has a modification* $\widehat{Z}_\gamma \in L^p(\Omega; C^{0,\tau'}([0,T]; \dot{H}_A^\sigma))$.

Proof By the spectral mapping theorem for fractional powers of operators, see e.g. [53, Section 5.3], we obtain that $A e_j = L^\beta e_j = \lambda_j^\beta e_j$ and $Q e_j = \widetilde{L}^{-\alpha} e_j = \widetilde{\lambda}_j^{-\alpha} e_j$. In particular, it follows that A inherits the self-adjointness and strict positive-definiteness from L. This readily implies that $0 \in \rho(A)$. By [42, Proposition 10.2.23] we see that $A_\mathbb{C}$ admits a bounded H^∞-calculus of angle $\omega_{H^\infty}(A_\mathbb{C}) = 0$, showing that Assumptions 3.1(i)–(iv) are satisfied for A.

Furthermore, we note that, for every $\sigma, s \in [0, \infty)$, we have that $\dot{H}_A^\sigma = \dot{H}_L^{\sigma\beta}$ and the spaces $\dot{H}_L^s$ and $\dot{H}_{\widetilde{L}}^s$ are isomorphic. The latter fact follows from the asymptotic behavior (5.1) of the eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ and $(\widetilde{\lambda}_j)_{j \in \mathbb{N}}$, since L and $\widetilde{L}$ have the same eigenfunctions. Thus, we obtain that $Q^{\frac{1}{2}} = \widetilde{L}^{-\frac{\alpha}{2}} \in \mathscr{L}(H; \dot{H}_L^\alpha) \subseteq \mathscr{L}(H; \dot{H}_A^r)$.

Since $\gamma \in \left(\frac{1}{2} + n, \infty\right) \cap \left[\frac{1}{2} + n + \frac{\sigma - r}{2}, \infty\right)$ is assumed, by Proposition 3.14 (see also Remark 3.15) the condition (3.19) of Theorem 3.12 is equivalent to requiring that $A^{n + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \in \mathscr{L}_2(H; \dot{H}_A^\sigma)$.

Since also $\gamma \in \left(\frac{\sigma - r}{2} + n, \infty\right) \cap \left[n + \tau + \frac{1}{2}, \infty\right)$, we therefore conclude with Theorem 3.12 that it suffices to check that the quantity

$$
\left\| A^{\frac{\sigma}{2} + n + \tau + \frac{1}{2} - \gamma} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 = \left\| L^{\beta\left(\frac{\sigma}{2} + n + \tau + \frac{1}{2} - \gamma\right)} \widetilde{L}^{-\frac{\alpha}{2}} \right\|_{\mathscr{L}_2(H)}^2
$$
$$
= \sum_{j=1}^{\infty} \left\| L^{\beta\left(\frac{\sigma}{2} + n + \tau + \frac{1}{2} - \gamma\right)} \widetilde{L}^{-\frac{\alpha}{2}} e_j \right\|_H^2 = \sum_{j=1}^{\infty} \lambda_j^{2\beta\left(\frac{\sigma}{2} + n + \tau + \frac{1}{2} - \gamma\right)} \widetilde{\lambda}_j^{-\alpha} \tag{5.3}
$$

is finite. Indeed, applying Weyl's law (5.1) to both L and $\widetilde{L}$, it follows that

$$
\sum_{j=1}^{\infty} \lambda_j^{2\beta\left(\frac{\sigma}{2} + n + \tau + \frac{1}{2} - \gamma\right)} \widetilde{\lambda}_j^{-\alpha} \eqsim_{(\alpha, \beta, \gamma, \sigma, n, \tau)} \sum_{j=1}^{\infty} j^{\frac{4}{d}\left[\beta\left(n + \tau + \frac{1 + \sigma}{2}\right) - \beta\gamma - \frac{\alpha}{2}\right]},
$$

so that (5.3) is finite if and only if (5.2) holds, as we assume. Then, for all $p \in [1, \infty)$, Theorem 3.8, Theorem 3.12 and Proposition 3.14 yield the existence of a mild solution $Z_\gamma \in C^{n,\tau}([0,T]; L^p(\Omega; \dot{H}_A^\sigma))$, which is unique up to modification. The last assertion for $n = 0$ and $\tau \in (0,1)$ follows from Corollary 3.13. $\square$

The spatial regularity obtained in Corollary 5.2 is measured by means of the spaces $\dot{H}_A^\sigma = \dot{H}_L^{\beta\sigma}$. It would be more practical to express this in terms of fractional-order Sobolev spaces $H^s(\mathcal{D})$, $s \geq 0$. This raises the question of how $\dot{H}_L^s$ and $H^s(\mathcal{D})$ relate. The answer to this question depends on the smoothness of the coefficients a, κ and of

the boundary $\partial\mathcal{D}$. We therefore introduce two additional sets of assumptions: Assumption 5.3 is only slightly more restrictive than the minimal conditions of Assumption 5.1, whereas Assumption 5.4 requires a high degree of smoothness.

Assumption 5.3 (Euclidean domain—$H^2(\mathcal{D})$-regular setting)

(i) $\mathcal{D}$ is convex.
(ii) $a\colon \overline{\mathcal{D}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is Lipschitz continuous, i.e.,

$$|a_{ij}(x) - a_{ij}(y)| \lesssim \|x - y\|_{\mathbb{R}^d} \quad \forall x, y \in \overline{\mathcal{D}}, \quad \forall i, j \in \{1, \dots, d\}.$$

Assumption 5.4 (Euclidean domain—smooth setting)

(i) The boundary $\partial\mathcal{D}$ is of class C^∞;
(ii) $a_{ij} \in C^\infty(\overline{\mathcal{D}})$ holds for all $i, j \in \{1, \dots, d\}$, i.e., for all entries of a;
(iii) $\kappa \in C^\infty(\overline{\mathcal{D}})$.

The results of the next lemma are taken from [21, Lemma 2] and [12, Lemma 4.3].

Lemma 5.5 *Let* $L\colon \mathsf{D}(L) \subseteq H^1_0(\mathcal{D}) \to L^2(\mathcal{D})$ *denote a symmetric second-order differential operator defined as above, cf. (1.3). Then, the following assertions hold:*

(a) *If Assumption 5.1 is satisfied, then* $\dot{H}^s_L \hookrightarrow H^s(\mathcal{D})$ *for all* $s \in [0, 1]$. *Moreover, the norms* $\|\cdot\|_{\dot{H}^s_L}$ *and* $\|\cdot\|_{H^s(\mathcal{D})}$ *are equivalent on* $\dot{H}^s_L$ *for* $s \in [0, 1]\setminus\{1/2\}$;
(b) *If Assumptions 5.1 and 5.3 are fulfilled, then*

$$\left(\dot{H}^s_L, \|\cdot\|_{\dot{H}^s_L}\right) \cong \left(H^s(\mathcal{D}) \cap H^1_0(\mathcal{D}), \|\cdot\|_{H^s(\mathcal{D})}\right) \quad \forall s \in [1, 2];$$

(c) *If Assumptions 5.1 and 5.4 are satisfied, then we have* $\dot{H}^s_L \hookrightarrow H^s(\mathcal{D})$ *for all* $s \in [0, \infty)$, *and the norms* $\|\cdot\|_{\dot{H}^s_L}$, $\|\cdot\|_{H^s(\mathcal{D})}$ *are equivalent on* $\dot{H}^s_L$ *for every* $s \in [0, \infty)\setminus\mathfrak{E}$, *where* $\mathfrak{E} := \{2k + 1/2 : k \in \mathbb{N}_0\}$ *is called the* exclusion set.

Combining Lemma 5.5 with the results of Corollary 5.2 shows that the mild solution Z_γ is an element of $C^{n,\tau}([0, T]; L^p(\Omega; H^{\beta\sigma}(\mathcal{D})))$, provided that $\sigma\beta \in [0, s']$, where $s' \in [1, \infty)$ is prescribed by the smoothness of the coefficients a, κ and the boundary $\partial\mathcal{D}$ via Lemma 5.5(a), (b) or (c). Note that we do not have to take the exclusion set $\mathfrak{E}$ into account, as we only need the embedding $\dot{H}^s_L \hookrightarrow H^s(\mathcal{D})$.

Lastly, we consider the covariance structure of the mild solution, as treated in the abstract setting in Sect. 4. The most illustrative results are the asymptotic formulas presented in Corollaries 4.2 and 4.3, which we translate to the current setting in Corollary 5.6. We see that (Whittle–)Matérn operators are recovered as marginal spatial or temporal covariance operators.

Corollary 5.6 *Consider the setting of Corollary 5.2 with* $L = \widetilde{L}$, *i.e.,* $Q := L^{-\alpha}$. *Let* $\alpha, \beta \in [0, \infty)$ *and* $\gamma \in (1/2, \infty)$ *be such that* $\beta\gamma > \frac{1}{2}(\frac{d}{2} - \alpha + \beta)$, *and let* Z_γ *be the mild solution in the sense of Definition 3.7. Then the asymptotic marginal spatial covariance of* Z_γ *satisfies*

$$\lim_{t\to\infty} Q_{Z_\gamma}(t, t) = \Gamma(\gamma - 1/2)\left[2\sqrt{\pi}\,\Gamma(\gamma)\right]^{-1} L^{\beta(1-2\gamma)-\alpha} \quad \text{in} \ \ \mathscr{L}(L^2(\mathcal{D})).$$

For $\beta = 0$, the covariance of Z_γ is separable in the sense that there exists a function $\varrho_{Z_\gamma} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that

$$Q_{Z_\gamma}(s, t) = \varrho_{Z_\gamma}(s, t) \, L^{-\alpha} \quad \forall s, t \in [0, \infty),$$

and for all $h \in \mathbb{R} \backslash \{0\}$ we have

$$\lim_{t \to \infty} Q_{Z_\gamma}(t, t + h) = 2^{\frac{1}{2} - \gamma} \big[\sqrt{\pi} \, \Gamma(\gamma) \big]^{-1} |h|^{\gamma - \frac{1}{2}} K_{\gamma - \frac{1}{2}}(|h|) \, L^{-\alpha} \quad in \;\; \mathscr{L}(L^2(\mathcal{D})).$$

Proof Existence and uniqueness of the mild solution Z_γ follow from Corollary 5.2 with $L = \widetilde{L}$ and $n = \tau = \sigma = 0$. Recall from its proof that A satisfies all of Assumptions 3.1(i)–(iv). Moreover, $A = L^\beta$ is self-adjoint and $Q = L^{-\alpha} \in \mathscr{L}(L^2(\mathcal{D}))$ commutes with A, so that it also commutes with $S(t)$ for all $t \in [0, \infty)$, cf. [37, Theorem 1.3.2(a)]. All assertions follow thus from Corollaries 4.2 and 4.3. $\square$

Remark 5.7 The asymptotic results obtained in Corollary 5.6 are in accordance with the marginal spatial and temporal covariance functions derived in [49, Section 3, Proposition 1 and Corollary 1] for the case of the differential operator $L = \gamma_s^2 - \Delta$ acting on functions defined on all of $\mathbb{R}^2$, where $\gamma_s \in (0, \infty)$. Note that, in order to exploit Fourier techniques, in [49] the "time" variable t is an element of the whole real axis, $t \in \mathbb{R}$, instead of only its non-negative part.

Remark 5.8 Corollaries 5.2 and 5.6 explain and justify the roles of the parameters α, β and γ. They control three important properties of spatiotemporal Whittle–Matérn fields. Besides the temporal and spatial smoothness, measured respectively by the quantities $n + \tau$ and σ, we identify a third degree of freedom: The *degree of separability*, expressed by the ratio $\frac{\alpha}{\beta} \in [0, \infty]$. Indeed, if $\frac{\alpha}{\beta} = \infty$, i.e. $\beta = 0$, we observe that the covariance of the field is separable and that its temporal and spatial behavior are exclusively governed by the parameters γ and α, respectively. In contrast, if $\frac{\alpha}{\beta} = 0$, i.e. $\alpha = 0$, the SPDE is driven by spatiotemporal Gaussian white noise and the "coloring" of its solution is fully determined by the fractional parabolic differential operator $\big(\partial_t + L^\beta\big)^\gamma$.

5.2 Surfaces

In this subsection, we provide a brief demonstration of how the above results can be extended to spatiotemporal Whittle–Matérn fields on more general spatial domains. More precisely, we consider a smooth, closed, connected, orientable and compact 2-surface $\mathcal{M}$ immersed in $\mathbb{R}^3$ and endowed with the positive surface measure $\nu_{\mathcal{M}}$ on $\mathcal{B}(\mathcal{M})$, induced by the first fundamental form. An important example of such a surface is given by the 2-sphere, $\mathcal{M} = \mathbb{S}^2$.

On $H := L^2(\mathcal{M})$, we consider the following analog of the symmetric, strongly elliptic second-order differential operator from Sect. 5.1, formally given by

$$Lu := -\nabla_{\mathcal{M}} \cdot (a \nabla_{\mathcal{M}} u) + \kappa^2 u, \qquad u \in \mathsf{D}(L) \subseteq L^2(\mathcal{M}),$$

where $\nabla_{\mathcal{M}} \cdot$ and $\nabla_{\mathcal{M}}$ denote the surface divergence and the surface gradient, respectively. We record the precise conditions on the surface $\mathcal{M}$ and on the coefficients a, κ in Assumption 5.9 below; with regard to smoothness, they are analogous to the setting of Assumption 5.4 in the case of a bounded Euclidean domain.

Assumption 5.9 (Surface—smooth setting)

(i) a is a symmetric tensor field, i.e., $a(x) \colon T_x \mathcal{M} \to T_x \mathcal{M}$ is linear and symmetric for all $x \in \mathcal{M}$, where $T_x \mathcal{M}$ denotes the tangent space of x. Moreover, a is smooth and strongly elliptic in the following sense:

$$\exists \theta > 0 : \quad \forall x \in \mathcal{M}, \ \forall \xi \in T_x \mathcal{M} : \quad \xi^\top a(x) \xi \geq \theta \|\xi\|_{\mathbb{R}^3}^2 .$$

(ii) The coefficient $\kappa \colon \mathcal{M} \to \mathbb{R}$ is smooth and bounded away from zero, i.e., there exists $\kappa_0 \in (0, \infty)$ such that $|\kappa(x)| \geq \kappa_0$ for all $x \in \mathcal{M}$.

The conditions in Assumption 5.9 are sufficient to ensure that $L \colon \dot{H}_L^1 \to (\dot{H}_L^1)^*$ is boundedly invertible, and has a compact inverse on $L^2(\mathcal{M})$. This allows us to find an orthonormal basis $(e_j)_{j \in \mathbb{N}}$ for $L^2(\mathcal{M})$ and a non-decreasing sequence of positive real eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ of L accumulating only at infinity, as in Sect. 5.1. Moreover, fractional powers L^β are well-defined for all $\beta \in \mathbb{R}$, the sequence of eigenvalues still satisfies Weyl's law (5.1) (with $d = 2$), and a spectral mapping theorem holds, cf. [72, Theorems XII.1.3 and XII.2.1]. These facts are sufficient to repeat the proofs of Corollaries 5.2 and 5.6 yielding the analogous results, with $d = 2$ and other obvious modifications to the conditions. In particular, the analog of Corollary 5.2 on the surface $\mathcal{M}$ implies regularity of the solution process in the space $C^{n,\tau}([0, T]; L^p(\Omega; H^{\beta\sigma}(\mathcal{M})))$.

An important difference from the (smooth) Euclidean setting of Assumption 5.4 is that under Assumption 5.9, the Sobolev space $H^s(\mathcal{M})$ and $\dot{H}_L^s$ are isomorphic for every $s \in [0, \infty)$, see [72, Example XII.2.1]. In other words, the absence of a boundary $\partial \mathcal{M}$ implies that one does not need to exclude the exception set $\mathfrak{E}$ from the admissible exponents s in the analog of Lemma 5.5(c).

Declarations

A Auxiliary results

Throughout this section, H denotes a separable Hilbert space which, if not specified otherwise, is considered over the real scalar field $\mathbb{R}$.

A.1 Bochner counterparts

The first lemma records relations between a linear operator $A\colon \mathsf{D}(A) \subseteq H \to H$ and its Bochner space counterpart $\mathcal{A}$ which is defined on a subspace of $L^2(0, T; H)$, where $T \in (0, \infty)$.

Lemma A.1 *Let $T \in (0, \infty)$ and $A\colon \mathsf{D}(A) \subseteq H \to H$ be a linear operator on a real or complex Hilbert space H. Consider the associated operator $\mathcal{A}$ on $L^2(0, T; H)$ as defined in* (3.4). *Then, the following hold:*

(a) *$\mathcal{A}$ is bounded if and only if A is bounded, and in that case we have*

$$\|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))} = \|A\|_{\mathscr{L}(H)};$$

(b) *$\mathcal{A}$ is closed if and only if A is.*

Proof If A is bounded, then the inequality $\|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))} \leq \|A\|_{\mathscr{L}(H)}$ is easily verified. Now suppose that $\mathcal{A}$ is bounded. Then for all $x \in H$ we have

$$\|Ax\|_H = \|T^{-1/2}\mathbf{1}_{(0,T)} \otimes Ax\|_{L^2(0,T;H)} = \|\mathcal{A}(T^{-1/2}\mathbf{1}_{(0,T)} \otimes x)\|_{L^2(0,T;H)}$$
$$\leq \|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}\|T^{-1/2}\mathbf{1}_{(0,T)} \otimes x\|_{L^2(0,T;H)} = \|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}\|x\|_H.$$

Here, given $f\colon (0, T) \to \mathbb{R}$ and $x \in H$, the function $f \otimes x\colon (0, T) \to H$ is defined by $[f \otimes x](t) := f(t)x$ for all $t \in (0, T)$. We thus find that A is bounded with operator norm $\|A\|_{\mathscr{L}(H)} \leq \|\mathcal{A}\|_{\mathscr{L}(L^2(0,T;H))}$, which finishes the proof of (a).

To prove part (b), first let A be closed and let the sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathsf{D}(\mathcal{A})$ be such that $v_n \to v$ and $\mathcal{A}v_n \to y$ in $L^2(0, T; H)$. We need to prove that $v \in \mathsf{D}(\mathcal{A})$ and $y = \mathcal{A}v$. Let $(v_{n_k})_{k\in\mathbb{N}}$ be a subsequence such that $v_{n_k} \to v$ and $\mathcal{A}v_{n_k} \to y$ in H, a.e. in $(0, T)$, so that by the closedness of A it follows that $v(\vartheta) \in \mathsf{D}(A)$ and $y(\vartheta) = Av(\vartheta)$ for a.a. $\vartheta \in (0, T)$. From the latter we obtain that $y = \mathcal{A}v$, which is meaningful since $v, y \in L^2(0, T; H)$ yields that $v \in \mathsf{D}(\mathcal{A})$.

Now let $\mathcal{A}$ be closed and let $(x_n)_{n\in\mathbb{N}}$ in $\mathsf{D}(A)$ be such that $x_n \to x$ and $Ax_n \to y$ in H. This implies the following convergences in $L^2(0, T; H)$:

$$\mathbf{1}_{(0,T)} \otimes x_n \to \mathbf{1}_{(0,T)} \otimes x,$$
$$\mathcal{A}(\mathbf{1}_{(0,T)} \otimes x_n) = \mathbf{1}_{(0,T)} \otimes Ax_n \to \mathbf{1}_{(0,T)} \otimes y.$$

Since $\mathcal{A}$ is closed, we deduce that $\mathbf{1}_{(0,T)} \otimes x \in \mathsf{D}(\mathcal{A})$ and $\mathbf{1}_{(0,T)} \otimes y = \mathcal{A}(\mathbf{1}_{(0,T)} \otimes x)$, from which we may conclude $x \in \mathsf{D}(A)$ and $y = Ax$. Hence A is closed. $\qquad\square$

The following lemma is generally useful for determining the domain of a generator of a given C_0-semigroup, and it will subsequently be used to show that the Bochner space counterpart of a C_0-semigroup is again a C_0-semigroup, see Proposition A.3.

Lemma A.2 *Let* $(S(t))_{t \geq 0}$ *be a* C_0-*semigroup on* H *with infinitesimal generator* $\widetilde{A} \colon \mathsf{D}(\widetilde{A}) \subseteq H \to H$. *If* $A \colon \mathsf{D}(A) \subseteq H \to H$ *is a linear operator satisfying* $A \subseteq \widetilde{A}$ *and* $\mathsf{D}(A)$ *is dense in* $\mathsf{D}(\widetilde{A})$ *with respect to the graph norm* $\| \cdot \|_{\mathsf{D}(\widetilde{A})}$, *then* $\widetilde{A} = \overline{A}$.

Proof Let $(x, \widetilde{A}x) \in \mathsf{G}(\widetilde{A})$ and choose a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathsf{D}(A)$ such that $x_n \to x$ in $\mathsf{D}(\widetilde{A})$. Using the fact that $A \subseteq \widetilde{A}$, we have $(x_n, Ax_n) = (x_n, \widetilde{A}x_n) \to (x, \widetilde{A}x)$ with respect to the product norm on $H \times H$, which shows that $(x, \widetilde{A}x) \in \overline{\mathsf{G}(A)}$. Conversely, for any $(x, y) \in \overline{\mathsf{G}(A)}$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathsf{D}(A)$ such that $(x_n, \widetilde{A}x_n) = (x_n, Ax_n) \to (x, y)$ in $H \times H$. Since $\widetilde{A}$ is closed as the generator of a C_0-semigroup, see [31, Theorem II.1.4], we find that $(x, y) \in \mathsf{G}(\widetilde{A})$. This proves $\overline{\mathsf{G}(A)} = \mathsf{G}(\widetilde{A})$. $\qquad\square$

Proposition A.3 *Let* $T \in (0, \infty)$ *and let Assumption* 3.1(i) *be satisfied. The family* $(\mathcal{S}(t))_{t \geq 0}$ *of operators on* $L^2(0, T; H)$ *given by* (3.5) *is a* C_0-*semigroup with infinitesimal generator* $-\mathcal{A}$, *as defined by* (3.4).

Proof First note that the operators $(\mathcal{S}(t))_{t \geq 0}$ are well-defined in the sense that they map elements in $L^2(0, T; H)$ to $L^2(0, T; H)$. In fact, Lemma A.1(a) shows that $\|\mathcal{S}(t)\|_{\mathscr{L}(L^2(0,T;H))} = \|S(t)\|_{\mathscr{L}(H)}$ holds for every $t \geq 0$. We now check that $(\mathcal{S}(t))_{t \geq 0}$ is a C_0-semigroup. Clearly, $\mathcal{S}(0) = I$ and the semigroup property holds. Let $M \geq 1$ and $w \in \mathbb{R}$ be as in (3.2), so that

$$\forall h \in [0, 1]: \quad \|S(h)\|_{\mathscr{L}(H)} \leq Me^{-wh} \leq Me^{(-w)\vee 0} =: \widetilde{M}.$$

To show strong continuity, let $x \in H$, $h \in (0, 1)$ and note that

$$\|S(h)x - x\|_H^2 \leq 2\|S(h)x\|_H^2 + 2\|x\|_H^2 \leq 2(\widetilde{M}^2 + 1)\|x\|_H^2.$$

By dominated convergence, $\lim_{h \downarrow 0} \|\mathcal{S}(h)v - v\|_{L^2(0,T;H)} = 0$ for $v \in L^2(0, T; H)$.

Next we investigate the infinitesimal generator of $(\mathcal{S}(t))_{t \geq 0}$, which we denote by $-\widetilde{\mathcal{A}}$ for the time being. We wish to show that $\widetilde{\mathcal{A}} = \mathcal{A}$. Let $x \in \mathsf{D}(A)$ and consider

$$\left\| \frac{1}{h}(S(h)x - x) + Ax \right\|_H^2 \leq 2 \left\| \frac{1}{h}(S(h)x - x) \right\|_H^2 + 2\|Ax\|_H^2.$$

To bound the first term, we use [61, Chapter 1, Theorem 2.4(d)] and note that, for every $h \in (0, 1)$, we obtain

$$\left\| \frac{1}{h}(S(h)x - x) \right\|_H^2 = \left\| \frac{1}{h} \int_0^h S(s)Ax \, ds \right\|_H^2 \leq \frac{1}{h^2} \left| \int_0^h \|S(s)Ax\|_H \, ds \right|^2 \leq \widetilde{M}^2 \|Ax\|_H^2.$$

The two previous displays show that, for $v \in L^2(0, T; D(A))$ and all $h \in (0, 1)$,

$$\int_0^T \left\| \frac{1}{h}(S(h)v(\vartheta) - v(\vartheta)) + Av(\vartheta) \right\|_H^2 d\vartheta \le 2(\widetilde{M}^2 + 1)\|Av\|_{L^2(0,T;H)}^2 < \infty.$$

This justifies the use of the dominated convergence theorem to conclude that

$$-\widetilde{\mathcal{A}}v = \lim_{h\downarrow 0} \frac{1}{h}(\mathcal{S}(h)v - v) = -\mathcal{A}v \quad \text{in } L^2(0, T; H),$$

i.e., $-\mathcal{A} \subseteq -\widetilde{\mathcal{A}}$ as $v \in D(\mathcal{A}) = L^2(0, T; D(A))$ was arbitrary. Since $D(\mathcal{A})$ is dense in $L^2(0, T; H)$ (by density of $D(A)$ in H), and $\mathcal{S}(t)$ maps $D(\mathcal{A})$ to itself for each $t \ge 0$, Proposition II.1.7 of [31] implies that $D(\mathcal{A})$ is dense in the domain $D(\widetilde{\mathcal{A}})$ of the generator of $(\mathcal{S}(t))_{t\ge 0}$ with respect to the graph norm $\|\cdot\|_{D(\widetilde{\mathcal{A}})}$. Applying Lemma A.2 and noting that $\mathcal{A}$ is closed by Lemma A.1(b) completes the proof. □

A.2 Translation operators

Lemma A.4 *Let U be a real and separable Hilbert space and let $J := (0, T)$ for some $T \in (0, \infty]$. For every $u \in L^2(J; U)$ we have that*

$$\lim_{h\to 0} \|u(\cdot + h) - u\|_{L^2(J_h;U)} = 0.$$

Here, we define for each $h \in \mathbb{R}$ the interval $J_h := ((-h) \vee 0, T \wedge (T - h)) \subseteq J$ and $u(\cdot + h) \colon J_h \to U$ denotes the function u shifted to the left by an increment h.

Proof Let $v \in C_c^\infty(J; U)$ and fix an arbitrary $\varepsilon \in (0, \infty)$. Choose a compact interval $[a, b] \subset [0, \infty)$ such that $\text{supp}(v(\cdot + h) - v|_{J_h}) \subseteq [a, b]$ for all $h \in [-1, 1]$. By the uniform continuity of v, there exists a $\delta \in (0, 1)$ such that, for all $h \in (-\delta, \delta)$ and every $t \in J_h$, the estimate $\|v(t + h) - v(t)\|_U < \sqrt{\varepsilon/(b - a)}$ holds. Thus,

$$\|v(t + h) - v(t)\|_{L^2(J_h;U)}^2 < \varepsilon \quad \forall h \in (-\delta, \delta).$$

This shows the desired convergence for functions in the space $C_c^\infty(J; U)$, which is dense in $L^2(J; U)$; indeed, since the set of U-valued measurable simple functions is dense in $L^2(J; U)$ [41, Lemma 1.2.19(1)], it suffices to note that the scalar-valued function space $C_c^\infty(J)$ is dense in $L^2(J)$ [2, Corollary 2.30]. Combined with the fact that the translation operator is contractive from $L^2(J; U)$ to $L^2(J_h; U)$ (and thus bounded, uniformly in h), the result extends to $L^2(J; U)$. □

Proposition A.5 *Suppose that $T \in (0, \infty)$. The family $(\mathcal{T}(t))_{t\ge 0} \subseteq \mathcal{L}(L^2(0, T; H))$ defined in (3.6) is a C_0-semigroup whose infinitesimal generator is given by $-\partial_t$, where ∂_t is the Bochner–Sobolev vector-valued weak derivative on $D(\partial_t) = H_{0,\{0\}}^1(0, T; H)$.*

Proof For each $t \ge 0$, it is clear that $\mathcal{T}(t)$ is a well-defined contractive linear map on $L^2(0, T; H)$. Furthermore, it follows readily from the definition (3.6) that $\mathcal{T}(0) = I$

and that the semigroup property is satisfied, since for all $s, t \geq 0$, $v \in L^2(0, T; H)$ and a.a. $\vartheta \in [0, T]$ we have that

$$[\mathcal{T}(t)\mathcal{T}(s)v](\vartheta) = [\widetilde{\mathcal{T}(s)v}](\vartheta - t) = \widetilde{v}(\vartheta - t - s) = [\mathcal{T}(t+s)v](\vartheta).$$

The strong continuity follows from Lemma A.4 for $h \uparrow 0$.

Next, we turn to the generator of $(\mathcal{T}(t))_{t \geq 0}$. To this end, let an arbitrary $v \in C_c^\infty((0, T]; H)$ be given and note that its extension by zero to $(-\infty, T]$, again denoted by $\widetilde{v}$, is continuously differentiable with classical (and hence weak) derivative $\partial_\vartheta \widetilde{v} = \widetilde{\partial_\vartheta v}$ by the compact support of v in $(0, T]$. Fix an arbitrary $\vartheta \in [0, T]$. The function $t \mapsto \widetilde{v}(\vartheta - t)$ is continuously differentiable on $[0, \infty)$ with derivative $t \mapsto -\widetilde{\partial_\vartheta v}(\vartheta - t)$ by the chain rule. Thus, the fundamental theorem of calculus gives

$$\mathcal{T}(t)v(\vartheta) - v(\vartheta) = \widetilde{v}(\vartheta - t) - \widetilde{v}(\vartheta) = -\int_0^t \widetilde{\partial_\vartheta v}(\vartheta - s)\,\mathrm{d}s = -\int_0^t [\mathcal{T}(s)\partial_\vartheta v](\vartheta)\,\mathrm{d}s$$

for every $t \geq 0$. It follows that

$$\mathcal{T}(t)v - v = -\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s.$$

Furthermore, we know from [61, Chapter 1, Theorem 2.4(b)] that if R denotes the generator of $(\mathcal{T}(t))_{t \geq 0}$, then we have

$$\mathcal{T}(t)v - v = R\int_0^t \mathcal{T}(s)v\,\mathrm{d}s,$$

hence, combining the previous two displays yields

$$R\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = -\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s. \tag{A.1}$$

Set $v_t := \frac{1}{t}\int_0^t \mathcal{T}(s)v\,\mathrm{d}s$ for $t \in (0, \infty)$. It follows that $v_t \to \mathcal{T}(0)v = v$ in $L^2(0, T; H)$ as $t \downarrow 0$, see e.g. [61, Chapter 1, Theorem 2.4(a)]. Dividing both sides of (A.1) by $t \in (0, \infty)$ and letting $t \downarrow 0$, one obtains

$$Rv_t = R\frac{1}{t}\int_0^t \mathcal{T}(s)v\,\mathrm{d}s = -\frac{1}{t}\int_0^t \mathcal{T}(s)\partial_\vartheta v\,\mathrm{d}s \to -\mathcal{T}(0)\partial_\vartheta v = -\partial_\vartheta v.$$

Since R is assumed to be the generator of a C_0-semigroup, it is in particular closed by [31, Proposition II.1.4]. Combined with the convergence $v_t \to v$ and $Rv_t \to -\partial_\vartheta v$ as $t \downarrow 0$, this yields $v \in D(R)$ and $Rv = -\partial_\vartheta v$, hence $-\partial_\vartheta|_{C_c^\infty((0,T];H)} \subseteq R$.

As $C_c^\infty((0, T]; H)$ is dense in $L^2(0, T; H)$ and

$$\mathcal{T}(t)C_c^\infty((0, T]; H) \subseteq C_c^\infty((0, T]; H)$$

for all $t \geq 0$, we have that $C_c^\infty((0,T];H)$ is dense in $D(R)$ with respect to the graph norm of R by [31, Proposition II.1.7]. It is evident from the respective definitions that $\| \cdot \|_{D(R)} \eqsim \| \cdot \|_{H^1(0,T;H)}$. These observations together imply

$$D(R) = \overline{C_c^\infty((0,T];H)}^{D(R)} = \overline{C_c^\infty((0,T];H)}^{H^1(0,T;H)} = H^1_{0,\{0\}}(0,T;H). \qquad \square$$

A.3 The proof of Lemma 3.6

Proof of Lemma 3.6 Analogously to [25, Proposition 5.9] it can be shown that the operator defined by the right-hand side of (3.12) maps functions in $L^2(0,T;H)$ to $C_{0,\{T\}}([0,T];H)$. Now we prove the identity in (3.12). Let $f,g \in L^2(0,T;H)$ be arbitrary. By (3.11) and continuity of the inner product $(\cdot,\cdot)_H$, we obtain

$$
\begin{aligned}
(\mathcal{B}^{-\gamma} f, g)_{L^2(0,T;H)} &= \int_0^T \left([\mathcal{B}^{-\gamma} f](t), g(t) \right)_H \mathrm{d}t \\
&= \int_0^T \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} S(t-s) f(s)\, \mathrm{d}s, g(t) \right)_H \mathrm{d}t \\
&= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \left(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s) f(s), g(t) \right)_H \mathrm{d}s\, \mathrm{d}t.
\end{aligned}
\tag{A.2}
$$

Next, we would like to use Fubini's theorem to exchange the order of integration. By (3.2) the semigroup $(S(t))_{t \geq 0}$ is uniformly bounded on the compact interval $[0,T]$,

$$\widetilde{M}_T := \sup_{t \in [0,T]} \|S(t)\|_{\mathscr{L}(H)} \leq M e^{(-wT)\vee 0} < \infty.$$

We then use the Cauchy–Schwarz inequality on H and on $L^2(0,T)$ as well as the fact that $\gamma > \frac{1}{2}$ to find

$$
\begin{aligned}
&\int_0^T \int_0^T \left| \left(\mathbf{1}_{(0,t)}(s)(t-s)^{\gamma-1} S(t-s) f(s), g(t) \right)_H \right| \mathrm{d}s\, \mathrm{d}t \\
&\leq \widetilde{M}_T \int_0^T \int_0^t (t-s)^{\gamma-1} \|f(s)\|_H\, \mathrm{d}s\, \|g(t)\|_H\, \mathrm{d}t \\
&\leq \widetilde{M}_T \|f\|_{L^2(0,T;H)} \int_0^T \left(\int_0^t (t-s)^{2\gamma-2}\, \mathrm{d}s \right)^{1/2} \|g(t)\|_H\, \mathrm{d}t \\
&= \frac{\widetilde{M}_T}{\sqrt{2\gamma-1}} \|f\|_{L^2(0,T;H)} \int_0^T t^{\gamma-\frac{1}{2}} \|g(t)\|_H\, \mathrm{d}t \\
&\leq \frac{\widetilde{M}_T T^\gamma}{\sqrt{2\gamma(2\gamma-1)}} \|f\|_{L^2(0,T;H)} \|g\|_{L^2(0,T;H)} < \infty.
\end{aligned}
$$

This justifies changing the order of integration in (A.2), which gives

$$
\begin{aligned}
(\mathcal{B}^{-\gamma} f, g)_{L^2(0,T;H)} &= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \left(\mathbf{1}_{(s,T)}(t)(t-s)^{\gamma-1} S(t-s) f(s), g(t)\right)_H \mathrm{d}t \, \mathrm{d}s \\
&= \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^T \left(f(s), \mathbf{1}_{(s,T)}(t)(t-s)^{\gamma-1} [S(t-s)]^* g(t)\right)_H \mathrm{d}t \, \mathrm{d}s \\
&= \int_0^T \left(f(s), \frac{1}{\Gamma(\gamma)} \int_s^T (t-s)^{\gamma-1} [S(t-s)]^* g(t) \, \mathrm{d}t\right)_H \mathrm{d}s,
\end{aligned}
$$

where we interchanged integrals and inner products as before in the last step. $\qquad\square$

A.4 Hölder continuity and weak derivatives

Recall from Sect. 2 that $(W(t))_{t\geq 0}$ denotes an H-valued cylindrical Wiener process.

Lemma A.6 *Let Assumptions* 3.1(i),(ii) *be satisfied. Suppose that* $a \in \left(-\frac{1}{2}, \infty\right)$, $b, \sigma \in [0, \infty)$ *and* $\tau \in \left(0, a + \frac{1}{2}\right] \cap (0, 1)$ *are given. If* $\sigma \neq 0$, *then suppose moreover that Assumption* 3.1(iv) *holds. Let* $\Phi_{a,b} \colon (0, \infty) \to \mathscr{L}(H; \dot{H}_A^\sigma)$ *be defined by* (3.22) *and let* $J := (0, T)$ *for some* $T \in (0, \infty]$. *Then, for all* $p \in [1, \infty)$, $t \in [0, T)$ *and* $h \in J$ *with* $h \leq T - t$,

$$
\left\| \int_0^t \left[\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\right] \mathrm{d}W(s) \right\|_{L^p(\Omega; \dot{H}_A^\sigma)}
$$
$$
\lesssim_{(p,a,\tau)} h^\tau \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H; \dot{H}_A^\sigma)}.
$$

Proof We first use the Burkholder–Davis–Gundy inequality (combined with nestedness of the L^p spaces if $p < 2$) to bound the quantity of interest $I_\star$,

$$
\begin{aligned}
I_\star &:= \left\| \int_0^t \left[\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\right] \mathrm{d}W(s) \right\|_{L^p(\Omega; \dot{H}_A^\sigma)} \\
&\lesssim_p \left[\int_0^t \|\Phi_{a,b}(t+h-s) - \Phi_{a,b}(t-s)\|^2_{\mathscr{L}_2(H; \dot{H}_A^\sigma)} \, \mathrm{d}s \right]^{1/2} \\
&= \left[\int_0^t \|\Phi_{a,b}(u+h) - \Phi_{a,b}(u)\|^2_{\mathscr{L}_2(H; \dot{H}_A^\sigma)} \, \mathrm{d}u \right]^{1/2}, \qquad\qquad (\text{A.3})
\end{aligned}
$$

where we also applied the change of variables $u := t - s$. For every $u \in (0, t)$, Lemma 3.20 implies that $\Phi_{a,b}(u + \cdot)$ is differentiable as a function from $(0, h)$ to $\mathscr{L}(H; \dot{H}_A^\sigma)$ with derivative $\Phi'_{a,b}(u + \cdot)$ and, moreover, $r \mapsto \|\Phi'_{a,b}(u+r)\|_{\mathscr{L}(H; \dot{H}_A^\sigma)}$ is bounded on $[0, h]$. We conclude that $\Phi_{a,b}(u + \cdot) \in H^1(0, h; \mathscr{L}(H; \dot{H}_A^\sigma))$, so that by [32, §5.9.2, Theorem 2] the identity

$$
\Phi_{a,b}(u+h) - \Phi_{a,b}(u) = \int_0^h \Phi'_{a,b}(u+r) \, \mathrm{d}r
$$

holds as operators in $\mathcal{L}(H; \dot{H}_A^\sigma)$. We now estimate (A.3) by exploiting this relation, moving the norm inside the integral, applying formula (3.25) for the derivative of $\Phi_{a,b}$ and using the triangle and Minkowski inequalities, which gives

$$I_\star \lesssim_p \left[\int_0^t \left| \int_0^h \|\Phi_{a,b}'(u+r)\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \, dr \right|^2 du \right]^{1/2} \leq \left[\int_0^t \left| |a| F(u) + G(u) \right|^2 du \right]^{1/2}$$

$$\leq |a| \left[\int_0^t |F(u)|^2 \, du \right]^{1/2} + \left[\int_0^t |G(u)|^2 \, du \right]^{1/2}, \tag{A.4}$$

where

$$F(u) := \int_0^h \|\Phi_{a-1,b}(u+r)\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \, dr \quad \text{and}$$

$$G(u) := \int_0^h \|\Phi_{a,b+1}(u+r)\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \, dr.$$

Using Minkowski's integral inequality (see e.g. [68, §A.1]), we obtain

$$\left[\int_0^t |F(u)|^2 \, du \right]^{1/2}$$

$$= \left[\int_0^t \left| \int_0^h \|\Phi_{a-1,b}(u+r)\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \, dr \right|^2 du \right]^{1/2}$$

$$\leq \int_0^h \left[\int_0^t \|\Phi_{a-1,b}(u+r)\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)}^2 \, du \right]^{1/2} dr$$

$$= \int_0^h \left[\int_0^t (u+r)^{2(a-1)} \left\| A^{a+\frac{1}{2}-\tau} S(u+r) A^{\frac{\sigma}{2}-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H)}^2 \, du \right]^{1/2} dr.$$

Since the semigroup $(S(t))_{t \geq 0}$ is assumed to be analytic, by (3.3) the estimate

$$\left\| A^{a+\frac{1}{2}-\tau} S(u+r) A^{\frac{\sigma}{2}-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H)}$$

$$\lesssim_{(a,\tau)} (u+r)^{-a-\frac{1}{2}+\tau} \left\| A^{\frac{\sigma}{2}-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H)} \tag{A.5}$$

follows, where we also used the assumption that $a + \frac{1}{2} - \tau \geq 0$. We conclude that

$$\left[\int_0^t |F(u)|^2 \, du \right]^{1/2} \lesssim_{(a,\tau)} \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \int_0^h \left[\int_0^t (u+r)^{2\tau-3} \, du \right]^{1/2} dr$$

$$\leq \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \int_0^h \left[\int_r^\infty \bar{u}^{2\tau-3} \, d\bar{u} \right]^{1/2} dr$$

$$= \frac{1}{\sqrt{2-2\tau}} \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(H;\dot{H}_A^\sigma)} \int_0^h r^{\tau-1} \, dr$$

$$= \frac{1}{\tau\sqrt{2-2\tau}} h^\tau \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}.$$

Similarly, we can bound the integral $\int_0^t |G(u)|^2 \, \mathrm{d}u$ in (A.4). Again by Minkowski's integral inequality and analogously to (A.5), noting that $a + \frac{3}{2} - \tau > a + \frac{1}{2} - \tau \geq 0$, we find that

$$\left[\int_0^t |G(u)|^2 \, \mathrm{d}u \right]^{1/2} = \left[\int_0^t \left| \int_0^h \| \Phi_{a,b+1}(u+r) \|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)} \, \mathrm{d}r \right|^2 \mathrm{d}u \right]^{1/2}$$

$$\leq \int_0^h \left[\int_0^t \| \Phi_{a,b+1}(u+r) \|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}u \right]^{1/2} \mathrm{d}r$$

$$= \int_0^h \left[\int_0^t (u+r)^{2a} \left\| A^{a+\frac{3}{2}-\tau} S(u+r) A^{\frac{\sigma}{2}-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \, \mathrm{d}u \right]^{1/2} \mathrm{d}r$$

$$\lesssim_{(a,\tau)} \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)} \int_0^h \left[\int_0^t (u+r)^{2\tau-3} \, \mathrm{d}u \right]^{1/2} \mathrm{d}r$$

$$\leq \frac{1}{\tau\sqrt{2-2\tau}} h^\tau \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}. \qquad \square$$

Lemma A.7 *Let Assumptions* 3.1(i),(ii) *be satisfied. Suppose that* $a \in \left(-\frac{1}{2}, \infty \right)$, $b, \sigma \in [0, \infty)$ *and* $\tau \in \left(0, 1 \wedge \left(a + \frac{1}{2} \right) \right]$ *are given. If* $\sigma \neq 0$, *then suppose furthermore that Assumption* 3.1(iv) *holds. Let* $J := (0, T)$ *for some* $T \in (0, \infty]$. *Then, for all* $p \in [1, \infty)$, $t \in [0, T)$ *and* $h \in J$ *with* $h \leq T - t$, *the function* $\Phi_{a,b} \colon (0, \infty) \to \mathscr{L}(H; \dot{H}_A^\sigma)$ *in* (3.22) *satisfies*

$$\left\| \int_t^{t+h} \Phi_{a,b}(t+h-s) \, \mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}_A^\sigma)} \lesssim_{(p,a,\tau)} h^\tau \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}.$$

Proof We apply the Burkholder–Davis–Gundy inequality (combined with nestedness of the L^p spaces if $p < 2$), the change of variables $u := t + h - s$, and obtain

$$\left\| \int_t^{t+h} \Phi_{a,b}(t+h-s) \, \mathrm{d}W(s) \right\|_{L^p(\Omega;\dot{H}_A^\sigma)}^2$$

$$\lesssim_p \int_t^{t+h} \| \Phi_{a,b}(t+h-s) \|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}s = \int_0^h \| \Phi_{a,b}(u) \|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \, \mathrm{d}u$$

$$= \int_0^h u^{2a} \left\| A^{a+\frac{1}{2}-\tau} S(u) A^{\frac{\sigma}{2}-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H)}^2 \, \mathrm{d}u$$

$$\lesssim_{(a,\tau)} \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2 \int_0^h u^{2\tau-1} \, \mathrm{d}u = \frac{h^{2\tau}}{2\tau} \left\| A^{-a-\frac{1}{2}+b+\tau} Q^{\frac{1}{2}} \right\|_{\mathscr{L}_2(H;\dot{H}_A^\sigma)}^2,$$

where we could proceed as in (A.5), since $a + \frac{1}{2} - \tau \geq 0$ is assumed. This completes the proof. $\qquad \square$

Proposition A.8 provides a useful relation between the weak derivative and the difference quotient.

Proposition A.8 *Let U be a real and separable Hilbert space and let $J := (0, T)$ for some $T \in (0, \infty]$. Suppose that $\Psi \in H^1(J; U)$ and let $\Psi' \in L^2(J; U)$ denote the weak derivative of Ψ. For $h \in \mathbb{R} \backslash \{0\}$, let $J_h \subseteq J$ be as in Lemma A.4 and define the difference quotient $D_h \Psi : J_h \to U$ of Ψ by*

$$[D_h \Psi](t) := \frac{\Psi(t+h) - \Psi(t)}{h} \quad \text{for a.a. } t \in J_h. \tag{A.6}$$

Then, we have $\lim_{h \to 0} \|D_h \Psi - \Psi'\|_{L^2(J_h; U)} = 0$.

Proof Suppose that $\Psi \in E$, where we set $E := C^\infty([0, T]; U)$ if $T < \infty$ and $E := C_c^\infty([0, \infty); U)$ if $T = \infty$. Then, fixing an arbitrary $h \in \mathbb{R} \backslash \{0\}$, we have that

$$[D_h \Psi](t) = \frac{1}{h} \int_0^h \Psi'(t+s) \, ds \quad \forall t \in J_h \tag{A.7}$$

holds by the fundamental theorem of calculus, where we use the convention that $\int_0^h = -\int_h^0$ whenever $h \in (-t, 0)$. Applying the Cauchy–Schwarz inequality gives

$$\|[D_h \Psi](t)\|_U^2 \leq \left| \frac{1}{h} \int_0^h \|\Psi'(t+s)\|_U \, ds \right|^2 \leq \left| \frac{1}{h} \int_0^h \|\Psi'(t+s)\|_U^2 \, ds \right|$$

$$= \frac{1}{h} \int_0^h \|\Psi'(t+s)\|_U^2 \, ds \quad \forall t \in J_h.$$

The absolute value can be removed in the last step by the integral sign convention. Integrating this expression over $t \in J_h$ and using Fubini's theorem, we obtain that

$$\|D_h \Psi\|_{L^2(J_h; U)}^2 = \int_{J_h} \|[D_h \Psi](t)\|_U^2 \, dt$$

$$\leq \frac{1}{h} \int_{J_h} \int_0^h \|\Psi'(t+s)\|_U^2 \, ds \, dt = \frac{1}{h} \int_0^h \int_{J_h} \|\Psi'(t+s)\|_U^2 \, dt \, ds. \tag{A.8}$$

For all $s \in (0, h)$ (resp., $s \in (h, 0)$ if $h < 0$), the change of variables $r := t + s$ gives

$$\int_{J_h} \|\Psi'(t+s)\|_U^2 \, dt = \int_{J_h + s} \|\Psi'(r)\|_U^2 \, dr \leq \int_J \|\Psi'(r)\|_U^2 \, dr = \|\Psi'\|_{L^2(J; U)}^2.$$

Hence, we can bound the inner integral in (A.8) independently of s, which implies

$$\|D_h \Psi\|_{L^2(J_h; U)}^2 \leq \|\Psi'\|_{L^2(J; U)}^2 \leq \|\Psi\|_{H^1(J; U)}^2. \tag{A.9}$$

This estimate shows that the linear operator D_h is bounded from $\left(E, \| \cdot \|_{H^1(J; U)} \right)$ to $L^2(J_h; U)$ for all $h \in \mathbb{R} \backslash \{0\}$. By density of E in $H^1(J; U)$ (see [26, XVIII.§1.2, Lemma 1]), the above estimate holds for all $\Psi \in H^1(J; U)$.

Suppose again that $\Psi \in E$. We recall (A.7) and find

$$[D_h \Psi](t) - \Psi'(t) = \frac{1}{h} \int_0^h \big(\Psi'(t+s) - \Psi'(t) \big) \, ds \quad \forall \, t \in J_h. \qquad (A.10)$$

By the compact support of $D_h \Psi$ and Ψ', there exists a bounded interval $K \subset [0, \infty)$ such that

$$\mathrm{supp}(D_h \Psi - \Psi'|_{J_h}) \subseteq K \quad \forall \, h \in [-1, 1].$$

Furthermore, by uniform continuity of $\Psi' \in C^\infty([0, T]; U)$ (respectively, of $\Psi' \in C_c^\infty([0, \infty); U))$, for every $\varepsilon \in (0, \infty)$, there exists some $\delta \in (0, 1)$ such that $\|\Psi'(\xi) - \Psi'(\eta)\|_U < \varepsilon$ if $|\xi - \eta| < \delta$. Thus,

$$\|[D_h \Psi](t) - \Psi'(t)\|_U < \varepsilon \quad \forall \, t \in J_h$$

follows for all $h \in (-\delta, \delta)$ by (A.10) and, consequently,

$$\|D_h \Psi - \Psi'\|_{L^2(J_h; U)} \lesssim_K \|D_h \Psi - \Psi'\|_{L^\infty(J_h; U)} \to 0 \quad \text{as } h \to 0.$$

This proves the assertion for functions $\Psi \in E$. The general case for $\Psi \in H^1(J; U)$ follows then from density of E and the h-uniform bound (A.9): Given $\varepsilon \in (0, \infty)$, we may choose $v \in E$ such that $\|\Psi - v\|_{H^1(J; U)} < \frac{\varepsilon}{3}$, and $h_0 \in (0, \infty)$ such that $\|D_h v - v'\|_{L^2(J_h; U)} < \frac{\varepsilon}{3}$ for all $h \in (-h_0, h_0)$. Thus, we obtain

$$\|D_h \Psi - \Psi'\|_{L^2(J_h; U)} \le \|D_h(\Psi - v)\|_{L^2(J_h; U)} + \|D_h v - v'\|_{L^2(J_h; U)} + \|v' - \Psi'\|_{L^2(J_h; U)}$$
$$\le 2\|\Psi - v\|_{H^1(J; U)} + \|D_h v - v\|_{L^2(J_h; U)} < \varepsilon,$$

for all $h \in (-h_0, h_0)$, which completes the proof. $\qquad \square$

Lemma A.9 *Let $J := (0, T)$ for some $T \in (0, \infty]$. Suppose that E and F are real separable Banach spaces satisfying $E \hookrightarrow F$. If $u \in H^1_{0, \{0\}}(J; F)$ and $u, u' \in L^2(J; E)$, where u' denotes the F-valued weak derivative of u, then $u \in H^1_{0, \{0\}}(J; E)$ and its E-valued weak derivative coincides with u' almost everywhere in J.*

Proof Let $\mathscr{I}_E \colon L^1(J; E) \to E$ and $\mathscr{I}_F \colon L^1(J; F) \to F$ denote, respectively, the E-valued and F-valued Bochner integrals over the interval J. Given an arbitrary $\phi \in C_c^\infty(J)$, the assumption $u \in H^1(J; F)$ implies $\mathscr{I}_F(\phi u') = -\mathscr{I}_F(\phi' u)$, and we wish to show $\mathscr{I}_E(\phi u') = -\mathscr{I}_E(\phi' u)$. To this end, we claim that $\mathscr{I}_E$ and $\mathscr{I}_F$ coincide on $L^1(J; E) \hookrightarrow L^1(J; F)$ and we apply this fact to $\phi u'$ and $\phi' u$. To verify the claim, fix an arbitrary $f \in L^1(J; E)$. By definition of $\mathscr{I}_E$, there exist E-valued measurable simple functions $(f_n)_{n \in \mathbb{N}}$ satisfying $f_n \to f$ in $L^1(J; E)$ and $\mathscr{I}_E(f_n) \to \mathscr{I}_E(f)$ in E. For all $n \in \mathbb{N}$, it follows from the respective definitions and the inclusion $E \subseteq F$ that f_n is an F-valued measurable simple function and that $\mathscr{I}_F(f_n) = \mathscr{I}_E(f_n)$. Since $E \hookrightarrow F$, we observe that $f_n \to f$ in $L^1(J; F)$ and $\mathscr{I}_F(f_n) = \mathscr{I}_E(f_n) \to \mathscr{I}_E(f)$ in

F, hence $\mathscr{I}_F(f) = \mathscr{I}_E(f)$. We conclude that $u \in H^1(J; E)$ and the E-valued weak derivative coincides with u' a.e. in J. Now it remains to prove that $u \in H_{0,\{0\}}^1(J; E)$. Note that $u \in H_{0,\{0\}}^1(J; F)$ is equivalent to the statement that the unique continuous representative $\widetilde{u} \in C(\overline{J}; F)$ of u, which exists by virtue of [32, §5.9.2, Theorem 2], vanishes at zero, cf. [32, §5.5, Theorem 2]. Similarly, from $u \in H^1(J; E)$ we obtain a function $\widehat{u} \in C(\overline{J}; E) \hookrightarrow C(\overline{J}; F)$ such that $u = \widehat{u}$ a.e., hence $\widehat{u} = \widetilde{u}$ by uniqueness. In particular, $\widehat{u}(0) = 0$ and thus $u \in H_{0,\{0\}}(J; E)$. □

B Sectorial linear operators and functional calculus

In this appendix, several definitions and results regarding sectorial linear operators, semigroups and functional calculus are recorded. We refer the reader to [31, 37, 42, 61] for more details on these topics.

Throughout this section, $A \colon \mathsf{D}(A) \subseteq H \to H$ denotes a linear operator whose negative $-A$ generates a C_0-semigroup $(S(t))_{t \geq 0}$ on a separable Hilbert space H. The corresponding scalar field is given by the complex numbers $\mathbb{C}$ in Sect. B.1 and the real numbers $\mathbb{R}$ in Sect. B.2.

B.1 Sectoriality and H^∞-calculus

Let H be a Hilbert space over the *complex* scalar field $\mathbb{C}$.

Definition B.1 We say that $\lambda \in \mathbb{C}$ belongs to the *resolvent set* $\rho(A)$ if and only if $R(\lambda, A) := (\lambda I - A)^{-1}$ exists in $\mathscr{L}(H)$. The set $\sigma(A) := \mathbb{C}\backslash\rho(A)$ is called the *spectrum*. A is said to be *sectorial* if there exists an $\omega \in (0, \pi)$ such that

$$\sigma(A) \subseteq \overline{\Sigma}_\omega \quad \text{and} \quad \sup\{\|\lambda R(\lambda, A)\|_{\mathscr{L}(H)} : \lambda \in \mathbb{C}\backslash\overline{\Sigma}_\omega\} < \infty, \qquad \text{(B.1)}$$

where $\Sigma_\omega := \{\lambda \in \mathbb{C}\backslash\{0\} : \arg \lambda \in (-\omega, \omega)\}$. The *angle of sectoriality* $\omega(A)$ is defined as the infimum of all ω for which (B.1) holds.

Theorem B.2 *The operator A is sectorial with $\omega(A) \in [0, \pi/2)$ if and only if $(S(t))_{t \geq 0}$ is (uniformly) bounded analytic.*

Proof The claim follows from [42, Theorem G.5.2], noting that $\mathsf{D}(A)$ is dense in H since $-A$ generates a C_0-semigroup, see [31, Theorem II.1.4]. □

Given $\varphi \in (0, \pi)$, we say that a holomorphic function $f \colon \Sigma_\varphi \to \mathbb{C}$ belongs to $H_0^\infty(\Sigma_\varphi)$ if and only if there exist constants $\alpha \in (0, \infty)$ and $M \in [0, \infty)$ such that $|f(z)| \leq M(|z|^\alpha \wedge |z|^{-\alpha})$ for all $z \in \Sigma_\varphi$. If A is sectorial and $\varphi \in (\omega(A), \pi)$, then

$$f(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(\zeta) R(\zeta, A) \, \mathrm{d}\zeta, \qquad f \in H_0^\infty(\Sigma_\varphi),$$

is well-defined, i.e., the $\mathscr{L}(H)$-valued Bochner integral is convergent and independent of $\nu \in (\omega(A), \varphi)$. We call the mapping $f \mapsto f(A)$ the *Dunford calculus* for A; it is an algebra homomorphism from $H_0^\infty(\Sigma_\varphi)$ to $\mathscr{L}(H)$, see [37, Lemma 2.3.1(a)].

Definition B.3 Let A be a sectorial operator and $\varphi \in (\omega(A), \pi)$. Then A is said to have a *bounded $H^\infty(\Sigma_\varphi)$-calculus* if there exists a constant $C \in (0, \infty)$ such that $\|f(A)\|_{\mathscr{L}(H)} \leq C \sup_{z \in \Sigma_\varphi} |f(z)|$ for all $f \in H_0^\infty(\Sigma_\varphi)$. The angle $\omega_{H^\infty}(A)$ is defined as the infimum over all admissible $\varphi \in (\omega(A), \pi)$ in the above definition.

For operators acting on a (complex) Hilbert space, the admissibility of a bounded H^∞-calculus can be characterized by the following theorem. It is taken from [37, Theorem 7.3.1]; see [42, Theorem 10.4.21] for a generalization to non-injective A.

Theorem B.4 *Let $A \colon \mathsf{D}(A) \subseteq H \to H$ be injective and sectorial. Then*

$$\|x\|_H^2 \eqsim_f \int_0^\infty \|f(tA)x\|_H^2 \, \frac{\mathrm{d}t}{t} \quad \forall x \in H$$

holds for all non-zero $f \in \bigcup_{\varphi \in (\omega(A), \pi)} H_0^\infty(\Sigma_\varphi)$ if and only if A admits a bounded $H^\infty(\Sigma_\varphi)$-calculus for some (or, equivalently, for all) $\varphi \in (\omega(A), \pi)$.

Remark B.5 Since $\omega_{H^\infty}(A)$ is defined as an infimum over angles contained in the interval $(\omega(A), \pi)$, any operator admitting a bounded H^∞-calculus satisfies that $\omega_{H^\infty}(A) \geq \omega(A)$. This inequality is also true for operators on a Banach space, defined analogously to Definition B.3. Theorem B.4 implies that the reverse inequality holds for operators on a Hilbert space with a bounded $H^\infty(\Sigma_\varphi)$-calculus for some $\varphi \in (\omega(A), \pi)$. Indeed, in this case, the same holds for all $\varphi \in (\omega(A), \pi)$, hence $\omega_{H^\infty}(A) \leq \omega(A)$ upon taking the infimum. We thus have $\omega_{H^\infty}(A) = \omega(A)$.

B.2 Complexifications, semigroups and fractional powers

In this subsection, H denotes a *real* Hilbert space.

B.2.1 Complexifications

The complexified Hilbert space $H_\mathbb{C}$ is defined by equipping the set $H \times H$ with component-wise addition and the respective scalar and inner products

$$(a + bi)(x, y) := (ax - by, bx + ay), \qquad\qquad x, y \in H; \; a, b \in \mathbb{R},$$
$$((x, y), (u, v))_{H_\mathbb{C}} := (x, u)_H + (y, v)_H + i[(y, u)_H - (x, v)_H], \quad x, y, u, v \in H. \quad \text{(B.2)}$$

In the sequel, we will write $x + iy := (x, y) \in H_\mathbb{C}$.

A linear operator A on H similarly gives rise to a complexified counterpart $A_\mathbb{C}$ on $H_\mathbb{C}$ by defining $A_\mathbb{C}(x + iy) := Ax + iAy$ on $\mathsf{D}(A_\mathbb{C}) = \{x + iy : x, y \in \mathsf{D}(A)\}$. It follows readily from the above definitions that $T \mapsto T_\mathbb{C} \in \mathscr{L}(\mathscr{L}(H); \mathscr{L}(H_\mathbb{C}))$ is an inverse-preserving and isometric algebra homomorphism. Analogous results hold for unbounded operators, taking natural domains into account. We have the following relation between semigroups and complexifications.

Lemma B.6 *The family $(S(t))_{t \geq 0} \subseteq \mathscr{L}(H)$ is a C_0-semigroup if and only if $(S_\mathbb{C}(t))_{t \geq 0} \subseteq \mathscr{L}(H_\mathbb{C})$ is a C_0-semigroup. In this case, their respective generators $-A \colon \mathsf{D}(A) \subseteq H \to H$ and $-\widehat{A} \colon \mathsf{D}(\widehat{A}) \subseteq H_\mathbb{C} \to H_\mathbb{C}$ satisfy $A_\mathbb{C} = \widehat{A}$.*

Proof If $(S(t))_{t\geq 0}$ is a C_0-semigroup, then clearly $S_{\mathbb{C}}(0) = I$ and

$$S_{\mathbb{C}}(t)S_{\mathbb{C}}(s) = [S(t)S(s)]_{\mathbb{C}} = S_{\mathbb{C}}(t+s), \quad \forall s, t \geq 0.$$

Moreover,

$$S_{\mathbb{C}}(t)S_{\mathbb{C}}(s) = [S(t)S(s)]_{\mathbb{C}} = S_{\mathbb{C}}(t+s) \quad \text{as } t \downarrow 0$$

for $\widehat{x} = x + iy \in H_{\mathbb{C}}$. The reverse implication is readily established by identifying every $x \in H$ with $x + i0 \in H_{\mathbb{C}}$.

Suppose that $(S(t))_{t\geq 0}$ and $(S_{\mathbb{C}}(t))_{t\geq 0}$ are C_0-semigroups with respective generators $-A$ and $-\widehat{A}$. Then $\widehat{x} = x + iy \in D(A_{\mathbb{C}})$ is equivalent to the existence of the two limits $-Ax = \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x)$ and $-Ay = \lim_{t\downarrow 0} \frac{1}{t}(S(t)y - y)$ in H. Thus,

$$A_{\mathbb{C}}\widehat{x} = Ax + iAy = \lim_{t\downarrow 0}\left[\frac{1}{t}\left(x - S(t)x\right) + \frac{i}{t}\left(y - S(t)y\right)\right]$$

$$= \lim_{t\downarrow 0}\frac{1}{t}\left(\widehat{x} - S_{\mathbb{C}}(t)\widehat{x}\right) = \widehat{A}\widehat{x},$$

where the limits in the previous display are taken with respect to $\|\cdot\|_{H_{\mathbb{C}}}$. □

B.2.2 Fractional powers

Let $\alpha \in (0, \infty)$ be given. If $(S(t))_{t\geq 0}$ is exponentially stable, that is, (3.2) holds for some $w \in (0, \infty)$, then we define negative fractional powers of A by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}S(t)\,dt, \tag{B.3}$$

non-negative powers by $A^\alpha := (A^{-\alpha})^{-1}$ and $A^0 := I$. By using Lemma B.6 and interchanging the bounded operator $[\,\cdot\,]_{\mathbb{C}}$ with the Bochner integral, we find

$$[A^{-\alpha}]_{\mathbb{C}} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}S_{\mathbb{C}}(t)\,dt = A_{\mathbb{C}}^{-\alpha} \quad \forall \alpha \in (0, \infty),$$

and the same relation can be derived for arbitrary powers $\alpha \in \mathbb{R}$. This definition of a fractional-order operator $A_{\mathbb{C}}^\alpha$ is adopted in [61, Chapter 2, Section 6] and is equivalent to the Dunford-type definition used in [37], see Corollary 3.3.6 therein.

B.2.3 A square function estimate

The following square function estimate is central to the proof of Proposition 3.14.

Lemma B.7 *Let A satisfy Assumptions* 3.1*(i),(iii),(iv). Then, for $a \in (0, \infty)$,*

$$\int_0^\infty \left\| t^{a-\frac{1}{2}} A^a S(t) x \right\|_H^2 dt \eqsim_a \|x\|_H^2 \quad \forall x \in H.$$

Proof Given $a \in (0, \infty)$ and $\varphi \in (\omega(A), \pi/2)$, the function $f(z) := z^a e^{-z}$ belongs to $H_0^\infty(\Sigma_\varphi)$ and we have the identity $f(tA_\mathbb{C}) = t^a A_\mathbb{C}^a S_\mathbb{C}(t) = [t^a A^a S(t)]_\mathbb{C}$; see the proof of [37, Proposition 3.4.3], which is applicable to our definition of fractional powers as remarked in Sect. B.2.2. By invoking Theorem B.4, we find

$$\int_0^\infty \left\| [t^a A^a S(t)]_\mathbb{C} x \right\|_{H_\mathbb{C}}^2 \frac{dt}{t} = \int_0^\infty \left\| f(tA_\mathbb{C}) x \right\|_{H_\mathbb{C}}^2 \frac{dt}{t} \eqsim_a \|x\|_{H_\mathbb{C}}^2 \quad \forall x \in H_\mathbb{C}.$$

Applying this equivalence to $x + i0$ for all $x \in H$ finishes the proof. □

References

1. Abatangelo, N., Dupaigne, L.: Nonhomogeneous boundary conditions for the spectral fractional Laplacian. Ann. Inst. H. Poincaré Anal. Non Linéaire **34**(2), 439–467 (2017)
2. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, Pure and Applied Mathematics (Amsterdam), vol. 140, 2nd edn. Elsevier/Academic Press, Amsterdam (2003)
3. Alexeeff, S.E., Nychka, D., Sain, S.R., Tebaldi, C.: Emulating mean patterns and variability of temperature across and within scenarios in anthropogenic climate change experiments. Clim. Change **146**(3), 319–333 (2018)
4. Angulo, J.M., Kelbert, M.Y., Leonenko, N.N., Ruiz-Medina, M.D.: Spatiotemporal random fields associated with stochastic fractional Helmholtz and heat equations. Stoch. Environ. Res. Risk Assess. **22**(suppl. 1), 3–13 (2008)
5. Antil, H., Pfefferer, J., Rogovs, S.: Fractional operators with inhomogeneous boundary conditions: analysis, control, and discretization. Commun. Math. Sci. **16**(5), 1395–1426 (2018)
6. Bakka, H., Vanhatalo, J., Illian, J.B., Simpson, D., Rue, H.: Non-stationary Gaussian models with physical barriers. Spat. Stat. **29**, 268–288 (2019)
7. Beguin, J., Fuglstad, G.-A., Mansuy, N., Paré, D.: Predicting soil properties in the Canadian boreal forest with limited data: comparison of spatial and non-spatial statistical approaches. Geoderma **306**, 195–205 (2017)
8. Bertino, L., Evensen, G., Wackernagel, H.: Sequential data assimilation techniques in oceanography. Int. Stat. Rev. **71**(2), 223–241 (2003)
9. Biswas, A., De León-Contreras, M., Stinga, P.R.: Harnack inequalities and Hölder estimates for master equations. SIAM J. Math. Anal. **53**(2), 2319–2348 (2021)
10. Biswas, A., Stinga, P.R.: Regularity estimates for nonlocal space-time master equations in bounded domains. J. Evol. Equ. **21**(1), 503–565 (2021)
11. Bolin, D., Kirchner, K.: The rational SPDE approach for Gaussian random fields with general smoothness. J. Comput. Graph. Stat. **29**(2), 274–285 (2020)
12. Bolin, D., Kirchner, K.: Equivalence of measures and asymptotically optimal linear prediction for Gaussian random fields with fractional-order covariance operators. Bernoulli **29**(2), 1476–1504 (2023)
13. Bolin, D., Kirchner, K., Kovács, M.: Weak convergence of Galerkin approximations for fractional elliptic stochastic PDEs with spatial white noise. BIT **58**(4), 881–906 (2018)
14. Bolin, D., Kirchner, K., Kovács, M.: Numerical solution of fractional elliptic stochastic PDEs with spatial white noise. IMA J. Numer. Anal. **40**(2), 1051–1073 (2020)
15. Bolin, D., Lindgren, F.: Spatial models generated by nested stochastic partial differential equations, with an application to global ozone mapping. Ann. Appl. Stat. **5**(1), 523–550 (2011)
16. Bolin, D., Lindgren, F.: A comparison between Markov approximations and other methods for large spatial data sets. Comput. Stat. Data Anal. **61**, 7–21 (2013)

17. Bonaccorsi, S.: Fractional stochastic evolution equations with Lévy noise. Differ. Integr. Equ. **22**(11–12), 1141–1152 (2009)
18. Cameletti, M., Lindgren, F., Simpson, D., Rue, H.: Spatio-temporal modeling of particulate matter concentration through the SPDE approach. AStA Adv. Stat. Anal. **97**(2), 109–131 (2013)
19. Carrizo Vergara, R., Allard, D., Desassis, N.: A general framework for SPDE-based stationary random fields. Bernoulli **28**(1), 1–32 (2022)
20. Cox, S.G., Hutzenthaler, M., Jentzen, A.: Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations. Preprint arXiv:1309.5595v3 (2021)
21. Cox, S.G., Kirchner, K.: Regularity and convergence analysis in Sobolev and Hölder spaces for generalized Whittle-Matérn fields. Numer. Math. **146**(4), 819–873 (2020)
22. Cressie, N., Huang, H.C.: Classes of nonseparable, spatio-temporal stationary covariance functions. J. Am. Stat. Assoc. **94**(448), 1330–1340 (1999)
23. Cressie, N., Wikle, C.K.: Statistics for Spatio-temporal Data. Wiley Series in Probability and Statistics. Wiley, Hoboken (2011)
24. Da Prato, G., Kwapień, S., Zabczyk, J.: Regularity of solutions of linear stochastic equations in Hilbert spaces. Stochastics **23**(1), 1–23 (1987)
25. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications, vol. 152, 2nd edn. Cambridge University Press, Cambridge (2014)
26. Dautray, R., Lions, J.L.: Mathematical Analysis and Numerical Methods for Science and Technology, vol. 5. Springer, Berlin (1992)
27. Davies, E.B.: Spectral Theory and Differential Operators. Cambridge Studies in Advanced Mathematics, vol. 42. Cambridge University Press, Cambridge (1995)
28. De Iaco, S., Palma, M., Posa, D.: Spatio-temporal geostatistical modeling for French fertility predictions. Spat. Stat. **14**(part B), 546–562 (2015)
29. Denk, R., Hieber, M., Prüss, J.: $\mathscr{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Am. Math. Soc. **166**(788), viii+114 (2003)
30. Desch, W., Londen, S.O.: Semilinear stochastic integral equations in L_p. In: Parabolic Problems. Progress in Nonlinear Differential Equations and Their Application, vol. 80, pp. 131–166. Birkhäuser/Springer Basel AG, Basel (2011)
31. Engel, K.J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, vol. 194. Springer, New York (2000)
32. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19, 2nd edn. American Mathematical Society, Providence (2010)
33. Fernández-Avilés, G., Montero, J.M.: Spatio-temporal modeling of financial maps from a joint multidimensional scaling-geostatistical perspective. Expert Syst. Appl. **60**, 280–293 (2016)
34. Fuentes, M., Chen, L., Davis, J.M.: A class of nonseparable and nonstationary spatial temporal covariance functions. Environmetrics **19**(5), 487–507 (2008)
35. Fuglstad, G.-A., Lindgren, F., Simpson, D., Rue, H.V.: Exploring a new class of non-stationary spatial Gaussian random fields with varying local anisotropy. Stat. Sin. **25**(1), 115–133 (2015)
36. Gneiting, T.: Nonseparable, stationary covariance functions for space-time data. J. Am. Stat. Assoc. **97**(458), 590–600 (2002)
37. Haase, M.: The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications, vol. 169. Birkhäuser Verlag, Basel (2006)
38. Handcock, M.S., Wallis, J.R.: An approach to statistical spatial-temporal modeling of meteorological fields. J. Am. Stat. Assoc. **89**(426), 368–390 (1994)
39. Harbrecht, H., Herrmann, L., Kirchner, K., Schwab, Ch.: Multilevel approximation of Gaussian random fields: covariance compression, estimation and spatial prediction. Preprint arXiv:2103.04424v1 (2021)
40. Herrmann, L., Kirchner, K., Schwab, Ch.: Multilevel approximation of Gaussian random fields: fast simulation. Math. Models Methods Appl. Sci. **30**(1), 181–223 (2020)
41. Hytönen, T., van Neerven, J., Veraar, M., Weis, L.: Analysis in Banach Spaces. Vol. I. Martingales and Littlewood-Paley Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 63. Springer, Cham (2016)
42. Hytönen, T., van Neerven, J., Veraar, M., Weis, L.: Analysis in Banach Spaces. Vol. II. Probabilistic Methods and Operator Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 67. Springer, Cham (2017)
43. Jost, G., Heuvelink, G.B.M., Papritz, A.: Analysing the space-time distribution of soil water storage of a forest ecosystem using spatio-temporal kriging. Geoderma **128**(3), 258–273 (2005)

44. Kelbert, M.Y., Leonenko, N.N., Ruiz-Medina, M.D.: Fractional random fields associated with stochastic fractional heat equations. Adv. Appl. Probab. **37**(1), 108–133 (2005)
45. Kirchner, K., Lang, A., Larsson, S.: Covariance structure of parabolic stochastic partial differential equations with multiplicative Lévy noise. J. Differ. Equ. **262**(12), 5896–5927 (2017)
46. Kovács, M., Larsson, S., Urban, K.: On wavelet-Galerkin methods for semilinear parabolic equations with additive noise. In: Monte Carlo and Quasi-Monte Carlo Methods 2012, Springer Proceedings in Mathematics and Statistics, vol. 65, pp. 481–499. Springer, Heidelberg (2013)
47. Kunstmann, P.C., Weis, L.: Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In: Functional Analytic Methods for Evolution Equations. Lecture Notes in Mathematics, vol. 1855, pp. 65–311. Springer, Berlin (2004)
48. Lawson, A.B.: Hierarchical modeling in spatial epidemiology. Wiley Interdiscip. Rev. Comput. Stat. **6**(6), 405–417 (2014)
49. Lindgren, F., Bakka, H., Bolin, D., Krainski, E., Rue, H.: A diffusion-based spatio-temporal extension of Gaussian Matérn fields. Preprint arXiv:2006.04917v3 (2023)
50. Lindgren, F., Rue, H., Lindström, J.: An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach. J. R. Stat. Soc. Ser. B Stat. Methodol. **73**(4), 423–498 (2011). With discussion and a reply by the authors
51. Litsgård, M., Nyström, K.: On local regularity estimates for fractional powers of parabolic operators with time-dependent measurable coefficients. J. Evol. Equ. **23**(1), Paper No. 3, 33 (2023)
52. Liu, W., Röckner, M.: Stochastic Partial Differential Equations: An Introduction. Universitext. Springer, Cham (2015)
53. Martínez Carracedo, C., Sanz Alix, M.: The Theory of Fractional Powers of Operators. North-Holland Mathematics Studies, vol. 187. North-Holland, Amsterdam (2001)
54. Matérn, B.: Spatial variation: stochastic models and their application to some problems in forest surveys and other sampling investigations. Meddelanden Från Statens Skogsforskningsinstitut, Band 49, Nr. 5, Stockholm (1960)
55. Mateu, J., Porcu, E., Gregori, P.: Recent advances to model anisotropic space-time data. Stat. Methods Appl. **17**(2), 209–223 (2008)
56. Mejia, A.F., Yue, Y., Bolin, D., Lindgren, F., Lindquist, M.A.: A Bayesian general linear modeling approach to cortical surface fMRI data analysis. J. Am. Stat. Assoc. **115**(530), 501–520 (2020)
57. Nyström, K., Sande, O.: Extension properties and boundary estimates for a fractional heat operator. Nonlinear Anal. **140**, 29–37 (2016)
58. Oberhettinger, F., Badii, L.: Tables of Laplace Transforms. Springer, New York (1973)
59. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): NIST handbook of mathematical functions. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge (2010)
60. Ouhabaz, E.M.: Analysis of Heat Equations on Domains. London Mathematical Society Monographs Series, vol. 31. Princeton University Press, Princeton (2005)
61. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, vol. 44. Springer, New York (1983)
62. Pereira, S., Turkman, K.F., Correia, L., Rue, H.: Unemployment estimation: spatial point referenced methods and models. Spat. Stat. **41**, Paper No. 100345 (2021)
63. Peszat, S., Zabczyk, J.: Stochastic Partial Differential Equations with Lévy Noise. Encyclopedia of Mathematics and its Applications, vol. 113. Cambridge University Press, Cambridge (2007)
64. Porcu, E., Furrer, R., Nychka, D.: 30 years of space-time covariance functions. Wiley Interdiscip. Rev. Comput. Stat. **13**(2), e1512, 24 (2021)
65. Porcu, E., Mateu, J., Bevilacqua, M.: Covariance functions that are stationary or nonstationary in space and stationary in time. Stat. Neerl. **61**(3), 358–382 (2007)
66. Sang, H., Jun, M., Huang, J.Z.: Covariance approximation for large multivariate spatial data sets with an application to multiple climate model errors. Ann. Appl. Stat. **5**(4), 2519–2548 (2011)
67. Sigrist, F., Künsch, H.R., Stahel, W.A.: Stochastic partial differential equation based modelling of large space-time data sets. J. R. Stat. Soc. Ser. B Stat. Methodol. **77**(1), 3–33 (2015)
68. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton (1970)
69. Stein, M.L.: Interpolation of Spatial Data. Springer Series in Statistics. Springer, New York (1999)
70. Stein, M.L.: Space-time covariance functions. J. Am. Stat. Assoc. **100**(469), 310–321 (2005)

71. Stinga, P.R., Torrea, J.L.: Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation. SIAM J. Math. Anal. **49**(5), 3893–3924 (2017)
72. Taylor, M.E.: Pseudodifferential Operators. Princeton Mathematical Series, No. 34. Princeton University Press, Princeton (1981)
73. Whittle, P.: Stochastic processes in several dimensions. Bull. Inst. Int. Stat. **40**, 974–994 (1963)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.