Ik Zie, Ik Zie, Wat Jij Niet Ziet…
(I Spy With My Little Eye…)

On how quantum mechanics uses indistinguishable states to express non-commutativity

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Preliminaries and Notation

The current thesis is presented to obtain a masters degree in applied mathematics with specialisation in analysis. The text presented here will thus assume some familiarity with the field of mathematical analysis, in particular the theory of functional analysis, measure and integration theory, real analysis and some probability theory of the level needed for and gained by completion of the master course ‘Applied Functional Analysis’ at the TU Delft. Furthermore, while the text is presented as a self-contained mathematical text, one can only appreciate the point made with at least some level of familiarity with the theory of quantum mechanics. Concretely, I can recommend the introductory lectures from the MIT course ‘8.04: Quantum Physics I’ by Barton Zweibach, which are publicially available on YouTube. A more mathematical introduction into quantum mechanics could then be provided by chapter 15 of Van Neerven’s ‘Functional Analysis’ [1] or chapters 2, 4 and 5 in Landsman’s ‘Foundation of Quantum Theory, From classical Concepts to Operator Algebras’ [2].

We aim to follow here mathematical convention where-ever possible. We hope to provide some explanation of the symbols used in the text, even if most will be explicitly defined there.

Following convention, we donete by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, the natural, whole, rational, real and complex numbers respectively. We will use $\mathbb{T}$ to denote the complex unit circle and $\mathbb{R}^+ = [0, \infty)$ will be used to denote the positive real numbers including 0. The natural numbers will not include $0 \notin \mathbb{N}$, unless specified otherwise. We will denote complex conjugation by a bar over the number $\overline{c}$ (for $c \in \mathbb{C}$). The aforementioned sets are assumed to be equipped with their standard topologies, which we be denoted by use of the letter τ.

We will use capital letters F, G to denote sets, $\mathcal{F}, \mathcal{G}$ for σ-algebras and $\mathscr{F}$ for an algebra of sets. The power set of a set Ω will be denoted by $\mathscr{P}(\Omega)$. We will use $(F_\lambda)_{\lambda \in \Lambda} \subseteq \mathscr{F}$ to denote as ordered family of sets lying with a larger set $\mathscr{F}$. We will use F^C to denote the complement of a set, $\cup$ for union, $\cap$ for intersection, $\setminus$ for the difference and $\sqcup$ for the disjoint union. The Cartesian product of two sets F, G will be denoted by $F \times G$. On a measurable space $(\Omega, \mathcal{F})$ the set of probability measure will be denoted by $M_1^+(\Omega)$. The indicator function of a set F will be denoted by 1_F. The topological closure will be denoted by writing a bar over a set $\overline{F}$. For a set X with topology τ_X we will use $\mathcal{B}(X)$ to denote its Borel σ-algebra. The Lebesgue measure will then be denoted by λ.

We use H to denote a *complex* Hilbert space with elements $h \in H$. We use $(\quad | \quad)$ to denote the inner product, when relevant we take complex conjugation in the second coordinate (i.e. for $L^2(\mathbb{R})$ $(f|g) = \int f\overline{g} d\lambda$). The othogonal complement of a subspace $G \subseteq H$ will be denoted by $G^\perp$. Direct sums of space are denoted by $H_1 \oplus H_2$. The set of bounded linear operators mapping from the Hilbert space H to K will be denoted by $\mathcal{L}(H, K)$, where $\mathcal{L}(H) = \mathcal{L}(H, K)$. A will be used to refer to a general or self-adjoint operator, U refers to a unitary operator, P an (orthogonal) projection and lastly, I refers to the identity operator. The set of projection operators will be denoted by $\mathcal{P}(H)$.

Introduction

In the current thesis we argue that quantum mechanics is best understood as a classical theory in which each measurement hides a set of its states. Our aim will however not be to replace quantum mechanical theory by a classical theory of hidden states or to avoid any of its possibly unfavourable metaphysical implications by replacing the current paradigm by a different formulation of the theory. The current thesis on hidden variables is rather guided by the question of how quantum mechanical theory differs from classical mechanics than any attempt to overcome this difference. In contrast to the standard approach to hidden variables, we argue that quantum mechanics in its current formulation itself is best understood as a hidden states theory.

The program of Bohrification

Any attempt at a mathematically structured description of the experiments of physics relies on assumptions and is guided by preferences. In this thesis, in our attempt to understand the set of experiments described by the theory of quantum mechanics, we follow the so-called program of Bohrification as outlined (in the introduction of) Landsman's recent work 'Foundation of Quantum Theory, From classical Concepts to Operator Algebras' [2, p. 1-19]. This program relies on the assumption that the language of (non-commutative) operator algebras - rather than for example the nonequivalent formalism of path integrals [2, p. 19] - is the right language to understand quantum phenomena. Moreover, it is founded on the key assumption that the physically relevant aspects of the non-commutative operator algebras of quantum mechanics are only accessible through the commutative algebras [2, p .10]. This axiom is motivated by the assumption that a measurement apparatus is to be understood classically [2, p. 8] and in some sense needs to be understood classically. This last claim is then motivated by the ideas that (i) as humans equipped with a human understanding of the world which simply grasps this world in classical terms, (ii) that it grands a form of objectification to the theory in the sense that it is both able to account for the classical results of a physical experimental set-up by specifying the commutative algebra and is able to grant the physical system an independent existence in the form of a the larger set of non-commutative elements existing independently of our inquiries and lastly, (iii) that all practical applications of quantum mechanical theory in experiments (and engineering) can (and practically are) understood in classical terms [2, p. 6,7]. As commutative algebras provide a natural classical understanding, it is thus exactly through the interplay of non-commutative algebras and their commutative subalgebras that the theory of quantum mechanics is to be understood.

We do however propose a slight modification to the program outlined in Landsman's work. We will work here with projection valued measures (and only marginally with their unsharp extension to positive operator valued measures) rather than the direct use of the operators in the commutative algebras. While we will operate under the same key assumption - that the physically relevant aspects of the non-commutative operator algebras of quantum mechanics are only accessible through commutative algebras -, we view that the projection valued measure allows for a more natural link experimental reality. While the difference between the two approaches boils down the conceptual difference between viewing the probability of event(s) A as the expectation of the(ir) indicator function(s) $\mathbb{E}(1_A)$ or viewing the expectation over a set of events as the weighted sum of their outcomes and is as such only marginal, the main gain of the projection valued measure is the flexibility it allows in formulating these events. In an experiment with outcomes labelled 'blue', 'red' and 'yellow', the first approach first has to construct a bijection between the set of event and some set of real-valued outcomes which can then be used to construct an operator with the required spectral measure. In contrast, the framework of the projection valued measure can directly assign measures to these (non-numerical) events. Moreover, the use of projection valued measures allows for a more natural approach to experiments with an unbounded set of numerical outcomes, which in a sense by-passes the convoluted construction of the unbounded operator and its spectral calculus. While we regard these differences as clear reasons to favour the use of projection valued measures over the use of self-adjoint operators, their difference is mostly conceptual. Namely, as we will aim to show in our first chapter, each commutative (i.e. Abelian) Von Neuammn algebra on a separable Hilbert space can be naturally associated with the spectral calculus of a projection valued measure and vice versa.

The contrast with the 'traditional' approach to hidden variables

It may perhaps seem strange to the reader that we will here will discuss hidden variables as a positive quality of quantum mechanical theory. Much of the discussion surrounding quantum mechanics deals exactly with the theoretical exclusion of quantum mechanical hidden variable theories. The central idea behind the 'traditional' approach to hidden variable theories is to find a formulation compatible with the predictions of quantum mechanics (at least within the known experimental bounds), which take more classical form. These attempts are motivated by a wish to move away from the deemed unfavourable metaphysical implications the standard theory of non-commutative algebras seems to imply. Already in his 1932 work 'Mathematical Foundation of Quantum Mechanics' (Mathematische Grundlagen der Quantenmechanik) ([3], cited from [2, p. 193]) Von Neumann proved a result excluding a class of classical theories (which are now known as the class of non-contextual dispersion-free normalized hidden variable theories) mirroring the predictions made by this theory of 'rings of operators' (now known as Von Neumann algebras) and a series of results, including the famous Bell and Kochen-Specker theorems, have made this original theorem by Von Neumann only stronger and more precise [2, p. 191]. These results excluding a classical formulation of quantum mechanical theory do however not exclude all possible classical theories of hidden variables. Even if Bohr may have believed it to be impossible, the formulation of De Broglie-Bohm theory shows at least the theoretical possibility of formulating a deterministic theory compatible with the outcomes of standard quantum mechanics [2, p. 7]. What is crucial for us here, is that the discussion surrounding the idea of hidden variable theories in the context of quantum mechanics (at least historically) seems to be founded on a mutual hostility. That is, entering the discussion is either motivated by a refusal to accept the philosophical implication of the standard theory of quantum mechanics on the side of the 'traditional' hidden variable theories (e.g. de Broglie and Bohm) or by an aversion to the unwillingness to accept these implications as they are on the side of Von Neumann and his followers.

To understand the non-commutative algebras themselves through hidden variables is thus already in its outset different from the traditional conception of hidden variable theories. Our insistence on the use of this term 'hidden variables', at least for the introduction here, is founded on the belief that it more accurately captures what the concept 'hidden' variables, or more precisely 'hidden' states (which are closely related to variables in the classical case), actually seems to imply. The fact of the matter is that - as strikingly enough Bricmont in his explanation on the ('hidden variable') De Broglie-Bohm theory also remarks [4, p. 148] - the term 'hidden variables' is somewhat misleading to describe the attempts made by these aforementioned theories. While they do aim to replace the more standard operator algebra formulation of quantum mechanics by a classical theory including as of yet undiscovered (and thus hidden) variables, these variables themselves are not hidden. The 'hidden' variables exactly express themselves in the recorded outcome of the experiment and as such are not hidden but revealed(!) by experimental results, albeit only retroactively. As such in our view the adjective 'undetermined' or 'unknown' would have been a more accurate concept to describe this class of theories. In contrast, our attempt here will deal with the states that actually remain unseen by experimental results. That is, we deal with those states (or variables) that exactly are not distinguishable by the experiment and as such are necessarily hidden. In order to avoid the inevitable confusion between the two projects however (the 'hidden variable' project and the current one), we have chosen to use somewhat forcefully the terminology of 'indistinguishable states' - even if we believe that 'hidden states' describes our current project here more accurately.

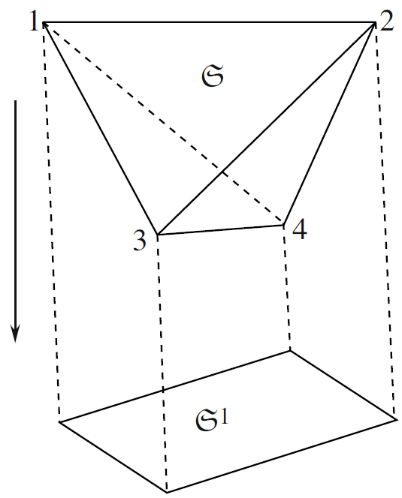

Figure 1: A measurement of the (x,y) position of the quadrangle, denoted by the symbol $\mathfrak{S}'$, compresses the states distinguished by the information of their z-coordinate of the original figure, denoted by $\mathfrak{S}$. Figure taken from [5, fig. 1.3, p.19].

Holevo's Hidden variables

How then do we aim to understand non-commutative algebras through hidden states? The current thesis is inspired by Holevo's recent work 'Probabilistic and Statistical Aspects of Quantum Theory' [5] in which he proposes to understand a quantum mechanical measurement as a classical model with 'restricted class of measurement'. The core idea, at least in our interpretation, is that analogue to an observable in the theory of statistics in which many states are compressed together in relevant observables (the average age as compression of each individual age), a quantum mechanical measurement also compresses the available state with its measurement. See figure 1. Quantum mechanics then differs from classical mechanics in the regard that no measurement can detect the whole phase space, but each measurement compresses the original phase space. As such it is thus of 'restricted class' compared to the classical case in which measurements of all states (in the phase space) is possible. A quantum mechanical measurement is then thus best understood as a classical measurement hiding variables, since each measurement compresses some information held by a set of variables (exactly as the information of the z-coordinate is hidden in the measurement displayed in figure 1).

Holevo has expressed this core idea in his through showing that 'any separated statistical model (...) is a reduction of a classical model with restricted class of measurements' [5, p. 29, th. 1.7.1], where then the quantum mechanical measurements are separated statistical models satisfying the conditions of the theorem in finite dimensions.[1] Van Neerven, in his recent work 'Functional Analysis' [1] has given an adapted proof of Holevo's statement and improved this result by including the infinite dimensional case [1, p. 574, th. 15.32].[2] It is on these results here that we aim to expand by aiming to clarify how the commutative algebra can be viewed a compression of states in the state space.

Outline of the thesis

Our aim in this thesis is thus to improve on the original idea of Holevo - to understand quantum mechanical measurements as a classical measurement in which states are compressed - by providing a systematic approach to these ideas. Concretely, our central claim will be to show that quantum mechanics and classical mechanics differ in kind as a (i) in the formalism of classical mechanics,

[1] We refrain from giving a precise formulation of the thoerem as this would require the exposition of all technical terms used by Holevo and lead us too far away from our main topic.

[2] Note that while the (current) printed edition contains only a sketch of proof, the online available corrections contain a full formal proof of the theorem.

at least theoretically, an observable exists that distinguishes all the states in its associated state space and (ii) in all non trivial cases of Hilbert spaces with dimensional larger than one any observable has associated indistinguishable states. It is then exactly through this difference that the non-commutativity of quantum mechanics can be understood in classical language.

The current thesis consists of two chapters. The first chapter provides the crucial mathematical theory for the aforementioned argument presented in the second chapter. As already mentioned above, the main aim of the first chapter, aside from this introduction into the theory of Abelian Von Neumann algebras, is to explain the tight relation between Abelian Von Neumann algebras and projection valued measures on a separable Hilbert space. In the second chapter our main argument is presented. There, an introduction into the difference between classical experiments and quantum experiments is introduced, the concept of indistinguishable states is defined and the above claimed difference is shown.

1. On Abelian Von Neumann Algebras and Projection Valued Measures

In this chapter we aim to clarify the precise relation between Abelian Von Neumann algebras and projection valued measure on a seperable Hilbert space. As we aim to understand the quantum mechanical observable through a projection valued measure rather than a self-adjoint operator, much of the current chapter can be read as an exercise in recovering the crucial insights on self-adjoint operators from the perspective of projection valued measures by passing through the Von Neumann algebra generated by this projection valued measure. The result of theorem 1.4.12 stands here as central achievement. Moreover, we aim to show that the use of projection valued measures is not only a conceptual improvement over the algebra of self-adjoint operators, but that it has also has mathematically more appealing structure. That is specifically, the case of the position operator on multiple dimensions, here introduced in example 1.3.7, not only presents no extra difficulty (over for example an observable defined on a compact outcome space), but also has the nice property of being countably generated.

In the current chapter we are mainly concerned with Von Neumann algebras. These Von Neumann however algebras fall in the larger class of $C^\star$-algebras, which we mention only briefly in the text below. The literature on these $C^\star$-algebras is however quite large and they come with their own set of techniques. As to avoid delving too deep is into the subject of $C^\star$-algebras, we have omitted the proofs referencing these techniques and use direct citations instead.

1.1 Projections on a Hilbert space

The smallest conceptual building block of nearly all our analysis will be the orthogonal projection. As such we start by introducing the projection.

Definition 1.1.1. ***projection.*** *Let H be a Hilbert space. A linear mapping $P \in \mathcal{L}(H)$ is a projection, if $P^2 = P$.*

We next define orthogonality.

Definition 1.1.2. ***Range and null space.*** *For a linear mapping $T \in \mathcal{L}(H)$, we define the range by $R(T) := \{Th \mid h \in H\}$ and the null space by $N(T) := \{h \mid h \in H, Th = 0\}$.*

Definition 1.1.3. ***Orthogonal projection.*** *A projection is orthogonal, if its null space is the orthogonal complement of its range, that is $R(P)^\perp = N(P)$.*

As orthogonal projections gives rise to orthogonal subspaces on the Hilbert space H, the converse is also possible. That is, we can also define projections using orthogonal subspaces.

Definition 1.1.4. ***Range projection.*** *For any closed subspace $S \subseteq H$, we define the range projection $P(S)$ as the operator which acts as the identity on S and as the zero operator on its orthogonal complement. For an operator $A \in \mathcal{L}(H)$, we define its range projection $P(A)$ as the range projection onto its closed image, that is, $P(A) = P(\overline{R(A)})$.*

For these projections we then have the following proposition, taken from [1, p. 258, prop. 8.8].

Proposition 1.1.5. *Let $P \in \mathcal{L}(H)$ be a projection. P is orthogonal if and only if P is self-adjoint.*

Proof. For 'if', we note that if P is orthogonal and clearly $h - Ph \in N(P)$ and $Ph \in R(P)$, we get $h - Ph \perp Ph'$ for all $h, h' \in H$. Therefore

$$(h|Ph') = (Ph|Ph') + (h - Ph|Ph') = (Ph|Ph') = (Ph|Ph') + (Ph|h' - Ph') = (Ph|h') \tag{1.1.1}$$

Thus P is self-adjoint.

For the 'only if' part, we note that if P is self-adjoint, then

$$(h - Ph|Ph') = (P^\star(h - Ph)|h') = (P(h - Ph)|h') = 0 \tag{1.1.2}$$

as $P(h - Ph) = Ph - Ph = 0$. Now every element in $N(P)$ is of the form $h - Ph$, we get that $N(P) \perp R(P)$ as required. □

The set of orthogonal projections on a Hilbert space will in the remainder be denoted by $\mathcal{P}(H)$. The set $\mathcal{P}(H)$ comes with a natural ordering (which it inherits from the larger set of positive operator as their spectrum $\sigma(P) \subseteq \{0, 1\}$, as $\sigma(P) = \sigma(P^2) = \sigma(P)^2$).

Definition 1.1.6. ***Partial ordering of the projections.*** *We define $P \leqslant P'$ if $P' - P$ is again an orthogonal projection.*

This ordering mirrors the partial ordering of sets given by the inclusion relation $\subseteq$, as $P \leqslant P'$ if and only if $R(P) \subseteq R(P')$. Clearly for all $P \in \mathcal{P}(H)$, we have $O \leqslant P \leqslant I$.

For commuting projections, we have the following result.

Proposition 1.1.7. *Orthogonal projections P and Q commute if and only if their product PQ is an orthogonal projection. In this case the range of PQ is the intersection of their respective ranges.*[1]

Proof. For our first claim we make twice use of proposition 1.1.5. For 'only if', we note that $PQ = QP = Q^\star P^\star = (PQ)^\star$. For the 'if' part, we note that $PQ = (PQ)^\star = Q^\star P^\star = QP$. For the last claim, we see that if $h \in N(P) \cup N(Q)$, then $PQh = QPh = 0$, thus $N(P) \cup N(Q) \subseteq N(PQ)$. If $h \notin N(P) \cup N(Q)$, then $R(PQ) = N(PQ)^\perp \subseteq (N(P) \cup N(Q))^\perp = R(P) \cap R(Q)$. Now for any $h \in H$ we get $PQh = P(Qh) \in R(P)$ and $PQh = QPh = Q(Ph) \in R(Q)$, thus $R(P) \cap R(Q) \subseteq R(PQ)$, proving our last claim. □

As a closing remark: in the above section we have specified that our projections are orthogonal projections. In the remainder we will often no longer do so: every projection, unless noted otherwise, is from now taken to be an orthogonal projection.

1.2 Von Neumann Algebras

If projections form the basic elements of our analysis, then Von Neumann algebras are the mathematical spaces in which they express their structure. Von Neumann algebras are not only useful for their application in quantum mechanics, but they form a class of mathematical objects which display a beautiful interplay between topological and algebraic concepts. We will take this observation as a leitmotiv when working towards the definition of these algebras.

In this section we will introduce Von Neumann algebras by first proving Von Neumann's famous double commutant theorem. We include a full proof here as we deem the proof insightful into the structure of the algebra. From there, motivated by the proof, we cite some initial results on the relation between projections and Von Neumann algebras. Lastly, and after a short intermezzo on Zorn's lemma, we move towards the special class of Abelian Von Neumann algebras.

1.2.1 Von Neumann's double commutant theorem

We start by making this 'interplay between topological and algebraic concepts' precise. First we introduce our topological concepts, then introduce our algebraic concepts and lastly prove the famous theorem.

[1] The following proof is adapted from the stackexchangepost: https://math.stackexchange.com/questions/2761101/condition-that-the-product-of-orthogonal-projections-is-orthogonal, accessed on 31-3-2023.

Topologies on the bounded operators

We start with our 'topological concepts'. We remind the reader of the following definition.

Definition 1.2.1. ***Norm on the bounded linear operators.*** *The norm on the space of linear operators $\mathcal{L}(H)$ is given by*

$$\|A\| := \sup_{h \in H, \|h\|=1} \|Ah\| < \infty. \tag{1.2.1}$$

As the above defines a norm, it induces a natural topology on the set of bounded operators:

Definition 1.2.2. ***Uniform topology.*** *The topology generated by the norm of equation* (1.2.1) *is called the uniform topology.*

As this definition suggests, this however is not the only topology possible on the set bounded linear operators. Two other frequently used topologies are the strong and the weak topologies. For our purposes here, the strong topology and its contrast to the uniform topology are of central importance, while the weak topology will only be used sparingly (but is mentioned here for completeness).

Definition 1.2.3. ***Strong (operator) topology.*** *The strong operator topology on $\mathcal{L}(H)$ is the smallest topology τ on $\mathcal{L}(H)$ with the property that for all $A \in \mathcal{L}(H)$ the map $A \to Ah$ is continuous for all $h \in H$. The topology is thus generated by sets of the form $\{B \mid B \in \mathcal{L}(H),\ \|(B - A)h\| < \epsilon\}$.*

Definition 1.2.4. ***Weak (operator) topology.*** *The weak operator topology on $\mathcal{L}(H)$ is the smallest topology τ on $\mathcal{L}(H)$ with the property that for all $A \in \mathcal{L}(H)$ the map $A \to (Ah|h')$ is continuous for all $h \in H$. The topology is thus generated by sets of the form $\{B \mid B \in \mathcal{L}(H),\ \|([B - A]h|h')\| < \epsilon\}$.*

The uniform and strong topology cannot be reduced to one another. Their relation can be described as follows.

Proposition 1.2.5. *Convergence in the uniform topology implies convergence in the strong topology, but not necessarily the other way around. This means that given a set of operators, its norm closure is contained is contained in its strong closure, but, again, not necessarily the other way around.*

Proof. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(H)$ be a sequence of operators. If $\lim_{n\to\infty} A_n = A$ for some $A \in \mathcal{L}(H)$ in norm, then for each $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\sup_{h\in H,\|h\|=1} \|(A_n - A)h\| < \epsilon$. In particular for each $h \in H$, we have for all $n \geqslant N$ that $\|(A_n - A)h\| < \epsilon$. Thus convergence in norm implies convergence in the strong topology.

For a - maybe the - counter example of the converse claim, let $l^2(\mathbb{N})$ be the Hilbert space of square summable sequences with basis e_n and define by P_n the projection onto the nth coordinate. Then $\lim_{n\to\infty} P_n = O$ strongly, but not in norm. To see this, let $h \in l^2(\mathbb{N})$ and write $h = \sum_{n=1}^{\infty} a_n e_n$. Then as by definition $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, $(|a_n|^2)_{n\in\mathbb{N}}$ converges to zero. That is, for each h there exists some N such that for all $n \geqslant N$, $\|P_n h\| = \|a_n e_n\| = |a_n|^2 < \epsilon$, showing the strong convergence. To see that $(P_n)_{n\in\mathbb{N}}$ does not converge in norm, note that $\|P_n\| \geqslant \|P_n e_n\| = 1$ for all $n \in \mathbb{N}$. □

While the strong topology in general turn $\mathcal{L}(H)$ into a locally convex space (a notion which we will not deal with any more as of now), in the special case of the unit ball on a separable Hilbert space a metric can be defined.

Definition 1.2.6. ***Unit ball of the bounded operators.*** *Let $\mathcal{L}(H)$ be the space of bounded operators. Its unit ball is defined as $S = \{A \mid A \in \mathcal{L}(H), \|A\| \leqslant 1\}$.*

The following proposition stems from [6, p. 134], where it is mentioned as an exercise.

Proposition 1.2.7. *Let H be a separable Hilbert space with a countable dense set of norm-one elements $(h_n)_{n\in\mathbb{N}}$. Then the unit ball of bounded operators S with the topology generated by the metric*

$$d(A, B) := \sum_{n=1}^{\infty} \frac{\|(A - B)h_n\|}{2^n}, \tag{1.2.2}$$

equals the strong topology. Consequently, the unit ball of bounded operators on a separable Hilbert space is strongly metrizable.

Proof. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(H)$ be a sequence of bounded operators and $A \in \mathcal{L}(H)$. We prove that $\lim_{n\to\mathbb{N}} d(A_n, A) = 0$ if and only if $\lim_{n\to\infty} A_n = A$ in the strong operator topology.

Let $\lim_{n\to\mathbb{N}} d(A_n, A) = 0$. Let $\epsilon > 0$ and $h \in H$ of norm one. As $(h_n)_{n\in\mathbb{N}}$ is a dense subset, we can write $h = \sum_{n=1}^{\infty} a_n h_n$ for some sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ with $\sum_{n=1}^{\infty} |a_n|^2 = 1$. As such there exists some $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |a_n|^2 \leqslant \frac{\epsilon}{4}$. Now as $\lim_{n\to\mathbb{N}} d(A_n, A) = 0$, there exists some $M \in \mathbb{N}$ such that for all $m \geqslant M$ we have $d(A_m, A) < \frac{\epsilon}{2^{N+1}}$. In this case we get for $n \leqslant N$ and $m \geqslant M$ that $\frac{1}{2^n}\|(A_m - A)h_n\| \leqslant \sum_{n=1}^{\infty} \frac{\|(A_n-A)h_m\|}{2^n} < \frac{\epsilon}{2^{N+1}}$ and so we get $\|(A_m - A)h_n\| < \frac{\epsilon}{2}$. Moreover, as $A_n, A \in S$ we get $\|(A_m - A)h_n\| \leqslant 2$ for all $n \in \mathbb{N}$, in particular $n \geqslant N$. Combining these results then gives for all $m \geqslant M$ that $\|(A_m - A)h\| = \sum_{n=1}^{\infty} |a_n|^2 \|(A_m - A)h_n\| = \sum_{n=1}^{N} |a_n|^2 \|(A_m - A)h_n\| + \sum_{n=N}^{\infty} |a_n|^2 \|(A_m - A)h_n\| < \sum_{n=1}^{N} |a_n|^2 \frac{\epsilon}{2} + \sum_{n=N}^{\infty} |a_n|^2 2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $\lim_{n\to\infty} A_n h = Ah$ for all $h \in H$, giving the required strong convergence.

Conversly, let A_n converge to A in the strong operator topology. Let $\epsilon > 0$. Now let $\frac{1}{2^{N+1}} < \epsilon$. As A_n converges strongly, we have for $h_1, h_2, \ldots h_N$ that there exists some $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\|(A_n - A)h_n\| < \frac{\epsilon}{2}$. Now, again note that $\|(A_m - A)h_n\| \leqslant 2$. This then gives for all $n > N$, that $d(A_n, A) = \sum_{n=1}^{\infty} \frac{\|(A_n-A)h_m\|}{2^n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ as required. □

The commutant

Next we turn to our algebraic perspective. We start by noting that the bounded operators have a natural algebraic structure.

Definition 1.2.8. ***(Bananch) Algebra, multiplication.*** *A normed algebra is a normed vector space with an associative submultiplicative bilinear map*

$$\begin{aligned} \cdot : A \times A &\to A \\ (a, b) &\to ab, \end{aligned} \tag{1.2.3}$$

where associtivity requires $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and submultiplicativity requires $\|ab\| \leqslant \|a\|\|b\|$ for all $a, b, c \in A$. The above map is called the multiplication map. A normed algebra is a Banach algebra if it is complete with respect to its norm.

Definition 1.2.9. ***$C^\star$-algebra, involution.*** *A $\star$-algebra is a (Banach) algebra A with a mapping*

$$\begin{aligned} {}^\star : A &\to A \\ a &\to a^\star, \end{aligned} \tag{1.2.4}$$

such that $(a^\star)^\star = a$ and $(ab)^\star = b^\star a^\star$ for all $a, b \in A$. The above map is called the involution. A $C^\star$-algebra $\mathcal{A}$ is a $\star$-algebra, if $\|a^\star a\| = \|a\|^2$ for all $a \in A$.

Remark 1.2.10. Note that the inequality $\|a^\star a\| \leqslant \|a\|^2$ follows from submultiplicativity required in the definition of a Banach algebra, the inequalitiy $\|a\|^2 \leqslant \|a^\star a\|$ is however non-trivial for the definition of a $C^\star$-algebra.

Definition 1.2.11. ***Banach-, $C^\star$-subalgebra.*** *A Banach subalgebra is a subspace of Banach algebra, which is closed under its multiplication and moreover topologically closed (making it again complete). A $C^\star$-subalgebra is a Banach subalgebra of a $C^\star$-algebra closed involution.*

Proposition 1.2.12. *The bounded linear operators on a Hilbert space $\mathcal{L}(H)$ with composition as its bilinear map, that is $\cdot = \circ$, and taking its adjoint as its involution is a $C^\star$-algebra.*

Proof. $\mathcal{L}(H)$ is a Banach space and complete with respect to its norm. The composition of two linear mappings $A, B \in \mathcal{L}(H)$ is a linear map, as $(A \circ B)(c_1h_1 + c_2h_2) = A(c_1Bh_1 + c_2Bh_2) = c_1ABh_1 + c_2ABh_2$. Moreover its clearly associative. The submultiplicativity follows from

$$\|AB\| = \sup_{h\in H, \|h\|=1} \|ABh\| \leqslant \sup_{h\in H, \|h\|=1} \|Ah\| \sup_{h\in H, \|h\|=1} \|Bh\| = \|A\|\|B\|, \tag{1.2.5}$$

which also show the boundedness of the composition, making the multiplication map well defined and continuous in both coordinates.

We define its involution as taking the adjoint. This defines an involution as for all $h, h' \in H$, we have $(Ah|h') = (h|A^\star h') = ((A^\star)^\star h|h')$ and $(ABh|h') = (Bh|A^\star h') = (h|B^\star A^\star h')$. That this involution then defines a $C^\star$-algebra can be seen by use of the Cauchy-Schwartz inequality. That is,

$$\begin{aligned} \|A\|^2 &= \sup_{h \in H, \|h\|=1} (Ah|Ah) = \sup_{h \in H, \|h\|=1} (A^\star Ah|h) \leqslant \sup_{h \in H, \|h\|=1} \|A^\star Ah\| \|h\| \\ &= \sup_{h \in H, \|h\|=1} \|A^\star Ah\| = \|A^\star A\|, \end{aligned} \tag{1.2.6}$$

as required. ☐

Remark 1.2.13. In the remainder we will write for $A, B \in \mathcal{L}(H)$ $A \circ B$ as AB (and thus we omit the composition symbol).

For this algebraic structure, we now define the double commutant as follows.

Definition 1.2.14. ***Commutator, Commuting operators.*** *For the bounded linear operators on a Hilbert space, we define the commutator by*

$$\begin{aligned} [\ \ ,\ \ ] : \mathcal{L}(H) \times \mathcal{L}(H) &\to \mathcal{L}(H) \\ (A, B) &\to AB - BA. \end{aligned} \tag{1.2.7}$$

If for $A, B \in \mathcal{L}(H)$ we have $[A, B] = 0$, we say A and B commute. Equivalently, we thus have $AB = BA$.

Using this structure of the commutator, we then define the commutant and double commutant.

Definition 1.2.15. ***(Double) commutant.*** *Let $\mathcal{A} \subseteq \mathcal{L}(H)$ be a set of bounded operators. Then the set $\mathcal{A}' := \{B \mid B \in \mathcal{L}(H), A \in \mathcal{A}, [A, B] = 0\}$ is called the commutant. We call the communant of the commutant its double commutant, that is $\mathcal{A}'' := (\mathcal{A}')'$.*

Note that almost directly by definition we have $\mathcal{A} \subseteq \mathcal{A}''$.

While the reader may have already been familair with the commutant and note that its definition does not need the level of abstraction of the algebraic definitions 1.2.8 and 1.2.14, we have chosen to include them here to explicate how the definition of a commutant arises from these purely algebraic concepts.

The double commutant theorem

In this section we sketch the mentioned connection between our algebraic and topological definitions and use this to define a Von Neumann algebra. The following proof is taken from [1, p. 303, th. 9.27]

Theorem 1.2.16. *Let H be a Hilbert space and $\mathcal{A} \subseteq \mathcal{L}(H)$ a $\star$-subalgebra containing the identity I. Then $\mathcal{A}$ is strongly dense in $\mathcal{A}''$.*

Proof. To prove our claim, we show that each strong neighborhood of a given operator $A_0 \in \mathcal{A}''$ intersects with $\mathcal{A}$. That is, let $\epsilon > 0$, we show that for any choice of $\{h_1, h_2, \ldots, h_N\} \subseteq H$ there exists some $A \in \mathcal{A}$, such that we have $\|(A_0 - A)h_i\| < \epsilon$ for each $i \in \{1, 2, \ldots, N\}$.

Let $h_0 \in H$ and $K = \overline{\{Ah_0 \mid A \in \mathcal{A}\}}$. Our first claim is that K is an invariant subspace for $\mathcal{A}$. Let $A \in \mathcal{A}$ and $h \in K$ that is $h = \lim_{n\to\infty} B_n h_0$ for some sequence $(B_n)_{n=1}^\infty \subseteq \mathcal{A}$ and some $h_0 \in H$. In this case clearly $Ah = \lim_{n\to\infty} AB_n h_0$ and thus $Ah \in K$, as K is closed and $\mathcal{L}(H)$ a Banach algebra. Similarly, we have for all $A \in \mathcal{A}$ and $h' \in K^\perp$, then for all $h'' \in K$ we have $(Ah'|h'') = (h'|A^\star h'') = 0$ and as $\mathcal{A}$ is a $\star$-algebra, we have $A^\star \in \mathcal{A}$, from which conclude $A^\star h' \in K^\perp$.

Now define $P = P(K)$ as the range projection onto K. For all $A \in \mathcal{A}$ and $h \in H$, we then clearly have $APh \in K$ and $A(I - P)h \in K^\perp$ and therefore $APh = PAPh = PA(Ph + (I - P)h) = PAh$. We conclude that $P \in \mathcal{A}'$.

As $P \in \mathcal{A}'$, we have $A_0 P = PA_0$, as $A_0 \in \mathcal{A}''$ and thus $A_0 Ph_0 = PA_0 h_0 \in K$. Therefore, we get that for each $\epsilon > 0$, we have that there exists some $A \in \mathcal{A}$, such that $\|(A_0 - A)h_0\| < \epsilon$ (as K was the closure of $\{Ah_0 \mid A \in \mathcal{A}\}$).

To extend our argument to any choice of $\{h_1, \ldots, h_N\} \subseteq H$, we fix some choice of $\{h_1, \ldots, h_N\}$. We then define

$$\begin{aligned} \rho : \mathcal{L}(H) &\to \mathcal{L}(H^N) \\ A &\to (A, A, \ldots, A) \end{aligned} \tag{1.2.8}$$

where $(A, A, \ldots, A)(h'_1, h'_2, \ldots, h'_N) := (Ah'_1, Ah'_2, \ldots, Ah'_N)$. We now claim that $\rho(A_0) \in (\rho(\mathcal{A}))''$. To see this, let $B = (B_1, B_2, \ldots, B_N) \in (\rho(\mathcal{A}))'$, that is $B\rho(A) = \rho(A)B$ for all $A \in \mathcal{A}$. Now by looking at the restriction mapping onto each coordinate separately, we see that thus $B_i A = AB_i$ for all $A \in \mathcal{A}$ and so $B_i \in \mathcal{A}'$ for all $i \in \{1, 2, \ldots, N\}$. From this we conclude that in particular $B_i A_0 = A_0 B_i$ for all $i \in \{1, 2, \ldots, N\}$ and so $\rho(A_0) \in (\rho(\mathcal{A}))''$. The proof is now finished by applying our original argument to H^N to obtain some operator in $\rho(\mathcal{A})$, such that $\|(\rho(A_0) - \rho(A))(h_1, h_2, \ldots, h_N)\| < \epsilon$. Thereby we get

$$\|(A_0 - A)h_i\| \leqslant \sum_{i=1}^{N} \|(A_0 - A)h_i\|^2 = \|(\rho(A_0) - \rho(A))(h_1, h_2, \ldots, h_N)\| < \epsilon \tag{1.2.9}$$

for all $i \in \{1, 2, \ldots, N\}$, as required. □

Note the crucial role played in the proof by the projection onto the space K above. This already indicates how projections stand central in the analysis of both the double commutant and the strong topology.

Corollary 1.2.17. ***Von Neumann's double commutant theorem.*** *Let H be a Hilbert space and $\mathcal{A} \subseteq \mathcal{L}(H)$ a $\star$-subalgebra containing the identity I. Then $\mathcal{A} = \mathcal{A}''$ if and only if $\mathcal{A}$ is strongly closed.*

Proof. Immediate from the above. □

It may be obvious to the reader that Von Neumann did more than only formulate the corollary.

This crucial result then motivates in the broad sense the following definition. That is, not only does it motivate its formulation, but it moreover motivates the interest is such objects.

Definition 1.2.18. ***Von Neumann Algebra.*** *Let H be a Hilbert space. Then $\mathcal{A} \subseteq \mathcal{L}(H)$ is Von Neumann algebra, if it is a $C^\star$-subalgebra of $\mathcal{L}(H)$ containing the identity and it is strongly closed or, equivalently, equals its double commutant.*

Remark 1.2.19. Note that in our definition we specify that every Von Neumann algebra consists of bounded operators acting on some Hilbert space H.

A Von Neumann Algebra is sometimes, as in [6] for example, defined a strongly closed $\star$-subalgebra of $\mathcal{L}(H)$. This definition is slightly broader then the result above as it allows for Von Neumann algebras on closed subspaces of H. We here deal with this setting in the following way.

Definition 1.2.20. ***Von Neumann Algebra on a subspace.*** *Let H be a Hilbert space and $K \subseteq H$ be a closed subspace. Then $\mathcal{A} \subseteq \mathcal{L}(H)$ is Von Neumann algebra on K, if it is a $C^\star$-subalgebra of $\mathcal{L}(H)$ such that the closed image of all elements in the strong closure of the algebra is contained in K and the algebra contains $P(K)$.*

1.2.2 Projections in a Von Neumann algebra

As already noted above, the proof of the double commutant theorem shows the fundamental role projections play in the analysis of Von Neumann algebras. In this section we cite some major results further clarifying this relation.

Theorem 1.2.21. *If $\mathcal{A}$ is a von Neumann algebra, then it contains the range projections of all if its elements. That is, for all $A \in \mathcal{A}$, there exists a $P \in \mathcal{A}$ such that $P = P(\overline{R(A)})$, where $\overline{R(A)}$ denotes the closure of the image of A.*

Proof. See [6, p. 119, th. 4.1.9]. □

The following results will be shown later on using the theory build there, but for now we mention them with citations.

Theorem 1.2.22. *Let $\mathcal{A}$ be a von Neumann algebra. Then it is the uniform closed linear span of its projections.*

Proof. For now see [6, p. 119, 120, th. 4.1.11]. □

Theorem 1.2.23. *Let $\mathcal{A}$ be non-zero a von Neumann algebra. Then for all $B \in \mathcal{H}$ we have $B \in \mathcal{A}$ if and only if B commutes with all of the projections in $\mathcal{A}'$.*

Proof. As $\mathcal{A}' = (\mathcal{A}')''$, $\mathcal{A}'$ is itself a Von Neumann algebra and thus by the above result it is the linear closed span of its projections. Thus if $B \in \mathcal{L}(H)$ commutes with all the projections in $\mathcal{A}'$, we see it commutes with all elements $\mathcal{A}'$ and so $B \in \mathcal{A}'' = \mathcal{A}$ as $\mathcal{A}$ is a Von Neumann algebra. □

1.2.3 Intermezzo: Zorn's lemma

In the next section we will discuss a special class of Von Neumann algebra's: maximally Abelian Von Neumann algebra's. As their name implies, in some sense they are 'maximal'. It may however not be intuitively clear in which way we deal with maximality here. Zorn's lemma is a way to make this precise. As this maximality will be crucial to our argument further on, we present here, without proof, the construction of Zorn's lemma. This section is based in full on [1, p. 621, appendix A].

Zorn's lemma provides a sufficient condition for the existence of maximal elements in partially ordered sets. We first introduce the relevant terminology and then the main result.

Definition 1.2.24. ***Relation.*** *Let S be a set. A relation $\leqslant$ is a subset R of the Cartesian product $S \times S$, where for $x, y \in S$ $x \leqslant y$ indicates that $(x, y) \in R$ i.e. that x and y are related.*

Definition 1.2.25. ***Partially ordered set.*** *A partially ordered set is a pair $(S, \leqslant)$ of a set S and a relation $\leqslant$, such that for all $x, y, z \in S$ we have (i) $x \leqslant x$, (ii) if $x \leqslant y$ and $y \leqslant x$, then $x = y$ and (iii) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.*

Remark 1.2.26. Note that in the above definitions not each pair of elements in $S \times S$ need to be related.

Example 1.2.27. *As mentioned above, the set of orthogonal projections $\mathcal{P}(H)$ is partially ordered, $P \leqslant P'$ if $P' - P$ is again an orthogonal projection.*

Definition 1.2.28. ***Totally ordered set.*** *A totally ordered set is partially ordered set in which each pair of elements is related.*

Definition 1.2.29. ***Chain.*** *A chain is subset of a partially ordered set that is totally ordered.*

Definition 1.2.30. ***Maximal element.*** *Let S be partially ordered. Then $x \in S$ is maximal if $x \leqslant y$ implies $x = y$.*

Definition 1.2.31. ***Upper bound.*** *Let $S' \subseteq S$ be a subset of a partially ordered set S. Then $x \in S$ is a upper bound for S' if $y \leqslant x$ for all $y \in S'$.*

Theorem 1.2.32. ***Zorn's lemma.*** *Let $(S, \leqslant)$ be a partially ordered set with the property that each chain as an upper bound. Then S contains at least one maximal element.*

While we have omitted the proof, it is important to note that the use of Zorn's lemma relies on the axiom of choice.

1.2.4 (Maximal) Abelian Von Neumann algebras

A special class of Von Neumann algebra are the (maximal) Abelain Von Neumann algebras. As we will show, all Von Neumann algebras generated by projection valued measures - which we will come central in this thesis and which are to be defined below - are Abelian and as such they are of special interest here. Abelian Von Neumann algebras have a lot of structure, which is best expressed in a representation theorem showing that these Abelian algebras are isometrically isomorphic to some space of measurable functions. Moreover, every Abelian Von Neumann algebra comes with a separating vector, which is a useful tool in dealing with the strong convergence. In the case of a maximal Abelian Von Neumann algebra, this vector is moreover cyclic, giving more structure and thereby allowing for the improvement on the previous representation theorem. All results, except proposition 1.2.38, in this section are taken from [6, p. 133-138, sec 4.4].

The maximalility of a maximal Abelian Von Neumann algebra

We start with our working definitions.

Definition 1.2.33. ***Abelian Von Neumann Algebra.*** *A Von Neumann algebra $\mathcal{A}$ is Abelian if all of its elements commute or, equivalently, $\mathcal{A} \cap \mathcal{A}' = \mathcal{A}$.*

Definition 1.2.34. ***Maximal Abelian Von Neumann Algebra.*** *An Abelian Von Neumann algebra acting on some Hilbert space H is maximal if it is not contained in any other Abelian Von Neumann algebra acting on the same Hilbert space H, or, equivalently $\mathcal{A} = \mathcal{A}'$.*

Remark 1.2.35. Note that in the definition of a maximal Abelian Von Neumann algebra, the inclusion $\mathcal{A} \subseteq \mathcal{A}'$ is trivial.

The following result, presented after recalling the Fuglede–Putnam–Rosenblum theorem, shows that Abelian Von Neumann algebras can be easily constructed. After this result we show how the adjective 'maximal' used in the above definition of maximal Abelian Von Neumann algebras is consistent with the idea of maximility in Zorn's lemma.

Definition 1.2.36. ***Normal operators.*** *An operator is normal if it commutes with its adjoint.*

Theorem 1.2.37. ***Fuglede–Putnam–Rosenblum.*** *Let $A \in \mathcal{L}(H)$ be normal and $B \in \mathcal{L}(H)$. Then if $AB = BA$, we get $A^\star B = BA^\star$.*

Proof. See [1, p. 265, th. 8.18]. □

Proposition 1.2.38. *Let $\mathcal{B} \subseteq \mathcal{L}(H)$ be some Abelian set of normal operators. Then $\mathcal{B}''$ is an Abelian Von Neumann algebra.*

Proof. We start by showing that the $C^\star$-algebra generated by $\mathcal{B}$, denoted by $C^\star(\mathcal{B})$, is an Abelian $C^\star$-algebra. Firstly, note that $\mathcal{B} \cup \{I\}$ is again Abelian. Secondly, we first claim that the norm closure of span of $\mathcal{B} \cup \{I\}$ is again Abelian, as the commutator is a linear map. Moreover, its completion with respect to the uniform topology is again Abelian as the commutator is continuous in the uniform topology, since it is linear and bounded ($\|[A, B]\| \leqslant 2\|A\|\|B\|$). As both the sum and the product of two commuting normal operators is again normal, the set generated contains only normal operators. As such, it is again Abelian under taking the adjoint by the above theorem. This proves our first claim.

Now note that the commutant of a set of operators $\mathcal{B}'$ is a $\star$-algebra containing the identity. Moreover, by following the definition of the commutant, we have $\mathcal{B}' = \mathcal{B}'''$. Thus $\mathcal{B}'$ is a Von Neumann algebra. Now as $\mathcal{B}$ is Abelian, we get $\mathcal{B} \subseteq \mathcal{B}'$. But then we get by the double commutant theorem that $\mathcal{B}'' = \overline{C^\star(\mathcal{B})}^{SOT} \subseteq \overline{\mathcal{B}'}^{SOT} = \mathcal{B}'$ and thus $\mathcal{B}''$ is an Abelian Von Neumann algebra. □

Proposition 1.2.39. *Every Abelian Von Neumann algebra is contained in a maximal Abelian Von Neumann algebra.*

Proof. Here we use Zorn's lemma. Let $A_{AVN}(H)$ be the set of Abelian Von Neumann algebras on a given Hilbert space H. We now use set inclusion $\subseteq$ as our partial order. We note that in this case each chain C has a natural upper bound in the form of the strong closure of $\{A \mid A \in \mathcal{A}, \mathcal{A} \in$

$C\} = \bigcup_{\mathcal{A}\in C} \mathcal{A}$. As each chain is ordered by inclusion we have that $\bigcup_{\mathcal{A}\in C} \mathcal{A}$ is Abelian, contains the identity, is closed under multiplication and involution and $\overline{\bigcup_{\mathcal{A}\in C} \mathcal{A}}^{SOT}$ is again norm complete, as the norm closure lies in the strong closure. $\overline{\bigcup_{\mathcal{A}\in C} \mathcal{A}}^{SOT}$ is thus a $\star$-algebra containing the identity and so $\overline{\bigcup_{\mathcal{A}\in C} \mathcal{A}}^{SOT} \in A_{AVN}(H)$ by (second argument of the) the above lemma. As such we have sufficient conditions for the application of Zorn's lemma and thereby we can assume the existence of a maximal element $\mathcal{A}_M$. As by maximality we have $\mathcal{A}_M \subseteq \mathcal{A}$ for some $\mathcal{A} \in A_{AVN}(H)$ implies $\mathcal{A}_M = \mathcal{A}$, we see that $\mathcal{A}_M$ is not contained in any other Abelian Von Neumann algebra and so we conclude that $\mathcal{A}_M$ is maximal Abelian. Note that by construction this implies $\mathcal{A}_M = \mathcal{A}'_M$, as if not then there exists some $A \in \mathcal{A}'_M \backslash \mathcal{A}_M$ and then $\mathcal{A}_M \subset \{A \cup \mathcal{A}_M\}''$, which contradicts the maximality of $\mathcal{A}_M$. □

A key example and classification

We next move towards (maybe) the example of a maximal Abelian Von Neumann algebra.

Definition 1.2.40. ***(μ-)bounded measurable functions.*** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We say a function is μ-essentially bounded if there exists some $R \in \mathbb{R}^+$ such that $\mu(\{\omega \mid |f| > R\}) = 0$. We define by $L^\infty(\Omega, \mu)$ the equivalence classes of μ-essentially bounded functions, where f and g are equivalent if $\mu(\{\omega \mid |f - g| > 0\}) = 0$. We define the norm on this space by*

$$\|f\|_\infty := \inf\{R \mid R \in \mathbb{R}^+, \mu(\{\omega \mid |f| > R\}) = 0\}. \tag{1.2.10}$$

We note that $L^\infty(\Omega, \mu)$ is a Banach space [1, p. 49, th. 2.20]. In the remainder we will often omit the specification μ and refer to such functions as bounded measurable functions.

Definition 1.2.41. ***(Algebra of) multiplication operators.*** *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, let $L^2(\Omega, \mu)$ be its associated Hilbert space of square integrable functions and let $L^\infty(\Omega, \mu)$ be its associated Banach space of bounded measurable functions. Then we define the multiplication operator M_f for $f \in L^\infty(\Omega, \mu)$, by*

$$\begin{aligned} M_f : L^2(\Omega, \mu) &\to L^2(\Omega, \mu) \\ h &\to fh. \end{aligned} \tag{1.2.11}$$

This algebra is referred to as the (Von Neumann) algebra of multiplication operators.

The following result is adapted from [2, p. 594, 595, th B.106-108].

Proposition 1.2.42. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with σ-finite measure μ, let $L^2(\Omega, \mu)$ be its associated Hilbert space of square integrable functions and let $L^\infty(\Omega, \mu)$ be its associated Banach space of bounded measurable functions. Then the multiplication algebra is a maximal Abelian Von Neumann algebra.*

Proof. Let $A \in \mathcal{M}'$. As μ is σ-finite, there exists some sequence of pairwise disjoint sets such that $(F_n)_{n\in\mathbb{N}}$ and $0 < \mu(F_n) < \infty$ for all $n \in \mathbb{N}$.

For now fix $n \in N$. As $\mu(F_n) < \infty$, we have $1_{F_n} \in L^2(\Omega, \mu)$. We now define $f_{A,n} = A1_{F_n}$ and claim that $f_{A,n} \in L^\infty(\Omega, \mu)$. We show this claim by contradiction. Assume $f_{A,n} \notin L^\infty(\Omega, \mu)$. Then for some $R > \|A\|$ there exists some $F_R \in \mathcal{F}$ such that $|f_{A,n}(\omega)| > R$ has $\mu(F_R) > 0$. Now as $f_{A.n}$ has support in F_n, we get that also $\mu(F_R \cap F_n) > 0$. But then $\frac{1}{\mu(F_R\cap F_n)} 1_{F_R\cap F_n} \in L^2(\Omega, \mu)$ and is of norm one and so

$$\|A\| < \left\| \frac{f_{A,n}1_{F_R\cap F_n}}{\mu(F_R \cap F_n)} \right\| = \left\| \frac{A1_{F_n}1_{F_R\cap F_n}}{\mu(F_R \cap F_n)} \right\| = \left\| \frac{A1_{F_R\cap F_n}}{\mu(F_R \cap F_n)} \right\| \leqslant \sup_{\|h\|\leqslant 1, h\in H} \|Ah\| = \|A\|, \tag{1.2.12}$$

giving us the contradiction and thereby proving our claim. Moreover, we see that $\|f_{A,n}\|_\infty \leqslant \|A\|$.

As $f_{A,n} \in L^\infty(\Omega, \mu)$, $M_{f_{A,n}} \in \mathcal{M}$. Now let $g \in L^2(\Omega, \mu)$ with ess supp$(g) \subseteq F_n$. As $L^2(\Omega, \mu) \subseteq L^\infty(\Omega, \mu)$, M_g is well defined as as such we get

$$\begin{aligned} Ag = Ag1_{F_n} &= AM_g1_{F_n} = M_gA1_{F_n}1_{F_n} = M_gf_{A,n}1_{F_n} \\ &= M_gM_{f_{A,n}}1_{F_n} = M_{f_{A,n}}M_g1_{F_n} = M_{f_{A,n}}g \end{aligned} \tag{1.2.13}$$

almost everywhere.

We now extend our argument to the whole of Ω. Set $f_A := \sum_{n=1}^{\infty} f_{A,n}$. Then $f_A \in L^\infty(\Omega,\mu)$ as

$$\begin{aligned}\|f_A\|_\infty &= \left\|\sum_{n=1}^{\infty} f_{A,n}\right\|_\infty \leqslant \left\|\sum_{n=1}^{\infty} (\sup_{n\in\mathbb{N}} f_{A,n}) 1_{F_n}\right\|_\infty = \left\|(\sup_{n\in\mathbb{N}} f_{A,n}) \sum_{n=1}^{\infty} 1_{F_n}\right\|_\infty \\ &= \left\|(\sup_{n\in\mathbb{N}} f_{A,n}) 1_\Omega\right\|_\infty = \sup_{n\in\mathbb{N}} \|f_{A,n}\|_\infty \leqslant \|A\| < \infty\end{aligned} \tag{1.2.14}$$

and as such M_{f_A} is well-defined. Using our relation above, we now get for general $g \in L^2(\Omega,\mu)$

$$\begin{aligned}Ag &= Ag1_\Omega = Ag\left(\sum_{n=1}^{\infty} 1_{F_n}\right) = \sum_{n=1}^{\infty} Ag1_{F_n} = \sum_{n=1}^{\infty} M_{f_{A,n}} g 1_{F_n} \\ &= \left(\sum_{n=1}^{\infty} f_{A,n} 1_{F_n}\right) g = M_{\sum_{n=1}^{\infty} f_{A,n} 1_{F_n}} g = M_{f_A} g\end{aligned} \tag{1.2.15}$$

almost everywhere. Thus $A = M_{f_A} \in \mathcal{M}$ as required. □

Theorem 1.2.43. *Let H be a separable infinite dimensional Hilbert space and $\mathcal{A}$ a maximal Abelian Von Neumann algebra, then $\mathcal{A}$ is unitarily equivalent to one the following cases:*

1. *the multiplication algebra on $L^2([0,1])$,*
2. *the multiplication algebra on $l^2(\mathbb{N})$,*
3. *the multiplication algebra on $L^2([0,1]) \oplus l^2(\mathbb{N})$,*
4. *the multiplication algebra $L^2([0,1]) \oplus D_N(\mathbb{C})$, with $D_N(\mathbb{C})$ the set of N-dimensional complex diagonal matrices.*

Moreover, these cases above are unitarily nonequivalent.

Proof. See [2, p. 601, th. B.118]. The idea of the proof is that two Von Neumann algebras $\mathcal{A}_1, \mathcal{A}_2$ on Hilbert spaces H_1, H_2 are unitarily equivalent if their projections say $P_1 \in \mathcal{A}_1$ and $P_2 \in \mathcal{A}_2$ can be written as the product of of a partial isometry $U : H_1 \to H_2$ such that $P_1 = UU^\star$ and $P_2 = U^\star U$. The possibility of these partials isometry existing is argued for on the basis of the cardinality of the rank of the image space of the projections. The four cases above then cover the four possible combinations of these cardinalities. That is, case (1) contains only projections of uncountable rank, (2) only projections of countable rank, (3) contains countably many sets of projections uncountable rank and (4) contains countable many sets of projections uncountable rank. □

Results for Abelian Von Neumann algebras

Next we move to our two central results on Abelian Von Neumann algebras.

Definition 1.2.44. ***Cyclic vector, closed image space.*** *For a Von Neumann algebra $\mathcal{A}$ a Hilbert space element $h \in H$ is called cyclic, if its closed image space equals H, that is $[\mathcal{A}h] = \overline{\{Ah \mid A \in \mathcal{A}\}} = H$.*

Definition 1.2.45. ***separating vector.*** *For a Von Neumann algebra $\mathcal{A}$ a Hilbert space element $h \in H$ is called separating, if for all $A \in \mathcal{A}$ $Ah = 0$ implies $A = 0$.*

Theorem 1.2.46. *Let $\mathcal{A}$ be a Abelian Von Neumann algebra on a separable Hilbert space, which has a separating vector. Then there exists some second countable compact Hausdorff space $\Omega_{\mathcal{A}}$, a positive measure $\mu_{\mathcal{A}}$ and unitary map $U_{\mathcal{A}} : H \to L^2(\Omega_{\mathcal{A}}, \mu_{\mathcal{A}})$, such that $U_{\mathcal{A}} \mathcal{A} U_{\mathcal{A}}^\star$ is the Von Neumann algebra of multiplication operators M_f on $L^2(\Omega_{\mathcal{A}}, \mu_{\mathcal{A}})$.*

Proof. See [6, p. 135, th. 4.4.3] □

Theorem 1.2.47. *If $\mathcal{A}$ is an Abelian Von Neumann algebra acting on a seperable Hilbert space, then there exists some $h \in H$ that is separating for $\mathcal{A}$.*

Proof. Let E be a maximal set in H of unit vectors such that the spaces $[\mathcal{A}h]$ with $h \in E$ are pairwise orthogonal. This E exists by Zorn's lemma. Now, if $y \in H$ is a unit vector orthogonal to all $[\mathcal{A}h]$, then $[\mathcal{A}y]$ is also orthogonal to all $[\mathcal{A}h]$, which contradicts the maximallity of E and the fact that $\mathcal{A}$ is Abelian (and so all its projection commute). Hence H is the orthogonal sum of the spaces $[\mathcal{A}h]$. As H is seperable, the set E is necessarily countable, so we may write $E = \{h_n \mid n \in \mathbb{N}\}$, where h_n is a sequence of unit vectors in H. Now choose $h := \sum_{i=1}^{\infty} \frac{h_n}{2^n}$. If $A \in \mathcal{A}$ and $Ah = 0$, then $Ah_n = 0$ for all n, because the sequence Ah_n consists of pairwise orthogonal elements. Hence, if $B \in \mathcal{A}$, then $ABh = BAh = 0$ as $\mathcal{A}$ is Abelian and so $A[\mathcal{A}h_n] = \overline{\{ABh \mid B \in \mathcal{A}_P\}} = \{0\}$ for all n. It follows that $A = 0$, so h is a separating vector for $\mathcal{A}$. □

Results for maximal Abelian Von Neumann algebras

These results are then improved for maximal Abelian Von Neumann Algebras.

Proposition 1.2.48. *Let $\mathcal{A}$ Von Neumann algebra and let $h \in H$ be a separating vector for $\mathcal{A}'$, then h is cyclic for $\mathcal{A}$.*

Proof. To see this, let P denote the projection of H onto $[\mathcal{A}h]$. As $[\mathcal{A}h]$ is an $\mathcal{A}$-invariant subspace, we have $P \in \mathcal{A}'$. Now as $I \in \mathcal{A}$, we have $h \in [\mathcal{A}h]$ and so $(I - P)h = 0$. But as h is separating for $\mathcal{A}'$, we get $I - P = 0$ and so $[\mathcal{A}h] = H$. □

Corollary 1.2.49. *If $\mathcal{A}$ is maximal Abelian, then it separating vector is also cyclic.*

Proof. By theorem 1.2.47 there exists a separating vector $h \in H$ for $\mathcal{A}$ and by proposition 1.2.48 h is also a cyclic vector for $\mathcal{A}'$. As $\mathcal{A}$ is maximally Abelian, we have $\mathcal{A} = \mathcal{A}'$ and thus h is both separating and cyclic. □

Corollary 1.2.50. *Let $\mathcal{A}$ be a maximal Abelian Von Neumann algebra and acting on a separable Hilbert space. Then there exists some second countable compact Hausdorff space $\Omega_{\mathcal{A}}$, a positive measure $\mu_{\mathcal{A}}$ and unitary map $U_{\mathcal{A}} : H \to L^2(\Omega_{\mathcal{A}}, \mu_{\mathcal{A}})$, such that $U_{\mathcal{A}}\mathcal{A}U_{\mathcal{A}}^{\star}$ is the Von Neumann algebra of multiplication operators on M_f on $L^2(\Omega_{\mathcal{A}}, \mu_{\mathcal{A}})$.*

Proof. By theorem 1.2.47 every Abelian Von Neumann algebra has a separating vector, which is also cyclic by corollary 1.2.49 and so theorem 1.2.46 can be applied. □

1.3 Projection valued measures

The above section, through theorem 1.2.46, already indicates a connection between Von Neumann algebras and measure spaces. The concept of a projection valued measure further expands on this idea. Projection valued measures form a collection of projections which create measure spaces by being applied to individual Hilbert space elements.

1.3.1 Definition and basic properties

The definition and the propositions are taken from [1, p. 285-287, sect. 9.2].

Definition 1.3.1. ***Projection valued measure.*** *A projection valued measure on a measurable space $(\Omega, \mathcal{F})$ is a mapping $P : \mathcal{F} \to \mathcal{P}(H)$ such that (i) $P(\Omega) = I$ and (ii) for all $h \in H$ the map*

$$\begin{aligned} P_h : \mathcal{F} &\to [0, \infty) \\ F &\to (P(F)h|h) \end{aligned} \tag{1.3.1}$$

defines a measure on $(\Omega, \mathcal{F})$.

Directly, we see that if $h \in H$ is of norm one, then P_h is a probability measure as $P_h(\Omega) = (P(\Omega)h|h) = (h|h) = 1$. It is exactly in this context that projection valued measures will be used here.

Remark 1.3.2. We will denote the image of a projection valued measure as

$$P(\mathcal{F}) := \{P(F) \mid F \in \mathcal{F}\} \tag{1.3.2}$$

Before moving to the key properties of the projection valued measure, we prove a helpful lemma taken from [1, p. 255, prop. 8.1].

Lemma 1.3.3. *If $A \in \mathcal{L}(H)$ satisfies $(Ah|h) = 0$ for all $h \in H$, then $A = 0$.*

Proof. Let $A \in \mathcal{L}(H)$ satisfy $(Ah|h) = 0$ for all $h \in H$. Then all $h, h' \in H$ we have $(A(h+h')|h+h') = 0$ and again using our assumption we thus get $(Ah|h')+(Ah'|h) = 0$. Now we replace h' by ih', to obtain $-i(Ah|h')+i(Ah'|h) = 0$. We multiply by i to gain the crucial result of $(Ah|h')-(Ah'|h) = 0$. When we now add our two equation $(Ah|h') + (Ah'|h) = 0$ and $(Ah|h') - (Ah'|h) = 0$, we obtain $(Ah|h') = 0$ for all $h, h' \in H$ giving $A = 0$ as required. ☐

While the definition of the projection valued measure is quite lean, they inhered a lot of structure from their measures. This is made precise in the following proposition.

Proposition 1.3.4. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a projection valued measure. Then the following assertions are true:*

1. *$P(\varnothing) = 0$.*
2. *If $F_1, F_2 \in \mathcal{F}$ are disjoint, then the ranges of $P(F_1)$ and $P(F_2)$ are orthogonal.*
3. *If $F_1, F_2, \cdots \in \mathcal{F}$ are disjoint, then $P(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} P(F_n)$, where the convergence $\lim_{N\to\infty} \sum_{n=1}^{N} P(F_n) = \sum_{n=1}^{\infty} P(F_n)$ is in the strong operator topology.*
4. *For all $F_1, F_2 \in \mathcal{F}$, we have $P(F_1 \cap F_2) = P(F_1)P(F_2) = P(F_2)P(F_1)$.*
5. *If $F_1, F_2 \in \mathcal{F}$ with $F_1 \subseteq F_2$, then $P(F_1) \leqslant P(F_2)$.*

Proof. Let $h \in H$.

(1) As P_h is an additive function, we get

$$(h|h) = P_h(\Omega) = P_h(\Omega \cup \varnothing) = P_h(\Omega \cup \varnothing) = P_h(\Omega) + P_h(\varnothing) = (h|h) + P_h(\varnothing) \tag{1.3.3}$$

and so $P_h(\varnothing) = (P(\varnothing)h|h) = 0$, which by lemma 1.3.3 implies $P(\varnothing) = 0$ as desired.

(2) Let $F_1, F_2 \in \mathcal{F}$ be disjoint. Now, using (2), we get

$$P(F_1)+P(F_2) = P(F_1 \cup F_2) = P(F_1 \cup F_2)^2 = P(F_1)+P(F_1)P(F_2)+P(F_2)P(F_1)+P(F_2). \tag{1.3.4}$$

and thus $P(F_1)P(F_2) + P(F_2)P(F_1) = 0$. Now using this result and $P^2 = P$ for all projections, then following equalities give us the desired result.

$$(P(F_1)P(F_2))^2 = P(F_1)[P(F_2)P(F_1)]P(F_2) = -P(F_1)P(F_1)P(F_2)P(F_2) = -P(F_1)P(F_2) \tag{1.3.5}$$

and so

$$(P(F_2)P(F_1))^2 = -P(F_2)P(F_1) \tag{1.3.6}$$

giving

$$(P(F_2)P(F_1))^2 + (P(F_1)P(F_2))^2 = 0, \tag{1.3.7}$$

then for any $h \in H$ we get

$$\begin{aligned}([P(F_2)P(F_1)]^2h|h) &= (P(F_2)P(F_1)h|P(F_1)P(F_2)h) = -(P(F_2)P(F_1)h|P(F_2)P(F_1)h) \\ &= -\|P(F_2)P(F_1)h\|\end{aligned} \tag{1.3.8}$$

and

$$([P(F_1)P(F_2)]^2h|h) = -\|P(F_1)P(F_2)h\|. \tag{1.3.9}$$

Combining these equation then finally gives us

$$\|P(F_1)P(F_2)h\| + \|P(F_2)P(F_1)h\| = 0 \tag{1.3.10}$$

from which conclude

$$\|P(F_1)P(F_2)h\| = \|P(F_2)P(F_1)h\| = 0 \tag{1.3.11}$$

as desired.

(3) Now for a sequence of disjoint sets $(F_n)_{n=1}^{\infty}$, we get clearly that $P(\bigcup_{n=1}^{N} F_n) = \sum_{n=1}^{N} P(F_n)$ for finite N. Again by additivity of their measures, we get for disjoint sets that

$$(P(\bigcup_{n=1}^{\infty} F_n)h|h) = P_h\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P_h(F_n) = \sum_{n=1}^{\infty}(P(F_n)h|h) \tag{1.3.12}$$

and so again $((P(\bigcup_{n=1}^{\infty} F_n) - \sum_{n=1}^{\infty} P(F_n))h|h) = 0$ for all h, so $P(\bigcup_{n=1}^{\infty}) = \sum_{n=1}^{\infty} P(F_n)$, with convergence in the strong topology.

(4) From (2) we gain that if $F_1, F_2 \in \mathcal{F}$ are disjoint, then $P(F_1 \cap F_2) = P(\varnothing) = 0 = P(F_1)P(F_2) = P(F_2)P(F_1)$. Using this we get for general sets that

$$\begin{aligned} P(F_1)P(F_2) &= [P(F_1\backslash F_2) + P(F_1 \cap F_2)][P(F_2\backslash F_1) + P(F_1 \cap F_2)] \\ &= P(F_1\backslash F_2)P(F_2\backslash F_1) + P(F_1 \cap F_2)P(F_2\backslash F_1) \\ &\quad + P(F_1\backslash F_2) + P(F_1 \cap F_2)P(F_2\backslash F_1) + P(F_1 \cap F_2)P(F_1 \cap F_2) \\ &= P(F_1 \cap F_2) = P(F_2 \cap F_1) = P(F_1)P(F_2) \end{aligned} \tag{1.3.13}$$

(5) Lastly, if $F_1 \subseteq F_2$, then $F_1 \cap F_2 = F_1$, so

$$P(F_2) = P(F_1\backslash F_2) + P(F_1 \cap F_2) = P(F_1\backslash F_2) + P(F_1) \tag{1.3.14}$$

and thus $P(F_2) - P(F_1) = P(F_1\backslash F_2)$, which is a projection by definition, so $P(F_1) \leqslant P(F_2)$. □

Furthermore, projection valued measures can be constructed using other projection valued measures.

Proposition 1.3.5. *Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable space and let $f : \Omega \to \Omega'$ be a measurable map. If $P : \mathcal{F} \to \mathcal{P}(H)$ is a projection valued measure, then*

$$\begin{aligned} Q : \mathcal{F}' &\to \mathcal{P}(H) \\ F' &\to P(f^{-1}(F')) \end{aligned} \tag{1.3.15}$$

is also a projection valued measure.

Proof. As for any $h \in H$, we have $(Q(\Omega')h|h) = (P(f^{-1}(\Omega'))h|h) = (P(\Omega)h|h) = (h|h)$, we get that $Q(\Omega') = I$. Moreover, $Q_h(F') = (Q(F')h|h) = (P(f^{-1}(F'))h|h) = P_h(f^{-1}(F'))$ is clearly a measure for all $h \in H$, as f is measurable. □

We present here two examples of projection valued measures, which, as we will show in the next section, differ in kind. This difference will form a leitmotif of the current thesis and we will come back many times to these examples.

Example 1.3.6. *As the natural numbers have discrete topology, the Borel σ-algebra is simply its power set. The following function is a projection valued measure:*

$$\begin{aligned} P^{\mathbb{N}} : \mathcal{B}(\mathbb{N}) &\to \mathcal{P}(l^2(\mathbb{N})) \\ \mathcal{I} &\to \sum_{i \in \mathcal{I}} e_{k_i} \bar{\otimes} e_{k_i}, \end{aligned} \tag{1.3.16}$$

where

$$e_i \bar{\otimes} e_i : H \to H \qquad (1.3.17)$$
$$h \to (h|e_i)e_i$$

and the convergence, in the case of $\mathcal{I}$ *containing infinely many elements, of* $\sum_{i \in \mathcal{I}} e_{k_i} \bar{\otimes} e_{k_i} = \lim_{N\to\infty} \sum_{j=1}^{N} e_{k_{i_j}} \bar{\otimes} e_{k_{i_j}}$ *is in the strong operator topology. That is,* $P(\mathcal{I})h = P(\mathcal{I}) \sum_{k=1}^{\infty} (h|e_k)e_k = \sum_{i=1}^{\infty} (h|e_{k_i})e_{k_i} \lim_{N\to\infty} \sum_{j=1}^{N} (e_{k_{i_j}} \bar{\otimes} e_{k_{i_j}})h$.

Example 1.3.7. *On* $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \lambda)$ *(with Lebesgue measure* λ *and standard topology) for some finite* $k \in \mathbb{N}$*, we define*

$$P^{\mathbb{R}^k} : \mathcal{B}(\mathbb{R}^k) \to \mathcal{P}(L^2(\mathbb{R}^k)) \qquad (1.3.18)$$
$$F \to 1_F,$$

where here 1_F*, with slight abuse of notation, is here the operator denoting a multiplication with the indicator function* 1_F

$$1_F : L^2(\mathbb{R}^k) \to L^2(\mathbb{R}^k) \qquad (1.3.19)$$
$$f \to 1_F f.$$

1.3.2 Intermezzo: atomic and non-atomic measures

To see how these two projection valued measures differ, we need the concept of atomic measures. We start by classifying atomic sets.

Definition 1.3.8. ***Atomic set.*** *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. A set* $F \in \mathcal{F}$ *is atomic, if* $\mu(F) > 0$ *and if* $F = F_1 \cup F_2$ *with* $F_1, F_2 \in \mathcal{F}$ *disjoint, then either* $\mu(F_1) = 0$ *or* $\mu(F_2) = 0$.

Using this notion we define atomic and non-atomic measures.

Definition 1.3.9. ***Atomic measure.*** *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. The measure* μ *is atomic, if every* $F \in \mathcal{F}$ *with* $\mu(F) > 0$ *contains some atomic set.*

Definition 1.3.10. ***Non-atomic measure.*** *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. The measure* μ *is non-atomic, if it has no atomic sets.*

Note that a measure can fall in neither category and that only the zero-measure is both atomic and non-atomic.

The above two examples coincide with these definitions in the following way.

Example 1.3.11. *For any* $h \in l^2(\mathbb{N})$*, the measure* $P_h^{\mathbb{N}}$ *is atomic.*

Proof. Now as there exist no decomposition of $\{n\}$ in disjoint sets and every set is the disjoint countable union of sets of this form, the above definition is directly satisfied. That is, if we write $h = \sum_{n=1}^{\infty} h_n e_n$, then for any $\{n\}$, we have that

$$P_h^{\mathbb{N}}(\{n\}) = \left((e_n \bar{\otimes} e_n) \sum_{n=1}^{\infty} h_n e_n \middle| \sum_{n=1}^{\infty} h_n e_n \right) = |h_n|^2. \qquad (1.3.20)$$

Now if $|h_n|^2 = 0$, then the definition is satisfied and if $|h_n|^2 > 0$, then there exists no subset of $\{n\}$ with smaller non-zero measure. □

Example 1.3.12. *For any* $f \in L^2(\mathbb{R}^k)$*, the measure* $P_h^{\mathbb{R}^k}$ *is non-atomic.*

Proof. Let $G =$ ess supp (f). Now for $F \in \mathcal{B}(\mathbb{R}^k)$ we have $P_f^{\mathbb{R}^k}(F) > 0$ if and only if $\lambda(F \cap G) > 0$. Now let $\lambda(F \cap G) > 0$ and moreover assume $\lambda(F \cap G) < \infty$. We define $F_{R,1} = (\infty, R] \times \mathbb{R}^{k-1}$. Using this set, we define $f_1 : \mathbb{R} \to \mathbb{R}, R \to \lambda((F \cap G) \cap F_{R,1})$. This function clearly maps surjectively onto $[0, \lambda(F \cap G)]$. Now pick $S = f_1^{-1}(\frac{1}{2})$. Then $\lambda((F \cap G) \cap F_{S,1}) = \frac{1}{2}$ as required. □

1.3.3 A projection valued measure on a separable Hilbert space has strongly closed image

In this section we prove the claim that the image space of a projection valued measure is strongly closed on a separable Hilbert space. The importance of this result is motivated by the following insight.

Definition 1.3.13. ***Set limits.*** *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be a sequence of sets. We then define its limit supremum by $\limsup_{n\to\infty} F_n := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_n$ and its limit inferior by $\liminf_{n\to\infty} F_n := \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} F_n$. If these two limits converge to the same set, we define this set as its limit, that is $\lim_{n\to\infty} F_n := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} F_n$.*

Definition 1.3.14. ***Continuous with respect to set theoretic limits for measures.*** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We say μ is continuous with respect to set theoretic limits, if for a sequence of sets $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ for which the set limit exists, we have $\mu(\lim_{n\to\infty} F_n) = \lim_{n\to\infty}\mu(F_n)$.*

Proposition 1.3.15. *Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space. Then μ is continuous with respect to set theoretic limits.*[2]

Proof. We first prove that $\mu(\liminf_{n\to\infty} A_n) \leqslant \liminf_{n\to\infty}\mu(A_n)$. Consider the sets $B_n = \bigcap_{k=n}^{\infty} F_k$. Obviously, $B_n \subseteq F_k$ for all $k \geqslant n$, so $\mu(B_n) \leqslant \mu(F_k)$ for all $k \geqslant n$. After taking the infimum we get $\mu(B_n) \leqslant \inf_{k\geqslant n}\mu(F_k)$. Since $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$ then the sequence $\{\mu(B_n) : n \in \mathbb{N}\}$ is non-decreasing, so there exist $\lim_{n\to\infty}\mu(B_n)$. Similarly the sequence $\{\inf_{k\geqslant n}\mu(A_k) : n \in \mathbb{N}\}$ is non-decreasing hence there exist $\lim_{n\to\infty}\inf_{k\geqslant n}\mu(A_k)$. Since existence of limits is justified we write $\lim_{n\to\infty}\mu(B_n) \leqslant \lim_{n\to\infty}\inf_{k\geqslant n}\mu(F_k) = \liminf_{n\to\infty}\mu(F_n)$. Now, again recall that $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$, so $\mu(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty}\mu(B_n)$. It is remains to note that $\liminf_{n\to\infty} F_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} F_k = \bigcup_{n=1}^{\infty} B_n$. Combining the above the gives $\mu(\liminf_{n\to\infty} F_n) \leqslant \liminf_{n\to\infty}\mu(F_n)$, as required.

Now in a similar way we can prove $\mu(\limsup_{n\to\infty} F_n) \geqslant \limsup_{n\to\infty}\mu(F_n)$. This then gives us the required result as by assumption we have

$$\begin{aligned}\mu(\lim_{n\to\infty} F_n) &= \mu(\liminf_{n\to\infty} F_n) \leqslant \liminf_{n\to\infty}\mu(F_n) \leqslant \lim_{n\to\infty}\mu(F_n) \leqslant \limsup_{n\to\infty}\mu(F_n) \leqslant \mu(\limsup_{n\to\infty} F_n) \\ &= \mu(\lim_{n\to\infty} F_n),\end{aligned} \tag{1.3.21}$$

giving $\mu(\lim_{n\to\infty} F_n) = \lim_{n\to\infty}\mu(F_n)$ as required. □

Definition 1.3.16. ***Continuous with respect to set theoretic limits for projection valued measures.*** *Let $(\Omega, \mathcal{F})$ be a measurable space with projection valued measure $P : \mathcal{F} \to \mathcal{P}(H)$. We say P is continuous with respect to set theoretic limits, if for a sequence of sets $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ for which the set limit exists, we have $P(\lim_{n\to\infty} F_n) = \lim_{n\to\infty} P(F_n)$.*

Example 1.3.17. *Projection valued measures are not continuous under set limits in the uniform topology.*

Proof. Let P^N as in example 1.3.6 and let e_n its associated orthonormal basis. Then set $F_n = \{k \in \mathbb{N} \mid k \geqslant n\}$. As F_n is non-increasing, we see that $\bigcup_{k=n}^{\infty} F_n = F_k$ and $\bigcap_{n=1}^{\infty} F_n = \bigcup_{k=n}^{\infty} F_n$, so $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n = \bigcap_{k=n}^{\infty} F_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} F_n$. As such we conclude that the set limit of F_n exists and $\lim_{n\to\infty} F_n = \bigcap_{n=1}^{\infty} F_n = \varnothing$. As such we get $P(\lim_{n\to\infty} F_n) = P(\varnothing) = 0$. However, this convergence is not satisfied in norm, as $\sup_{\|h\|=1}\|P(F_n)h\| \geqslant 1$ as $P(F_n)e_n = e_n$ for all $n \in \mathbb{N}$.

□

Proposition 1.3.18. *Projection valued measures are continuous under set limits in the strong operator topology.*

[2]This result is an adaptation of the following stack-exchange post, https://math.stackexchange.com/q/171523, accessed on 17-03-2023 .

Proof. The strong continuity with respect to set theoretic limits follows from proposition 1.3.4 property (5), that is, if $A, B \in \mathcal{F}$ then $P(A) \leqslant P(B)$. Let $(F_n)_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of sets such that its set theoretic limit $\lim_{n\to\infty} F_n = F$ exists. Following the argument we see that for all $h \in H$, we have $P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right) \leqslant P(F_n)$ for all $n \in \mathbb{N}$ and so $P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right) - P(F_n)$ is again a projection, giving

$$\begin{aligned}\left\|\left[P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right) - P(F_n)\right]h\right\| &= \left(\left[P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right) - P(F_n)\right]h|h\right) \\ &= \left(P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right)h|h\right) - (P(F_n)h|h) \\ &= P_h\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right) - P_h(F_n),\end{aligned} \tag{1.3.22}$$

which by the above proposition gives $P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} F_k\right)h = P(F)h$ for every $h \in H$. By similar argument we can see that $P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} F_k\right)h = P(F)h$, giving the required. □

The above example and proposition show that the norm topology is insufficient for dealing with sequences of events. A larger topology, such as the strong topology needs to be considered.

It remains however to be shown that the strong topology does not add more projections than we have events for. That is, we will show that in a separable Hilbert space the strong closure does not add more projections which cannot be represented by an event in the σ-algebra $\mathcal{F}$. Most of the remainder of this section will contain a proof of the claim that the image space of a bounded measure is closed. Our claim will then be shown from there.

We start with two propositions about the nature of atomic and non-atomic measures defined in the above section.

Proposition 1.3.19. *Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Then μ has at most countably many pair-wise disjoint atomic sets.*

Proof. If not, then there exists uncountably many pair-wise disjoint atomic sets. In particular there will exist some $N \in \mathbb{N}$ such that there exists uncountably many pair-wise disjoint atomic sets $(A_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{F}$ with Λ some uncountable set with $\mu(A_\lambda) > \frac{1}{N}$ for all $\lambda \in \Lambda$. But as $(A_\lambda)_{\lambda\in\Lambda}$ are pair-wise disjoint and uncountable, each countable cover of Ω will contain uncountably many A_λ and thus have infinite measure (by the additivity of the measure). This contradicts our assumption that $(\Omega, \mathcal{F}, \mu)$ was a σ-finite measure space. □

Corollary 1.3.20. *The image of a finite atomic measure is closed.*

Proof. From the above proof follows that its image consists of discrete points, with the exception of 0. As per definition $\varnothing \in \mathcal{F}$ and $\mu(\varnothing) = 0$, we conclude that the image is closed. □

Now for non-atomic measure we can also prove that its image is closed.

Proposition 1.3.21. *Let $(\Omega, \mathcal{F}, \mu)$ be a bounded measure space. Then μ is non-atomic if and only if its image maps surjectively onto* $[0, \mu(\Omega)]$.

Proof. We first show that μ is non-atomic if and only if its image lies dense in $[0, \mu(\Omega)]$. As μ is closed under countable unions, this is sufficient to show the above claim.

The 'if' direction, we prove by contraposition. Let the image space not be dense. Then there exists some $\epsilon > 0$ and some $x \in [0, \mu(\Omega) - \epsilon]$, such that $[x-\epsilon, x+\epsilon]$ is not in the image space of μ. As B is the least upperbound, there must exists some $F_u \in \mathcal{F}$ such that $\mu(F_u) = \inf\{\mu(F) \mid \mu(F) > x\}$ and some $F_l \in \mathcal{F}$ such that $\mu(F_l) = \sup\{\mu(F) \mid \mu(F) > x\}$. We now claim that $F := F_u \backslash F_l$ is atomic. To see this, note that $\mu(F) > 2\epsilon$ and that for any disjoint $F_1 \cup F_2 = F$ we have $\mu(F_1) = 0$ or $\mu(F_2) = 0$, as if not this would contradict the definition of F_l or F_u.

For the 'only if' direction, we again use contraposition. Let A be some atomic set of measure $\mu(A) = \alpha > 0$. Then, note that $[\mu(\Omega) - 2\frac{\alpha}{3}, \mu(\Omega) - \frac{\alpha}{3}]$ does not lie in the image space of μ and as such its is image is clearly not dense in $[0, \mu(\Omega)]$ (as no sequence can approximate $\mu(\Omega) - \frac{\alpha}{2}$). □

To show that the image space of any bounded measure is closed, we show that all measures allow for a decomposition into their atomic and non-atomic parts. This decomposition unique if the two measures are required to fullfil some orthogonalility requirement. This paragraph is based on [7] except for the last corollary.

Definition 1.3.22. ***Singular measures.*** *Let $(\Omega, \mathcal{F})$ be a measurable space with measures μ_1, μ_2. Then μ_1 is singular with respect to μ_2, if for every $F \in \mathcal{F}$*

$$\mu_1(F) = \sup\{\mu_1(F \cap G) \mid G \in \mathcal{F}, \mu_2(G) = 0\}. \tag{1.3.23}$$

Definition 1.3.23. ***Orthogonal singular measures.*** *Let $(\Omega, \mathcal{F})$ be a σ-finite measurable space with measures μ_1, μ_2. Then μ_1 and μ_2 are orthogonal singular if μ_1 is singular with respect to μ_2 and μ_2 is singular with respect to μ_1.*

For these measures we have the following result.

Theorem 1.3.24. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Then there exists a unique pair of orthogonal singular measures μ_1, μ_2 such that $\mu = \mu_1 + \mu_2$ with μ_1 atomic and μ_2 non-atomic.*

Proof. Let $\mathcal{M}$ be the family of countable unions of atomic sets of the measure μ (clearly $\mathcal{M} \subseteq \mathcal{F}$). We then define for each $F \in \mathcal{F}$

$$\begin{aligned} \mu_1(E) &:= \sup\{\mu(E \cap M) \mid M \in \mathcal{M}\} \text{ and} \\ \mu_2(E) &:= \sup\{\mu(E \cap N) \mid \mu_1(N) = 0\}. \end{aligned} \tag{1.3.24}$$

In this case μ_1 and μ_2 are clearly measures and orthogonal singular.

We show that μ_1 is atomic. Let $F \in \mathcal{F}$ with $\mu_1(F) > 0$. Note that if no such F exists, our claim holds. If such F exists, then there also exists some $M \in \mathcal{M}$ such that $\mu(F \cap M) > 0$. Now by lemma 1.3.19, we write $M = \bigcup_{n=1}^{\infty} M_n$, with M_n atomic for μ. As M_n being an atom implies that $F \cap M_n$ is an atom for μ and as $\mu_1(G) \leqslant \mu(G)$ for all $G \in \mathcal{F}$, we get that $F \cap M_n \subseteq F$ is also an atom for μ_1 as required.

We now show that μ_2 is non-atomic. Let $F \in \mathcal{F}$ with $\mu_2(F) > 0$. Note that if no such F exists, again our claim holds. Then $\mu_2(F \cap N) > 0$ for some $N \in \mathcal{F}$ with $\mu_1(N) = 0$. Now clearly N is non-atomic and thus so is $F \cap N$. As $F \cap N$ is non-atomic, there exists some $E \in \mathcal{F}$, such that $\mu(N \cap (F \cap E)) = \mu((F \cap N) \cap E) > 0$ and $\mu(N \cap (F \backslash E)) = \mu((F \cap N) \backslash E) > 0$, from which we get by definition that $\mu_2(F \cap E) > 0$ and $\mu_2(F \backslash E) > 0$ as required.

For the uniqueness claim we refer to [7, th. 2.5] (will not be used here). ☐

Corollary 1.3.25. *The image space of a finite measure is closed.*

Proof. By the above theorem we can split our measure in an atomic and non-atomic part, which are both clearly also bounded. As the union of two closed sets is again closed, we get our result by corollary 1.3.20 and proposition 1.3.21. ☐

This then leads to our desired result.

Theorem 1.3.26. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a projection valued measure acting on a separable Hilbert space H. Then its image $P(\mathcal{F})$ is strongly closed.*

Proof. Let $\mathcal{A}_P = \{P(\mathcal{F})\}''$ be the Von Neumann algebra generated by $P(\mathcal{F})$. As, by property (4) of proposition 1.3.4, $P(\mathcal{F})$ is an Abelian set (of normal operators) and thus $\mathcal{A}_P$ is an Abelian Von Neumann algebra acting on a separable Hilbert by proposition 1.2.38. Now by theorem 1.2.47 there exists a separating vector $k \in H$ for $\mathcal{A}_P$, which is cyclic for $\mathcal{A}_P'$ by proposition 1.2.48.

Now let $A \in \overline{P(\mathcal{F})}^{SOT}$. As $P(\mathcal{F})$ is contained in the unit ball of bounded operators, its strong limit also lies in this strong ball. As H is separable, this unit ball is metrizable by proposition 1.2.7. As such there exists a sequence of projections $(P(F_n))_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} P(F_n)k = Ak$ for our separating vector k. The sequence $P(F_n)k$ is then a Cauchy sequence (in the Hilbert space norm) and as such $P_k(F_n)$ will converge. As by corollary 1.3.20, the image of the measure $P_k(\;\;)$ is closed and thus there exists some $F \in \mathcal{F}$ such that $\lim_{n \to \infty} P_k(F_n) = P_k(F)$.

We now show that $\lim_{n\to\infty} P(F_n)h = P(F)h$ for all $h \in H$. Let $\epsilon > 0$ and $h \in H$ with $\|h\| > 0$ (the $h = 0$ case is trivial). As k is cyclic for $\mathcal{A}'_P$ there exists some sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}'$ such that $\lim_{n\to\infty} A_n k = h$. In particular, there exists some $N \in \mathbb{N}$ such that $\|A_N k - h\| < \frac{\epsilon}{12\|h\|}$ and $\|A_N k - h\| \leqslant \|h\|$ (such that $\|A_N k\| \leqslant 2\|h\|$). Furthermore, as $\lim_{n\to\infty} P_k(F_n) = P_k(F)$, there exists some $M \in \mathbb{N}$ such that for all $n \geqslant M$ we have $|P_k(F_n) = P_k(F)| < \frac{\epsilon}{2\|A_N^\star A_N k\|}$. We then get the following (rather long) calculation to show that for all $n \geqslant M$ we have

$$
\begin{aligned}
|([P(F_n) - P(F)]h|h)| &= |([P(F_n) - P(F)](h - A_N k + A_N k)|h)| \\
&\leqslant |([P(F_n) - P(F)]A_N k|h)| + |([P(F_n) - P(F)](h - A_N k)|h)| \\
&\leqslant |([P(F_n) - P(F)]A_N k|h)| + \|[P(F_n) - P(F)](h - A_N k)\|\|h\| \\
&\leqslant |([P(F_n) - P(F)]A_N k|h))| + \|P(F_n) - P(F)\|\|(h - A_N k)\|\|h\| \\
&\leqslant |([P(F_n) - P(F)]A_N k|(h - A_N k + A_N k))| + 2\|(h - A_N k)\|\|h\| \\
&< |([P(F_n) - P(F)]A_N k|A_N k)| + |([P(F_n) - P(F)]A_N k|h - A_N k)| + \tfrac{2\epsilon}{12} \\
&\leqslant |([P(F_n) - P(F)]A_N k|A_N k)| + \|P(F_n) - P(F)\|\|A_N k\|\|h - A_N k\| + \tfrac{\epsilon}{6} \\
&\leqslant |([P(F_n) - P(F)]A_N k|A_N k)| + 4\|h\|\|h - A_N k\| + \tfrac{\epsilon}{6} \\
&< |([P(F_n) - P(F)]A_N k|A_N k)| + \tfrac{4\epsilon}{12} + \tfrac{\epsilon}{6} \\
&= |(A_N[P(F_n) - P(F)]k|A_N k)| + \tfrac{\epsilon}{2} \\
&= |([P(F_n) - P(F)]k|A_N^\star A_N k)| + \tfrac{\epsilon}{2} \\
&= |(P(F_n)k|A_N^\star A_N k)| - |(P(F)k|A_N^\star A_N k)| + \tfrac{\epsilon}{2} \\
&\leqslant \|P(F_n)k\|\|A_N^\star A_N k\| - \|(P(F)k\|\|A_N^\star A_N k\| + \tfrac{\epsilon}{2} \\
&= \|A_N^\star A_N k\||(P_k(F_n) - P_k(F))| + \tfrac{\epsilon}{2} \\
&< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.
\end{aligned} \tag{1.3.25}
$$

This then gives for all $h \in H$ that

$$
|([A - P(F)]h|h)| = \lim_{n\to\infty} |([P(F_n) - P(F)]h|h)| = 0 \tag{1.3.26}
$$

and so $A = P(F)$ by lemma 1.3.3. □

1.3.4 Carathéodory's extension theorem for projection valued measures

This subsection is, except for the formulation of the classical Carathéodory taken from [1, p. 653, th.E.9], in full based on [8, sec. 5].

We start by shortly recalling the classical formulation of Carathéodory's extension theorem.

Definition 1.3.27. ***An algebra of sets.*** *A algebra of sets $\mathscr{F}$ is a collection of sets that (i) contains the empty set, (ii) is closed under taking the complement and (iii) is closed under taking finite unions or finite intersections.*

Definition 1.3.28. ***σ-finite measure.*** *A measure on measurable space $(\Omega, \mathcal{F})$ is σ-finite if Ω can be written as the countable union of sets of finite measure.*

Definition 1.3.29. ***μ-measurable sets.*** *Let Ω be a set and let $\mu : \mathscr{P}(\Omega) \to [0, \infty]$ be a map satisfying $\mu(\varnothing) = 0$. Then a set $S \in \mathscr{P}(\Omega)$ is called μ-measurable, if $\mu(A) = \mu(S \cap A) + \mu(S \cap A^c)$ for all $A \in \mathscr{P}(\Omega)$.*

Theorem 1.3.30. *Let $\mathscr{F} \subseteq \mathscr{P}(\Omega)$ be a collection of sets in Ω and let $\mu : \mathscr{F} \to [0, \infty]$ be a map satisfying $\mu(\varnothing) = 0$. Then for $S \in \mathscr{P}(\Omega)$ define*

$$
\tilde{\mu}(S) := \inf \left\{ \sum_{n=1}^{\infty} \mu(F_n) \mid S \subseteq \bigcup_{n=1}^{\infty} F_n \text{ with } (F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F} \right\}, \tag{1.3.27}
$$

where $\tilde{\mu}(S) = \infty$ if the above set is empty. Lastly, let $\mathcal{S}_\mu$ be the collection of μ-measurable sets. Then $\mathcal{S}_\mu$ is a σ-algebra and $\tilde{\mu}$ is a measure on this σ-algebra.

Proof. See [1, p. 652, 653, th. E.6, E.7] $\square$

Theorem 1.3.31. *Let $\mathscr{F}$ be an algebra of sets and $\mu : \mathscr{F} \to \mathbb{R}^+$ a countably additive mapping that satisfies $\mu(\varnothing) = 0$. Then the measure $\tilde{\mu}$ of equation (1.3.27) above extends μ to the smallest σ-algebra containing $\mathscr{F}$. Moreover, if $\tilde{\mu}$ is σ-finite then its extension is the unique σ-finite measure extending $\mathscr{F}$.*

Proof. See [1, p. 655, th. E.9]. $\square$

The main challenge of extending this result to projection valued measures is that our extension has to be consistent both on the level of the projections and on the level of the measure spaces the different projection valued measure give rise to. That is, if we extend our projection valued measure on the level of its projections, we should check that all measures are consistent and if we extend our projection valued measure on the level of its measures, we should check that these measures give rise to a projection. The following constructions takes the second approach and shows the consistency of the level of projections.

We first prove three lemmas, then our main result follows. For this recall that the range projection of closed subspace $S \subseteq H$ is denoted by $P(S)$.

Definition 1.3.32. ***Countable additivity for projection valued measures.*** *Let Ω be some set and let $\mathscr{F}$ be an algebra of subsets of Ω. Futhermore, let $P : \mathscr{F} \to \mathcal{P}(H)$ be an additive mapping sending sets to projections. Then P is countably additive, if $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ sequence of pair wise disjoint sets, then $P(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} P(F_n)$, where the convergence of the sum $\sum_{n=1}^{\infty} P(F_n)$ is in the strong operator topology.*

Lemma 1.3.33. *Let $(S_i)_{i\in\mathbb{N}}$ be a sequence of subspaces. (i) If this sequence is increasing, that is $S_i \subset S_{i+1}$, then $\lim_{n\to\infty} P(S_i)h = P(\overline{\bigcup_{n=1}^{\infty} S_i})h$ for all $h \in H$. (ii) If the sequence is decreasing, that is $S_{i+1} \subset S_i$, then $\lim_{n\to\infty} P(S_i)h = P(\bigcap_{n=1}^{\infty} S_i)h$ for all $h \in H$.*

Proof. For the first claim let $\epsilon > 0$ and denote $S = \overline{\bigcup_{n=1}^{\infty} S_i}$. As clearly $P(S)h \in S$, $\bigcup_{n=1}^{\infty} S_i$ lies dense in S and as the sequence of subspaces are increasing, we have that there exists some $N \in \mathbb{N}$ such that $h' \in S_N$ with $\|P(S)h - h'\| < \frac{\epsilon}{2}$. Again, as the subspaces are increasing we have that for any $n > N$, that

$$\begin{aligned}\|P(S)h - P(S_n)h\| &= \left\|P(S)h - h' + h' - P(S_n)(P(S)h + h' - h')\right\| \\ &\leqslant \left\|P(S)h - h'\right\| + \left\|h' - P(S_n)h'\right\| + \left\|P(S_n)(h' - P(S)h)\right\| \\ &< \tfrac{\epsilon}{2} + 0 + \tfrac{\epsilon}{2} = \epsilon,\end{aligned} \tag{1.3.28}$$

proving our first claim. The second claim follows by taking complements of the first claim. $\square$

Lemma 1.3.34. *Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of subspace closed under taking finite intersections. Then for any $h \in H$ we have*

$$\left\|P\Big(\bigcap_{\lambda\in\Lambda} S_\lambda\Big)h\right\| = \inf_{\lambda\in\Lambda} \|P(S_\lambda)h\|. \tag{1.3.29}$$

Proof. For the infimum there must exist some sequence $(S_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|P(S_n)h\| = \inf_{\lambda\in\Lambda} \|P(S_\lambda)h\|$. Now define recursively $\tilde{S}_1 = S_1$ and $\tilde{S}_n = \tilde{S}_{n-1} \cap S_n$, then by the second claim of the above lemma, we get

$$\inf_{\lambda\in\Lambda} \|P(S_\lambda)h\| = \lim_{n\to\infty} \left\|P(\tilde{S}_n)h\right\| \leqslant \lim_{n\to\infty} \|P(S_n)h\| = \inf_{\lambda\in\Lambda} \|P(S_\lambda)h\|. \tag{1.3.30}$$

Now as $\bigcap_{\lambda\in\Lambda} S_\lambda \subseteq \bigcap_{n=1}^{\infty} \tilde{S}_n$, we thus clearly have

$$\left\|P\Big(\bigcap_{\lambda\in\Lambda} S_\lambda\Big)h\right\| \leqslant \left\|P\Big(\bigcap_{n=1}^{\infty} \tilde{S}_n\Big)h\right\| = \inf_{\lambda\in\Lambda} \|P(S_\lambda)h\|. \tag{1.3.31}$$

Furthermore, by definition we have $\inf_{\lambda\in\Lambda} \|P(S_\lambda)h\| \leqslant \lim_{n\to\infty} \left\|P(\tilde{\tilde{S}}_n)h\right\|$ for any sequence of subspaces $(\tilde{\tilde{S}}_n)_{n\in\mathbb{N}}$. Combining this with the above gives us the required result. $\square$

Theorem 1.3.35. ***Carathéodory for projection valued measures.*** *Let Ω be some set and let $\mathscr{F}$ be an algebra of subsets of Ω. Futhermore, let $P : \mathscr{F} \to \mathcal{P}(H)$ be an additive mapping sending sets to projections, such that $P(\varnothing) = 0$, $P(\Omega) = I$ and P is a countably additive mapping on $\mathscr{F}$. Then there exists a unique projection valued measure $P^* : \sigma(\mathscr{F}) \to \mathcal{P}(H)$ extending P.*

Proof. We define $P^* : \mathscr{P}(\Omega) \to \mathcal{P}(H)$, by

$$P^*(S) := P\left(\bigcap \left\{ \overline{\bigoplus_{n=1}^{\infty} P(F_n)H} \;\middle|\; S \subseteq \bigcup_{n=1}^{\infty} F_n \text{ with } (F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F} \right\}\right) \tag{1.3.32}$$

for each set $S \subseteq \Omega$, where the second P denotes a range projection.

Now let $h \in H$. Then $P_h : \mathscr{F} \to [0, \infty)$ defines a countably additive map satisfying $P_h(\varnothing) = 0$. We now define $\mathcal{S}_h$, by

$$\mathcal{S}_h := \{F \mid P_h(A) = P_h(F \cap A) + P_h(F \cap A^c) \text{ for all } A \in \mathscr{P}(\Omega)\}. \tag{1.3.33}$$

As P is a countably additive map sending set to projections, we get that P_h is a countably additive map. As such the map

$$\begin{aligned} \tilde{P}_h &: \mathscr{P}(\Omega) \to [0, \infty] \\ S &\to \inf\left\{ \sum_{n=1}^{\infty} P_h(F_n) \mid S \subseteq \bigcup_{n=1}^{\infty} F_n \text{ with } (F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F} \right\} \end{aligned} \tag{1.3.34}$$

defines as outer measure for P_h, which is a measure on the σ-algebra given by $\mathcal{S}_h$.

Now for every $S \in \mathscr{P}(\Omega)$, we have

$$\begin{aligned} \tilde{P}_h(S) &= \inf\left\{ \sum_{n=1}^{\infty} P_h(F_n) \;\middle|\; S \subseteq \bigcup_{n=1}^{\infty} F_n \text{ with } (F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F} \right\} \\ &\overset{(1)}{=} \inf\left\{ \left\| P\left(\overline{\bigoplus_{n=1}^{\infty} P(F_n)H}\right) h \right\| \;\middle|\; S \subseteq \bigcup_{n=1}^{\infty} F_n \text{ with } (F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F} \right\} \\ &\overset{(2)}{=} \left\| P\left(\bigcap \left\{ \overline{\bigoplus_{n=1}^{\infty} P(F_n)H} \;\middle|\; S \subseteq \bigcup_{n=1}^{\infty} F_n \text{ with } (F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F} \right\}\right) \right\| \\ &= \|P^*(S)\|. \end{aligned} \tag{1.3.35}$$

The numbered equalities are then justified as follows. For (1), note that for every sequence of sets $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$, we have that there exists a pair-wise disjoint sequence $(G_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$, such that $\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} F_n$. We define $(G_n)_{n\in\mathbb{N}}$ by $G_1 := F_1$, $G_2 := F_2 \backslash F_1$, $G_3 := F_3 \backslash (F_2 \cup F_3)$, etc.. Now by argument similar to proposition 1.3.4, we have that disjoint sets have disjoint ranges, which by Parsevals equality then gives $\sum_{n=1}^{\infty} P_h(G_n) = \sum_{n=1}^{\infty} \|P(G_n)h\|^2 = \left\|\sum_{n=1}^{\infty} P(G_n)h\right\|^2 = \left\|P(\overline{\bigoplus_{n=1}^{\infty} P(G_n)H})h\right\|$. Now as clearly $S \subseteq \bigcup_{n=1}^{\infty} F_n$ implies $S \subseteq \bigcup_{n=1}^{\infty} G_n$ our equality of infimums follows from the above. For (2) we apply lemma 1.3.34, for which we note that if $S \subseteq \bigcup_{n=1}^{\infty} F_n$ and $S \subseteq \bigcup_{n=1}^{\infty} G_n$ for $(F_n)_{n\in\mathbb{N}}, (G_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$, then $S \subseteq (\bigcup_{n=1}^{\infty} F_n \cap \bigcup_{n=1}^{\infty} G_n)$ and thus $S \subseteq \bigcup_{n=1}^{\infty}(\bigcup_{m=1}^{n} F_n \cap \bigcup_{m=1}^{n} G_n) = (\bigcup_{n=1}^{\infty} F_n \cap \bigcup_{n=1}^{\infty} G_n)$ and $(\bigcup_{m=1}^{n} F_n \cap \bigcup_{m=1}^{n} G_n)_{m\in\mathbb{N}\subseteq\mathscr{F}}$.

Our proof is now completed by the Caratheordory theorem for classical measures. That is, as $\mathcal{S}_h$ is a σ-algebra containing $\mathscr{F}$ for every $h \in H$, we have that $\sigma(\mathscr{F}) \subseteq \mathcal{S}_h$ for every $h \in H$. Therefore, when we now restrict our map P^* to $\sigma(\mathscr{F})$, we get by equation (1.3.35) that this map gives the Caratheodory extension for every P_h with $h \in H$. Thus $P : \sigma(\mathscr{F}) \to \mathcal{P}(H)$ is a map such that (i) $P(\Omega) = I$ and (ii) for all $h \in H$ the map $P_h^* : \sigma(\mathscr{F}) \to [0, \infty)$ defines a measure on $(\Omega, \sigma(\mathscr{F}))$. The uniqueness claim then follows from the fact that P_h^* is a bounded measure for every $h \in H$ and the uniqueness of the classical Caratheordory extension. $\square$

1.3.5 Kolmogorov's extension theorem for projection valued measures

The famous extension theorem by Kolmogorov deals with a problem concerning the next two definitions.[3]

[3]The following section is, except for the application to projection valued measures, in full an adaptation of the online lecture notes by K.C. Border, see:

Definition 1.3.36. ***Kolmogorov consistent.*** *A sequence $\{(\mathcal{F}_n, \mathbb{P}_n)\}_{n\in\mathbb{N}}$ of a pair of σ-algebra's $\mathcal{F}_n$ and probability measures $\mathbb{P}_n$ is Kolmogorov consistent, if for all n we have $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\mathbb{P}_{n+1}\restriction_{\mathcal{F}_n} = \mathbb{P}_n$.*

Remark 1.3.37. For each such sequence, we set $\mathcal{F}_0$ as the trivial σ-algebra, that is $\mathcal{F}_0 = \{\varnothing, \Omega\}$, and $\mathbb{P}_0$ as the trivial probability measure.

Definition 1.3.38. ***Kolmogorov extension.*** *Let $\{(\mathcal{F}_n, \mathbb{P}_n)\}_{n\in\mathbb{N}}$ be a Kolmogorov consistent sequence, then a measure $\mathbb{P}$ on the combined σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$ is a Kolmogorov extension $\{(\mathcal{F}_n, \mathbb{P}_n)\}_{n\in\mathbb{N}}$ if $\mathbb{P}\restriction_{\mathcal{F}_n} = \mathbb{P}_n$ for all $n \in \mathbb{N}$.*

The problem is now: when does such an extension exist? Kolmogorov's famous theorem shows that the existence of a compact subclass in the union of all σ-algebras is a sufficient condition to assume the existence of a Kolmogorov extension.

The condition of being compact class if defined as follows.

Definition 1.3.39. ***Compact class.*** *A collection $\mathcal{C}$ of sets in Ω is of compact class, if for all sequences $(K_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ with $\bigcap_{n=1}^{\infty} K_n = \varnothing$, there exists some N such that $\bigcap_{n=1}^{N} K_n = \varnothing$.*

For these sets of compact class, we have the following crucial lemma.

Lemma 1.3.40. *If $\mathbb{P}$ is a finite additive set of probability functions on some algebra $\mathscr{A}$ of subsets with a compact subclass $\mathcal{C}$, with*

$$\mathbb{P}(E) = \sup\{\mathbb{P}(C) \mid C \in \mathcal{C}, C \subset E\} \tag{1.3.36}$$

for all $E \in \mathscr{A}$, then $\mathbb{P}$ is countably additive.

Proof. For our claim it is sufficient to show that if $\{E_n\}_{n\in\mathbb{N}}$ has $E_n \downarrow \varnothing$, then $\lim_{n\to\infty} \mathbb{P}(E_n) = 0$ [1, p. 654, prop. E. 10].

Let $\epsilon > 0$ and choose $C_n \subset E_n$ with $\mathbb{P}(E_n) \leqslant \mathbb{P}(C_n) + \frac{\epsilon}{2^n}$. As we have $C_n \subset E_n$, we get

$$\Big(\bigcap_{m=1}^{n} E_m\Big)\backslash\Big(\bigcap_{m=1}^{n} C_m\Big) \subseteq \Big(\bigcup_{m=1}^{n} E_m \backslash C_m\Big) \tag{1.3.37}$$

and clearly

$$E_n = \Big[\Big(\bigcap_{m=1}^{n} E_m\Big)\backslash\Big(\bigcap_{m=1}^{n} C_m\Big)\Big] \cup \Big(\bigcap_{m=1}^{n} C_m\Big) \tag{1.3.38}$$

as $\{E_n\}_{n\in\mathbb{N}}$ is a non-increasing sequence. Now as $\bigcap_{m=1}^{n} C_m \subset \bigcap_{m=1}^{n} E_m = E_n \downarrow \varnothing$, we have, by $\mathcal{C}$ being of compact class, that there exists some N such that $\bigcap_{m=1}^{N} C_m = \varnothing$. But then for all $n \geqslant N$, we have

$$\begin{aligned}\mathbb{P}(E_n) \leqslant \mathbb{P}(E_N) &= \mathbb{P}\Big(\Big[\Big(\bigcap_{m=1}^{n} E_m\Big)\backslash\Big(\bigcap_{m=1}^{n} C_m\Big)\Big] \cup \Big(\bigcap_{m=1}^{n} C_m\Big)\Big) \\ &\leqslant \mathbb{P}(\bigcup_{m=1}^{n} E_m \backslash C_m) + \mathbb{P}(\varnothing) \leqslant \sum_{m=1}^{N} (\mathbb{P}(E_m) - \mathbb{P}(C_m)) = \sum_{m=1}^{N} \tfrac{\epsilon}{2^m} < \epsilon\end{aligned} \tag{1.3.39}$$

as required. $\square$

With this lemma we can show the desired result.

Theorem 1.3.41. *Let $\{(\mathcal{F}_n, \mathbb{P}_n)\}_{n\in\mathbb{N}}$ be a Kolmogorov consistent sequence and let $\mathcal{C} \subseteq \bigcup_{n=1}^{\infty} \mathcal{F}_n$ be of compact class. Moreover, let for every $n \in \mathbb{N}$ and every $F \in \mathcal{F}_n$*

$$\mathbb{P}_n(F) = \sup\{\mathbb{P}_n(C) \mid C \in \mathcal{C}, C \subset E\}. \tag{1.3.40}$$

Then there exists a unique Kolmogorov extension to the σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$.

https://web.archive.org/web/20150226030616/http://people.hss.caltech.edu/%7Ekcb/Notes/Kolmogorov.pdf, accessed on 31-3-2023.

Proof. Let $F \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, then there exists some N such that $F \in \mathcal{F}_N$. We now define the function $\mathbb{P}$, by

$$\mathbb{P} : \bigcup_{n=1}^{\infty} \mathcal{F}_n \to [0,1] \tag{1.3.41}$$
$$F \to \mathbb{P}_N(F).$$

We show that $\mathbb{P}$ can be extended to a measure over the σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$.

For this, we firstly note that as $\Omega \in \mathcal{F}_0 = \{\Omega, \varnothing\}$ and $\mathbb{P}_0$ is a probability measure, we see that $\mathbb{P}(\Omega) = \mathbb{P}_0(\Omega) = 1$. Moreover, $\mathbb{P}$ is finitely additive, as for any finite disjoint union $\bigcup_{n=1}^{k} F_n$, there exists some $N = max\{N_1, \ldots, N_k\}$ such that $\mathbb{P}(\bigcup_{n=1}^{k} F_n) = \mathbb{P}_N(\bigcup_{n=1}^{k} F_n) = \sum_{n=1}^{k} \mathbb{P}_N(F_k) = \sum_{n=1}^{k} \mathbb{P}(F_k)$. Lemma 1.3.40 now shows, that P is countably additive on $\bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then by, Carathéodory's extension theorem we get that $\mathbb{P}$ has unique extension to the σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$. □

We will now use this result to construct an analogue for projection valued measures. As can be seen in the proof above, much of the result is preparing for and then applying Carathéodory's theorem. As we have already proven this crucial result in theorem 1.3.35, adapting the proof above to the setting of projection valued measures should be an easy step. We start with adapting the above two definition to the setting of projection valued measures and then the main result is proven.

Definition 1.3.42. ***Kolmogorov consistent sequence of projection valued measures*** *A sequence $\{(\mathcal{F}_n, P^n)\}_{n\in\mathbb{N}}$ of a pair of σ-algebra's $\mathcal{F}_n$ and projection valued measures P^n is Kolmogorov consistent, if for all n we have $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $P_{n+1}\restriction_{\mathcal{F}_n} = P_n$.*

Definition 1.3.43. ***Kolmogorov extension for projection valued measures*** *Let $\{(\mathcal{F}_n, P^n)\}_{n\in\mathbb{N}}$ be a Kolmogorov consistent sequence of projection valued measures, then a projection valued measure P on the combined σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$ is a Kolmogorov extension $\{(\mathcal{F}_n, P^n)\}_{n\in\mathbb{N}}$ if $P\restriction_{\mathcal{F}_n} = P^n$ for all $n \in \mathbb{N}$.*

The above theorem now becomes.

Theorem 1.3.44. ***Kolmogorov's extension theorem for projection valued measures.*** *Let $\{(\mathcal{F}_n, P^n)\}_{n\in\mathbb{N}}$ be Kolmogorov consistent sequence of projection valued measures and let $\mathcal{C} \subseteq \bigcup_{n=1}^{\infty} \mathcal{F}_n$ be of compact class. Moreover, let for every $n \in \mathbb{N}$, every $F \in \mathcal{F}_n$ and every $h \in H$*

$$P_h^n(E) = \sup\{P_h^n(C) \mid C \in \mathcal{C}, C \subseteq E\}. \tag{1.3.42}$$

Then there exists a kolmogorov extension P on the σ-algebra $\mathcal{F} := \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$.

Proof. Let $F \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, then there exists some N such that $F \in \mathcal{F}_N$. We now define the map P, by

$$\mathbb{P} : \bigcup_{n=1}^{\infty} \mathcal{F}_n \to \mathcal{P}(H) \tag{1.3.43}$$
$$F \to P_N(F).$$

We show that P can be extended to a projection valued measure over the σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$.

For this, we firstly note that as $\Omega \in \mathcal{F}_0 = \{\Omega, \varnothing\}$ and P_0 is a projection valued measure, we see that $P(\Omega) = P_0(\Omega) = I$. Moreover, P is finitely additive, as for any finite disjoint union $\bigcup_{n=1}^{k} F_n$, there exists some $N = max\{N_1, \ldots, N_k\}$ such that $\mathbb{P}(\bigcup_{n=1}^{k} F_n) = P_N(\bigcup_{n=1}^{k} F_n) = \sum_{n=1}^{k} P_N(F_k) = \sum_{n=1}^{k} P(F_k)$. Lemma 1.3.40 now shows, that P_h is countably additive on $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ for every $h \in H$. By an argument similar to proposition 1.3.4 property (3), P is countably additive. Then by theorem 1.3.35, we get that P has unique extension to the σ-algebra $\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$. □

1.3.6 Application of Kolmogorov's extension theorem to example 1.3.7.

In this section we will show that $P^{\mathbb{R}^k}$ of example 1.3.7 is the Kolmogorov extension of a sequence of finitely generated projection valued measures and as such is countably generated. That is

Definition 1.3.45. ***Finitely generated.*** *A σ-algebra $\mathcal{F}$ is finitely generated if there exists a finite collection of sets such that $\sigma(\{F_1, \dots, F_N\}) = \mathcal{F}$. A projection valued measure is finitely generated, if its σ-algebra is finitely generated.*

Definition 1.3.46. ***Countably generated.*** *A σ algebra $\mathcal{F}$ is countably generated if there exists a countably collection of sets such that $\sigma((F_n)_{n\in\mathbb{N}}) = \mathcal{F}$. A projection valued measure is countably generated, if its σ-algebra is countably generated.*

Again, we first prove a helpful lemma, after which we present our main result.

Lemma 1.3.47. *Let Ω be a topological Hausdorff space with topology τ_Ω. Then*

$$\mathcal{C} = \{K \mid K \in \tau_\Omega \text{ and } K \text{ compact}\} \tag{1.3.44}$$

is of compact class.[4]

Proof. Let $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ be some sequence such that $\bigcap_{n=1}^{\infty} K_n = \varnothing$. Define $L_k = \bigcap_{n=1}^{k} C_n$. Now define τ_{K_1} as the subspace topology of K_1, where we note that K_1 is still a compact set. Moreover, as (Ω, τ_Ω) is Hausdorff, so is (K_1, τ_{K_1}) and thus each L_k will be closed as it is compact. As such the sequence $O_k = K_1 \backslash L_{k+1}$ will a sequence of open sets, as each complement is closed, and form a cover as

$$\bigcup_{k=1}^{\infty} O_k = \bigcup_{k=1}^{\infty} K_1 \backslash L_{k+1} = K_1 \backslash \bigcap_{k=1}^{\infty} L_{k+1} = K_1 \tag{1.3.45}$$

as $\bigcap_{k=1}^{\infty} L_{k+1} = \bigcap_{n=1}^{\infty} K_n = \varnothing$ by assumption. As K_1 is compact, this open cover will have a finite subcover. But as $O_k = K_1 \backslash L_{k+1} \subseteq K_1 \backslash L_{k+2} = O_{k+1}$, we have that $\bigcap_{n=1}^{N} O_k = O_N = K_1 \backslash L_{N+1} = K_1$. As such $\bigcap_{n=1}^{N+1} K_n = L_{N+1} = \varnothing$ as required. ☐

Proposition 1.3.48. *$P^{\mathbb{R}^k}$ is the Kolmogorov extension of a sequence of finitely generated projection valued measures.*

Proof. Note that $\mathbb{Q}^k$ lies dense in $\mathbb{R}^k$ and that $\mathbb{Q}^k$ is countable. Let $(q_n)_{n\in\mathbb{N}}$ be a counting of $\mathbb{Q}^k$. We now construct our Kolmogorov sequence of projection valued measures $\{(\mathcal{F}_n, P^n)\}_{n\in\mathbb{N}}$ as follows. Let us with slight abuse of notation denote by $[0, a]$ the interval with endpoints $0, a$, even if $a < 0$. Now define recursively $\mathcal{F}_1 = \sigma([0, q_1])$ and $\mathcal{F}_n = \sigma(F_{n-1}, [0, q_n])$. Each $\mathcal{F}_n$ has atmost 2^n elements and is thus finite. We furthermore define $P^n = P^{\mathbb{R}^k}\restriction_{\mathcal{F}_n}$.

Let $\tau_{\mathbb{Q}^k}$ be the restricted topology of $\tau_{\mathbb{R}^k}$, which is then still clearly Hausdorff. Let $\mathcal{C}$ be the collection of compacts sets, which by lemma 1.3.47 is of compact class. Moreover, we clearly have

$$P_h^n(E) = \sup\{P_h^n(C) \mid C \in \mathcal{C}, C \subseteq E\}. \tag{1.3.46}$$

As such, theorem 1.3.44 can be applied and thus the existence of a Kolmogorov extension P can be assumed.

As we clearly have $\mathcal{B}(\mathbb{Q}^k) \subseteq \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$ and $P(\sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n)$ is strongly closed, we see that P is defined on $\mathcal{B}(\mathbb{R}^k)$. Moreover, as $P^n = P^{\mathbb{R}^k}\restriction_{\mathcal{F}_n}$ on this dense subset, we conclude that $P = P^{\mathbb{R}^k}$. ☐

While a more direct construction for the above proposition may be possible (as we will show below), the proposition rather shows the broad application of theorem 1.3.44. That is, it shows that not all countably generated are atomic and that moreover, even projection valued measures for which P_h is non-atomic for every $h \in H$ can be countably generated.

[4]The following proof was adapted from the following stack-exchange discussion https://math.stackexchange.com/questions/1724984/help-with-proof-that-a-set-of-compact-spaces-is-a-compact-class, accessed on 21-3-2023.

1.4 Spectral calculus

The key connection between projection valued measures and the normal operators of a Von Neumann algebra's is given by spectral calculus.

1.4.1 Bounded functional calculus

This section will aim to clarify this connection by showing both how normal operators can be constructed from projection valued measures and, conversely, how normal operators naturally give rise to a projection valued measure.

Definition 1.4.1. ***Bounded measurable functions.*** *Let $(\Omega, \mathcal{F})$ be a measurable space. We define $\|f\|_\infty = \sup_{\omega\in\Omega} |f(\omega)|$*

$$B_b(\Omega, \mathbb{C}) = \{f : \Omega \to \mathbb{C} \mid f \text{ measurable and } \|f\|_\infty < \infty\}. \tag{1.4.1}$$

For this space $\| \ \|_\infty$ is a complete norm.

Remark 1.4.2. The above definition differs from definition 1.2.40 in that it is defined point-wise. Furthermore, in the remainder we will from now no longer specify the complex image of the bounded measurable functions, that is $B_b(\Omega) := B_b(\Omega, \mathbb{C})$.

The following result is taken from [1, p. 287, 288, th. 9.8].

Theorem 1.4.3. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a projection valued measure. Then there exists a unique linear mapping $\Phi_P : B_b(\Omega) \to \mathcal{L}(H)$ with the following properties:*

1. *for all $F \in \mathcal{F}$, we have $\Phi(1_F) = P(F)$,*
2. *for all $f, g \in B_b(\Omega)$, we have $\Phi(fg) = \Phi(f)(g)$,*
3. *for all $f \in B_b(\Omega)$, we have $\Phi(\overline{f}) = \Phi(f)^\star$,*
4. *for all $f \in B_b(\Omega)$, we have $\|\Phi(f)\| \leqslant \|f\|_\infty$,*
5. *for all bounded sequences $(f_n)_{n\in\mathbb{N}} \subseteq B_b(\Omega)$ with $f_n \to f$ pointwise, we have $\lim_{n\to\infty} \Phi(f_n)h = \Phi(f)h$ for all $h \in H$,*
6. *for all $h \in H$ and $f \in B_b(\Omega)$, we have $(\Phi(f)h|h) = \int_\Omega f dP_h$ and $\|\Phi(f)x\| = \int_\Omega |f|^2 dP_h$.*

Proof. We construct the mapping Φ first for indicator functions, then for simple functions and lastly for measurable functions. We start by setting $\Phi(1_F) = P(F)$ for all $F \in \mathcal{F}$. We then extend this definition to simple functions by linearity. That is for a finite partition $(F_n)_{n=1}^N \subset \mathcal{F}$ let $f = \sum_{n=1}^N c_n 1_{F_n}$. Then we have $\Phi(f) = \Phi(\sum_{n=1}^N c_n 1_{F_n}) = \sum_{n=1}^N c_n P(F)$. As the sets in a partition are disjoint, their projections will have orthogonal range and thus for all $h \in H$ we have

$$\|\Phi(f)h\|^2 = \sum_{n=1}^N |c_n|^2 \|P(F_n)h\|^2 \leqslant \max_{n\in\{1,\dots,N\}} |c_n|^2 \|h\| = \|f\|_\infty^2 \|h\|^2. \tag{1.4.2}$$

Now let f be a measurable functions and let f_n be a sequence of simple functions such that $\lim_{n\to\infty} f_n = f$. As $(f_n)_{n\in\mathbb{N}}$ is clearly Cauchy and $\|\Phi(f_n) - \phi(f_m)\| \leqslant \|f_n - f_m\|_\infty$ by the above relation, the sequence $(\Phi(f_n))_{n\in\mathbb{N}}$ is also Cauchy. We define $\Phi(f)$ as its limit. The uniqueness of this limit again follows from the uniqueness of the limit in $B_b(\Omega)$ and equation (1.4.2).

The claimed properties of this construction now follow routinely by finding approximating sequences of simple functions, showing the property there and then concluding that their limit maintains this property. As an example, we will show here the last claim of property 6. Let $f \in B_b(\Omega)$ and $h \in H$. Let $(\Phi(f_n))_{n\in\mathbb{N}}$ be a sequence of approximating simple functions for f. Then for $f_n = \sum_{m=1}^N c_m 1_{F_m}$, we have

$$\|\Phi(f_n)h\|^2 = \sum_{m=1}^N |c_m|^2 \|P(F_m)h\|^2 = \sum_{m=1}^N |c_m|^2 P_h(F_m)^2 = \int_\Omega |f_n|^2 dP_h. \tag{1.4.3}$$

As this equality holds for all $n \in \mathbb{N}$, it follows from the uniqueness of limits that $\lim_{n\to\infty} \|\Phi(f_n)h\|^2 = \lim_{n\to\infty} \int_\Omega |f_n|^2 dP_h$. □

combining property (2) and (3) of the above proposition shows how every operator created by functional calculus is normal.

The result of property (5) can be improved if we introduce the following definition. The next definition and proposition are taken from [1, p. 290, def.9.10 and prop. 9.11].

Definition 1.4.4. ***P-essentially bounded functions.*** *Let P be a projection valued measure. A measurable function $f : \Omega \to \mathbb{C}$ is P-essentially bounded if there exists some $R \in \mathbb{R}^+$ such that $P(\{\omega \mid |f(\omega)| > R\}) = 0$ (where 0 here denotes the 0 operator). We define the space of equivalence classes of P-essentially bounded functions by $L^\infty(\Omega, P)$, where we identify measurable functions f and g when $P(\{\omega \mid |(f(\omega) - g(\omega)| > 0\}) = 0$. Lastly, we define the norm for the P-essentially bounded functions by $\|f\|_{L^\infty(\Omega,P)} := \inf_{\mathbb{R}^+}\{R \mid P(\{\omega \mid |f(\omega)| > R\}) = 0\}$. With respect to this norm, $L^\infty(\Omega, P)$ is a Banach space.*

Note that the above definition mirrors that of the μ-essentially bounded functions given in definition 1.2.40, but on the level of operators. This space of P-essentially bounded functions can now be used to improve property (5) to an equality of norms.

Proposition 1.4.5. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a projection valued measure. Then for all $f \in L^\infty(\Omega, P)$ we have $\|\Phi(f)\| = \|f\|_{L^\infty(\Omega,P)}$.*

Proof. We show $\|\Phi(f)\| \leqslant \|f\|_{L^\infty(\Omega,P)}$. Let $\epsilon > 0$. We now define $F_\epsilon := \{\omega \mid |f(\omega)| > (1 - \epsilon)\|f\|_{L^\infty(\Omega,P)}\}$. Then for all $h \in R(P(F_\epsilon))$, we get

$$\begin{aligned} \|\Phi(f)h\| &= \int_\Omega |f|^2 dP_h \geqslant (1-\epsilon)^2\|f\|_{L^\infty(\Omega,P)} \int_\Omega 1_{F_\epsilon} dP_h \\ &= (1-\epsilon)^2\|f\|_{L^\infty(\Omega,P)}\|P(F_\epsilon)h\| = (1-\epsilon)^2\|f\|_{L^\infty(\Omega,P)}\|h\| \end{aligned} \tag{1.4.4}$$

and so $\|\Phi(f)h\| \geqslant (1-\epsilon)^2\|f\|_{L^\infty(\Omega,P)}$. As this bound holds for any $\epsilon > 0$, we conclude $\|\Phi(f)\| \leqslant \|f\|_{L^\infty(\Omega,P)}$, as required. ☐

The above theorem 1.4.3 shows that a projection valued measure gives rise to a natural set of normal operators. We now show that the converse also holds, that is each normal operator gives naturally rise to a projection valued measure.

Theorem 1.4.6. ***Spectral theorem for bounded normal operators.*** *Let $A \in \mathcal{L}(H)$ be a normal operator. Then there exists a unique projection valued measure P on $\sigma(A)$ such that*

$$A = \int_{\sigma(A)} f_\lambda dP(\lambda). \tag{1.4.5}$$

where $f_\lambda : \sigma(A) \to \sigma(A)$ with $f_\lambda(z) = z$.

Proof. See [1, p. 293-296, th. 9.14]. ☐

While we have omitted the proof here, the importance of the above result should not be underestimated - its proof is simply very long. The result will be fundamental in the remaining construction.

Using the spectral theorem we define plentiful operators. We define for $A \in \mathcal{L}(H)$ normal and $f \in L^\infty(\sigma(A))$

$$f(A) := \int_{\sigma(A)} f(\lambda)dP_A(\lambda) = \Phi_{P_a}(f), \tag{1.4.6}$$

where Φ_{P_a} denotes the functional calculus defined above.

1.4.2 Spectral calculus and Abelian Von Neumann algebras on seperable Hilbert spaces

In section 1.2.2 we have already mentioned some results hinting at the deep connection between Von Neumann algebras and its projections. Here we expand on this idea, by showing how Abelian Von Neumann algebras are the functional calculus of some projection valued measure and vice versa.

We start by making our first claim precise.

Proposition 1.4.7. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a projection valued measure acting on a separable Hilbert space and let $\mathcal{A}_P = (P(\mathcal{F}))''$ the double commutant of all projections in the image of this projection valued measure. Furthermore, let Proj$(\mathcal{A}) := \{P \mid P \in \mathcal{A}, P$ a projection$\}$ be the set of projections in a Von Neumann algebra. Then $P(\mathcal{F}) =$ Proj$(\mathcal{A}_P)$.*

Proof. Clearly, we have $P(\mathcal{F}) \subseteq \mathrm{Proj}(P(\mathcal{F})'')$. Now for the converse claim, note that by theorem 1.2.21, we get that $\mathrm{Proj}(\mathcal{A}) = \{P(A) \mid A \in \mathcal{A}, R(A) = N(A)^\perp\}$. Now as $P(\Omega) \in P(\mathcal{F})$, we get by theorem 1.2.16 that the $\star$-algebra generated by $P(\mathcal{F})$ is strongly dense in $\mathcal{A}_P$. Lastly, note that by theorem 1.3.26 $P(\mathcal{F})$ is strongly closed when acting on a separable Hilbert space. Combining these claims, we see that $\mathrm{Proj}(\mathcal{A}) = \{P(A) \mid A \in \mathcal{A}, R(A) = N(A)^\perp\} \subseteq \overline{P(\mathcal{F})}^{\mathrm{SOT}} = P(\mathcal{F})$, as required. □

Theorem 1.4.8. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a projection valued measure acting on a separable Hilbert space, then*

$$\mathcal{A}_P = \{\Phi_P(f) \mid f \in B_b(\Omega)\} = \overline{span_{\mathbb{C}}(P(\mathcal{F}))}, \tag{1.4.7}$$

where in the second presentation Φ_P is the mapping associated with bounded functional calculus, using the projection valued measure P, and the third presentation denotes the norm-closure of the span of projections.

Proof. By the above proposition we have $P(\mathcal{F}) = \mathrm{Proj}(\mathcal{A}_P)$. We start by showing the first equality of equation (1.4.7). The inclusion $\{\Phi_P(f) \mid f \in B_b(\Omega, \mathbb{C})\} \subseteq \mathcal{A}_P$ is directly clear. For the converse let $A \in \mathcal{A}_P$. As $A \in \mathcal{A}_P$ is Abelian, we have that A is both normal and bounded. Then by the spectral theorem for bounded normal operators, we have

$$A = \int_{\sigma(A)} f_\lambda d\tilde{P}(\lambda) \tag{1.4.8}$$

for some projection valued measure $\tilde{P}$, $\sigma(A)$ the spectrum of A and $f_\lambda : \sigma(A) \to \sigma(A)$ with $f_\lambda(z) = z$ as in equation (1.4.5). Now let $\tilde{F} \in \mathcal{B}(\sigma(A))$ be some measurable set. Then $\int_{\sigma(A)} 1_{\tilde{F}} d\tilde{P}(\lambda)$ forms a projection in $\mathcal{A}_P$. As this projection is clearly in $\mathcal{A}_P$, we have that it is in the double commutant of all projections. Now, as $P(\mathcal{F}) = \mathrm{Proj}(\mathcal{A}_P)$, we get that $\int_{\sigma(A)} 1_{\tilde{F}} d\tilde{P}(\lambda) \in P(\mathcal{F})$. That is, for any $\tilde{F} \in \mathcal{B}(\sigma(A))$ there exists some $F \in \mathcal{F}$ such that $P(F) = \int_{\sigma(A)} 1_{\tilde{F}} dP(\lambda) \in P(\mathcal{F})$.

Now as A is bounded, $\sigma(A)$ is compact and thus there exists some sequence of simple functions $0 \leqslant f_n \uparrow f_\lambda$ uniformly. Therefore, there exists some $f_A \in B_b(\Omega, \mathbb{C})$ such that

$$\begin{aligned} A &= \int_{\sigma(A)} f_\lambda d\tilde{P}(\lambda) = \int_{\sigma(A)} \lim_{n\to\infty} f_n d\tilde{P}(\lambda) \overset{(1)}{=} \lim_{n\to\infty} \int_{\sigma(A)} f_n d\tilde{P}(\lambda) \\ &= \lim_{n\to\infty} \int_{\sigma(A)} \sum_{i=1}^{n} c_i 1_{\tilde{F}_i} d\tilde{P}(\lambda) = \lim_{n\to\infty} \sum_{i=1}^{n} c_i \int_{\sigma(A)} 1_{\tilde{F}_i} d\tilde{P}(\lambda) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} c_i P(F_i) = \lim_{n\to\infty} \int_\Omega \sum_{i=1}^{n} c_i 1_{F_i} dP \overset{(2)}{=} \int_\Omega f_A dP, \end{aligned} \tag{1.4.9}$$

where the numbered equalities are justified as follows. (1) The changing of limit and integral is justified by functional calculus, that is property (4) of theorem 1.4.3. That is, as $f_n \uparrow f_\lambda$ uniformly and $\|\Phi(f_n - f_\lambda)\| \leqslant \|f_n - f\|_\infty$, we get $\Phi(f_n) \to \Phi(f_\lambda)$ uniformly. (2) The existence of f_A is justified by moving to the P-essentially bounded functions. That is, by proposition 1.4.5 we get

$$\begin{aligned} \left\| \sum_{i=1}^{n} c_i 1_{F_i} - \sum_{i=j}^{m} c_j 1_{F_j} \right\|_{L^\infty(\Omega, P)} &= \left\| \int_\Omega \sum_{i=1}^{n} c_i 1_{F_i} dP - \int_\Omega \sum_{j=1}^{n} c_j 1_{F_j} dP \right\| \\ &= \left\| \sum_{i=1}^{n} c_i P(F_i) - \sum_{j=1}^{m} c_j P(F_j) \right\|. \end{aligned} \tag{1.4.10}$$

Now, as $\sum_{i=1}^n c_i P(F_i)$ converges in norm, the sequence is Cauchy in norm and thus by the above equality we get that $\sum_{i=1}^n c_i 1_{F_i}$ is a Cauchy sequence. As $L^\infty(\Omega, P)$ is a Banach space, this sequence will converge to some $\tilde{f}_A \in L^\infty(\Omega, P)$. Now choose some $f_A \in B_b$ such that f_A is an element of the equivalence class of $\tilde{f}_A$. Then clearly $\lim_{n\to\infty} \Phi(f_n) = \Phi(f_A)$, as required.

The second equality follows directly from the above argument. □

From this result we deduce the following corollaries directly.

Corollary 1.4.9. *Let P be a projection valued measure acting on a separable Hilbert space and $\mathcal{A}_P$ be its Von Neumann algebra. Then for all $A \in \mathcal{A}_P$, if we have $\int_\Omega f_A dP = A$, then $im(f_A) = \sigma(A)$.*

Proof. This follows directly from equation (1.4.9). □

Corollary 1.4.10. *Let $A \in \mathcal{L}(H)$ be normal operator acting on a separable Hilbert space H. Then*

$$\{A\}'' = \{\Phi_{P_A}(f) \mid f \in B_b(\Omega)\}. \tag{1.4.11}$$

Proof. Let P_A be the projection valued measure associated with A by the spectral theorem. Then clearly

$$\{A\}'' = \{P_A(\mathcal{B}(\sigma(A)))\}'' = \mathcal{A}_{P_A} = \{\Phi_{P_A}(f) \mid f \in B_b(\Omega)\}, \tag{1.4.12}$$

where in the last equality we use the above result. □

We now move towards the converse claim. That is, we aim to show that there exists some projection valued measure such that its double commutant equals our Abelian Von Neumann algebra. The following result by Von Neumann himself makes this an easy step.

Theorem 1.4.11. *Every Abelian Von Neumann algebra acting on a separable Hilbert space $\mathcal{A}$ there exists a self-adjoint operator A such that $\{A\}'' = \mathcal{A}$.*

Proof. See [2, p. 599, th. B.117]. □

Theorem 1.4.12. *A Von Neumann algebra $\mathcal{A}$ acting on a separable Hilbert space H is Abelian if and only if there exists some projection valued measure P_A such that $\mathcal{A} = \{\Phi_{P_A}(f) \mid f \in B_b(\Omega)\}$.*

Proof. The 'only if' claim is shown in theorem 1.4.8. For the 'only if' claim, we use the theorem above. Let $\mathcal{A}$ be an Abelian Von Neumann algebra acting on a separable Hilbert space H and let A be the self-adjoint operator such that $\{A\}'' = \mathcal{A}$. Then by corollary 1.4.10, we get $\mathcal{A} = \{A\}'' = \{\Phi_{P_A}(f) \mid f \in B_b(\Omega)\}$, where P_A denotes the spectral measure of self-adjoint operator A. □

The above then forms a variation of theorem 1.2.46.

1.4.3 A projection valued measure on a separable Hilbert spaces is countably generated

In this section we prove the claim that projection valued measures on separable Hilbert spaces are countably generated up to its non-zero projections. We here then apply this claim to example 1.3.7. This then contrasts our earlier approach of subsection 1.3.6, as this argument moves from the topology of the measurable space $(\Omega, \mathcal{F}) = (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and uses this to show that $P^{\mathbb{R}^k}$ is a countably generated projection valued measure. The current argument starts from the image space of this projection valued measure, namely the separability of the Hilbert space H, to show that it is countably generated.

The following proof is adapted from [2, p. 599, th. B.117].

Proposition 1.4.13. *Let $\mathcal{A}$ be an Abelian Von Neumann algebra acting on a seperable Hilbert space. Then there exists a countable set of projections $(P_n)_{n\in\mathbb{N}}$ that is strongly dense in Proj($\mathcal{A}$).*

Proof. As H is seperable there exists a countable subset S such that its linear span is dense in H. Then as $\mathcal{A}$ is Abelian, by theorem 1.2.47, there exists a separating vector $k \in H$ that, by proposition 1.2.48 is cyclic for $\mathcal{A}'$. Now as H is separable, so is $\text{Proj}(\mathcal{A})k = \{Pk | P \in \text{Proj}(\mathcal{A})\}$, and thus there exists some countable subset $S = (P_n)_{n\in\mathbb{N}} \subseteq \text{Proj}(\mathcal{A})$ such that $\overline{\text{span}_{\mathbb{C}} Sk} = \text{Proj}(\mathcal{A})k$.

We now claim that S is strongly dense in $\text{Proj}(\mathcal{A})$. Let $h \in H$, $P \in \text{Proj}(\mathcal{A})$ and $\epsilon > 0$. Now as k is cyclic for $\mathcal{A}'$, we have that there exists some sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}'$ such that $\lim_{n\to\infty} A_n k = h$.

In particular there exists some N_0 such that $\|A_{N_0}nk - h\| < \frac{\epsilon}{3}$. As $\overline{\operatorname{span}_{\mathbb{C}} Sk} = \operatorname{Proj}(\mathcal{A})k$, there exists some subsequence $(P_{n_k})_{k\in\mathbb{N}}$ such that for some $N \geqslant N_0$ we have that for all $n_k \geqslant N$ that $\|Pk - P_{n_k}\| < \frac{\epsilon}{3\|A_{N_0}\|}$. In this case we have for all $n_k \geqslant N$ that

$$\begin{aligned}\|Ph - P_{n_k}h\| &= \|Ph - PA_{N_0}k + PA_{N_0}k - P_{n_k}A_{N_0}k + P_{n_k}A_{N_0}k - P_{n_k}h\| \\ &\leqslant \|Ph - PA_{N_0}k\| + \|PA_{N_0}k - P_{n_k}A_{N_0}k\| + \|P_{n_k}A_{N_0}k - P_{n_k}h\| \\ &\leqslant \|P\|\|h - A_{N_0}k\| + \|A_{N_0}\|\|Pk - P_{n_k}k\| + \|P_{n_k}\|\|A_{N_0}k - h\| \\ &< \frac{\epsilon}{3} + \|A_{N_0}\|\frac{\epsilon}{3\|A_{N_0}\|} + \frac{\epsilon}{3} = \epsilon,\end{aligned} \tag{1.4.13}$$

as required. ☐

Proposition 1.4.14. *Any projection valued measure $P : \mathcal{F} \to \mathcal{P}(H)$ acting on a seperable Hilbert space H is countably generated up to non-zero projections. That is there exists some sequence of sets $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that $P(\mathcal{F}) = P(\sigma(F_n))$.*

Proof. Let $\mathcal{A}_P$ be the Von Neumann algebra generated by P, which is clearly Abelian, and let $(P_n)_{n\in\mathbb{N}}$ be a countable set of projections that is strongly dense in $\operatorname{Proj}(\mathcal{A})$, which exists by proposition 1.4.13. Now by proposition 1.4.7, we have that there exists a sequence of sets $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$, such that $P_n = P(F_n)$ for all $n \in \mathbb{N}$. Then we have $P(\mathcal{F}) = \operatorname{Proj}(\mathcal{A}) = \overline{(P_n)_{n\in\mathbb{N}}}^{SOT} \subseteq \overline{(P(F_n))_{n\in\mathbb{N}}}^{SOT} \subseteq P(\sigma((F_n)_{n\in\mathbb{N}})) \subseteq P(\mathcal{F})$, as required. ☐

Our claim now follows as a simple corollary.

Corollary 1.4.15. *$P^{\mathbb{R}^k}$ is countably generated up to non-zero projections.*

Proof. $L^2(\mathbb{R}^k)$ is a seperable Hilbert space and thus by the above proposition, $P^{\mathbb{R}^k}$ is countably generated up to non-zero sets. ☐

2. The indistinguishable States of a Quantum Observable

In this chapter we turn to the main topic of this thesis: understanding the theory of quantum mechanics through its indistinguishable states. Our argument here will take the following form. We will start by sketching the structure we expect a mathematical theory related to physical experiments to take. We will argue here that such a theory should, by a combination of a state - i.e. the state in which a system 'is' - and an observable - i.e. a question posed to the set of possible states of the system -, produce a probability measure on a set of outcomes by which the frequency of measured events can be predicted. Concretely, we will express this structure through the construction of the 'assignment function' of a given observable. Given some observable, the assignment function will assign states to their relevant outcome measures. We will then use this assignment function to make our central point. That is, we show that (i) the structure of the classical assignment function, at least theoretically, allows for the existence of an observable that distinguishes all the states in its associated state space and (ii) the structure of the quantum mechanical assignment function does not allow such an observable to exists. That is, in all non trivial cases of Hilbert spaces with dimension larger than one any observable has associated indistinguishable states.

This main argument is presented in sections 2.1 and 2.2. Section 2.3 then shortly touches on the relation between indistinguishable states and non-commuting observables. The last two sections are in some sense dedicated to Holevo's original idea. Section 2.4 presents a quantum analogue to the classical assignment function, vaguely mirroring Holevo's result that 'any separated statistical model (...) is a reduction of a classical model with restricted class of measurements' [5, p. 29, th. 1.7.1]. The last paragraph applies the results of section 2.1 and 2.2 to a finite dimensional case and aims to construct a (higher dimensional) geometrical object for which the quantum mechanical measurement can be understood as a compression analogue to the idea of figure 1 from the introduction.

2.1 Measurements and hidden states

As mentioned above, we start by comparing measurements in classical theory and quantum theory.

2.1.1 Measurements in measure theoretic language

Before moving towards the more mathematically precise definitions in the next section, we first take a moment to clarify the basic structure of an experiment in physics. The application of our mathematical definitions to these experiment then is argued for on the basis of said structure.

A physics experiment

The following section is loosely based on [9, p. 1-3].

The setting of a physics experiment is (not accidentally so), comparable to that of a (criminal) interrogation. That is, a physical system is so to say forcefully brought into a set state, a specific quality is investigated and an outcome is recorded. Our mathematical theory should thus consist of three distinct elements: (1) a space to represent the ways in which the system can be prepared, or more generally, 'brought into forcefully', (2) a space to represent the different types of qualities the state could possesses and (3) a connection between the two spaces by which we catalogue the status of these qualities. As an example, we could have (1) a dice with six numbered sides (one to six), (2) the quality of whether the outcome of dice, the number pointing upwards, is 'even' or 'uneven' and (3) a connection between the two states describing how the numbers one, three and five are 'uneven' outcomes and how the numbers two, four and six are 'even'. While this example might seem somewhat redundant, this is mostly a consequence of the simplicity of the dice. When we aim to describe a more complex system, say a set of interacting fundamental particles, the interplay between the aforementioned elements becomes clearer: the space of states and its connection to the recorded outcomes can only retroactively be constructed based on these recorded outcomes, but these states and their connection to the outcomes are then proactively used to predict recorded results. This back-and-forth is the exactly the predicament of (experimental) physics.

The 'function' combing these three elements to model (many consecutive) measurements, may abstractly be denoted by

$$w_s^O(F) = \lim_{N\to\infty} \frac{n_F}{N}, \tag{2.1.1}$$

where s denotes some state of the system, O denotes some observable - some way of connecting the state to the set of outcomes - and F some concrete set of outcomes. This, for now unspecified, function connecting said elements should equal the fraction $\frac{n_F}{N}$ for large N. Here n_F denotes the number of times the outcome F is recorded and N is the number of times the experiment is performed. As we can only base our prediction on before recorded results, the fraction $\frac{n_F}{N}$ will be a stochastic of which we assume convergence in a manner analogue to the law of large numbers (of which retroactively demand that our model should satisfy the demands of this theorem).

As our prediction of the recorded outcomes of an experiment depends on the state of our system, it depends on the practical manner in which we are able to control the conditions of preparing this state. As such we distinguish between a sharp measurement and an unsharp measurement. In the first case we (theoretically) demand perfect control over our system, in the second case we assume that the distribution of possible states of the system is known. Between these cases we see the same interplay as before: while perfect control of our system is practically almost never possible and the sharp case is as such retroactively reconstructed by approximating with unsharp measurements, the unsharp case is predicatively understood as a collection of sharp measurements of which the distribution of states is known. So even if the setting of a sharp measurement may be practically too demanding, it harbours the theoretical essence of how theory in physics comes to its predictions.

The assignment functions

If our aim is to understand the difference in the mathematical structure of classical and quantum mechanics, then we need to understand the way in which both theories give rise to the measurement 'function' $w_s^O(F)$. We will here call these measurement 'functions' assignment functions.

We start with the structure of the space of outcomes. This structure will be the same for both classical and quantum mechanics. From the description above, we could reasonably expect that our experimental 'function' $w_s^O(F)$ takes the form of a probability measure on some σ-algebra of events. This choice is here loosely justified by noting that the operations in an algebra of sets correspond to our standard logical operations: union corresponds to 'or', taking the complement to 'not', etc.. A σ-algebra then allows us to perform these logical operations countably many times, while keeping the more deviant set theoretic constructions in check (see [2, ap. D, p. 777-804] for a more in depth discussion on the link between set-theory and logic). We will therefore denote our outcome space as the measurable space $(\Omega, \mathcal{F})$. A probability measure is then used to assign to a collection of events F a ratio corresponding to the ratio $0 \leqslant \frac{n_F}{N} \leqslant 1$. The space of probability measures on this outcome space will here be denoted by $M_1^+(\Omega)$.

How the theories of classical and quantum mechanics give rise to these outcome measures $M_1^+(\Omega)$ differs. We will now turn to this topic. While we have motivated that a theory in physics should be seen as a way to construct probability measures on the space of outcomes on the basis of the structure of an experiment, the question of why these constructions assigning observables and states to measure are as they are will not be addressed here. This topic is simply too broad and complex to address here in any meaningful way: the question goes to the heart of physics as such.

We start with our description of the classical assignment function. The definitions are taken from [1, ch. 15, p. 543, 544].

Definition 2.1.1. ***Classical state space.*** *Let $(X, \mathcal{X})$ be a measurable space. A classical state is a probability measure on $(X, \mathcal{X})$. We denote this space of probability measures by $M_1^+(X)$.*

As in the case of the sharp measurement the state of the system is perfectly controlled, we refer to such state as a pure state, or say that the system is in a pure state. As we understand an unsharp measurement as a collection of sharp measurements of which the distribution is known, the pure states, correspondingly, are taken to be the extreme points of the set of states. Recall that in the previous chapter, we defined an atomic measure as follows.

Definition 2.1.2. ***Atomic measure.*** *Let $(X, \mathcal{X})$ be a measurable space. For a measure $\mu \in M_1^+(X)$ a set $A \in \mathcal{X}$ is atomic, if $\mu(A) > 0$ and if $A = A_1 \cup A_2$ with $A_1, A_2 \in \mathcal{X}$ disjoint, then either $\mu(A_1) = 0$ or $\mu(A_2) = 0$. The measure μ is atomic, if every $A \in \mathcal{X}$ with $\mu(A) > 0$ contains some atomic set.*

Proposition 2.1.3. *The extreme points of the set of probability measures are the atomic measures.*

Proof. See [1, p. 144, ex. 4.35]. □

Therefore we define the pure states as follows.

Definition 2.1.4. ***Classical pure states.*** *Let $(X, \mathcal{X})$ be a measurable space. A classical pure state is an atomic probability measure on $(X, \mathcal{X})$. We denote the set of atomic measures on $(X, \mathcal{X})$ by $s(M_1^+(X))$.*

This space of states is now connected to the space of outcomes by a measurable function.

Definition 2.1.5. ***Classical observable.*** *Let $(X, \mathcal{X})$ and $(\Omega, \mathcal{F})$ be measurable spaces. A classical observable is a measurable function $f : X \to \Omega$.*

Definition 2.1.6. ***Classical (sharp) assignment function.*** *Let $(X, \mathcal{X})$ and $(\Omega, \mathcal{F})$ be measurable spaces and $f : X \to \Omega$ be a measurable function. Then the classical assignment function is given by the map*

$$\begin{aligned} \Phi_f : M_1^+(X) &\to M_1^+(\Omega) \\ \mu(\ \ ) &\to \mu(f^{-1}(\ \ )). \end{aligned} \tag{2.1.2}$$

A classical sharp measurement is now given by this map restricted to the pure states. We define the classical sharp assignment function by

$$\begin{aligned} \Phi_f : s(M_1^+(X)) &\to M_1^+(\Omega) \\ \mu_A(\ \ ) &\to \mu_A(f^{-1}(\ \ )). \end{aligned} \tag{2.1.3}$$

This assignment function is *interpreted* in physic as our abstract measurement function by

$$\mu(f^{-1}(F)) = w_\mu^f(F) = w_s^O(F) = \lim_{N\to\infty} \frac{n_F}{N}. \tag{2.1.4}$$

Next, we turn to the quantum mechanical assignment function. The definitions are again taken from [1, ch. 15, p. 543, 544]. We start with the state. For our next definition we have to deal with a slight inconvenience, as the term state has an already standard definition in the theory on Von Neumann algebras, which almost, but not exactly corresponds to its use in physics.

Definition 2.1.7. ***Normal bounded positive linear functional.*** *Let H be a Hilbert space. A bounded linear positive functional is normal, if $\phi(sup_{\nu\in N} A_\nu) = sup_{\nu\in N}\phi(A_\nu)$ for any ordered set N.*

Definition 2.1.8. ***Quantum state.*** *Let H be a Hilbert space. A quantum state is a normal bounded positive linear functional satisfying $\phi(I) = 1$.*

These state are thus not to be confused with their common mathematical definition of states as bounded positive linear functionals. If we here refer to a state in the context of Hilbert spaces, we always take our state to be normal. For these states, the pure states are again given as their extreme points.

Proposition 2.1.9. *The extreme points of the set of normal bounded positive linear functionals with $\phi(I) = 1$ are the rank one projections.*

Proof. See [1, p. 553, prop. 15.12]. □

The rank-one projections are of the form $h\bar{\otimes}h$, where $h \in H$ a norm-one vector and

$$\begin{aligned} h\bar{\otimes}h : H &\to H \\ h' &\to (h'|h)h. \end{aligned} \tag{2.1.5}$$

As such there exists a correspondence between the pure quantum states and the norm-one vectors. However, as we have $h\bar{\otimes}h = e^{i\phi}h\bar{\otimes}e^{i\phi}h$ for any $\phi \in [0, 2\pi)$, the norm-one vectors h and $e^{i\phi}h$ give rise to the same state. We now define a global equivalence relation on the norm one vectors by

$$h \sim h' \text{ if there exists some } c \in \mathbb{T} \text{ such that } h = ch'. \tag{2.1.6}$$

This equivalence is referred to as the global phase in most introductory physics books on quantum mechanics. Let $B_1(H)$ be the set of norm-one vectors. Using this notation, we define the equivalence class of norm-1 vectors by

$$s(H) = B_1(H)/\sim, \tag{2.1.7}$$

which consists of the sets

$$[h] := \{ch \mid \|h\| = 1, |c| = 1, c \in \mathbb{C}\}. \tag{2.1.8}$$

As this set gives, given a Hilbert space H, a direct grasp on the set of pure states, we often use these equivalence classes of norm-one vectors as representation of the normal bounded positive linear functionals.

We now turn to the observables and the assignment function.

Definition 2.1.10. ***Quantum observable.*** *Let H be a Hilbert space and let $(\Omega, \mathcal{F})$ be a measurable space. Then we define a quantum observable as a projection valued measure $P : \mathcal{F} \to \mathcal{P}(H)$.*

Definition 2.1.11. ***Quantum mechanical (sharp) assignment function.*** *Let H be a Hilbert space and let $(\Omega, \mathcal{F})$ be a measurable space. Let $\Xi(H)$ be the set of normal positive linear bounded functionals on H, then the quantum mechanical assignment function is given by*

$$\begin{aligned} \Phi_P : \Xi(H) &\to M_1^+(\Omega) \\ \phi &\to \phi(P(\quad)). \end{aligned} \tag{2.1.9}$$

A quantum mechanical sharp measurement is again given by the restriction of this map to the state space, however using the identification mentioned above our assignment function is written as

$$\begin{aligned} \Phi_P : s(H) &\to M_1^+(\Omega) \\ [h] &\to (P(\quad)h|h)(= \phi_h(P(\quad))). \end{aligned} \tag{2.1.10}$$

Again our quantum assignment function of the sharp measurement is used to interpret our abstract measurement function by

$$(P(F)h|h) = w^P_{[h]}(F) = w^O_s(F) = \lim_{N\to\infty} \frac{n_F}{N}. \tag{2.1.11}$$

Remark 2.1.12. From now on, unless stated otherwise, every state, classical or quantum mechanical, is taken to be a pure state. Moreover, again both in the classical and quantum mechanical case, when we use the term assignment function, we from now on refer to the sharp assignment function.

2.1.2 The equivalence classes of states

Central to our argument will be a comparison between equations (2.1.3) and (2.1.10). We now turn towards our main tool for our comparison between these two assignment functions: the equivalence classes of indistinguishable states.

We again start with our construction for classical mechanics.

Definition 2.1.13. ***Classical indistinguishable states.*** *Let $(X, \mathcal{X})$ a measurable space and let $s(M_1^+(X))$ be the set of atomic measures. Let f be a measurable function. Then the states μ_{A_1}, μ_{A_2} are indistinguishable if $\mu_{A_1}(f^{-1}(F)) = \mu_{A_2}(f^{-1}(F))$ for all $F \in \mathcal{F}$. We denote this property by $\mu_{A_1} \sim_f \mu_{A_2}$*

It can easily be seen that the indistinguishability relation $\sim_f$ defines an equivalence relation on the set of classical states. As such we can define the equivalence class of indistinguishable states by taking the quotient over the indistinguishable states. That is, we define

$$s_f(M_1^+(X)) := s(M_1^+(X))/\sim_f, \tag{2.1.12}$$

with elements

$$[\mu_A]_f = \{\mu_B \mid \mu_B \in s(M_1^+(X), \mu_B(f^{-1}(F)) = \mu_A(f^{-1}(F)) \text{ for all } F \in \mathcal{F}\}. \tag{2.1.13}$$

When we now restrict our classical assignment function, with slight abuse of notation, to the equivalence classes of indistinguishable states, then $\Phi_f \restriction_{s_f(M_1^+(X))}$ becomes an injective map. That is, if $\mu_{A_1} \sim_f \mu_{A_2}$ then $\Phi_f(\mu_{A_1}) = \Phi_f(\mu_{A_2})$. As such, in an experimental setting, each state produces a different measure on the outcome space and as such, possibly after many repeated experiments, can be distinguished.

For quantum mechanics we present a similar construction.

Definition 2.1.14. ***Quantum mechanical indistinguishable states.*** *Let H be a hilbert space and $s(H)$ the set of states. The states $[h_1], [h_2]$ are indistinguishable, if $(P(F)h_1|h_1) = (P(F)h_2|h_2)$ for all $F \in \mathcal{F}$. We denote this property by $h_1 \sim_P h_2$.*

Again this produces an equivalence relation on the set of states for which we define equivalence classes by

$$s_P(H) := s(H)/\sim_P, \tag{2.1.14}$$

with sets of the form

$$[h]_P = \{[h'] \mid [h'] \in s(H) \text{ s.t. } (P(F)h|h) = (P(F)h'|h') \text{ for all } F \in \mathcal{F}\}. \tag{2.1.15}$$

Again when we now restrict our quantum assignment function, with slight abuse of notation, to the equivalence classes of indistinguishable states, then $\Phi_P \restriction_{s_P(H)}$ becomes an injective map. As such, in a experimental setting, each state produces a different measure on the outcome space and can thus be distinguished (possibly after many repeated experiments) by these outcome measures.

These equivalence classes thus group in a sense what is 'seen' by an observable. That is, the difference between two states is either detected or remains hidden and as a such we will also say that indistinguishable states are hidden states. It is on the basis of these equivalence classes that we will argue that quantum mechanics hides states. More precisely, we will aim to show that there exists a classical observable, where each equivalence class $[\mu_A]_f = \{\mu_A\}$. Moreover, we will aim to show that in all non-trivial cases, this is not the case in quantum mechanics, as in all non-trivial cases there will exist some states $h, h' \in s(H)$ such that $h \not\sim h'$ (recall equation (2.1.7)) but $h \sim_P h'$.

2.2 Maximal distinguishing observables

In order to show the two claims (put in the languague introduced above) - that (1) in the classical case there exists some observable such that $[\mu_A]_f = \{\mu_A\}$ and (2) in the quantum case for every (non-trivial) observable exist some states $h, h' \in s(H)$ such that $h \not\sim h'$ but $h \sim_P h'$ - we will develop the notion of a maximal distinguishing observable. This notion aims to specify the set of observables that have minimal indistinguishable states. Using this concept we will then show our claim.

2.2.1 Definition

In our attempt to show that the classical case there exists some observable such that $[\mu_A]_f = \{\mu_A\}$, we note that the crucial logical quantifier is here "some". That is, not all classical measurements have this property, but there exists a class of measurements that does so. We will here call these observables the maximal distinguishing observables, as these measurements clearly distinguish as many states as they could possibly distinguish.

To be precise we formally define this property as follows.

Definition 2.2.1. ***maximal distinguishing classical observable.*** *We call a measurable function $f : X \to \Omega$ maximal distinguishing, if for every $\mu_{A_1}, \mu_{A_2} \in s(M_1^+(X))$, with $\mu_{A_1} \neq \mu_{A_2}$, there exists some $F \in \mathcal{F}$, such that $f(A_1) \subseteq F$ and $f(A_2) \subseteq \Omega \backslash F$.*

Our claim (1) then follows almost directly from our definition.

Proposition 2.2.2. *Let f be a maximal distinguishing observable. Then for all $\mu_A \in s(M_1^+(X))$ we have $[\mu_A]_f = \{\mu_A\}$.*

Proof. Let $\mu_{A_1}, \mu_{A_2} \in s(M_1^+(X))$, with $\mu_{A_1} \neq \mu_{A_2}$. Then as f is maximal distinguishing there exists some $F \in \mathcal{F}$, such that $f(A_1) \subseteq F$ and $f(A_2) \subseteq \Omega \backslash F$ and thereby we get $\mu_{A_1}(f^{-1}(F)) \neq \mu_{A_2}(f^{-1}(F))$ and thus $\mu_{A_1} \not\sim_f \mu_{A_2}$, as required. □

For classical observables we then have the following result showing that this definition arises in a context relevant to physics experiments. That is, as the classical state spaces in physics are given by differentiable manifolds [2, p. 84, 88], which are topologically Hausdorff, the following proposition provides sufficient and necessary conditions for the existence of a maximal distinguishing observable.

Proposition 2.2.3. *Let X be some topological Hausdorff space with measurable sets $\mathcal{B}(X)$ and let $f : X \to \Omega$ be a measurable function. Moreover, Let $(\Omega, \mathcal{F})$ be a sufficiently large space such that for each disjoint $G_1, G_2 \in \mathcal{X}$ with $f(G_1), f(G_2)$ disjoint, there exists some disjoint F_1, F_2 such that $f(G_1) \subseteq F_1$ and $f(G_2) \subseteq F_2$. Then f is injective if and only if it is maximal distinguishing.*

Proof. Let f be injective and let $\mu_{A_1}, \mu_{A_2} \in s(M_1^+(X))$, with $\mu_{A_1} \neq \mu_{A_2}$. Now, direct from the definition of an atomic measure, we see that in this case we have $A_1 \cap A_2 = \varnothing$. Now as f in injective we thus have $f(A_1) \cap f(A_2) = f(A_1 \cap A_2) = \varnothing$. Then by assumption we get that there exists some $F \in \mathcal{F}$ such that $f(A_1) \subseteq F$ and $f(A_2) \subseteq \Omega \backslash F$.

Now if f is not injective, then there exist some $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Let $A_1, A_2 \in \mathcal{X}$ be some sets such that $x_1 \in A_1$ and $x_2 \in A_2$ and $A_1 \cap A_2 = \varnothing$, which exist as X is Hausdorff. Moreover, we can take A_1, A_2 sufficiently small such that μ_{A_1} and μ_{A_2} are atomic probability measures. In this case, $f(A_1) \cap f(A_2) \neq \varnothing$ and thus there exists no $F \in \mathcal{F}$ such that $f(A_1) \subseteq F$ and $f(A_2) \subseteq \Omega \backslash F$.

□

Thus as soon as such an injective function $f : X \to \Omega$ to such a sufficiently large space of outcomes exists, all the states in the phase space can be distinguished and, as a results, the classical setting has no indistinguishable states. This proposition thus directly shows the possibility of constructing, at least theoretically, a classical measurement that is maximally distinguishing as if we pick the same measurable space as outcome and state space, that is $(\Omega, \mathcal{F}) = (X, \mathcal{B}(X))$, then the identity function clearly satisfies the conditions of the above proposition. This shows the first part of our claim.

In the classical case the idea of maximal distinguishing measurement can be defined with relative ease as the case exists where equivalence classes of states consist simply of one element. The observable thus, put bluntly, distinguishes all there is to distinguish. In the quantum case such an observable will only exist in the trivial case of a one dimension (as there is only one state in this case). As such it is not at all trivial to define a sense of 'maximal distinguishing'.

Intuively, however that if a classical observable f is injective (and thus maximal distinguihsing), it has in a sense the maximal amount of different points in its image. By analogy, it is then natural to demand of a maximal distinguishing quantum observable that it has maximal amount of projections in its image space. However, not every set of projections gives rise to a projection valued measure. Under this restriction, we could define a maximal distinguishing projection valued measure $P : \mathcal{F} \to \mathcal{P}(H)$ as a projection valued measure such that its image is not contained in the image of any other projection valued measure. That is, for any projection valued measure $\tilde{P} : \tilde{\mathcal{F}} \to \mathcal{P}(H)$, if $P(\mathcal{F}) \subseteq (\tilde{\mathcal{F}})$ then $P(\mathcal{F}) = (\tilde{\mathcal{F}})$. Now as we have seen in the previous chapter, by theorem 1.4.12, projection valued measures are deeply linked to Abelian Von Neumann algebras and for such Abelian Von Neumann algebras a sense of maximallity is well-defined. Using this identification, we define our maximal distinguishing observables as follows. We use the following notation for a projection valued measure $P : \mathcal{F} \to \mathcal{P}(H)$:

$$P(\mathcal{F}) := \{P(F) \mid F \in \mathcal{F}\} \tag{2.2.1}$$

and

$$P(\mathcal{F})^{\circ} := \{P \mid P \in \mathcal{P}(H),\ PP(F) = P(F)P \text{ for all } F \in \mathcal{F}\}. \tag{2.2.2}$$

Here $P(\mathcal{F})^{\circ}$ can be understood as the analogue of the commutator set in the set of projections on our Hilbert space. We now present our definition as follows.

Definition 2.2.4. ***Maximal distinguishing quantum mechanical observable.*** *We call a quantum mechanical observable $P : \mathcal{F} \to \mathcal{P}(H)$ maximally distinguishing, if $P(\mathcal{F}) = P(\mathcal{F})^{\circ}$.*

Remark 2.2.5. As is the case with Abelian Von Neumann algebras, the inclusion $P(\mathcal{F}) \subseteq P(\mathcal{F})^{\circ}$ is trivial, but the converse inclusion is not.

This definition has the following, now obvious, connection with maximal distinguishing Von Neumann algebras.

Proposition 2.2.6. *A quantum mechanical observable P is maximal distinguishing if and only if the Von Neumann algebra it generates $\mathcal{A}_P$ is maximal Abelian.*

Proof. For the 'only if' part, note that by theorem 1.4.8, we have $\mathcal{A}'_P = \text{span}_{\mathbb{C}}\{\text{Proj}(\mathcal{A}'_P)\} = \text{span}_{\mathbb{C}}\{\text{Proj}(P(\mathcal{F})^{\circ})\} = \text{span}_{\mathbb{C}}\{\text{Proj}(P(\mathcal{F}))\} = \text{span}_{\mathbb{C}}\{\text{Proj}(\mathcal{A}_P)\} = \mathcal{A}_P$. Conversly, for the 'if' part, note that if we have $\mathcal{A}_P = \mathcal{A}'_P$, then $\mathcal{A}_P = \{\Phi_P(f) \mid f \in B_b(\Omega)\} = \mathcal{A}'_P$. Thus $P \in P(\mathcal{F})^{\circ}$ if only if there exists some $1_F \in B_b(\Omega)$ such that $\Phi_P(1_F) = P$ if and only there exists some $\tilde{F} \in \mathcal{F}$ such that $\Phi_P(1_F) = P(\tilde{F})$, so $P(\mathcal{F}) = P(\mathcal{F})^{\circ}$. ☐

From this result follows the connection of our previous 'intuitive definition' of P being maximal distinguishing if, for any projection valued measure $\tilde{P} : \tilde{\mathcal{F}} \to \mathcal{P}(H)$, if $P(\mathcal{F}) \subseteq \tilde{P}(\tilde{\mathcal{F}})$ then $P(\mathcal{F}) = (\tilde{\mathcal{F}})$.

Corollary 2.2.7. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be a quantum observable. Then P is maximal distinguishing if and only if for any projection valued measure $\tilde{P} : \tilde{\mathcal{F}} \to \mathcal{P}(H)$, if $P(\mathcal{F}) \subseteq \tilde{P}(\tilde{\mathcal{F}})$ then $P(\mathcal{F}) = (\tilde{\mathcal{F}})$.*

Proof. As for any maximal Abelian Von Neumann $\mathcal{A}$, we have that if $\tilde{\mathcal{A}}$ Abelian and $\mathcal{A} \subseteq \tilde{\mathcal{A}}$, then $\mathcal{A} \subseteq \tilde{\mathcal{A}}$. Thus if for some Abelian Von Neumann algebra $\tilde{\mathcal{A}}$, we have $\text{Proj}(\mathcal{A}) \subseteq \text{Proj}(\tilde{\mathcal{A}})$, then $\text{Proj}(\mathcal{A}) = \text{Proj}(\tilde{\mathcal{A}})$. Our claim now follows from the proposition above. ☐

2.2.2 Key examples

In this section we prove that examples 1.3.6 and 1.3.7 from the previous chapter are both maximal distinguishing.

We start with example 1.3.6, which can be directly shown using our definition 2.2.4.

Proposition 2.2.8. *The operator defined by*

$$\begin{aligned} P^{\mathbb{N}} : \mathcal{B}(\mathbb{N}) &\to \mathcal{P}(l^2(\mathbb{N})) \\ I &\to \sum_{i \in I} e_{k_i} \bar{\otimes} e_{k_i}, \end{aligned} \tag{2.2.3}$$

where

$$\begin{aligned} e_i \bar{\otimes} e_i : l^2(\mathbb{N}) &\to l^2(\mathbb{N}) \\ h &\to (h|e_i)e_i \end{aligned} \tag{2.2.4}$$

is maximal distinguishing.

Proof. Let $P \in P^{\mathbb{N}}(\mathcal{B}(\mathbb{N}))^{\circ}$, we have for each $F_i := \{i\}$ with $i \in \mathbb{N}$ that P and $P(F_i)$ commute and thus that range of $P(F_i)$ either lies with the range of P or not. Now as $P(F_i)$ has one dimensional range and as it is the unique projection onto its range. Thus $PP(F_i) = P(F_i)$ or $PP(F_i) = O$. Choose $G := \bigcup\{F_i | PP(F_i) = P(F_i)\} \in \mathscr{P}(\mathbb{N}) = \mathcal{B}(\mathbb{N})$, then clearly

$$\begin{aligned} P &= PP(G) + PP(\Omega \backslash G) = P \sum_{F_i \in G} P(F_i) + P \sum_{F_i \in \Omega \backslash G} P(F_i) \\ &= \sum_{F_i \in G} PP(F_i) + \sum_{F_i \in \Omega \backslash G} PP(F_i) = \sum_{F_i \in G} P(F_i) = P(G) \end{aligned} \tag{2.2.5}$$

and so $P \in P^{\mathbb{N}}(\mathcal{B}(\mathbb{N}))$. □

For our second example, we will use proposition 2.2.6.

Proposition 2.2.9. *The projection valued measure*

$$\begin{aligned} P^{\mathbb{R}^k} : \mathcal{B}(\mathbb{R}^k) &\to \mathcal{P}(L^2(\mathbb{R}^k)) \\ F &\to 1_F, \end{aligned} \tag{2.2.6}$$

defined on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \lambda)$ *(with Lebesgue measure* λ *and standard topology) for some finite* $k \in \mathbb{N}$*, where here* 1_F*, with slight abuse of notation, is here the operator denoting a multiplication with the indicator function* 1_F

$$\begin{aligned} 1_F : L^2(\mathbb{R}^k) &\to L^2(\mathbb{R}^k) \\ f &\to 1_F f, \end{aligned} \tag{2.2.7}$$

is maximal distinguishing.

Proof. It is clear that the Von Neumann algebra generated by $P^{\mathbb{R}^k}$ is the multiplication algebra $\mathcal{M}$ on $L^2(\mathbb{R}^k)$ and as $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \lambda)$ is a σ-finite measure space its multiplication algebra is maximal Abelian by the proposition 1.2.42. By proposition 2.2.6 $P^{\mathbb{R}^k}$ is then maximal distinguishing. □

These observables are key examples, as they cover all the possible projection valued measures associated with maximal Abelian Von Neumann algebras by theorem 1.2.43. Recall that these four unitarily nonequivalent maximal Abelian Von Neumann algebras are given by: (1) the multiplication algebra on $L^2([0,1])$, (2) the multiplication algebra on $l^2(\mathbb{N})$, (3) the multiplication algebra on $L^2([0,1]) \oplus l^2(\mathbb{N})$ and (4) the multiplication algebra $L^2([0,1]) \oplus D_N(\mathbb{C})$, with $D_N(\mathbb{C})$ the set of N-dimensional complex diagonal matrices. In the finite dimensional case all Abelian Von Neumann (matrix) algebras are (obviously) equivalent to the multiplication algebra on $D_N(\mathbb{C})$. The above examples now cover these cases as follows. Case (2) is directly given by proposition 2.2.8 and the final dimensional case can be seen with the same argument. Case (3) is explicitly shown in proposition 1.2.42, case (1) is directly contained in the proof of the same proposition and case (4) is only a slight modification (wherein one moves from infinite to finite making the argument only more simple).

2.2.3 The existence of quantum mechanical indistinguishable states

We now move to the second part of our first initial claim: we state that even a maximal distinguishing quantum mechanical observable will in all non-trivial cases, that is for Hilbert space of dimension larger than two, hide some of its states.

We start with the following definition and result on direct sums of closed subspaces.

Definition 2.2.10. ***Direct sums of projection valued measures.*** *Let* $H_1, H_2 \subseteq H$ *be two closed orthogonal subspaces and let* $P_1 : \mathcal{F}_1 \to \mathcal{P}(H_1)$ *be a projection valued measure on the first subspace and* $P_2 : \mathcal{F}_2 \to \mathcal{P}(H_2)$ *be a projection valued measure onto the second. We define* $\mathcal{F}_1 \sqcup \mathcal{F}_2 := \{F_1 \sqcup F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$ *on* $\Omega_1 \sqcup \Omega_2$*, where* $\sqcup$ *denotes the disjoint union. We now define* $P_1 \oplus P_2 : \mathcal{F}_1 \sqcup \mathcal{F}_2 \to \mathcal{P}(H_1 \oplus H_2)$ *by* $P_1 \oplus P_2(F_1 \sqcup F_2) = P(F_1) + P(F_2)$.

Proposition 2.2.11. *Let $\mathcal{A}_1, \mathcal{A}_2$ be maximal Abelian Von Neumann algerbras on closed orthogonal subspaces H_1, H_2 respectively. Then $\mathcal{A}_1 \oplus \mathcal{A}_2 := \{A_1 \oplus A_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$ is maximal Abelian. Consequently, if P_1, P_2 are maximal distinguishing projection valued measures on H_1, H_2 respectively, their direct sum is again maximal distinguishing.*

Proof. $\mathcal{A}_1 \oplus \mathcal{A}_2$ is clearly Abelian, so $\mathcal{A}_1 \oplus \mathcal{A}_2 \subseteq (\mathcal{A}_1 \oplus \mathcal{A}_2)'$ is directly clear. Now let $A \in (\mathcal{A}_1 \oplus \mathcal{A}_2)'$ and let $P(H_1), P(H_2)$ be the projections onto H_1, H_2 in $H_1 \oplus H_2$ respectively. Then as $P(H_1) + P(H_2) = I_{H_1 \oplus H_2}$, we get $A = AP(H_1) + AP(H_2) = P(H_1)AP(H_1) + P(H_2)AP(H_2)$ as $A \in (\mathcal{A}_1 \oplus \mathcal{A}_2)'$. Now clearly $P(H_1)AP(H_1) \in \mathcal{A}_1'$ and so $P(H_1)AP(H_1) \in \mathcal{A}_1$ as $\mathcal{A}_1$ is maximal Abelian. Now by the same argument for $P(H_2)AP(H_2)$ we get $A = P(H_1)AP(H_1) + P(H_2)AP(H_2) \in \mathcal{A}_1 \oplus \mathcal{A}_2$, as required. The second claim then follows directly from proposition 2.2.6. $\square$

Example 2.2.12. *Let $h', h \in H$, with $\|h\| = \|h'\| = 1$ and $h' \neq -h$. There exists a projection with one-dimensional range such that for this projection and any projection orthogonal to it, we have*

$$(Ph|h) = (Ph'|h'). \tag{2.2.8}$$

Proof. As $h' \neq -h$, we have $h' + h \neq 0$. We now set

$$h_{bis} := \frac{h + h'}{\|h + h'\|} \tag{2.2.9}$$

and use this to define $P_{bis} = h_{bis} \overline{\otimes} h_{bis}$. By construction this projection will have one dimensional range. We now claim that P_{bis} had the desired property. To see this, firstly note that

$$(P_{bis}h|h) = |(h|h_{bis})|^2 = \frac{|(h|h+h')|^2}{\|h+h'\|} = \frac{|(h'|h+h')|^2}{\|h+h'\|} = \left|(h'|h_{bis})\right|^2 = (P_{bis}h'|h'), \tag{2.2.10}$$

which proves our first claim. For our second claim, we write $h = (h|h_{bis})h_{bis} + h - (h|h_{bis})h_{bis}$ and the same for h'. Now note that $h - (h|h_{bis})h_{bis} = -(h' - (h'|h_{bis})h_{bis})$. As any projection $\tilde{P}$ orthogonal to P_{bis} acts only on this second part, we have

$$(\tilde{P}h|h) = (\tilde{P}(h - (h|h_{bis})h_{bis})|\tilde{P}h) = (-\tilde{P}(h' - (h|h_{bis})h_{bis})| - \tilde{P}(h' - (h|h_{bis})h_{bis})) = (\tilde{P}h'|h'). \tag{2.2.11}$$

Thereby, we conclude that P_{bis} acts as required. $\square$

We now shot the existence of indistinguishable states for a quantum mechanical observable. Recall that by construction an observable P will hide states $h_1, h_2 \in s(H)$ if $(P(F)h_1|h_1) = (P(F)h_2|h_2)$ for all $F_1, F_2 \in \mathcal{F}$, which we denoted by $h_1 \sim_P h_2$. Furthermore, recall that the norm one vectors $h_1, h_2 \in H$ repesent the same state if $h_1 = ch_2$ for some $c \in \mathbb{T}$, which we denoted by $h_1 \sim h_2$.

Proposition 2.2.13. *If $dim(H) \geqslant 2$, there exists an maximal distinguishing observable $P : \mathcal{F} \to \mathcal{P}(H)$ such that for some $h_1 \nsim h_2$ we have $h_1 \sim_P h_2$.*

Proof. If $\dim(H) \geqslant 2$, there exist some norm one vectors $h_1, h_2 \in H$ such that $h_1 \neq ch_2$ for some $c \in \mathbb{T}$, that is $h_1 \nsim h_2$. then in particular $h_1 \neq -h_2$ and thus $h_{bis} := \frac{h+h'}{\|h+h'\|}$ is non-zero. Now $H_1 := h_{bis} \overline{\otimes} h_{bis} H$ is a closed subspace and so is its orthogonal complement $H_2 := H_1^{\perp}$. As H_1 is of dimension one, its trivial projection valued measure, from now on denoted by P_1 is maximal distinguishing. Let P_2 be some maximal distinguishing projection valued measure onto H_2. Let $P := P_1 \oplus P_2$, which by proposition 2.2.11 is again maximal distinguishing. Now, As any projection onto H_2 is by definition orthogonal to P_{bis}, we see by example 2.2.12, that then $(P(F)h_1|h_1) = (P(F)h_2|h_2)$ for all $F_1, F_2 \in \mathcal{F}$, as required. $\square$

2.2.4 Classification of the indistinguishable states of a quantum observable

The above section shows the existence of indistinguishable states for a quantum observable. Here we prove a natural identification between the indistinguishable states and the unitary operators in the Von Neumann algebra generated by the observable.

Theorem 2.2.14. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be maximal distinguishing acting on a seperable Hilbert space H. Then $h' \in [h]_P$ if and only if there exists some $U \in \mathcal{A}_P$ unitary such that $Uh = h'$ and $U^\star P(F)U = P(F)$ for all $F \in \mathcal{F}$.*

Proof. We start with the 'if' part. This part is easy to see as, this follows from the fact that for all $F \in \mathcal{F}$ we have

$$(P(F)h'|h') = (P(F)Uh|Uh) = (U^\star P(F)Uh|h) = (P(F)h|h), \tag{2.2.12}$$

as $U^\star P(F)U = P(F)$, so $h' \in [h]_P$.

Now for the 'only if' part, we use the representation of corollary 1.2.50. Let $h' \in [h]_P$. As P is maximally distinguishing, its associated Von Neumann algebra $\mathcal{A}_P$ is maximally Abelian on a seperable Hilbert space. Now, by use of the notation and result of corollary 1.2.50, we define $f_h := U_{\mathcal{A}_P}h$ and let $f_{h'} := U_{\mathcal{A}_P}h'$. As $f_h : \Omega_{\mathcal{A}_P} \to \mathbb{C}$, we define $R_h : \Omega_{\mathcal{A}_P} \to [0, \infty)$ as its radius and $\Phi_h : \Omega_{\mathcal{A}_P} \to \mathbb{T}$ as its complex phase, where if $\{\omega \mid R_h(\omega) = 0\}$ is measurable, we set $\Phi_h(\omega) = 1$ on this set. Similarly let $f_{h'} = R_{h'}\Phi_{h'}$.

Now let $\phi = \Phi_{h'}\overline{\Phi_h}$ and, as clearly $\phi \in L^\infty(\Omega_{\mathcal{A}_P}, \mu_{\mathcal{A}_P})$ define $U := U^\star_{\mathcal{A}_P}\phi U_{\mathcal{A}_P}$. Now we show that $Uh = h'$. Now recall that as $h' \in [h]_P$, we have that $(P(F)h|h) = (P(F)h'|h')$ and recall that as $P(\mathcal{F})$ generates the Von Neumann algebra $\mathcal{A}_P$, we have that for any $F' \in \mathcal{B}(\Omega_{\mathcal{A}})$ there exists some $F \in \mathcal{F}$ such that $1_{F'} = UP(F)U^\star$. So for any $F' \in \mathcal{B}(\Omega_{\mathcal{A}})$ we have

$$\begin{aligned}\int_{\Omega_{\mathcal{A}_P}} 1_{F'}R_h^2 d\mu_{\mathcal{A}_P} &= \int_{\Omega_{\mathcal{A}_P}} 1_{F'}f_h\overline{f_h}d\mu_{\mathcal{A}_P} = (U_{\mathcal{A}_P}P(F)U^\star_{\mathcal{A}_P}f_h|f_h) = (P(F)h|h) = (P(F)h'|h') \\ &= (U_{\mathcal{A}_P}P(F)U^\star_{\mathcal{A}_P}f_{h'}|f_{h'}) = \int_{\Omega_{\mathcal{A}_P}} 1_{F'}f_{h'}\overline{f_{h'}}d\mu_{\mathcal{A}_P} = \int_{\Omega_{\mathcal{A}_P}} 1_{F'}R_{h'}^2 d\mu_{\mathcal{A}_P}\end{aligned} \tag{2.2.13}$$

and so $R_h^2 = R_{h'}^2$ $\mu_{\mathcal{A}_P}$-almost everywhere and so we have $R_h = R_{h'}$ $\mu_{\mathcal{A}_P}$-almost everywhere. This then gives us

$$Uh = U_{\mathcal{A}_P}\phi U^\star_{\mathcal{A}_P}h = U_{\mathcal{A}_P}\phi f_h = U_{\mathcal{A}_P}\Phi_{h'}\overline{\Phi_h}R_h\Phi_h = U_{\mathcal{A}_P}R_{h'}\Phi_{h'} = U_{\mathcal{A}_P}f_{h'} = h', \tag{2.2.14}$$

as $\overline{\Phi_h}\Phi_h = 1_{\Omega_{\mathcal{A}_P}}$. Furthermore, as $U \in \mathcal{A}_P$, which is Abelian, we have $UP(F) = P(F)U$ for all $F \in \mathcal{F}$ and, lastly, as $\mathrm{Im}(\phi) \subseteq \mathbb{T}$ we have $\sigma(U) \subseteq \mathbb{T}$ and as such U is unitary. This concludes the 'only if' part. ☐

Corollary 2.2.15. *Let $P : \mathcal{F} \to \mathcal{P}(H)$ be some observable. Then $h' \in [h]_P$ if and only if there exists some $U \in \mathcal{A}'_P$ unitary such that $Uh = h'$ and $U^\star P(F)U = P(F)$ for all $F \in \mathcal{F}$.*

Proof. Again the 'if' claim is trivial. For the 'only if' part, we note that as the Von Neumann algebra $\mathcal{A}_P$ is Abelian, it is contained in some maximal Abelian Von Neumann algebra $\mathcal{A}_M$. Then by theorem 1.4.12 and the above theorem there exists some $U \in \mathcal{A}_M$ unitary, such that $Uh = h'$ and $U^\star P_{\mathcal{A}_M}(F)U = P_{\mathcal{A}_M}(F)$ for all $F \in \mathcal{F}_{\mathcal{A}_M}$. Now as $\mathcal{A}_P \subseteq \mathcal{A}_M$, we have that $P(\mathcal{F}) \subseteq P_{\mathcal{A}_M}(\mathcal{F}_{\mathcal{A}_M})$ and so in particular $U^\star P(F)U = P(F)$ for all $F \in \mathcal{F}$. Moreover, as $\mathcal{A}_M \subseteq \mathcal{A}'_P$, we get $U \in \mathcal{A}'_P$. ☐

These results give us a firm grasp on the hidden variables through the unitary operators in their Von Neumann algebras, which can again be described through functional calculus.

2.3 Non-commuting observables

Next, we compare the way in which these assignment functions allow for the combining of two observables. That is, allow for the combination of two observables into a larger observable. We then relate this to the way in which they interact with their respective classes of indistinguishable states.

We again start with the classical case.

Example 2.3.1. *Let $f_1 : X \to \Omega_1$ and $f_2 X \to \Omega_2$ be classical observables. Then we define $f_{1,2} : X \to \Omega_1 \times \Omega_2$ by $f_1, 2(x) = (f_1(x), f_2(x))$. $f_{1,2}$ is measurable for the σ-algebra $\mathcal{F}_1 \times \mathcal{F}_2 := \{F_1 \times F_2 \mid F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. Moreover, we clearly have for $\mu_{A_1}, \mu_{A_2} \in s(M_1^+(X))$ that if $\mu_{A_1} \sim_{f_{1,2}} \mu_{A_2}$, we have both $\mu_{A_1} \sim_{f_1} \mu_{A_2}$ and $\mu_{A_1} \sim_{f_2} \mu_{A_2}$.*

In the classical case, we thus have that every pair of observables acting on the same state space X can be combined into a classical observable which distinguish more or equal the amount of states which made up the combination. Thus each combination of observables only allows us to see (equal or) more of the state space.

In analogy to the classical case, we have for the quantum case that two observables can be combined if they commute.

Definition 2.3.2. ***Commuting quantun observables.*** *Let P_1, P_2 be two quantum mechanical observables. We say these observables commute if all of the projections in their respective images commute.*

Example 2.3.3. *Let $P_1 : \mathcal{F}_1 \to \mathcal{P}(H)$, $P_2 : \mathcal{F}_2 \to \mathcal{P}(H)$ be commuting quantum mechanical observables acting on a separable Hilbert space. Then their images form a set of commuting normal operators and thus by proposition 1.2.38, we get that $\{P_1(\mathcal{F}_1), P_2(\mathcal{F}_2)\}''$ generates an Abelian Von Neumann algebra. As such, by theorem 1.4.12, there exists a projection valued measure $P : \mathcal{F} \to \mathcal{P}(H)$ such that $P_1(\mathcal{F}_1) \subseteq P(\mathcal{F})$ and $P_2(\mathcal{F}_2) \subseteq P(\mathcal{F})$. As such if for $h, h' \in H$ we have $h \sim_P h'$, then both $h \sim_{P_1} h'$ and $h \sim_{P_2} h'$.*

In constract to the classical case such a natural construction only exists in the case of commuting observables. In the case of non-commuting observables the outcome of these observable may accidentally still be describable with some measure on the outcome space, but there exist cases in which the construction of such a measure is explicitly excluded. As a full discussion of this topic would take us to far astray, Landsman's recent work on the foundations of quantum mechanics contains an (excellent) introduction into these results on the exclusion of classical measures describing the outcome space of non-commuting observable [2, chapter 6, p. 191-245] (especially [2, esp. sec 6.5, p. 213-220] where in the discussion of Bell's theorem an explicit case of a set of observables whose resulting measures can not be unified into a single measure is given).

What we wish to expand on here further is the relation between the non-commuting observables and their respective hidden states.

Proposition 2.3.4. *Let $P_1 : \mathcal{F}_1 \to \mathcal{P}(H), P_2 : \mathcal{F}_2 \to \mathcal{P}(H)$ be maximal distinguishing quantum mechanical observables on a separable Hilbert space H. Then P_1 and P_2 commute if and only if they give rise to the same projections, that is $P_1(\mathcal{F}_1) = P(\mathcal{F}_2)$.*

Proof. The 'if' claim is trivial, as the image of a projection valued measure is Abelian. For the only if claim, $\mathcal{A}_{P_1} = \mathcal{A}'_{P_1}$ as P_1 maximal distinguishing and thus $\mathcal{A}_{P_1}$ is maximal Abelian by proposition 2.2.6. Thus $A \in \mathcal{A}_{P_1}$ if and only if it commutes with all projections in $\overline{P_1(\mathcal{F}_1) \text{ by theorem}}$ 1.2.23.

Let $A \in \mathcal{A}_{P_2}$. Now as again by theorem 1.2.50 we have that $\mathcal{A}_{P_2} = \overline{\text{span}_{\mathbb{C}}\{P_2(\mathcal{F}_2)\}}$, we see that $A = \lim_{N\to\infty} \sum_{n=1}^{N} a_n P_2(F_n)$ for some sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ and $(F_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}_2$. Now we thus have for every $F_1 \in \mathcal{F}_1$ that $[A, P_1(F_1)] = \lim_{N\to\infty} \sum_{n=1}^{N} a_n[P_2(F_n), P_1(F_1)] = 0$ by our assumption that P_1 and P_2 commute and thus A commutes with all projections in $P_1(\mathcal{F}_1)$. By the above argument we thus get $A \in \mathcal{A}_{P_1}$. By reversing the roles of P_1, P_2 in this argument, we get $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$.

Our claim now follows from proposition 1.4.7, that is, $P_1(\mathcal{F}_1) = \text{Proj}(\mathcal{A}_{P_1}) = \text{Proj}(\mathcal{A}_{P_2}) = P_2(\mathcal{F}_2)$ as claimed. ☐

Proposition 2.3.5. *Let P_1, P_2 be maximal distinguishing quantum mechanical observables on a separable Hilbert space H with $\dim(H) \geqslant 2$. Then $h_1 \sim_{P_1} h_2$ but $h_1 \nsim_{P_2} h_2$ for some $h_1, h_2 \in s(H)$ if and only if P_1, P_2 fail to commute.*

Proof. If $h_1 \sim_{P_1} h_2$ but $h_1 \nsim_{P_2} h_2$ for some $h_1, h_2 \in s(H)$, then clearly there exists some $P \in P_2(\mathcal{F}_2)$ such that $P \notin P_1(\mathcal{F}_1)$. By the above theorem we then get that P_1, P_2 fail to commute. Conversely, if P_1, P_2 fail to commute, then $P_1(\mathcal{F}_1) \neq P_2(\mathcal{F}_2)$ and so there exists some projection such that $P \in P_2(\mathcal{F}_2)$ but not in $P_1(\mathcal{F}_1)$. Consequently, there exists some unitary $U \in \mathcal{A}_{P_1}$ such that $U \neq cI$ for some $c \in \mathbb{T}$ and U does not commute with P (which will exist as $\dim(H) \geqslant 2$). Then as $U^\star PU - P \neq 0$ (and $U^\star PU + P \neq 0$), there exists some $h \in H$ with $\|h\| = 1$ such that $|\|U^\star PUh\| - \|Ph\|| \neq 0$. Then for h clearly $(P_1(F_1)Uh|Uh) = (P_1(F_1)h|h)$ for all $F_1 \in \mathcal{F}_1$ as $U \in \mathcal{A}_{P_1} = \mathcal{A}'_{P_1}$, but $(PUh|Uh) \neq (Ph|h)$. Thereby, $h \sim_{P_1} Uh$ but $h \not\sim_{P_2} Uh$, as required. □

From the perspective of indistinguishable states, the interpretation of non-commuting observables is thus that they reveal what was previously indistinguishable (and as consequence will also lose the ability to distinguish other states).

2.4 A quantum analogue of the classical assignment function

Recall the equivalence classes $[h]_P$ of section 2.1.2. We now use these equivalence classes to construct a map from the equivalence classes $s_P(H)$ to the measures on outcome space $M_1^+(\Omega)$. We will argue that this map is comparable to classical observable in a certain regard (specified below).

For classical analogue we define the following map

$$\begin{aligned} \pi_P^1 : s(H) &\to s_P(H) \\ [h] &\to [h]_P. \end{aligned} \tag{2.4.1}$$

Using this above function π_1, we can define by

$$\begin{aligned} f_P : s_P(H) &\to M_1^+(\Omega) \\ [h]_P &\to (P(\quad)h|h) \text{ for some } h \in [h]_P, \end{aligned} \tag{2.4.2}$$

a map such that the following diagram commutes

$$\begin{tikzcd} \mathrm{s}(H) \arrow[r, "\pi_P^1"] \arrow[dr, "\Phi_P"'] & s_P(H) \arrow[d, "f_P"] \\ & M_1^+(\Omega). \end{tikzcd}$$

For this construction we then get the following structure for f_P. Let $s^+(H)$ denote the closed set of convex combinations of $s(H)$.

Proposition 2.4.1. *The map f_P is injective and has natural convexity preserving extension to the space of unsharp states $s^+(H)$.*

Proof. For injectivity, we assume that if $f_P([h]_P^1) = f_P([h]_P^2)$ for some $[h]_P^1, [h]_P^2 \in s_P(H)$. In this case we will have $(P(\quad)h_1|h_1) = f_P([h]_P^2) = f_P([h]_P^2) = (P(\quad)h_2|h_2)$ for some $h_1 \in [h]_P^1$ and $h_2 \in [h]_P^2$. As such we however get $h_1 \sim_P h_2$ as then $(P(F)h_1|h_1) = (P(F)h_1|h_1)$ for all $F \in \mathcal{F}$. But then $[h]_P^1 = \pi_P^1([h_1]) = \pi_P^2([h_2]) = [h]_P^2$, showing that f_P is injective.

Now for the convexity of the extension we use the commutation of the above diagram. Let $p_1 + p_2 = 1$ for some $p_1, p_2 \in [0,1]$. We then can simply note that $p_1\Phi_P([h_1]) + p_2\Phi_P([h_2]) = p_1 f_P([h_1]) + p_2 f_P([h_2])$ will hold and so f_P will have a well-defined natural extension to unsharp states, matching the extension of the assignment function Φ_P. □

While the above construction allows only in some more trivial cases for the construction of a true pull-back measure using f_P analogue to the classical assignment function, making it a complete analogue function. (This construction can fail in infinite dimensional cases as orthogonal complements are no longer necessarily closed.) That being said, it does posses some properties justifying its comparison to a classical function. As we have seen in proposition 2.2.3, classical observables are characterized by the fact that (at least theoretically) a classical observable exists such that $[\mu_A]_f = \{\mu_A\}$ for all states $\mu_A \in s(M_1^+(X))$. The function f_P mirrors the classical observable in this regard as, with slight abuse of terminology, we have $[[h]_P]_{f_P} = \{[h]_P\}$, where $[h]_P \sim_{f_P} [h']_P$ if $\Phi_P([h]_P) = \Phi_P([h]_P)$. Moreover, like a classical sharp state, it can be naturally extended to the unsharp measurements. We therefore see this function as the proper classical mechanical analogue to a quantum observable.

2.5 Application to a finite dimensional Hilbert space

As much of this thesis relies for its inspiration on figure 1 presented in Holevo's work, we would like to present the relation of the construction presented above to this idea. This we will do by applying our construction to the setting of a observable with finite distinct outcomes acting on a finite dimensional Hilbert space. Our goal is to, using the theory presented above, construct a geometrical figure such that a quantum mechanical measurement can be understood as the geometrical compression of this figure. This figure can then be understood as a higher dimensional analogue to the famous two dimensional Bloch sphere. See figure 2.1.

The setting of a finite dimensional observable is the following. Let us define our outcome space by $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$, where each $\omega_i, i \in \{1, \ldots, N\}$, denotes an outcome. In this case $\mathcal{F} := \mathcal{B}(\Omega) = \mathscr{P}(\Omega)$, where we write $F_i = \{\omega_i\}$ for each $i \in \{1, \ldots, N\}$. We now assume that our observable is maximal distinguishing. Like in example 1.3.6, in this case our observable is defined by

$$P^N : \{F_1, F_2, ..., F_N\} \to \mathcal{P}(H) \tag{2.5.1}$$

$$F_i \to P(F_i) = f_i \bar{\otimes} f_i \tag{2.5.2}$$

where

$$\begin{aligned} f_i \bar{\otimes} f_i : H &\to H \\ h &\to (h|f_i) f_i \end{aligned} \tag{2.5.3}$$

and f_i is some orthonormal basis of the finite dimensional Hilbert space H.

As we have only finitely many measurement outcomes, the space of all possible probability distributions over these measurement outcomes can be described as the n-simplex $\mathbf{S}_n := \{\mathbf{p} := (p_1, p_2, ..., p_N) \mid 0 \leqslant p_1, p_2, ..., p_N \leqslant 1, \sum_{i=1}^{n} p_i = 1\}$, where p_1 represents the probability of finding ω_1, p_2 the probability of finding ω_2, etc.. Onto this space our assignment function now becomes

$$\begin{aligned} \Phi_{P^N} : s(H) &\to \mathbf{S}_n \\ [h] &\to \mathbf{p} = ((P(F_1)h|h), (P(F_2)h|h), ..., (P(F_n)h|h)). \end{aligned} \tag{2.5.4}$$

Now as each projection $P(F_i)$ with $i \in \{1, 2, \ldots, N\}$ is of range one and $(f_i)_{i \in \{1,2,\ldots,N\}}$ forms an orthonormal basis the convexity of simplex will match the linearity on the set of states. That is, $\sum_{i=1}^{\infty} a_i f_i \in s(H)$, then $\Phi_{P^N}(\sum_{i=1}^{\infty} a_i f_i) = (|a_1|^2, |a_2|^2, \ldots, |a_N|^2)$. Therefore, we get for $\sum_{i=1}^{\infty} a_i f_i, \sum_{i=1}^{\infty} b_i f_i \in s(H)$, that $\sum_{i=1}^{\infty} a_i f_i \sim_{P^N} \sum_{i=1}^{\infty} b_i f_i$ if and only if $(|a_1|^2, |a_2|^2, \ldots, |a_N|^2) = (|b_1|^2, |b_2|^2, \ldots, |b_N|^2)$.

Using this identification, we can write out the 'shape' of the space of indistinguishable states. We will use theorem 2.2.14. Now by noting that the Von Neumann generated P^N is unitarily equivalent to the multiplication algebra on $D_(\mathbb{C})$ and as such the unitaries are isomorphic to $\mathbb{T}^N$ (with point-wise multiplication). That is, the unitaries in the Von Neumann generated P^N are given by the set $\mathcal{U}_P := \{\sum_{i=1}^{N} c_i P(F_i) \mid c_i \in \mathbb{T}, \forall i \in \{1, ..., n\}\}$ and this bijection is given by

$$\begin{aligned} \phi : \mathbb{T}^N &\to \mathcal{U}(\mathcal{A}_P) \\ (c_1, c_2, \ldots, c_N) &\to \sum_{i=1}^{N} c_i P(F_i) \end{aligned} \tag{2.5.5}$$

Now as we have to account for the equivalence of states due to the global phase, we get that the unitaries changing the state are isomorphic to $\mathbb{T}^N / \mathbb{T} = \mathbb{T}^{N-1}$. Furthermore, we can track whether a unitary operator changes a given state or not. That is, for $\sum_{i=1}^{\infty} a_i f_i \in s(H)$ with $a_j = 0$ for some $j \in \{1, 2, \ldots, N\}$, then this coordinate is in a sense 'not affected' by $c_j P(F_j)$ for c_j. Combining all these insight we define any ordered set $(i_1, i_2, \ldots, i_k)$, with $i_1, \ldots i_k \in \{1, \ldots, N-1\}$ and $i_1 < i_2 < \cdots < i_k$ for some $k \leqslant N-1$

$$\begin{aligned} \xi_{(i_1, i_2, \ldots, i_k)} : \mathbb{T}^k &\to \mathbb{T}^{N-1} \\ (\alpha_1, \alpha_2, \ldots, \alpha_k) &\to \begin{cases} \alpha_{i_j} = \alpha_j, & j \in \{1, \ldots, k\} \\ 0, & \text{else,} \end{cases} \end{aligned} \tag{2.5.6}$$

such that for example if $N-1=3$, then $\xi_{(1,3)}(i,-1)=(i,0,-1)$. Let $\mathbf{S}_n$ be the n-simplex as above and let $\partial\mathbf{S}_n$ be its boundary. Using this function we can then construct the shape of the state space with the indistinguishable states attached to points on the simplex:

- if Φ_{P^N} maps h to a vertex (a 0-face) (p_i), $i \in \{1,...,n\}$, then $\Psi_{[h]_{P^n}}([h]_{P^N}) = \{[h]\}$.
- if Φ_{P^N} maps h to an edge (a 1-face) (p_i,p_j), with $i,j \in \{1,...,n\}$ and $i<j$, then $\Psi_{[h]_{P^n}}([h]_{P^N}) \cong \xi_i(\mathbb{T})$.
- if Φ_{P^N} maps h to a face (a 2-face) (p_i,p_j,p_k), with $i,j,k \in \{1,...,n\}$ and $i<j<k$, then $\Psi_{[h]_{P^n}}([h]_{P^N}) \cong \xi_{(i,j)}(\mathbb{T}^2)$.
- ...
- if Φ_{P^N} maps h to an (n-1)-face $(p_{i_1},p_{i_2},\dots,p_{i_{n-1}})$, with $i_j \in \mathcal{I} \subseteq \{1,2,\dots n\}\backslash\{l\}$ some ordered labelling, then in $\Psi_{[h]_{P^n}}([h]_{P^N}) \cong \xi_{\mathcal{I}}(\mathbb{T}^{N-2})$.
- if Φ_{P^N} maps h to $\mathbf{S}_n\backslash\partial\mathbf{S}_n$, then $\Psi_{[h]_{P^n}}([h]_{P^N}) \cong \mathbb{T}^{N-1}$.

While this approach may seem somewhat convoluted, its outcomes are actually quite concrete: from the construction we can recover the shape of the bloch sphere.

Example 2.5.1. ***A qubit.*** *The space of hidden variables of a maximal distinguishing two dimensional Hilbert space, that is for a spin-$\frac{1}{2}$ particle or a qubit, is congruent to a sphere. That is a line segment, with circles on its non-end points. See figure 2.1.*

Moreover, we can use the above result to speculate the shape a higher dimensional geometric representation should take.

Example 2.5.2. ***Two qubits.*** *The space of hidden variables of a maximal distinguishing four dimensional Hilbert space, that is for two spin-$\frac{1}{2}$ particles or two qubits by $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$, is congruent to the figure constructed above, that is a tetrahedron $\mathbf{S}_4$, with circles on the vertices, spheres on the faces and 3-spheres on the centre points from an object in 7-dimensional space.*

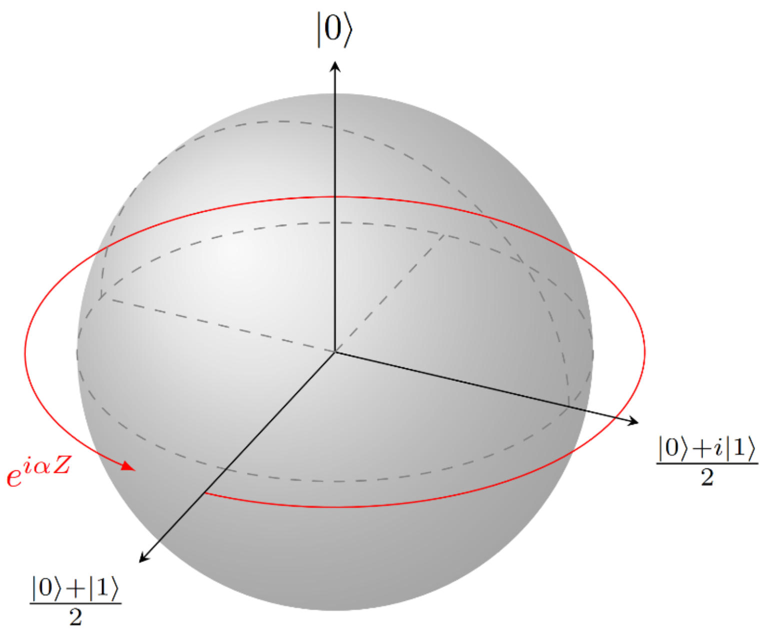

Figure 2.1: Graphic representation of the hidden states of the projection valued measure associated to the Pauli-σ_Z operator. When projecting on the set of states, represented by the Bloch sphere, along its central axis by applying this projection valued measure, we lose the ability to distinguish its original location along the vertical circle. Except for the north and south poles of the sphere, where there is no information lost, the space of indistinguishable states is thus congruent to a circle $S^1 \cong \mathbb{T}$.

Conclusion

Our comparison between quantum and classical mechanics, was here motivated by the abstract structure we expect a measurement to take. That is, a measurement can abstractly be understood a function assigning states and observables to different measures on the relevant space of outcomes. We have called these functions the assignment functions. This then tied in to our original 'program of Bohrification' as exactly the projection valued measures, which, on a separable Hilbert space, stand in a direct correspondence to commutative algebras as shown in theorem 1.4.12, form the basis of the quantum mechanical assignment function. By applying the the projection valued measure to a state, we gain a probability measure on the set of outcomes predicting recorded frequencies of events. As such we can, now retroactively, argue that indeed the physically relevant aspects of the non-commutative operator algebras of quantum theory are accessible only through their commutative subalgebras.

In order to then compare quantum mechanics and classical mechanics, we have compared the structure of their relevant assignment functions. The quantum mechanical assignment function, as we have seen, differs from a classical function with respect to the fact that classical functions allow for an assignment function in which the observable assigns a different outcome measure to each classical state, which in the (non-trivial) quantum mechanical case is not possible. Even quantum mechanical observables which hide the maximal amount of states, have a set of indistinguishable states associated to their assignment function. The quantum mechanical assignment function, in contrast to the classical one, thus hides states.

How then does this difference help us to understand the non-commutative aspect of quantum mechanics? While we have been mostly concerned with the commutative aspect of the projection valued measure, it is exactly the fact that each quantum mechanical assignment function hides states that allows for a non-commutative physics theory. We can see this point in an argument from the absurd. If there was only one quantum mechanical experiment, then one maximal distinguishing quantum mechanical observable would give rise to all the possible outcome measures. As a consequence, the indistinguishable states of the observable would form a theoretical excess or residue. That is, as a shorter, more concise theory needing less states would exist (possibliy constructed using our classical analogue of equation (2.4.2)), we would, by an occam's razor type argument, simply argue for a theory predicting the same outcomes, but assigning these measures directly to their states (exactly as the classical assignment function does). However, exactly as different two quantum observables only allow for a natural combination extending the two observables if they commute, such an observable does not exsist. In some rarer cases - as the theory on hidden variables has shown - non-commuting observables even exclude the formation of a classical measure all together. It is exactly as one quantum mechanical observable (or rather one classical observable) cannot tell the complete tale that quantum mechanics is non-commutative. The non-commutative observables can thus exactly be understood in classical terms due to the fact that different measurements hide and detect different sets of states.

As a closing note, we would like to point in some directions for further research. Firstly, we like to note the possibility for some technical improvement of some of the proofs presented. In particular, the prove of theorem 2.2.14, makes use of the representation of a Von Neumann algebra as a multiplication algebra, but the approach of Chapter 1, with theorem 1.4.12 as a main result, seems to suggest that more direct approach should be possible. Moreover, the result of theorem 1.4.12 makes use of Von Neumann's theorem showing that an Abelian Von Neumann algebra acting on a separable Hilbert space is singly generated. As an Abelian Von Neumann algebra acting on a separable Hilbert space has a countable strongly dense set of projections an opening for a direct proof of the existence of suitable projection valued measure using Kolmogorov's extension theorem seems possible, improving the directness of the obtained result. Lastly, there is the possibility of the integration of symmetries into the argument. The relation between the hidden states and the unitaries in (the commutant of) its Von Neumann algebra (which have natural conneection to conserved symmetries) seems to point in a direction deeply linked with the specific use of representation theory in quantum mechanics (the repesentation of local Abelian groups as in [1, chapter 15]), but this is, as of yet, insufficiently understood.

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