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MULTISCALE ANALYSIS FOR TRAVELING-PULSE SOLUTIONS TO THE STOCHASTIC FITZHUGH–NAGUMO EQUATIONS

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We investigate the stability of traveling-pulse solutions to the stochastic FitzHugh–Nagumo equations with additive noise. Special attention is given to the effect of small noise on the classical deterministically stable fast traveling pulse. Our method is based on adapting the velocity of the traveling wave by solving a scalar stochastic ordinary differential equation (SODE) and tracking perturbations to the wave meeting a system of a scalar stochastic partial differential equation (SPDE) coupled to a scalar ordinary differential equation (ODE). This approach has been recently employed by Krüger and Stannat (*Nonlinear Anal.* **162** (2017) 197–223) for scalar stochastic bistable reaction–diffusion equations such as the Nagumo equation. A main difference in our situation of an SPDE coupled to an ODE is that the linearization has essential spectrum parallel to the imaginary axis and thus only generates a strongly continuous semigroup. Furthermore, the linearization around the traveling wave is not self-adjoint anymore, so that fluctuations around the wave cannot be expected to be orthogonal in a corresponding inner product. We demonstrate that this problem can be overcome by making use of Riesz instead of orthogonal spectral projections as recently employed in a series of papers by Hamster and Hupkes in case of analytic semigroups. We expect that our approach can also be applied to traveling waves and other patterns in more general situations such as systems of SPDEs with linearizations only generating a strongly continuous semigroup. This provides a relevant generalization as these systems are prevalent in many applications.

CONTENTS

1. Introduction.

1.1. *The stochastic FitzHugh–Nagumo equations.* We consider the *stochastic FitzHugh–Nagumo equations*

(1.1a) $\mathrm{d}\tilde{u}(t, x) = \left(\nu\partial_x^2\tilde{u}(t, x) + f\left(\tilde{u}(t, x)\right) - \tilde{v}(t, x)\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t, x)$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

(1.1b) $\mathrm{d}\tilde{v}(t, x) = \varepsilon\left(\tilde{u}(t, x) - \gamma\tilde{v}(t, x)\right)\mathrm{d}t$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

in which the independent variables t and x denote time and position on a neural axon, respectively. The dependent variable $\tilde{u}$ denotes the *electric potential* and $\tilde{v}$ is a *gating variable*. The parameter $\nu > 0$ determines the strength of the *diffusion* $\partial_x^2\tilde{u}$, while the nonlinearity $f(\tilde{u})$ is a reaction term which typically has the form $f(\tilde{u}) = \chi(\tilde{u})\tilde{u}(1-\tilde{u})(\tilde{u}-a)$ with $0 < a < 1$ and is suitably cut off by the factor χ. The parameter $\sigma > 0$ determines the strength of the noise W. Here, we assume that W is an infinite-dimensional Wiener process taking values in a Hilbert space to be specified in what follows. The parameter $\varepsilon > 0$ determines the strength of the coupling of the electric potential $\tilde{u}$ to the gating variable $\tilde{v}$ and is assumed to be sufficiently small. The parameter $\gamma > 0$ determines the decay of the gating variable $\tilde{v}$.

The *classical FitzHugh–Nagumo* [34, 73] *partial differential equations* (PDEs) obtained for $\sigma = 0$ form a simplified, yet qualitatively very similar, model for the *Hodgkin–Huxley equations* [45], which was a key part of Hodgkin's and Huxley's Nobel prize awarded in 1963. By now, the FitzHugh–Nagumo system is a standard model for the generation and transmission of electrical signals in neuroscience [26, 47]. The literature on the PDE version of the FitzHugh–Nagumo system ($\sigma = 0$) is very large; see, for example, the recent papers [15, 38] and detailed references therein. For the ODE version ($\sigma = 0$, $\nu = 0$), the literature is vast [76], mostly due to the crucial role played by bistable nonlinearities in all areas of nonlinear science and the commonly found multiple time scale structure of the FitzHugh–Nagumo system [58]. Also the SODE variant for $\nu = 0$ is quite well studied, mainly due to a flurry of activity since the mid 1990s; see, for example, [3, 7, 9, 63, 64, 72]. Yet, the full SPDE variant (1.1) has only attracted major attention quite recently, including a large number of numerical studies [67, 80, 81, 83, 86–88] as well as analytical studies regarding existence, regularity, invariant measures and attractors [4, 8, 12, 61, 62, 90]. The question regarding stochastic stability of pulses for additive noise is far less studied. We refer to [41] for multiplicative noise with a regularized equation (diffusion in the second variable) and to the review [60] for the stochastic Nagumo case ($\varepsilon = 0$). For further biophysical motivation regarding various noise terms in the FitzHugh–Nagumo equation we refer to the review [63].

Here we contribute to a more detailed understanding of stochastic pulse stability of the FitzHugh–Nagumo equations (1.1) exploiting the *multiscale* nature of the problem. More precisely, our aim is to understand the dynamics of (1.1) near a deterministically stable pulse in the regime, where the parameters $\sigma > 0$ (strength of the noise) and $\varepsilon > 0$ (coupling to the gating variable) are small. The subsequent analysis generalizes the recent analysis of Krüger and Stannat [55] for corresponding scalar stochastic bistable reaction–diffusion equations

such as the Nagumo equation. We remark that the idea of tracking small noise fluctuations for SPDEs around traveling waves via a multiscale SODE approximation goes back at least to the early 1980s and works by Ebeling, Mikhailov, and Schimansky–Geier [70, 71]. From the viewpoint of applications the extension from Nagumo to FitzHugh–Nagumo is a crucial generalization as the FitzHugh–Nagumo model (1.1) is far more realistic than the one studied in [55] since only (1.1) allows for deterministically stable *traveling-pulse solutions*. In fact, deterministically stable localized pulses model far better the real action potentials generated in neurons in comparison to the deterministically stable traveling fronts appearing in the Nagumo equation. From a mathematical viewpoint, our generalization is important as it does extend beyond the setting of treating the equation in a Hilbert space in which the linearization of the traveling wave is self-adjoint (this is the case in [55]) or generates an analytic semigroup and thereby makes the methods of multiscale approximation available to a broad class of stochastic SPDE-ODE reaction–diffusion systems.

1.2. *Traveling-pulse solutions and their stability.*   We recall some of the well-known results on existence of traveling-pulse solutions to the deterministic version of (1.1) where $\sigma = 0$. These solutions have the form $\tilde{u}(t, x) = \hat{u}(\xi)$ and $\tilde{v}(t, x) = \hat{v}(\xi)$, where $\xi = x + st$ and $s \in \mathbb{R}$ is the velocity of the traveling wave. The tuple $(\hat{u}, \hat{v})$ therefore fulfills the set of equations

$$(1.2a) \qquad \frac{d\hat{u}}{d\xi} = \hat{u}' \quad \text{for } \xi \in \mathbb{R},$$

$$(1.2b) \qquad \frac{d\hat{u}'}{d\xi} = \frac{1}{\nu}(s\hat{u}' - f(\hat{u}) + \hat{v}) \quad \text{for } \xi \in \mathbb{R},$$

$$(1.2c) \qquad \frac{d\hat{v}}{d\xi} = \frac{\varepsilon}{s}(\hat{u} - \gamma\hat{v}) \quad \text{for } \xi \in \mathbb{R}.$$

Traveling-pulse solutions to (1.2) are solutions $(s, \hat{u}, \hat{v})^{\mathsf{t}}$ such that $(\hat{u}, \hat{u}', \hat{v})^{\mathsf{t}} \to (b_1, b_2, b_3)^{\mathsf{t}}$ as $\xi \to \pm\infty$ where $(b_1, b_2, b_3)^{\mathsf{t}} \in \mathbb{R}^3$ is a stationary solution to (1.2) fulfilling

$$b_2 = 0, \qquad f(b_1) = b_3 \quad \text{and} \quad b_1 = \gamma b_3,$$

that is, they are homoclinic orbits of the dynamical system (1.2). In what follows we will also have the convention that we mean nontrivial traveling pulses. Furthermore, we are only interested in solutions $(\hat{u}, \hat{v})^{\mathsf{t}}$ such that $(\hat{u}, \hat{v})^{\mathsf{t}} \to (0, 0)^{\mathsf{t}}$ as $\xi \to \pm\infty$. Indeed, this situation will occur if the equilibrium point $(b_1, b_2, b_3)^{\mathsf{t}} = (0, 0, 0)^{\mathsf{t}}$ is unique, which is the case, for example, if $\gamma \geq 0$ is sufficiently small [76]. Further note that the velocity s of the traveling wave is not a parameter but a functional of f and ε.

Next, we give a very brief overview of the existing literature on the existence of homoclinic orbits of (1.2) corresponding to traveling-pulse solutions. For a wave speed $s = 0$ and $\varepsilon = 0$, we recover the planar ODE associated to traveling waves of the Nagumo equation. Using the resulting Hamiltonian structure of the ODE, it is easy to see that a homoclinic orbit exists for $s = 0$ and $\varepsilon = 0$. A singular perturbation argument in combination with Melnikov's method [44, 58, 85] yields the existence of a *slow pulse* with wave speed $s \approx 0$. Yet, an application of the Sturm–Liouville theory [52, 59] shows that the slow pulse is unstable. As it is deterministically already unstable, considering this pulse under the influence of noise is not expected to be biophysically relevant as the noisy small perturbations will be amplified exponentially near the slow pulse. Yet, there is also a *fast pulse* corresponding to much higher wave speeds $s = s(f, \varepsilon)$. Carpenter [13] and Conley [21] constructed these homoclinic orbits to (1.2) employing the method of isolating blocks of the fast and slow subsystems [20]. See

also [35], where the methods developed in [21] have been further improved. Then a generalization of the pulse construction to large classes of FitzHugh–Nagumo-like models has been provided in [33]. Later, a fully geometric construction of the fast pulse via the exchange lemma [49, 50, 58] was proved by Jones, Kopell, and Langer [51] using differential forms. The connection in parameter space between slow and fast pulses has been proved in [56]. Then the parametric bifurcation structure has been analyzed in a more refined way in [15, 16, 38, 39]. A nonperturbative approach to construct traveling-pulse solutions of (1.2) has been carried out by Arioli and Koch [2] for the value $\varepsilon = 0.01$ using computer-assisted proofs.

Stability of the fast traveling-pulse solutions to the deterministic version of (1.1) with $\sigma = 0$ in the space of bounded uniformly continuous functions has been obtained by Jones [48] for the case of a cubic polynomial, in the sense that solutions starting sufficiently close to a wave profile decay to a translate of

$$\left(\hat{u}(\cdot + st), \hat{v}(\cdot + st) \right)^{\mathsf{t}}.$$

The proof relies on analysis developed by Evans [27–32], where it is proved that in the space of bounded uniformly continuous functions that linear stability implies nonlinear stability and that the point spectrum of the linearization around the traveling pulse is determined by the roots of a function $D(\lambda)$, the *Evans function*. Jones proves that instability can only occur due to eigenvalues of the linear operator near $\lambda = 0$. By calculating the winding number of $D(\lambda)$ for a small circle around $\lambda = 0$, it is shown that only two eigenvalues lie in it, one of which is $\lambda = 0$ (related to the translation invariance of the problem) and the other is negative because $\frac{dD}{d\lambda}(0) > 0$. For a more recent stream-lined stability analysis allowing for more general reaction terms f but still restricted to the space of bounded uniformly continuous functions, we refer to [2, 91] while stability in $L^2(\mathbb{R}; \mathbb{R}^2)$ and $H^1(\mathbb{R}; \mathbb{R}^2)$ is proved in [36, 77, 92] (see Section 2.2 for further details). Nonlinear stability of the fast FitzHugh–Nagumo pulse with oscillatory tails (instead of monotone tails as considered in this paper) has been proved by Carter, de Rijk, and Sandstede in [14]. For general introductions into the subject, we refer to [19, 79].

1.3. *An approach for computing the velocity correction.*

Our aim is to investigate the following decomposition:

$$(1.3a) \qquad \tilde{u}(t, x) = \hat{u}\big(x + st + \varphi(t)\big) + u_\varphi(t, x),$$

$$(1.3b) \qquad \tilde{v}(t, x) = \hat{v}\big(x + st + \varphi(t)\big) + v_\varphi(t, x)$$

of solutions to (1.1), where the function $\varphi(t)$ is a random correction to the position of the wave front, and $u_\varphi(t, x)$ and $v_\varphi(t, x)$ denote lower-order fluctuations that are uniquely defined through (1.3) for any choice of $\varphi = \varphi(t)$. Ideally, we would like to choose φ to minimize the distance in the direction of the traveling wave between the solution $\tilde{X} := (\tilde{u}, \tilde{v})^{\mathsf{t}}$ of the FitzHugh–Nagumo SPDEs (1.1) and the suitably translated traveling wave $\hat{X} = (\hat{u}, \hat{v})^{\mathsf{t}}$, that is,

$$(1.4) \qquad \varphi(t) \in \underset{\varphi \in \mathbb{R}}{\operatorname{argmin}} \big\| \Pi^0_{st+\varphi} \big(\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \varphi) \big) \big\|^2_H \quad \text{with } \tilde{X} := \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \hat{X} := \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix},$$

where $\|\cdot\|_H$ is the norm in the spatial variable of a suitable underlying Hilbert space H and $\Pi^0_{st+\varphi}$ is a suitable projection operator onto the traveling wave such that $\Pi^0_{st+\varphi} \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi) = \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi)$ (see Proposition 2.8(d) and (3.1) further below). However, as the minimization problem (1.4) is not necessarily convex, uniqueness of a minimizer is not ensured.

We follow the approach in [55] and replace (1.4) by the weaker condition for a critical point of finding $\varphi = \varphi(t)$ such that

$$(1.5) \qquad 0 = \left(\Pi_{st+\varphi}^0 (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \varphi)), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi) \right)_H,$$

where $(\cdot, \cdot)_H$ denotes the inner product of H. This approach has been employed by Inglis and MacLaurin in [46] for more general classes of SPDE systems with the drawback that results only hold up to the first stopping time when the local minimum turns into a saddle. Here, we follow the work around proposed in [55] in the sense that $\varphi(t)$ is approximated by a process $\varphi^m(t)$, which in our case fulfills the random ordinary differential equation (RODE)

$$(1.6) \qquad \frac{d\varphi^m}{dt}(t) = m \left(\Pi_{st+\varphi^m(t)}^0 (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \varphi^m(t))), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi^m(t)) \right)_H$$

for given initial condition and a relaxation parameter $m > 0$ that is chosen sufficiently large. Notably, approximating φ with φ^m through (1.6) implies differentiability while this is in general not true for (1.5). By further analyzing (1.6) we will construct φ as a solution of an SODE and $(u_\varphi, v_\varphi)^{\mathsf{t}}$ as the solution of a system of an SPDE coupled to an SODE in such a way that $(u_\varphi, v_\varphi)^{\mathsf{t}}$ fulfill estimates in suitable function spaces, making precise what the notion "lower order" means.

In [55], Section 3.4, it has been pointed out that with the strategy described above the first-order fluctuations around the traveling wave are orthogonal in their suitably chosen inner product. In fact, in the case of second-order bistable reaction–diffusion equations, the spatial linearization around the traveling wave, the *frozen-wave operator*, is of Sturm–Liouville type. Hence, one can always find a weighted L^2-inner product in which it is self-adjoint, so that one can replace the projection operator used in (1.4) with an orthogonal projection induced by this inner product. One cannot expect a similar approach to be applicable in our situation (1.1) of an SPDE coupled to an ODE or other more complicated systems of SPDEs. Instead, we will build up our analysis on Riesz spectral projections of the frozen-wave operator that do not require a self-adjoint structure and yield a partition into two subspaces invariant under the linearized flow. Note that nonorthogonal Riesz spectral projections have also been employed by Hamster and Hupkes in [40–43] by projecting onto the eigenvector of the adjoint of the frozen-wave operator. Their SODE to determine φ is more involved compared to (1.6). Their method applies to a variety of (systems) of reaction–diffusion equations with special forms of multiplicative noise (partially only including a one-dimensional Wiener process). However, in all situations treated there, the frozen-wave operator is sectorial with spectral angle larger than $\frac{\pi}{2}$ and therefore generates an analytic semigroup. Specifically, in [41], (1.2), [42], (1.3), [43], (1.17), the second component of the FitzHugh–Nagumo system is regularized by adding $c\partial_x^2 \tilde{v}$ with some $c > 0$ to the right-hand side of (1.1b) (see [17, 22] for existence and stability of the pulse in this case), which is different from the only partly parabolic FitzHugh–Nagumo system without diffusion in the second component as treated for instance in [2, 13, 21, 35, 48, 51, 91]. In our setting, the frozen-wave operator has essential spectrum parallel to the imaginary axis (see Section 2.2 and Appendix A.2 below) and therefore is so far not covered by this approach.

Historically, one can track back stability and fluctuation analysis of bistable scalar SPDEs, such as the Nagumo SPDE, at least to early works by Ebeling, Mikhailov, and Schimansky–Geier in [70, 71], where it was recognized that the deterministic reference wave speed should be corrected by a stochastic term, which in turn satisfies an SODE. Many further works followed, for example, using a more rigid/frozen stochastic frame in combination with numerical simulations by Lord and Thümmler in [68], employing functional inequalities to establish stability bounds by Stannat in [84], the adaptation of a rigorous multiscale expansion by

Krüger and Stannat in [55] originally motivated by work on the related problem of stochastic traveling waves for bistable neural field equations by the same authors in [54] and by Inglis and MacLaurin in [46], and work on long-time stochastic stability tracking for suitable multiplicative noise by Hamster and Hupkes in [40] via stochastic convolution estimates by the sameauthors in [41]. We also remark that in case of stochastic dispersive PDEs, the random modulation of soliton solutions (i.e., standing or traveling wave packages) has been investigated by de Bouard and Debussche for the stochastic Korteweg–de Vries equation in [24] and by de Bouard and Fukuizumi for the Gross–Pitaevskii equation in [25]. Furthermore, multiscale expansions also frequently appear in the context of SPDE amplitude equations at bifurcation points as treated for instance by Blömker in [10] and by Blömker, Hairer, and Pavliotis in [11].

1.4. *Outline.* We continue with the setting and auxiliary results in Section 2, followed by the main results in Section 3. The proofs of auxiliary results are contained in the Appendix while the proofs of the main results can be found in Section 4. In Section 2.1 and Appendix A.1 we construct solutions to (1.1) using the variational approach for equations with locally monotone coefficients [65, 66]. In Section 2.2 and Appendix A.2, we then formulate the SPDE of perturbations around the traveling wave which to leading-order is governed by the linearization around the deterministically-stable fast FHN pulse. Results on the deterministic linearized evolution of perturbations around the traveling wave are provided in Proposition 2.6 and Proposition 2.8 below. Afterwards, in Section 3.1 and Section 4.1, we derive an SODE approximating the correction of the wave velocity (cf. Proposition 3.2 below). The leading-order part of this SODE is an Ornstein–Uhlenbeck-type process with a linear damping in the drift due to the relaxation method of the frame and additive stochastic fluctuations obtained from projecting the infinite-dimensional noise onto the deterministic translation-invariant mode. Practically, this entails that the deterministic reference wave has phase diffusion along the translation direction. Subsequently, in Section 3.2 and Section 4.2, we prove a multiscale expansion in terms of the linearized evolution (cf. Theorem 3.3 below), which is further investigated in Section 3.3 and Section 4.3 in the limit as $m \to \infty$ of immediate relaxation (cf. Theorem 3.4 and Proposition 3.5 below). In particular, our results yield bounds on:

- the first exit time where the multiscale decomposition cannot be guaranteed to hold anymore and
- on the second moment of fluctuations transverse to the traveling wave mode after correcting the wave velocity.

Concluding remarks and an outlook on future research can be found in Section 5.

2. Setting and auxiliary results. For what follows, we fix a *stochastic basis*, that is, a *complete filtered probability space*

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}\right),$$

with a complete and right-continuous filtration $(\mathcal{F}_t)_{t \in [0,T]}$, where $T \in (0, \infty)$ is arbitrary.

2.1. *Existence and uniqueness of solutions using the variational approach.* In line with the classical findings for the deterministic FitzHugh–Nagumo PDE discussed above, we make the following assumption on (1.1) or (1.2), respectively. In particular, we are only going to study stochastic perturbations to the deterministically stable fast pulse solution.

GLOBAL ASSUMPTION 2.1.   In a right-neighborhood of $\varepsilon = 0$ the system (1.2) has a nontrivial homoclinic orbit of the point $(0, 0, 0)^{\mathsf{t}}$ in phase space. The corresponding traveling-pulse solution $(\hat{u}, \hat{v})^{\mathsf{t}}$ of the FitzHugh–Nagumo equations (1.1) with $\sigma = 0$ is locally asymptotically stable up to translation.

We will assume certain properties on the reaction term $f(w)$ that are fulfilled for instance by the choice $f(w) = \chi(w)w(1 - w)(w - a)$, where $a \in (0, 1)$ and $\chi \in C^\infty(\mathbb{R})$ meets $\chi_{|[-c_1, \infty)} \equiv 1$ and $\chi(w) = \frac{c_2^2}{w^2}$ for $w \in (-\infty, -c_2]$, where $1 < c_1 < c_2$ sufficiently large (cutoff at $-\infty$), for which existence of a traveling-pulse solution (cf. Global Assumption 2.1) is guaranteed provided $\varepsilon > 0$ is small. Here, we directly make the usual abstract bi-stability assumptions [18, 55, 91] for f so that the nonlinearity effectively behaves like the classical cubic nonlinearity chosen for the Nagumo and FitzHugh–Nagumo equations.

GLOBAL ASSUMPTIONS 2.2.   We have $f \in C^3(\mathbb{R})$ and there exist $a \in (0, 1)$ and $d > 0$ with

$$(2.1a) \qquad f(0) = f(a) = f(1) = 0,$$

$$(2.1b) \qquad f(w) < 0 \quad \text{for } w \in (0, a) \cup (1, \infty),$$

$$(2.1c) \qquad f(w) > 0 \quad \text{for } w \in (-\infty, 0) \cup (a, 1),$$

$$(2.1d) \qquad f'(0) < 0, \qquad f'(a) > 0, \qquad f'(1) < 0,$$

$$(2.1e) \qquad f(w) - \frac{w}{d} \neq 0 \quad \text{for } w \in \mathbb{R} \setminus \{0\},$$

$$(2.1f) \qquad \int_0^1 f(w)\, dw > 0,$$

$$(2.1g) \qquad \eta_1 := \sup_{w \in \mathbb{R}} f'(w) < \infty,$$

$$(2.1h) \qquad \big| f(w_1 + w_2) - f(w_1) - f'(w_1)w_2 \big| \leq \eta_2 \big(1 + |w_1| + |w_2|\big)|w_2|^2$$
$$\text{for } w_1, w_2 \in \mathbb{R},$$

$$(2.1i) \qquad \big| f'(w) \big| \leq \eta_3 \big(1 + |w|^2\big) \quad \text{for } w \in \mathbb{R},$$

$$(2.1j) \qquad \big| f(w_1) - f(w_2) \big| \leq \eta_4 |w_1 - w_2| \big(1 + |w_1|^2 + |w_2|^2\big)$$
$$\text{for } w_1, w_2 \in \mathbb{R},$$

$$(2.1k) \qquad \big| f'(w_1 + w_2) - f'(w_1) - f''(w_1)w_2 \big| \leq \eta_5 |w_2|^2$$
$$\text{for } w_1, w_2 \in \mathbb{R},$$

$$(2.1l) \qquad \big| f'(w_1) - f'(w_2) \big| \leq \eta_6 |w_1 - w_2| \big(1 + |w_1| + |w_2|\big)$$
$$\text{for } w_1, w_2 \in \mathbb{R},$$

$$(2.1m) \qquad \big| f''(w_1) - f''(w_2) \big| \leq \eta_7 |w_1 - w_2| \quad \text{for } w_1, w_2 \in \mathbb{R},$$

where $\eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7 < \infty$.

Note that the linearization of (1.2) in $(0, 0, 0)^{\mathsf{t}}$ only depends on $f'(0)$ and because of (2.1d) this point is hyperbolic and hence $\frac{d^j \hat{u}}{d\xi^j}$ for $j \in \{0, 1, 2, 3, 4, 5\}$ and $\frac{d^j \hat{v}}{d\xi^j}$ for $j \in \{0, 1, 2, 3, 4\}$ are exponentially decaying as $|\xi| \to \pm\infty$. In particular,

$$\left\| \frac{d^j \hat{u}}{d\xi^j} \right\|_{L^\infty(\mathbb{R})} < \infty \quad \text{and} \quad \left\| \frac{d^j \hat{u}}{d\xi^j} \right\|_{L^2(\mathbb{R})} < \infty \quad \text{for all } j \in \{0, 1, 2, 3, 4, 5\},$$

as well as

$$\left\|\frac{d^j \hat{v}}{d\xi^j}\right\|_{L^\infty(\mathbb{R})} < \infty \quad \text{and} \quad \left\|\frac{d^j \hat{v}}{d\xi^j}\right\|_{L^2(\mathbb{R})} < \infty \quad \text{for all } j \in \{0, 1, 2, 3, 4\}.$$

For $\varphi \equiv 0$, we may use (1.3) as a definition of $(u, v) := (u_0, v_0)$ and employing (1.2), we obtain the system

$$
\begin{aligned}
(2.2a) \quad du(t, x) &= \left(\nu\partial_x^2 u(t, x) + f\left(u(t, x) + \hat{u}(x + st)\right) - f\left(\hat{u}(x + st)\right) - v(t, x)\right) dt \\
&\quad + \sigma\, dW(t, x),
\end{aligned}
$$

$$(2.2b) \quad dv(t, x) = \varepsilon\left(u(t, x) - \gamma v(t, x)\right) dt.$$

In order to analyze (2.2), we use the following Hilbert space setting. Suppose that $L^2(\mathbb{R})$ is the standard L^2-space of $\mathbb{R}$-valued functions on the real line,

$$H^1(\mathbb{R}) := \{u \in L^2(\mathbb{R}) : \partial_x u \in L^2(\mathbb{R})\},$$

where $\partial_x u$ denotes the distributional derivative, and $H^{-1}(\mathbb{R})$ denotes the topological dual of $H^1(\mathbb{R})$ relative to $L^2(\mathbb{R})$. The Laplacian $\partial_x^2 u$ for $u \in H^1(\mathbb{R})$ can be defined via its bilinear form as

$$(2.3) \qquad {}_{H^1(\mathbb{R})}\langle w, \partial_x^2 u\rangle_{H^{-1}(\mathbb{R})} := -\int_{\mathbb{R}} (\partial_x w)(\partial_x u)\, dx.$$

Then, we may recast the system (2.2) in form of the abstract stochastic evolution equation

$$(2.4) \qquad dX(t, \cdot) = \mathcal{A}(t, X(t, \cdot))\, dt + \mathcal{B}(t, X(t, \cdot))\, dW_U(t, \cdot)$$

and introduce the *rigged space triple* (*Gelfand triple*)

$$(2.5) \qquad \underbrace{H^1(\mathbb{R})_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R})}_{=:V} \hookrightarrow \underbrace{L^2(\mathbb{R})_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R})}_{=:H=H^*} \hookrightarrow \underbrace{H^{-1}(\mathbb{R})_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R})}_{=V},$$

where

$$(2.6a) \qquad (Y_1, Y_2)_H := Z\int_{\mathbb{R}} (\varepsilon w_1 w_2 + q_1 q_2)\, dx,$$

$$(2.6b) \qquad (Y_1, Y_2)_V := Z\int_{\mathbb{R}} (\varepsilon w_1 w_2 + \varepsilon(\partial_x w_1)(\partial_x w_2) + q_1 q_2)\, dx,$$

with

$$(2.7) \qquad Y_j := \begin{pmatrix} w_j \\ q_j \end{pmatrix} \quad \text{and} \quad Z := \left(\int_{\mathbb{R}}\left(\varepsilon\left(\frac{d\hat{u}}{d\xi}\right)^2 + \left(\frac{d\hat{v}}{d\xi}\right)^2\right) d\xi\right)^{-1},$$

and where

$$(2.8a) \qquad X := \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\mathcal{A}: [0, T] \times V \times \Omega \to V^*,$$

$$(2.8b) \qquad (t, X, \omega) \mapsto \mathcal{A}(t, X) := \begin{pmatrix} \nu\partial_x^2 u + f\left(u + \hat{u}(\cdot + st)\right) - f\left(\hat{u}(\cdot + st)\right) - v \\ \varepsilon(u - \gamma v) \end{pmatrix},$$

$$\mathcal{B}: [0, T] \times V \times \Omega \to L_2(U; H),$$

$$(2.8c) \qquad (t, X, \omega) \mapsto \left(U \to H, N \mapsto \mathcal{B}(t, X)N := \begin{pmatrix} \sigma\sqrt{Q}N \\ 0 \end{pmatrix}\right).$$

Here, we assume that $U := L^2(\mathbb{R})$, $Q \in L(U)$ is a symmetric nonnegative-definite bounded linear operator in U with finite trace, $L_2(U; H)$ denotes the space of *Hilbert–Schmidt operators* $U \to H$, and W_U is a U-valued $(\mathcal{F}_t)_{t\in[0,T]}$-adapted *cylindrical* id_U-*Wiener process*. Note that $W = \sqrt{Q}W_U$ is in fact a U-valued $(\mathcal{F}_t)_{t\in[0,T]}$-adapted Q-Wiener process. We remark that the assumption of Q having finite trace leads to a noise term in (2.4) which is not translation invariant. However, since we study stability of a pulse with exponential tails traveling with finite speed on an axon (which in reality has finite length), this assumption appears to be acceptable regarding the relevance of our results in terms of applications. Further note that the scaling of the first component with $\sqrt{\varepsilon}$ in (2.6) changes the geometry of the orthogonal sums indicated by the symbol $\sqrt{\varepsilon}\oplus$ rather than $\oplus$ in (2.5). On the other hand, the normalization with Z merely implies the convenient property that $\|\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\|_H = 1$ but leaves all angles between vectors unchanged.

In order to show existence of solutions to (2.4), one could use the concept of mild solutions as employed for the FitzHugh–Nagumo SPDE with additive noise in [61, 90]. The approach presented in [55], which we build upon, uses variational solutions [57, 66, 75] for equations with locally monotone coefficients [65, 66]. As we will show in Proposition 2.5 below, the variational solution (constructed in Proposition 2.4 below) also turns out to be a mild solution. We remark that the authors of [41–43] use [65] to construct solutions, too, but as previously mentioned, their system is regularized by adding $c\partial_x^2 u$ with $c > 0$ to the second component in (2.8b), which also changes the Gelfand triple to $H^1(\mathbb{R}; \mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}; \mathbb{R}^2) \hookrightarrow H^{-1}(\mathbb{R}; \mathbb{R}^2)$. Since the variational approach has not been directly worked out for the FitzHugh–Nagumo SPDE with additive noise and no regularization of the second component, we use this opportunity to fill this gap within this work as an auxiliary step. Therefore, we use the concept of *variational solutions* (cf. [65], Definition 1.1, and [66], Definition 5.1.2):

DEFINITION 2.3.   A variational solution to (2.4) is a continuous H-valued $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(X(t, \cdot))_{t\in[0,T]}$ such that the $\mathrm{d}t \otimes \mathbb{P}$-equivalence class $\check{X}$ meets $\check{X} \in L^2([0, T] \times \Omega, \mathrm{d}t \otimes \mathbb{P}; V)$ and such that the solution formula

$$(2.9) \quad X(t, \cdot) = X(0, \cdot) + \int_0^t \mathcal{A}(t', \bar{X}(t', \cdot))\,\mathrm{d}t' + \int_0^t \mathcal{B}(t', \bar{X}(t', \cdot))\,\mathrm{d}W_U(t', \cdot) \quad \text{for } t \in [0, t]$$

is satisfied, $\mathbb{P}$-almost surely, where $\bar{X}$ denotes any V-valued progressively measurable $\mathrm{d}t \otimes \mathbb{P}$-version of $\check{X}$.

Under the given hypothesis, we can prove existence of solutions and regularity in space under additional assumptions.

PROPOSITION 2.4.   *For $p \in [6, \infty)$ and $T > 0$, the stochastic evolution equation* (2.4) *has for any $X(0, \cdot) = (u(0, \cdot), v(0, \cdot))^{\mathsf{t}} = X^{(0)} = (u^{(0)}, v^{(0)})^{\mathsf{t}} \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ a unique variational solution. The solution further satisfies $X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; C^0([0, T]; H))$.*

For the subsequent result, we define the stronger space

$$(2.10a) \qquad\qquad\qquad \mathcal{V} := H^1(\mathbb{R})\,_{\sqrt{\varepsilon}}\oplus H^1(\mathbb{R})$$

with inner product

$$(2.10b) \quad (Y_1, Y_2)_{\mathcal{V}} := Z \int_{\mathbb{R}} \left(\varepsilon w_1 w_2 + \varepsilon(\partial_x w_1)(\partial_x w_2) + q_1 q_2 + (\partial_x q_1)(\partial_x q_2)\right)\mathrm{d}x$$

$$\overset{(2.6a)}{=} (Y_1, Y_2)_H + (\partial_x Y_1, \partial_x Y_2)_H,$$

where Y_j and Z are as in (2.7).

PROPOSITION 2.5. *In the situation of Proposition* 2.4, *the solution X is also a mild solution in the sense of* [23] *with linear operator*

$$\begin{pmatrix} \nu\partial_x^2 & 0 \\ 0 & -\varepsilon\gamma \end{pmatrix} : H \supseteq H^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R}) \to H.$$

If additionally $\sqrt{Q} \in L_2(U; H^1(\mathbb{R}))$, $u^{(0)} \in L^2(\mathbb{R})$, *and* $v^{(0)} \in H^1(\mathbb{R})$, *then*

$$tu, v \in C^0([0, T]; H^1(\mathbb{R})), \quad \mathbb{P}\text{-almost surely.}$$

If further $u^{(0)} \in H^1(\mathbb{R})$, *that is,* $X^{(0)} \in \mathcal{V}$, *then* $X \in C^0([0, T]; \mathcal{V})$, $\mathbb{P}$-*almost surely.*

We will give the proof of Proposition 2.4 and Proposition 2.5 in Appendix A.1.

2.2. *Linearization around the traveling wave.* We recall the global Assumptions 2.1 and write $\hat{X} := (\hat{u}, \hat{v})^{\mathsf{t}}$ for the deterministically stable fast traveling pulse. Then we define

(2.11) $$\tilde{X}(t, x) := \left(\tilde{u}(t, x), \tilde{v}(t, x)\right)^{\mathsf{t}} = X(t, x) + \hat{X}(x + st),$$

and in line with (1.3) set

(2.12)
$$X_\varphi(t, x) := \left(u_\varphi(t, x), v_\varphi(t, x)\right)^{\mathsf{t}} := \tilde{X}(t, x) - \hat{X}(x + st + \varphi)$$
$$= X(t, x) + \hat{X}(x + st) - \hat{X}(x + st + \varphi),$$

where $\varphi = \varphi(t)$ is yet to be determined. We split the stochastic evolution equation (2.4) around the traveling-wave profile $\hat{X}$ into a linear and nonlinear part using (1.2). This yields

(2.13)
$$dX_\varphi(t, \cdot) = \left(\mathcal{L}_{st+\varphi(t)} X_\varphi(t, \cdot) + R_{\varphi(t)}(t, X_\varphi(t, \cdot), \cdot) - \dot{\varphi}(t) \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi(t)) \right) dt$$
$$+ \begin{pmatrix} \sigma \\ 0 \end{pmatrix} dW(t, \cdot),$$

where

(2.14) $$\mathcal{L}_{st+\varphi} Y := \begin{pmatrix} \nu\partial_x^2 w + f'(\hat{u}(\cdot + st + \varphi))w - q \\ \varepsilon(w - \gamma q) \end{pmatrix}, \quad Y := \begin{pmatrix} w \\ q \end{pmatrix},$$

and

(2.15) $$R_\varphi(t, Y, \cdot) := \begin{pmatrix} f(w + \hat{u}(\cdot + st + \varphi)) - f(\hat{u}(\cdot + st + \varphi)) - f'(\hat{u}(\cdot + st + \varphi))w \\ 0 \end{pmatrix}.$$

We have the following result, which is proved in Appendix A.2.1.

PROPOSITION 2.6 (Linearized evolution). *The family of operators* $(\mathcal{L}_{st})_{t\geq 0}$, *with*

$$\mathcal{L}_{st} : D(\mathcal{L}_{st}) = H^3(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R}) \to \mathcal{V} \overset{(2.10)}{=} H^1(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R}),$$

generates an evolution family $(P_{st,st'})_{t\geq t'\geq 0}$ *in* $\mathcal{V}$ *with*

(2.16a) $$\|P_{st,st'}\|_{L(\mathcal{V})} \leq e^{\beta(t-t')},$$

where

(2.16b) $$\beta := \|f'(\hat{u}) - f'(0)\|_{W^{1,\infty}(\mathbb{R})} - \min\{-f'(0), \varepsilon\gamma\},$$

and we have used the norm in the $\xi = x + st$ *coordinate in the last expression.*

We also write

$$(2.17) \quad \mathcal{L}^{\#}Y := \begin{pmatrix} \nu\partial_\xi^2 w + f'(\hat{u})w - q - s\partial_\xi w \\ \varepsilon(w - \gamma q) - s\partial_\xi q \end{pmatrix} = \begin{pmatrix} \nu\partial_\xi^2 + f'(\hat{u}) - s\partial_\xi & -1 \\ \varepsilon & -\varepsilon\gamma - s\partial_\xi \end{pmatrix} Y$$

for the linearized evolution in the moving frame, that is, with respect to $\xi = x + st$, and call $\mathcal{L}^{\#}$ the *frozen-wave operator*. Defining the *translation operator* $\mathcal{T}_c$ by

$$(2.18) \qquad\qquad\qquad \mathcal{T}_c Y := Y(\cdot + c) \quad \text{for any } c \in \mathbb{R},$$

one may readily check that

$$(2.19) \qquad\qquad\qquad \partial_t - \mathcal{L}_{st} = \mathcal{T}_{st}(\partial_t - \mathcal{L}^{\#})\mathcal{T}_{-st}.$$

Furthermore, by differentiating (1.2) it follows

$$(2.20) \qquad\qquad\qquad \mathcal{L}^{\#}\frac{d\hat{X}}{d\xi} = 0 \quad \text{where } \hat{X} := \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}.$$

Equation (2.20) simply means that the derivative of the traveling wave is an eigenvector of the frozen-wave operator, which actually arises due to translation invariance [59]. Here and in what follows, we use the conventions of [52], Section 2.2.4, Section 2.2.5, Definition 2.2.3, (4.1.11):

DEFINITION 2.7.   For a complex Banach space E and a linear operator $\mathcal{B}: E \supseteq D(\mathcal{B}) \to E$ we define:

(a) the *resolvent set* $\rho(\mathcal{B})$ as the set of all $\lambda \in \mathbb{C}$ such that $\lambda\,\mathrm{id}_E - \mathcal{B}$ is invertible and $(\lambda\,\mathrm{id}_E - \mathcal{B})^{-1}$ is bounded;

(b) the *spectrum* $\sigma(\mathcal{B}) := \mathbb{C} \setminus \rho(\mathcal{B})$;

(c) the *point spectrum* $\sigma_{\mathrm{p}}(\mathcal{B})$ as the set of all $\lambda \in \mathbb{C}$ such that the *index* $\mathrm{ind}(\lambda\,\mathrm{id}_E - \mathcal{B})$ of $\lambda\,\mathrm{id}_E - \mathcal{B}$ satisfies $\mathrm{ind}(\lambda\,\mathrm{id}_E - \mathcal{B}) = 0$ but $\lambda\,\mathrm{id}_E - \mathcal{B}$ is not invertible;

(d) the *essential spectrum* $\sigma_{\mathrm{ess}}(\mathcal{B})$ as the set of all $\lambda \in \mathbb{C}$ such that $\lambda\,\mathrm{id}_E - \mathcal{B}$ is not a Fredholm operator with $\mathrm{ind}(\lambda\,\mathrm{id}_E - \mathcal{B}) = 0$;

(e) for a $\mathbb{C}$-Hilbert space E with inner product $(\cdot, \cdot)_E$ the *numerical range*

$$\mathcal{R}_E(\mathcal{B}) := \{(\mathcal{B}Y, Y)_E : Y \in D(\mathcal{B}), \|Y\|_E = 1\}.$$

Next, we are going to list (spectral) properties of the frozen-wave operator in the statements below, which make use of a complex Hilbert space

$$(2.21\mathrm{a}) \qquad\qquad\qquad H_{\mathbb{C}} := L^2(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R}; \mathbb{C}),$$

endowed with the inner product

$$(2.21\mathrm{b}) \qquad (Y_1, Y_2)_{H_{\mathbb{C}}} := Z\int_{\mathbb{R}} (\varepsilon\overline{w_1}w_2 + \overline{q_1}q_2)\,d\xi, \qquad Y_j := \begin{pmatrix} w_j \\ q_j \end{pmatrix},$$

$$Z \overset{(2.7)}{=} \left(\int_{\mathbb{R}} \left(\varepsilon\left(\frac{d\hat{u}}{d\xi}\right)^2 + \left(\frac{d\hat{v}}{d\xi}\right)^2 \right) d\xi \right)^{-1}.$$

As mentioned in Section 1.2 already, the spectral properties of $\mathcal{L}^{\#}$ have in fact been studied by Jones [48] (based on the framework developed in [27–32]) with shorter proofs given by Yanagida in [91] and by Arioli and Koch in [2], Section 3, in the space of bounded uniformly continuous functions. Furthermore, in [2], Section 4.1, bounds on the eigenvalues for square integrable functions on the torus have been derived by a scaling of the components analogous

to the one used in (2.6) or (2.21b). Here, we adapt these approaches in order to give corresponding results in $H_\mathbb{C}$ in Appendix A.2.2, a choice being compatible with the Hilbert-space setting of Section 2.1. Directly applying the deterministic results [2, 48, 91] would require to adapt our stochastic framework to Banach spaces, which we briefly comment on in Section 5.

We emphasize that the following stability result, Proposition 2.8, in the space of square-integrable functions is (up to a scaling of the components of Y) in more generality proved by Ghazaryan, Latushkin, and Schecter in [36] and by Yurov in [92]. These proofs mainly rely on applying the Gearhart–Prüss theorem (see, e.g., [52], Theorem 4.1.5) and Palmer's theorem [5] and are more complicated than our relatively elementary arguments provided in Appendix A.2.2. See also [77] for an alternative proof using the Laplace transform by Rottmann–Matthes and [78] by the same author, where the method has been applied to first-order hyperbolic PDEs.

PROPOSITION 2.8 (Properties of the frozen-wave operator $\mathcal{L}^\#$). *For the frozen-wave operator*

$$\mathcal{L}^\#: D(\mathcal{L}^\#) := H^2(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R}; \mathbb{C}) \to H_\mathbb{C} \overset{(2.21a)}{=} L^2(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R}; \mathbb{C})$$

it holds:

(a) $\mathcal{L}^\#$ $[\mathcal{L}^\#|_{D(\mathcal{L}^\#) \cap H}]$ *generates a C_0-semigroup* $(P_{st}^\#)_{t \geq 0}$ $[(P_{st}^\#|_H)_{t \geq 0}]$ *in $H_\mathbb{C}$ [H]*.

(b) *We have* $P_{st,st'} = \mathcal{T}_{st} P_{s(t-t')}^\#|_V \mathcal{T}_{-st'}$ *with* $P_{st,st'}$ *as in Proposition 2.6*.

(c) *The* spectrum $\sigma(\mathcal{L}^\#) = \sigma_{\mathrm{ess}}(\mathcal{L}^\#) \dot{\cup} \sigma_\mathrm{p}(\mathcal{L}^\#)$ *consists of*:

   (i) essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L}^\#) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\, \lambda \leq -\kappa\}$ *with*

(2.22) $$\kappa := \min\{-f'(0), \varepsilon\gamma\}$$

   and

   (ii) point spectrum $\sigma_\mathrm{p}(\mathcal{L}^\#)$ *consisting of an isolated eigenvalue* 0 *of multiplicity* 1 *with eigenvector* $\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}$ *such that* $\sigma_\mathrm{p}(\mathcal{L}^\#) \setminus \{0\} \subseteq \{\lambda \in \mathbb{C}: \mathrm{Re}\, \lambda \leq \lambda^*(\varepsilon)\}$ *for an eigenvalue* $\lambda^*(\varepsilon) < 0$, *which we will call the* Jones eigenvalue [48] (*see* [91] *for more general reaction terms f as treated here*).

(d) *Defining the* Riesz spectral projections *using* Dunford calculus *as*

(2.23) $$\Pi^{\#,0} := \frac{1}{2\pi i} \oint_{|\lambda|=r} (\lambda \, \mathrm{id}_{H_\mathbb{C}} - \mathcal{L}^\#)^{-1} \, \mathrm{d}\lambda \quad and \quad \Pi^\# := \mathrm{id}_{H_\mathbb{C}} - \Pi^{\#,0},$$

   where $r := \frac{1}{2}\min\{|\lambda^*(\varepsilon)|, \kappa\}$, *it follows that for any* $\vartheta < \min\{\kappa, -\lambda^*(\varepsilon)\}$ *there exists* $C_\vartheta \in [0, \infty)$ *independent of* $t \in [0, \infty)$ *such that*

(2.24) $$\|P_{st}^\# \Pi^\#\|_{L(H_\mathbb{C})} \leq C_\vartheta e^{-\vartheta t},$$

   where $\|\cdot\|_{L(H_\mathbb{C})}$ *denotes the* operator norm *of bounded linear operators*

$$L^2(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R}; \mathbb{C}) \to L^2(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R}; \mathbb{C}).$$

(e) *The operators* $.\Pi^{\#,0}|_H, .\Pi^\#|_H \colon H \to H$ *are well defined and bounded, too, with* $\|\Pi^{\#,0}\|_{L(H)} \leq \|\Pi^{\#,0}\|_{L(H_\mathbb{C})}$, $\|\Pi^\#\|_{L(H)} \leq \|\Pi^\#\|_{L(H_\mathbb{C})}$, *and* (2.24) *holds true with $H_\mathbb{C}$ replaced by H*.

3. Main results.   For the following results, we can choose $T \in [0, \infty)$ arbitrarily.

3.1. *Correction of the wave velocity.*   In accordance with (1.6) we define the *translated spectral Riesz projections*

$$(3.1) \qquad \Pi^0_{st+\varphi} := \mathcal{T}_{st+\varphi} \Pi^{\#,0} \mathcal{T}_{-st-\varphi}, \qquad \Pi_{st+\varphi} := \mathcal{T}_{st+\varphi} \Pi^{\#} \mathcal{T}_{-st-\varphi},$$

and postulate the RODE as an approximation for the noise-induced wave velocity change

$$(3.2a) \qquad \dot{\varphi}^m(t) = \frac{d\varphi^m}{dt}(t) = m \left(\Pi^0_{st+\varphi^m(t)} X^m(t, \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi^m(t)) \right)_H,$$

where $m > 0$ and

$$X^m(t, x) := X_{\varphi^m}(t, x) \overset{(2.12)}{=} \tilde{X}(t, x) - \hat{X}(x + st + \varphi^m(t))$$

$$\overset{(2.12)}{=} X(t, x) + \hat{X}(x + st) - \hat{X}(x + st + \varphi^m(t)).$$

REMARK 3.1.   Indeed, from (1.4) with the choices (3.1) and $H \overset{(2.5)}{=} L^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R})$, for $X^{(0)} \in \mathcal{V}$ we obtain the necessary condition for a minimum to be

$$
\begin{aligned}
0 &= \partial_\varphi \| \Pi^0_{st+\varphi} X_\varphi(t, \cdot) \|^2_H \\
&\overset{(2.12)}{=} \partial_\varphi \| \Pi^0_{st+\varphi} (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \varphi)) \|^2_H \\
&\overset{(3.1)}{=} \partial_\varphi \| \Pi^{\#,0} (\tilde{X}(t, \cdot - st - \varphi) - \hat{X}) \|^2_H \\
&= -2 (\Pi^{\#,0} (\tilde{X}(t, \cdot - st - \varphi) - \hat{X}), \Pi^{\#,0} ((\partial_x \tilde{X})(t, \cdot - st - \varphi)))_H \\
&= \underbrace{- \int_\mathbb{R} \partial_\xi |\mathrm{diag}(\sqrt{\varepsilon}, 1) \cdot (\Pi^{\#,0} (\tilde{X}(t, \cdot - st - \varphi) - \hat{X}))(\xi)|^2 \, d\xi}_{=0} \\
&\quad - 2 \left(\Pi^{\#,0} (\tilde{X}(t, \cdot - st - \varphi) - \hat{X}), \frac{d\hat{X}}{d\xi} \right)_H \\
&= -2 \left(\Pi^0_{st+\varphi} (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \varphi)), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi) \right)_H \\
&\overset{(2.12)}{=} -2 \left(\Pi^0_{st+\varphi} X_\varphi(t, \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi) \right)_H,
\end{aligned}
$$

so that (1.5) holds true, while (1.6) or (3.2a) correspond to a relaxation of this condition with numerical parameter m.

By choosing φ^m from (3.2a) as the velocity adaption, the SPDE (2.13) becomes

$$
\begin{aligned}
dX^m(t, \cdot) &= \left(\mathcal{L}_{st+\varphi^m(t)} X^m(t, \cdot) + R^m(t, X^m(t, \cdot), \cdot) - \dot{\varphi}^m(t) \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi^m(t)) \right) dt \\
&\quad + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} dW(t, \cdot),
\end{aligned}
$$
(3.2b)

where

(3.3a)
$$R^m(t, Y, \cdot) := R_{\varphi^m}(t, Y, \cdot)$$
$$\overset{(2.15)}{=} \begin{pmatrix} f(w + \hat{u}(\cdot + st + \varphi^m)) - f(\hat{u}(\cdot + st + \varphi^m)) - f'(\hat{u}(\cdot + st + \varphi^m))w \\ 0 \end{pmatrix},$$

(3.3b)
$$Y = \begin{pmatrix} w \\ q \end{pmatrix}.$$

The proof of the following proposition is contained in Section 4.1.

PROPOSITION 3.2. *Suppose $\sqrt{Q} \in L_2(L^2(\mathbb{R}); H^1(\mathbb{R}))$ and $u^{(0)}, v^{(0)} \in H^1(\mathbb{R})$.*

(a) $\mathbb{P}$-*almost surely, there exists a unique $(\mathcal{F}_t)_{t \geq 0}$-adapted solution $\varphi^m \in C^1([0, T])$ to the pathwise ODE* (3.2a) *subject to $\varphi^m(0) = 0$.*

(b) *The correction $\dot{\varphi}^m$ to the velocity s of the traveling pulse satisfies the SODE*

$$
\begin{aligned}
d\dot{\varphi}^m(t) = {}&-m\dot{\varphi}^m(t)\left(1 + \left(\Pi^0_{st+\varphi^m(t)}\partial_x X^m(t, \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi^m(t))\right)_H\right) dt \\
&+ \sigma m\left(\Pi^0_{st+\varphi^m(t)}(1, 0)^\mathsf{t} dW(t, \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi^m(t))\right)_H \\
&+ m\left(\Pi^0_{st+\varphi^m(t)} R^m(t, X^m(t, \cdot), \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st + \varphi^m(t))\right)_H dt.
\end{aligned}
$$

(3.4)

3.2. *Reduced stochastic dynamics and multiscale analysis.* Since for small values of σ, X^m and $\dot{\varphi}^m$ are expected to be small, we may also consider the reduced set of equations (in which by linearity σ can be scaled out)

(3.5a)
$$d\dot{\varphi}_0^m(t) = -m\dot{\varphi}_0^m(t) \, dt + m\left(\Pi^0_{st}(1, 0)^\mathsf{t} dW(t, \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st)\right)_H,$$

(3.5b)
$$dX_0^m(t, \cdot) = \left(\mathcal{L}_{st} X_0^m(t, \cdot) - \dot{\varphi}_0^m(t)\frac{d\hat{X}}{d\xi}(\cdot + st)\right) dt + (1, 0)^\mathsf{t} \, dW(t, \cdot).$$

Note that the SPDE (3.5b) describes the stochastic fluctuations around the stable fast pulse with stochastically adapted wave velocity. Due to the construction (cf. the definition of $\mathcal{L}_{st}$ in (2.14)), the aforementioned SPDE arises from a multiscale argument and represents a leading-order approximation to track the fluctuations around the deterministically stable fast pulse.

The following statements establish existence of solutions to (3.5) and yield a multiscale expansion of the solution $\tilde{X} = (\tilde{u}, \tilde{v})^\mathsf{t}$ to the original equation (1.1). Their proof is contained in Section 4.2.

THEOREM 3.3. *For*

$$\sqrt{Q} \in L_2(L^2(\mathbb{R}); H^1(\mathbb{R})), \qquad X_0^m(0, \cdot) = X_0^{(0)} \in \mathcal{V},$$

$$\dot{\varphi}_0^m(0) = m\left(\Pi^{\#,0} X^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H, \qquad \varphi_0^m(0) = 0,$$

we have:

(a) *The system* (3.5) *has a unique mild solution* [82], *Definition* 1.1, *given by*

(3.6a)
$$\varphi_0^m(t) = (1 - e^{-mt})\left(\Pi^{\#,0}X^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H$$
$$+ \int_0^t (1 - e^{-m(t-t')})\left(\Pi_{st'}^0(1,0)^{\mathsf{t}}dW(t',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st')\right)_H,$$

(3.6b) $\qquad X_0^m(t,\cdot) = P_{st,0}X_0^{(0)} - \varphi_0^m(t)\frac{d\hat{X}}{d\xi}(\cdot + st) + \int_0^t P_{st,st'}(1,0)^{\mathsf{t}}\,dW(t',\cdot),$

$\mathbb{P}$-*almost surely, where* $(P_{st,st'})_{t \geq t' \geq 0}$ *is given by Proposition* 2.6.

(b) *Let* X *be defined as in* (2.12) *using Proposition* 2.4 *and Proposition* 2.5. *Define the stopping times*

(3.7a) $\qquad\qquad\qquad \tau_{q,\sigma} := \inf(\{t \in [0,T] : \|X(t,\cdot)\|_{\mathcal{V}} \geq \sigma^{1-q}\} \cup \{T\}),$

(3.7b) $\qquad\qquad\qquad \tau_{q,\sigma}^m := \inf(\{t \in [0,T] : |\varphi_0^m(t)| \geq \sigma^{-q}\} \cup \{T\}),$

where $q \in [0, \frac{1}{2})$. *Then, on* $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^m\} = T\}$ *we have*

(3.8) $\qquad\qquad \tilde{X}(t,\cdot) =: \hat{X}(\cdot + st + \sigma\varphi_0^m(t)) + \sigma X_0^m(t,\cdot) + \sigma S^m(t,\cdot),$

where the remainder S^m *meets the estimate*

(3.9) $\qquad\qquad\qquad \|S^m(t,\cdot)\|_{\mathcal{V}} \leq C\sigma^{1-2q}(1 + \sigma^{1-q}), \quad \mathbb{P}$-*almost surely,*

with $C < \infty$ *being independent of* σ.

(c) *For* $q \in [0, \frac{1}{2})$ *it holds* $\mathbb{P}[\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^m\} = T] \geq 1 - C\sigma^{2q} \to 1$ *as* $\sigma \searrow 0$ *for a constant* $C < \infty$.

Note that by scaling out σ beforehand, we have $X_0^{(0)} := \sigma^{-1}X^{(0)}$, where $X^{(0)}$ is the initial condition for the unscaled equations (2.4) describing the fluctuations around the traveling wave.

3.3. *Immediate relaxation.* In the limit as $m \to \infty$ (*immediate relaxation*), the system (3.5) further simplifies and we expect the solution (X_0^m, φ_0^m) to (3.5) to converge to

(3.10a) $\quad \varphi_0^\infty(t) := \left(\Pi^{\#,0}X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H + \int_0^t \left(\Pi_{st'}^0(1,0)^{\mathsf{t}}dW(t',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st')\right)_H,$

(3.10b) $X_0^\infty(t,\cdot) := P_{st,0}\Pi^{\#}X_0^{(0)} + \int_0^t P_{st,st'}\Pi_{st'}(1,0)^{\mathsf{t}}\,dW(t',\cdot),$

where $(P_{st,st'})_{t \geq t' \geq 0}$ is given by Proposition 2.6.

The proof of the following statement is contained in Section 4.3.

THEOREM 3.4. *Suppose* $\sqrt{Q} \in L_2(L^2(\mathbb{R}); H^1(\mathbb{R}))$ *and* $X_0^{(0)} \in \mathcal{V}$. *Let* φ^m *be given as in Proposition* 3.2(a), $X^m := X_{\varphi^m}$ *be defined as in* (2.12) *using Proposition* 2.4 *and Proposition* 2.5, *and* φ_0^m *and* X_0^m *be given by Theorem* 3.3.

(a) *We have* $\Pi_{st}^0 X_0^\infty(t,\cdot) = 0$ *or equivalently* $\Pi_{st}X_0^\infty(t,\cdot) = X_0^\infty(t,\cdot)$ *for all* $t \in [0,T]$, $\mathbb{P}$-*almost surely.*

(b) *For any $\delta > 0$, we have*

(3.11a) $$\mathbb{E}\Big[\sup_{t\in[\delta,T]}|\varphi_0^m(t) - \varphi_0^\infty(t)|\Big] \to 0 \quad as\ m \to \infty,$$

(3.11b) $$\mathbb{E}\Big[\sup_{t\in[\delta,T]}\|X_0^m(t,\cdot) - X_0^\infty(t,\cdot)\|_{\mathcal{V}}\Big] \to 0 \quad as\ m \to \infty.$$

(c) *Suppose $q \in [0,\frac{1}{2})$, let $\tau_{q,\sigma}$ be defined as in* (3.7a), *and*

(3.12) $$\tau_{q,\sigma}^\infty := \inf(\{t \in [0,T]\colon |\varphi_0^\infty(t)| \geq \sigma^{-q}\} \cup \{T\}).$$

On $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\} = T\}$ the stochastic traveling wave has the multiscale decomposition

(3.13) $$\tilde{X}(t,\cdot) =: \hat{X}(\cdot + st + \sigma\varphi_0^\infty(t)) + \sigma X_0^\infty(t,\cdot) + \sigma S^\infty(t,\cdot),$$

with

(3.14) $$\|S^\infty(t,\cdot)\|_{\mathcal{V}} \leq C\sigma^{1-2q}(1 + \sigma^{1-q}), \quad \mathbb{P}\text{-almost surely},$$

where $C < \infty$ is independent of σ.

(d) *For $q \in [0,\frac{1}{2})$ it holds $\mathbb{P}[\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\} = T] \geq 1 - C\sigma^{2q} \to 1$ as $\sigma \searrow 0$ for some $C < \infty$.*

(e) *On $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\} = T\}$, where $q \in [0,\frac{1}{2})$, the function*

$$\mathbb{R} \ni \varphi \mapsto \|\Pi_{st}^0(\tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma\varphi))\|_H^2 \in \mathbb{R}$$

is for $0 \leq t \leq T$ fixed, $\mathbb{P}$-almost surely, locally approximately minimal at $\varphi = \varphi_0^\infty(t)$ in the sense that

(3.15a) $$\partial_\varphi\|\Pi_{st}^0(\tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma\varphi))\|_H^2\big|_{\varphi=\varphi_0^\infty(t)} = \mathcal{O}(\sigma^{3-2q}) = o(\sigma^2),$$

and

(3.15b) $$\partial_\varphi^2\|\Pi_{st}^0(\tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma\varphi))\|_H^2\big|_{\varphi=\varphi_0^\infty(t)} = \sigma^2\Big(2\Big\|\frac{d\hat{X}}{d\xi}\Big\|_H^2 + \mathcal{O}(\sigma^{1-2q})\Big) > 0$$

as $\sigma \searrow 0$, $\mathbb{P}$-almost surely.

Note that Theorem 3.4(a) is a generalization of the orthogonality property in [55], Section 3.3. Theorem 3.4(c) and Theorem 3.4(d) yield a multiscale expansion in the immediate-relaxation limit. Finally, Theorem 3.4(e) relates to our original motivation in (1.4) to obtain the correction of the translation of the wave through a minimization argument. The following result is proved in in Section 4.3.

PROPOSITION 3.5.   *Suppose that $\sqrt{Q} \in L_2(L^2(\mathbb{R}); L^2(\mathbb{R}))$ and $X_0^{(0)} \in H$. Then the second moment of X_0^∞ satisfies the bound*

(3.16) $$\mathbb{E}[\|X_0^\infty(t,\cdot)\|_H^2] \leq 2C_\vartheta^2 e^{-2\vartheta t}\|X_0^{(0)}\|_H^2 + C_\vartheta^2 \frac{1 - e^{-2\vartheta t}}{\vartheta}\|\Pi^\#\|_{L(H)}^2\varepsilon Z\|\sqrt{Q}\|_{L_2(L^2(\mathbb{R}))}^2$$

with $\vartheta < \min\{\kappa, -\lambda^*(\varepsilon)\}$, where κ and the Jones eigenvalue $\lambda^*(\varepsilon)$ have been introduced in Proposition 2.8, and the constant $C_\vartheta \in [0,\infty)$ only depends on ϑ (and the parameters γ, ε, f, and ν of the deterministic FitzHugh–Nagumo system (1.1) with $\sigma = 0$) but is independent of t, T, and σ.

The inequality (3.16) is crucial for theoretical and application purposes as it bounds the expected size of fluctuations/deviations from the deterministic pulse. It shows that the second moment of the remaining deviations from the pulse, after correcting the wave velocity, are bounded by a sum of two contributions. The first term is simply due to the initial data $X_0^{(0)}$ and is exponentially decaying as in (2.24) of Proposition 2.8(d). The second term is due to noise around the traveling wave, where once more the decay constant of (2.24) enters. From a theoretical viewpoint the multiscale estimate for the second moment can then be useful in Doob/Markov-type inequalities to control the probabilities of individual sample paths [1, 6, 37].

Note that the noise contribution is of lower order compared to the second moment of fluctuations around the corresponding deterministic traveling wave (i.e., without stochastic velocity adaptation) because we have shifted appropriately to minimize the deviations in direction of the derivative of the traveling wave (which corresponds to the eigenvector with eigenvalue 0 of the frozen-wave operator). This is made more precise in the following proposition, in which we compute the second moment of the fluctuations in direction of the derivative of the traveling wave on the event $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^{\infty}\} = T\}$ where the multiscale decomposition holds true. The proof can be found in Section 4.3.

PROPOSITION 3.6. *There exists a sequence $(Q_N)_{N \in \mathbb{N}}$ with $\sqrt{Q_N} \in L_2(L^2(\mathbb{R}); H^1(\mathbb{R}))$ such that for the deviations without adapting the wave velocity $X := X_0$ of (2.12) (i.e., with $\varphi = 0$) with respect to Q_N in the definition of the noise (2.8c), the second moment of deviations in direction of the traveling wave satisfies*

$$
\mathbb{E}\left[\left(\Pi_{st}^0 X(t, \cdot), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st) \right)_H^2 \mathbb{1}_{\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^{\infty}\} = T\}} \right]
$$

$$
= \sigma^2 \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H^2 + \sigma^2 \varepsilon Z\left(t \left\| (1,0)^{\mathsf{t}} (\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H^2 + o(N^0) \right) + o(\sigma^2),
$$

where $\| (1,0)^{\mathsf{t}} (\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \|_H^2 > 0$, $o(N^0)$ can depend on t but not on σ, and $o(\sigma^2)$ can depend on t and N. In particular, asymptotically the in σ leading contribution of this moment grows linearly in time.

4. Proofs of main results.

4.1. *Correction of the wave velocity.* In this section, we prove Propositions 3.2.

PROOF OF PROPOSITION 3.2(a). We use the pathwise definition

$$
F \colon [0, T] \times \mathbb{R} \mapsto \mathbb{R}, \qquad (t, \varphi) \mapsto m\left(\Pi_{st+\varphi}^0 (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \varphi)), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st + \varphi) \right)_H
$$

and note that by employing translation invariance of integrals, we have

$$
F(t, \varphi) \overset{(3.1)}{=} m\left(\Pi^{\#,0} (\tilde{X}(t, \cdot - st - \varphi) - \hat{X}), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \quad \text{for } (t, \varphi) \in [0, T] \times \mathbb{R}.
$$

Then, we can compute for $(t_1, \varphi_1), (t_2, \varphi_2) \in [0, T] \times \mathbb{R}$,

$$
\left| F(t_1, \varphi_1) - F(t_2, \varphi_2) \right| \leq m \left\| \Pi^{\#,0} \right\|_{L(H)} \left\| \tilde{X}(t_1, \cdot - st_1 - \varphi_1) - \tilde{X}(t_2, \cdot - st_2 - \varphi_2) \right\|_H \left\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H
$$

and on noting that

$$
\begin{aligned}
\big\| \tilde{X}(t_1, \cdot - st_1 - \varphi_1) &- \tilde{X}(t_2, \cdot - st_2 - \varphi_2) \big\|_H \\
&\leq \big\| \tilde{X}(t_1, \cdot - st_1 - \varphi_1) - \tilde{X}(t_2, \cdot - st_1 - \varphi_1) \big\|_H \\
&\quad + \big\| \tilde{X}(t_2, \cdot - st_1 - \varphi_1) - \tilde{X}(t_2, \cdot - st_2 - \varphi_2) \big\|_H,
\end{aligned}
$$

with

$$
\begin{aligned}
\big\| \tilde{X}(t_1, \cdot &- st_1 - \varphi_1) - \tilde{X}(t_2, \cdot - st_1 - \varphi_1) \big\|_H \\
&\overset{(2.11)}{\leq} \big\| X(t_1, \cdot - st_1 - \varphi_1) - X(t_2, \cdot - st_1 - \varphi_1) \big\|_H \\
&\quad + \big\| \hat{X}(\cdot - \varphi_1) - \hat{X}(\cdot + s(t_2 - t_1) - \varphi_1) \big\|_H \\
&= \big\| X(t_1, \cdot) - X(t_2, \cdot) \big\|_H + \bigg\| \int_{s(t_1 - t_2) + \varphi_1}^{\varphi_1} \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot - \xi) \mathrm{d}\xi \bigg\|_H \\
&\leq \big\| X(t_1, \cdot) - X(t_2, \cdot) \big\|_H + |s||t_1 - t_2| \bigg\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \bigg\|_H
\end{aligned}
$$

and

$$
\begin{aligned}
\big\| \tilde{X}(t_2, \cdot &- st_1 - \varphi_1) - \tilde{X}(t_2, \cdot - st_2 - \varphi_2) \big\|_H \\
&= \bigg\| \int_{st_1 + \varphi_1}^{st_2 + \varphi_2} \partial_\xi \tilde{X}(t_2, \cdot - \xi) \mathrm{d}\xi \bigg\|_H \\
&\leq \bigg| \int_{st_1 + \varphi_1}^{st_2 + \varphi_2} \big\| \partial_\xi \tilde{X}(t_2, \cdot - \xi) \big\|_H \mathrm{d}\xi \bigg| \\
&\leq \big(|s||t_1 - t_2| + |\varphi_1 - \varphi_2| \big) \sup_{t \in [0,T]} \big\| \partial_\xi \tilde{X}(t, \cdot) \big\|_H \\
&\overset{(2.11)}{\leq} \big(|s||t_1 - t_2| + |\varphi_1 - \varphi_2| \big) \bigg(\sup_{t \in [0,T]} \big\| \partial_\xi X(t, \cdot) \big\|_H + \bigg\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \bigg\|_H \bigg),
\end{aligned}
$$

and applying Proposition 2.5, we obtain $\sup_{t \in [0,T]} \| \partial_\xi X(t, \cdot) \|_H < \infty$, $\mathbb{P}$-almost surely, so that F is, $\mathbb{P}$-almost surely, continuous and globally Lipschitz continuous in the second component. Making use of the Picard–Lindelöf theorem finishes the proof. $\square$

PROOF OF PROPOSITION 3.2(b).   We can rewrite (3.2a) according to

$$
\begin{aligned}
\dot{\varphi}^m(t) &\overset{(3.2a)}{=} m\bigg(\Pi_{st + \varphi^m(t)}^0 X^m(t, \cdot), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st + \varphi^m(t)) \bigg)_H \\
&\overset{(3.1)}{=} m\bigg(\Pi^{\#,0} X^m(t, \cdot - st - \varphi^m(t)), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \bigg)_H, \quad \mathbb{P}\text{-almost surely.}
\end{aligned}
$$

Taking the differential yields

$$
\begin{aligned}
\mathrm{d}\dot{\varphi}^m(t) &= m\bigg(\Pi^{\#,0}(\mathrm{d}X^m)(t, \cdot - st - \varphi^m(t)), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \bigg)_H \\
&\quad - (s + \dot{\varphi}^m(t)) m\bigg(\Pi^{\#,0} \partial_\xi X^m(t, \cdot - st - \varphi^m(t)), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \bigg)_H \mathrm{d}t
\end{aligned}
$$

$$\overset{(3.1)}{=} m\left(\Pi^0_{st+\varphi^m(t)}\mathrm{d}X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t$$

$$-(s+\dot{\varphi}^m(t))m\left(\Pi^0_{st+\varphi^m(t)}\partial_\xi X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t$$

$$\overset{(3.2\mathrm{b})}{=} m\left(\Pi^0_{st+\varphi^m(t)}\mathcal{L}_{st+\varphi^m(t)}X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t$$

$$+m\left(\Pi^0_{st+\varphi^m(t)}R^m(t,X^m(t,\cdot),\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t$$

$$-m\dot{\varphi}^m(t)\left(\Pi^0_{st+\varphi^m(t)}\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t)),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t$$

$$+\sigma m\left(\Pi^0_{st+\varphi^m(t)}(1,0)^{\mathrm{t}}\mathrm{d}W(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H$$

$$-m(s+\dot{\varphi}^m(t))\left(\Pi^0_{st+\varphi^m(t)}\partial_x X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t,$$

$\mathbb{P}$-almost surely. Now, we note that

$$\left(\Pi^0_{st+\varphi^m(t)}\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t)),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H$$

$$\overset{(3.1)}{=}\left(\Pi^{\#,0}\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi},\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H \overset{(2.20),(2.23)}{=}\left\|\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right\|^2_H \overset{(2.6\mathrm{a})}{=} 1,$$

so that we obtain the simplification

$$\mathrm{d}\dot{\varphi}^m(t)=m\left(\Pi^0_{st+\varphi^m(t)}(\mathcal{L}_{st+\varphi^m(t)}-s\,\partial_x)X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t$$

$$-m\dot{\varphi}^m(t)\left(1+\left(\Pi^0_{st+\varphi^m(t)}\partial_x X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H\right)\mathrm{d}t$$

$$+\sigma m\left(\Pi^0_{st+\varphi^m(t)}(1,0)^{\mathrm{t}}\mathrm{d}W(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H$$

$$+m\left(\Pi^0_{st+\varphi^m(t)}R^m(t,X^m(t,\cdot),\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H \mathrm{d}t,$$

$\mathbb{P}$-almost surely. Next, we use (2.14), (2.17), (2.18), (2.23), and (3.1) to conclude that

$$\Pi^0_{st+\varphi^m(t)}(\mathcal{L}_{st+\varphi^m(t)}-s\,\partial_x)=\mathcal{T}_{st+\varphi^m(t)}\Pi^{\#,0}\mathcal{L}^{\#}\overset{(2.23)}{=}\mathcal{T}_{st+\varphi^m(t)}\mathcal{L}^{\#}\Pi^{\#,0}$$

and therefore

$$\left(\Pi^0_{st+\varphi^m(t)}(\mathcal{L}_{st+\varphi^m(t)}-s\,\partial_x)X^m(t,\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st+\varphi^m(t))\right)_H$$

$$=\left(\mathcal{L}^{\#}\Pi^{\#,0}X^m(t,\cdot-st-\varphi^m(t)),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H$$

$$\overset{(2.6\mathrm{a}),(2.7)}{=}\left(\mathcal{L}^{\#}\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi},\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H\left(\Pi^{\#,0}X^m(t,\cdot-st-\varphi^m(t)),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H \overset{(2.20)}{=} 0,$$

$\mathbb{P}$-almost surely, so that we end up with (3.4). $\quad\square$

4.2. *Reduced stochastic dynamics and multiscale analysis.* In this section, we prove Theorem 3.3. We note that the existence and uniqueness of mild solutions for all nonautonomous linear SPDEs appearing in the next proof is guaranteed by standard results, see for instance [82] or the very general approach in [89], Theorem 1.3.

PROOF OF THEOREM 3.3(a). Since equation (3.5a) decouples from (3.5b), it forms a linear SDE for which we have a unique mild solution given by

$$\dot{\varphi}_0^m(t) = m e^{-mt}\left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H + m \int_0^t e^{-m(t-t')}\left(\Pi_{st'}^0(1,0)^{\mathsf{t}} dW(t',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st')\right)_H,$$

$\mathbb{P}$-almost surely. Another integration using $\varphi_0^m(0) = 0$ yields

$$\varphi_0^m(t) = (1 - e^{-mt})\left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H$$

$$+ m \int_0^t \int_0^{t'} e^{-m(t'-t'')}\left(\Pi_{st''}^0(1,0)^{\mathsf{t}} dW(t'',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st'')\right)_H dt',$$

$\mathbb{P}$-almost surely. Utilizing

$$m \int_0^t \int_0^{t'} e^{-m(t'-t'')}\left(\Pi_{st''}^0(1,0)^{\mathsf{t}} dW(t'',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st'')\right)_H dt'$$

$$= \int_0^t \int_0^{t-t''} m e^{-mt'''} dt'''\left(\Pi_{st''}^0(1,0)^{\mathsf{t}} dW(t'',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st'')\right)_H$$

$$= \int_0^t (1 - e^{-m(t-t'')})\left(\Pi_{st''}^0(1,0)^{\mathsf{t}} dW(t'',\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st'')\right)_H,$$

we end up with (3.6a).

Since φ_0^m is already uniquely defined, we may uniquely solve (3.5b) with the mild-solution formula

$$X_0^m(t,\cdot) = P_{st,0} X_0^{(0)} - \int_0^t \dot{\varphi}_0^m(t') P_{st,st'} \frac{d\hat{X}}{d\xi}(\cdot + st') dt' + \int_0^t P_{st,st'}(1,0)^{\mathsf{t}} dW(t',\cdot),$$

$\mathbb{P}$-almost surely. With the help of Proposition 2.8(a) and (b) it follows that

$$P_{st,st'} \frac{d\hat{X}}{d\xi}(\cdot + st') \stackrel{(2.18)}{=} \mathcal{T}_{st} P_{s(t-t')}^{\#} \frac{d\hat{X}}{d\xi} \stackrel{(2.18),(2.20)}{=} \frac{d\hat{X}}{d\xi}(\cdot + st),$$

so that

$$\int_0^t \dot{\varphi}_0^m(t') P_{st,st'} \frac{d\hat{X}}{d\xi}(\cdot + st') dt' = \int_0^t \dot{\varphi}_0^m(t') dt' \frac{d\hat{X}}{d\xi}(\cdot + st) = \varphi_0^m(t) \frac{d\hat{X}}{d\xi}(\cdot + st)$$

and we arrive at (3.6b). □

PROOF OF THEOREM 3.3(b). In what follows, we restrict ourselves to paths on $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^m\} = T\}$, which ensures smallness of $X(t,\cdot)$ and $\varphi_0^m(t)$.

We consider the remainder

$$\sigma S^m(t,\cdot) \stackrel{(3.8)}{=} \tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma \varphi_0^m(t)) - \sigma X_0^m(t,\cdot)$$

$$\stackrel{(2.11)}{=} X(t,\cdot) + \hat{X}(\cdot + st) - \hat{X}(\cdot + st + \sigma \varphi_0^m(t)) - \sigma X_0^m(t,\cdot),$$

so that with

$$dX(t, \cdot) \overset{(2.13),(2.15)}{=} \mathcal{L}_{st} X(t, \cdot) \, dt + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} dW(t, \cdot),$$

$$+ \begin{pmatrix} f\big(u(t, \cdot) + \hat{u}(\cdot + st)\big) - f\big(\hat{u}(\cdot + st)\big) - f'(\hat{u}(\cdot + st))u(t, \cdot) \\ 0 \end{pmatrix} dt$$

$$d\hat{X}(\cdot + st) \overset{(1.2),(2.14)}{=} \mathcal{L}_{st} \hat{X}(\cdot + st) \, dt + \begin{pmatrix} f\big(\hat{u}(\cdot + st)\big) - f'(\hat{u}(\cdot + st))\hat{u}(\cdot + st) \\ 0 \end{pmatrix} dt,$$

as well as

$$d\hat{X}\big(\cdot + st + \sigma \varphi_0^m(t)\big)$$

$$\overset{(1.2),(2.14)}{=} \mathcal{L}_{st} \hat{X}\big(\cdot + st + \sigma \varphi_0^m(t)\big) \, dt$$

$$+ \begin{pmatrix} f\big(\hat{u}(\cdot + st + \sigma \varphi_0^m(t))\big) - f'(\hat{u}(\cdot + st))\hat{u}(\cdot + st + \sigma \varphi_0^m(t)) \\ 0 \end{pmatrix} dt$$

$$+ \sigma \dot{\varphi}_0^m(t) \frac{d\hat{X}}{d\xi}\big(\cdot + st + \sigma \varphi_0^m(t)\big) \, dt$$

and

$$dX_0^m(t, \cdot) \overset{(3.5b)}{=} \mathcal{L}_{st} X_0^m(t, \cdot) \, dt - \dot{\varphi}_0^m(t) \frac{d\hat{X}}{d\xi}(\cdot + st) \, dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} dW(t, \cdot),$$

we get with

$$(4.1) \qquad \tilde{u}_0^m(t, \cdot) := u(t, \cdot) + \hat{u}(\cdot + st) - \hat{u}\big(\cdot + st + \sigma \varphi_0^m(t)\big)$$

that

$$dS^m(t, \cdot) - \mathcal{L}_{st} S^m(t, \cdot) \, dt = S_1^m(t, \cdot) + S_2^m(t, \cdot) + S_3^m(t, \cdot),$$

where

$$(4.2a) \qquad \begin{aligned} S_1^m(t, \cdot) := \sigma^{-1} \begin{pmatrix} f\big(\hat{u}(\cdot + st + \sigma \varphi_0^m(t)) + \tilde{u}_0^m(t, \cdot)\big) - f\big(\hat{u}(\cdot + st + \sigma \varphi_0^m(t))\big) \\ 0 \end{pmatrix} \\ - \sigma^{-1} \begin{pmatrix} f'(\hat{u}(\cdot + st + \sigma \varphi_0^m(t)))\tilde{u}_0^m(t, \cdot) \\ 0 \end{pmatrix}, \end{aligned}$$

$$(4.2b) \qquad S_2^m(t, \cdot) := \sigma^{-1}\big(f'(\hat{u}(\cdot + st + \sigma \varphi_0^m(t))) - f'(\hat{u}(\cdot + st))\big) \begin{pmatrix} \tilde{u}_0^m(t, \cdot) \\ 0 \end{pmatrix},$$

$$(4.2c) \qquad S_3^m(t, \cdot) := \dot{\varphi}_0^m(t) \Big(\frac{d\hat{X}}{d\xi}(\cdot + st) - \frac{d\hat{X}}{d\xi}\big(\cdot + st + \sigma \varphi_0^m(t)\big) \Big).$$

Since $(\mathcal{L}_{st})_{t \geq 0}$ generates an evolution family $(P_{st,st'})_{t \geq t' \geq 0}$ (cf. Proposition 2.6), we find the mild-solution representation

$$(4.3) \qquad S^m(t, \cdot) = \int_0^t P_{st,st'}\big(S_1^m(t', \cdot) + S_2^m(t', \cdot) + S_3^m(t', \cdot)\big) \, dt', \quad \mathbb{P}\text{-almost surely.}$$

Note that (4.3) can be rigorously justified by following the arguments detailed at the beginning of the proof of Proposition 2.5. We continue by treating the terms $\int_0^t P_{st,st'} S_j^m(t', \cdot) \, dt'$ for $j \in \{1, 2, 3\}$ separately.

First observe that we have the pointwise estimate

$$
\left|S_1^m(t,x)\right| \overset{(2.1h),(4.2a)}{\leq} \sigma^{-1}\eta_2\big(1 + \left|\hat{u}(x + st + \sigma\varphi_0^m(t))\right| + \left|\tilde{u}_0^m(t,x)\right|\big)\left|\tilde{u}_0^m(t,x)\right|^2
$$
$$
\leq \quad \sigma^{-1}\eta_2\big(1 + \|\hat{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\big)\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\left|\tilde{u}_0^m(t,x)\right|,
$$

where the Sobolev embedding in form of $\|\tilde{u}_0^m(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}$ has been applied. This already yields

$$
(4.4) \quad \begin{aligned}
&\|S_1^m(t,\cdot)\|_H \\
&\overset{(2.6a)}{\leq} \sigma^{-1}\sqrt{\varepsilon Z}\eta_2\big(1 + \|\hat{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\big)\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\|\tilde{u}_0^m(t,\cdot)\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

On the other hand, a direct computation gives (suppressing the arguments in $S_1^m = S_1^m(t,x)$, $u_0^m = u_0^m(t,x)$, $\hat{u} = \hat{u}(x + st + \sigma\varphi_0^m(t))$, and $\tilde{u}_0^m = \tilde{u}_0^m(t,x)$)

$$
\sigma\,\partial_x S_1^m \overset{(4.2a)}{=} \left(f'(\hat{u} + \tilde{u}_0^m)\Big(\partial_x\tilde{u}_0^m + \frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}\Big) - f'(\hat{u})\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi} - f''(\hat{u})\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}\tilde{u}_0^m - f'(\hat{u})\partial_x\tilde{u}_0^m \right)(1,0)^\mathsf{t}
$$
$$
= \left((f'(\hat{u} + \tilde{u}_0^m) - f'(\hat{u}) - f''(\hat{u})\tilde{u}_0^m)\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi} + (f'(\hat{u} + \tilde{u}_0^m) - f'(\hat{u}))\partial_x\tilde{u}_0^m \right)(1,0)^\mathsf{t}.
$$

Now, we note that

$$
\left| (f'(\hat{u} + \tilde{u}_0^m) - f'(\hat{u}) - f''(\hat{u})\tilde{u}_0^m)\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi} \right| \overset{(2.1k)}{\leq} \eta_5\big(\tilde{u}_0^m(t,x)\big)^2\left|\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}(x + st + \sigma\varphi_0^m(t))\right|
$$
$$
\leq \eta_5\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\left\|\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}\right\|_{L^\infty(\mathbb{R})}\left|\tilde{u}_0^m(t,x)\right|,
$$

$$
\left| (f'(\hat{u} + \tilde{u}_0^m) - f'(\hat{u}))\partial_x\tilde{u}_0^m \right| \overset{(2.1l)}{\leq} \eta_6\left|\tilde{u}_0^m(t,x)\right|\left|\partial_x\tilde{u}_0^m(t,x)\right|
$$
$$
\big(1 + 2\left|\hat{u}(x + st + \sigma\varphi_0^m(t))\right| + \left|\tilde{u}_0^m(t,x)\right|\big)
$$
$$
\leq \eta_6\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\left|\partial_x\tilde{u}_0^m(t,x)\right|
$$
$$
\big(1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\big),
$$

where $\|\tilde{u}_0^m(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}$ has been utilized once again. Hence,

$$
\begin{aligned}
&\|\partial_x S_1^m(t,\cdot)\|_H \\
&\overset{(2.6a)}{\leq} \sigma^{-1}\sqrt{\varepsilon Z}\eta_5\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\left\|\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}\right\|_{L^\infty(\mathbb{R})}\|\tilde{u}_0^m(t,\cdot)\|_{L^2(\mathbb{R})} \\
&\quad + \sigma^{-1}\sqrt{\varepsilon Z}\eta_6\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\|\partial_x\tilde{u}_0^m(t,\cdot)\|_{L^2(\mathbb{R})}\big(1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})} + \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\big).
\end{aligned}
$$

The combination with (4.4) yields

$$
(4.5) \quad \|S_1^m(t,\cdot)\|_{\mathcal{V}} \overset{(2.10b)}{\leq} \sigma^{-1}C_1\|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}^2\big(1 + \|\tilde{u}_0^m(t,\cdot)\|_{H^1(\mathbb{R})}\big),
$$

where $C_1 < \infty$ is independent of σ.

Next, we estimate S_2^m. Notice that on one hand we have

$$
\left| f'(\hat{u}(x + st + \sigma\varphi_0^m(t))) - f'(\hat{u}(x + st)) \right| \overset{(2.1l)}{\leq} \eta_6\left|\hat{u}(x + st + \sigma\varphi_0^m(t)) - \hat{u}(x + st)\right|
$$
$$
\times \big(1 + \left|\hat{u}(x + st + \sigma\varphi_0^m(t))\right| + \left|\hat{u}(x + st)\right|\big)
$$
$$
\leq \eta_6\left\|\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}\right\|_{L^\infty(\mathbb{R})}\big(1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}\big)\sigma\left|\varphi_0^m(t)\right|,
$$

so that

$$(4.6) \quad \left\| S_2^m(t,\cdot) \right\|_H \overset{(2.6a),(4.2b)}{\leq} \sqrt{\varepsilon Z} \eta_6 \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^\infty(\mathbb{R})} (1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}) |\varphi_0^m(t)| \left\| \tilde{u}_0^m(t,\cdot) \right\|_{L^2(\mathbb{R})}.$$

On the other hand, for the derivative we get

$$\partial_x \left[\left(f'\left(\hat{u}(x + st + \sigma\varphi_0^m(t)) \right) - f'\left(\hat{u}(x + st) \right) \right) \tilde{u}_0^m(t,x) \right]$$

$$= \left[f''\left(\hat{u}(x + st + \sigma\varphi_0^m(t)) \right) \frac{d\hat{u}}{d\xi}(x + st + \sigma\varphi_0^m(t)) - f''\left(\hat{u}(x + st) \right) \frac{d\hat{u}}{d\xi}(x + st) \right] \tilde{u}_0^m(t,x)$$

$$+ \left[f'\left(\hat{u}(x + st + \sigma\varphi_0^m(t)) \right) - f'\left(\hat{u}(x + st) \right) \right] \partial_x \tilde{u}_0^m(t,x)$$

$$= \left[f''\left(\hat{u}(x + st + \sigma\varphi_0^m(t)) \right) - f''\left(\hat{u}(x + st) \right) \right] \frac{d\hat{u}}{d\xi}(x + st + \sigma\varphi_0^m(t)) \tilde{u}_0^m(t,x)$$

$$+ f''\left(\hat{u}(x + st) \right) \left[\frac{d\hat{u}}{d\xi}(x + st + \sigma\varphi_0^m(t)) - \frac{d\hat{u}}{d\xi}(x + st) \right] \tilde{u}_0^m(t,x)$$

$$+ \left[f'\left(\hat{u}(x + st + \sigma\varphi_0^m(t)) \right) - f'\left(\hat{u}(x + st) \right) \right] \partial_x \tilde{u}_0^m(t,x),$$

so that

$$\left| \partial_x \left[\left(f'\left(\hat{u}(x + st + \sigma\varphi_0^m(t)) \right) - f'\left(\hat{u}(x + st) \right) \right) \tilde{u}_0^m(t,x) \right] \right|$$

$$\overset{(2.1l),(2.1m)}{\leq} \eta_7 |\hat{u}(x + st + \sigma\varphi_0^m(t)) - \hat{u}(x + st)| \left| \frac{d\hat{u}}{d\xi}(x + st + \sigma\varphi_0^m(t)) \right| |\tilde{u}_0^m(t,x)|$$

$$+ |f''(\hat{u}(x + st))| \left| \frac{d\hat{u}}{d\xi}(x + st + \sigma\varphi_0^m(t)) - \frac{d\hat{u}}{d\xi}(x + st) \right| |\tilde{u}_0^m(t,x)|$$

$$+ \eta_6 |\hat{u}(x + st + \sigma\varphi_0^m(t)) - \hat{u}(x + st)| (1 + |\hat{u}(x + st + \sigma\varphi_0^m(t))| + |\hat{u}(x + st)|)$$

$$\times |\partial_x \tilde{u}_0^m(t,x)|$$

$$\leq \eta_7 \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^\infty(\mathbb{R})}^2 \sigma |\varphi_0^m(t)| |\tilde{u}_0^m(t,x)| + \|f''(\hat{u})\|_{L^\infty(\mathbb{R})} \left\| \frac{d^2\hat{u}}{d\xi^2} \right\|_{L^\infty(\mathbb{R})} \sigma |\varphi_0^m(t)| |\tilde{u}_0^m(t,x)|$$

$$+ \eta_6 \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^\infty(\mathbb{R})} (1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}) \sigma |\varphi_0^m(t)| |\partial_x \tilde{u}_0^m(t,x)|.$$

This gives

$$\left\| \partial_x \left[\left(f'\left(\hat{u}(\cdot + st + \sigma\varphi_0^m(t)) \right) - f'\left(\hat{u}(\cdot + st) \right) \right) \tilde{u}_0^m(t,\cdot) \right] \right\|_H$$

$$\overset{(2.6a)}{\leq} \sqrt{\varepsilon Z} \eta_7 \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^\infty(\mathbb{R})}^2 \sigma |\varphi_0^m(t)| \left\| \tilde{u}_0^m(t,\cdot) \right\|_{L^2(\mathbb{R})}$$

$$+ \sqrt{\varepsilon Z} \|f''(\hat{u})\|_{L^\infty(\mathbb{R})} \left\| \frac{d^2\hat{u}}{d\xi^2} \right\|_{L^\infty(\mathbb{R})} \sigma |\varphi_0^m(t)| \left\| \tilde{u}_0^m(t,\cdot) \right\|_{L^2(\mathbb{R})}$$

$$+ \sqrt{\varepsilon Z} \eta_6 \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^\infty(\mathbb{R})} (1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}) \sigma |\varphi_0^m(t)| \left\| \partial_x \tilde{u}_0^m(t,\cdot) \right\|_{L^2(\mathbb{R})}$$

and the combination with (4.6) yields

$$(4.7) \qquad \left\| S_2^m(t,\cdot) \right\|_{\mathcal{V}} \overset{(2.10b)}{\leq} C_2 |\varphi_0^m(t)| \left\| \tilde{u}_0^m(t,\cdot) \right\|_{H^1(\mathbb{R})},$$

where $C_2 < \infty$ is independent of σ.

For estimating $\|\int_0^t P_{st,st'} S_3^m(t', \cdot) dt'\|_{\mathcal{V}}$, observe that

$$(\partial_t - s\partial_x)\left(\hat{X}(x + st + \sigma\varphi_0^m(t)) - \hat{X}(x + st) - \frac{d\hat{X}}{d\xi}(x + st)\sigma\varphi_0^m(t)\right)$$

$$= \sigma\dot{\varphi}_0^m(t)\left(\frac{d\hat{X}}{d\xi}(x + st + \sigma\varphi_0^m(t)) - \frac{d\hat{X}}{d\xi}(x + st)\right) \stackrel{(4.2c)}{=} -\sigma S_3^m(t, \cdot).$$

Using integration by parts, this gives

$$\sigma \int_0^t P_{st,st'} S_3^m(t', \cdot) dt'$$

$$= -\int_0^t P_{st,st'}(\partial_{t'} - s\partial_x)$$

$$\times \left(\hat{X}(\cdot + st' + \sigma\varphi_0^m(t')) - \hat{X}(\cdot + st') - \frac{d\hat{X}}{d\xi}(\cdot + st')\sigma\varphi_0^m(t')\right) dt'$$

$$= -\hat{X}(\cdot + st + \sigma\varphi_0^m(t)) + \hat{X}(\cdot + st) + \frac{d\hat{X}}{d\xi}(\cdot + st)\sigma\varphi_0^m(t)$$

$$+ \int_0^t ((\partial_{t'} P_{st,st'}) + P_{st,st'} s\partial_x)$$

$$\times \left(\hat{X}(\cdot + st' + \sigma\varphi_0^m(t')) - \hat{X}(\cdot + st') - \frac{d\hat{X}}{d\xi}(\cdot + st')\sigma\varphi_0^m(t')\right) dt'$$

$$= -\hat{X}(\cdot + st + \sigma\varphi_0^m(t)) + \hat{X}(\cdot + st) + \frac{d\hat{X}}{d\xi}(\cdot + st)\sigma\varphi_0^m(t)$$

$$- \int_0^t P_{st,st'}(\mathcal{L}_{st'} - s\partial_x)$$

$$\times \left(\hat{X}(\cdot + st' + \sigma\varphi_0^m(t')) - \hat{X}(\cdot + st') - \frac{d\hat{X}}{d\xi}(\cdot + st')\sigma\varphi_0^m(t')\right) dt',$$

$\mathbb{P}$-almost surely, and therefore by boundedness of

$$(\mathcal{L}_{st} - s\partial_x)|_{H^3(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^2(\mathbb{R})} \colon \mathcal{U} := H^3(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^2(\mathbb{R}) \to \mathcal{V}$$

(cf. (2.14)) and employing Proposition 2.6, we have

$$\sigma \left\| \int_0^t P_{st,st'} S_3^m(t', \cdot) dt' \right\|_{\mathcal{V}}$$

$$\stackrel{(2.16a)}{\leq} \left\| \hat{X}(\cdot + st + \sigma\varphi_0^m(t)) - \hat{X}(\cdot + st) - \frac{d\hat{X}}{d\xi}(\cdot + st)\sigma\varphi_0^m(t) \right\|_{\mathcal{V}}$$

$$+ C \int_0^t e^{\beta(t-t')} \left\| \hat{X}(\cdot + st' + \sigma\varphi_0^m(t')) - \hat{X}(\cdot + st') - \frac{d\hat{X}}{d\xi}(\cdot + st')\sigma\varphi_0^m(t') \right\|_{\mathcal{U}} dt',$$

$\mathbb{P}$-almost surely, where $C < \infty$ and $\beta \stackrel{(2.16b)}{=} \|f'(\hat{u}) - f'(0)\|_{W^{1,\infty}(\mathbb{R})} - \min\{\nu, -f'(0), \varepsilon\gamma\}$. Now, we note that

$$\left\| \partial_x^j \left(\hat{u}(\cdot + st + \sigma\varphi_0^m(t)) - \hat{u}(\cdot + st) - \frac{d\hat{u}}{d\xi}(\cdot + st)\sigma\varphi_0^m(t) \right) \right\|_{L^2(\mathbb{R})} \leq \frac{1}{2} \left\| \frac{d^{j+2}\hat{u}}{d\xi^{j+2}} \right\|_{L^2(\mathbb{R})} (\sigma\varphi_0^m(t))^2$$

for $j \in \{0, 1, 2, 3\}$ and in the same way

$$\left\| \partial_x^j \left(\hat{v}(\cdot + st + \sigma \varphi_0^m(t)) - \hat{v}(\cdot + st) - \frac{\mathrm{d}\hat{v}}{\mathrm{d}\xi}(\cdot + st) \sigma \varphi_0^m(t) \right) \right\|_{L^2(\mathbb{R})} \leq \frac{1}{2} \left\| \frac{\mathrm{d}^{j+2}\hat{v}}{\mathrm{d}\xi^{j+2}} \right\|_{L^2(\mathbb{R})} (\sigma \varphi_0^m(t))^2$$

for $j \in \{0, 1, 2\}$. As a result, we obtain

$$(4.8) \quad \begin{aligned} \left\| \int_0^t P_{st,st'} S_3^m(t', \cdot) \mathrm{d}t' \right\|_{\mathcal{V}} &\leq \sigma^{-1} C_3 \left((\sigma \varphi_0^m(t))^2 + \int_0^t e^{\beta(t-t')} (\sigma \varphi_0^m(t'))^2 \, \mathrm{d}t' \right) \\ &\leq \sigma^{-1} C_3 \frac{e^{\beta t} + \beta - 1}{\beta} \max_{t' \in [0,t]} (\sigma \varphi_0^m(t'))^2, \end{aligned}$$

$\mathbb{P}$-almost surely, where $C_3 < \infty$ is independent of σ.

We collect (4.5), (4.7), and (4.8), and obtain because of (2.16) of Proposition 2.6 and (4.3),

$$(4.9) \quad \begin{aligned} &\| S^m(t, \cdot) \|_{\mathcal{V}} \\ &\leq \sigma^{-1} C_1 \int_0^t e^{\beta(t-t')} (\| \tilde{u}_0^m(t', \cdot) \|_{H^1(\mathbb{R})}^2 + \| \tilde{u}_0^m(t', \cdot) \|_{H^1(\mathbb{R})}^3) \, \mathrm{d}t' \\ &\quad + \sigma^{-1} C_2 \int_0^t e^{\beta(t-t')} \sigma |\varphi_0^m(t')| \| \tilde{u}_0^m(t', \cdot) \|_{H^1(\mathbb{R})} \, \mathrm{d}t' \\ &\quad + \sigma^{-1} C_3 \frac{e^{\beta t} + \beta - 1}{\beta} \max_{t' \in [0,t]} (\sigma \varphi_0^m(t'))^2 \\ &\leq \sigma^{-1} C_4 \max_{t' \in [0,t]} ((\| \tilde{u}_0^m(t', \cdot) \|_{H^1(\mathbb{R})} + \sigma |\varphi_0^m(t')|)^2 + \| \tilde{u}_0^m(t', \cdot) \|_{H^1(\mathbb{R})}^3), \end{aligned}$$

$\mathbb{P}$-almost surely, where $C_4 < \infty$ is independent of σ. Now, because of (4.1) we have

$$\begin{aligned} \| \tilde{u}_0^m(t, \cdot) \|_{H^1(\mathbb{R})} &\leq \| u(t, \cdot) \|_{H^1(\mathbb{R})} + \| \hat{u}(\cdot + st) - \hat{u}(\cdot + st + \sigma \varphi_0^m(t)) \|_{H^1(\mathbb{R})} \\ &\leq \| u(t, \cdot) \|_{H^1(\mathbb{R})} + \left\| \frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi} \right\|_{L^\infty(\mathbb{R})} \sigma |\varphi_0^m(t)|, \end{aligned}$$

which in combination with (3.7) and (4.9) gives (3.9). $\square$

PROOF OF THEOREM 3.3(c).   We fix an element $\omega \in \Omega$. If for $\sigma \in (0, 1]$ we have $t := \tau_{q,\sigma}(\omega) < T$, then

$$\begin{aligned} \sigma^{1-q} &\stackrel{(3.7a)}{=} \| X(t, \cdot) \|_{\mathcal{V}} \stackrel{(2.11)}{=} \| \tilde{X}(t, \cdot) - \hat{X}(\cdot + st) \|_{\mathcal{V}} \\ &\stackrel{(3.8)}{\leq} \sigma \| S^m(t, \cdot) \|_{\mathcal{V}} + \| \hat{X}(\cdot + st + \sigma \varphi_0^m(t)) - \hat{X}(\cdot + st) + \sigma X_0^m(t, \cdot) \|_{\mathcal{V}} \\ &\stackrel{(3.9)}{\leq} C \sigma^{2-2q} + \| \hat{X}(\cdot + st + \sigma \varphi_0^m(t)) - \hat{X}(\cdot + st) + \sigma X_0^m(t, \cdot) \|_{\mathcal{V}}, \end{aligned}$$

where $C < \infty$ is independent of $\sigma \in (0, 1]$. This gives with help of Markov's inequality

$$\begin{aligned} &\mathbb{P}[\tau_{q,\sigma} < T] \\ &\leq \mathbb{P}\left[\max_{t \in [0,T]} \| X(t, \cdot) \|_{\mathcal{V}} \geq \sigma^{1-q} \right] \\ &\leq \mathbb{P}\left[\max_{t \in [0,T]} \| \hat{X}(\cdot + st + \sigma \varphi_0^m(t)) - \hat{X}(\cdot + st) + \sigma X_0^m(t_0, \cdot) \|_{\mathcal{V}} \geq \sigma^{1-q}(1 - C\sigma^{1-q}) \right] \\ &\leq \frac{2\sigma^{2q-2}}{(1 - C\sigma^{1-q})^2} \mathbb{E}\left[\max_{t \in [0,T]} \| \hat{X}(\cdot + st + \sigma \varphi_0^m(t)) - \hat{X}(\cdot + st) \|_{\mathcal{V}}^2 \right] \end{aligned}$$

$$+ \frac{2\sigma^{2q}}{(1 - C\sigma^{1-q})^2} \mathbb{E}\Big[\max_{t \in [0,T]} \|X_0^m(t, \cdot)\|_{\mathcal{V}}^2 \Big]$$

$$\leq \frac{\tilde{C}\sigma^{2q}}{(1 - C\sigma^{1-q})^2} \Big(\mathbb{E}\Big[\max_{t \in [0,T]} |\varphi_0^m(t)|^2 \Big] + \mathbb{E}\Big[\max_{t \in [0,T]} \|X_0^m(t, \cdot)\|_{\mathcal{V}}^2 \Big] \Big),$$

where $\tilde{C} < \infty$ is independent of $\sigma \in (0, 1] \cap (0, C^{-\frac{1}{1-q}})$. Hence, in order to conclude that $\mathbb{P}[\tau_{q,\sigma} < T] \leq C\sigma^{2q} \to 0$ as $\sigma \searrow 0$ for some $C < \infty$, it suffices to prove that

$$\mathbb{E}\Big[\max_{t \in [0,T]} |\varphi_0^m(t)|^2 \Big] < \infty \quad \text{and} \quad \mathbb{E}\Big[\max_{t \in [0,T]} \|X_0^m(t, \cdot)\|_{\mathcal{V}}^2 \Big] < \infty.$$

This follows from the representations (3.6) and the Burkholder–Davis–Gundy inequality [53], Theorem 3.28, or [23], Lemma 7.7, to estimate the martingale. Indeed, we have

$$\mathbb{E}\Big[\max_{t \in [0,T]} |\varphi_0^m(t)|^2 \Big]$$

$$\leq 2\Big| \Big(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \Big)_H \Big|^2$$

$$+ 2\mathbb{E} \max_{t \in [0,T]} \Big| \int_0^t (1 - e^{-m(t-t')}) \Big(\Pi_{st'}^0(1,0)^{\mathsf{t}} \mathrm{d}W(t', \cdot), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st') \Big)_H \Big|^2$$

$$\overset{(3.1)}{\leq} 2\|\Pi^{\#,0}\|_{L(H)}^2 \|X_0^{(0)}\|_H^2 \Big\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \Big\|_H^2 + 2C_1 T \varepsilon Z \|\Pi^{\#,0}\|_{L(H)}^2 \Big\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \Big\|_H^2 \|\sqrt{Q}\|_{L_2(L^2(\mathbb{R}))}^2 < \infty,$$

as well as

$$\mathbb{E}\Big[\max_{t \in [0,T]} \|X_0^m(t, \cdot)\|_{\mathcal{V}}^2 \Big]$$

$$\leq 3 \max_{t \in [0,T]} \|P_{st,0} X_0^{(0)}\|_{\mathcal{V}}^2 + 3\Big\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \Big\|_{\mathcal{V}}^2 \mathbb{E} \max_{t \in [0,T]} |\varphi_0^m(t)|^2$$

$$+ 3\mathbb{E} \max_{t \in [0,T]} \Big\| \int_0^t P_{st,st'}(1,0)^{\mathsf{t}} \mathrm{d}W(t', \cdot) \Big\|_{\mathcal{V}}^2$$

$$\overset{(2.16)}{\leq} 3e^{2\beta T} \|X_0^{(0)}\|_{\mathcal{V}}^2 + 3\Big\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \Big\|_{\mathcal{V}}^2 \mathbb{E} \max_{t \in [0,T]} |\varphi_0^m(t)|^2 + 3C_2 \frac{e^{2\beta T} - 1}{2\beta} \varepsilon Z \|\sqrt{Q}\|_{L_2(L^2(\mathbb{R}))}^2 < \infty,$$

where $C_1, C_2 < \infty$ are independent of σ and Proposition 2.6 has been employed.

In the same way, we obtain

$$\mathbb{P}\big[\tau_{q,\sigma}^m < T \big] \overset{(3.7b)}{\leq} \mathbb{P}\Big[\max_{t \in [0,T]} |\varphi_0^m(t)| \geq \sigma^{-q} \Big] \leq \sigma^{2q} \mathbb{E}\Big[\max_{t \in [0,T]} |\varphi_0^m(t)|^2 \Big] \to 0 \quad \text{as } \sigma \searrow 0. \quad \square$$

4.3. *Immediate relaxation.* Here, we give the proofs of Theorem 3.4 and Proposition 3.5.

PROOF OF THEOREM 3.4(a). Utilizing Proposition 2.8(b) and (d) yields

$$\Pi_{st}^0 X_0^\infty(t, \cdot) \overset{(3.10b)}{=} \Pi_{st}^0 P_{st,0} \Pi^\# X_0^{(0)} + \int_0^t \Pi_{st}^0 P_{st,st'} \Pi_{st'}(1,0)^{\mathsf{t}} \, \mathrm{d}W(t', \cdot)$$

$$\overset{(3.1)}{=} \mathcal{T}_{st} \Pi^{\#,0} P_{st}^\# \Pi^\# X_0^{(0)} + \int_0^t \mathcal{T}_{st} \Pi^{\#,0} P_{s(t-t')}^\# \Pi^\# \mathcal{T}_{-st'}(1,0)^{\mathsf{t}} \, \mathrm{d}W(t', \cdot)$$

$$\overset{(2.23)}{=} \mathcal{T}_{st} P_{st}^\# \underbrace{\Pi^{\#,0} \Pi^\#}_{\overset{(2.23)}{=}0} X_0^{(0)} + \int_0^t \mathcal{T}_{st} P_{s(t-t')}^\# \underbrace{\Pi^{\#,0} \Pi^\#}_{\overset{(2.23)}{=}0} \mathcal{T}_{-st'}(1,0)^{\mathsf{t}} \, \mathrm{d}W(t', \cdot) = 0,$$

$\mathbb{P}$-almost surely, so that because of (2.23) and (3.1) the claim follows. $\square$

PROOF OF THEOREM 3.4(b).   Using Theorem 3.3(a), we obtain for the difference of φ_0^m and φ_0^∞ the following form:

$$
(4.10) \quad (\varphi_0^\infty - \varphi_0^m)(t) \overset{(3.6a),(3.10a)}{=} e^{-mt} \left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi} \right)_H
$$
$$
+ \int_0^t e^{-m(t-t')} \left(\Pi_{st'}^0 (1,0)^t dW(t', \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st') \right)_H,
$$

$\mathbb{P}$-almost surely. On the other hand, we obtain with (2.23), (3.1), (3.6b), and (3.10b)

$$
(X_0^\infty - X_0^m)(t, \cdot) = -P_{st,0} \Pi^{\#,0} X_0^{(0)} - \int_0^t P_{st,st'} \Pi_{st'}^0 (1,0)^t \, dW(t', \cdot) + \varphi_0^m(t) \frac{d\hat{X}}{d\xi}(\cdot + st)
$$

and with help of Proposition 2.8(b), (cii), and (d) we conclude that

$$
P_{st,0} \Pi^{\#,0} X_0^{(0)} \overset{(2.20),(2.23)}{=} \left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi} \right)_H \mathcal{T}_{st} P_{st}^\# \frac{d\hat{X}}{d\xi}
$$
$$
\overset{(2.18),(2.20)}{=} \left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi} \right)_H \frac{d\hat{X}}{d\xi}(\cdot + st),
$$

$$
P_{st,st'} \Pi_{st'}^0 (1,0)^t \, dW(t', \cdot) \overset{(2.20),(2.23),(3.1)}{=} \left(\Pi_{st'}^0 (1,0)^t dW(t', \cdot - st'), \frac{d\hat{X}}{d\xi} \right)_H \mathcal{T}_{st} P_{s(t-t')}^\# \frac{d\hat{X}}{d\xi}
$$
$$
\overset{(2.18),(2.20)}{=} \left(\Pi_{st'}^0 (1,0)^t dW(t', \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st') \right)_H \frac{d\hat{X}}{d\xi}(\cdot + st),
$$

and thus

$$
(X_0^\infty - X_0^m)(t, \cdot)
$$
$$
= \left(-\left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi} \right)_H \right.
$$
$$
(4.11) \qquad \left. - \int_0^t \left(\Pi_{st'}^0 (1,0)^t dW(t', \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st') \right)_H + \varphi_0^m(t) \right) \frac{d\hat{X}}{d\xi}(\cdot + st)
$$
$$
\overset{(3.10a)}{=} -(\varphi_0^\infty(t) - \varphi_0^m(t)) \frac{d\hat{X}}{d\xi}(\cdot + st), \quad \mathbb{P}\text{-almost surely.}
$$

Hence, once (3.11a) is established, (3.11b) is immediate from (4.11).

In order to prove (3.11a), notice that obviously

$$
\lim_{m \to \infty} \sup_{t \in [\delta, T]} e^{-mt} \left| \left(\Pi^{\#,0} X_0^{(0)}, \frac{d\hat{X}}{d\xi} \right)_H \right| = 0
$$

for any $\delta > 0$. For the remaining term we apply the simplification

$$
\int_0^t e^{-m(t-t')} \left(\Pi_{st'}^0 (1,0)^t dW(t', \cdot), \frac{d\hat{X}}{d\xi}(\cdot + st') \right)_H
$$
$$
\overset{(2.18),(3.1)}{=} \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0} \mathcal{T}_{-st'} (1,0)^t dW(t', \cdot), \frac{d\hat{X}}{d\xi} \right)_H
$$
$$
\overset{(2.18)}{=} \left(\Pi^{\#,0} (1,0)^t W(t, \cdot - st), \frac{d\hat{X}}{d\xi} \right)_H
$$

$$(4.12) \quad -m \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} W(t', \cdot - st'), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t'$$

$$+ s \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (\partial_x W)(t', \cdot - st'), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t'$$

$$= e^{-mt} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} W(t, \cdot - st), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H$$

$$+ m \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (W(t, \cdot - st) - W(t', \cdot - st')), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t'$$

$$+ s \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (\partial_x W)(t', \cdot - st'), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t', \quad \mathbb{P}\text{-almost surely.}$$

Now, note that $[0,T] \ni t \mapsto W(t, \cdot) \in H^1(\mathbb{R})$ is, $\mathbb{P}$-almost surely, Hölder continuous with exponent $\alpha < \frac{1}{2}$, so that in particular

$$\sup_{t,t' \in [0,T]} \frac{|(\Pi^{\#,0}(1,0)^{\mathrm{t}} (W(t, \cdot - st) - W(t', \cdot - st)), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi})_H|}{|t - t'|^\alpha}$$

$$\overset{(2.6a)}{\leq} \|\Pi^{\#,0}\|_{L(H)} \left\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H \sqrt{\varepsilon Z} \sup_{t,t' \in [0,T]} \frac{\|W(t, \cdot) - W(t', \cdot)\|_{L^2(\mathbb{R})}}{|t - t'|^\alpha} < \infty,$$

$\mathbb{P}$-almost surely. Furthermore,

$$\left| \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (W(t, \cdot - st) - W(t, \cdot - st')), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \right|$$

$$= \left| s \int_{t'}^t \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (\partial_x W)(t, \cdot - st''), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t'' \right|$$

$$\overset{(2.6a)}{\leq} |s| \|\Pi^{\#,0}\|_{L(H)} \sqrt{\varepsilon Z} \|(\partial_x W)(t, \cdot)\|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H |t - t'|,$$

so that we may conclude

$$M := \sup_{t,t' \in [0,T]} \frac{|(\Pi^{\#,0}(1,0)^{\mathrm{t}} (W(t, \cdot - st) - W(t', \cdot - st')), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi})_H|}{|t - t'|^\alpha} < \infty,$$

$\mathbb{P}$-almost surely, that is,

$$\left| m \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (W(t, \cdot - st) - W(t', \cdot - st')), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t' \right|$$

$$\leq m^{-\alpha} M \int_0^{mT} e^{-\tau} \tau^\alpha \, \mathrm{d}\tau \leq m^{-\alpha} M \Gamma(1 + \alpha) \to 0 \quad \text{as } m \to \infty, \mathbb{P}\text{-almost surely.}$$

By continuity, furthermore

$$\sup_{t \in [\delta, T]} e^{-mt} \left| \left(\Pi^0_{st}(1,0)^{\mathrm{t}} W(t, \cdot), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st) \right)_H \right|$$

$$\overset{(3.1)}{\leq} e^{-m\delta} \|\Pi^{\#,0}\|_{L(H)} \sqrt{\varepsilon Z} \sup_{t \in [\delta, T]} \|W(t, \cdot)\|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H$$

$$\to 0 \quad \text{as } m \to \infty, \mathbb{P}\text{-almost surely,}$$

and

$$\sup_{t\in[\delta,T]}\left|s\int_0^t e^{-m(t-t')}\left(\Pi^{\#,0}(1,0)^{\mathsf{t}}(\partial_x W)(t',\cdot-st'),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H \mathrm{d}t'\right|$$

$$\leq m^{-1}|s|\|\Pi^{\#,0}\|_{L(H)}\sqrt{\varepsilon Z}\sup_{t\in[0,T]}\|(\partial_x W)(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right\|_H$$

$$\to 0 \quad \text{as } m\to\infty, \mathbb{P}\text{-almost surely.}$$

As a result, we infer that

$$(4.13) \qquad \sup_{t\in[\delta,T]}\left|(\varphi_0^\infty-\varphi_0^m)(t)\right|\to 0 \quad \text{as } m\to\infty, \mathbb{P}\text{-almost surely.}$$

From (4.10) and (4.12) we further deduce that $\sup_{t\in[0,T]}|(\varphi_0^\infty-\varphi_0^m)(t)|\leq g$, where

$$g := \|\Pi^{\#,0}\|_{L(H)}\|X_0^{(0)}\|_H\left\|\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right\|_H + \|\Pi^{\#,0}\|_{L(H)}\sqrt{\varepsilon Z}\sup_{t\in[0,T]}\|W(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right\|_H$$

$$+ |s|T\|\Pi^{\#,0}\|_{L(H)}\sqrt{\varepsilon Z}\sup_{t\in[0,T]}\|(\partial_x W)(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right\|_H.$$

With help of the Burkholder–Davis–Gundy inequality [23], Lemma 7.7, we infer that

$$\mathbb{E}\sup_{t\in[0,T]}\|W(t,\cdot)\|_{H^1(\mathbb{R})}\leq C\sqrt{T}\|\sqrt{Q}\|_{L_2(L^2(\mathbb{R});H^1(\mathbb{R}))}<\infty,$$

that is, $\mathbb{E}[g]<\infty$. Hence, with (4.13) and dominated convergence, we conclude that (3.11a) holds true. $\square$

PROOF OF THEOREM 3.4(c).   We introduce the auxiliary stopping time

$$\tau_{q,\sigma,c}^\infty := \inf(\{t\in[0,T]:|\varphi_0^\infty(t)|\geq\sigma^{-q}-c\}\cup\{T\}),$$

where $c\in(0,\sigma^{-q}]$, and start by proving that we can use the multiscale decomposition (3.8) of Theorem 3.3(b) on $\{\min\{\tau_{q,\sigma},\tau_{q,\sigma,c}^\infty\}=T\}\cap E^p$, $\mathbb{P}$-almost surely, where $E^p\in\mathcal{F}$ is an event with $\mathbb{P}[E^p]\geq 1-p$, and where $p\in(0,1]$ is arbitrary. On $\{\min\{\tau_{q,\sigma},\tau_{q,\sigma,c}^\infty\}=T\}$ we have

$$\varphi_0^\infty(0)+\frac{c}{2}\overset{(3.10a)}{=}\left|\left(\Pi^{\#,0}X_0^{(0)},\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H\right|+\frac{c}{2}<\sigma^{-q}.$$

Now, note that for any $m>0$ we have

$$\varphi_0^m(t) \overset{(3.6a)}{=} (1-e^{-mt})\left(\Pi^{\#,0}X_0^{(0)},\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H$$

$$+\int_0^t(1-e^{-m(t-t')})\left(\Pi_{st'}^0(1,0)^{\mathsf{t}}\mathrm{d}W(t',\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot+st')\right)_H$$

$$\overset{(2.18),(3.1)}{=}(1-e^{-mt})\left(\Pi^{\#,0}X_0^{(0)},\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H$$

$$+\int_0^t(1-e^{-m(t-t')})\left(\Pi^{\#,0}\mathcal{T}_{-st'}(1,0)^{\mathsf{t}}\mathrm{d}W(t',\cdot),\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H$$

$$\overset{(2.18)}{=}(1-e^{-mt})\left(\Pi^{\#,0}X_0^{(0)},\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}\right)_H$$

$$- m \int_0^t e^{-m(t-t')} \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} W(t', \cdot - st'), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t'$$

$$+ s \int_0^t (1 - e^{-m(t-t')}) \left(\Pi^{\#,0}(1,0)^{\mathrm{t}} (\partial_x W)(t', \cdot - st'), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \mathrm{d}t',$$

$\mathbb{P}$-almost surely, where we have integrated by parts in the last step. Hence, for $\delta > 0$ it holds $\sup_{t \in [0,\delta]} |\varphi_0^m(t)| \le g(\delta)$, $\mathbb{P}$-almost surely, where

$$g(t) := \left| \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \right| + \|\Pi^{\#,0}\|_{L(H)} \sup_{t' \in [0,t]} \|W(t', \cdot)\|_H \left\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H$$

$$+ |s| t \|\Pi^{\#,0}\|_{L(H)} \sqrt{\varepsilon Z} \sup_{t' \in [0,t]} \|(\partial_x W)(t', \cdot)\|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H$$

is m-independent. It is obvious that g is, $\mathbb{P}$-almost surely, nondecreasing and by the Burkholder–Davis–Gundy inequality [23], Lemma 7.7, as in the proof of Theorem 3.4(b), we see that

$$\mathbb{E}\left[\sup_{t \in [0,\delta]} |\varphi_0^m(t)| \right] \le \mathbb{E}[g(\delta)] \to \left| \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \right| \quad \text{as } \delta \searrow 0.$$

Since

$$g(\delta) \ge g(0) = \left| \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \right|,$$

we conclude that

$$\mathbb{P}\left[\sup_{t \in [0,\delta]} |\varphi_0^m(t)| \ge \left| \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \right| + \frac{c}{2} \right] \to 0 \quad \text{as } \delta \searrow 0$$

and hence, for any $p \in (0, 1]$ there exists $\delta \in [0, T]$ small enough such that

$$\sup_{t \in [0,\delta]} |\varphi_0^m(t)| \le \left| \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H \right| + \frac{c}{2} < \sigma^{-q}$$

on an event $E_1^p \in \mathcal{F}$ with $\mathbb{P}[E_1^p] \ge 1 - \frac{p}{2}$. Using the convergence (3.11a), we infer that for any $p \in (0, 1]$ we find $m \in (0, \infty)$ sufficiently large with

$$\sup_{t \in [\delta, T]} |\varphi_0^m(t)| \le \sup_{t \in [\delta, T]} |\varphi_0^\infty(t)| + c \le \sigma^{-q}$$

on an event $E_2^p \in \mathcal{F}$ with $\mathbb{P}[E_2^p] \ge 1 - \frac{p}{2}$. Define $E^p := E_1^p \cap E_2^p$, then

$$\mathbb{P}[E^p] = 1 - \mathbb{P}[\Omega \setminus E^p] \ge 1 - \mathbb{P}[\Omega \setminus E_1^p] - \mathbb{P}[\Omega \setminus E_2^p] = -1 + \mathbb{P}[E_1^p] + \mathbb{P}[E_2^p] \ge 1 - p.$$

In total, on $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma,c}^\infty\} = T\} \cap E^p$ we can apply the multiscale decomposition (3.8) of Theorem 3.3(b) in order to obtain

$$\sigma \|S^\infty(t, \cdot)\|_{\mathcal{V}}$$

$$\overset{(3.13)}{=} \|\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \sigma\varphi_0(t)) - \sigma X_0^\infty(t, \cdot)\|_{\mathcal{V}}$$

$$\overset{(3.8)}{\le} \sigma \|S^m(t, \cdot)\|_V + \sigma \|X_0^\infty(t, \cdot) - X_0^m(t, \cdot)\|_{\mathcal{V}}$$

$$+ \|\hat{X}(\cdot + st + \sigma\varphi_0^\infty(t)) - \hat{X}(\cdot + st + \sigma\varphi_0^m(t))\|_{\mathcal{V}}$$

$$\overset{(3.9)}{\leq} C\sigma^{2-2q}(1+\sigma^{1-q}) + \sigma\|X_0^\infty(t,\cdot) - X_0^m(t)\|_{\mathcal{V}}$$

$$+ \sqrt{\sigma|\varphi_0^\infty(t) - \varphi_0^m(t)|}\sqrt{Z}\sqrt{2\varepsilon\left\|\frac{d\hat{u}}{d\xi}\right\|_{W^{1,\infty}(\mathbb{R})}\|\hat{u}\|_{W^{1,1}(\mathbb{R})} + 2\left\|\frac{d\hat{v}}{d\xi}\right\|_{W^{1,\infty}(\mathbb{R})}\|\hat{v}\|_{W^{1,1}(\mathbb{R})}}$$

$$\to C\sigma^{2-2q}(1+\sigma^{1-q}) \quad \text{as } m \to \infty, \mathbb{P}\text{-almost surely,}$$

that is, (3.14) holds true on $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma,c}^\infty\} = T\} \cap E^p$, where $C < \infty$ is independent of σ, p, and c. Setting $E := \bigcup_{p \in (0,1]} E^p$, it follows $\mathbb{P}[E] = 1$ and we infer that the pointwise estimate (3.14) holds true on $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma,c}^\infty\} = T\}$, too. Since

$$\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\} = T\} = \bigcup_{c \in (0,\sigma^{-q}]} \{\min\{\tau_{q,\sigma}, \tau_{q,\sigma,c}^\infty\} = T\},$$

we infer that (3.14) is also valid on $\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\} = T\}$. $\square$

PROOF OF THEOREM 3.4(d).   Because of Theorem 3.3(c), it suffices to prove

$$\mathbb{P}[\tau_{q,\sigma}^\infty < T] \leq C\sigma^{2q} \to 0 \quad \text{as } \sigma \searrow 0$$

for some $C < \infty$. Indeed, by Markov's inequality, we obtain

$$\mathbb{P}[\tau_{q,\sigma}^\infty < T] \overset{(3.12)}{\leq} \mathbb{P}\Big[\max_{t \in [0,T]}|\varphi_0^\infty(t)| \geq \sigma^{-q}\Big] \leq \sigma^{2q}\mathbb{E}\Big[\max_{t \in [0,T]}|\varphi_0^\infty(t)|^2\Big].$$

Now note that $\mathbb{E}[\max_{t \in [0,T]}|\varphi_0^\infty(t)|^2] < \infty$ because of the representation (3.10a), where the martingale can be estimated using the Burkholder–Davis–Gundy inequality [53], Theorem 3.28, or [23], Lemma 7.7. $\square$

PROOF OF THEOREM 3.4(e).   We use

$$\partial_\varphi\|\Pi_{st}^0(\tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma\varphi))\|_H^2\big|_{\varphi = \varphi_0^\infty(t)}$$

$$= -2\sigma\left(\Pi_{st}^0(\tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma\varphi_0^\infty(t))), \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t))\right)_H$$

$$\overset{(3.13)}{=} -2\sigma^2\left(\Pi_{st}^0(X_0^\infty(t,\cdot) + S^\infty(t,\cdot)), \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t))\right)_H$$

$$= -2\sigma^2\left(\Pi_{st}^0 S^\infty(t,\cdot), \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t))\right)_H, \quad \mathbb{P}\text{-almost surely,}$$

where we have used Theorem 3.4(a) in the last step. Hence,

$$\left|\partial_\varphi\|\Pi_{st+\varphi}^0(\tilde{X}(t,\cdot) - \hat{X}(\cdot + st + \sigma\varphi))\|_H^2\big|_{\varphi = \varphi_0^\infty(t)}\right| \overset{(3.1)}{\leq} 2\sigma^2\|\Pi^{\#,0}\|_{L(H)}\left\|\frac{d\hat{X}}{d\xi}\right\|_H\|S^\infty(t,\cdot)\|_H$$

$$\overset{(3.14)}{\leq} C\sigma^{3-2q}(1+\sigma^{1-q}),$$

$\mathbb{P}$-almost surely, where $C < \infty$ is independent of σ. This proves (3.15a).

For the second derivative, we obtain

$$
\partial_\varphi^2 \big\| \Pi_{st}^0 (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \sigma\varphi)) \big\|_H^2 \big|_{\varphi=\varphi_0^\infty(t)}
$$

$$
= -2\sigma \, \partial_\varphi \Big(\Pi_{st}^0 (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \sigma\varphi)), \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi) \Big)_H \Big|_{\varphi=\varphi_0^\infty(t)}
$$

$$
= 2\sigma^2 \Big(\Pi_{st}^0 \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t)), \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t)) \Big)_H
$$

$$
\quad - 2\sigma^2 \Big(\Pi_{st}^0 (\tilde{X}(t, \cdot) - \hat{X}(\cdot + st + \sigma\varphi_0^\infty(t))), \frac{d^2\hat{X}}{d\xi^2}(\cdot + st + \sigma\varphi_0^\infty(t)) \Big)_H
$$

$$
\overset{(3.13)}{=} 2\sigma^2 \Big(\Pi_{st}^0 \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t)), \frac{d\hat{X}}{d\xi}(\cdot + st + \sigma\varphi_0^\infty(t)) \Big)_H
$$

$$
\quad - 2\sigma^2 \Big(\Pi_{st}^0 (X_0^\infty(t, \cdot) + S^\infty(t, \cdot)), \frac{d^2\hat{X}}{d\xi^2}(\cdot + st + \sigma\varphi_0^\infty(t)) \Big)_H
$$

$$
\overset{(3.1)}{=} 2\sigma^2 \Big(\Pi^{\#,0} \frac{d\hat{X}}{d\xi}(\cdot + \sigma\varphi_0^\infty(t)), \frac{d\hat{X}}{d\xi}(\cdot + \sigma\varphi_0^\infty(t)) \Big)_H
$$

$$
\quad - 2\sigma^2 \Big(\Pi_{st}^0 S^\infty(t, \cdot), \frac{d^2\hat{X}}{d\xi^2}(\cdot + st + \sigma\varphi_0^\infty(t)) \Big)_H, \quad \mathbb{P}\text{-almost surely,}
$$

where Theorem 3.4(a) was used in the last step once more. Further utilizing

$$
2\sigma^2 \Big(\Pi^{\#,0} \frac{d\hat{X}}{d\xi}(\cdot + \sigma\varphi_0^\infty(t)), \frac{d\hat{X}}{d\xi}(\cdot + \sigma\varphi_0^\infty(t)) \Big)_H
$$

$$
= 2\sigma^2 \Big\| \frac{d\hat{X}}{d\xi} \Big\|_H^2 + 2\sigma^3 \int_0^{\varphi_0^\infty(t)} \Big(\Pi^{\#,0} \frac{d^2\hat{X}}{d\xi^2}(\cdot + \sigma\varphi), \frac{d\hat{X}}{d\xi} \Big)_H d\varphi
$$

$$
\quad + 2\sigma^3 \int_0^{\varphi_0^\infty(t)} \Big(\frac{d\hat{X}}{d\xi}, \frac{d^2\hat{X}}{d\xi^2}(\cdot + \sigma\varphi) \Big)_H d\varphi
$$

$$
\quad + 2\sigma^4 \int_0^{\varphi_0^\infty(t)} \int_0^{\varphi_0^\infty(t)} \Big(\Pi^{\#,0} \frac{d^2\hat{X}}{d^2\xi}(\cdot + \sigma\varphi), \frac{d^2\hat{X}}{d\xi^2}(\cdot + \sigma\varphi') \Big)_H d\varphi \, d\varphi',
$$

we can estimate

$$
\Big| \int_0^{\varphi_0^\infty(t)} \Big(\Pi^{\#,0} \frac{d^2\hat{X}}{d\xi^2}(\cdot + \sigma\varphi), \frac{d\hat{X}}{d\xi} \Big)_H d\varphi \Big| \leq \sigma^{-q} \big\| \Pi^{\#,0} \big\|_{L(H)} \Big\| \frac{d^2\hat{X}}{d\xi^2} \Big\|_H \Big\| \frac{d\hat{X}}{d\xi} \Big\|_H,
$$

$$
\Big| \int_0^{\varphi_0^\infty(t)} \Big(\frac{d\hat{X}}{d\xi}, \frac{d^2\hat{X}}{d\xi^2}(\cdot + \sigma\varphi) \Big)_H d\varphi \Big| \leq \sigma^{-q} \Big\| \frac{d\hat{X}}{d\xi} \Big\|_H \Big\| \frac{d^2\hat{X}}{d\xi^2} \Big\|_H,
$$

and

$$
\Big| \int_0^{\varphi_0^\infty(t)} \int_0^{\varphi_0^\infty(t)} \Big(\Pi^{\#,0} \frac{d^2\hat{X}}{d^2\xi}(\cdot + \sigma\varphi), \frac{d^2\hat{X}}{d\xi^2}(\cdot + \sigma\varphi') \Big)_H d\varphi \, d\varphi' \Big|
$$

$$
\leq \sigma^{-2q} \big\| \Pi^{\#,0} \big\|_{L(H)} \Big\| \frac{d^2\hat{X}}{d\xi^2} \Big\|_H^2,
$$

as well as

$$\left|\left(\Pi_{st}^0 S^\infty(t,\cdot), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}(\cdot + st + \sigma\varphi_0^\infty(t))\right)_H\right| \overset{(3.1)}{\leq} \left\|\Pi^{\#,0}\right\|_{L(H)} \left\|\frac{\mathrm{d}^2\hat{X}}{\mathrm{d}\xi^2}\right\|_H \|S^\infty(t,\cdot)\|_H$$

$$\overset{(3.14)}{\leq} C\sigma^{1-2q}(1+\sigma^{1-q}),$$

$\mathbb{P}$-almost surely, where $C < \infty$ is independent of σ, we conclude that (3.15b) holds true, too.
□

PROOF OF PROPOSITION 3.5.   From (3.10b) we obtain with help of (2.18), Proposition 2.8(b) and (d), and (3.1)

$$\|X_0^\infty(t,\cdot)\|_H \leq \|P_{st}^\#\Pi^\# X_0^{(0)}\|_H + \left\|\int_0^t P_{s(t-t')}^\#\Pi^\#\mathcal{T}_{-st'}(1,0)^{\mathsf{t}}\mathrm{d}W(t',\cdot)\right\|_H$$

$$\overset{(2.24)}{\leq} C_\vartheta e^{-\vartheta t}\|X_0^{(0)}\|_H + \left\|\int_0^t P_{s(t-t')}^\#\Pi^\#\mathcal{T}_{-st'}(1,0)^{\mathsf{t}}\mathrm{d}W(t',\cdot)\right\|_H,$$

$\mathbb{P}$-almost surely, where ϑ has been introduced in (2.22). Therefore, using Itô's isometry, the second moment can be bounded as follows

$$\mathbb{E}\big[\|X_0^\infty(t,\cdot)\|_H^2\big] \leq 2C_\vartheta^2 e^{-2\vartheta t}\|X_0^{(0)}\|_H^2 + 2\int_0^t \|P_{s(t-t')}^\#\Pi^\#\mathcal{T}_{-st'}(1,0)^{\mathsf{t}}\sqrt{Q}\|_{L_2(L^2(\mathbb{R});H)}^2\,\mathrm{d}t'.$$

For an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$ we can write

$$\|P_{s(t-t')}^\#\Pi^\#\mathcal{T}_{-st'}(1,0)^{\mathsf{t}}\sqrt{Q}\|_{L_2(L^2(\mathbb{R});H)}^2 = \sum_{k=1}^\infty \|P_{s(t-t')}^\#\Pi^\#\mathcal{T}_{-st'}(1,0)^t\sqrt{Q}e_k\|_H^2$$

$$\leq C_\vartheta^2 e^{-2\vartheta(t-t')}\|\Pi^\#\|_{L(H)}^2\varepsilon Z\sum_{k=1}^\infty \|\sqrt{Q}e_k\|_{L^2(\mathbb{R})}^2$$

$$\leq C_\vartheta^2 e^{-2\vartheta(t-t')}\|\Pi^\#\|_{L(H)}^2\varepsilon Z\|\sqrt{Q}\|_{L_2(L^2(\mathbb{R}))}^2,$$

so that eventually we arrive at (3.16).   □

PROOF OF PROPOSITION 3.6.   Suppose $Q = Q_N \in L_2(L^2(\mathbb{R}); H^1(\mathbb{R}))$ to be specified further below. With (2.12)–(2.15), the deviations around the traveling wave without stochastic velocity adaption ($\varphi = 0$) satisfy the following mild-solution formula:

$$X(t,\cdot) = \sigma P_{st,0}X_0^{(0)} + \int_0^t P_{st,st'}R_0(t', X(t',\cdot), \cdot)\,\mathrm{d}t' + \sigma\int_0^t P_{st,st'}(1,0)^{\mathsf{t}}\,\mathrm{d}W(t',\cdot),$$

which can be justified with arguments analogous to those given at the beginning of the proof of Proposition 2.5. Thanks to Proposition 2.8(b) and (2.18), in the moving frame this corresponds to

$$X(t, \cdot - st) = \sigma P_{st}^\# X_0^{(0)} + \int_0^t P_{s(t-t')}^\# R_0(t', X(t', \cdot - st'), \cdot - st')\,\mathrm{d}t'$$

$$+ \sigma\int_0^t P_{s(t-t')}^\#(1,0)^{\mathsf{t}}\,\mathrm{d}W(t', \cdot - st').$$

We are interested in the deviations in direction of the derivative of the traveling wave which is given by the projection $\Pi^{\#,0}$ defined in (2.23). Note that by (2.20) we have $\Pi^{\#,0}P_{st}^\# =$

$P_{st}^{\#}\Pi^{\#,0} = \Pi^{\#,0}$, so that

$$\Pi^{\#,0}X(t,\cdot - st) = \sigma\Pi^{\#,0}X_0^{(0)} + \int_0^t \Pi^{\#,0}R_0(t', X(t',\cdot - st'),\cdot - st')\,dt'$$

$$+ \sigma\int_0^t \Pi^{\#,0}(1,0)^{\mathsf{t}}\,dW(t',\cdot - st').$$

Hence,

$$\left(\Pi^{\#,0}X(t,\cdot - st), \frac{d\hat{X}}{d\xi}\right)_H = \sigma\left(\Pi^{\#,0}X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H$$

$$+ \underbrace{\int_0^t \left(\Pi^{\#,0}R_0(t', X(t',\cdot - st'),\cdot - st'), \frac{d\hat{X}}{d\xi}\right)_H dt'}_{=:I_1}$$

$$+ \underbrace{\sigma\int_0^t \left(\Pi^{\#,0}(1,0)^{\mathsf{t}}\,dW(t',\cdot - st'), \frac{d\hat{X}}{d\xi}\right)_H}_{=:I_2}.$$

In order to compute $\mathbb{E}[(\Pi^{\#,0}X(t,\cdot - st), \frac{d\hat{X}}{d\xi})_H^2 \mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}=T\}}]$ we develop the square and consider the terms separately. For the first term note that

$$\mathbb{E}\left[\sigma^2\left(\Pi^{\#,0}X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H^2 \mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}=T\}}\right]$$

$$= \sigma^2\left(\Pi^{\#,0}X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H^2 - \mathbb{P}(\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\} < T)\sigma^2\left(\Pi^{\#,0}X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H^2$$

$$= \sigma^2\left(\Pi^{\#,0}X_0^{(0)}, \frac{d\hat{X}}{d\xi}\right)_H^2 + o(\sigma^2),$$

since by Theorem 3.4(d) we have $\mathbb{P}(\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\} < T) \leq C\sigma^{2q}$ for some $C < \infty$ depending on t and N but independent of σ.

Now note that using (2.15), with the same argumentation as in the estimate of S_1^m (cf. (4.2a)) in the proof of Theorem 3.3(b) it holds that

$$\mathbb{E}[I_1^2\mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}=T\}}] \leq C^2\sigma^2\sigma^{2-4q}(1+\sigma^{1-q})^2$$

for a constant $C < \infty$ depending on t and N but independent of σ. For the third term we write again

$$\mathbb{E}[I_2^2\mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}=T\}}] = \mathbb{E}[I_2^2] - \mathbb{E}[I_2^2\mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}<T\}}],$$

and we will recognize that $\mathbb{E}[I_2^2\mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}<T\}}] = o(\sigma^2)$ depending on t and N by dominated convergence together with Theorem 3.4(d) and the fact that $\sigma^{-2}I_2^2\mathbb{1}_{\{\min\{\tau_{q,\sigma},\tau_{q,\sigma}^\infty\}<T\}} \leq \sigma^{-2}I_2^2$, where the latter is a random variable independent of σ whose expectation is finite by the computations that follow.

Take an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$ with $e_k \in C_c^\infty(\mathbb{R})$ for each $k \in \mathbb{N}$ (existence of such a basis follows by applying the Gram–Schmidt algorithm to a countable dense subset of $L^2(\mathbb{R})$ in $C_c^\infty(\mathbb{R})$). With Itô's isometry we obtain

$$\mathbb{E}[I_2^2] = \sigma^2\int_0^t \sum_{k=1}^\infty \left(\Pi^{\#,0}\mathcal{T}_{-st'}(1,0)^{\mathsf{t}}\sqrt{Q_N}e_k, \frac{d\hat{X}}{d\xi}\right)_H^2 dt'.$$

Now take the sequence of Hilbert–Schmidt operators $(Q_N)_{N\in\mathbb{N}}$ with

$$\sqrt{Q_N}e_k = \begin{cases} e_k, & k \leq N, \\ 0, & k > N, \end{cases}$$

and compute the with N increasing second moment of I_2 for each $N \in \mathbb{N}$

$$\mathbb{E}[I_2^2] = \sigma^2 \int_0^t \sum_{k=1}^N \left(\Pi^{\#,0}(1,0)^{\mathsf{t}} e_k(\cdot - st'), \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H^2 \mathrm{d}t'$$

$$\overset{(2.6a)}{=} \sigma^2 \varepsilon^2 Z^2 \int_0^t \sum_{k=1}^N \left(e_k(\cdot - st'), \left((\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_1 \right)_{L^2(\mathbb{R})}^2 \mathrm{d}t',$$

where $((\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi})_1$ denotes the first component of $(\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}$. Taking the limit $N \to \infty$, we obtain with Parseval's identity

$$\mathbb{E}[I_2^2] \to \sigma^2 \varepsilon^2 Z^2 \int_0^t \left\| \left((\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_1 \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}t' = \sigma^2 \varepsilon Z t \left\| (1,0)^{\mathsf{t}} (\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right\|_H^2.$$

By the previous estimates, the mixed terms $\mathbb{E}[2\sigma(\Pi^{\#,0}X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi})_H I_1 \mathbb{1}_{\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\}=T\}}]$ and $\mathbb{E}[2I_1 I_2 \mathbb{1}_{\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\}=T\}}]$ are of order $o(\sigma^2)$ depending on t and N by the Cauchy–Schwarz inequality. For the third mixed term, we write again

$$\mathbb{E}\left[2\sigma \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H I_2 \mathbb{1}_{\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\}=T\}} \right]$$

$$= \mathbb{E}\left[2\sigma \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H I_2 \right] - \mathbb{E}\left[2\sigma \left(\Pi^{\#,0} X_0^{(0)}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_H I_2 \mathbb{1}_{\{\min\{\tau_{q,\sigma}, \tau_{q,\sigma}^\infty\}<T\}} \right]$$

$$= o(\sigma^2)$$

depending on t and N since the first term in the second line is equal to 0 because I_2 is a martingale (and equal to 0 for $t = 0$) and the second term in the second line is of order $o(\sigma^2)$ by the Cauchy–Schwarz inequality in conjunction with dominated convergence as $\sigma \searrow 0$.

In order to ensure linear growth in time, we prove that $(1,0)^{\mathsf{t}} (\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \neq 0$ by contradiction. Assume that $(1,0)^{\mathsf{t}} (\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} = 0$. Then for $\phi \in L^2(\mathbb{R})$ we have

$$0 = \varepsilon Z \left(\left((\Pi^{\#,0})^* \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_1, \phi \right)_{L^2(\mathbb{R})} = \left(\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}, \Pi^{\#,0}(1,0)^{\mathsf{t}}\phi \right)_H.$$

Since $\Pi^{\#,0}$ is a projection on the linear subspace generated by $\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}$ (see (2.23)), we obtain $\Pi^{\#,0}(1,0)^{\mathsf{t}}\phi = 0$. Hence, for any $Y \in H$ it holds that

$$0 = (Y, \Pi^{\#,0}(1,0)^{\mathsf{t}}\phi)_H = \varepsilon Z(((\Pi^{\#,0})^* Y)_1, \phi)_{L^2(\mathbb{R})}.$$

The above equality is in particular valid for a nontrivial eigenfunction $Y = (w,q)^{\mathsf{t}} \neq 0$ of $(\mathcal{L}^\#)^*$ with eigenvalue 0, that is, $((\Pi^{\#,0})^* Y)_1 = 0$. Note that such an eigenfunction exists because of (2.20) or Proposition 2.8(cii), respectively. Indeed, we have $\mathrm{ind}(\mathcal{L}^\#) = \mathrm{ind}((\mathcal{L}^\#)^*) = 0$ by Definition 2.7(c) and because for any $\tilde{Y} \in H_\mathbb{C}$ we have

$$0 = \left(\tilde{Y}, \mathcal{L}^\# \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_{H_\mathbb{C}} = \left((\mathcal{L}^\#)^* \tilde{Y}, \frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi} \right)_{H_\mathbb{C}},$$

the range of $(\mathcal{L}^{\#})^*$ is orthogonal to $\frac{d\hat{X}}{d\xi}$, so that the dimension of the kernel of $(\mathcal{L}^{\#})^*$ is at least 1. Interchanging the roles of $\mathcal{L}^{\#}$ and $(\mathcal{L}^{\#})^*$ shows that the range of $\mathcal{L}^{\#}$ is orthogonal to the kernel of $(\mathcal{L}^{\#})^*$. Since the range of $\mathcal{L}^{\#}$ has codimension 1 we conclude that the dimension of the kernel of $(\mathcal{L}^{\#})^*$ is 1.

For such Y we additionally have

$$(\Pi^{\#,0})^*Y \overset{(2.23)}{=} \left(-\frac{1}{2\pi i} \oint_{|\lambda|=r} (\overline{\lambda}\,\mathrm{id}_{H_{\mathbb{C}}} - (\mathcal{L}^{\#})^*)^{-1}\,\mathrm{d}\overline{\lambda} \right)Y$$

$$= \left(\frac{1}{2\pi i} \oint_{|\lambda|=r} (\lambda\,\mathrm{id}_{H_{\mathbb{C}}} - (\mathcal{L}^{\#})^*)^{-1}\,\mathrm{d}\lambda \right)Y = Y.$$

With $((\Pi^{\#,0})^*Y)_1 = 0$ this implies immediately that $w = 0$. In order to determine the second component, we compute $(\mathcal{L}^{\#})^*Y$ explicitly. Let $Y_j = (w_j, q_j)^{\mathsf{t}} \in C_c^{\infty}(\mathbb{R};\mathbb{C})^2$ for $j = 1, 2$, then by (2.17) we obtain through integration by parts and regrouping terms

$$(\mathcal{L}^{\#}Y_1, Y_2)_{H_{\mathbb{C}}} = \varepsilon Z\nu \int_{\mathbb{R}} (\partial_\xi^2 \overline{w_1})w_2\,\mathrm{d}\xi + \varepsilon Z \int_{\mathbb{R}} f'(\hat{u})\overline{w_1}w_2\,\mathrm{d}\xi - \varepsilon Z \int_{\mathbb{R}} \overline{q_1}w_2\,\mathrm{d}\xi$$

$$- \varepsilon Z s \int_{\mathbb{R}} (\partial_\xi \overline{w_1})w_2\,\mathrm{d}\xi + \varepsilon Z \int_{\mathbb{R}} \overline{w_1}q_2\,\mathrm{d}\xi$$

$$- \varepsilon Z\gamma \int_{\mathbb{R}} \overline{q_1}q_2\,\mathrm{d}\xi - Z s \int_{\mathbb{R}} (\partial_\xi \overline{q_1})q_2\,\mathrm{d}\xi$$

$$= \left(Y_1, \begin{pmatrix} \nu\partial_\xi^2 + f'(\hat{u}) + s\partial_\xi & 1 \\ -\varepsilon & -\varepsilon\gamma + s\partial_\xi \end{pmatrix} Y_2 \right)_{H_{\mathbb{C}}}$$

$$= (Y_1, (\mathcal{L}^{\#})^*Y_2)_{H_{\mathbb{C}}}.$$

Now $((\Pi^{\#,0})^*Y)_1 = 0$ is equivalent to

$$\nu\partial_\xi^2 w + f'(\hat{u})w + s\partial_\xi w + \varepsilon q = 0.$$

With $w = 0$ this implies $q = 0$, a contradiction to Y being nontrivial. $\square$

5. Conclusions and outlook.

In this work, we have shown how to derive a multiscale decomposition near a deterministically stable traveling wave for the FitzHugh–Nagumo SPDE-ODE system. This decomposition into a component along the translation invariant family and into its complement exploits the small noise and small time scale separation parameters to derive leading-order dynamics. More precisely, the stochastically adjusted wave speed is given by an SODE to account for stochastic phase dynamics along the deterministically neutral translation mode. The fluctuations in the infinitely many remaining modes are captured by an SPDE system. Locally, near the wave, these two equations can be linearized to provide a relatively explicit solution representation which approximates well the deviations in the corresponding modes. In particular, our approach does not require an analytic semigroup generated by the linearization and it applies to much wider classes of SPDE systems as well as other patterns.

Natural generalizations of our work could be to treat the case of a cylindrical Wiener process allowing for translation-invariant (white) noise or to allow for multiplicative noise. Furthermore, it would be a desirable goal to obtain optimal estimates on the relevant stopping times, where the approach breaks down which has been addressed for the FitzHugh–Nagumo system with a regularizing Laplacian in the second component in [43] and for stochastic neural field equations in [69]. These could then be compared with high-accuracy numerical

simulations; cf. the references in numerics in the Introduction. Of course, other classes of traveling wave patterns and effects could also be tackled via multiscale decomposition, for example, periodic wave trains, deterministically chaotic waves, stochastic pulse splitting effects induced by large deviations or propagation failure, just to name a few. Many of these effects are somewhat understood for special examples of scalar SPDEs (see [60] and the references therein) but it is clear that the dynamics can be far more complicated for SPDE systems. From a technical viewpoint, we have already pointed out that applying stochastic slow manifold methods and related fast-slow sample path estimates would be natural directions for future research. Furthermore, a generalization from the Hilbert-space setting to Banach spaces appears possible as the variational approach also works if V and V^* are Banach spaces (cf. Assumptions A.1), we do not require any orthogonality, and Riesz spectral projections are available in Banach spaces, too. Notably, related techniques in the deterministic setting, such as Lyapunov–Schmidt or center-manifold reductions work in Banach spaces as well. In summary, it seems evident that rigorous multiscale methods for pattern formation in SPDEs have already been very successful but still need additional development.

APPENDIX: PROOFS OF AUXILIARY RESULTS

A.1. Existence and uniqueness of solutions using the variational approach. In this section, we prove Proposition 2.4 and Proposition 2.5. Note that the formulation (2.4) for general operators $\mathcal{A}\colon [0, T] \times V \times \Omega \to V^*$ and $\mathcal{B}\colon [0, T] \times V \times \Omega \to L_2(U; H)$ can be found in [65] and existence of variational solutions has been established under the following conditions.

ASSUMPTIONS A.1. The Hilbert space H is separable, V is a reflexive Banach space which is continuously and densely embedded into H, and $(W_U(t), t \geq 0)$ is a cylindrical Wiener process on a separable Hilbert space U with respect to a complete filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$$

with a complete and right-continuous filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Furthermore, there exist $\alpha > 1$, $\beta \geq 0, \theta > 0, C_1, C_2 < \infty$, a positive $(\mathcal{F}_t)_{t \in [0, T]}$-adapted process $F \in L^{\frac{p}{2}}([0, T] \times \Omega; \mathrm{d}t \times \mathbb{P})$ with $p \geq \beta + 2$, and $g\colon V \to [0, \infty)$ measurable and locally bounded, such that the following conditions (LR1)–(LR6) hold for all $Y, Y_1, Y_2 \in V$ and $(t, \omega) \in [0, T] \times \Omega$:

(LR1) *Hemicontinuity*: the map $\mathbb{R} \ni \tau \mapsto {}_{V^*}\langle \mathcal{A}(t, Y_1 + \tau Y_2), Y \rangle_V \in \mathbb{R}$ is continuous.
(LR2) *Local monotonicity*:

$$2_{V^*}\langle \mathcal{A}(t, Y_1) - \mathcal{A}(t, Y_2), Y_1 - Y_2 \rangle_V + \|\mathcal{B}(t, Y_1) - \mathcal{B}(t, Y_2)\|_{L_2(U;H)}^2$$

$$\leq (C_1 + g(Y_2)) \|Y_1 - Y_2\|_H^2.$$

(LR3) *Coercivity*: $2_{V^*}\langle \mathcal{A}(t, Y), Y \rangle_V + \|\mathcal{B}(t, Y)\|_{L_2(U;H)}^2 + \theta \|Y\|_V^\alpha \leq F(t) + C_1 \|Y\|_H^2$.
(LR4) *Growth*: $\|\mathcal{A}(t, Y)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (F(t) + C_1 \|Y\|_V^\alpha)(1 + \|Y\|_H^\beta)$.
(LR5) $\|\mathcal{B}(t, Y)\|_{L_2(U;H)}^2 \leq C_2(F(t) + \|Y\|_H^2)$.
(LR6) $g(Y) \leq C_2(1 + \|Y\|_V^\alpha)(1 + \|Y\|_H^\beta)$.

Note that conditions (LR1) and (LR3) are the same as the classical ones from [57], while conditions (LR2) and (LR4) are weaker. A main theorem presented in [65] reads as follows:

THEOREM A.2 (Liu and Röckner [65]).   *Under Assumptions* A.1, *for any initial datum*

$$X(0) = X^{(0)} \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H),$$

equation (2.4) *has a unique variational solution* $(X(t, \cdot))_{t \in [0,T]}$ (*cf. Definition* 2.3), *which additionally satisfies*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|X(t, \cdot)\|_H^p + \int_0^T \|X(t, \cdot)\|_V^\alpha \, dt \right] < \infty.$$

We will use Theorem A.2 to prove Proposition 2.4 by verifying Assumptions A.1.

PROOF OF PROPOSITION 2.4.   The Hilbert spaces

$$U = L^2(\mathbb{R}) \quad \text{and} \quad H \overset{(2.5)}{=} L^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R})$$

are obviously separable, $V \overset{(2.5)}{=} H^1(\mathbb{R})_{\sqrt{\varepsilon}} \oplus L^2(\mathbb{R})$ is reflexive, and as the test functions are dense in H and V, also V is dense in H. Next, we concentrate on verifying (LR1)–(LR6) with the choices

(A.1a) $\alpha := 2,$

(A.1b) $\beta := 4,$

(A.1c) $\theta := \nu,$

(A.1d) $C_1 := \max\left\{ 2\eta_1, 1, \eta_1 + \nu, 4\nu^2, \dfrac{64\eta_4^2}{\varepsilon^2 Z^2} \right\},$

(A.1e) $C_2 := 1,$

(A.1f) $g := 0,$

(A.1g) $F(t) := \max\{ 1 + (2\sqrt{\varepsilon} + \varepsilon\gamma + \eta_4(1 + 3\|\hat{u}\|_{L^\infty(\mathbb{R})}^2))^4, \varepsilon Z \sigma \|\sqrt{Q}\|_{L_2(U)}^2 \},$

(A.1h) $p \geq 6.$

Proof of (LR1). We have for $Y = (w, q)^{\mathsf{t}}$, $Y_1 = (w_1, q_1)^{\mathsf{t}}$, $Y_2 = (w_2, q_2)^{\mathsf{t}} \in V$,

$$\begin{aligned}
&{}_{V^*}\langle \mathcal{A}(t, Y_1 + \tau Y_2), Y \rangle_V \\
&\quad \overset{(2.8b)}{=} \varepsilon Z\,{}_{H^{-1}(\mathbb{R})}\langle \nu \partial_x^2(w_1 + \tau w_2), w \rangle_{H^1(\mathbb{R})} \\
&\qquad + \varepsilon Z\,{}_{H^{-1}(\mathbb{R})}\langle f(w_1 + \tau w_2 + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st)), w \rangle_{H^1(\mathbb{R})} \\
&\qquad - \varepsilon Z\,{}_{H^{-1}(\mathbb{R})}\langle q_1 + \tau q_2, w \rangle_{H^1(\mathbb{R})} \\
&\qquad + \varepsilon Z\,{}_{L^2(\mathbb{R})}\langle w_1 + \tau w_2 - \gamma(q_1 + \tau q_2), q \rangle_{L^2(\mathbb{R})} \\
&\quad = \tau \varepsilon Z\big(-\nu(\partial_x w_2, \partial_x w)_{L^2(\mathbb{R})} - (q_2, w)_{L^2(\mathbb{R})} + (w_2 - \gamma q_2, q)_{L^2(\mathbb{R})}\big) \\
&\qquad + \varepsilon Z\big(f(w_1 + \tau w_2 + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st)), w\big)_{L^2(\mathbb{R})} \\
&\qquad - \varepsilon Z \nu(\partial_x w_1, \partial_x w)_{L^2(\mathbb{R})} - \varepsilon Z(q_1, w)_{L^2(\mathbb{R})} + \varepsilon Z(w_1 - \gamma q_1, q)_{L^2(\mathbb{R})}.
\end{aligned}$$

Hence, hemicontinuity follows if

$$h \colon \mathbb{R} \to \mathbb{R}, \qquad \tau \mapsto h(\tau) := \varepsilon Z\big(f(w_1 + \tau w_2 + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st)), w\big)_{L^2(\mathbb{R})}$$

is continuous. This is true because for $\tau_1, \tau_2 \in \mathbb{R}$ we have

$$
\begin{aligned}
|h(\tau_1) - h(\tau_2)| \;=\;& \varepsilon Z |(f(w_1 + \tau_1 w_2 + \hat{u}(\cdot + st)) - f(w_1 + \tau_2 w_2 + \hat{u}(\cdot + st)), w)_{L^2(\mathbb{R})}| \\
\overset{(2.1j)}{\leq}\;& \varepsilon Z \eta_4 (1 + 6\|\hat{u}\|^2_{L^\infty(\mathbb{R})} + 6\|w_1\|^2_{L^\infty(\mathbb{R})} + 3(\tau_1^2 + \tau_2^2)\|w_2\|^2_{L^\infty(\mathbb{R})}) \\
& \times \|w_2\|_{L^2(\mathbb{R})} |\tau_1 - \tau_2| \|w\|_{L^2(\mathbb{R})} \\
\overset{(2.6b)}{\leq}\;& \eta_4 \left(1 + 6\|\hat{u}\|^2_{L^\infty(\mathbb{R})} + \frac{6}{\varepsilon Z}\|Y_1\|^2_V + \frac{3(\tau_1^2 + \tau_2^2)}{\varepsilon Z}\|Y_2\|^2_V\right) \\
& \times \|Y_2\|_H \|Y\|_H |\tau_1 - \tau_2|,
\end{aligned}
$$

where the Sobolev embedding theorem in form of $\|w_j\|_{L^\infty(\mathbb{R})} \leq \|w_j\|_{H^1(\mathbb{R})}$ has been applied.

Proof of (LR2). For proving local monotonicity, observe that for $Y_1 = (w_1, q_1)^{\mathsf{t}}$, $Y_2 = (w_2, q_2)^{\mathsf{t}} \in V$,

$$
\begin{aligned}
2_{V*}\langle \mathcal{A}(t, Y_1) - \mathcal{A}(t, Y_2), Y_1 - Y_2 \rangle_V \\
\overset{(2.8b)}{=}\; & -2\varepsilon Z \nu \|\partial_x(w_1 - w_2)\|^2_{L^2(\mathbb{R})} \\
& + 2\varepsilon Z (f(w_1 + \hat{u}(\cdot + st)) - f(w_2 + \hat{u}(\cdot + st)), w_1 - w_2)_{L^2(\mathbb{R})} \\
& - 2\varepsilon Z \gamma \|q_1 - q_2\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

On noting that

$$
2\varepsilon Z (f(w_1 + \hat{u}(\cdot + st)) - f(w_2 + \hat{u}(\cdot + st)), w_1 - w_2)_{L^2(\mathbb{R})} \overset{(2.1g)}{\leq} 2\varepsilon Z \eta_1 \|w_1 - w_2\|^2_{L^2(\mathbb{R})}
$$

$$
\overset{(2.6b)}{\leq} 2\eta_1 \|Y_1 - Y_2\|^2_H,
$$

and $\mathcal{B}(t, Y_1) - \mathcal{B}(t, Y_2) \overset{(2.8c)}{=} 0$, this results in

$$
2_{V*}\langle \mathcal{A}(t, Y_1) - \mathcal{A}(t, Y_2), Y_1 - Y_2 \rangle_V + \|\mathcal{B}(t, Y_1) - \mathcal{B}(t, Y_2)\|^2_{L_2(U;H)} \;\leq\; 2\eta_1 \|Y_1 - Y_2\|^2_H
$$

$$
\overset{(A.1d)}{\leq} C_1 \|Y_1 - Y_2\|^2_H.
$$

Note that we have verified the classical monotonicity assumption in [57], that is, local monotonicity is not needed here.

Proof of (LR3). For proving coercivity, we note that for $Y = (w, q)^{\mathsf{t}} \in V$,

$$
_{V*}\langle \mathcal{A}(t, Y), Y \rangle_V \overset{(2.8b)}{=} -\varepsilon Z \nu \|\partial_x w\|^2_{L^2(\mathbb{R})} + \varepsilon Z (f(w + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st)), w)_{L^2(\mathbb{R})}
$$

$$
- \varepsilon Z \gamma \|q\|^2_{L^2(\mathbb{R})}.
$$

Using

$$
\varepsilon Z (f(w + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st)), w)_{L^2(\mathbb{R})} \overset{(2.1g)}{\leq} \varepsilon Z \eta_1 \|w\|^2_{L^2(\mathbb{R})} \overset{(2.6a)}{\leq} \eta_1 \|Y\|^2_H
$$

and $\|\mathcal{B}(t, Y)\|^2_{L_2(U;H)} \overset{(2.6a),(2.8c)}{=} \varepsilon Z \sigma \|\sqrt{Q}\|^2_{L_2(U)}$, we obtain the estimate

$$
\begin{aligned}
_{V*}\langle \mathcal{A}(t, Y), Y \rangle_V + \|\mathcal{B}(t, Y)\|^2_{L_2(U;H)} \;\leq\;& \varepsilon Z \sigma \|\sqrt{Q}\|^2_{L_2(U)} - \varepsilon Z \nu \|\partial_x w\|^2_{L^2(\mathbb{R})} + \eta_1 \|Y\|^2_H \\
\overset{(2.6)}{=}\;& \varepsilon Z \sigma \|\sqrt{Q}\|^2_{L_2(U)} + (\eta_1 + \nu)\|Y\|^2_H - \nu \|Y\|^2_V \\
\overset{(A.1)}{\leq}\;& F(t) + C_1 \|Y\|^2_H - \theta \|Y\|^2_V,
\end{aligned}
$$

which yields (LR3).

Proof of (LR4). We have for $Y = (w, q)^{\mathsf{t}}$, $\tilde{Y} = (\tilde{w}, \tilde{q})^{\mathsf{t}} \in V$,

$$
{}_{V*}\langle \mathcal{A}(t, Y), \tilde{Y} \rangle_V
$$

$$
\overset{(2.8b)}{=} -\varepsilon Z \nu (\partial_x w, \partial_x \tilde{w})_{L^2(\mathbb{R})} + \varepsilon Z \big(f\big(w + \hat{u}(\cdot + st)\big) - f\big(\hat{u}(\cdot + st)\big), \tilde{w} \big)_{L^2(\mathbb{R})}
$$

$$
- \varepsilon Z (q, \tilde{w})_{L^2(\mathbb{R})} + \varepsilon Z (w, \tilde{q})_{L^2(\mathbb{R})} - \varepsilon Z \gamma (q, \tilde{q})_{L^2(\mathbb{R})}
$$

$$
\overset{(2.1j)}{\leq} \varepsilon Z \nu \| \partial_x w \|_{L^2(\mathbb{R})} \| \partial_x \tilde{w} \|_{L^2(\mathbb{R})}
$$

$$
+ \varepsilon Z \eta_4 \big(1 + 3 \| \hat{u} \|_{L^\infty(\mathbb{R})}^2 + 2 \| w \|_{L^\infty(\mathbb{R})}^2 \big) \| w \|_{L^2(\mathbb{R})} \| \tilde{w} \|_{L^2(\mathbb{R})}
$$

$$
+ \varepsilon Z \| q \|_{L^2(\mathbb{R})} \| \tilde{w} \|_{L^2(\mathbb{R})} + \varepsilon Z \| w \|_{L^2(\mathbb{R})} \| \tilde{q} \|_{L^2(\mathbb{R})} + \varepsilon Z \gamma \| q \|_{L^2(\mathbb{R})} \| \tilde{q} \|_{L^2(\mathbb{R})}
$$

$$
\overset{(2.6)}{\leq} \left(\| Y \|_V \left(\nu + \frac{4 \eta_4}{\varepsilon Z} \| Y \|_H^2 \right) + \| Y \|_H \big(2 \sqrt{\varepsilon} + \varepsilon \gamma + \eta_4 \big(1 + 3 \| \hat{u} \|_{L^\infty(\mathbb{R})}^2 \big) \big) \right) \| \tilde{Y} \|_V,
$$

where we have used that

$$
\| w \|_{L^\infty(\mathbb{R})}^2 \leq 2 \| w \|_{L^2(\mathbb{R})} \left\| \frac{\mathrm{d}w}{\mathrm{d}x} \right\|_{L^2(\mathbb{R})} \overset{(2.6)}{\leq} \frac{2}{\varepsilon Z} \| Y \|_H \| Y \|_V.
$$

This implies

$$
\big\| \mathcal{A}(t, Y) \big\|_{V^*}^{\frac{\alpha}{\alpha - 1}}
$$

$$
\overset{(A.1a)}{=} \big\| \mathcal{A}(t, Y) \big\|_{V^*}^2
$$

$$
\leq 2 \| Y \|_V^2 \left(\nu + \frac{4 \eta_4}{\varepsilon Z} \| Y \|_H^2 \right)^2 + 2 \big(2 \sqrt{\varepsilon} + \varepsilon \gamma + \eta_4 \big(1 + 3 \| \hat{u} \|_{L^\infty(\mathbb{R})}^2 \big) \big)^2 \| Y \|_H^2
$$

$$
\leq \left(1 + \big(2 \sqrt{\varepsilon} + \varepsilon \gamma + \eta_4 \big(1 + 3 \| \hat{u} \|_{L^\infty(\mathbb{R})}^2 \big) \big)^4 + 4 \max \left\{ \nu^2, \frac{16 \eta_4^2}{\varepsilon^2 Z^2} \right\} \| Y \|_V^2 \right) \big(1 + \| Y \|_H^4 \big)
$$

$$
\overset{(A.1)}{\leq} \big(F(t) + C_1 \| Y \|_V^\alpha \big) \big(1 + \| Y \|_H^\beta \big),
$$

that is, (LR4). Note that here we have not obtained the classical growth condition in [57].

Proof of (LR5). We have

$$
\| \mathcal{B}(t, Y) \|_{L_2(U; H)}^2 \overset{(2.6a), (2.8c)}{=} \varepsilon Z \sigma \| \sqrt{Q} \|_{L_2(U)}^2 \leq C_2 \big(F(t) + \| Y \|_H^2 \big).
$$

Proof of (LR6). This trivially holds because $g = 0$. $\quad\square$

PROOF OF PROPOSITION 2.5. Denote by $X = (u, v)^{\mathsf{t}}$ the solution from Proposition 2.4. We first apply [66], Proposition G.0.5 (i), and verify the conditions there to conclude that X meets a mild-solution representation. Notably, $\mathcal{B}(t, X(t, \cdot))$ (cf. (2.8c)) does for $t \in [0, T]$ neither depend on t nor on X and takes values in $L_2(U; H)$. Hence, $\mathcal{B}(t, X(t, \cdot))$ is deterministic and in particular

$$
\mathbb{P} \left[\int_0^T \| \mathcal{B}(t, X(t, \cdot)) \|_{L_2(U; H)} \, \mathrm{d}t < \infty \right] = 1
$$

trivially holds true. Furthermore, by the regularity of the variational solution stated in Definition 2.3, we have

$$
\mathbb{E} \left[\int_0^T \| X(t, \cdot) \|_V \, \mathrm{d}t \right] \leq \sqrt{T} \left(\mathbb{E} \left[\int_0^T \| X(t, \cdot) \|_V^2 \, \mathrm{d}t \right] \right)^{\frac{1}{2}} < \infty,
$$

which is why obviously also

$$\mathbb{P}\left[\int_0^T \|X(t,\cdot)\|_H \, dt < \infty\right] = 1$$

is valid. In view of the above estimate, the mixing of u and v in the two components of $\mathcal{A}(t, X(t))$ (cf. (2.8b)) is immaterial, so that it remains show that

(A.2) $$\mathbb{P}\left[\int_0^T \|f(u(t,\cdot) + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st))\|_{L^2(\mathbb{R})} \, dt < \infty\right] = 1$$

holds true. Indeed, we can estimate

$$\mathbb{E}\left[\int_0^T \|f(u(t,\cdot) + \hat{u}(\cdot + st)) - f(\hat{u}(\cdot + st))\|_{L^2(\mathbb{R})} \, dt\right]^{\frac{1}{2}}$$

$$\overset{(2.1j)}{\leq} \eta_4 \mathbb{E}\left[\sup_{t \in [0,T]} \|u(t,\cdot)\|_{L^2(\mathbb{R})} \int_0^T \left(1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}^2 + 3\|u(t,\cdot)\|_{H^1(\mathbb{R})}^2\right) dt\right]^{\frac{1}{2}}$$

$$\leq \frac{\eta_4}{2} \mathbb{E}\left[\sup_{t \in [0,T]} \|u(t,\cdot)\|_{L^2(\mathbb{R})}\right] + \frac{\eta_4 T}{2}(1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}^2) + \frac{3\eta_4}{2} \mathbb{E}\left[\int_0^T \|u(t,\cdot)\|_{H^1(\mathbb{R})}^2 \, dt\right],$$

where the Sobolev embedding in form of $\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u(t,\cdot)\|_{H^1(\mathbb{R})}$ has been used and finiteness of the terms in the last line follows from the regularity of the variational solution stated in Definition 2.3 and Proposition 2.4. Hence, also (A.2) is satisfied.

Next, we prove additional regularity provided $\sqrt{Q} \in L_2(U; H^1(\mathbb{R}))$, $u^{(0)} \in L^2(\mathbb{R})$, and $v^{(0)} \in H^1(\mathbb{R})$. From the first component of (2.4) or (2.9) we derive the mild-solution formula

$$tu(t,\cdot) = \int_0^t K_{t-t'} * \left(f(u(t',\cdot) + \hat{u}(\cdot + st')) - f(\hat{u}(\cdot + st')) - v(t',\cdot)\right)t' \, dt'$$

$$+ \int_0^t K_{t-t'} * u(t',\cdot) \, dt' + \int_0^t K_{t-t'} * t' \, dW(t',\cdot), \quad \mathbb{P}\text{-almost surely,}$$

where $K_t(x) = \frac{e^{-\frac{x^2}{4\nu t}}}{\sqrt{4\pi \nu t}}$ denotes the heat kernel generated by $\nu \partial_x^2$ and $*$ the convolution on the real line. For the second component of (2.4) or (2.9) we get analogously

$$v(t,\cdot) = e^{-\varepsilon\gamma t} v^{(0)} + \varepsilon \int_0^t e^{-\varepsilon\gamma(t-t')} u(t',\cdot) \, dt', \quad \mathbb{P}\text{-almost surely.}$$

Differentiation in space yields

$$t(\partial_x u)(t,\cdot) = \int_0^t K_{t-t'} * \left(f'(u(t',\cdot) + \hat{u}(\cdot + st'))\left(\partial_x u(t',\cdot) + \frac{d\hat{u}}{d\xi}(\cdot + st')\right)\right)t' \, dt'$$

(A.3) $$- \int_0^t K_{t-t'} * \left(f'(\hat{u}(\cdot + st'))\frac{d\hat{u}}{d\xi}(\cdot + st') + \partial_x v(t',\cdot)\right)t' \, dt'$$

$$+ \int_0^t K_{t-t'} * \partial_x u(t',\cdot) \, dt' + \int_0^t K_{t-t'} * (t' \, d(\partial_x W)(t',\cdot)),$$

$\mathbb{P}$-almost surely, and

(A.4) $$\partial_x v(t,\cdot) = e^{-\varepsilon\gamma t}(\partial_x v^{(0)}) + \varepsilon \int_0^t e^{-\varepsilon\gamma(t-t')}(\partial_x u)(t',\cdot) \, dt', \quad \mathbb{P}\text{-almost surely.}$$

We estimate the three lines on the right-hand side of (A.3) separately:

Using $\|K_{t-t'}\|_{L^1(\mathbb{R})} = 1$, we obtain for the first term on the right-hand side of (A.3) with Young's convolution inequality

$$\left\| \int_0^t K_{t-t'} * \left(f'(u(t', \cdot) + \hat{u}(\cdot + st')) \left(\partial_x u(t', \cdot) + \frac{d\hat{u}}{d\xi}(\cdot + st') \right) \right) t' dt' \right\|_{L^2(\mathbb{R})}$$

$$\overset{(2.1i)}{\leq} \eta_3 \int_0^t (1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}^2 + 2\|u(t', \cdot)\|_{H^1(\mathbb{R})}^2) \left(\|\partial_x u(t', \cdot)\|_{L^2(\mathbb{R})} + \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^2(\mathbb{R})} \right) t' dt'$$

$$\leq \frac{\eta_3 T}{2} (1 + 2\|\hat{u}\|_{L^\infty(\mathbb{R})}^2) \left(T \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^2(\mathbb{R})} + \frac{2T^{\frac{1}{2}}}{\sqrt{3}} \|\partial_x u\|_{L^2([0,T] \times \mathbb{R})} \right)$$

$$+ 2\eta_3 T \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^2(\mathbb{R})} \|u\|_{L^2([0,T];H^1(\mathbb{R}))}^2 + \frac{2\eta_3 T^{\frac{3}{2}}}{\sqrt{3}} \|u\|_{C^0([0,T];L^2(\mathbb{R}))}^2 \|\partial_x u\|_{L^2([0,T] \times \mathbb{R})}$$

$$+ 2\eta_3 \int_0^t \|\partial_x u(t', \cdot)\|_{L^2(\mathbb{R})}^2 t' \|\partial_x u(t', \cdot)\|_{L^2(\mathbb{R})} dt',$$

where the Sobolev embedding on the real line in the form of $\|u(t', \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u(t', \cdot)\|_{H^1(\mathbb{R})}$ has been applied.

For the second line of (A.3), we obtain similarly

$$\left\| \int_0^t K_{t-t'} * \left(f'(\hat{u}(\cdot + st')) \frac{d\hat{u}}{d\xi}(\cdot + st') + \partial_x v(t', \cdot) \right) t' dt' \right\|_{L^2(\mathbb{R})}$$

$$\overset{(2.1i)}{\leq} \frac{\eta_3 T^2}{2} \left(1 + \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^\infty(\mathbb{R})}^2 \right) \left\| \frac{d\hat{u}}{d\xi} \right\|_{L^2(\mathbb{R})} + \frac{T^{\frac{3}{2}}}{\sqrt{3}} \|\partial_x v\|_{L^2([0,T] \times \mathbb{R})},$$

where we may use

(A.5) $\|\partial_x v\|_{L^2([0,T] \times \mathbb{R})} \overset{(A.4)}{\leq} \frac{1}{\sqrt{2\varepsilon\gamma}} \|\partial_x v^{(0)}\|_{L^2(\mathbb{R})} + \frac{1}{\gamma} \|\partial_x u\|_{L^2([0,T] \times \mathbb{R})}, \quad \mathbb{P}\text{-almost surely.}$

Finally, the first integral on third line of (A.3) yields

$$\left\| \int_0^t K_{t-t'} * \partial_x u(t', \cdot) dt' \right\|_{L^2(\mathbb{R})} \leq \sqrt{T} \|\partial_x u\|_{L^2([0,T] \times \mathbb{R})},$$

while the second integral gives with help of Itô's isometry

$$\mathbb{E}\left[\left\| \int_0^t K_{t-t'} * (t' d(\partial_x W)(t', \cdot)) \right\|_{L^2(\mathbb{R})}^2 \right] \leq \int_0^t (t')^2 \|(K_{t-t'}*) \partial_x \sqrt{Q}\|_{L_2(U)}^2 dt'$$

$$\leq \frac{T^3}{3} \|\sqrt{Q}\|_{L_2(U;H^1(\mathbb{R}))}^2.$$

Using $X = (u, v)^{\mathtt{t}} \in L^2([0, T]; V) \cap C^0([0, T]; H)$, $\mathbb{P}$-almost surely, and therefore

$$u \in L^2([0, T]; H^1(\mathbb{R})) \cap C^0([0, T]; L^2(\mathbb{R})), \quad \mathbb{P}\text{-almost surely,}$$

we conclude that there exists a constant $C < \infty$, $\mathbb{P}$-almost surely, with $C \to 0$ as $T \searrow 0$, $\mathbb{P}$-almost surely, such that

$$\|t(\partial_x u)(t, \cdot)\|_{L^2(\mathbb{R})} \leq C + 2\eta_3 \int_0^t \|\partial_x u(t', \cdot)\|_{L^2(\mathbb{R})}^2 t' \|\partial_x u(t', \cdot)\|_{L^2(\mathbb{R})} dt, \quad \mathbb{P}\text{-almost surely.}$$

Grönwall's inequality implies

$$\|t(\partial_x u)(t, \cdot)\|_{L^2(\mathbb{R})} \leq C \exp\left(2\eta_3 \int_0^t \|\partial_x u(t', \cdot)\|_{L^2(\mathbb{R})}^2 dt' \right) < \infty, \quad \mathbb{P}\text{-almost surely,}$$

whence $t(\partial_x u) \in L^\infty([0,T]; L^2(\mathbb{R}))$, $\mathbb{P}$-almost surely. Furthermore, since $C \to 0$ as $T \searrow 0$, $\mathbb{P}$-almost surely, $[0,T] \ni t \mapsto t\partial_x u(t, \cdot) \in L^2(\mathbb{R})$ is, $\mathbb{P}$-almost surely, continuous in $t = 0$.

For proving continuity, observe that for $T \geq t_2 \geq t_1 > 0$ we have

$$t_2 \partial_x u(t_2, \cdot) - t_1 \partial_x u(t_1, \cdot) = (t_2 - t_1)\partial_x u(t_2, \cdot) + t_1\big(\partial_x u(t_2, \cdot) - \partial_x u(t_1, \cdot)\big).$$

Then, it follows that $(t_2 - t_1)\partial_x u(t_2, \cdot) \to 0$ as $t_2 \to t_1$ in $L^2(\mathbb{R})$ by the reasoning before if one translates in time and uses uniqueness. For the remaining term, we derive once more from the first component of (2.4) or (2.9)

$$
\begin{aligned}
(A.6) \quad & \partial_x u(t_2, \cdot) - \partial_x u(t_1, \cdot) \\
& = K_{t_2 - t_1} * (\partial_x u)(t_1, \cdot) - \partial_x u(t_1, \cdot) \\
& \quad + \int_{t_1}^{t_2} K_{t_2 - t'} * \left(f'\big(u(t', \cdot) + \hat{u}(\cdot + st')\big)\left(\partial_x u(t', \cdot) + \frac{d\hat{u}}{d\xi}(\cdot + st')\right)\right) dt' \\
& \quad - \int_{t_1}^{t_2} K_{t_2 - t'} * \left(f'(\hat{u}(\cdot + st'))\frac{d\hat{u}}{d\xi}(\cdot + st') - \partial_x v(t', \cdot)\right) dt' \\
& \quad + \int_{t_1}^{t_2} K_{t_2 - t'} * d(\partial_x W)(t', \cdot), \quad \mathbb{P}\text{-almost surely.}
\end{aligned}
$$

For the first line of (A.6) observe that

$$\big\| K_{t_2 - t_1} * (\partial_x u)(t_1, \cdot) - \partial_x u(t_1, \cdot)\big\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } t_2 \to t_1, \mathbb{P}\text{-almost surely,}$$

follows from $tu \in L^\infty([0,T]; L^2(\mathbb{R}))$, $\mathbb{P}$-almost surely, and the fact that $(K_t*)_{t \geq 0} \subset L(L^2(\mathbb{R}))$ is strongly continuous in $t = 0$. The other lines of (A.6) can be estimated as before. Altogether, we obtain

$$\big\| t_2 \partial_x u(t_2, \cdot) - t_1 \partial_x u(t_1, \cdot)\big\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } t_2 \to t_1, \mathbb{P}\text{-almost surely,}$$

which implies $tu \in C^0([0,T]; H^1(\mathbb{R}))$, $\mathbb{P}$-almost surely. Finally, by (A.4) and (A.5) it follows that $v \in C^0([0,T]; H^1(\mathbb{R}))$, $\mathbb{P}$-almost surely.

If $u^{(0)} \in H^1(\mathbb{R})$, an analogous reasoning without time weight yields that

$$u \in C^0([0,T]; H^1(\mathbb{R})), \quad \mathbb{P}\text{-almost surely,}$$

too. $\quad\square$

A.2. Linearization around the traveling wave. In this section, we give the proofs of Proposition 2.6 and Proposition 2.8.

A.2.1. *Fixed frame.* We focus on investigating the linearized evolution generated by the family of operators $(\mathcal{L}_{st})_{t \geq 0}$ defined in (2.14).

PROOF OF PROPOSITION 2.6.   Note that in view of (2.14) we may write

$$
(A.7) \qquad \mathcal{L}_{st} = \mathcal{L}_{\pm\infty} + \mathcal{R}_{st} \quad \text{with} \quad
\begin{cases}
\mathcal{L}_{\pm\infty} := \begin{pmatrix} \nu\partial_x^2 + f'(0) & -1 \\ \varepsilon & -\varepsilon\gamma \end{pmatrix}, \\[2ex]
\mathcal{R}_{st} := \begin{pmatrix} f'(\hat{u}(\cdot + st)) - f'(0) & 0 \\ 0 & 0 \end{pmatrix}.
\end{cases}
$$

For $Y = (w, q)^t \in (C_c^\infty(\mathbb{R}))^2$ we have

$$
\begin{aligned}
(\mathcal{L}_{\pm\infty} Y, Y)_\mathcal{V} &= \varepsilon Z \left(v \frac{d^2 w}{dx^2} + f'(0)w - q, w \right)_{H^1(\mathbb{R})} + \varepsilon Z(w - \gamma q, q)_{H^1(\mathbb{R})} \\
&= \varepsilon Z \left(-v \left\| \frac{d^2 w}{dx^2} \right\|_{L^2(\mathbb{R})}^2 - (v - f'(0)) \left\| \frac{dw}{dx} \right\|_{L^2(\mathbb{R})}^2 + f'(0) \|w\|_{L^2(\mathbb{R})}^2 \right) \\
&\quad - \varepsilon Z \gamma \|q\|_{H^1(\mathbb{R})}^2 \\
&\leq -\min\{-f'(0), \varepsilon\gamma\} \|Y\|_\mathcal{V}^2 - \varepsilon Z v \left\| \frac{d^2 w}{dx^2} \right\|_{L^2(\mathbb{R})}^2 .
\end{aligned}
$$

By density of $(C_c^\infty(\mathbb{R}))^2$ in $D(\mathcal{L}_{\pm\infty}) = H^3(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R})$, we infer that $\mathcal{L}_{\pm\infty} + \kappa \, \mathrm{id}_\mathcal{V}$ with

$$
\kappa := \min\{-f'(0), \varepsilon\gamma\} \overset{(2.1d)}{\geq} 0
$$

is dissipative.

In order to prove that $\mathcal{L}_{\pm\infty} + (\kappa - 1) \, \mathrm{id}_\mathcal{V} : D(\mathcal{L}_{\pm\infty}) \to \mathcal{V}$ is a bijection, we define

$$
\begin{aligned}
\mathcal{M}(Y_1, Y_2) &:= -\varepsilon Z v \left(\frac{dw_1}{dx}, \frac{dw_2}{dx} \right)_{H^1(\mathbb{R})} + \varepsilon Z \left((f'(0) + \kappa - 1)w_1 - q_1, w_2 \right)_{H^1(\mathbb{R})} \\
&\quad + \varepsilon Z \left(w_1 + (\kappa - 1 - \gamma)q_1, q_2 \right)_{H^1(\mathbb{R})}
\end{aligned}
$$

for

$$
Y_j := (w_j, w_j)^t \in H^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R})
$$

and we recognize that $\mathcal{M} : (H^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R}))^2 \to \mathcal{V}$ is bilinear, continuous, and $-\mathcal{M}$ is coercive, so that by the Lax–Milgram theorem for any $Y_2 \in \mathcal{V}$ there exists $Y_1 \in H^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R})$ such that $\mathcal{M}(Y_1, \tilde{Y}) = (Y_2, \tilde{Y})_\mathcal{V}$ for all $\tilde{Y} \in H^2(\mathbb{R})_{\sqrt{\varepsilon}} \oplus H^1(\mathbb{R})$. This implies $\mathcal{L}_{\pm\infty} Y_1 + (\kappa - 1)Y_1 = Y_2$ distributionally and hence in particular

$$
v \frac{d^2 w_1}{dx^2} = w_2 + (1 - \kappa - f'(0))w_1 + q_1 \in H^1(\mathbb{R}),
$$

giving $w_1 \in H^3(\mathbb{R})$ and thus $Y_1 \in D(\mathcal{L}_{\pm\infty})$. The arguments have shown that

$$
\mathcal{L}_{\pm\infty} + (\kappa - 1) \, \mathrm{id}_\mathcal{V} : D(\mathcal{L}_{\pm\infty}) \to \mathcal{V}
$$

is a bijection.

The Lumer–Philips theorem [74], Chapter 1, Theorem 4.3, yields that $\mathcal{L}_{\pm\infty} + \kappa \, \mathrm{id}_\mathcal{V}$ generates a C_0-semigroup of contractions in $\mathcal{V}$. Hence, $\mathcal{L}_{\pm\infty}$ generates a C_0-semigroup $(e^{t\mathcal{L}_{\pm\infty}})_{t \geq 0}$ in $\mathcal{V}$ with bound

$$
\left\| e^{t\mathcal{L}_{\pm\infty}} \right\|_{L(\mathcal{V})} \leq e^{-\kappa t}.
$$

Now, since $\hat{u}$ and $\frac{d\hat{u}}{d\xi}$ are bounded and $f \in C^1(\mathbb{R})$, the family $(\mathcal{R}_{st})_{t \geq 0}$ is uniformly bounded $\mathcal{V} \to \mathcal{V}$ with $\|\mathcal{R}_{st}\|_{L(\mathcal{V})} \leq \|f'(\hat{u}) - f'(0)\|_{W^{1,\infty}(\mathbb{R})} < \infty$. By [74], Chapter 5, Theorem 2.3, the family $(\mathcal{L}_{st})_{t \geq 0}$ of linear operators generates an evolution family $(P_{st,st'})_{t \geq t' \geq 0}$ of bounded linear operators $\mathcal{V} \to \mathcal{V}$ meeting estimate (2.16), that is,

$$
\|P_{st,st'}\|_{L(\mathcal{V})} \leq e^{\beta(t-t')} \quad \text{with } \beta := \|f'(\hat{u}) - f'(0)\|_{W^{1,\infty}(\mathbb{R})} - \min\{-f'(0), \varepsilon\gamma\}. \qquad \square
$$

A.2.2. *Moving frame.*   In this section, we prove spectral properties of the frozen-wave operator $\mathcal{L}^\#$ (cf. (2.17)), as stated in Proposition 2.8.

We view $\mathcal{L}^\#$ as a perturbation of the limiting operator $\mathcal{L}^\#_{\pm\infty}$ (cf. (2.17))

(A.8a) $$\mathcal{L}^\# = \mathcal{L}^\#_{\pm\infty} + \mathcal{R}^\#,$$

where

(A.8b) $$\mathcal{L}^\#_{\pm\infty} := \begin{pmatrix} \nu\partial_\xi^2 + f'(0) - s\partial_\xi & -1 \\ \varepsilon & -\varepsilon\gamma - s\partial_\xi \end{pmatrix} \quad \text{and} \quad \mathcal{R}^\# := \begin{pmatrix} f'(\hat{u}) - f'(0) & 0 \\ 0 & 0 \end{pmatrix}.$$

LEMMA A.3.   *For $Y \in D(\mathcal{L}^\#) = H^2(\mathbb{R};\mathbb{C})_{\sqrt{\varepsilon}}\oplus H^1(\mathbb{R};\mathbb{C})$ we have*

(A.9) $$(\mathcal{L}^\#_{\pm\infty}Y, Y)_{H_\mathbb{C}} \leq -\kappa\|Y\|^2_{H_\mathbb{C}} - \varepsilon Z\nu\left\|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right\|^2_{L^2(\mathbb{R})} \quad \text{with } \kappa \overset{(2.22)}{=} \min\{-f'(0), \varepsilon\gamma\}.$$

In particular, $\mathcal{L}^\#_{\pm\infty}$ generates a C_0-semigroup $(P^{\pm\infty}_{st})_{t\geq 0}$ of contractions in $H_\mathbb{C}$ satisfying

(A.10) $$\|P^{\pm\infty}_{st}\|_{L(H_\mathbb{C})} \leq e^{-\kappa t}.$$

PROOF.   By density, we may assume $Y = (w, q)^{\mathrm{t}} \in (C^\infty_{\mathrm{c}}(\mathbb{R};\mathbb{C}))^2$ and obtain

$$(\mathcal{L}^\#_{\pm\infty}Y, Y)_{H_\mathbb{C}} = \varepsilon Z\left(\nu\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} + f'(0)w - s\frac{\mathrm{d}w}{\mathrm{d}\xi} - q, w\right)_{L^2(\mathbb{R};\mathbb{C})}$$

$$+ Z\left(\varepsilon(w - \gamma q) - s\frac{\mathrm{d}q}{\mathrm{d}\xi}, q\right)_{L^2(\mathbb{R};\mathbb{C})}$$

$$= \varepsilon Z\left(-\nu\left\|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right\|^2_{L^2(\mathbb{R};\mathbb{C})} + f'(0)\|w\|^2_{L^2(\mathbb{R};\mathbb{C})}\right) - \varepsilon Z\gamma\|q\|^2_{L^2(\mathbb{R};\mathbb{C})}$$

$$+ 2i\varepsilon Z\,\mathrm{Im}((w, q)_{L^2(\mathbb{R};\mathbb{C})}) - \varepsilon Zs\int_\mathbb{R}\left(\overline{\frac{\mathrm{d}w}{\mathrm{d}\xi}}\right)w\,\mathrm{d}\xi - Zs\int_\mathbb{R}\left(\overline{\frac{\mathrm{d}q}{\mathrm{d}\xi}}\right)q\,\mathrm{d}\xi.$$

This implies

$$\mathrm{Re}((\mathcal{L}^\#_{\pm\infty}Y, Y)_{H_\mathbb{C}}) \overset{(2.21b)}{\leq} -\min\{-f'(0), \varepsilon\gamma\}\|Y\|^2_{H_\mathbb{C}} - \frac{Zs}{2}\int_\mathbb{R}\frac{\mathrm{d}}{\mathrm{d}\xi}(\varepsilon|w|^2 + |q|^2)\,\mathrm{d}\xi$$

$$- \varepsilon Z\nu\left\|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right\|^2_{L^2(\mathbb{R})}$$

$$= -\kappa\|Y\|^2_{H_\mathbb{C}} - \varepsilon Z\nu\left\|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right\|^2_{L^2(\mathbb{R})},$$

which is (A.9). From (A.9), we recognize that

$$\mathcal{L}^\#_{\pm\infty} + \kappa\,\mathrm{id}_{H_\mathbb{C}}\colon D(\mathcal{L}^\#) = H^2(\mathbb{R};\mathbb{C})_{\sqrt{\varepsilon}}\oplus H^1(\mathbb{R};\mathbb{C}) \to H_\mathbb{C} \overset{(2.21a)}{=} L^2(\mathbb{R};\mathbb{C})_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R};\mathbb{C})$$

is dissipative, so that, as in the proof of Proposition 2.6, we deduce with the Lax–Milgram and the Lumer–Philips theorem [74], Chapter 1, Theorem 4.3, that $\mathcal{L}^\#_{\pm\infty}$ generates a C_0-semigroup of contractions $(P^{\pm\infty}_{st})_{t\geq 0}$ in $H_\mathbb{C}$ meeting (A.10). $\square$

PROOF OF PROPOSITION 2.8(a).   Since $\mathcal{R}^\#$ is a bounded operator in $H_\mathbb{C}$ with

$$\|\mathcal{R}^\#\|_{L(H_\mathbb{C})} \leq \|f'(\hat{u}) - f'(0)\|_{L^\infty(\mathbb{R})},$$

we conclude with Lemma A.3 and [74], Chapter 5, Theorem 2.3, that $\mathcal{L}^{\#}$ generates a C_0-semigroup $(P_{st}^{\#})_{t\geq 0}$ in $H_{\mathbb{C}}$ meeting the bound

$$(A.11) \qquad \|P_{st}^{\#}\|_{L(H_{\mathbb{C}})} \leq e^{-\rho t} \quad \text{with } \rho := \kappa - \|f'(\hat{u}) - f'(0)\|_{L^{\infty}(\mathbb{R})}.$$

Since $\mathcal{L}^{\#}$ has real coefficients (cf. (2.17)), the statement for H instead of $H_{\mathbb{C}}$ is immediate. $\qquad\square$

PROOF OF PROPOSITION 2.8(b).   We have

$$(\partial_t - \mathcal{L}^{\#})\mathcal{T}_{-st}P_{st,st'}\mathcal{T}_{st'} \overset{(2.19)}{=} \mathcal{T}_{-st}(\partial_t - \mathcal{L}_{st})P_{st,st'}\mathcal{T}_{st'} = 0,$$

so that with

$$\mathcal{T}_{-st}P_{st,st'}\mathcal{T}_{st'}|_{t=t'} = \mathcal{T}_{-st'}\,\mathrm{id}_{\mathcal{V}}\,\mathcal{T}_{st'} = \mathcal{T}_{-st'}\mathcal{T}_{st'}|_{\mathcal{V}} = \mathrm{id}_{\mathcal{V}}$$

the claim follows by uniqueness of the evolution family. $\square$

Note that (A.11) just provides a rough estimate on the action of the semigroup $(P_{st}^{\#})_{t\geq 0}$. In what follows, we provide a more detailed spectral analysis of $\mathcal{L}^{\#}$, leading to the proofs of Proposition 2.8(c) and (d).

LEMMA A.4.   *We have*

$$\sigma_{\mathrm{ess}}(\mathcal{L}_{\pm\infty}^{\#}) \subseteq \{\lambda \in \mathbb{C}: \operatorname{Re}\lambda \leq -\kappa\} \quad \text{where } \kappa \overset{(2.22)}{=} \min\{-f'(0), \varepsilon\gamma\}.$$

In particular, $\sigma_{\mathrm{ess}}(\mathcal{L}_{\pm\infty}^{\#})$ lies to the left of the imaginary axis.

PROOF.   We first use that $\sigma_{\mathrm{ess}}(\mathcal{L}_{\pm\infty}^{\#}) \subseteq \mathcal{R}_{H_{\mathbb{C}}}(\mathcal{L}_{\pm\infty}^{\#})$ (cf. Definition 2.7(d) and (e), and [52], Lemma 4.1.9). Then the result is immediate from (2.1d), Definition 2.7(e), and Lemma A.3. $\square$

Lemma A.4 can be lifted to obtain the essential spectrum of $\mathcal{L}^{\#}$ using the following compactness argument.

LEMMA A.5 (Compactness).   *The operator (cf. (A.8b))*

$$\mathcal{R}^{\#}: H_{\mathbb{C}} \supset H^1(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R}; \mathbb{C}) \to H_{\mathbb{C}}$$

is compact. In particular,

$$\mathcal{L}^{\#} \overset{(A.8)}{=} \mathcal{L}_{\pm\infty}^{\#} + \mathcal{R}^{\#}: H_{\mathbb{C}} \supset D(\mathcal{L}^{\#}) = H^2(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}}\oplus H^1(\mathbb{R}; \mathbb{C}) \to H_{\mathbb{C}}$$

is a relatively compact perturbation of $\mathcal{L}_{\pm\infty}^{\#}: H_{\mathbb{C}} \supset D(\mathcal{L}^{\#}) \to H_{\mathbb{C}}$.

PROOF.   Though the proof is standard, for the sake of self-contained presentations we provide all necessary details. Suppose that $(Y_n)_n \in (H^1(\mathbb{R}; \mathbb{C})_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R}; \mathbb{C}))^{\mathbb{N}}$ with $Y_n = (w_n, q_n)^{\mathsf{t}}$ and $\sup_{n\in\mathbb{N}}\|w_n\|_{H^1(\mathbb{R};\mathbb{C})} < \infty$. Because of

$$\mathcal{R}^{\#}Y_n \overset{(A.8b)}{=} \begin{pmatrix} (f'(\hat{u}) - f'(0))w_n \\ 0 \end{pmatrix},$$

we first prove that $((f'(\hat{u}) - f'(0))w_n)_n$ has a convergent subsequence in $L^2(\mathbb{R}; \mathbb{C})$. Therefore, note that $f''(\hat{u}(\xi))\frac{d\hat{u}}{d\xi}$ and $|f'(\hat{u}(\xi)) - f'(0)| \leq \sup_{w\in[0,\hat{u}(\xi)]\cup[\hat{u}(\xi),0]}|f''(w)||\hat{u}(\xi)|$ decay

exponentially as $|\xi| \to \pm\infty$. Now, setting $y := \arctan\xi$ and $\phi_n(y) := \sqrt{1 + \xi^2}(f'(\hat{u}(\xi)) - f'(0))w_n(\xi)$, we may compute that $\mathrm{d}y = \frac{1}{1+\xi^2}\,\mathrm{d}\xi$, $\frac{\mathrm{d}}{\mathrm{d}y} = (1 + \xi^2)\frac{\mathrm{d}}{\mathrm{d}\xi}$,

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}y} = \xi\sqrt{1 + \xi^2}(f'(\hat{u}(\xi)) - f'(0))w_n(\xi) + (1 + \xi^2)^{\frac{3}{2}}f''(\hat{u}(\xi))\frac{\mathrm{d}\hat{u}}{\mathrm{d}\xi}(\xi)w_n(\xi)$$

$$+ (1 + \xi^2)^{\frac{3}{2}}(f'(\hat{u}(\xi)) - f'(0))\frac{\mathrm{d}w_n}{\mathrm{d}\xi}(\xi),$$

and therefore

$$\|\phi_n\|^2_{H^1((-\frac{\pi}{2}, \frac{\pi}{2}); \mathbb{C})} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(|\phi_n|^2 + \left|\frac{\mathrm{d}\phi_n}{\mathrm{d}y}\right|^2\right)\mathrm{d}y$$

$$\leq C\int_{\mathbb{R}}\left(|w_n|^2 + \left|\frac{\mathrm{d}w_n}{\mathrm{d}\xi}\right|^2\right)\mathrm{d}\xi = C\|w_n\|^2_{H^1(\mathbb{R}; \mathbb{C})}$$

for an n-independent constant $C < \infty$. By the Rellich–Kondrachov theorem, $(\phi_n)_n$ has a convergent subsequence in $L^2((-\frac{\pi}{2}, \frac{\pi}{2}); \mathbb{C})$ and because

$$\|\phi_n - \phi_m\|_{L^2((-\frac{\pi}{2}, \frac{\pi}{2}); \mathbb{C})} = \|(f'(\hat{u}(\xi)) - f'(0))(w_n - w_m)\|_{L^2(\mathbb{R}; \mathbb{C})},$$

we infer that $((f'(\hat{u}(\xi)) - f'(0))w_n)_n$ has a convergent subsequence in $L^2(\mathbb{R}; \mathbb{C})$.

Suppose that $(Y_n)_n \in (D(\mathcal{L}^\#))^{\mathbb{N}}$ meets $\sup_{n\in\mathbb{N}}(\|Y_n\|_{H_{\mathbb{C}}} + \|\mathcal{L}^\#_{\pm\infty}Y_n\|_{H_{\mathbb{C}}}) < \infty$. Then, it suffices to show that $\sup_{n\in\mathbb{N}}\|w_n\|_{H^1(\mathbb{R}; \mathbb{C})} < \infty$. Observe that by interpolation, we have for $Y = (w, q)^{\mathrm{t}} \in (C_{\mathrm{c}}^\infty(\mathbb{R}; \mathbb{C}))^2$

$$\|\mathcal{L}^\#_{\pm\infty}Y\|^2_{H_{\mathbb{C}}} = \varepsilon\left\|v\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} + f'(0)w - s\frac{\mathrm{d}w}{\mathrm{d}\xi} - q\right\|^2_{L^2(\mathbb{R}; \mathbb{C})} + \left\|\varepsilon w - \varepsilon\gamma q - s\frac{\mathrm{d}q}{\mathrm{d}\xi}\right\|^2_{L^2(\mathbb{R}; \mathbb{C})}$$

$$\geq C_1\left\|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right\|^2_{L^2(\mathbb{R}; \mathbb{C})} - C_2(\|w\|^2_{L^2(\mathbb{R}; \mathbb{C})} + \|q\|^2_{L^2(\mathbb{R}; \mathbb{C})}),$$

with $C_1 > 0$ and $C_2 < \infty$ independent of n. Since

$$\|Y_n\|^2_{H_{\mathbb{C}}} \overset{(2.21b)}{=} \varepsilon Z\|w_n\|^2_{L^2(\mathbb{R}; \mathbb{C})} + Z\|q_n\|^2_{L^2(\mathbb{R}; \mathbb{C})},$$

we may conclude with (2.1d) that indeed $\sup_{n\in\mathbb{N}}\|w_n\|_{H^1(\mathbb{R}; \mathbb{C})} < \infty$ holds true. $\square$

PROOF OF PROPOSITION 2.8(ci). If

$$\sigma_{\mathrm{ess}}(\mathcal{L}^\#) = \sigma_{\mathrm{ess}}(\mathcal{L}^\#_{\pm\infty}) \tag{A.12}$$

holds true, it follows in particular that

$$\sigma_{\mathrm{ess}}(\mathcal{L}^\#) \subseteq \{\lambda \in \mathbb{C}\colon \mathrm{Re}\,\lambda \leq -\kappa\} \quad \text{where } \kappa \overset{(2.22)}{=} \min\{-f'(0), \varepsilon\gamma\}$$

by Lemma A.4. The equality (A.12), on the other hand, follows by Weyl's essential spectrum theorem [52], Theorem 2.2.6, because the operators

$$\mathcal{L}^\#, \mathcal{L}^\#_{\pm\infty}\colon D(\mathcal{L}^\#) = H^2(\mathbb{R}; \mathbb{C})\,_{\sqrt{\varepsilon}}\oplus H^1(\mathbb{R}; \mathbb{C}) \to H_{\mathbb{C}} \overset{(2.21a)}{=} L^2(\mathbb{R}; \mathbb{C})\,_{\sqrt{\varepsilon}}\oplus L^2(\mathbb{R}; \mathbb{C})$$

are closed (which follows by an interpolation argument as in the proof of Lemma A.5) and $\mathcal{L}^\#$ is a compact pertubation of $\mathcal{L}^\#$ by Lemma A.5. $\square$

PROOF OF PROPOSITION 2.8(cii). It suffices to prove that for $\lambda \in \sigma_{\mathrm{p}}(\mathcal{L}^\#)$ any corresponding eigenvector $Y = (w, q)^{\mathrm{t}} \in D(\mathcal{L}^\#)$ is bounded and has infinitely many bounded

derivatives to conclude that in $H_{\mathbb{C}}$ there are no additional eigenvalues compared to the ones obtained by Yanagida [91], Section 5.1 (cf. [48], Section 1, Theorem, in the case of the cubic polynomial without cut off).

Indeed, for Y as above satisfying $\mathcal{L}^{\#}Y = \lambda Y$, we have

$$\nu \frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} \overset{(2.17)}{=} \underbrace{\lambda w - f'(\hat{u})w + s\frac{\mathrm{d}w}{\mathrm{d}\xi} + q}_{\in H^1(\mathbb{R};\mathbb{C})},$$

$$s\frac{\mathrm{d}q}{\mathrm{d}\xi} \overset{(2.17)}{=} \underbrace{-\lambda q + \varepsilon w - \varepsilon\gamma q}_{\in H^1(\mathbb{R};\mathbb{C})},$$

which immediately yields $w \in H^3(\mathbb{R};\mathbb{C})$ and $q \in H^2(\mathbb{R};\mathbb{C})$. Inductively, we obtain that $w, q \in H^k(\mathbb{R};\mathbb{C})$ for any $k \in \mathbb{N}$, giving smoothness and boundedness of w, q, and all derivatives.

With the stability analysis in [91], Section 5.1, we conclude that, except for the simple eigenvalue 0, all eigenvalues are to the left of the imaginary axis with real part bounded by $\lambda^*(\varepsilon)$. We further remark that in (2.20) it has already been noted that $\frac{\mathrm{d}\hat{X}}{\mathrm{d}\xi}$ is an eigenvector to the eigenvalue 0. $\quad\square$

In what follows, we need a regularizing effect of the semigroup $(P_{st}^{\pm\infty})_{t\geq 0}$ of Lemma A.3 in the first component.

LEMMA A.6. *For $Y^{(0)} = (w^{(0)}, q^{(0)})^{\mathrm{t}} \in H_{\mathbb{C}}$ we write*

$$Y_{\pm\infty}(t, \cdot) := (w_{\pm\infty}(t, \cdot), q_{\pm\infty}(t, \cdot))^{\mathrm{t}} := P_{st}^{\pm\infty}Y^{(0)}.$$

Then, it holds that $w_{\pm\infty}, q_{\pm\infty}, \frac{\sqrt{t}\partial_\xi w_{\pm\infty}}{1+\sqrt{t}} \in L^\infty([0,\infty); L^2(\mathbb{R};\mathbb{C}))$ with

$$\|w_{\pm\infty}\|_{L^\infty([0,\infty); L^2(\mathbb{R};\mathbb{C}))} + \|q_{\pm\infty}\|_{L^\infty([0,\infty); L^2(\mathbb{R};\mathbb{C}))} + \left\|\frac{\sqrt{t}\partial_\xi w_{\pm\infty}}{1+\sqrt{t}}\right\|_{L^\infty([0,\infty); L^2(\mathbb{R};\mathbb{C}))}$$

$$\leq C\|Y^{(0)}\|_{H_{\mathbb{C}}},$$

where $C < \infty$ is independent of $Y^{(0)}$.

PROOF. First, assume $Y^{(0)} = (w^{(0)}, q^{(0)})^{\mathrm{t}} \in (C_c^\infty(\mathbb{R};\mathbb{C}))^2$. Then, by construction

(A.13) $$\partial_t Y_{\pm\infty} - \mathcal{L}_{\pm\infty}^{\#}Y_{\pm\infty} = 0,$$

which implies

$$\partial_t^j (\mathcal{L}_{\pm\infty}^{\#})^k Y_{\pm\infty} = (\mathcal{L}_{\pm\infty}^{\#})^{j+k} Y_{\pm\infty} = P_{st}^{\pm\infty}(\mathcal{L}_{\pm\infty}^{\#})^{j+k} Y^{(0)} \quad \text{for all } j, k \in \mathbb{N}_0$$

and hence $(\mathcal{L}_{\pm\infty}^{\#})^k Y_{\pm\infty} \in C^\infty([0,\infty); H_{\mathbb{C}})$ for any $k \in \mathbb{N}_0$. The coercivity/dissipativity estimate (A.9) of Lemma A.3 then yields $w_{\pm\infty} \in C^\infty([0,\infty); H^k(\mathbb{R}))$ for any $k \in \mathbb{N}_0$.

Having established qualitative regularity for regular initial data, we next prove a priori estimates. Indeed, by testing (A.13) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_{\pm\infty}\|_{H_{\mathbb{C}}}^2 - (\mathcal{L}_0^{\#}Y_{\pm\infty}, Y_{\pm\infty})_{H_{\mathbb{C}}} = 0,$$

so that with (A.9) of Lemma A.3 we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t, \cdot)\|_{H_{\mathbb{C}}}^2 + \kappa\|Y(t, \cdot)\|_{H_{\mathbb{C}}}^2 + \varepsilon Z\nu\|\partial_\xi w(t, \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2 \leq 0.$$

Integrating in time yields

$$(A.14) \quad \begin{aligned} &\frac{1}{2} \|Y_{\pm\infty}(t, \cdot)\|_{H_{\mathbb{C}}}^2 + \int_0^t \left(\kappa \|Y_{\pm\infty}(t', \cdot)\|_{H_{\mathbb{C}}}^2 + \varepsilon Z \nu \|\partial_\xi w_{\pm\infty}(t', \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2 \right) \mathrm{d}t' \\ &\leq \frac{1}{2} \|Y^{(0)}\|_{H_{\mathbb{C}}}^2. \end{aligned}$$

To obtain a pointwise bound in time on $\|\partial_\xi w_{\pm\infty}(t, \cdot)\|_{L^2(\mathbb{R})}^2$, observe that from (A.8b) and (A.13) it follows that

$$\partial_t \partial_\xi w_{\pm\infty} - \nu \partial_\xi^3 w_{\pm\infty} - f'(0) \partial_\xi w_{\pm\infty} + s \partial_\xi^2 w_{\pm\infty} + \partial_\xi q_{\pm\infty} = 0.$$

Testing with $t \partial_\xi w_{\pm\infty}$ gives with

$$\mathrm{Re}\big(\partial_\xi^2 w_{\pm\infty}(t, \cdot), \partial_\xi w_{\pm\infty}(t, \cdot)\big)_{L^2(\mathbb{R};\mathbb{C})} = \frac{1}{2} \mathrm{Re} \int_{\mathbb{R}} \partial_\xi \big|\partial_\xi w_{\pm\infty}(t, \xi)\big|^2 \mathrm{d}\xi = 0$$

and

$$\mathrm{Re}\big(\partial_\xi q_{\pm\infty}(t, \cdot), \partial_\xi w_{\pm\infty}(t, \cdot)\big)_{L^2(\mathbb{R})} \leq \frac{\nu}{2} \|\partial_\xi^2 w_{\pm\infty}(t, \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2 + \frac{1}{2\nu} \|q_{\pm\infty}(t, \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2$$

that

$$\begin{aligned} &\frac{t}{2} \|\partial_\xi w_{\pm\infty}(t, \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2 + \int_0^t t' \left(\frac{\nu}{2} \|\partial_\xi^2 w_{\pm\infty}(t', \cdot)\|_{L^2(\mathbb{R})}^2 - f'(0) \|\partial_\xi w_{\pm\infty}(t', \cdot)\|_{L^2(\mathbb{R})}^2 \right) \mathrm{d}t' \\ &\leq \frac{t}{2\nu} \int_0^t \|q_{\pm\infty}(t', \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2 \mathrm{d}t' + \frac{1}{2} \int_0^t \|\partial_\xi w_{\pm\infty}(t', \cdot)\|_{L^2(\mathbb{R};\mathbb{C})}^2 \mathrm{d}t'. \end{aligned}$$

The combination with (2.21b) and (A.14) yields that there exists a constant $C < \infty$ such that

$$\underset{t \geq 0}{\mathrm{ess\text{-}sup}} \frac{\sqrt{t}}{1 + \sqrt{t}} \|\partial_\xi w_{\pm\infty}(t, \cdot)\|_{L^2(\mathbb{R};\mathbb{C})} \leq C \|Y^{(0)}\|_{H_{\mathbb{C}}}^2.$$

The statement of the lemma now follows by density of $(C_c^\infty(\mathbb{R}; \mathbb{C}))^2$ in $H_{\mathbb{C}}$. $\square$

PROOF OF PROPOSITION 2.8(d).   We loosely follow the approach in [2], Section 3.2, Proposition 3.5. First observe that by Lemma A.5 and Lemma A.6 the operator $\mathcal{R}^\# P_{st}^{\pm\infty} \Pi^\# \in L(H_{\mathbb{C}})$ is compact for any $t > 0$. This compactness implies thanks to [2], Proposition 3.4, that for every $t \geq 0$ the operator $P_{st}^\# \Pi^\# - P_{st}^{\pm\infty} \Pi^\#$ is compact as well. By estimate (A.10) of Lemma A.3 and the Neumann series, we recognize that the operator $P_{st}^{\pm\infty} \Pi^\#$ has no spectrum outside the disc $\{\mu \in \mathbb{C} : |\mu| \leq e^{-\kappa t} \|\Pi^\#\|_{L(H_{\mathbb{C}})}\}$. Now, since $P_{st}^\# \Pi^\#$ is a compact perturbation of $P_{st}^{\pm\infty} \Pi^\#$, the spectrum of $P_{st}^\# \Pi^\#$ in

$$\big\{\mu \in \mathbb{C} : |\mu| > e^{-\kappa t} \|\Pi^\#\|_{L(H_{\mathbb{C}})}\big\}$$

only contains point spectrum $\sigma_{\mathrm{p}}(P_{st}^\# \Pi^\#)$ (cf. Definition 2.7 and [52], Theorem 2.2.6). Using [74], Chapter 2, Theorem 2.4, we infer that

$$\sigma_{\mathrm{p}}\big(P_{st}^\# \Pi^\#\big) \subseteq \{0\} \cup \{e^{\lambda t} : \lambda \in \sigma_{\mathrm{p}}(\mathcal{L}^\# \Pi^\#)\} \subseteq \{\mu \in \mathbb{C} : |\mu| \leq e^{\max\{-\kappa, \lambda^*(\varepsilon)\} t}\},$$

where we have used Proposition 2.8(cii) in the last inclusion and that

$$\sigma_{\mathrm{p}}(\mathcal{L}^\# \Pi^\#) = \sigma_{\mathrm{p}}(\mathcal{L}^\#) \setminus \{0\}$$

since 0 is not an eigenvalue of $\mathcal{L}^\# \Pi^\#$. Altogether, we have

$$\sigma\big(P_{st}^\# \Pi^\#\big) \subseteq \{\mu \in \mathbb{C} : |\mu| \leq e^{\max\{-\kappa, \lambda^*(\varepsilon)\} t} \|\Pi^\#\|_{L(H_{\mathbb{C}})}\}$$

because $\|\Pi^\#\|_{L(H_{\mathbb{C}})} \geq 1$.

Now, let $\max\{-\kappa, \lambda^*(\varepsilon)\} < -\vartheta < 0$. Since the spectral radius $\lim_{n\to\infty}\|(P_{st}^\#\Pi^\#)^n\|_{L(H_\mathbb{C})}^{\frac{1}{n}}$ of $P_{st}^\#\Pi^\#$ meets

$$\lim_{n\to\infty}\|(P_{st}^\#\Pi^\#)^n\|_{L(H_\mathbb{C})}^{\frac{1}{n}} = \lim_{n\to\infty}\|P_{stn}^\#\Pi^\#\|_{L(H_\mathbb{C})}^{\frac{1}{n}}$$

and

$$\lim_{n\to\infty}\|(P_{st}^\#\Pi^\#)^n\|_{L(H_\mathbb{C})}^{\frac{1}{n}} = \lim_{n\to\infty}\max\{|\mu|: \mu \in \sigma(P_{st}^\#\Pi^\#)\} \leq e^{\max\{-\kappa,\lambda^*(\varepsilon)\}t}\|\Pi^\#\|_{L(H_\mathbb{C})},$$

we have $\|P_{stn}^\#\Pi^\#\|_{L(H_\mathbb{C})}^{\frac{1}{n}} \leq e^{-\vartheta t}$ for t and n large enough. Thus there exists $C_\vartheta < \infty$ such that (2.24) holds true. $\square$

PROOF OF PROPOSITION 2.8(e). Suppose that $Y \in H$ is real-valued. Then

$$\Pi^{\#,0}Y \stackrel{(2.23)}{=} \frac{1}{2\pi i}\oint_{|\lambda|=r} X_\lambda \, d\lambda = \frac{r}{2\pi}\int_0^{2\pi} X_{re^{i\tau}}e^{i\tau}\, d\tau \quad \text{where } X_\lambda := (\lambda\,\mathrm{id}_H - \mathcal{L}^\#)^{-1}Y.$$

From the equation

$$\lambda X_\lambda - \mathcal{L}^\# X_\lambda = Y \quad \text{where } |\lambda| = r,$$

and because the coefficients of $\mathcal{L}^\#$ are real (cf. (2.17)), it follows that

$$\overline{\lambda X_\lambda} - \mathcal{L}^\#\overline{X_\lambda} = Y.$$

Because $r > 0$ is chosen sufficiently small we have $\lambda \in \rho(\mathcal{L}^\#)$ (cf. Definition 2.7(a)) and it follows due to uniqueness $\overline{X_\lambda} = X_{\overline{\lambda}}$. Hence,

$$\overline{\Pi^{\#,0}Y} = \frac{r}{2\pi}\int_0^{2\pi}\overline{X_{re^{i\tau}}}e^{-i\tau}\, d\tau = \frac{r}{2\pi}\int_0^{2\pi}X_{re^{-i\tau}}e^{-i\tau}\, d\tau = \frac{r}{2\pi}\int_{-2\pi}^0 X_{re^{i\tau}}e^{i\tau}\, d\tau$$

$$= \frac{r}{2\pi}\int_0^{2\pi}X_{re^{i\tau}}e^{i\tau}\, d\tau = \Pi^{\#,0}Y.$$

By (2.23) also $\overline{\Pi^\# Y} = \Pi^\# Y$, that is, $\Pi^{\#,0}, \Pi^\#: H \to H$ are well defined and the estimates of the operator norms are trivial, too. $\square$

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