The Semilinear Cahn-Hilliard-Gurtin System in Critical Spaces

by
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To obtain the degree of Master of Science in Applied Mathematics
at Delft University of Technology,
to be defended publicly on Monday 27-9-2021, at 14:00.

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Abstract

In this thesis, the semilinear Cahn-Hilliard-Gurtin equation is studied using the method of Maximal Regularity. In 2012, Wilke developed a linear theory in L^p-spaces, and achieved a local and global well-posedness result for large p. In 2013, Denk and Kaip developed a linear theory in mixed integrability $L^p L^q$-spaces, using the method of Newton polygons. In this thesis, we connected the recent weighted anisotropic Mikhlin multiplier theorem of Lorist with the method of Newton polygons, leading to a linear theory in time-weighted $L^p w_\alpha L^q$-spaces, which is novel. By a postulation that the linear theory also holds in domains, we are able to treat the local well-posedness in the recently developed critical space setting of Prüss et al. This approach draws upon recent advances in interpolation theory in the setting of fractional Sobolev spaces with power weights in time, such as exhibited in the work of Agresti and Veraar. By adapting the global well-posedness result of Wilke, we are able to treat the semilinear equation in less regular spaces, i.e. smaller integrability parameters p and q, and with rough initial data.

1 Introduction

In this thesis, we will consider the generalized Cahn-Hilliard-Gurtin equation using the methods of maximal L^p-regularity. This method was found to be useful for the analysis of many nonlinear partial differential equations, and even though the concept is classical, some of the main achievements in the abstract theory were only obtained in the 1990's and at the beginning of the 21st century. In essence, the idea is to first consider a linearization of the given nonlinear p.d.e., and try to solve this in some optimal way before treating the full nonlinear problem with a Banach fixed point argument. To make this more precise, let $p \in (1, \infty)$, X_0 and X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$, i.e. X_1 embeds densely into X_0. Consider an abstract quasilinear p.d.e. of the form

$$\begin{aligned} \partial_t u(t) + A(u(t))u(t) &= f(t), \qquad t \in J, \\ u(0) &= u_0, \qquad t = 0. \end{aligned} \tag{1}$$

Here $J := [0, T]$ is a bounded time interval. Assume that $A : X^{\mathrm{Tr}} \to B(X_1, X_0)$ is continuous, where X^{Tr} is the so-called trace space, which we shall introduce later in the thesis. Furthermore, we assume that the function f is a vector-valued function in the Bochner space $L^p(J; X_0)$. For an introductory treatment of vector-valued functions and Bochner integrals, we refer the reader to Chapter 1 of [HvVW16]. Now denote $\tilde{A} := A(\psi)$ for a fixed function $\psi \in X^{\mathrm{Tr}}$. The linearization of this system is then given by

$$\begin{aligned} \partial_t u(t) + \tilde{A}u(t) &= f(t), \qquad t \in J, \\ u(0, x) &= u_0(x), \quad t = 0. \end{aligned} \tag{2}$$

Typically, $\tilde{A}$ can be an elliptic differential operator, such as the Laplacian, on a Banach space $X_0 = L^q(\mathbb{R}^n)$ with $q \in (1, \infty)$. In the maximal regularity approach, the aim is to solve the linearized equation (2) in appropriate function spaces, and to show that the solution has the 'optimal' regularity one could reasonably expect. In the case of this abstract Cauchy problem, it is reasonable to expect that $\partial_t u \in L^p(J; X_0)$ and $\tilde{A}u \in L^p(J; X_1)$. An even stronger assumption would be that $u \in L^p(J; X_0)$, which would give an 'optimal' space for the solution u,

$$u \in Z := H^{1,p}(J; X_0) \cap L^p(J; X_1). \tag{3}$$

In this case, we will see that the initial condition u_0 is determined by the colloquially called Trace Theorem, which states that $Z \hookrightarrow C(J; X^{\mathrm{Tr}})$. For this reason, we are interested in finding a solution operator

$$\mathcal{S} : L^p(J; X_0) \times X^{\mathrm{Tr}} \longrightarrow Z : (f, u_0) \mapsto u$$

that induces an isomorphism between the Banach space associated to the solution $u \in Z$, and the Banach spaces associated to the data $f \in L^p(J; X_0)$ and $u_0 \in X^{\mathrm{Tr}}$. If such a solution operator exists, then the solution of the linearized problem is bounded by the data, as then

$$||u||_Z \leq ||\mathcal{S}||(||u_0||_{X^{\mathrm{Tr}}} + ||f||_{L^p(J;X_0)}). \tag{4}$$

At this point, a Banach fixed-point argument can be used to find a solution to the nonlinear equation (1). Typically, this gives a local well-posedness result, i.e. the existence of solutions on small time intervals. To achieve global well-posedness, other methods than maximal regularity need to be used, such as a priori energy estimates. For a standard approach in elliptical p.d.e. theory, we refer the reader to [DHP03] and [KW04]. For a more recent introduction to a maximal regularity approach for parabolic evolution equations, we refer the reader to [Den20].

In this thesis the Cahn-Hilliard equation is studied, named after John W. Cahn and John E. Hilliard. It describes the process of spontaneous phase separation in a binary fluid (see [CH58]). In the 90's Gurtin proposed a generalization of the classical Cahn-Hilliard equation in an attempt to develop a more complete theory (see [Gur96]). Several shortcomings of the classical equation were addressed. By considering a balance of microscopic forces in conjunction with constitutive equations consistent with the second law of thermodynamics, Gurtin's framework can account for processes such as deformation and heat transfer. Let u denote the density – or concentration – of diffusing species of atoms, and μ the so-called chemical potential. Then the semilinear version of the Cahn-Hilliard-Gurtin equations on $\mathbb{R}^n$ are given by

$$\begin{aligned} \partial_t u - \operatorname{div}(B\nabla\mu) &= \operatorname{div}(a\partial_t u) + f, & t \in J, x \in \mathbb{R}^n, \\ \mu - c \cdot \nabla\mu &= \beta\partial_t u - \Delta u + \Phi'(u) + g, & t \in J, x \in \mathbb{R}^n, \\ u &= u_0, & t = 0, x \in \mathbb{R}^n. \end{aligned} \tag{5}$$

Here $a, c \in \mathbb{R}^n$, $\beta > 0$ and $B \in \mathbb{R}^{n\times n}$ are constant, and satisfy some technical condition for a solutions to exist. The non-linearity $\Phi : \mathbb{R} \to \mathbb{R}$ denotes the potential, and is classically assumed to be in $C_b^3(\mathbb{R})$. It is clear that (5) is not a parabolic evolution equation, as it has a more complicated structure. By naively inspecting the equations, it is not straightforward to see how to solve (u, μ) given the data (f, g, u_0), illustrating the difficulty of this problem.

In 2012, Wilke developed maximal L^p-regularity theory on the whole space $\mathbb{R}^n$, the half-space $\mathbb{R}^n_+$ and on bounded domains (see [Wil12]). Using a localization argument, Wilke proved a local well-posedness result of a quasilinear Cahn-Hilliard-Gurtin equation, under the assumption that p is large. In 2013, Denk and Kaip developed linear theory for the Cahn-Hilliard-Gurtin equations in mixed-integrability L^pL^q-spaces, using the methods of Newton polygons (see [DK13], Section 4.3). Compared to Wilke, this method is more structured, as the complicated structure of the equation is captured by a general approach.

In this thesis, we will connect the Newton polygon approach of Denk and Kaip to recent advances in Fourier multiplier theory. By checking Mikhlin type conditions for Newton polygons, the weighted anisotropic mixed-norm Mikhlin multiplier theorem (see [Lor20], Section 7) will allow us to build linear theory in weighted mixed-norm spaces. This allows us to consider spaces with Muckenhaupt weights, such as a power weights. In the field of harmonic analysis Muckenhaupt weights posses desirable features, such as extrapolation of weighted norm estimates. Furthermore, it is possible to treat rough initial data with Muckenhaupt weights.

Due to time constraints we were not able to rigorously consider linear theory for the half-space $\mathbb{R}^n_+$ or domains. Instead, for the purpose of obtaining new local and global well-posedness results for the semilinear Cahn-Hilliard-Gurtin equation, the linear theory for domains, as well as the localization argument were postulated.

Using the maximal regularity result for time-weighted L^pL^q-spaces, we are able to consider the Cahn-Hilliard-Gurtin equations in Critical Spaces. The setting of Critical Spaces is a recent invention of Prüss et al. (see [PW17] and [PSW18]). Drawing upon recent advances in interpolation theory in the setting of fractional Sobolev spaces with power weights in time, as exhibited in the work of [AV20], we are in a position to study the semilinear equation in spaces with lower regularity.

Outline

In Chapter 2 we introduce the concepts of homogeneous functions, anisotropic distance functions, inhomogeneous symbol classes, and Newton polygons. We introduce the reader to the weighted mixed-norm Mikhlin multiplier theorem, sectorial operators and the H^∞-calculus. Then, by considering the scaling of N-parameter elliptic symbols, we give conditions for a Fourier multiplier operator to be a sectorial operator with a bounded H^∞-calculus in Proposition 2.46. Furthermore, we characterize the domain of a certain class of Fourier multiplier operators in Proposition 2.48. Then, utilizing these results, together with the Trace Theorem and a Paley-Wiener argument, we give a Maximal regularity result for the heat equation in Proposition 2.60.

In Chapter 3 we develop linear theory for the Cahn-Hilliard-Gurtin problem on $\mathbb{R}^n$, and postulate a maximal regularity result for domains together with a localization argument in $\mathbb{R}^n$.

In Chapter 4 we first adapt the classical local well-posedness result of Wilke to a weighted L^pL^q-setting. Then we consider local well-posedness using the method of Critical Spaces.

In Chapter 5 we adapt the global well-posedness argument of Wilke, such that it becomes compatible, to some degree, with the local well-posedness result of the critical spaces.

Acknowledgments

I would like to take this opportunity to thank my supervisors Mark Veraar and Jonas Sauer for introducing me to the subject of this thesis, and much more mathematics. I'm grateful for all their help and patience they provided during our many online meetings. Also I would like to thank them for all their useful comments on earlier versions of this thesis. With the situation in the world, it has been a difficult year for all of us. Yet I always felt part of the (S)PDE group, with the weekly symposiums, social zoom meetings, and even occasional bike ride. Finally, I would like to thank Johan Dubbeldam for completing the thesis committee.

2 Preliminaries

2.1 Homogeneous Functions

A subset L of a field $\mathbb{R}^n$ or $\mathbb{C}^n$ is called a cone if $az \in L$ for all $z \in L$ and $a \geq 0$. Prototypical examples of this are the whole field $\mathbb{R}^n$ or $\mathbb{C}^n$, a sector $\Sigma_\sigma := \{z \in \mathbb{C};\, |\arg(z)| < \sigma\}$, and a bi-sector $\Sigma_\sigma^{\mathrm{bi}} := \Sigma_\sigma \cup (-\Sigma_\sigma)$. In the next subsection we shall use the letter L exclusively for a cone.

Definition 2.1. A function $f \in C(L;\mathbb{C})$ is called homogeneous of degree $N \in \mathbb{C}$ if the identity

$$\delta^\lambda f(x) = \lambda^N f(x) \tag{6}$$

holds for all $x \in L \setminus \{0\}$ and $\lambda > 0$, where $\delta^\lambda f(x) := f(\lambda x)$ denotes the dilation operator. The set of homogeneous functions f of degree N such that $f(x) \neq 0$ for all $x \in L \setminus \{0\}$ will be denoted by $S^{(N)}(L)$.

Proposition 2.2. *Suppose $f \in S^{(N)}(L)$, then there exists a constant $C_0 \in [0,\infty)$ such that*

$$|f(x)| \leq C_0 |x|^{Re(N)}, \quad C_0 = \max_{|y|=1} |f(y)|.$$

If $\min_{|y|=1} |f(y)| > 0$, then there also exists a constant $C_1 \in [0,\infty)$ such that

$$|x|^{Re(N)} \leq C_1 |f(x)|, \quad C_1 = \max_{|y|=1} |f(y)|^{-1}.$$

Furthermore, if f is also holomorphic on the interior of L, i.e. $f \in S^{(N)}(L) \cap H(\mathring{L})$, then $\partial^\alpha u \in S^{(N-|\alpha|)}(L) \cap H(\mathring{L})$ for all $\alpha \in \mathbb{N}^n$.

Proof. This follows directly from the following calculation

$$|f(x)| = \left|\delta^{|x|} f\left(\frac{x}{|x|}\right)\right| = \left||x|^N f\left(\frac{x}{|x|}\right)\right| = |x|^{\mathrm{Re}(N)} \left|f\left(\frac{x}{|x|}\right)\right| \leq |x|^{\mathrm{Re}(N)} \max_{|y|=1} |f(y)|.$$

Assuming $\min_{|y|=1} |f(y)| > 0$ the second statement follows from

$$|x|^{\mathrm{Re}(N)} = |f(x)| \left|f\left(\frac{x}{|x|}\right)\right|^{-1} \leq |f(x)| \max_{|y|=1} |f(y)|^{-1}.$$

For the third statement let $\lambda > 0$, then by a calculation we see

$$\lambda^N \partial^\alpha f(x) = \partial^\alpha \delta^\lambda f(x) = \partial^\alpha f(\lambda x) = \lambda^{|\alpha|} \delta^\lambda \partial^\alpha f(x).$$

As we now have that $\lambda^{N-|\alpha|} \partial^\alpha f = \delta^\lambda \partial^\alpha f$, the claim follows. $\square$

Definition 2.3. For $\mathbf{a} = (a_1, ..., a_n) \in (0,\infty)^n$ let $|\cdot|_{\mathbf{a}}$ denote the anisotropic distance function on $\mathbb{R}^n$, which is defined as

$$|x|_{\mathbf{a}} := \left(\sum_{k=1}^n |x_k|^{2/a_k}\right)^{1/2}, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n. \tag{7}$$

Let $\lambda > 0$, then the anisotropic scaling will be denoted by

$$\lambda^{\mathbf{a}} x := (\lambda^{a_1} x_1, ..., \lambda^{a_n} x_n), \quad x \in \mathbb{R}^n,$$
$$\lambda^{t\mathbf{a}} x := (\lambda^t)^{\mathbf{a}}, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \setminus \{0\}.$$

And the element-wise powers of vectors will be denoted by

$$x^\lambda := (x_1^\lambda, ..., x_n^\lambda), \quad x \in \mathbb{R}^n.$$

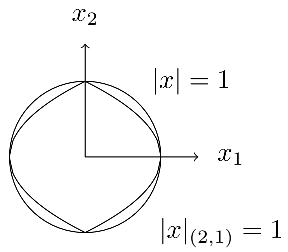


Figure 1: Prototypical example of a unit ball in $(\mathbb{R}^2, |\cdot|)$ and $(\mathbb{R}^2, |\cdot|_{\mathbf{a}})$.

Remark 2.4. There exists other types of distance functions in literature for working with anisotropic spaces. In the recent book of Amann (see [Ama19], Section 3.2), the distance function (7) is called the natural ν-quasinorm. This work, together with the work of Denk and Kaip (see [DK13], Definition 3.15) also considers the Euclidean ν-quasinorm $|x|_{\mathrm{eucl},\mathbf{a}}$, which is defined as

$$\sum_{k=1}^{n} \frac{x_k^2}{|x|_{\mathrm{eucl}}^{2a_k}} = 1, \text{ for } x \neq 0,$$

and $|0|_{\mathrm{eucl},\mathbf{a}} := 0$. Notice that these two distance functions are not equivalent, in the sense that there do not exists constants $C_1, C_2 > 0$ such that $C_1|x|_{\mathrm{eucl},\mathbf{a}} \leq |x|_{\mathbf{a}} \leq C_2|x|_{\mathrm{eucl},\mathbf{a}}$ for all $x \in \mathbb{R}^n$.

Proposition 2.5. *The anisotropic distance function $|\cdot|_{\mathbf{a}}$ has the following properties:*

(i) $|\lambda^{\boldsymbol{a}} x| = \lambda |x|_{\boldsymbol{a}}$

(ii) For $\boldsymbol{a} \in (\mathbb{N} \setminus \{0\})^n$ and $x, y \in \mathbb{R}^n$ the triangle inequality $|x + y|_{\boldsymbol{a}} \leq |x|_{\boldsymbol{a}} + |y|_{\boldsymbol{a}}$ holds.

Proof. (i) For $\lambda > 0$ we have

$$|\lambda^{\mathbf{a}} x|_{\mathbf{a}} = \left(\sum_{k=1}^{n} |\lambda^{a_k} x_k|^{2/a_k} \right)^{1/2} = \lambda \left(\sum_{k=1}^{n} |x_k|^{2/a_k} \right)^{1/2} = \lambda |x|_{\mathbf{a}}.$$

(ii) Let $x, y \in \mathbb{R}^n$ and notice that for all $k \in \mathbb{N} \setminus \{0\}$ using the binomial expansion

$$|x| + |y| \leq (|x|^{1/k} + |y|^{1/k})^k = \sum_{j=0}^{k} \binom{k}{j} |x|^{j/k} |y|^{(k-j)/k}.$$

Taking the k-th root on both sides yields

$$(|x| + |y|)^{1/k} \leq |x|^{1/k} + |y|^{1/k}. \tag{8}$$

Now the triangle inequality for $|\cdot|_{\mathbf{a}}$ follows by an application of the Minkowski inequality:

$$\begin{aligned} |x + y|_{\mathbf{a}} &= \left(\sum_{k=1}^{n} |x_k + y_k|^{2/a_k} \right)^{1/2} \\ &\overset{(8)}{\leq} \left(\sum_{k=1}^{n} (|x_k|^{1/a_k} + |y_k|^{1/a_k})^2 \right)^{1/2} \\ &\overset{\text{Minkowski}}{\leq} \left(\sum_{k=1} |x_k|^{2/a_k} \right)^{1/2} + \left(\sum_{k=1}^{n} |y_k|^{2/a_k} \right)^{1/2} \\ &= |x|_{\mathbf{a}} + |y|_{\mathbf{a}} \end{aligned}$$

□

Definition 2.6. Let $\mathbf{a} = (a_1, ..., a_n) \in (0, \infty)^n$. We say a function $f \in C(\mathbb{R}^n)$ is homogeneous of order/degree $N \in \mathbb{C}$ w.r.t. the quasi-norm $|\cdot|_{\mathbf{a}}$ if for all $x \in L \setminus \{0\}$ and $\lambda > 0$ the identity

$$\delta_{\mathbf{a}}^{\lambda} f(x) = \lambda^N f(x) \tag{9}$$

holds, where $\delta_{\mathbf{a}}^{\lambda} f(x) := f(\lambda^{\mathbf{a}} x) = (\lambda^{a_1} x_1, ..., \lambda^{a_n} x_n)$ denotes the anisotropic dilation operator.

Remark 2.7. For $\mathbf{a} = (1, ..., 1)$ this definition indeed coincides with the classical definition of homogeneous functions, see Definition 2.1. Furthermore, notice that the anisotropic quasi-norm is itself a homogeneous function of order 1, cf. Proposition 2.5.

Proposition 2.8. *Let $\boldsymbol{a} = (a_1, ..., a_n) \in (0, \infty)^n$. If $f \in C(\mathbb{R}^n)$ is homogeneous of order $N \in \mathbb{C}$ w.r.t. $|\cdot|_{\boldsymbol{a}}$, then*

$$|f(x)| \leq \left(\max_{|y|_{\boldsymbol{a}}=1} |f(y)| \right) |x|_{\boldsymbol{a}}^{Re(N)}.$$

Proof. Let $x \in \mathbb{R}^n \setminus \{0\}$ arbitrarily and set $\lambda := |x|_{\mathbf{a}}$, then

$$|f(x)| = |f(\lambda^{\mathbf{a}}\lambda^{-\mathbf{a}}x)| = |\delta_{\mathbf{a}}^{\lambda} f(\lambda^{-\mathbf{a}}x)| = |\lambda^N f(\lambda^{-\mathbf{a}}x)| = \lambda^{\mathrm{Re}(N)}|f(\lambda^{-\mathbf{a}}x)| \leq \left(\max_{|y|_{\mathbf{a}}=1} |f(y)| \right) |x|_{\mathbf{a}}^{\mathrm{Re}(N)}.$$

□

2.2 Combinatorics of Partial Derivatives

For multi-indices $\alpha, \beta \in \mathbb{N}^n$ let $\alpha \leq \beta$ denote their canonical ordering, i.e.

$$\alpha \leq \beta :\Longleftrightarrow (\forall k \in \{1, ..., n\} : \alpha_k \leq \beta_k).$$

Suppose $|\alpha| > 1$, then we call $\{\beta^1, ..., \beta^k\} \subset \mathbb{N}^n$ a partition of α if $\beta^1 + ... + \beta^k = \alpha$ and $|\beta^j| \geq 1$ for all $j \in \{1, ..., k\}$. E.g., $\{(1, 0), (0, 1)\}$ and $\{(1, 1)\}$ are all the partitions of $\alpha = (1, 1)$.

Proposition 2.9. *Let $\alpha \in \mathbb{N}^n$ be a multi-index such that $|\alpha| \geq 1$, and suppose $f : \mathbb{C} \to \mathbb{C}$, $u, v : \mathbb{R}^n \to \mathbb{C}$ are sufficiently smooth. Then the following generalization of the chain rule holds, which is sometimes attributed to Faà di Bruno,*

$$\partial^\alpha (f \circ u)(x) = \sum_{\beta^1+...+\beta^k=\alpha} (f^{(k)} \circ u) \prod_{j=1}^{k} \partial^{\beta^j} u. \tag{10}$$

Here the multi-indices $\beta^j \in \mathbb{N}^n$ such that $|\beta^j| > 0$ for all $j \in \{1, ..., k\}$, and the summation runs over all partitions of α. Also the generalization of the product rule, called Leibniz rule, holds for partial derivatives,

$$\partial^\alpha (u \cdot v) = \sum_{\beta \leq \alpha} (\partial^{\alpha-\beta} u)(\partial^\beta v). \tag{11}$$

Proof. See [Har06], Proposition 1 and Proposition 5. □

Remark 2.10. From this, we can also recover a generalization of the quotient rule. Let $u, v : \mathbb{R}^n \to \mathbb{C}$ be sufficiently smooth and assume $v(x) \neq 0$ for all $x \in \mathbb{R}^n$. Set $f(z) := z^{-1}$ and notice its derivatives are given by $f^{(k)}(z) = (-1)^k k! z^{-k-1}$. Then for $\alpha \in \mathbb{N}^n$ with $|\alpha| > 0$ we can combine the Faà di Bruno rule and the Leibniz rule,

$$\begin{aligned} \partial^\alpha \left(\frac{u}{v} \right)(x) &= \sum_{\beta \leq \alpha} (\partial^{\alpha-\beta} u(x))(\partial^\beta (f \circ v)(x)) \\ &= \sum_{0<\beta\leq\alpha} (\partial^{\alpha-\beta} u(x)) \sum_{\gamma^1+...+\gamma^k=\beta} (f^{(k)} \circ v)(x) \prod_{j=1}^{k} \partial^{\gamma^j} v(x) + (\partial^\alpha u(x))(f \circ v)(x). \end{aligned} \tag{12}$$

Here $\gamma^j \in \mathbb{N}^n$ such that $|\gamma^j| > 0$ for all $j \in \{1, ..., k\}$ and the inner summation runs over all partitions of β, in the same way as before.

2.3 Operator Sum Theorems

Definition 2.11 (Positive operator)**.** Let X be a Banach space, then a linear operator $(A, D(A))$ is said to be a positive operator if its resolvent $\rho(A)$ contains $(-\infty, 0]$ and there exists $C > 0$ such that for all $\lambda \in (-\infty, 0]$ the inequality

$$||R(\lambda, A)||_{L(X)} \leq \frac{C}{1 + |\lambda|} \tag{13}$$

holds, where the resolvent mapping is defined as $R(\lambda, A) := (\lambda I - A)^{-1}$.

Lemma 2.12. *Let A be a positive operator. Then the resolvent of A contains the set*

$$\Lambda = \left\{ \lambda \in \mathbb{C}; Re(\lambda) \leq 0, |Im(\lambda)| \leq \frac{|Re(\lambda)| + 1}{C} \right\} \cup B_{1/C}(0), \tag{14}$$

where C is the constant from formula (13), see Figure 2.

Proof. See [Lun99], Lemma 4.1.2. □

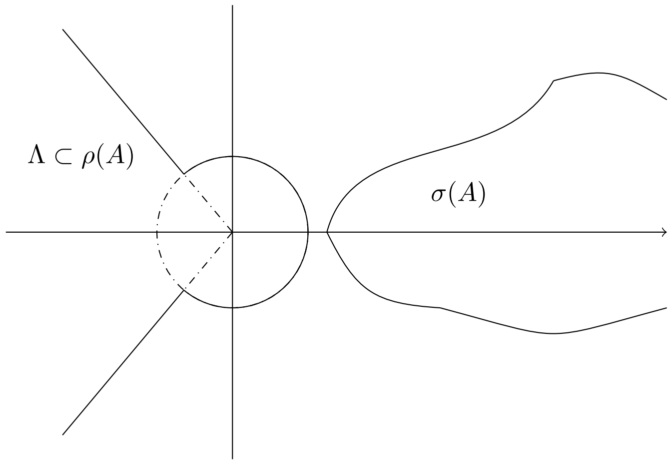


Figure 2: Illustration of the situation in Lemma 2.12.

Definition 2.13 (BIP)**.** Let X be a Banach space, then a linear operator $(A, D(A))$ is said to have Bounded Imaginary Powers (BIP) if A^{is} belongs to $L(X)$ for all $s \in \mathbb{R}$, the group $s \mapsto A^{is}$ is strongly continuous, and there exists a constant $C \geq 0$ and angle $\theta_A \in (0, \pi)$ such that

$$||A^{is}|| \leq Ce^{\theta_A |s|}, \quad s \in \mathbb{R}. \tag{15}$$

Theorem 2.14 (Dore-Venni)**.** *Suppose X is a non-trivial complex Banach space and $A : D(A) \to X$, $B : D(B) \to X$ are closed linear operators, with domains dense in X. Furthermore, suppose that the following three properties hold:*

(i) A and B are positive operators.

(ii) If $\lambda \in \rho(A)$, $\mu \in \rho(B)$, then $(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}$.

(iii) A and B have the BIP property such that $\theta_A + \theta_B < \pi$.

If X is a UMD space, then $A + B$ is closed and $(A + B)^{-1} \in L(X)$.

Proof. See [DV87], Theorem 2.1. □

Remark 2.15. Let $X = L^p([0, T]; Y)$, where Y is UMD Banach space, then it can be seen from direct computation that ∂_t is a positive operator on X with the BIP property (see [DV87], Theorem 3.1). These calculation are somewhat tedious, which is the motivation for providing sufficient conditions for positive operators in section 2.6.4.

2.4 Mikhlin multiplier theorem

Classically, the Mikhlin multiplier theorem gives condition for the boundedness of a Fourier multiplier operator $T_m : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) : f \mapsto \mathcal{F}^{-1}[m\mathcal{F}f]$ for $m \in L^\infty(\mathbb{R}^n)$. Let $N := \lfloor \frac{n}{2} \rfloor + 1$ and suppose $m \in C^N(\mathbb{R}^n \setminus \{0\})$. If there exists a constant $C > 0$ such that $\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{|\alpha|} |\partial^\alpha m(\xi)| \leq C$ for all $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| \leq N$, then the classical Mikhlin multiplier states that $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator. At the beginning of the century, Weis was the first to prove an operator-valued Mikhlin multiplier theorem (see [Wei01]), which has been extended in many directions ever since. In this section we will cite a weighted anisotropic mixed-norm Mikhlin multiplier theorem from a recent work of Lorist ([Lor20]), which will play a central roll in achieving maximal regularity results in this thesis.

Let X and Y be Banach spaces, and denote the space of X-valued Schwartz functions by $\mathscr{S}(\mathbb{R}^n; X)$, and the space of Y-valued tempered distributions by $\mathscr{S}'(\mathbb{R}^n; Y)$. In this section we will consider an Fourier multiplier operator

$$T_m : \mathscr{S}(\mathbb{R}^n; X) \to \mathscr{S}'(\mathbb{R}^n; Y) : f \mapsto \mathcal{F}^{-1}[m\mathcal{F}[f]],$$

where $m \in L^\infty(\mathbb{R}^n; L(X, Y))$.

Definition 2.16. Let $p \in (1, \infty)$, a weight w is said to be of class A_p if the Muchenhaupt characteristic constant $[w]_{A_p}$ is finite, which is defined as

$$[w]_{A_p} := \sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} dx \right)^{p-1}. \tag{16}$$

A weight w is called an A_∞ weight on $\mathbb{R}^n$ if

$$[w]_{A_\infty} := \sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q \log w(x)^{-1} dx \right) < \infty.$$

Remark 2.17. The power function $x \mapsto |x|^a$ is an A_p weight on $\mathbb{R}^n$ if and only if $a \in (-n, n(p-1))$. This can be seen by dividing the balls $B_R(x_0)$ of $\mathbb{R}^n$ into two categories, balls of type I that satisfy $|x_0| \geq 3R$ and type II that satisfy $|x_0| < 3R$, and by using the doubling criterium. For details, see [Gra14], Example 7.1.7. It follows by Jensen's inequality that $[w]_{A_\infty} \leq [w]_{A_p}$. Therefore the power weight $x \mapsto |x|^a$ is an A_∞ weight on $\mathbb{R}^n$ provided that $a \in (-n, \infty)$.

Definition 2.18. Let (X, ρ) be a quasi-metric space, i.e. a set X equipped with a quasi-metric ρ, which instead of satisfying the triangle inequality, satisfies

$$\rho(x, y) \leq K(\rho(x, z) + \rho(z, y)), \quad x, y, z \in X$$

for some $K \geq 1$. Let μ be a Borel measure on X satisfying the doubling criterium, i.e. there exists a constant $C_\mu > 0$ such that for all balls $B(x, r) \subset X$ the inequality $\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$. If furthermore also all balls have finite measure, i.e. $\mu(B(x, r)) < \infty$ for all $B(x, r) \subset X$, then the triple (X, ρ, μ) is called a space of homogeneous type.

Definition 2.19. For $l \in \mathbb{N} \setminus \{0\}$ and $\mathbf{n} = (n_1, ..., n_l) \in (\mathbb{N} \setminus \{0\})^l$ let $\mathbb{R}^n_{\mathbf{n}} := \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_l}$ denote the $\mathbf{n}$-decomposition of $\mathbb{R}^n$, and its elements by $t = (t_1, ..., t_l) \in \mathbb{R}^n_{\mathbf{n}}$. Similarly, for $\mathbf{a} \in (0, \infty)^n$ we identify $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_l) \in (0, \infty)^{n_1} \times ... \times (0, \infty)^{n_l}$. Let $\mathbb{R}^n_{\mathbf{a}} := (\mathbb{R}^n, |\cdot - \cdot|_{\mathbf{a}}, dx)$ denote a space of homogeneous type, where dx denotes the Lebesgue measure, and $|\cdot|_{\mathbf{a}}$ the anisotropic distance function (see Definition 2.3). For $\mathbf{p} = (p_1, ..., p_l) \in [1, \infty)^l$, a vector of weights $\mathbf{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{n_j}_{\mathbf{a}_j})$, and a Banach space X, define the anisotropic mixed-norm space $L^{\mathbf{p}}(\mathbb{R}^n_{\mathbf{n}}, \mathbf{w}; X)$ as the space of all strongly measurable functions $f : \mathbb{R}^n_{\mathbf{n}} \to \mathbb{C}$ such that

$$||f||_{L^{\mathbf{p}}(\mathbb{R}^n_{\mathbf{n}}, \mathbf{w}; X)} := \left(\int_{\mathbb{R}^{\mathbf{n}_1}} ... \left(\int_{\mathbb{R}^{\mathbf{n}_l}} |f|^{p_l} \mathbf{w}_l dx_l \right)^{p_{l-1}/p_l} ... \; w_1 dx_1 \right)^{1/p_1} < \infty. \tag{17}$$

Remark 2.20. In the simplest unweighted case we can identify $L^{(p,q)}(\mathbb{R}^n \times \mathbb{R}^m)$ as the Bochner space $L^p(\mathbb{R}^n; L^q(\mathbb{R}^m))$. Notice that in general $L^{(p,q)}(\mathbb{R}^n \times \mathbb{R}^m) \neq L^{(q,p)}(\mathbb{R}^m \times \mathbb{R}^n)$.

A corollary of the weighted anisotropic, mixed-norm Mikhlin multiplier from the recent work of Lorist is described in the following corollary.

Corollary 2.21. *Let X and Y be UMD Banach spaces. Let $\boldsymbol{a} \in (0, \infty)^n, m \in L^\infty(\mathbb{R}^n; L(X, Y))$, and denote $N := |\boldsymbol{a}|_{\ell^1(\mathbb{R})} + |\boldsymbol{a}|_{\ell^\infty(\mathbb{R})} + 1$. Suppose there exists $C_m > 0$ such that for all $\theta \in \mathbb{N}^n$ with $\boldsymbol{a} \cdot \theta \leq N$ the distributional derivative $\partial^\theta m$ coincides with a continuous function on $\mathbb{R}^n \setminus \{0\}$ and*

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| |\xi|_{\boldsymbol{a}}^{\boldsymbol{a} \cdot \theta} \cdot \partial^\theta m(\xi) \right| \leq C_m. \tag{18}$$

Then for all $\boldsymbol{p} \in (1, \infty)^l$ and $\boldsymbol{w} \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{n_j}_{\boldsymbol{a}_j})$ we have

$$||T_m||_{L^p(\mathbb{R}^n_{\boldsymbol{n}}, \boldsymbol{w}; X) \to L^p(\mathbb{R}_{\boldsymbol{n}}, \boldsymbol{w}; Y)} \lesssim_{X, Y, \boldsymbol{n}, \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{w}} C_m.$$

Proof. See [Lor20], Theorem 7.1. ☐

2.5 H-calculus

In this section we will recall some elementary properties of the H^∞-calculus. We refer the interested reader to [HvVW17], Chapter 10.

For $\sigma \in (0, \pi)$ the open sector of angle ω is defined as

$$\Sigma_\omega := \{z \in \mathbb{C} \setminus \{0\}; \, |\arg(z)| < \omega\},$$

where the argument is taken in $(-\pi, \pi)$.

Definition 2.22 (Sectorial operator)**.** A linear operator $(A, D(A))$ is said to be sectorial if there exists $\omega \in (0, \pi)$ such that the spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$ and

$$M_{\omega,A} := \sup_{z \in (\overline{\Sigma_\omega})^C} ||zR(z, A)|| < \infty. \tag{19}$$

In this situation we say that A is ω-sectorial with constant $M_{\omega,A}$. The infimum of all $\omega \in (0, \pi)$ such that A is ω-sectorial is called the angle of sectoriality of A and is denoted by $\omega(A)$.

Remark 2.23. Notice that sectorial operators are positive operators, see Lemma 2.12.

Definition 2.24 (The Hardy spaces $H^p(\Sigma_\sigma)$)**.** . Let $p \in [1, \infty]$ and $\sigma \in (0, \pi)$. The Banach space of all holomorphic functions $f : \Sigma_\sigma \to \mathbb{C}$ for which

$$||f||_{H^p(\Sigma_\sigma)} := \sup_{|\nu|<\sigma} ||t \mapsto f(e^{i\nu}t)||_{L^p(\mathbb{R}_+, \frac{dt}{t})} < \infty$$

is denoted by $H^p(\Sigma_\sigma)$.

Definition 2.25. Let A be a sectorial operator on a Banach space X and let $\omega(A) < \sigma < \pi$. The operator A is said to have a bounded $H^\infty(\Sigma_\sigma)$-calculus if there exists $C \geq 0$ such that

$$||f(A)|| \leq C||f||_\infty, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma).$$

Let A be a sectorial operator on X and let $f \in H^\infty(\Sigma_\sigma)$ with $\omega(A) < \sigma < \pi$. For the regulariser $\zeta(z) = z(1 + z)^{-2}$ we have $\zeta(A) = A(I + A)^{-2}$ and $D(A) \cap R(A) = R(\zeta(A))$. If $x \in D(A) \cap R(A)$, say $x = \zeta(A)y$ with $y \in X$, define

$$\Psi_A(f)x := (f\zeta)(A)y$$

using the Dunford calculus of A applied to $f\zeta \in H^1(\Sigma_\sigma)$. For $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ we have $\Psi_A(f)x = (f\zeta)(A)y = f(A)\zeta(A)y = f(A)x$ by the multiplicativity of the Dunford calculus. This shows that

$$\Psi_A(f) = f(A) \text{ on } D(A) \cap R(A).$$

As it turns out, whether or not A has a bounded H^∞-calculus is completely determined by the part of A in $\overline{D(A) \cap R(A)}$, as can be seen in the next Proposition.

Proposition 2.26. *Let A be a sectorial operator on X and let $\omega(A) < \sigma < \pi$. Then the following assertions are equivalent:*

(i) A has a bounded $H^\infty(\Sigma_\sigma)$-calculus, i.e. there exists a constant $C \geq 0$ such that for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$,

$$||f(A)x|| \leq C||f||_\infty||x||, \quad x \in X.$$

(ii) There exists a constant $C \geq 0$ such that for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$,

$$||f(A)x|| \leq C||f||_\infty||x||, \quad x \in D(A) \cap R(A).$$

(iii) There exists a constant $C \geq 0$ such that for all $f \in H^\infty(\Sigma_\sigma)$,

$$||\Psi_A(f)x|| \leq C||f||_\infty||x||, \quad x \in D(A) \cap R(A).$$

If $C_{(1)}, C_{(2)}$, and $C_{(3)}$ denote the respective best constants, then $C_{(2)} \leq C_{(1)} \leq 2M_{\sigma,A}C_{(3)}$ and $C_{(3)} \leq C_{(2)}/\sin(\sigma \vee \pi/2)$, and if $D(A) \cap R(A)$ is dense, then $C_{(1)} \leq C_{(3)}$.

Proof. See [HvVW17], Proposition 10.2.11. □

Definition 2.27. Let A be a sectorial operator admitting a bounded $H^\infty(\Sigma_\sigma)$-calculus. The mapping

$$H^\infty(\Sigma_\sigma) \to L(\overline{D(A) \cap R(A)})$$
$$f \mapsto f(A)$$

defined above is called the $H^\infty(\Sigma_\sigma)$-calculus of A.

2.6 Newton Polygons

2.6.1 Inhomogeneous Symbol Classes

Definition 2.28. Let $L_t \subset \mathbb{C}$ and $L_x \subset \mathbb{C}^n$ be closed cones. The set of homogeneous functions $f \in C(L_t \times L_x; \mathbb{C})$ of degree N w.r.t. $|\cdot|_{(\rho,1_n)}$ is denoted by $S^{(\rho,N)}(L_t \times L_x)$ and are called the ρ-homogeneous symbols.

Definition 2.29. Let $f \in S^{(\rho,N)}$ and $\gamma > 0$, then we define the γ-order of f by

$$d_\gamma(f) := \max\left\{N, \frac{N}{\rho}\gamma\right\},$$

and its γ-principal part by

$$\pi_\gamma(f) : L_t \times L_x \to \mathbb{C} : (\tau,\xi) \mapsto \lim_{\eta\to\infty} \eta^{-d_\gamma(f)} \delta^\eta_{(\gamma,1_n)} f(\tau,\xi) := \lim_{\eta\to\infty} \eta^{-d_\gamma(f)} f(\eta^\gamma\tau, \eta\xi).$$

In a similar way we define the ∞-order of f by

$$d_\infty(f) := \frac{N}{\rho},$$

and its ∞-principal part by

$$\pi_\infty(f) : (L_t \times L_x) \to \mathbb{C} : (\tau,\xi) \mapsto \lim_{\eta\to\infty} \eta^{-d_\infty(f)} \delta^\eta_{(1,0_n)} f(\tau,\xi) := \lim_{\eta\to\infty} \eta^{-d_\infty(f)} f(\eta\tau, \xi).$$

Example 2.30. Consider the symbol $m(\tau,\xi) := |\tau|^{N/\rho} + |\xi|^N$, with $N > 0$. Then $m \in S^{(N)}(L_t \times L_x)$, and this function is known as the canonical ρ-homogeneous function.

Definition 2.31. For $\rho > 0$ we define $\tilde{S}(L_t \times L_x)$ as the symbols m of the form

$$m : L_t \times L_x \to \mathbb{C} : (\tau,\xi) \mapsto \sum_{j\in I_m} f_j(\tau,\xi) g_j(\tau) h_j(\xi), \tag{20}$$

where I_m is an arbitrary finite index set and for all $j \in I_m$ the terms f_j, g_j, h_j are homogeneous in the following sense:

$$\begin{aligned} &f_j \in S^{(\rho,N_j)}(L_t \times L_x) \cap H(\mathring{L}_t \times \mathring{L}_x),\ N_j \geq 0, \\ &g_j \in S^{(M_j)}(L_t) \cap H(\mathring{L}_t),\ M_j \geq 0, \\ &h_j \in S^{(L_j)}(L_x) \cap H(\mathring{L}_x),\ L_j \geq 0. \end{aligned}$$

Notice that $\rho > 0$ is not included in the notation as it is fixed in all applications.

Definition 2.32. Let $m \in \tilde{S}(L_t \times L_x)$ and $\gamma > 0$, then the γ-order of m is defined by

$$d_\gamma(m) := \max_{j\in I_m}(d_\gamma(f_j) + \gamma M_j + L_j),$$

and for all $(\tau,\xi) \in L_t \times L_x$ the γ-principal part of m is defined by

$$[\pi_\gamma m](\tau,\xi) := \lim_{\eta\to\infty} \eta^{-d_\gamma(m)} \delta^\eta_{(\gamma,1_n)} m = \sum_{j\in I_\gamma} [\pi_\gamma f_j](\tau,\xi) g_j(\tau) h_j(\xi),$$

where $I_\gamma := \{j \in I_m;\ d_\gamma(f_j) + \gamma M_j + L_j = d_\gamma(m)\}$. Similarly we define the ∞-order of m by

$$d_\infty(m) := \max_{j\in I_m}(M_j + N_j/\rho),$$

and for all $(\tau,\xi) \in L_t \times L_x$ the ∞-principal part is defined by

$$[\pi_\infty m](\tau,\xi) := \lim_{\eta\to\infty} \eta^{-d_\infty(m)} \delta^\eta_{(1,0_n)} m = \sum_{j\in I_\infty} f_j(\tau,0) g_j(\tau) h_j(\xi),$$

where $I_\infty := \{j \in I;\ M_j + N_j/\rho = d_\infty(m)\}$.

Definition 2.33. The representation (20) of a symbol $m \in \tilde{S}(L_t \times L_x)$ is said to be regular if $[\pi_\gamma m] \neq 0$ for all $\gamma \in (0,\infty]$. The subset of symbols $\tilde{S}(L_t \times L_x)$ for which a regular representation exists is denoted by $S(L_t \times L_x)$, and we tacitly assume that a given representation $m \in S(L_t \times L_x)$ is always regular.

Example 2.34. The function $m(\tau,\xi) := \sqrt{-\xi^2} - \sqrt{\tau - \xi^2}$ is in $\tilde{S}(L_t \times L_x)$, but the given representation is not regular as

$$\pi_\gamma m(\tau,\xi) = \sqrt{-\xi^2} - \sqrt{-\xi^2} = 0 \quad \text{for} \quad \gamma < 2.$$

Example 2.35. Consider the operator $\partial_t - \Delta\sqrt{\partial_t - \Delta}$, which is associated to the analysis of the Stefan problem (see equation (1.1) in [DK13]). By taking a Fourier transform in both the temporal and spatial variable it can be seen that its symbol is given by

$$m(\tau,\xi) = i\tau + |\xi|^2\sqrt{i\tau + |\xi|^2}.$$

Then notice $m \in S(L_t \times L_x)$ with the underlying scaling $\rho = 2$.

2.6.2 Newton Polygons

In this section we consider Newton polygons which can be seen as a geometrical description of the γ-principal parts and the γ-order. We shall see later that, under some technical conditions, the vertices of a Newton polygon associated to an N-parametric symbol $m \in S(L_t \times L_x)$ describes the domain of Fourier multiplier operator T_m, see Proposition 2.48.

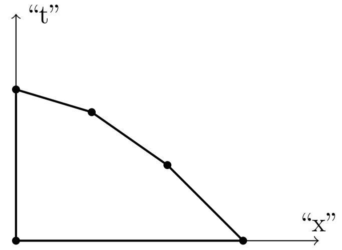


Figure 3: Example of a Newton polygon.

Definition 2.36. Let $\nu \subset [0,\infty)^2$ be a finite set, then the Newton polygon $N(\nu)$ is defined as the convex hull of the set

$$\nu \cup \{(0,0)\} \cup \bigcup_{(a,b)\in\nu} \{(a,0),(0,b)\}.$$

Similarly, a convex set $N \subset [0,\infty)^2$ is called a Newton polygon if there exists a finite set $\nu \subset [0,\infty)^2$ such that $N = N(\nu)$.

Definition 2.37. Let $m \in S(L_t \times L_x)$ with regular representation (20) and set

$$\nu(m) := \bigcup_{j\in I_m} \{(N_j + L_j, M_j), (L_j, N_j/\rho + M_j)\}.$$

Then $N(\nu(m))$ is said to be the Newton polygon associated to m.

Definition 2.38. Let N be a Newton polygon and let $N_V \subset [0,\infty)^2$ denotes its vertices. Then the corresponding weight function W_N of N is defined by

$$W_N : \mathbb{C} \times \mathbb{C}^n \to [0,\infty) : (\tau,\xi) \mapsto \sum_{(r,s)\in N_V} |\xi|^r|\tau|^s.$$

Remark 2.39. A natural question would be what happens for points $(\tilde{r},\tilde{s})$ in the interior of a Newton polygon N, and whether the terms $|\xi|^{\tilde{r}}|\tau|^{\tilde{s}}$ can be estimated by the weight function $W_N(\tau,\xi)$. It turns out this is the case, as by a convexity argument (see [GV92], Lemma 2.1.1) we can see that for a point $(\tilde{r},\tilde{s})$ in the convex hull of N_V we have the inequality

$$|\xi|^{\tilde{r}}|\tau|^{\tilde{s}} \leq \sum_{(r,s)\in N_V} |\xi|^r|\tau|^s.$$

2.6.3 N-parameter ellipticity

Definition 2.40. Let $m \in S(L_t \times L_x)$, then m is called N-parameter elliptic in $\mathring{L}_t \times \mathring{L}_x$ if there exists $C_1, C_2 > 0$ and $\tau_0 \geq 0$ such that

$$C_1 W_m(\tau,\xi) \leq |m(\tau,\xi)| \leq C_2 W_m(\tau,\xi)$$

holds for all $(\tau,\xi) \in \mathring{L}_t \times \mathring{L}_x$ with $|\tau| \geq \tau_0$. The subset of symbols from $S(L_t \times L_x)$ which are N-parameter elliptic will be denoted by $S_N(L_t \times L_x)$.

Remark 2.41. Notice that for every symbol $m(\tau,\xi) \in S(L_t \times L_x)$ we have the one-sided estimate $|m(\tau,\xi)| \leq C_2 W_m(\tau,\xi)$ for all $(\tau,\xi) \in \mathring{L}_t \times \mathring{L}_x$ (see [DK13], Lemma 2.30).

The next Theorem gives sufficient conditions for N-parameter ellipticity.

Theorem 2.42. *Let $m \in S(L_t \times L_x)$ by a symbol satisfying*

$$[\pi_\gamma m](\tau,\xi) \neq 0, \ [\pi_\infty m](\tau,0) \neq 0 \tag{21}$$

for all $(\tau,\xi) \in (L_t \setminus \{0\}) \times (L_x \setminus \{0\})$ and all $\gamma \in (0,\infty)$. Then m is N-parameter-elliptic in $\mathring{L}_t \times \mathring{L}_x$.

Proof. See [DK13], Theorem 2.56. □

2.6.4 Sufficient Conditions H^∞-calculus

In this section we connect the weighted anisotropic Mikhlin multiplier theorem with the methods of Newton polygons.

Proposition 2.43. *Suppose $\alpha \in \mathbb{N} \times \mathbb{N}^n$ and $m(\tau,\xi) \in S(L_t \times L_x)$. For notational convenience denote the operator associated to the Mikhlin-style bound as*

$$\mathfrak{B}^\alpha m(\tau,\xi) := |(\tau,\xi)|_{(\rho,1_n)}^{(\rho,1_n)\cdot(\alpha_t,\alpha_x)} \partial^\alpha m(\tau,\xi). \tag{22}$$

Then $\mathfrak{B}^\alpha m(\tau,\xi) \in S(L_t \times L_x)$, moreover there exists a constant $C_{|\alpha|} \geq 0$ such that

$$|\mathfrak{B}^\alpha m(\tau,\xi)| \leq C_{|\alpha|} W_m(\tau,\xi).$$

Proof. Let $\alpha := (\alpha_t,\alpha_x) \in \mathbb{N} \times \mathbb{N}^n$ such that $|\alpha| > 0$. As $m \in S(L_t \times L_x)$ it is of the form (20). Fix a single term $j \in I_m$, then by applying the product rule (11) twice we obtain the identity

$$\partial^{(\alpha_t,\alpha_x)} f_j(\tau,\xi) g_j(\tau) h_j(\xi) = \sum_{\beta_t \leq \alpha_t} \sum_{\beta_x \leq \alpha_x} (\partial^{(\beta_t,\beta_x)} f_j)(\partial_\tau^{\alpha_t-\beta_t} g_j)(\partial_\xi^{\alpha_x-\beta_x} h_j).$$

Now we can carefully examine the scaling of the separate functions using Proposition 2.2.

$$\begin{aligned}
\partial^{(\beta_t,\beta_x)} f_j &\in S^{(\rho,N_j-(\beta_t,\beta_x)\cdot(\rho,1_n))}(L_t \times L_x) \cap H(\mathring{L}_t \times \mathring{L}_x) \\
\partial_t^{\alpha_t-\beta_t} g_j &\in S^{(M_j-|\alpha_t-\beta_t|)}(L_t) \cap H(\mathring{L}_t) \\
\partial_x^{\alpha_x-\beta_x} h_j &\in S^{(L_j-|\alpha_x-\beta_x|)}(L_x) \cap H(\mathring{L}_x)
\end{aligned}$$

Of course, these derivatives may be zero, but regardless they follow the above scaling. Taking this scaling into account shows that a term

$$(\partial^{(\beta_t,\beta_x)} f_j)(\partial_\tau^{\alpha_t-\beta_t} g_j)(\partial_\xi^{\alpha_x-\beta_x} h_j)$$

is homogeneous with degree $N_j + \rho\cdot M_j + L_j - (\alpha_t,\alpha_x)\cdot(\rho,1_n)$ with respect to $|(\tau,\xi)|_{(\rho,1_n)}$. Therefore, the term $\mathfrak{B}^\alpha[f_j g_j h_j]$ and $f_j g_j h_j$ are homogeneous with degree $N_j + \rho \cdot M_j + L_j$ with respect to $|(\tau,\xi)|_{(\rho,1_n)}$, and $\mathfrak{B}^\alpha[f_j g_j h_j] \in S(L_t \times L_x)$. As this is true for all terms $j \in I_m$, it follows that $\mathfrak{B}^\alpha m \in S(L_t \times L_x)$, and the γ-orders of $\mathfrak{B}^\alpha m$ and m are identical, indeed

$$\begin{aligned}
d_\gamma(m) &:= \max_{j \in I_m}(d_\gamma(f_j) + \gamma M_j + L_j) = d_\gamma(\mathfrak{B}^\alpha m), \\
d_\infty(m) &:= \max_{j \in I_m}(M_j + N_j/\rho) = d_\infty(\mathfrak{B}^\alpha m).
\end{aligned}$$

Similarly,

$$\nu(m) := \bigcup_{j \in I_m} \{(N_j + L_j, M_j), (L_j, N_j/\rho + M_j)\} = \nu(\mathfrak{B}^\alpha m),$$

therefore the Newton polygons of m and $\mathfrak{B}^\alpha m$ are identical, i.e. $N(\nu(m)) = N(\nu(\mathfrak{B}^\alpha m))$. □

Proposition 2.44. *Suppose $m(\tau,\xi) \in S(L_t \times L_x)$ and $\hat{u}(\tau,\xi) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$ with X a complex Banach space. Then $m(\tau,\xi)\hat{u}(\tau,\xi) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$.*

Proof. Recall that the Schwartz class of X-valued functions on $\mathbb{R}^n$ is defined as

$$\mathscr{S}(\mathbb{R}^n;X) := \{f \in C^\infty(\mathbb{R}^n;X); ||f||_{\alpha,\beta} < \infty \text{ for all } \alpha,\beta \in \mathbb{N}^n\},$$

where $||f||_{\alpha,\beta} := ||x \mapsto x^\alpha \partial^\beta f(x)||_\infty$. Let $\hat{u}(\tau,\xi) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n)$. As $m \in S(L_t \times L_x)$, it is of the form (20). Fix a single term $j \in I_m$. We now want to show that

$$||(\tau,\xi) \mapsto f_j(\tau,\xi)g_j(\tau)h_j(\xi)\hat{u}(\tau,\xi)||_{\alpha,\beta} < \infty$$

for all $\alpha := (\alpha_t,\alpha_x) \in \mathbb{N}\times\mathbb{N}^n$ and $\beta := (\beta_t,\beta_x) \in \mathbb{N}\times\mathbb{N}^n$. By Proposition 2.8 we see that there exits a constant $C > 0$ such that

$$|f_j(\tau,\xi)g_j(\tau)h_j(\xi)| \leq C|(\tau,\xi)|_{(\rho,1_n)}^{N_j}|\tau|^{M_j}|\xi|^{L_j}.$$

This shows that for $\tilde{\alpha_t} = \rho N_j + M_j$ and $\tilde{\alpha_x} = (N_j + L_j)1_n$, we have the bound

$$\begin{aligned} |f_j(\tau,\xi)g_j(\tau)h_j(\xi)\hat{u}(\tau,\xi)| &\leq C\left(|\tau|^{2/\rho} + |\xi|^2\right)^{N_j/2} |\tau|^{M_j}|\xi|^{L_j}|\hat{u}(\tau,\xi)| \\ &\leq C||\hat{u}||_{(\tilde{\alpha_t},\tilde{\alpha_x}),(0,0_n)}. \end{aligned}$$

Similarly, using the product rule (11), it may be seen that for all $\alpha,\beta \in \mathbb{N}\times\mathbb{N}^n$ there exists $C > 0$ and $\tilde{\alpha},\tilde{\beta} \in \mathbb{N}\times\mathbb{N}^n$ such that $||f_jg_jh_j\hat{u}||_{\alpha,\beta} \leq C||\hat{u}||_{\tilde{\alpha},\tilde{\beta}} < \infty$, from which the claim follows. By the triangle inequality it now follows that $m(\tau,\xi)\hat{u}(\tau,\xi) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$. ☐

Remark 2.45. At this point it can be helpful to comment on the difference between the approach of Denk and Kaip [DK13] and the methods in this thesis. Denk and Kaip take a Laplace transform in time, instead of a Fourier transform. In this case, it is enough to require that the half-line $\mathbb{R}_{>0}$ is in the interior of the cone $\mathring{L}_t$. Suppose that $L_t = \Sigma_\sigma$ with $\sigma \in (0,\pi)$. Then it is possible to assume without loss of generality that $\tau_0 = 0$. Indeed, let $m(\tau,\xi) \in S(L_t \times L_x)$ be an arbitrary symbol that is N-parameter-elliptic with $\tau_0 > 0$. Notice that there exists a constant $c > 0$ such that $c|\tau| \leq |\tau+\tau_0| \leq |\tau| + \tau_0$ for $\tau \in L_t$, see Figure 4. Now consider the shifted symbol $\tilde{m}(\tau,\xi) := m(\tau+\tau_0,\xi)$. Then $\tilde{m}$ is N-parameter-elliptic with associated constant $\tilde{\tau}_0 = 0$. This can be seen as follows. Consider a vertex $(r,s) \in N_V \subset [0,\infty)^2$. If $0 < t < 1$, then $|\tau+\tau_0|^t \leq |\tau|^t + \tau_0$. For $n \in \mathbb{N}$ we see using the Binomial formula that $(|\tau| + \tau_0)^n = \sum_{k=0}^n \binom{n}{k}|\tau|^k\tau_0^{n-k}$. Now write $s = n + t \in [0,\infty)$, then

$$|\xi|^r|\tau+\tau_0|^s \leq |\xi|^r \sum_{k=0}^n \binom{n}{k}\left(|\tau|^{k+t}\tau_0^{n-k} + |\tau|^k\tau_0^{n-k+1}\right).$$

As $(r,k+t),(r,k) \in \operatorname{conv}(N_V)$ for all $k \in \{0,...,n\}$ it follows from Remark 2.39 that $|\xi|^r|\tau+\tau_0|^s \lesssim W_m(\tau,\xi)$, and therefore there exists $C > 0$ such that

$$cW_m(\tau,\xi) \leq W_{\tilde{m}}(\tau,\xi) = W_m(\tau+\tau_0,\xi) \leq CW_m(\tau,\xi).$$

So indeed, the shifted symbol is N-parameter-elliptic with associated constant $\tau_0 = 0$.

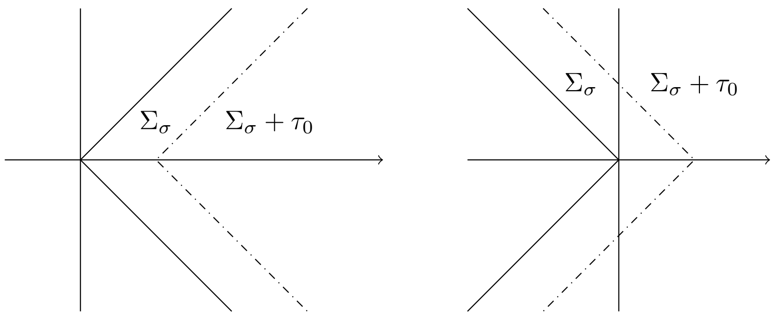


Figure 4: Illustration of the elementary inequality $c|\tau| \leq |\tau+\tau_0| \leq |\tau| + \tau_0$ for $\tau \in \Sigma_\sigma$ with $\sigma \in (0,\pi)$ and $\tau_0 > 0$. From this we can see that $c \geq 1$ for $\sigma \in (0,\pi/2)$ and $0 < c < 1$ for $\sigma \in (\pi/2,\pi)$.

Now suppose given a function $f \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^n)$ we are interested in solving an equation

$$T_m u := \mathcal{L}_t^{-1} \mathcal{F}_x^{-1} [m(\tau, \xi) \mathcal{L}_t \mathcal{F}_x u] = f.$$

If we are able to find a solution $v \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^n)$ of a shifted equation

$$\mathcal{L}_t^{-1} \mathcal{F}_x^{-1} [m(\tau \pm i\tau_0, \xi) \mathcal{L}_t \mathcal{F}_x v] = g := f e^{\mp \tau_0 t},$$

then we are able to recover $u \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^n)$ via the transformation

$$u(t, x) = v(t, x) e^{\mp \tau_0 t}.$$

In the approach of this thesis, we take a Fourier transform in time, and therefore we require that $\mathbb{R} \subset \mathring{L}_t$. Then instead, we must consider a shift of the form $\tilde{m}(\tau, \xi) := m(\tau \pm i, \xi)$, for any hope of the shifted symbol to be N-parameter elliptic with $\tau_0 = 0$. Probably, one can find sufficient conditions that this is true. But in practice, it is easy to check if a shifted symbol is N-parameter elliptic or not. For this reason, all results using N-parameter-ellipticity in this thesis will have the extra condition, compared to Denk and Kaip, that $\tau_0 = 0$.

Proposition 2.46. *Let $L_t \subset \mathbb{C}$ and $L_x \subset \mathbb{C}^n$ be closed cones such that $\mathbb{R} \subset \mathring{L}_t$ and $\mathbb{R}^n \subset \mathring{L}_x$. Let X be a UMD Banach space, and let $m(\tau, \xi) \in S(L_t \times L_x)$ be N-parameter elliptic in $\mathring{L}_t \times \mathring{L}_x$ with $\tau_0 = 0$. If $m[\mathbb{R} \times \mathbb{R}^n] \subset \overline{\Sigma_\omega}$, then T_m is a sectorial operator on $L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n, w_t \times w_x; X)$ for all $p, q \in (1, \infty)$, $w_t \in A_p(\mathbb{R})$, $w_x \in A_q(\mathbb{R}^n)$, and with angle of sectoriality $\omega(T_m) = \omega$. Furthermore, T_m admits a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $\sigma \in (\omega, \pi)$ and has the BIP property.*

Proof. Let $p, q \in (1, \infty)$ and $w_t \in A_p(\mathbb{R}), w_x \in A_q(\mathbb{R}^n)$ arbitrarily, and suppose $m[\mathbb{R} \times \mathbb{R}^n] \subset \overline{\Sigma_\omega}$. Let $\sigma \in (\omega, \pi)$ and let Γ denote the contour $\partial \Sigma_\sigma$ in positive sense, i.e. $\Gamma := \partial \Sigma_\sigma = -\mathbb{R}_+ e^{i\sigma} \oplus \mathbb{R}_+ e^{-i\sigma}$. As the anisotropic Mikhlin multiplier theorem will be invoked multiple times, it is beneficial to do the required calculation first for an arbitrary $f \in H^\infty(\Sigma_\sigma)$. By the Cauchy integration formula we have

$$|(f \circ m)(\tau, \xi)| \leq \frac{1}{2\pi} \int_\Gamma \frac{|f(z)|}{|z - m(\tau, \xi)|} |dz| \leq \frac{1}{2\pi} \int_{\Gamma'} \frac{\sup_{z \in \Gamma} |f(z)|}{|m(\tau, \xi)(\zeta - 1)|} |m(\tau, \xi)| \, |d\zeta|.$$

Here $\Gamma' = \partial(m(\tau, \xi)\Sigma_\sigma)$. As $m(\tau, \xi) \in \overline{\Sigma_\omega}$ for all (τ, ξ), write $m(\tau, \xi) = re^{i\theta}$ with $\theta \in (-\sigma, \sigma)$. Therefore it can be seen that the point 1 is enclosed by $\Gamma' = -\mathbb{R}_+ e^{i(\sigma + \theta)} \oplus \mathbb{R}_+ e^{-i(\sigma - \theta)}$ for all (τ, ξ). An application of the Cauchy theorem therefore yields

$$|(f \circ m)(\tau, \xi)| \leq \frac{\sup_{z \in \Gamma} |f(z)|}{2\pi} \int_\Gamma \frac{1}{|z - 1|} |dz|. \tag{23}$$

Let $\alpha := (\alpha_t, \alpha_x) \in \mathbb{N} \times \mathbb{N}^n$ with $|\alpha| > 0$, then by the chain rule (10) we have the following identity,

$$\begin{aligned} |\mathfrak{B}^\alpha (f \circ m)(\tau, \xi)| &\leq |(\tau, \xi)|_{(\rho, 1_n)}^{(\alpha_t, \alpha_x) \cdot (\rho, 1_n)} \sum_{\beta^1 + \ldots + \beta^k = \alpha} |(f^{(k)} \circ m)(\tau, \xi)| \prod_{j=1}^k |\partial^{\beta^j} m(\tau, \xi)| \\ &= \sum_{\beta^1 + \ldots + \beta^k = \alpha} |(f^{(k)} \circ m)(\tau, \xi)| \prod_{j=1}^k |\mathfrak{B}^{\beta^j} m(\tau, \xi)|. \end{aligned} \tag{24}$$

Here the notation $\mathfrak{B}^\alpha$ comes from Proposition 2.43. Now using Cauchy-differentiation formula we obtain a bound for the k-th derivative of f.

$$\begin{aligned} |(f^{(k)} \circ m)(\tau, \xi)| &\leq \frac{k!}{2\pi} \int_\Gamma \frac{|f(z)|}{|z - m(\tau, \xi)|^{k+1}} |dz| \leq \frac{k!}{2\pi} \int_{\Gamma'} \frac{\sup_{z \in \Gamma} |f(z)|}{|m(\tau, \xi)(\zeta - 1)|^{k+1}} |m(\tau, \xi)| \, |d\zeta| \\ &= \frac{k!}{2\pi} \frac{\sup_{z \in \Gamma} |f(z)|}{|m(\tau, \xi)|^k} \int_{\Gamma'} \frac{1}{|\zeta - 1|^{k+1}} |d\zeta| \end{aligned}$$

Here $\Gamma' = \partial(m(\tau, \xi)\Sigma_\sigma)$, and as we have already seen the point 1 is enclosed by Γ', therefore an application of the Cauchy theorem yields

$$|(f^{(k)} \circ m)(\tau, \xi)| \leq \frac{k!}{2\pi} \frac{\sup_{z \in \Gamma} |f(z)|}{|m(\tau, \xi)|^k} \int_\Gamma \frac{1}{|\zeta - 1|^{k+1}} |d\zeta|.$$

By Proposition 2.43, the terms $\mathfrak{B}^{\beta^j}m(\tau,\xi)$ can all be estimated by the same weight function $W_m(\tau,\xi)$. Using this together with the N-parameter ellipticity and absorbing the constants, we are able to find an estimate for (24).

$$|\mathfrak{B}^{\alpha}(f\circ m)(\tau,\xi)| \leq \sum_{\beta^1+\ldots+\beta^k=\alpha} C_k \frac{\sup_{z\in\Gamma}|f(z)|}{|W_m(\tau,\xi)|^k}\prod_{j=1}^{k}|W_m(\tau,\xi)| \leq C\sup_{z\in\Gamma}|f(z)| \tag{25}$$

Now that we have the Mikhlin style bounds for arbitrary $f\in H^\infty(\Sigma_\sigma)$, let us return to the statement. In order to show that T_m is a sectorial operator, we first have to show that $\sigma(T_m)\subset\overline{\Sigma_\sigma}$. This we do by showing that $\mathbb{C}\setminus\overline{\Sigma_\sigma}\subset\rho(T_m)$, where $\rho(T_m)$ denotes the resolvent of T_m. For this purpose, let $\lambda\in\mathbb{C}\setminus\overline{\Sigma_\sigma}$ and set $f_\lambda(z)=(\lambda-z)^{-1}$. As $\lambda-z\neq 0$ for all $z\in\overline{\Sigma_\sigma}$ we see that $f_\lambda(\cdot)\in H(\Sigma_\sigma)$, and $\sup_{z\in\Gamma}|f_\lambda(z)|<\infty$. Then by the bounds (23) and (25) and an application of the anisotropic Mikhlin multiplier theorem (see Corollary 2.21), we see that

$$||R(\lambda,T_m)||_{L(L^{(p,q)}(\mathbb{R}\times\mathbb{R}^n,(w_t,w_x);X))}<\infty \tag{26}$$

for all $\lambda\in\mathbb{C}\setminus\overline{\Sigma_\sigma}$, so indeed $\sigma(T_m)\subset\overline{\Sigma_\sigma}$.

Let $\nu\in(\sigma,\pi)$. Now we want to show that $\sup_{\lambda\in\mathbb{C}\setminus\overline{\Sigma_\nu}}||\lambda R(\lambda,T_m)||<\infty$. For this purpose denote $g_\lambda(z):=\lambda(\lambda-z)^{-1}$ for $\lambda\in\mathbb{C}\setminus\overline{\Sigma_\nu}$. We claim there exists a constant $C_g>0$ such that for all $\lambda\in\mathbb{C}\setminus\overline{\Sigma_\nu}$ the following inequality holds,

$$\sup_{z\in\Gamma}|g_\lambda(z)|\leq C_g<\infty.$$

For this purpose, suppose that $\lambda=re^{i\theta}$ and $z=Re^{i\sigma}$ with $\theta\in(\nu,\pi)$ and $r,R>0$. In this case, we can estimate $|g_\lambda(z)|$ uniformly by

$$|g_\lambda(z)|=\left|\frac{re^{i\theta}}{re^{i\theta}-Re^{i\sigma}}\right|=\left|\frac{1}{1-\frac{R}{r}e^{i(\sigma-\theta)}}\right|\leq\frac{1}{|1-e^{i(\sigma-\nu)}|}<\infty.$$

Suppose that $(z_n;n\in\mathbb{N})\subset\mathbb{C}\setminus\overline{\Sigma_\nu}$ such that $\lim_{n\to\infty}z_n=0$, then we have

$$\begin{aligned}|g_\lambda(z_n)|&=\left|\frac{\lambda}{\lambda-z_n}\right|\to\left|\frac{\lambda}{\lambda-0}\right|=1 \text{ as } n\to\infty\ (\lambda\neq 0), \text{ and}\\ |g_0(z_n)|&=\left|\frac{0}{0-z_n}\right|=0\to 0 \text{ as } n\to\infty.\end{aligned}$$

This shows that taking $C_g:=\max\{1,|1-e^{i(\sigma-\nu)}\}<\infty$ satisfies the inequality. Now again by the bounds (23) and (25) and an application of the anisotropic Mikhlin multiplier theorem, we see that

$$\sup_{\lambda\in\mathbb{C}\setminus\overline{\Sigma_\nu}}||\lambda R(\lambda,T_m)||_{L(L^{(p,q)}(\mathbb{R}\times\mathbb{R}^n,(w_t,w_x);X))}<\infty. \tag{27}$$

This shows that T_m is a ν-sectorial operator on $L^{(p,q)}(\mathbb{R}\times\mathbb{R}^n,(w_t,w_x);X)$. As $\omega<\sigma<\nu<\pi$, the infimum of all $\nu\in(0,\pi)$ such that T_m is ν-sectorial is ω. Therefore indeed the angle of sectoriality of T_m is $\omega(T_m)=\omega$.

To see that T_m has a bounded $H^\infty(\Sigma_\sigma)$ calculus on $L^{(p,q)}(\mathbb{R}\times\mathbb{R}^n,(w_t,w_x);X)$, let $f\in H^\infty(\Sigma_\sigma)$. Suppose $\hat{u}\in\mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$, and by Proposition 2.44 we see that $m(\tau,\xi)\hat{u}(\tau,\xi)\in\mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$. As the Fourier transform is an homeomorphism from $\mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$ to $\mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)$, it follows that $\mathscr{S}(\mathbb{R}\times\mathbb{R}^n;X)\subset D(T_m)\cap R(T_m)$. As $R(z,T_m)u=\mathcal{F}^{-1}[(z-m(\tau,\xi))^{-1}\hat{u}(\tau,\xi)]$ we obtain by the Cauchy theorem

$$\begin{aligned}T_m(f)u:=\Psi_{T_m}(f)u&=\frac{1}{2\pi i}\int_{\partial\Sigma_\nu}f(z)R(z,T_m)udz=\frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\mathcal{F}^{-1}\left[\frac{f(z)}{z-m}\hat{u}\right]dz\\ &=\mathcal{F}^{-1}\left[\frac{1}{2\pi i}\int_{\partial\Sigma_\nu}\frac{f(z)}{z-m(\tau,\xi)}dz\hat{u}(\tau,\xi)\right]=\mathcal{F}^{-1}\left[f(m(\tau,\xi))\hat{u}(\tau,\xi)\right]\end{aligned}$$

Then again by (23), (25) and an application of the anisotropic Mikhlin multiplier theorem, we see that

$$||f(T_m)||_{L^{(p,q)}(\mathbb{R}\times\mathbb{R}^n,(w_t,w_x);X)}\leq C\sup_{z\in\Gamma}|f(z)|\leq C||f||_{H^\infty(\Sigma_\sigma)}.$$

By Proposition 2.26 this is enough to show that T_m admits a bounded $H^\infty(\Sigma_\sigma)$-calculus.

Now we want to show that T_m has the BIP property, for this purpose consider $g_s(z) := z^{is}$ with $s \in \mathbb{R}$. Notice that $g_s(\cdot) \in H^\infty(\Sigma_\sigma)$, and

$$\sup_{z\in\Gamma} |g_s(z)| = \sup_{r\in[0,\infty)} \max\{|(re^{+i\sigma})^{is}|, |(re^{-i\sigma})^{is}|\} = e^{\sigma|s|}.$$

This shows that $||g_s(T_m)||_{L(L^{(p,q)}(\mathbb{R}\times\mathbb{R}^n,(w_t,w_x);X))} \leq Ce^{\sigma|s|}$. Now we must check that $s \mapsto g_s(T_m)$ is a strongly continuous group. It is clear that $g_0(T_m) = I$ and $g_{s+r}(T_m) = (g_s g_r)(T_m) = g_s(T_m)g_r(T_m)$. Now let $u \in L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n)$ arbitrarily, and consider a sequence $(s_n; n \in \mathbb{N}) \subset \mathbb{R}$ converging to 0. As $g_{s_n}(z) \to g_0(z)$ for all $z \in \Sigma_\sigma$, it follows from the convergence property of the H^∞-calculus that

$$\lim_{n\to\infty} g_{s_n}(T_m)u = g_0(T_m)u = u.$$

As the sequence $(s_n; n \in \mathbb{N})$ was arbitrary, this shows that the mapping $s \mapsto g_s(T_m)u$ is sequentially convergent at $s = 0$. Therefore $s \mapsto g_s(T_m)$ is continuous at $s = 0$, and we may conclude that

$$\lim_{s\to 0} ||g_s(T_m)u - u|| = 0.$$

This shows that indeed $s \mapsto g_s(T_m)$ is a strongly continuous group, hence T_m has the BIP property. □

Remark 2.47. Notice that the shifted operator $\partial_t + 1$, with associated symbol $m(\tau, \xi) = i\tau + 1$, fits the requirements of the previous Proposition. Indeed, $m[\mathbb{R} \times \mathbb{R}^n] \subset \Sigma_\sigma$ for $\sigma \in [\pi/2, \pi)$. Define the bi-sector $\Sigma^{\text{bi}}_\omega := \Sigma_\omega \oplus -\Sigma_\omega$. Let $L_t = \bar{\Sigma}^{\text{bi}}_{\omega_t}$ and $L_x = (\bar{\Sigma}^{\text{bi}}_{\omega_t})^n$ for $\omega_t, \omega_x \in (0, \pi/4)$. Then $\mathbb{R} \subset \mathring{L}_t$, $\mathbb{R}^n \subset \mathring{L}_x$ and m is N-parameter-elliptic in $\mathring{L}_t \times \mathring{L}_x$ as clearly there exists $C \in (0, 1)$ such that the two-sided inequality

$$C(1 + |\tau|) \leq |m(\tau, \xi)| \leq 1 + |\tau|$$

holds for all $(\tau, \xi) \in \mathring{L}_t \times \mathring{L}_x$. Therefore, the shifted operator $T_m = \partial_t + 1$ is a sectorial operator on $L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n, w_t \times w_x; X)$ for all $p, q \in (1, \infty)$, $w_t \in A_p(\mathbb{R})$, $w_x \in A_q(\mathbb{R}^n)$ and with angle of sectoriality $\omega(T_m) = \frac{\pi}{2}$. Furthermore, T_m admits a bounded H^∞-calculus and has the BIP property.

Notice also that the operator ∂_t does not fit the requirements of the previous Proposition, as its symbol $m(\tau, \xi) = i\tau$ is not N-parameter-elliptic in $\mathring{L}_t \times \mathring{L}_x$. It is however well known that this is a sectorial operator on $L^p(\mathbb{R}; X)$ (see [HvVW17], Example 10.1.4). The previous Proposition could also be adjusted in such a way that it includes this operator, as (26) and (27) are bounded for this particular symbol. However, as we only work on compact time intervals in this thesis, it is often possible and enough to consider shifted operators. This justifies the N-parameter ellipticity condition in the previous Proposition.

Similarly, we can see that also the shifted Laplacian operator, $T_m = -\Delta + 1$ with symbol $m(\tau, \xi) = |\xi|^2 + 1$, fits the requirements of the previous Proposition. However, the Laplacian operator $-\Delta$ does not fit the requirements, as its symbol is not N-parameter elliptic in $\mathring{L}_t \times \mathring{L}_x$ either.

Proposition 2.48. *Suppose $m \in S(L_t \times L_x)$ is N-parameter elliptic with $\tau_0 = 0$, $\mathbb{R} \subset \mathring{L}_t$ and $\mathbb{R}^n \subset \mathring{L}_x$. If $P \in S(L_t \times L_x)$ such that it can be bounded by the weight function of the Newton polygon associated to m (see Definition 2.38), i.e. $|P(\tau, \xi)| \leq C|W_m(\tau, \xi)|$, then the Fourier multiplier operator*

$$T_{P/m}u := \mathcal{F}^{-1}\left[\frac{P(\tau, \xi)}{m(\tau, \xi)}\mathcal{F}[u]\right]$$

is bounded on $L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n, (w_t, w_x))$ for all $p, q \in (1, \infty)$, $w_t \in A_p(\mathbb{R})$, $w_x \in A_q(\mathbb{R}^n)$. Assume that the vertices of the Newton polygon are integer, i.e. $\nu(m) \subset \mathbb{N}^2$, then in particular we have that the range of $T_{m^{-1}}$ is characterized as

$$R(T_{m^{-1}}) = \bigcap_{(r,s)\in\nu(m)} H^{s,p}(\mathbb{R}, w_t; H^{r,q}(\mathbb{R}^n, w_q)).$$

Proof. Denote $f(z) := z^{-1}$ and observe $f^{(k)}(z) = (-1)^k k! z^{-(k+1)}$. As m is N-parameter elliptic with $\tau_0 = 0$, it follows that $0 < C_0 W_m(\tau, \xi) \leq |m(\tau, \xi)|$ for all $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$, and therefore

$$\left|\frac{P(\tau, \xi)}{m(\tau, \xi)}\right| \leq C\left|\frac{W_m(\tau, \xi)}{W_m(\tau, \xi)}\right| \leq C < \infty.$$

Now suppose $\alpha \in \mathbb{N} \times \mathbb{N}^n$ such that $|\alpha| > 0$. Then by using the product rule (11) and recalling (22), we see that

$$|\mathfrak{B}^\alpha P(\tau,\xi) \cdot (f \circ m)(\tau,\xi)| \le \sum_{\beta \le \alpha} |\mathfrak{B}^{\alpha-\beta} P| \cdot |\mathfrak{B}^\beta (f \circ m)|.$$

Now by Proposition 2.43 it follows that there exists constants $C_{|\alpha-\beta|} \ge 0$ for all α, β such that

$$|\mathfrak{B}^{\alpha-\beta} P(\tau,\xi)| \le C_{|\alpha-\beta|} W_m(\tau,\xi).$$

By the chain rule (10), and again the N-parameter ellipticity, we see that

$$\begin{aligned}|\mathfrak{B}^\beta (f \circ m)| &\le \sum_{\gamma^1+\ldots+\gamma^k=\beta} \frac{k!}{|m(\tau,\xi)|^{k+1}} \prod_{j=1}^k |\mathfrak{B}^{\gamma^j} m(\tau,\xi)| \\ &\le \sum_{\gamma^1+\ldots+\gamma^k=\beta} \frac{k!}{W_m(\tau,\xi)^{k+1}} \prod_{j=1}^k C_{|\gamma^j|} W_m(\tau,\xi) \le C_{|\beta|} \frac{1}{W_m(\tau,\xi)}.\end{aligned}$$

Therefore, there exists a constant $C_{|\alpha|} \ge 0$ such that $|\mathfrak{B}^\alpha P(\tau,\xi) \cdot (f \circ m)(\tau,\xi)| \le C_{|\alpha|}$. Then from weighted anisotropic Mikhlin multiplier theorem (Corollary 2.21) we see that $T_{p/m}$ defines a bounded linear operator on $L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n; (w_t, w_x))$ for all $p, q \in (1,\infty)$, $w_t \in A_p(\mathbb{R})$, and $w_x \in A_q(\mathbb{R}^n)$.

To characterize the range of the operator $T_{m^{-1}}$, consider the functions $P_{r,s}(\tau,\xi) := (i\tau)^r (i\xi)^s$, for all vertices (r,s) of the Newton polygon associated to the symbol m. Then by definition of the weight function W_m we have $|P_{r,s}(\tau,\xi)| \le W_m(\tau,\xi)$. Therefore we see that $T_{P_{r,s}/m}$ defines a bounded linear operator on $L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n; (w_t, w_x))$, and hence $T_{m^{-1}}$ defines a bounded linear operator on $H^{s,p}(\mathbb{R}, w_t; H^{s,q}(\mathbb{R}^n, w_x))$. By taking intersections over all vertices of the Newton polygon, and using the convexity (see Remark 2.39), the desired result is obtained. □

2.7 Trace Theorem

In this section we consider the Trace Theorem, which in this thesis is important for determining the space of the initial condition. The trace method is a form of real interpolation, and is treated in more detail in the texts of for instance [Lun99], [BL76], [Tri78]. As we are only a 'user' of these methods, we shall not introduce them to their full extend here, but choose to only cite important results.

Definition 2.49. For $p \in [1,\infty)$ we denote by $L^p_*(\mathbb{R}_{\ge 0})$ the space of measurable functions $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ with respect to the measure dt/t endowed with its natural norm

$$||f||_{L^p_*(\mathbb{R}_{\ge 0})} := \left(\int_0^\infty |f(t)|^p \frac{dt}{t} \right)^{1/p}.$$

If X is a Banach space then $L^p_*(\mathbb{R}_{\ge 0}; X)$ is the set of all Bochner measurable functions $f : \mathbb{R}_{\ge 0} \to X$ such that $t \mapsto ||f(t)||_X$ is in $L^p_*(\mathbb{R}_{\ge 0})$. This space is endowed with the natural norm

$$||f||_{L^p_*(\mathbb{R}_{\ge 0};X)} := ||t \mapsto ||f(t)||_X||_{L^p_*(\mathbb{R}_{\ge 0})}.$$

Definition 2.50. For $0 < \theta < 1$ and $1 \le p \le \infty$ define $V(p,\theta,Y,X)$ as the set of all functions $u : \mathbb{R}_{\ge 0} \to X + Y$ such that $u \in W^{1,p}(a,b; X+Y)$ for every $0 < a < b < \infty$, and

$$\begin{aligned} u_\theta(t) = t^\theta u(t) \in L^p_*(\mathbb{R}_{\ge 0}; Y) \\ v_\theta(t) = t^\theta u'(t) \in L^p_*(\mathbb{R}_{\ge 0}; X) \end{aligned}$$

with associated norm

$$||u||_{V(p,1-\theta,Y,X)} := ||u_\theta||_{L^p_*(\mathbb{R}_{\ge 0};Y)} + ||v_\theta||_{L^p_*(\mathbb{R}_{\ge 0};X)}$$

Proposition 2.51. *For $(\theta,p) \in (0,1) \times [1,\infty]$, $(X,Y)_{\theta,p}$ is the set of the traces at $t = 0$ of the functions in $V(p, 1-\theta, Y, X)$, and the norm*

$$||x||^{Tr}_{\theta,p} := \inf\{||u||_{V(p,1-\theta,Y,X)}; u(0) = x, u \in V(p,1-\theta,Y,X)\}$$

is an equivalent norm in $(X,Y)_{\theta,p}$.

Proof. See [Lun99], Proposition 1.2.2. □

Proposition 2.52. *Let $J = [0,T] \subset \mathbb{R}$ and $\{X_0, X_1\}$ be an interpolation couple. Denote $w_\alpha(t) := t^\alpha$ the exponential weight in time. Then we have the following embedding, which is colloquially called the Trace Theorem,*

$$H^{1,p}(J, w_t; X_0) \cap L^p(J, w_t; X_1) \hookrightarrow C(J; (X_0, X_1)_{1-(\alpha+1)/p,p}),$$

where $p \in (1,\infty)$ and $\alpha \in [0, p-1)$, and $(X_0, X_1)_{\theta,p}$ denotes the real interpolation space. In particular, choosing $\alpha = 0$ we have the unweighted embedding

$$H^{1,p}(J; X_0) \cap L^p(J; X_1) \hookrightarrow C(J, (X_0, X_1)_{1-1/p,p}).$$

Proof. Suppose $u \in H^{1,p}(J, w_\alpha; X_0) \cap L^p(J, w_\alpha; X_1)$, then with the aid of Proposition 2.51 and by extending u with 0 outside the interval J, we see that

$$||u(t)||_{(X_0,X_1)_{1-\theta,p}} \leq ||t \mapsto t^\theta u(t)||_{L^p_*(\mathbb{R}_{\geq 0}; X_1)} + ||t \mapsto t^\theta u'(t)||_{L^p_*(\mathbb{R}_{\geq 0}; X_0)}.$$

Now by writing out we see that

$$||t \mapsto t^\theta u(t)||^p_{L^p_*(\mathbb{R}_{\geq 0}; X_1)} = \int_0^\infty ||u(t)||^p_{X_1} t^{\theta p - 1} dt = ||u||_{L^p(\mathbb{R}_{\geq 0}, w_\alpha; X_1)},$$

where $\alpha := \theta p - 1 \in [0, p-1)$. And similarly we see that

$$||t \mapsto t^\theta u'(t)|| = ||u'||_{L^p(\mathbb{R}_{\geq 0}, w_\alpha; X_0)} \leq ||u||_{H^{1,p}(\mathbb{R}_{\geq 0}, w_\alpha; X_0)}.$$

Therefore we have the uniform bound

$$\sup_{t \in J} ||u(t)||_{(X_0,X_1)_{1-(\alpha+1)/p,p}} \leq ||u||_{L^p(J, w_\alpha; X_1)} + ||u||_{H^{1,p}(J, w_\alpha; X_0)}.$$

Now in order to show that $u \in C(J, (X_0, X_1)_{1-(\alpha+1)/p,p})$ we have to show that there exists a constant $C > 0$ such that

$$||u(t+s) - u(t)||_{(X_0,X_1)_{1-(\alpha+1)/p,p}} \leq C|s|, \quad \text{for all } t \in J.$$

Notice that the translation operator $(T_s u)(t) := u(s+t)$ defines a contraction on $L^p(\mathbb{R}_{\geq 0}, w_\alpha; X)$ as

$$\begin{aligned} ||T_s u||^p_{L^p(\mathbb{R}_{\geq 0}, w_\alpha; X)} &= \int_0^\infty ||u(t+s)||^p_X t^\alpha dt = \int_s^\infty ||u(t)||^p_X (t-s)^\alpha dt \\ &\leq \int_s^\infty ||u(t)||^p_X t^\alpha dt \leq \int_0^\infty ||u(t)||^p_X t^\alpha dt = ||u||^p_{L^p(\mathbb{R}_{\geq 0}, w_\alpha; X)}. \end{aligned}$$

Let $\varepsilon > 0$. Using the density of Schwartz functions in L^p-spaces with an A_∞-weight (see [Lin14], Lemma 2.2.3), we see that $\mathscr{S}(\mathbb{R}; X) \hookrightarrow L^p(\mathbb{R}, w_\alpha; X)$. Therefore there exists $v \in \mathscr{S}(\mathbb{R}; X)$ such that $||u - v||_{L^p(J, w_\alpha; X)} < \varepsilon$, which we can use to bound $(T_s - I)u$,

$$\begin{aligned} ||(T_s - I)u||_{L^p(J, w_\alpha; X)} &\leq ||(T_s - I)(u - v)||_{L^p(J, w_\alpha; X)} + ||(T_s - I)v||_{L^p(\mathbb{R}_{\geq 0}, w_\alpha; X)} \\ &\leq (||T_s|| + ||I||) \underbrace{||u - v||_{L^p(J, w_\alpha; X)}}_{\leq \varepsilon} + ||(T_s - I)v||_{L^p(J, w_\alpha; X)}. \end{aligned}$$

Notice that as $v \in \mathscr{S}(\mathbb{R}; X)$, we have $\sup_{t \in J} ||v(t+s) - v(t)||_X \to 0$ as $s \to 0$. Therefore by the compactness of J, we can choose $s > 0$ such that

$$||(T_s - I)v||_{L^p(J, w_\alpha; X)} \leq ||1||_{L^p(J, w_\alpha)} \sup_{t \in J} ||v(t+s) - v(t)||_X \leq \varepsilon.$$

This shows that indeed $u \in C(J, (X_0, X_1)_{1-(\alpha+1)/p,p})$. □

Corollary 2.53. *Let $J = [0,T] \subset \mathbb{R}$, $p, q \in (1,\infty)$, and $w_\alpha(t) = t^\alpha$ with $\alpha \in [0, p-1)$. In particular we have*

$$H^{1,p}(J, w_\alpha; L^q(\mathbb{R}^n)) \cap L^p(J, w_\alpha; H^{2,q}(\mathbb{R}^n)) \hookrightarrow C(J; B^{2-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n)),$$

where $B^{2-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n) = (L^q(\mathbb{R}^n), H^{2,q}(\mathbb{R}^n))_{1-(1+\alpha)/p,p}$. Similarly, we have

$$H^{1,p}(J, w_\alpha; H^{1,q}(\mathbb{R}^n)) \cap L^p(J, w_\alpha; H^{3,q}(\mathbb{R}^n)) \hookrightarrow C(J; B^{3-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n)),$$

where $B^{3-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n) = (H^{1,q}(\mathbb{R}^n), H^{3,q}(\mathbb{R}^n))_{1-(1+\alpha)/p,p}$. Finally, for a bounded domain $\Omega \subset \mathbb{R}^n$ with a boundary that is Lipschitz continuous we have

$$H^{1,p}(J, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J, w_\alpha; H^{3,q}(\Omega)) \hookrightarrow C(J; B^{3-2(1+\alpha)/p}_{q,p}(\Omega)).$$

Proof. For $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$, $p \in (1,\infty)$, $1 \leq q_0, q_1, q \leq \infty$, and $\theta \in (0,1)$, we have by real interpolation (see [Tri78], Theorem 2.4.2) that

$$(H^{s_0,p}(\mathbb{R}^n), H^{s_1,p}(\mathbb{R}^n))_{\theta,q} = B^s_{p,q}(\mathbb{R}^n), \quad \text{where } s = (1-\theta)s_0 + \theta s_1.$$

From this fact we can see that

$$B^{2-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n) = (L^q(\mathbb{R}^n), H^{2,q}(\mathbb{R}^n))_{1-(1+\alpha)/p,p}, \text{ and}$$
$$B^{3-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n) = (H^{1,q}(\mathbb{R}^n), H^{3,q}(\mathbb{R}^n))_{1-(1+\alpha)/p,p}.$$

The interpolation result on domains with Lipschitz continuous boundaries can be achieved using appropriate extension operators (see for example [Leo17], Section 17.3). □

2.8 Paley-Wiener

Definition 2.54. The Hardy-Lebesgue class $H^2(0)$ are the functions $f : \mathbb{C} \to \mathbb{C}$ such that

i) f is holomorphic in the right-half plane $\mathrm{Re}(z) > 0$, and

ii) for each fixed $x > 0$ the function $y \mapsto f(x+iy) \in L^2(\mathbb{R})$ such that

$$\sup_{x>0} \int_{\mathbb{R}} |f(x+iy)|^2 dy < \infty.$$

Definition 2.55. For a function $f \in L^2((0,\infty))$ denote its one-sided Laplace transformation as

$$\mathcal{L}[f(t)](\tau) := (2\pi)^{-1/2} \int_0^\infty f(t) e^{-t\tau} dt, \quad \mathrm{Re}(\tau) > 0.$$

If a function f has both a temporal and spatial component, such as $f(t,x) \in L^p(\mathbb{R}; L^q(\mathbb{R}^n))$, then the one-sided Laplace transformation on its temporal variable will be notated as

$$(\mathcal{L}_1 f)(\tau, x) := \mathcal{L}_1[f(\cdot, x)](\tau) := (2\pi)^{-1/2} \int_0^\infty f(t,x) e^{-t\tau} dt, \quad \mathrm{Re}(\tau) > 0.$$

Proposition 2.56. *If $f \in L^2((0,\infty))$, then the one-sided Laplace transform $\mathcal{L}[f]$ belongs to the Hardy-Lebesgue class $H^2(0)$.*

Proof. See [Yos95], Theorem VI.4.1. □

Theorem 2.57 (Paley-Wiener)**.** *If $f \in H^2(0)$, then the boundary function $f(iy) \in L^2(\mathbb{R})$ of $f(x+iy)$ exists in the sense that*

$$\lim_{x \to 0+} \int_{\mathbb{R}} |f(iy) - f(x+iy)|^2 dy = 0$$

in such a way that the inverse Fourier transform

$$g(t) = \int_{\mathbb{R}} f(2\pi i y) e^{2\pi i t y} dy$$

vanishes for $t < 0$ and $f(z)$ may be obtained as the one-sided Laplace transform of $g(t)$.

Proof. See [Yos95], Theorem VI.4.2. □

Proposition 2.58. *Let X be a Banach space, then the vector-valued Schwartz functions $\mathscr{S}(\mathbb{R} \times \mathbb{R}^n; X)$ are dense in $L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n, (w_t, w_x); X)$ for all $p, q \in [1,\infty)$, $w_t \in A_\infty(\mathbb{R})$, and $w_x \in A_\infty(\mathbb{R}^n)$.*

Proof. See [Lin14], Lemma 2.2.3. □

Proposition 2.59. *Let $J = [0,T]$, $p, q \in (1,\infty)$, $w_t \in A_\infty(\mathbb{R})$, and $w_x \in A_\infty(\mathbb{R}^n)$. Suppose $m \in \mathscr{S}(L_t \times L_x)$ is an N-parameter-elliptic Laplace-Fourier symbol with $\{z \in \mathbb{C};\ Re(z) > 0\} \subset L_t$, and denote the associated Laplace-Fourier multiplier operator by*

$$T_m f := \mathcal{L}_1 \mathcal{F}_2 \left[m \mathcal{L}_1 \mathcal{F}_2 f \right].$$

Suppose $u \in L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n, (w_t, w_x))$ is the unique solution of $f \in L^{(p,q)}(\mathbb{R} \times \mathbb{R}^n, (w_t, w_x))$ satisfying the equation

$$\begin{aligned} T_m u &= f, \quad t \in J, x \in \mathbb{R}^n, \\ u &= 0, \quad t = 0, x \in \mathbb{R}^n. \end{aligned} \tag{28}$$

If $supp(f) \subset J \times \mathbb{R}^n$, then the solution $u(t,x)$ vanishes for times $t < 0$.

Proof. Suppose $f \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^n)$ such that $\mathrm{supp}(f) \subset J \times \mathbb{R}^n$. As the Fourier transform is an homeomorphism on the Schwartz functions, also $\mathcal{F}_2[f] \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^n)$. And as $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ is dense, we see that $\hat{f} := \mathcal{L}_1\mathcal{F}_2[f](\cdot, \xi)$ belongs to the Hardy-Lebesgue class $H^2(0)$ by Proposition 2.56 for all $\xi \in \mathbb{R}^n$. Now we want to show that $m^{-1}(\cdot, \xi)\hat{f}(\cdot, \xi)$ also belongs to the Hardy-Lebesgue class. Now observe that by the N-parameter-ellipticity there exists a constant $C > 0$ such that

$$\frac{1}{|m(\tau, \xi)|} \leq \frac{1}{CW_m(\tau, \xi)} \leq \frac{1}{C}.$$

As, by hypothesis, $m(\cdot, \xi)$ is holomorphic in the right half plane for all $\xi \in \mathbb{R}^n$ and $m(\cdot, \xi) \neq 0$, it follows that $m^{-1}(\cdot, \xi)$ is also holomorphic in the right-half plane. Now, from the N-parameter-ellipticity of m and the fact that $\hat{f} \in H^2(0)$ we also see that

$$\sup_{r>0} \int_{\mathbb{R}} \left| \frac{1}{m(r + is, \xi)} \hat{f}(r + is, \xi) \right|^2 ds \leq \frac{1}{C^2} \int_{\mathbb{R}} |\hat{f}(r + is, \xi)|^2 ds < \infty.$$

This shows that $m^{-1}(\cdot, \xi)\hat{f}(\cdot, \xi) \in H^2(0)$ for all $\xi \in \mathbb{R}^n$. Now by Theorem 2.57 we see that u vanishes for $t < 0$. □

2.9 The Heat Equation

Proposition 2.60. *Denote $J = [0, T] \subset \mathbb{R}$, let $p, q \in (1, \infty)$, and $w_\alpha(t) = t^\alpha$ with $\alpha \in [0, p - 1)$. Consider the heat equation*

$$\begin{aligned} \partial_t u - \Delta u &= f, \quad t \in J, x \in \mathbb{R}^n, \\ u(0, x) &= u_0, \quad t = 0, x \in \mathbb{R}^n. \end{aligned} \tag{29}$$

There exists a unique solution

$$u \in Z^1 := H^{1,p}(J, w_\alpha; L^q(\mathbb{R}^n)) \cap L^p(J, w_\alpha; H^{2,q}(\mathbb{R}^n))$$

if and only if

$$f \in X^1 := L^p(J, w_\alpha; L^q(\mathbb{R}^n)) \text{ and } u_0 \in X^{Tr} := B^{3-2(1+\alpha)/p}_{q,p}(\mathbb{R}^n).$$

In particular there exists a constant $C > 0$ independent of T such that

$$||u||_{H^{1,p}(J,w_\alpha;L^q(\mathbb{R}^n))} + ||\partial_t u||_{L^p(J,w_\alpha;H^{1,q}(\mathbb{R}^n))} \leq C||f||_{L^p(J,w_\alpha;L^q(\mathbb{R}^n))}. \tag{30}$$

Proof. The necessity of f and u_0 become clear after substitution of $u \in Z^1$ into the equation. Indeed, we see that $f \in X^1$, and from the Trace Theorem (Corollary 2.53) we see that $Z^1 \hookrightarrow C(J; X^{\mathrm{Tr}})$. From this we obtain the initial condition $u_0(\cdot) = \lim_{t \to 0+} u(t, \cdot)$ with the desired regularity.

Now suppose $f \in X^1$ and $u_0 \in X^{\mathrm{Tr}}$. Without loss of generality, we may assume that $u_0 = 0$, as a nonzero u_0 can always be added to the solution of the homogeneous heat equation. Notice that after extending u and f to the whole real line, the equation can be rewritten as $T_m u = f$, where T_m is a Fourier multiplier with associated symbol $m(\tau, \xi) = i\tau + |\xi|^2$. Now we can consider the shifted symbol $\tilde{m}(\tau, \xi) := m(\tau - i, \xi) = i\tau + 1 + |\xi|^2$, which corresponds to the shifted p.d.e. $(\partial_t + 1 - \Delta)\tilde{u} = \tilde{f}$, where $\tilde{f} = e^{-t} f$ and $\tilde{u} = e^{-t} u$. Let $L_t = \bar{\Sigma}^{\mathrm{bi}}_{\omega_t}$ and $L_x = (\bar{\Sigma}^{\mathrm{bi}}_{\omega_x})^n$ for $\omega_t, \omega_x \in (0, \pi/4)$. We can see directly that this shifted symbol is N-parameter elliptic for all $\tau \in \mathring{L}_t$ and $\xi \in \mathring{L}_x$, as

$$C_0 W_{\tilde{m}}(\tau, \xi) \leq |\tilde{m}(\tau, \xi)| \leq C_1 W_{\tilde{m}}(\tau, \xi),$$

where we recall that $W_m(\tau, \xi) = 1 + |\tau| + |\xi|^2$, see also Figure 5 for the associated Newton polygon.

Then by Proposition 2.48 we see that the solution of the equation $T_{\tilde{m}}\tilde{u} = \tilde{f}$ is in the space

$$D(T_{\tilde{m}}) = H^{1,p}(\mathbb{R}, w_\alpha; L^q(\mathbb{R}^n)) \cap L^p(\mathbb{R}, w_\alpha; H^{2,q}(\mathbb{R}^n)).$$

Now from Paley-Wiener (Proposition 2.59) it can be seen that $\tilde{u}$ vanishes for $t < 0$, as the Laplace-Fourier symbol $M(\lambda, \xi) = 1 + \lambda + |\xi|^2$ is holomorphic on the right half-plane. Therefore, we see that $\tilde{u}$ solves the equation

$$\begin{aligned} (\partial_t + 1)\tilde{u} - \Delta\tilde{u} &= \tilde{f}, \quad t \geq 0, x \in \mathbb{R}^n, \\ \tilde{u}(0, x) &= 0, \quad t = 0, x \in \mathbb{R}^n. \end{aligned}$$

Therefore we see that $u(t, x) = e^t \tilde{u}(t, x) 1_{[0,T]}(t)$ solves the original heat equation (29), and $u \in Z^1$ as $[0, T]$ is compact. □

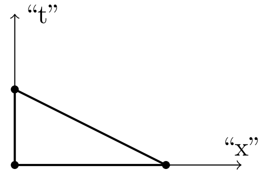


Figure 5: The Newton Polygon N for the heat equation $m(\tau,\xi) = i\tau + |\xi|^2$, with vertices $\{(0,0),(2,0),(0,1)\}$.

Remark 2.61. Notice that the altered heat equation $\partial_t + 1 + \Delta$ cannot be solved, as its Laplace-Fourier symbol $M(\lambda,\xi) = 1 + \lambda - |\xi|^2$ is not N-parameter elliptic, and therefore the Paley-Wiener Theorem cannot be invoked.

Remark 2.62. It is also possible to solve the heat equation with an operator-sum method. Indeed, consider the shifted p.d.e.

$$\begin{aligned}(\partial_t + 1)\tilde{u} + (1-\Delta)\tilde{u} = \tilde{f}, &\quad t \geq 0, x \in \mathbb{R}^n,\\ \tilde{u}(0,x) = 0, &\quad t = 0, x \in \mathbb{R}^n.\end{aligned}$$

Consider the operators $T_{m_1} = (\partial_t + 1)$ with associated symbol $m_1(\tau,\xi) = i\tau + 1$, and $T_{m_2} = 1 - \Delta$ with associated symbol $m_2(\tau,\xi) = 1 + |\xi|^2$. Then by Proposition 2.46, we see that T_{m_1} is a sectorial operator on $L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ with the BIP property (for $\theta_1 > \frac{\pi}{2}$). And similarly, T_{m_2} is a sectorial operator on $L^p(\mathbb{R}; L^q(\mathbb{R}^n))$ with the BIP property (for $\theta_2 > 0$). Then using Dore-Venni (Theorem 2.14) we see that $(T_{m_1} + T_{m_2})^{-1} \in L(L^p(\mathbb{R}; L^q(\mathbb{R}^n)))$. From $D(T_{m_1}) = H^{1,p}(\mathbb{R}; L^q(\mathbb{R}^n))$ and $D(T_{m_2}) = L^q(\mathbb{R}; H^{2,q}(\mathbb{R}^m))$, we see that

$$R((T_{m_1} + T_{m_2})^{-1}) = D(T_{m_1} + T_{m_2}) = D(T_{m_1}) \cap D(T_{m_2}) = Z^1.$$

3 Linear Cahn-Hilliard-Gurtin Problem

3.1 Linear Model in $\mathbb{R}^n$

In this section we will consider the linear Cahn-Hilliard-Gurtin equations on $\mathbb{R}^n$, which is given by

$$\begin{aligned} \partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(B\nabla\mu) + f, \qquad t \in J, x \in \mathbb{R}^n \\ \mu - c\cdot\nabla\mu &= \beta\partial_t u - \Delta u + g, \qquad t \in J, x \in \mathbb{R}^n \\ u(x,0) &= u_0(x), \qquad\qquad\quad t = 0, x \in \mathbb{R}^n. \end{aligned} \tag{31}$$

Here $J = [0,T]$, and $\beta > 0$, $a, c \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times n}$. In order for a solution to exists, the matrix

$$\begin{pmatrix} \beta & c^T \\ a & B \end{pmatrix} \tag{32}$$

is required to be positive definite (see [Gur96], Appendix B and also compare to [Wil12], Section 1), which is equivalent with the statement

$$\beta z_0^2 + ((a+c)\cdot z_1)z_0 + Bz_1\cdot z_1 > 0 \quad \text{for all } (z_0,z_1) \in (\mathbb{R}\setminus\{0\})\times(\mathbb{R}\setminus\{0\})^n, \tag{33}$$

Proposition 3.1. *Suppose the matrix (32) is positive-definite, then also the matrix*

$$A := \beta B - \frac{1}{2}(a\otimes c + c\otimes a) \in \mathbb{R}^{n\times n}$$

is positive definite, where $a\otimes c \in \mathbb{R}^{n\times n}$ is defined by $(a\otimes c)_{i,j} = a_i c_j$ for $i,j \in \{1,...,n\}$.

Proof. Denote $d := a + c$, and notice that we can rewrite (33) as

$$\left(\beta t + \frac{1}{2\sqrt{\beta}}(d|x)\right)^2 + \left(\left(B - \frac{1}{4\beta}(d\otimes d)\right)x \mid x\right) > 0$$

As we are after an equality containing only a spatial variable, we have the freedom to choose the temporal variable. Therefore, for a given $x \in \mathbb{R}^n\setminus\{0\}$ choose $t \in \mathbb{R}\setminus\{0\}$ such that the squared bracket vanishes. This yields the estimate

$$\beta(Bx|x) - \frac{1}{4}((d\otimes d)x|x) > 0,$$

which holds for arbitrary $x \in \mathbb{R}^n$. Recall that for tensor products we have,

$$\begin{aligned} d\otimes d &= a\otimes c + c\otimes a + a\otimes a + c\otimes c, \\ (a-c)\otimes(a-c) &= a\otimes a - a\otimes c - c\otimes a + c\otimes c. \end{aligned}$$

From this we see that

$$d\otimes d + (a-c)\otimes(a-c) = 2(a\otimes c + c\otimes a).$$

As $(a-c)\otimes(a-c)$ is positive semi-definite, we finally obtain

$$\begin{aligned} \beta(Bx|x) - \frac{1}{2}((a\otimes c + c\otimes a)x|x) &= \beta(Bx|x) - \frac{1}{4}((d\otimes d + (a-c)\otimes(a-c))x|x) \\ &\geq \beta(Bx|x) - \frac{1}{4}((d\otimes d)x|x) > 0. \end{aligned}$$

□

Proposition 3.2. *Let $M \in \mathbb{R}^{n\times n}$ be symmetric and positive-definite.*

(i) There exists $\theta > 0$ and $C > 0$ such that

$$|z^T M z| \geq C|z|^2, \quad z \in \left(\overline{\Sigma}_\theta^{bi}\setminus\{0\}\right)^n.$$

(ii) For all $\varepsilon > 0$ there exists $\theta > 0$ such that

$$|\arg(z^T M z)| \leq \varepsilon, \quad z \in \left(\overline{\Sigma}_\theta^{bi}\setminus\{0\}\right)^n.$$

Proof. *i)* As M is positive-definite, all its eigenvalues are strictly positive. Consider its diagonalization $M = P^{-1}DP$, with

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Let $\theta \in (0, \pi/4)$ and $\xi \in \left(\overline{\Sigma}_\theta^{\text{bi}} \setminus \{0\}\right)^n$. As $P \in \mathbb{R}^{n\times n}$ we have $\eta := P\xi \in \left(\overline{\Sigma}_\theta^{\text{bi}} \setminus \{0\}\right)^n$.

$$|\xi^T M\xi| = |\langle M\xi, \xi\rangle| = |\langle P^{-1}DP\xi, \xi\rangle| = |\langle D\eta, \eta\rangle| = |\sum_{k=1}^n \lambda_k \eta_k^2|$$

Denote $\eta_k = \pm r_k e^{i\theta_k}$ for $|\theta_k| \leq \theta$, and notice that $\eta_k^2 = r_k^2 e^{i2\theta_k} \in \overline{\Sigma}_{2\theta} \setminus \{0\}$. Therefore we see

$$|\xi^T M\xi| \geq \sum_{k=1}^n \lambda_k \cos(2\theta_k) r_k^2 \geq C_0|\xi|^2$$

ii) From above, we also see that for $\xi \in (\overline{\Sigma}_\theta^{\text{bi}} \setminus \{0\})^n$ we have

$$\xi^T M\xi = \sum_{k=1}^n \lambda_k \eta_k^2 \in \overline{\Sigma}_{2\theta} \setminus \{0\}.$$

□

Theorem 3.3. *Let $J = [0, T]$, $p, q \in (1, \infty)$, $\alpha \in [0, p-1)$, and $w_\alpha(t) := t^\alpha$. Assume that β, a, B, c satisfy the hypothesis (33). Then the linear Cahn-Hilliard-Gurtin equation (31) admits a unique solution*

$$\begin{aligned} u &\in Z^1 := H^{1,p}(J, w_\alpha; H^{1,q}(\mathbb{R}^n)) \cap L^p(J, w_\alpha; H^{3,q}(\mathbb{R}^n)), \\ \mu &\in Z^2 := L^p(J, w_\alpha; H^{2,q}(\mathbb{R}^n)) \end{aligned}$$

if and only if the data is subject to the conditions

$$\begin{aligned} f &\in X_1 := L^p(J, w_\alpha; L^q(\mathbb{R}^n)), \\ g &\in X_2 := L^p(J, w_\alpha; H^{1,q}(\mathbb{R}^n)), \text{ and} \\ u_0 &\in X^{Tr} := B_{q,p}^{3-2(\alpha+1)/p}(\mathbb{R}^n). \end{aligned}$$

Moreover, there exist a constant $C > 0$ such that

$$||u||_{Z^1} \leq C(||f||_{X_1} + ||g||_{X_2} + ||u_0||_{X^{Tr}}).$$

Proof. Necessity is clear by substituting the solution $(u, \mu) \in Z^1 \times Z^2$ into the system (31) which yields the desired regularity $f \in X_1$ and $g \in X_2$. The regularity of the initial value u_0 follows by the Trace Theorem (see Corollary 2.53) as $Z^1 \hookrightarrow C(J; X^{\text{Tr}})$. Now we consider the sufficiency of $f \in X_1$, $g \in X_2$, and $u_0 \in X^{\text{Tr}}$.

Step i) Without loss of generality we can assume $u_0 = 0$ and $g = 0$. For this purpose, consider the heat equation

$$\begin{aligned} \beta\partial_t v^* - \Delta v^* &= -g, \quad t \in J, x \in \mathbb{R}^n, \\ v^*(0, x) &= u_0, \quad t = 0, x \in \mathbb{R}^n. \end{aligned}$$

By Proposition 2.60 we know that given $g \in X_1$ and $u_0 \in X^{\text{Tr}}$, there exists a unique solution $v^* \in H^{1,p}(J, w_\alpha; L^q(\mathbb{R}^n)) \cap L^p(J, w_\alpha; H^{2,q}(\mathbb{R}^n))$. Now suppose we are able to find a solution (v, μ) of the homogeneous system

$$\begin{aligned} \partial_t v - \operatorname{div}(a\partial_t v) &= \operatorname{div}(B\nabla\mu) + F, \quad && t \in J, x \in \mathbb{R}^n \\ \mu - c \cdot \nabla\mu &= \beta\partial_t v - \Delta v, && t \in J, x \in \mathbb{R}^n \\ v(x, 0) &= 0, && t = 0, x \in \mathbb{R}^n. \end{aligned} \tag{34}$$

If we set $F := f + \partial_t v^* - \operatorname{div}(a\partial_t v^*) \in X_1$, then $(u, \mu) := (v + v^*, \mu)$ is a solution of the original system (31). Therefore we may indeed assume that $u_0 = 0$ and $g = 0$.

Step ii) In this step we do a shift of our system and Fourier transform it. Consider $v := e^{-\kappa t}u$ and $\nu := e^{-\kappa t}\mu$. Observe that $\partial_t u = e^{\kappa t}(\partial_t + \kappa)v$, therefore the shifted version of (31) becomes

$$\begin{aligned} (\partial_t+\kappa)v - \operatorname{div}(a(\partial_t+\kappa)v) &= \operatorname{div}(B\nabla\nu) + e^{-\kappa t}f, && t\in J, x\in\mathbb{R}^n \\ \nu - c\cdot\nabla\nu &= \beta(\partial_t+\kappa)v - \Delta v, && t\in J, x\in\mathbb{R}^n, \\ v(0,x) &= 0, && t=0, x\in\mathbb{R}^n. \end{aligned} \tag{35}$$

Observe that by taking the Fourier transform in both the temporal and the spatial variable we obtain the identities

$$\mathcal{F}[\operatorname{div}(a\partial_t v)] = \sum_{j=1}^n \mathcal{F}[\partial_j a_j \partial_t v] = \sum_{j=1}^n i\xi_j a_j i\tau v = (i\xi\cdot a)i\tau\hat{v}, \text{ and}$$
$$\mathcal{F}[\operatorname{div}(B\nabla\eta)] = \sum_{j=1}^n \mathcal{F}[\partial_j (B\nabla\eta)_j] = \sum_{j=1}^n i\xi_j (Bi\xi\hat{\eta})_j = -(B\xi\cdot\xi)\hat{\eta}.$$

Therefore, the Fourier transform (35) becomes

$$\begin{aligned} (i\tau+\kappa)(1-i(a\cdot\xi))\hat{v} &= -(B\xi\cdot\xi)\hat{\eta} + \hat{F}, \\ (1-i(c\cdot\xi))\hat{\eta} &= (\beta(i\tau+\kappa)+|\xi|^2)\hat{v}. \end{aligned}$$

Notice that this system of algebraic equations can be written in matrix form as

$$M(\tau,\xi)\begin{bmatrix}\hat{v}\\ \hat{\eta}\end{bmatrix} := \begin{bmatrix}(i\tau+\kappa)(1-i(a\cdot\xi)) & (B\xi\cdot\xi) \\ -(\beta(i\tau+\kappa)+|\xi|^2) & (1-i(c\cdot\xi))\end{bmatrix}\begin{bmatrix}\hat{v}\\ \hat{\eta}\end{bmatrix} = \begin{bmatrix}\hat{F}\\ 0\end{bmatrix}$$

Step iii) Now consider the symbol $m(\tau,\xi) := \det(M)$,

$$m(\tau,\xi) := \det M(\tau,\xi) = (i\tau+\kappa)(1-i(a\cdot\xi))(1-i(c\cdot\xi)) + (B\xi\cdot\xi)(\beta(i\tau+\kappa)+|\xi|^2).$$

The aim is to show that $m(\tau,\xi)$ is N-parameter elliptic. Denote $A := \beta B - \frac{1}{2}(a\otimes c + c\otimes a)$, which is positive definite by Proposition 3.1. Using this, we can rewrite our symbol,

$$m(\tau,\xi) = (i\tau+\kappa)(1-i(a+c)\cdot\xi + \xi^T A\xi) + (\xi^T B\xi)|\xi|^2. \tag{36}$$

We claim that $m[\mathbb{R}\times\mathbb{R}^n] \subset \Sigma_{\theta+\pi/2}$ for some $\theta\in(0,\pi/2)$.
By Proposition 3.2 we see that $\xi^T A\xi \geq C|\xi|^2$ for $\xi\in\mathbb{R}^n$. This shows that there exists a $\theta\in(0,\pi/2)$ such that $(1-i(a+c)\cdot\xi+\xi^T A\xi)\in\Sigma_\theta$ for all $\xi\in\mathbb{R}^n$. Furthermore, $\xi^T B\xi|\xi|^2 \geq 0$ for all $\xi\in\mathbb{R}^n$, and $(i\tau+\kappa)\in\Sigma_{\frac{\pi}{2}}$ for all $\tau\in\mathbb{R}$. Therefore, $(i\tau+\kappa)(1-i(1+c)\cdot\xi+\xi^T A\xi)\in\Sigma_{\theta+\pi/2}$ for all $\xi\in\mathbb{R}^n$ and $\tau\in\mathbb{R}$, which justifies the claim.

We set the underlying scaling for the temporal variable to $\rho = 1$. Then the degree of m is given by $d_\gamma(m) = \max\{4, \gamma+2\}$, and the γ-principal order of m is given by,

$$\pi_\gamma m(\tau,\xi) = \begin{cases} \xi^T B\xi|\xi|^2, & \gamma\in(0,2), \\ \xi^T B\xi|\xi|^2 + i\tau\xi^T A\xi, & \gamma = 2, \\ i\tau\xi^T A\xi, & \gamma\in(2,\infty), \\ i\tau(1-i(a+c)\cdot\xi+\xi^T A\xi), & \gamma=\infty. \end{cases}$$

By Proposition 3.2, there exists $\sigma_x, \sigma_t\in(0,\pi)$ such that for $L_t := \overline{\Sigma}^{\mathrm{bi}}_{\sigma_t}\setminus\{0\}$ and $L_x := (\overline{\Sigma}^{\mathrm{bi}}_{\sigma_x}\setminus\{0\})^n$ we have

(i) $\pi_\gamma \neq 0$ for all $(\tau,\xi)\in(\bar{L}_t\setminus\{0\})\times(\bar{L}_x\setminus\{0\})$, and

(ii) $\pi_\infty m(\tau,0) = i\tau \neq 0$ for all $\tau\in L_t\setminus\{0\}$.

Therefore, by Theorem 2.42, we see that m is N-parameter elliptic in $\mathring{L}_t\times\mathring{L}_x$ with $\tau_0 = 0$. The Newton polygon associated to m can be seen in Figure 6, and the associated weight function of m is given by

$$W_m(\tau,\xi) = 1 + |\tau| + |\tau|\,|\xi|^2 + |\xi|^4.$$

As by N-parameter ellipticity we have that $|m(\tau,\xi)| > 0$ for all $\tau\in\mathbb{R}$ and $\xi\in\mathbb{R}^n$, we can invert the matrix $M(\tau,\xi)$. Therefore, the algebraic solution of the system is given by

$$\begin{bmatrix}\hat{v}\\ \hat{\eta}\end{bmatrix} = \frac{1}{m(\tau,\xi)}\begin{bmatrix}(1-ic\cdot\xi) & (\beta(i\tau+\kappa)+|\xi|^2) \\ -(B\xi\cdot\xi) & (i\tau+\kappa)(1-ia\cdot\xi)\end{bmatrix}\begin{bmatrix}\hat{F}\\ 0\end{bmatrix}.$$

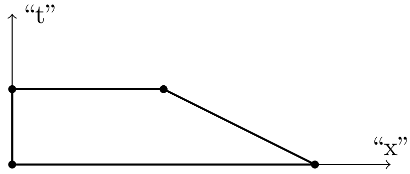


Figure 6: The Newton Polygon N associated to the symbol (36), with vertices $\{(0,0),(4,0),(2,1),(0,1)\}$.

This leads to an equation for $\hat{v}$,

$$m(\tau,\xi)\hat{v} = [1 - i(c\cdot\xi)]\hat{F}. \tag{37}$$

By extending F from $[0,T]$ to the $\mathbb{R}$, we see by Proposition 2.48 that

$$v \in H^{1,p}(\mathbb{R}, w_\alpha; H^{1,q}(\mathbb{R}^n)) \cap L^p(\mathbb{R}, w_\alpha; H^{3,q}(\mathbb{R}^n)).$$

Consider the Laplace-Fourier symbol

$$P(\lambda,\xi) = \frac{(\lambda+\kappa)(1 - i(a+c)\cdot\xi + \xi^T A\xi) + (\xi^T B\xi)|\xi|^2}{1 - ic\cdot\xi}.$$

Notice that $\lambda \mapsto P(\lambda,\xi)$ is holomorphic in the right-half plane $\{\lambda \in \mathbb{C}; \mathrm{Re}(\lambda) > 0\}$. Therefore, by Paley-Wiener (see Proposition 2.59) we see that v vanishes for $t < 0$, hence the restriction of v to $[0,T]$ solves the shifted system (35) and

$$v \in H^{1,p}(J, w_\alpha; H^{1,q}(\mathbb{R}^n)) \cap L^p(J, w_\alpha; H^{3,q}(\mathbb{R}^n)).$$

Now we can also find a solution for η using the equation

$$(1 - ic\cdot\xi)\hat{\eta} = (\beta i\tau + |\xi|^2)\hat{v} \longrightarrow \frac{m(\tau,\xi)}{\beta i\tau + |\xi|^2}\hat{\eta} = \hat{F}.$$

By the obtained regularity of v we see, again by an application of Proposition 2.48, that

$$\eta \in L^p(\mathbb{R}; H^{2,q}(\mathbb{R}^n)).$$

As also the Laplace-Fourier symbol

$$Q(\lambda,\xi) = \frac{(\lambda+\kappa)(1 - i(a+c)\cdot\xi + \xi^T A\xi) + (\xi^T B\xi)|\xi|^2}{\beta\lambda + |\xi|^2}$$

is holomorphic on the right half-plane in the λ variable, we see that η vanishes for $t < 0$ as well. Therefore the restriction of v to $[0,T]$ solves the shifted system (35) and

$$\eta \in L^p(J; H^{2,q}(\mathbb{R}^n)).$$

By shifting back, we are able to solve the original system (34). □

Remark 3.4. For $p = q$ the spaces are the same as in [Wil12], Theorem 2.2. For $p, q \in (1,\infty)$ and $\alpha = 0$, the spaces are the same as in [DK13], Theorem 4.18, except for the fact that Denk and Kaip work on the half-line instead of a compact interval $J = [0,T]$. The linear theory for $\alpha > 0$ is novel.

3.2 Localization in $\mathbb{R}^n$

In this section we will postulate a maximal regularity result for the Cahn-Hilliard-Gurtin system on $\mathbb{R}^n$ with C^1 coefficients. This result can likely be achieved by performing a standard localization procedure on the result of Theorem 3.3, which we hope to do rigorously in future work. In the L^p-setting of Wilke, the localization procedure is proven for domains (see [Wil12], Section 4), which is some justification for the postulation.

Consider the Cahn-Hilliard-Gurtin system on the full space $\mathbb{R}^n$.

$$\begin{aligned} \partial_t u - \mathrm{div}(a\partial_t u) &= \mathrm{div}(b\nabla\mu) + f, && t \in J_0, x \in \mathbb{R}^n, \\ \mu - c\cdot\nabla\mu &= \beta\partial_t u - \Delta u + g, && t \in J_0, x \in \mathbb{R}^n, \\ u(0,\cdot) &= u_0, && t = 0, x \in \mathbb{R}^n. \end{aligned} \tag{38}$$

Here the coefficients $a, c \in [C^1(\bar{\Omega})]^n$ and $b \in C^1(\bar{\Omega})$. We suppose furthermore that $\mathrm{div}(a(x)) = \mathrm{div}(c(x)) = 0$ for $x \in \mathbb{R}^n$ and (β, a, c, B) satisfy (33).

Theorem 3.5 (Maximal Regularity full space). *Denote $J_0 = [0, T_0]$ with $T_0 < \infty$. Let $p, q \in (1, \infty)$ and $\alpha \in [0, p-1)$. Then the linearized system (38) admits a unique solution*

$$\begin{aligned} u \in Z^1 &:= H^{1,p}(J_0, w_\alpha; H^{1,q}(\mathbb{R}^n)) \cap L^p(J_0, w_\alpha; H^{3,q}(\mathbb{R}^n)), \text{ and} \\ \mu \in Z^2 &:= L^p(J_0, w_\alpha; H^{2,q}(\mathbb{R}^n)), \end{aligned} \tag{39}$$

if and only if the data is subject to

$$\begin{aligned} f \in X^1 &:= L^p(J_0, w_\alpha; L^q(\mathbb{R}^n)), \\ g \in X^2 &:= L^p(J_0, w_\alpha; H^{1,q}(\mathbb{R}^n)), \text{ and} \\ u_0 \in X^{Tr} &:= B_{qp}^{3-2(\alpha+1)/p}(\mathbb{R}^n). \end{aligned} \tag{40}$$

Moreover, there exists a constant $C > 0$ independent of T_0 such that

$$||u||_{Z^1} + ||\mu||_{Z^2} \le C\left(||f||_{X^1} + ||g||_{X^2} + ||u_0||_{X^{Tr}}\right).$$

3.3 Bounded Domains

In this section we will postulate a maximal regularity result for domains, which we hope to prove rigorously in future work. For this purpose, the linear model first would need to be considered in the half-space $\mathbb{R}^n_+$, and then a localization procedure would need to be applied. Notice that Wilke did this in an L^p-setting (see [Wil12], Sections 4 and 5), which is some justification for the postulation.

Consider the semilinear version of the Cahn-Hilliard-Gurtin system.

$$\begin{aligned} \partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(b\nabla\mu) + f, && t \in J_0, x \in \Omega, \\ \mu - c \cdot \nabla\mu &= \beta\partial_t u - \Delta u + g, && t \in J_0, x \in \Omega, \\ b\partial_\nu \mu &= h_1, && t \in J_0, x \in \partial\Omega, \\ \partial_\nu u &= h_2, && t \in J_0, x \in \partial\Omega, \\ u(0, \cdot) &= u_0, && t = 0, x \in \Omega. \end{aligned} \tag{41}$$

Here $a, c \in [C^1(\bar{\Omega})]^n$, $b \in C^1(\bar{\Omega})$ and (β, a, c, B) satisfy (33). We suppose furthermore that $\operatorname{div}(a(x)) = \operatorname{div}(c(x)) = 0$ for all x in the domain Ω. Furthermore $a(x) \cdot \nu(x) = c(x) \cdot \nu(x) = 0$ for all x on the boundary $\partial\Omega$ and where ν denotes the outward normal on $\partial\Omega$.

Theorem 3.6 (Maximal Regularity on Domains). *Denote $J_0 = [0, T_0]$ with $T_0 < \infty$. Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $\partial\Omega \in C^3$. Let $p, q \in (1, \infty)$ and $\alpha \in [0, p-1)$. Then the linear system (41) admits a unique solution*

$$\begin{aligned} u \in Z^1 &:= H^{1,p}(J_0, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J_0, w_\alpha; H^{3,q}(\Omega)), \text{ and} \\ \mu \in Z^2 &:= L^p(J_0, w_\alpha; H^{2,q}(\Omega)), \end{aligned}$$

if and only if the data is subject to

$$\begin{aligned} f \in X^1 &:= L^p(J_0, w_\alpha; L^q(\Omega)), \\ g \in X^2 &:= L^p(J_0, w_\alpha; H^{1,q}(\Omega)), \\ h_1 \in Y^1 &:= L^p(J_0, w_\alpha; W^{1-1/q,q}(\partial\Omega)), \\ h_2 \in Y^2 &:= F_{p,q}^{1-\frac{1}{2q}}(J_0, w_\alpha; L^q(\partial\Omega)) \cap L^p(J_0, w_\alpha; W^{2-\frac{1}{q},q}(\partial\Omega)), \\ u_0 \in X^{Tr} &:= B_{qp}^{3-2(\alpha+1)/p}(\Omega), \text{ and} \\ \partial_\nu u_0 &= h_2|_{t=0} \text{ if } 1 - \frac{1}{2q} > \frac{1+\alpha}{p}. \end{aligned} \tag{42}$$

Moreover, there exists a constant $C > 0$ independent of T_0 such that

$$||u||_{Z^1} + ||\mu||_{Z^2} \le C\left(||f||_{X^1} + ||g||_{X^2} + ||h_1||_{Y^1} + ||h_2||_{Y^2} + ||u_0||_{X^{Tr}}\right).$$

Remark 3.7. Even though we won't prove sufficiency of the spaces X^1, X^2, Y^1, Y^2 and X^{Tr} in this thesis, the necessity can be seen relatively easily by substituting the solution $(u, \mu) \in Z^1 \times Z^2$ into the equation (41). Indeed, from this we see that $f \in X^1$ and $g \in X^2$. Also, we see that $u_0 \in X^{\text{Tr}}$, as $Z^1 \hookrightarrow C(J_0; X^{\text{Tr}})$ by Proposition 2.52. The necessity of Y^1 can be seen from the fact that $\partial_\nu \mu = \nabla \mu \cdot \nu \in L^p(J_0, w_\alpha; H^{1,q}(\Omega))$ and recalling that the trace operator is bounded from $H^{1,q}(\Omega)$ to $W^{1-1/q,q}(\partial\Omega)$ (see for example [Leo17], Theorem 18.51). Seeing the necessity of Y^2 requires more modern tools. From a trace embedding result on anisotropic mixed-norm Triebel-Lizorkin spaces (see [HL21], Corollary 3.10 and the representation Theorem 2.5) we see that the trace operator is a bounded linear operator in the sense that

$$H^{1,p}(J_0, w_\alpha; L^q(\Omega)) \cap L^p(J_0, w_\alpha; H^{2,q}(\Omega)) \to Y^2.$$

The compatibility condition $\partial_\nu u_0 = h_2|_{t=0}$ follows from Theorem 1.2 of [ALV21], which states

$$Y^2 \hookrightarrow C(J_0; B_{q,p}^{2-\frac{1}{q}-2\frac{1+\alpha}{p}}(\partial\Omega)) \text{ provided that } 1 - \frac{1}{2q} > \frac{1+\alpha}{p}.$$

4 Local Well-Posedness

4.1 Classical Local Well-Posedness

In this section we consider the quasilinear Cahn-Hilliard-Gurtin system on the whole space and on smooth domains. For the purpose of having efficient notation in this section, we use the symbol Ω for both the whole space and domains. We will prove local well-posedness under the assumption that p and q are large enough such that the trace space is sufficiently smooth. This is considered a classical way of proving local well-posedness, as the methods are similar to the work of Prüss at the beginning of the century (see [Prü02], Section 3). We will generalize the proof of Wilke (see [Wil12], Section 5) to a time weighted L^pL^q-setting, which allows us to deal with rough initial data.

Recall that the quasilinear version of the Cahn-Hilliard-Gurtin system on $\Omega = \mathbb{R}^n$ is given by,

$$\begin{aligned} \partial_t u - \operatorname{div}(a(x,u,\nabla u)\partial_t u) &= \operatorname{div}(b(x,u,\nabla u)\nabla\mu) + f, && t \in J, x \in \Omega,\\ \mu - c(x,u,\nabla u)\cdot\nabla\mu &= \beta\partial_t u - \Delta u + \Phi'(u) + g, && t \in J, x \in \Omega,\\ u(0) &= u_0, && t = 0, x \in \Omega. \end{aligned} \tag{43}$$

Here we assume that the coefficients are sufficiently smooth and bounded, i.e. $a, c \in C_b^1(\bar{\Omega}; C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n))$ and $b \in C_b^1(\bar{\Omega}; C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}))$, with $B = bI$. If we are working on a domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^3$, then we consider the following two quasilinear boundary conditions

$$\begin{aligned} b(x,u,\nabla u)\partial_\nu\mu &= h_1, && t \in J, x \in \partial\Omega,\\ \partial_\nu u &= h_2, && t \in J, x \in \partial\Omega. \end{aligned} \tag{44}$$

We suppose the data is in the spaces (40) or (42), if we work in the whole space or domain respectively. By the Sobolev embedding theorem on Besov spaces (see [Tri78], Theorem 2.8.1), we have that the trace space X^{Tr} embeds into C^2 if p and q are large enough, i.e.

$$X^{\mathrm{Tr}} = B_{q,p}^{3-2(\alpha+1)/p}(\Omega) \hookrightarrow C^2(\Omega), \quad \text{provided that } 3 - 2\frac{1+\alpha}{p} > \frac{n}{q} + 2,$$

where once again $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ is an arbitrary domain. Under this 'largeness' assumption, enough regularity is obtained in order to fix coefficients on the initial condition u_0,

$$\begin{aligned} a_0(x) &:= a(x, u_0(x), \nabla u_0(x)),\\ b_0(x) &:= b(x, u_0(x), \nabla u_0(x)), \text{ and}\\ c_0(x) &:= c(x, u_0(x), \nabla u_0(x)). \end{aligned}$$

We suppose that these fixed coefficients satisfy $\operatorname{div}(a_0(x)) = \operatorname{div}(c_0(x)) = 0$ for $x \in \Omega$, and if we work on domains we assume additionally that $a_0(x)\cdot\nu(x) = c_0(x)\cdot\nu(x) = 0$ for $x \in \partial\Omega$, where ν denotes the outward normal on $\partial\Omega$. Now the aim is to solve the quasilinear system (43) via a Banach fixed-point argument.

First we give the construction for $\Omega = \mathbb{R}^n$. Define the following operator, which is associated to the linearized system (38),

$$\mathbb{L} : \mathbb{E}_1 \to \mathbb{E}_0 : (u,\mu) \mapsto \begin{bmatrix} \partial_t u - \operatorname{div}(a_0\partial_t u) - \operatorname{div}(b_0\nabla\mu) \\ \mu - c_0\cdot\nabla\mu - \beta\partial_t u + \Delta u \end{bmatrix}.$$

Here $\mathbb{E}_0$ and $\mathbb{E}_1$ are the data and solution spaces respectively, defined by

$$\mathbb{E}_1 := Z^1 \times Z^2, \quad {}_0\mathbb{E}_1 := \{(u,v) \in \mathbb{E}_1; u|_{t=0} = 0\}, \text{ and } \mathbb{E}_0 := X^1 \times X^2. \tag{45}$$

Notice that the spaces $\mathbb{E}_0$ and $\mathbb{E}_1$ equipped with canonical norms $||\cdot||_{\mathbb{E}_0}$ and $||\cdot||_{\mathbb{E}_1}$ respectively are Banach spaces. Furthermore, notice that $\mathbb{L}$ is an isometry due to Theorem 3.5. Now let (u^*,μ^*) denote the unique solution of the linearized system (38) with coefficients a_0, b_0, and c_0, which exists due to Theorem 3.5. To capture the non-linear behavior of the quasilinear system, we define the mapping

$$\mathbb{G} : {}_0\mathbb{E}_1 \times \mathbb{E}_1 \to \mathbb{E}_0 : ((u,\mu),(u^*,\mu^*)) \mapsto \begin{bmatrix} G_1(u,u^*) + G_2((u,\mu),(u^*,\mu^*)) \\ G_3((u,\mu),(u^*,\mu^*)) + G_4(u,u^*) \end{bmatrix},$$

where the mappings G_1, G_2, G_3, and G_4 are defined by

$$\begin{aligned}
G_1 &: (u, u^*) \mapsto \operatorname{div}((a(x, u+u^*, \nabla(u+u^*)) - a_0)\partial_t(u+u^*)),\\
G_2 &: ((u,\mu),(u^*,\mu^*)) \mapsto \operatorname{div}((b(x, u+u^*, \nabla(u+u^*)) - b_0)\nabla(\mu+\mu^*)),\\
G_3 &: ((u,\mu),(u^*,\mu^*)) \mapsto (c(x, u+u^*, \nabla(u+u^*)) - c_0)\nabla(\mu+\mu^*), \text{ and}\\
G_4 &: (u, u^*) \mapsto \Phi'(u+u^*).
\end{aligned}$$

Now notice that (u, μ) is a solution of the quasilinear system (43) if and only (u, μ) is a fixed point of the equation

$$\mathbb{L}(u,\mu) = \mathbb{G}((u,\mu),(u^*,\mu^*)).$$

The construction when we work on a smooth domain $\Omega \subset \mathbb{R}^n$ is similar. The solution space $\mathbb{E}_1$ remains unchanged, the data space is defined as

$$\mathbb{E}_0 := X^1 \times X^2 \times Y^1 \times Y^2, \text{ and } {}_0\mathbb{E}_0 := \{(f, g, h_1, h_2, u_0) \in \mathbb{E}_0;\ h_2|_{t=0} = 0\}. \tag{46}$$

The operator associated to the linearized system (41) becomes

$$\mathbb{L} : \mathbb{E}_1 \to \mathbb{E}_0 : (u,\mu) \mapsto \begin{bmatrix} \partial_t u - \operatorname{div}(a_0 \partial_t u) - \operatorname{div}(b_0 \nabla \mu) \\ \mu - c_0 \cdot \nabla\mu - \beta\partial_t u + \Delta u \\ b\nabla u \cdot \nu \\ \partial_\nu u \end{bmatrix}. \tag{47}$$

And the operator associated to the non-linear behavior of the quasilinear system becomes

$$\mathbb{G} : {}_0\mathbb{E}_1 \times \mathbb{E}_1 \to \mathbb{E}_0 : ((u,\mu),(u^*,\mu^*)) \mapsto \begin{bmatrix} G_1(u,u^*) + G_2((u,\mu),(u^*,\mu^*)) \\ G_3((u,\mu),(u^*,\mu^*)) + G_4(u,u^*) \\ G_5((u,\mu),(u^*,\mu^*)) \\ 0 \end{bmatrix},$$

where G_5 is defined as

$$G_5 : ((u,\mu),(u^*,\mu^*)) \mapsto (b_0 - b(x, u+u^*, \nabla u + \nabla u^*))\nabla(\mu+\mu^*) \cdot \nu.$$

The following lemma shows that the operator $\mathbb{G}$ is bounded for a small enough time interval.

Lemma 4.1. *Let $n \in \mathbb{N} \setminus \{0\}$, $p, q \in (1,\infty)$, and $\alpha \in [0, p-1)$ such that $3 - 2(1+\alpha)/p > n/q + 2$. Let $\Omega = \mathbb{R}^n$ or let $\Omega \subset \mathbb{R}^n$ a domain with $\partial\Omega \in C^3$. Set $J_0 = [0, T_0]$, and suppose $\Phi \in C^{3-}(\mathbb{R})$, $a, c \in C_b^1(\bar\Omega; C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n))$ and $b \in C_b^1(\bar\Omega; C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}))$. Define the ball $\mathbb{B}_r \subset {}_0\mathbb{E}_1$ by*

$$\mathbb{B}_r := \{(u,\mu) \in {}_0\mathbb{E}_1; ||(u,\mu)||_{\mathbb{E}_1} < r\}, \text{ with } r \in (0,1).$$

Then there exists a constant $C > 0$, independent of T and r, and functions $\zeta_k(T)$ with $\zeta_k(T) \to 0$ as $T \to 0$, $k \in \{1, ..., 4\}$ such that for all $(u_1,\mu_1), (u_2,\mu_2) \in \mathbb{B}_r \subset {}_0\mathbb{E}_1$ the following estimates hold,

(i) $||G_1(u_1,u^*) - G_1(u_2,u^*)||_{X^1} \le C(r + \zeta_1(T))||(u_1,\mu_1) - (u_2,\mu_2)||_{\mathbb{E}_1}$,

(ii) $||G_2((u_1,\mu_1),(u^*,\mu^*)) - G_2((u_2,\mu_2),(u^*,\mu^*))||_{X^1} \le C(r + \zeta_2(T))||(u_1,\mu_1) - (u_2,\mu_2)||_{\mathbb{E}_1}$,

(iii) $||G_3((u_1,\mu_1),(u^*,\mu^*)) - G_3((u_2,\mu_2),(u^*,\mu^*))||_{X^2} \le C(r + \zeta_3(T))||(u_1,\mu_1) - (u_2,\mu_2)||_{\mathbb{E}_1}$,

(iv) $||G_4(u_1,u^*) - G_4(u_2,u^*)||_{X^2} \le C\zeta_4(T)||(u_1,\mu_1) - (u_2,\mu_2)||_{\mathbb{E}_1}$, *and*

(v) $||G_5((u_1,\mu_1),(u^*,\mu^*)) - G_5((u_2,\mu_2),(u^*,\mu^*))||_{Y^1} \le C(r + \zeta_5(T))||(u_1,\mu_1) - (u_2,\mu_2)||_{\mathbb{E}_1}$.

Proof. For notational convenience denote $v_1 := u_1 + u^*$, $v_2 := u_1 + u^*$. By the embeddings $Z^1 \hookrightarrow C(J_0; X^{\mathrm{Tr}}) \hookrightarrow C(J_0; C^2(\Omega))$ we have

$$||v_1||_{\infty,C^2} = ||u_1||_{\infty,C^2} + ||u^*||_{\infty,C^2} \lesssim ||u_1||_{\infty,X^{\mathrm{Tr}}} + ||u^*||_{\infty,X^{\mathrm{Tr}}} \lesssim r + ||u^*||_{\infty,X^{\mathrm{Tr}}} =: R.$$

We also have

$$||\partial_t v_1||_{X^1} \le ||\partial_t u_1||_{X^1} + ||\partial_t u^*||_{X^1} \le ||\partial_t u_1||_{Z^1} + ||\partial_t u^*||_{X^1} \le r + ||\partial_t u^*||_{X^1}.$$

As $||\partial_t u^*||^p_{X^1} := \int_0^T ||\partial_t u^*(s,\cdot)||^p_{L^q(\Omega)} ds \le ||u^*||_{Z^1(J_0)} < \infty$, we see by the Dominated Convergence Theorem that

$$||\partial_t u^*||^p_{L^p(J;L^q(\Omega))} = \int_0^T ||\partial_t u^*(s,\cdot)||^p_{L^q(\Omega)} ds \to 0 \text{ as } T \to 0.$$

Similarly we see that

$$||\partial_t \nabla u^*||_{L^p(J;L^q(\Omega))} \to 0 \text{ as } T \to 0.$$

'(i)' Denote $a_1 := a(x, v_1(x), \nabla v_1(x))$ and $a_2 := a(x, v_2(x), \nabla v_2(x))$, then we have

$$\begin{aligned} G_1(u_1, u^*) - G_1(u_2, u^*) &= \operatorname{div}((a_1 - a_0)\partial_t v_1) - \operatorname{div}((a_2 - a_0)\partial_t v_2) \\ &= \underbrace{\operatorname{div}(a_1 - a_0)\partial_t v_1 - \operatorname{div}(a_2 - a_0)\partial_t v_2}_{\boxed{\text{A}}} + \underbrace{(a_1 - a_0) \cdot \nabla\partial_t v_1 - (a_2 - a_0) \cdot \nabla\partial_t v_2}_{\boxed{\text{B}}}. \end{aligned}$$

We will first bound the 'A'-part. By linearity of the divergence operator we see that for the first two terms we have,

$$\begin{aligned} \boxed{\text{A}} &= \boxed{\text{A}} + \operatorname{div}(a_2 - a_0)\partial_t v_1 - \operatorname{div}(a_2 - a_0)\partial_t v_1 \\ &= \underbrace{\operatorname{div}(a_1 - a_2)\partial_t v_1}_{\boxed{\text{A1}}} + \underbrace{\operatorname{div}(a_2 - a_0)(\partial_t v_1 - \partial_t v_2)}_{\boxed{\text{A2}}}. \end{aligned}$$

For the purpose of obtaining an identity for $\operatorname{div}(a(x, u(x), \nabla u(x)))$ we introduce the notation $a = a(\xi, \phi, \psi)$ with $\psi := [\psi_1, ..., \psi_n]$.

$$\begin{aligned} \operatorname{div}(a(x, u, \nabla u)) &:= \sum_{j=1}^n \partial_{x_j}(a(x, u(x), \nabla u(x)))_j \\ &= \sum_{j=1}^n \left((\partial_\xi a(x, u, \nabla u))_j + (\partial_\phi a(x, u, \nabla u))_j \partial_{x_j} u + \sum_{k=1}^n (\partial_{\psi_k} a(x, u, \nabla u))_j \partial_{x_j}\partial_{x_k} u \right) \\ &= \operatorname{div}_\xi a(x, u, \nabla u) + \partial_\phi a(x, u, \nabla u) \cdot \nabla u + \partial_\psi a(x, u, \nabla u) : \nabla^2 u. \end{aligned}$$

Here the notation $\partial_\psi a(x, u, \nabla u) : \nabla^2 u$ is an abbreviation for

$$\partial_\psi a(x, u, \nabla u) : \nabla^2 u := \sum_{j,k=1}^n (\partial_{\psi_k} a(x, u, \nabla u))_j \partial_{x_j}\partial_{x_k} u.$$

We estimate $\operatorname{div}(a_1 - a_2)$ in $L^\infty(J; L^\infty(\Omega))$ as follows,

$$\begin{aligned} ||\operatorname{div}(a_1) - \operatorname{div}(a_2)||_{\infty,\infty} &= ||(\partial_\phi a_1 \cdot \nabla v_1 + \partial_\psi a_1 : \nabla^2 v_1) - (\partial_\phi a_2 \cdot \nabla v_2 + \partial_\psi a_2 : \nabla^2 v_2)||_{\infty,\infty} \\ &\leq ||\partial_\phi(a_1 - a_2)||_{\infty,\infty} ||\nabla v_1||_{\infty,\infty} + ||\partial_\phi a_2||_{\infty,\infty} ||\nabla v_1 - \nabla v_2||_{\infty,\infty} \\ &\quad + ||\partial_\psi(a_1 - a_2) : \nabla^2 v_1||_{\infty,\infty} + ||\partial_\psi a_2 : (\nabla^2 v_1 - \nabla^2 v_2)||_{\infty,\infty}. \end{aligned}$$

Now as $a \in C_b^2(\mathbb{R}^n)$ we see that $||\partial_\phi a_2||_{\infty,\infty} \leq ||a||_{C_b^2}$. And from the embeddings $Z^1 \hookrightarrow C(J; X^{\text{Tr}}) \hookrightarrow C(J; C^2(\mathbb{R}^n))$ we see that

$$\begin{aligned} ||\partial_\phi(a_1 - a_2)||_{\infty,\infty} &\leq ||a||_{C_b^2} ||v_1 - v_2||_{C^1} \lesssim ||a||_{C_b^2} ||u_1 - u_2||_{Z^1} \\ ||\partial_\phi(a_1 - a_2) : \nabla^2 v_1||_{\infty,\infty} &\leq \sum_{j,k=1}^n ||(\partial_{\psi_k}(a_1 - a_2))_j||_{\infty,\infty} ||\partial_{x_j}\partial_{x_k} v_1||_{\infty,\infty} \\ &\leq ||a||_{C_b^2} ||v_1 - v_2||_{\infty,C^1} ||v_1||_{\infty,C^2} \lesssim ||a||_{C_b^2} ||u_1 - u_2||_{Z^1} R \\ ||\partial_\phi a_2 : (\nabla^2 v_1 - \nabla^2 v_2)||_{\infty,\infty} &\leq \sum_{j,k=1}^n ||(\partial_{\psi_k} a_2)_j||_{\infty,\infty} ||\partial_{x_j}\partial_{x_k}(v_1 - v_2)||_{\infty,\infty} \\ &\leq ||a||_{C_b^2} ||u_1 - u_2||_{\infty,C^2} \lesssim ||a||_{C_b^2} ||u_1 - u_2||_{Z^1} \end{aligned}$$

Putting this all together, we can estimate A1 in X^1 by

$$||\text{A1}||_{X^1} \leq ||\operatorname{div}(a_1 - a_2)||_{\infty,\infty} ||\partial_t v_1||_{X^1} \lesssim ||a||_{C_b^2} ||u_1 - u_2||_{Z^1} \underbrace{||\partial_t v_1||_{X^1}}_{\to 0 \text{ as } T \to 0}.$$

For the purpose of bounding A2, consider the term $\operatorname{div}(a_2 - a_0)$ in $L^\infty(J; L^\infty(\Omega))$, similarly as before we have

$$\begin{aligned} ||\operatorname{div}(a_2 - a_0)||_{\infty,\infty} &\leq ||\partial_\phi(a_2 - a_0)||_{\infty,\infty} ||\nabla v_2||_{\infty,\infty} + ||\partial_\phi a_0||_{\infty,\infty} ||\nabla v_2 - \nabla u_0||_{\infty,\infty} \\ &\quad + ||\partial_\psi(a_2 - a_0) : \nabla^2 v_2||_{\infty,\infty} + ||\partial_\psi a_0 : (\nabla^2 v_2 - \nabla^2 u_0)||_{\infty,\infty}. \end{aligned}$$

In order to proceed further, we compare a_2 and a_0 to $a^* := a(u^*, \nabla u^*)$.

$$\begin{aligned} ||\nabla v_2 - \nabla u_0||_{\infty,\infty} &\lesssim ||v_2 - u_0||_{\infty, X^{\mathrm{Tr}}} \\ &\leq ||v_2 - u^*||_{\infty, X^{\mathrm{Tr}}} + ||u^* - u_0||_{\infty, X^{\mathrm{Tr}}} \\ &\lesssim \underbrace{||u_2||_{Z^1}}_{\leq r} + ||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}. \end{aligned}$$

Because $u^*|_{t=0} = u_0$ and $u^* - u_0 \in C(J; X^{\mathrm{Tr}}(\Omega))$, we see that $||u^* - u_0||_{L^\infty(J;X^{\mathrm{Tr}})} \to 0$ as $T \to 0$. And similarly,

$$\begin{aligned} ||\partial_\phi(a_2 - a_0)||_{\infty,\infty} &\lesssim ||a||_{C_b^2} ||v_2 - u_0||_{\infty, X^{\mathrm{Tr}}} \\ &\lesssim ||a||_{C_b^2} (\underbrace{||u_2||_{Z^1}}_{\leq r} + \underbrace{||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}}_{\to 0 \text{ as } T \to 0}) \\ ||\partial_\psi(a_2 - a_0) : \nabla^2 v_2||_{\infty,\infty} &\lesssim ||a||_{C_b^2} ||v_2 - u_0||_{\infty, X^{\mathrm{Tr}}} ||\nabla^2 v_2||_{\infty,\infty} \\ &\lesssim ||a||_{C_b^2} (\underbrace{||u_2||_{Z^1}}_{\leq r} + \underbrace{||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}}_{\to 0 \text{ as } T \to 0}) R \\ ||\partial_\psi a_0 : (\nabla^2 v_2 - \nabla^2 u_0)||_{\infty,\infty} &\lesssim ||a||_{C_b^2} ||v_2 - u_0||_{\infty, X^{\mathrm{Tr}}} \\ &\lesssim ||a||_{C_b^2} (\underbrace{||u_2||_{Z^1}}_{\leq r} + \underbrace{||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}}_{\to 0 \text{ as } T \to 0}) \end{aligned}$$

Hence, A2 can be bounded by

$$\begin{aligned} ||\mathrm{A2}||_{X^1} &\leq ||\mathrm{div}(a_2 - a_0)||_{\infty,\infty} ||\partial_t v_1 - \partial_t v_2||_{X^1} \\ &\lesssim ||a||_{C_b^2} (r + ||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}) ||u_1 - u_2||_{Z^1}. \end{aligned}$$

Therefore we can bound A by

$$||A||_{X^1} \lesssim (r + ||\partial_t v_1||_{X^1} + ||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}) ||u_1 - u_2||_{Z^1}.$$

Now we will estimate part 'B', which we will rewrite in a similar way as part 'A'.

$$\boxed{\mathrm{B}} = \underbrace{(a_1 - a_2) \cdot \nabla \partial_t v_1}_{\boxed{\mathrm{B1}}} + \underbrace{(a_2 - a_0) \cdot (\nabla \partial_t v_1 - \nabla \partial_t v_2)}_{\boxed{\mathrm{B2}}}.$$

Estimating the first term,

$$||\mathrm{B1}||_{X^1} \leq ||a_1 - a_2||_{\infty,\infty} ||\nabla \partial_t v_1||_{X^1} \lesssim ||a||_{C_b^2} ||u_1 - u_2||_{Z^1} ||\nabla \partial_t v_1||_{X^1},$$

and the second term

$$\begin{aligned} ||\mathrm{B2}||_{X^1} &\leq ||a_2 - a_0||_{\infty,\infty} ||\nabla \partial_t v_1 - \nabla \partial_t v_2||_{X^1} \\ &\lesssim ||a||_{C_b^2} ||v_2 - u_0||_{\infty, X^{\mathrm{Tr}}} ||u_1 - u_2||_{Z^1} \\ &\lesssim ||a||_{C_b^2} (r + ||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}) ||u_1 - u_2||_{Z^1}. \end{aligned}$$

So from this we see that also 'B' can be bounded as

$$||\mathrm{B1}||_{X^1} \lesssim (r + ||\nabla \partial_t v_1||_{X^1} + ||u^* - u_0||_{\infty, X^{\mathrm{Tr}}}) ||u_1 - u_2||_{Z^1},$$

from which the first statement follows.
The second and third inequality are proven in an analogous way as the first inequality.

'(iv)' Notice that as $Z^1 \hookrightarrow C(J, X^{\mathrm{Tr}}) \hookrightarrow C(J; C^2(\Omega))$, we have for all $t \in J$ that

$$||\Phi'(v_1) - \Phi'(v_2)||_{L^q(\Omega)} \leq [\Phi''] \, ||v_1 - v_2||_{L^q(\Omega)}.$$

Similarly we see

$$\begin{aligned} ||\partial_{x_j}(\Phi'(v_1) - \Phi'(v_2))||_{L^q(\Omega)} &= ||\Phi''(v_1)\partial_{x_j} v_1 - \Phi''(v_2)\partial_{x_j} v_2||_{L^q(\Omega)} \\ &\leq ||\Phi''(v_1)\partial_{x_j}(v_1 - v_2)||_{L^q(\Omega)} + ||(\Phi''(v_1) - \Phi''(v_2))\partial_{x_j} v_2||_{L^q(\Omega)}. \end{aligned}$$

We have

$$||\Phi''(v_1)||_{\infty,\infty} \leq ||\Phi''(v_1) - \Phi''(0)||_{\infty,\infty} + |\Phi''(0)| \leq [\Phi'''] \, ||v_1||_{\infty,\infty} + |\Phi''(0)| \\ \leq [\Phi''']R + |\Phi''(0)| < \infty.$$

Using this

$$||\partial_{x_j}(\Phi'(v_1) - \Phi'(v_2))||_{L^q(\Omega)} \leq ||\Phi''(v_1)||_{\infty,\infty}||v_1 - v_2||_{H^{1,q}(\Omega)} \\ + [\Phi''']||v_1 - v_2||_{H^{1,q}(\Omega)} \underbrace{||\partial_{x_j} v_2||_{\infty,\infty}}_{\leq R}.$$

Combining this we've shown that for all $t \in J$ the following inequality holds,

$$||\Phi'(v_1) - \Phi'(v_2)||_{H^{1,q}(\Omega)} \lesssim_{\Phi,R} ||v_1 - v_2||_{H^{1,q}(\Omega)}.$$

Then we see that

$$||\Phi'(v_1) - \Phi'(v_2)||_{L^p(J;H^{1,q}(\Omega))} \lesssim_{\Phi,R} ||v_1 - v_2||_{L^p(J;H^{1,q}(\Omega))}.$$

'(v)' This inequality can be proved using the proof of[1] the second inequality, and trace theory. Notice that by the divergence theorem we have

$$\int_{\partial\Omega} B\nabla u \cdot \nu dS = \int_{\Omega} \operatorname{div}(B\nabla u)dx,$$

and as the trace map is bounded from $H^{1,q}(\Omega)$ to $H^{1-1/q,q}(\partial\Omega)$ we get

$$||G_5((u_1,\mu_1),(u^*,\mu^*)) - G_5((u_2,\mu_2),(u^*,\mu^*))||_{W^{1-1/q,q}(\partial\Omega)} \\ = ||[(b_0 - b_1)\nabla(\mu_1 + \mu^*) - (b_0 - b_2)\nabla(\mu_2 + \mu^*)] \cdot \nu||_{W^{1-1/q,q}(\partial\Omega)} \\ \lesssim ||G_2((u_1,\mu_1),(u^*,\mu^*)) - G_2((u_2,\mu_2),(u^*,\mu^*))||_{H^{1,q}(\Omega)}.$$

Therefore, by integrating over time we obtain

$$||G_5((u_1,\mu_1),(u^*,\mu^*)) - G_5((u_2,\mu_2),(u^*,\mu^*))||_{Y^1} \lesssim ||G_2((u_1,\mu_1),(u^*,\mu^*)) - G_2((u_2,\mu_2),(u^*,\mu^*))||_{X^2}.$$

□

Theorem 4.2. *Let $n \in \mathbb{N}\setminus\{0\}$, $p,q \in (1,\infty)$, and $\alpha \in [0,p-1)$ such that $3-2(1+\alpha)/p > n/q+2$. Let $\Omega = \mathbb{R}^n$ or let $\Omega \subset \mathbb{R}^n$ a domain with $\partial\Omega \in C^3$. Set $J_0 = [0,T_0]$, and suppose $\Phi \in C^{3-}(\mathbb{R})$, $a,c \in C_b^1(\bar{\Omega}; C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n))$ and $b \in C_b^1(\bar{\Omega}; C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}))$.*

Then there exists an interval $J = [0,T] \subset J_0$ such that the quasilinear system (43), and additionally (44) in case $\Omega \subset \mathbb{R}^n$ is a domain, admits a unique solution

$$u \in H^{1,p}(J, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J, w_\alpha; H^{3,q}(\Omega)) =: Z^1 \text{ and } \mu \in L^p(J, w_\alpha; H^{3,q}(\Omega)) =: Z^2$$

if the data are subject to (40) for the case that $\Omega = \mathbb{R}^n$ or (42) for the case $\Omega \subset \mathbb{R}^n$ is a domain.

Proof. It can be seen that (u,μ) is a solution of the quasilinear system (43) if and only if

$$\mathbb{L}(u,\mu) = \mathbb{G}((u,\mu),(u^*,\mu^*)).$$

We aim to show that such a fixed point exists with a Banach contraction argument. For this purpose, consider the mapping

$$\mathcal{T} : {}_0\mathbb{E}_1 \to \mathbb{E}_1 : (u,\mu) \mapsto \mathbb{L}^{-1}\mathbb{G}((u,\mu),(u^*,\mu^*)). \tag{48}$$

Consider a ball $\mathbb{B}_r := \{(\phi,\psi) \in {}_0\mathbb{E}_1; ||(\phi,\psi)||_{\mathbb{E}_1} < r\}$ with radius $r \in (0,1)$, which will be fixed later. We have to show that $\mathcal{T}[\mathbb{B}_r] \subset \mathbb{B}_r$, and that there exists a constant $\kappa \in [0,1)$ such that the contractive inequality

$$||\mathcal{T}(u_1,\mu_1) - \mathcal{T}(u_2,\mu_2)||_{\mathbb{E}_1} \leq \kappa||(u_1,\mu_1) - (u_2,\mu_2)||_{\mathbb{E}_1}$$

[1]Of course, G_2 was estimated in X^1 and not in X^2. But as $X^{\text{Tr}} \hookrightarrow C^2$ and $B \in C_b^2(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^{n\times n})$ this goes analogously.

holds for all $(u_1,\mu_1),(u_2,\mu_2)\in\mathbb{B}_r$. Indeed, by the previous Lemma we see that we can bound $\mathcal{T}$ in $\mathbb{E}_1$ by

$$\begin{aligned}||\mathcal{T}(u_1,\mu_1)-\mathcal{T}(u_2,\mu_2)||_{\mathbb{E}_1} &\leq ||\mathbb{L}^{-1}||_{\mathbb{E}_0\to\mathbb{E}_1}||\mathbb{G}((u_1,\mu_1),(u^*,\mu^*))-\mathbb{G}((u_2,\mu_2),(u^*,\mu^*))||_{\mathbb{E}_0}\\ &\leq C(r+\zeta(T))||(u_1,\mu_1)-(u_2,\mu_2)||_{\mathbb{E}_1},\end{aligned}$$

where $\zeta(T)\to 0$ as $T\to 0$. This allows us to choose $r\in(0,1)$ and $T>0$ sufficiently small such that the contractive inequality is satisfied. Also notice that

$$\begin{aligned}||\mathcal{T}(u,\mu)||_{\mathbb{E}_1} &\leq ||\mathcal{T}(u,\mu)-\mathcal{T}(0,0)||+||\mathcal{T}(0,0)||_{\mathbb{E}_1}\\ &\leq C(r+\zeta(T))||(u_1,\mu_1)||+||\mathbb{G}((0,0),(u^*,\mu^*))||_{\mathbb{E}_1})\\ &\leq C(r+\zeta(T))r+C(r+\zeta(T))||(u^*,\mu^*)||_{\mathbb{E}_1}.\end{aligned}$$

Therefore choosing $r\in(0,1)$ and $T>0$ sufficiently small, we also have that $\mathcal{T}[\mathbb{B}_r]\subset\mathbb{B}_r$, which concludes the proof. □

4.2 Critical Spaces

In this section we will consider the semilinear Cahn-Hilliard-Gurtin system on domains in a so-called critical spaces setting. This is a relatively modern approach, and was first introduced by Prüss and Wilke at the end of last decade in their work about parabolic evolution equations (see [PW17] and [PSW18]). As the Cahn-Hilliard-Gurtin system is of course not parabolic, we unfortunately cannot follow this work directly. Instead we will draw upon the work of Agresti and Veraar, and in particular their exposition on vector-valued fractional Sobolev spaces with power weights in time (see [AV20], Section 2.2).

Proposition 4.3 (Sobolev embedding). *Let X be a UMD Banach space, and let $T\in(0,\infty]$. Assume $1<p_0\leq p_1<\infty$, $s_0,s_1\in(0,1)$ and $\alpha_i\in(-1,p_i-1)$ for $i\in(0,1)$. Assume that*

$$\frac{\alpha_1}{p_1}\leq\frac{\alpha_0}{p_0}\ \text{and}\ s_0-\frac{1+\alpha_0}{p_0}\geq s_1-\frac{1+\alpha_1}{p_1}.$$

Then there is a constant C independent of T such that for all $f\in {}_0H^{s_0,p_0}([0,T],w_{\alpha_0};X)$,

$$||f||_{{}_0H^{s_1,p_1}([0,T],w_{\alpha_1};X)}\leq C||f||_{{}_0H^{s_0,p_0}([0,T],w_{\alpha_0};X)}.$$

The same holds for ${}_0H^{s_i,p_i}([0,T],w_{\alpha_i};X)$ replaced by $H^{s_i,p_i}([0,T],w_{\alpha_i};X)$ with a constant C which depends on T.

Proof. See Proposition 2.7 [AV20]. □

Theorem 4.4 (Mixed-derivative inequality). *Let (X_0,X_1) be an interpolation couple such that both X_0 and X_1 are UMD spaces. Let $p_0,p_1\in(1,\infty)$, $\alpha_0\in(-1,p_0-1)$, $\alpha_1\in(-1,p_1-1)$, and $s_0,s_1\in(0,1)$. For $\theta\in(0,1)$ denote*

$$s:=s_0(1-\theta)+s_1\theta,\quad \frac{1}{p}:=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \alpha:=(1-\theta)\frac{p}{p_0}\alpha_0+\theta\frac{p}{p_1}\alpha_1.$$

Assume $T\in(0,\infty]$ and $s\neq\frac{1+\alpha}{p}$. Then there exists a constant $C>0$ independent of T such that for all $f\in{}_0H^{s_0,p_0}([0,T],w_{\alpha_0};X_0)\cap{}_0H^{s_1,p_1}([0,T],w_{\alpha_1};X_1)$,

$$||f||_{{}_0H^{s,p}([0,T],w_\alpha;[X_0,X_1]_\theta)}\leq C||f||^{1-\theta}_{{}_0H^{s_0,p_0}([0,T],w_{\alpha_0};X_0)}||f||^{\theta}_{{}_0H^{s_1,p_1}([0,T],w_{\alpha_1};X_1)}.$$

Here $[X_0,X_1]_\theta$ denotes the complex interpolation space. The same holds with ${}_0H^{s_i,p_i}([0,T],w_{\alpha_i};X_i)$ replaced by $H^{s_i,p_i}([0,T],w_{\alpha_i};X_i)$ with a constant C which depends on T in which case $s=\frac{1+\alpha}{p}$ is also allowed.

Proof. See Proposition 2.8 [AV20]. □

The following Lemma is a weakened version of Lemma 4.9 from [AV20], for the purpose of presenting the main idea without too much notation. This version is also sufficient for our needs in this thesis.

Lemma 4.5. *Let $p \in (1,\infty)$, $\alpha \in [0, p-1)$ and $\rho \geq 0$ such that*

$$\rho\left(\varphi - 1 + \frac{1+\alpha}{p}\right) + \beta \leq 1, \tag{49}$$

where $\varphi \in (1 - \frac{1+\alpha}{p}, 1)$ and $\beta \in (1-\frac{1+\alpha}{p}, \varphi]$. Furthermore, let $\rho^* \in [\rho, \infty)$ such that equality holds in (49). Denote

$$\frac{1}{r'} := \frac{\rho^*(\varphi - 1 + (1+\alpha)/p)}{(1+\alpha)/p}, \quad \text{and} \quad \frac{1}{r} := \frac{\beta - 1 + (1+\alpha)/p}{(1+\alpha)/p}, \tag{50}$$

such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then we have the embedding, which is independent of T,

$${}_0H^{1,p}(J, w_\alpha; X_0) \cap L^p(J, w_\alpha; X_1) \hookrightarrow \mathfrak{X}(J), \tag{51}$$

where $\mathfrak{X}(J)$ is defined as

$$\mathfrak{X}(T) := L^{pr}(J, w_\alpha; X_\beta) \cap L^{\rho^* pr'}(J, w_\alpha; X_\varphi). \tag{52}$$

Proof. By hypothesis we see that the inequality $1 - \beta - \frac{1+\alpha}{p} \geq -\frac{1+\alpha}{pr}$ holds, therefore from the Sobolev embedding we have

$${}_0H^{1-\beta,p}(J, w_\alpha; X_\beta) \hookrightarrow L^{pr}(J, w_\alpha; X_\beta).$$

From the mixed derivative inequality we see that for all $\beta \in (0,1)$ we have

$${}_0H^{1,p}(J, w_\alpha; X_0) \cap {}_0L^p(J, w_\alpha; X_1) \hookrightarrow {}_0H^{1-\beta,p}(J, w_\alpha; X_\beta).$$

Similarly, as $1 - \varphi - \frac{1+\alpha}{p} \geq -\frac{1+\alpha}{\rho^* pr'}$ we have

$${}_0H^{1,p}(J, w_\alpha; X_0) \cap L^p(J, w_\alpha; X_1) \hookrightarrow {}_0H^{1-\varphi,p}(J, w_\alpha; X_\varphi) \hookrightarrow L^{\rho^* pr'}(J, w_\alpha; X_\varphi).$$

Combining the above embeddings now immediately gives the result. □

Remark 4.6. Notice that for all $p \in (1,\infty)$, $\rho \in (0,\infty)$ and $\alpha \in [0, p-1)$, there exists $\varphi \in (1 - \frac{1+\alpha}{p}, 1)$ and $\beta \in (1 - \frac{1+\alpha}{p}, \varphi]$ such that (49) holds. This can be seen easily by rewriting (49), i.e.

$$\frac{1+\alpha}{p} \leq \frac{1-\beta}{\rho} + 1 - \varphi.$$

Notice that $1 - \varphi \leq 1 - \beta < \frac{1+\alpha}{p}$ and $0 < 1 - \varphi < \frac{1+\alpha}{p}$, which shows the existence of such φ and β.

Proposition 4.7. *Let $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^3$ and $q \in (1,\infty)$. Consider the non-linearity $\Phi(u) = \frac{1}{k+1}u^{k+1}$ with $k \in \mathbb{N}$ such that $k \geq 2$. If $\frac{k-1}{k}\frac{n}{2q} < 1$, then Φ' satisfies the spatial bounds*

$$||\Phi'(u)||_{H^{1,q}(\Omega)} \precsim ||u||^k_{H^{1,kq}(\Omega)} \precsim ||u||^k_{X_\beta}, \quad \text{and} \tag{53}$$

$$||\Phi'(u_1) - \Phi'(u_2)||_{H^{1,q}(\Omega)} \precsim (1 + ||u_1||^{k-1}_{X_\beta} + ||u_2||^{k-1}_{X_\beta})||u_1 - u_2||_{X_\beta}. \tag{54}$$

Here X_β denotes the complex interpolation space $[H^{1,q}(\Omega), H^{3,q}(\Omega)]_\beta = H^{1+2\beta}(\Omega)$ with $\beta \in [\frac{k-1}{k}\frac{n}{2q}, 1)$. Similarly, if $\frac{n}{3q} < 1$, then the so-called double-well potential $\Psi(u) := \frac{1}{4}(u^2-1)^2$ satisfies the spatial bounds

$$||\Psi'(u)||_{H^{1,q}(\Omega)} \precsim ||u||^3_{X_\theta}, \quad \text{and} \tag{55}$$

$$||\Psi'(u_1) - \Psi'(u_2)||_{H^{1,q}(\Omega)} \precsim (1 + ||u_1||^2_{X_\theta} + ||u_2||^2_{X_\theta})||u_1 - u_2||_{X_\theta}, \tag{56}$$

where $\theta \in [\frac{n}{3q}, 1)$.

Proof. By definition we see that $||u^k||_{L^q(\Omega)} = ||u||^k_{L^{kq}(\Omega)} \leq ||u||^k_{H^{1,kq}(\Omega)}$. And using Hölder's inequality we see that

$$||\partial_j u^k||_{L^q(\Omega)} = ||ku^{k-1}\partial_j u||_{L^q(\Omega)} \leq k||u||^{k-1}_{L^{kq}(\Omega)}||\partial_j u||_{L^{kq}(\Omega)} \leq k||u||^k_{H^{1,kq}(\Omega)}.$$

So indeed, we see that $||u^k||_{L^q(\Omega)} \precsim ||u||^k_{H^{1,kq}(\Omega)}$. Now the Sobolev embedding on Bessel potential spaces states that $H^{s,q}(\Omega) \hookrightarrow H^{1,kq}(\Omega)$ if $s - \frac{n}{q} \geq 1 - \frac{n}{kq}$. Therefore we see that $s = 1 + 2\beta \geq$

$1+\frac{k-1}{k}\frac{n}{q}$, from which we obtain (53). In order to treat the difference estimate, we use the identity $u_1^k - u_2^k = (u_1-u_2)\sum_{j=1}^k u_1^{k-j}u_2^{j-1}$. Then by Hölder's inequality we have

$$||u_1^k - u_2^k||_{L^q(\Omega)} \le ||u_1-u_2||_{L^{kq}(\Omega)} \sum_{j=1}^k ||u_1||_{L^{kq}(\Omega)}^{k-j} ||u_2||_{L^{kq}(\Omega)}^{j-1}.$$

Now using the inequality $\sum_{j=1}^k |x|^{k-j}|y|^{j-1} \lesssim 1+|x|^{k-1}+|y|^{k-1}$, which holds for $x,y\in\mathbb{R}$, and applying the Sobolev embedding as before, we see that

$$\begin{aligned} ||u_1^k - u_2^k||_{L^q(\Omega)} &\le ||u_1-u_2||_{H^{1,kq}(\Omega)} \left(1+||u||_{H^{1,kq}(\Omega)}^{k-1} + ||u||_{H^{1,kq}(\Omega)}^{k-1}\right) \\ &\lesssim ||u_1-u_2||_{X_\beta}\left(1+||u_1||_{X_\beta}^{k-1}+||u_2||_{X_\beta}^{k-1}\right). \end{aligned}$$

Notice that by the product rule we have for all $j\in\{1,...,k\}$ that

$$\begin{aligned} \partial_i(u_1^k-u_2^k) &= \partial_i(u_1-u_2)u_1^{k-j}u_2^{j-1} \\ &= (\partial_i(u_1-u_2))u_1^{k-j}u_2^{j-1} + (u_1-u_2)(\partial_i u_1^{k-j})u_2^{j-1} + (u_1-u_2)u_1^{k-j}\partial_i u_2^{j-1}. \end{aligned}$$

From this and the same estimates as used previously we see that

$$\begin{aligned} ||\partial_i(u_1^k-u_2^k)||_{L^q(\Omega)} &\le \sum_{j=1}^n ||u_1-u_2||_{H^{1,kq}(\Omega)}||u_1||_{H^{1,kq}(\Omega)}^{k-j}||u_2||_{H^{1,kq}(\Omega)}^{j-1} \\ &\lesssim ||u_1-u_2||_{X_\beta}\left(1+||u_1||_{X_\beta}^{k-1}+||u_2||_{X_\beta}^{k-1}\right). \end{aligned}$$

This shows that the estimate (54) holds true.

Now we consider the double-well potential. Notice that $\Psi'(u)=(u^2-1)u$. Then using (53) and $H^{3,q}(\Omega)\hookrightarrow X_\theta \hookrightarrow H^{1,q}(\Omega)$ we see that

$$||\Psi'(u)||_{H^{1,q}(\Omega)} \le ||u^3||_{H^{1,q}(\Omega)} + ||u||_{H^{1,q}(\Omega)} \lesssim ||u||_{X_\theta}^3 + ||u||_{X_\theta},$$

where $\theta\in[\frac{n}{3q},1)$. Similarly, from (54) we see that

$$\begin{aligned} ||\Psi'(u_1)-\Psi'(u_2)||_{H^{1,q}(\Omega)} &\le ||u_1^3-u_2^3||_{H^{1,q}(\Omega)} + ||u_1-u_2||_{H^{1,q}(\Omega)} \\ &\lesssim (1+||u_1||_{X_\theta}^2+||u_2||_{X_\theta}^2)||u_1-u_2||_{X_\theta}. \end{aligned}$$

□

Remark 4.8. For the double-well potential $\Phi(u)=(u^2-1)^2$ we have the condition that

$$2\left(\beta-1+\frac{1+\alpha}{p}\right)+\beta\le 1, \text{ with } \beta\in(1-\frac{1+\alpha}{p},1)\cap[\frac{n}{3q},1).$$

To satisfy this condition we require that $\frac{3}{2}\left(1-\frac{n}{3q}\right) > \frac{1+\alpha}{p}$, which can be rewritten as

$$\frac{2}{3}\frac{1+\alpha}{p}+\frac{n}{3q}<1.$$

In comparison, the 'classical' framework for the semilinear equation has the condition $3-2\frac{1+\alpha}{p} > 1+\frac{n}{q}$, cf. Theorem 4.2, which can be rewritten as

$$\frac{n}{2q}+\frac{1+\alpha}{p}<1.$$

In the next theorem we are in the position to solve the semilinear Cahn-Hilliard-Gurtin system,

$$\begin{aligned} \partial_t u - \operatorname{div}(a\partial_t u) &= \operatorname{div}(b\nabla\mu)+f, & t\in J, x\in\Omega, \\ \mu - c\cdot\nabla\mu &= \beta\partial_t u - \Delta u + \Phi'(u) + g, & t\in J, x\in\Omega, \\ b\partial_\nu\mu &= h_1, & t\in J, x\in\partial\Omega, \\ \partial_\nu u &= h_2, & t\in J, x\in\partial\Omega \\ u(0,\cdot) &= u_0, & t=0, x\in\Omega. \end{aligned} \tag{57}$$

Here $a,c\in\mathbb{R}^n$, $b\in\mathbb{R}$ and (β,a,c,b) satisfy (33).

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega \in C^3$. Let $p, q \in (1, \infty)$, $\alpha \in [0, p-1)$, and denote $J_0 = [0, T_0]$. Consider a non-linearity $\Phi' : H^{3,q}(\Omega) \to H^{1,q}(\Omega)$ with constant $\rho > 0$ such that following spatial bounds hold for all $u, u_1, u_2 \in H^{3,q}(\Omega)$,*

$$||\Phi'(u)||_{H^{1,q}(\Omega)} \lesssim ||u||^{\rho}_{X_\varphi} ||u||_{X_\beta}, \text{ and} \tag{58}$$

$$||\Phi'(u_1) - \Phi'(u_2)||_{H^{1,q}(\Omega)} \lesssim (1 + ||u_1||^{\rho}_{X_\varphi} + ||u_2||^{\rho}_{X_\varphi})||u_1 - u_2||_{X_\beta}. \tag{59}$$

Here $X_\beta = [H^{1,q}(\Omega), H^{3,q}(\Omega)]_\beta$ and $X_\varphi = [H^{1,q}(\Omega), H^{3,q}(\Omega)]_\varphi$ with p, α, ρ, φ and β satisfying the same conditions as in Lemma 4.5. If the data are subject to (42), then there exists a time interval $J = [0, T] \subset J_0$ – depending on the data – such that the semilinear system (57) admits a unique solution

$$u \in Z^1 := H^{1,p}(J, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J_0, w_\alpha; H^{3,q}(\Omega)),$$
$$\mu \in Z^2 := L^p(J, w_\alpha; H^{2,q}(\Omega)).$$

Proof. Let $J_0 = [0, T_0]$ be a time interval that remains constant in the following argument, and let $J = [0, T] \subset J_0$ which we shall adjust until the semilinear equation becomes well-posed. Similarly as in the proof of Theorem 4.2, this theorem will be proved using a Banach fixed-point argument. Let $(u^*, \mu^*) \in Z^1(J_0) \times Z^2(J_0) =: \mathbb{E}_1(J_0)$ be the solution associated to the linearized equation (41), which exists due to the maximal regularity result achieved in Theorem 3.6. To capture the non-linear behavior of the semilinear Cahn-Hilliard-Gurtin system, define the mapping

$$\mathbb{G} : {}_0\mathbb{E}_1 \times \mathbb{E}_1 \to {}_0\mathbb{E}_0 : ((u, \mu), (u^*, \mu^*)) \mapsto \begin{pmatrix} 0 \\ \Phi'(u + u^*) \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbb{E}_0$ defined in (46) is the space associated to data. Now notice that $(u, \mu) \in \mathbb{E}_1$ is a solution of the semilinear system (57) if and only if it is a fixed point of the equation

$$\mathbb{L}(u, \mu) = \mathbb{G}((u, \mu), (u^*, \mu^*)),$$

where $\mathbb{L}$ is as defined in (47). Consider the mapping

$$\mathcal{T} : {}_0\mathbb{E}_1 \to \mathbb{E}_1 : (u, \mu) \mapsto \mathbb{L}^{-1}\mathbb{G}((u, \mu), (u^*, \mu^*)). \tag{60}$$

Let $\mathbb{B}_R \subset {}_0\mathbb{E}_1$ be a ball with radius $R \in (0, 1)$, which will be fixed later. We have to show that $\mathcal{T}[\mathbb{B}_R] \subset \mathbb{B}_R$, and that there exists a constant $\kappa \in [0, 1)$ such that the contractive inequality

$$||\mathcal{T}(u_1, \mu_1) - \mathcal{T}(u_2, \mu_2)||_{\mathbb{E}_1} \leq \kappa ||(u_1, \mu_1) - (u_2, \mu_2)||_{\mathbb{E}_1}$$

holds for all $(u_1, \mu_1), (u_2, \mu_2) \in \mathbb{B}_R$. For this, we must estimate Φ' in $X^2(J)$, i.e.

$$||\Phi'(u_1 + u^*)||_{L^p(J, w_\alpha; H^{1,q}(\Omega))} \leq R, \text{ and}$$
$$||\Phi'(u_1 + u^*) - \Phi'(u_2 + u^*)||_{L^p(J, w_\alpha; H^{1,q}(\Omega))} \leq \kappa ||u_1 - u_2||_{Z^1(J)}.$$

Suppose that X^{Tr} is critical, i.e. $\rho = \rho^*$ such that equality holds in (49), and let $u \in \mathbb{B}_R$. Using the hypothesis on the non-linearity Φ we can estimate,

$$\begin{aligned} ||\Phi'(u + u^*)||_{L^p(J, w_\alpha; H^{1,q}(\Omega))} &\leq C_0 \left\| ||u + u^*||^{\rho}_{X_\varphi} ||u + u^*||_{X_\beta} \right\|_{L^p(J, w_\alpha)} \\ &\leq C_0 \left\| u + u^* \right\|^{\rho}_{L^{\rho p r'}(J, w_\alpha; X_\varphi)} \left\| u + u^* \right\|_{L^{pr}(J, w_\alpha; X_\beta)}. \end{aligned}$$

In the last step Hölder's inequality was used with r and r' as defined in (50), and the constant $C_0 > 0$ associated to the spatial estimate of Φ' is independent of time. Now using the triangle inequality we have

$$||u + u^*||^{\rho}_{L^{\rho p r'}(J, w_\alpha; X_\varphi)} \leq (||u||_{L^{\rho p r'}(J, w_\alpha; X_\varphi)} + ||u^*||_{L^{\rho p r'}(J, w_\alpha; X_\varphi)})^{\rho}.$$

Notice that u^* is bounded in $L^{\rho p r'}(J, w_\alpha; X_\varphi)$, as by Lemma 4.5 we find

$$||u^*||_{L^{\rho p r'}(J, w_\alpha; X_\varphi)} \leq ||u^*||_{L^{\rho p r'}(J_0, w_\alpha; X_\varphi)} \leq ||u^*||_{\mathfrak{X}(J_0)} \leq C(J_0)||u^*||_{Z^1(J_0)} < \infty.$$

Furthermore, by the Dominated Convergence Theorem, we see that $||u^*||_{L^{\rho pr'}(J,w_\alpha;X_\varphi)} \to 0$ as $T \to 0$. Therefore, choose $T > 0$ such that $||u^*||_{L^{\rho pr'}(J,w_\alpha;X_\varphi)} \leq R$. Similarly, we can further restrict $T > 0$ such that $||u^*||_{L^{pr}(J,w_\alpha;X_\beta)} \leq R$. Then using the definition of $\mathfrak{X}(J)$ and applying Lemma 4.5, our estimate for $\Phi'(u+u^*)$ in X^2 becomes

$$||\Phi'(u+u^*)||_{L^p(J,w_\alpha;H^{1,q}(\Omega))} \leq C_0(||u||_{\mathfrak{X}(J)} + R)^{\rho+1} \leq C_0(C_1||u||_{Z^1(J)} + R)^{\rho+1}.$$

Here the constant $C_1 > 0$ is independent of T as $u|_{t=0} = 0$ by virtue of $u \in \mathbb{B}_R \subset {}_0\mathbb{E}_1$. As $\rho > 0$, it is now possible to choose $R \in (0,1)$ such that $C_0(C_1R + R)^{\rho+1} < R$. With this constructed T and R we now have $\mathcal{T}[\mathbb{B}_R] \subset \mathbb{B}_R$.

Now we consider the contractive inequality. Using the hypothesis on the non-linearity Φ we have

$$||\Phi'(u_1+u^*)-\Phi'(u_2+u^*)||_{L^p(J,w_\alpha;H^{1,q}(\Omega))} \precsim \left\|(1 + ||u^* + u_1||^\rho_{X_\varphi} + ||u_2 + u^*||^\rho_{X_\varphi})||u_1 - u_2||_{X_\beta}\right\|_{L^p(J,w_\alpha)}$$

The first term can be estimated using Hölder's inequality,

$$||u_1 - u_2||_{L^p(J,w_\alpha;X_\beta)} \leq \underbrace{||1||_{L^{pr'}(J,w_\alpha)}}_{\to 0 \text{ as } T\to 0} ||u_1 - u_2||_{L^{pr}(J,w_\alpha;X_\beta)}$$

For the remaining two terms we have by Hölder's inequality together with the choice of $T > 0$ and $R \in (0,1)$ from before,

$$\begin{aligned}\left\| ||u_1 + u^*||^\rho_{X_\varphi} ||u_1 - u_2||_{X_\beta} \right\|_{L^p(J,w_\alpha)} &\leq ||u_1 + u^*||^\rho_{L^{\rho pr'}(J,w_\alpha;X_\varphi)} ||u_1 - u_2||_{L^{pr}(J,w_\alpha;X_\beta)} \\ &\leq (C_1R + R)^\rho ||u_1 - u_2||_{\mathfrak{X}(J)}\end{aligned}$$

By applying Lemma 4.5 we obtain

$$\begin{aligned}||u_1 - u_2||_{L^p(J,w_\alpha;X_\beta)} &\leq \left(||1||_{L^{pr'}(J,w_\alpha)} + (C_1R + R)^\rho\right) ||u_1 - u_2||_{\mathfrak{X}(J)} \\ &\leq C_1 \left(||1||_{L^{pr'}(J,w_\alpha)} + (C_1R + R)^\rho\right) ||u_1 - u_2||_{Z^1(J)}.\end{aligned}$$

Now we can further restrict $R \in (0,1)$ and $T > 0$ such that

$$\kappa := C_1 \left(||1||_{L^{pr'}(J,w_\alpha)} + (C_1R + R)^\rho\right) < 1.$$

This concludes the Banach fixed-point argument for the case that X^{Tr} is critical.

Now suppose that X^{Tr} is not critical, i.e. $\rho^* > \rho$ as defined in (49). Then we can estimate using Hölder's inequality, and we get an extra term that vanishes as $T \to 0$, making the estimates 'easier' in some sense. Indeed, define $\lambda > 0$ such that $\frac{1}{\rho} = \frac{1}{\rho^*} + \frac{1}{\lambda}$, then

$$\begin{aligned}||\Phi'(u+u^*)||_{L^p(J,w_\alpha;H^{1,q}(\Omega))} &\precsim \left\| ||u+u^*||^\rho_{X_\varphi} ||u+u^*||_{X_\beta} \right\|_{L^p(J,w_\alpha)} \\ &\leq \underbrace{||1||_{L^{\lambda pr'}(J)}}_{\to 0 \text{ as } T\to 0} ||u+u^*||^\rho_{L^{\rho^* pr'}(J,w_\alpha;X_\varphi)} ||u+u^*||_{L^{pr}(J,w_\alpha;X_\varphi)}.\end{aligned}$$

The same trick can be applied to the contractive inequality. □

Corollary 4.10. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^3$, and denote $J_0 = [0,T_0]$. Let $p, q \in (1,\infty)$ and $\alpha \in [0, p-1)$ such that*

$$\frac{2}{3}\frac{1+\alpha}{p} + \frac{1}{q} < 1.$$

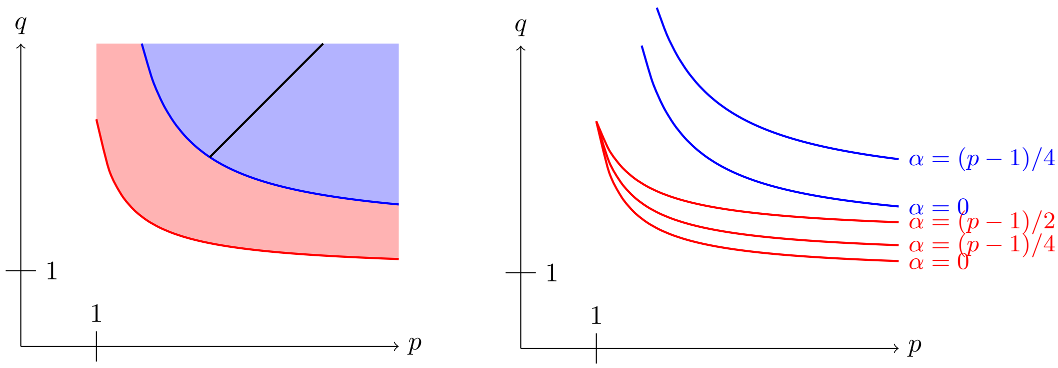
If the data are subject to (42), then there exists a time interval $J = [0,T] \subset J_0$ – depending on the data – such that the semilinear system (57) with the double well-potential $\Phi(u) = \frac{1}{4}(u^2 - 1)$ admits a unique solution

$$\begin{aligned}&u \in Z^1 := H^{1,p}(J, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J_0, w_\alpha; H^{3,q}(\Omega)),\\ &\mu \in Z^2 := L^p(J, w_\alpha; H^{2,q}(\Omega)).\end{aligned}$$

Proof. The condition for the spatial bounds follows from Proposition 4.7 and Remark 4.8. □

5 Global well-posedness

In this section we will adapt the global well-posedness argument from Wilke (see [Wil12], Section 6) in such a way that it becomes compatible, to some degree, with the local well-posedness results for the semilinear Cahn-Hilliard-Gurtin system from the previous section. The model non-linearity we have in mind for this is the double-well potential $\Phi(u) = \frac{1}{4}(u^2-1)^2$. To make a comparison between the gained flexibility of the parameters p, q and α clear, we shall fix the spatial dimension to 3. Then we can clearly see in Figure 7 that the critical space setting yields the most amount of allowed states for p, q and α. In fact, the allowed states of α, p and q from the classical setting are a subset of the allowed states of the critical spaces setting. In Wilke's paper, the integrability factor is assumed to be large. Under this assumption the trace space $X^{\mathrm{Tr}} = B^{3-2/p}_{p,p}(\Omega)$ embeds into $C^2(\Omega)$. The main challenge in this section shall be to work with smaller p, q and α's, whereby this type of smoothness becomes unavailable.



(a) Allowed states p, q for $\alpha = 0$.

(b) Allowed states p, q for several $\alpha(p)$.

Figure 7: (a) This illustrates the allowed states of integrability factors p and q for the semilinear CHG system with a double-well potential, $\alpha = 0$ and $n = 3$. Above the red line are all the allowed states of the critical space setting (see Corollary 4.10). Above the blue line are all the allowed states of the classical setting (see Theorem 4.2). The black line represents the allowed states obtained in the work from Wilke (see [Wil12], Theorem 5.2). (b) Illustrates the 'characteristic lines', above of which lie the allowed states, for different choices of α. Qualitatively, increasing the weight α makes the characteristic line higher. Furthermore, notice for the classical setting that letting $p \to 1$ implies $q \to \infty$, in contrast to the critical space setting, cf. Remark 4.8.

Now let us briefly sketch the global well-posedness argument. First, by a successive application of the local well-posedness result, we obtain a maximal time interval on which the solution exists, say $J_{\max} = [0, T_{\max})$. Next, we can reduce the time-weighted $L^p w_\alpha L^q$-spaces back to unweighted $L^p L^q$-spaces through a norm equivalency. Indeed, for a time interval $[a, b] \subset (0, T)$ that is away from the origin, it can be seen that $||\cdot||_{L^p([a,b],w_\alpha;X)} \asymp ||\cdot||_{L^p([a,b];X)}$. From this, we see that our local solution $u \in H^{1,p}(J_{\max}, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J_{\max}, w_\alpha; H^{3,q}(\Omega))$ instantaneously gains regularity,

$$u \in H^{1,p}_{\mathrm{loc}}((0,T); H^{1,q}(\Omega)) \cap L^p_{\mathrm{loc}}((0,T); H^{3,q}(\Omega)) \hookrightarrow C((0,T); X^{\mathrm{Tr}}).$$

With this gained regularity, it suffices to consider unweighted $L^p L^q$-spaces. In Lemma 5.3 we will show that, under certain technical conditions, the local solution $u \in L^\infty(J_{\max}; H^{1,2}(\Omega))$. Now by the maximal regularity result, there exists a constant $C > 0$ such that for all $T \in (0, T_{\max})$ we have

$$\begin{aligned} ||u||_{Z^1(T)} &\leq ||u||_{Z^1(T)} + ||\mu||_{Z^2(T)} \\ &\leq C(||\Phi'||_{X^2(T)} + ||f||_{X^1(T)} + ||g||_{X^2(T)} + ||h_1||_{Y^1(T)} + ||h_2||_{Y^2(T)} + ||u_0||_{X^{\mathrm{Tr}}}). \end{aligned}$$

If we can bound $||\Phi'||_{X^2(T)}$ for all $T \in [0, T_{\max})$, then the right hand side becomes bounded, as by assumption the data f, g, h_1, h_2 is bounded on a global time interval. Therefore we will show in Lemma 5.5 that there exists $\kappa \in (0,1)$ and $m > 0$ such that the bound

$$||\Phi'(u)||_{X^2(T)} \leq C(T) ||u||^{\kappa}_{X^2(T)} ||u||^m_{L^\infty((0,T_{\max}); H^{1,2}(\Omega))}$$

holds for all $T \in J_{\max}$ and where $\sup_{T \in J_{\max}} C(T) < \infty$. But then we can derive a contradiction with the maximality of the interval $J_{\max}$. Indeed, by substitution we see that there exists a constant $M = M(f, g, h_1, h_2, u_0) > 0$ which depends on the data, such that for all $T \in J_{\max}$ we have

$$||u||_{Z^1(T)} \leq M(1 + ||u||_{Z^1(T)}).$$

But as $\kappa \in (0,1)$ we see that $||u||_{Z^1(T_{\max})}$ is bounded, and therefore $u(T_{\max},\cdot) \in X^{\mathrm{Tr}}$ is well defined. Then we can do another iteration of the local well-posedness result, which is indeed a contradiction with the maximality of the interval $J_{\max}$.

Proposition 5.1. *If u and μ are solutions of the semilinear Cahn-Hilliard-Gurtin equations (57) and assume additionally that (β, a, c, b) satisfy the slightly stronger condition*

$$\beta z_0^2 + ((a+c)\cdot z_1)z_0 + Bz_1 \cdot z_1 \geq \varepsilon(z_0^2 + |z_1|^2) \quad \textit{for all } (z_0, z_1) \in \mathbb{R}\times\mathbb{R}^n, \tag{61}$$

for some $\varepsilon > 0$. Then u and μ satisfy the bound

$$\partial_t E(u) + \varepsilon(||\partial_t u||^2_{L^2(\Omega)} + ||\nabla\mu||^2_{L^2(\Omega)}) \leq \int_\Omega \mu f dx + \int_{\partial\Omega} \mu h_1 dS + \int_{\partial\Omega} (\partial_t u) h_2 dS - \int_\Omega (\partial_t u) g dx, \tag{62}$$

where $E(u) := \frac{1}{2}||\nabla u||^2_{L^2(\Omega)} + \int_\Omega \Phi(u)dx$. Moreover, we have

$$\int_\Omega u dx = \int_\Omega u_0 dx + \int_0^t \int_{\partial\Omega} h_1 dS d\tau + \int_0^t \int_\Omega f dx d\tau. \tag{63}$$

$$\int_\Omega \mu dx = \beta \int_\Omega \partial_t u dx - \int_{\partial\Omega} h_2 dS + \int_\Omega \Phi'(u)dx + \int_\Omega g dx, \ \textit{and} \tag{64}$$

Remark 5.2. For the moment we will assume that the integrals over the boundary

$$\int_{\partial\Omega} \mu h_1 dS \ \text{ and } \ \int_{\partial\Omega} (\partial_t u) h_2 dS$$

make sense. They will be justified in the proof of the next Lemma using Trace Theory.

Proof. Multiplying the first equation of (57) with μ yields,

$$(\partial_t u)\mu - \mathrm{div}(a\partial_t u)\mu = \mathrm{div}(b\nabla\mu)\mu + f\mu.$$

Now we use the divergence theorem, $\int_\Omega \mathrm{div}(F)dx = \int_{\partial\Omega} F\cdot\nu dS$, to the vector field $F = (0,0,...,uv,...,0)$, from which we see that the following integration by parts formula holds,

$$\int_\Omega u\partial_j v dx = \int_{\partial\Omega} uv\nu_j dS - \int_\Omega (\partial_j u)v dx.$$

Integrating the term $\mathrm{div}(a\partial_t u)$ over the domain Ω, and using the boundary condition $a\cdot\nu = 0$ on $\partial\Omega$ gives

$$\begin{aligned}\int_\Omega \mathrm{div}(a\partial_t u)\mu dx &= \sum_{j=1}^n \int_\Omega (\partial_j a_j \partial_t u)\mu dx = \sum_{j=1}^n \int_{\partial\Omega} a_j(\partial_t u)\mu\nu_j dS - \sum_{j=1}^n \int_\Omega a_j(\partial_t u)\partial_j\mu dx \\ &= \int_{\partial\Omega} (\partial_t u)\mu(a\cdot\nu)dS - \int_\Omega (a\cdot\nabla\mu)\partial_t u dx = -\int_\Omega (a\cdot\nabla\mu)\partial_t u dx.\end{aligned}$$

Similarly, by integrating $\mathrm{div}(b\nabla\mu)$ over the domain Ω, and using $b\partial_\nu\mu = b(\nu\cdot\nabla\mu) = h_1$ on $\partial\Omega$ we see that

$$\begin{aligned}\int_\Omega \mathrm{div}(b\nabla\mu)\mu dx &= \sum_{j=1}^n \int_\Omega [\partial_j(b\nabla\mu)_j]\mu dx = \sum_{j=1}^n \int_{\partial\Omega} (b\nabla\mu)_j\mu\nu_j dS - \sum_{j=1}^n \int_\Omega (b\nabla\mu)_j \partial_j\mu dx \\ &= \int_{\partial\Omega} b\mu(\nu\cdot\nabla\mu)dS - \int_\Omega b(\nabla\mu\cdot\nabla\mu)dx = \int_{\partial\Omega} \mu h_1 dS - \int_\Omega (B\nabla\mu\cdot\nabla\mu)dx.\end{aligned}$$

Therefore it follows that

$$\int_\Omega [(\partial_t u)\mu + (a\cdot\nabla\mu)\partial_t u + (b\nabla\mu\cdot\nabla\mu)]\, dx = \int_{\partial\Omega} \mu h_1 dS + \int_\Omega f\mu dx \tag{65}$$

Multiplying the second equation of (57) with $-\partial_t u$ yields

$$-\mu\partial_t u + c\cdot\nabla\mu\partial_t u = -\beta(\partial_t u)^2 + (\Delta u)\partial_t u - \Phi'(u)\partial_t u - g\partial_t u$$

Now notice that as $\partial_\nu u = h_2$ on $\partial\Omega$ and $\mathrm{div}(\nabla u\partial_t u) = (\Delta u)\partial_t u + \nabla u\cdot\nabla\partial_t u$. From this we obtain

$$\int_\Omega (\Delta u)\partial_t u dx = \int_{\partial\Omega} (\partial_t u)\nabla u\cdot\nu dS - \int_\Omega \nabla u\cdot\nabla\partial_t u dx = \int_{\partial\Omega} (\partial_t u)h_2 dS - \int_\Omega \frac{1}{2}\partial_t|\nabla u|^2 dx.$$

Since $\partial_t \Phi(u) = \Phi'(u)\partial_t u$ we see that

$$\int_\Omega \left[-\mu \partial_t u + (c \cdot \nabla \mu)\partial_t \mu + \beta(\partial_t u)^2 + \frac{1}{2}\partial_t |\nabla u|^2 + \partial_t \Phi(u) \right] dx = \int_{\partial\Omega} h_2 \partial_t u dS - \int_\Omega g \partial_t u dx \quad (66)$$

Adding equations (65) and (66) to each other yields

$$\int_\Omega \left[((a+c)\cdot \nabla\mu)\partial_t u + b\nabla\mu \cdot \nabla\mu + \beta|\partial_t u|^2 + \frac{1}{2}\partial_t|\nabla u|^2 + \partial_t \Phi(u) \right] dx$$
$$= \int_\Omega [\mu f - g\partial_t u]\, dx + \int_{\partial\Omega} [\mu h_1 + h_2 \partial_t u]\, dS$$

Now denote $E(u) := \frac{1}{2}||\nabla u||^2_{L^2(\Omega)} + \int_\Omega \Phi(u)dx$. From the assumption (61) with $z_0 = \partial_t u$ and $z_1 = \nabla \mu$ it follows that

$$\partial_t E(u) + \varepsilon(||\partial_t u||^2_{L^2(\Omega)} + ||\nabla\mu||^2_{L^2(\Omega)}) \le \partial_t E(u) + \int_\Omega \left[((a+c)\cdot\nabla\mu)\partial_t u + B\nabla\mu\cdot\nabla\mu + \beta|\partial_t u|^2 \right] dx$$
$$\le \int_\Omega \mu f dx + \int_{\partial\Omega} \mu h_1 dS + \int_{\partial\Omega} (\partial_t u) h_2 dS - \int_\Omega (\partial_t u) g dx.$$

By integrating the first equation of (57) over both time and space, and using the boundary condition and applying Fubini we get

$$\int_\Omega u dx = \int_\Omega \left(u_0 + \int_0^t \partial_t u d\tau \right) dx = \int_\Omega u_0 dx + \int_0^t \int_\Omega (\mathrm{div}(b\nabla\mu) + f) dx\tau$$
$$= \int_\Omega u_0 dx + \int_0^t \int_{\partial\Omega} h_1 dS d\tau + \int_0^t \int_\Omega f dx d\tau.$$

By integrating the second equation of (57) over the domain Ω, using the boundary condition $c\cdot\nu = 0$ on $\partial\Omega$, we get

$$\int_\Omega \mu dx = \beta \left[\int_{\partial\Omega} h_1 dS + \int_\Omega f dx \right] - \int_\Omega h_2 dS + \int_\Omega \Phi'(u)dx + \int_\Omega g dx.$$

□

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^3$. Let the data f, g, h_1, u_0 be in the usual spaces (42), and suppose additionally that there exists $\varepsilon > 0$ such that*

$$h_2 \in F^{1+\varepsilon}_{p,q}(J; L^q(\partial\Omega)) \cap L^p(J; W^{2-1/q,q}(\partial\Omega)).$$

Suppose there exists constants $c_0, ..., c_3 > 0$ and $\theta \in (0,1)$ such that the non-linearity $\Phi : \mathbb{R} \to \mathbb{R}$ of the semilinear system (57) satisfies the following two bounds for all $s \in \mathbb{R}$,

(i) $\Phi(s) \ge -\frac{\eta}{2}s^2 - c_0$, with $0 < \eta < \lambda_1$ and where λ_1 is the first non-trivial eigenvalue of the Neumann Laplacian,

(ii) $|\Phi'(s)| \le (c_1\Phi(s) + c_2 s^2 + c_3)^\theta$.

If $p \in [2,\infty)$ and $q \in [3/2, \infty)$, then $u \in L^\infty(J_{max}; H^{1,2}(\Omega))$.

Proof. We use equation (62) from Proposition 5.1 as a starting point. For the purpose of estimating the right hand side of (62), we shall first consider the integrals $|\int_\Omega u dx|$ and $|\int_\Omega \mu dx|$. From (63) we can estimate

$$\left| \int_\Omega u dx \right| \le ||u_0||_{L^1(\Omega)} + ||h_1||_{L^1(J;L^1(\partial\Omega))} + ||f||_{L^1(J;L^1(\Omega))}. \quad (67)$$

Notice that the L^1-norms of the data are bounded by an application of Hölder's inequality, as Ω is compact and $\partial\Omega \in C^3$. Next, from (64) and the first assumption on the non-linearity Φ we have

$$\left| \int_\Omega \mu dx \right| \le \int_\Omega (c_1\Phi(u) + c_2|u|^2 + c_3)^\theta dx + \beta||f||_{L^1(\Omega)} + ||g||_{L^1(\Omega)} + \beta||h_1||_{L^1(\partial\Omega)} + ||h_2||_{L^1(\partial\Omega)} \quad (68)$$

Now we shall consider the terms appearing on the right hand side of (62) one by one.

(i) Suppose $q \in [2, \infty)$, then by Hölder's inequality and Poincaré we have

$$\begin{aligned}\left|\int_\Omega \mu f dx\right| &\leq ||\mu f||_{L^1(\Omega)} \leq ||\mu||_{L^2(\Omega)}||f||_{L^2(\Omega)} \\ &\lesssim \left(||\nabla \mu||_{L^2(\Omega)} + \frac{1}{|\Omega|}\left|\int_\Omega \mu dx\right|\right)||f||_{L^2(\Omega)} \\ &\overset{(68)}{\leq} ||\nabla \mu||_{L^2(\Omega)}||f||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}\int_\Omega (c_1\Phi(u) + c_2|u|^2 + c_3)^\theta dx + C(f, g, h_1, h_2, \beta).\end{aligned}$$

Now let $\delta > 0$, then by Young's product inequality we see that

$$||\nabla \mu||_{L^2(\Omega)}||f||_{L^2(\Omega)} \leq \delta||\nabla \mu||^2_{L^2(\Omega)} + \frac{1}{4\delta}||f||^2_{L^2(\Omega)}, \text{ and}$$

$$\int_\Omega ||f||_{L^2(\Omega)}(c_1\Phi(u) + c_2|u|^2 + c_3)^\theta dx \leq |\Omega|(1-\theta)||f||^{1/(1-\theta)}_{L^2(\Omega)} + \int_\Omega \theta(c_1\Phi(u) + c_2u^2 + c_3)dx.$$

Now we estimate $||u||_{L^2(\Omega)}$ that appears in the second inequality using Poincaré,

$$||u||_{L^2(\Omega)} \lesssim ||\nabla u||_{L^2(\Omega)} + \frac{1}{|\Omega|}\left|\int_\Omega u dx\right| \overset{(67)}{\leq} ||\nabla u||_{L^2(\Omega)} + C(u_0, f, g).$$

Now recall that $E(u) := \frac{1}{2}||\nabla u||^2_{L^2(\Omega)} + \int_\Omega \Phi(u)dx$, so by carefully examining we see that

$$\left|\int_\Omega \mu f dx\right| \leq \delta||\nabla \mu||^2_{L^2(\Omega)} + CE(u) + C(u_0, f, g, h_1, h_2, \beta, \theta, \delta). \tag{69}$$

Now we consider what happens for $q < 2$. Notice that by the Sobolev embedding theorem we have $H^{1,2}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ where $1 = 1/q + 1/q^*$, provided that $q \geq 6/5$. Using this, we can estimate $|\int_\Omega \mu f dx|$ with Hölder's inequality and Poincare,

$$\begin{aligned}||\mu f||_{L^1(\Omega)} &\leq ||\mu||_{L^{q*}(\Omega)}||f||_{L^q(\Omega)} \leq ||\mu||_{H^{1,2}(\Omega)}||f||_{L^q(\Omega)} \\ &\leq \left(||\nabla \mu||_{L^2(\Omega)} + \frac{1}{|\Omega|}\left|\int_\Omega \mu dx\right|\right)||f||_{L^q(\Omega)}.\end{aligned}$$

From this point on, we can continue as before, and obtain an estimate of the form (69).

(ii) Suppose $q \in [2, \infty)$, then we have

$$\left|\int_{\partial\Omega} \mu h_1 dS\right| \leq ||\mu h_1||_{L^1(\partial\Omega)} \leq ||\mu||_{L^2(\partial\Omega)}||h_1||_{L^2(\partial\Omega)}.$$

Now as the trace mapping $T : H^{1,q}(\Omega) \to W^{1-1/q,q}(\partial\Omega)$ is bounded, we see that by using Poincaré we have

$$||\mu h_1||_{L^1(\partial\Omega)} \leq ||\mu||_{H^{1,2}(\Omega)}||h_1||_{L^2(\partial\Omega)} \leq \left(||\nabla \mu||_{L^2(\Omega)} + \left|\int_\Omega \mu dx\right|\right)||h_1||_{L^2(\partial\Omega)}.$$

From this point on, we can estimate analogously as in step (i), and obtain an estimate of the form

$$||\mu h_1||_{L^1(\partial\Omega)} \lesssim \delta||\nabla \mu||^2_{L^2(\Omega)} + E(u) + C(f, g, h_1, h_2, u_0, \theta, \delta, u_0). \tag{70}$$

Now we consider $q < 2$. Notice that $H^{1,2}(\Omega) \hookrightarrow W^{1-1/2,2}(\partial\Omega) \hookrightarrow L^{q^*}(\partial\Omega)$ where $1 = 1/q + 1/q^*$, provided that $q \geq 3/2$. From this we can estimate as follows,

$$\begin{aligned}||\mu h_1||_{L^1(\partial\Omega)} &\leq ||\mu||_{L^{q^*}(\partial\Omega)}||h_1||_{L^q(\partial\Omega)} \lesssim ||\mu||_{H^{1,2}(\Omega)}||h_1||_{W^{1-1/q,q}(\partial\Omega)} \\ &\lesssim \left(||\nabla \mu||_{L^2(\Omega)} + \left|\int_\Omega \mu dx\right|\right)||h_1||_{W^{1-1/q,q}(\partial\Omega)}.\end{aligned}$$

From this point on, we can estimate in the same way as before, and obtain an estimate of the form (70).

(iii) For the third term, we suppose that h_2 has more time regularity than is strictly given from the local well-posedness result. That is, we suppose there exists $\varepsilon > 0$ such that

$$h_2 \in F^{1+\varepsilon}_{p,q}(J; L^q(\partial\Omega)) \cap L^p(J; W^{2-1/q,q}(\partial\Omega)).$$

As $F^{1+\varepsilon}_{p,q}(J; L^q(\partial\Omega)) \hookrightarrow H^{1,p}(J; L^q(\partial\Omega))$ (see [Tri83], Section 2.3.2), we see that by Proposition 2.52 we have

$$h_2 \in H^{1,p}(J; L^q(\partial\Omega)) \cap L^p(J; W^{2-1/q,q}(\partial\Omega)) \hookrightarrow C(J; B^{(2-1/q)(1-1/p)}_{q,p}(\partial\Omega)).$$

Notice that for u we have the following regularity on the boundary $\partial\Omega$,

$$u \in H^{1,p}(J; W^{1-1/q,q}(\partial\Omega)) \cap L^p(J; W^{3-1/q,q}(\partial\Omega)) \hookrightarrow C(J; B^{3-1/q-2/p}_{q,p}(\partial\Omega)).$$

Then if we integrate the third term in both time and space, we see that by applying Fubini and integration by parts we get

$$\int_0^t \int_{\partial\Omega} (\partial_t u) h_2 dS d\tau = \int_{\partial\Omega} (u(t,\cdot)h_2(t,\cdot) - u_0(\cdot)h_2(0,\cdot))dS - \int_{\partial\Omega}\int_0^t u\partial_t h_2 d\tau dS.$$

Suppose $q \in [2,\infty)$, then using Hölder's inequality we can estimate

$$\left|\int_{\partial\Omega} u_0(\cdot)h_2(0,\cdot)dS\right| \le ||u_0||_{L^2(\partial\Omega)}||h_2||_{L^2(\partial\Omega)} \lesssim ||u_0||_{L^q(\partial\Omega)}||h_2||_{L^q(\partial\Omega)}.$$

Similarly we have

$$\begin{aligned}\left|\int_{\partial\Omega} u(t,\cdot)h_2(t,\cdot)dS\right| &\le ||u(t,\cdot)||_{L^2(\partial\Omega)}||h_2||_{L^2(\partial\Omega)} \le ||u(t,\cdot)||_{H^{1,2}(\Omega)}||h_2(t,\cdot)||_{L^2(\partial\Omega)} \\ &\le \delta||u(t,\cdot)||^2_{H^{1,2}(\Omega)} + \frac{1}{4\delta}||h_2(t,\cdot)||_{L^2(\partial\Omega)} \\ &\overset{(67)}{\le} \delta||\nabla u(t,\cdot)||^2_{L^2(\Omega)} + C(f,g,u_0) + \frac{1}{4\delta}||h_2(t,\cdot)||^2_{L^2(\partial\Omega)}.\end{aligned}$$

Now the remaining term, after applying Fubuni one last time, can be estimated by Hölder's inequality and Young's multiplication inequality

$$\begin{aligned}\left|\int_0^t\int_{\partial\Omega} u\partial_t h_2 dS d\tau\right| &\le \int_0^t ||u\partial_t h_2||_{L^1(\partial\Omega)}d\tau \le \int_0^t ||u||_{L^2(\partial\Omega)}||\partial_t h_2||_{L^2(\partial\Omega)}d\tau \\ &\le \int_0^t \left(\delta||u||_{L^2(\partial\Omega)} + \frac{4}{\delta}||h_2||_{L^2(\partial\Omega)}\right)d\tau.\end{aligned}$$

Now by using Poincaré we get

$$\delta||u||^2_{L^2(\partial\Omega)} \le \delta||u||^2_{H^{1,2}(\Omega)} \le \delta\left(||\nabla u||_{L^2(\Omega)} + \left|\int_\Omega u dx\right|\right)^2 \overset{(67)}{\le} \delta||\nabla u||^2_{L^2(\Omega)} + C(u_0,f,g).$$

So therefore we finally obtain

$$\left|\int_0^t\int_{\partial\Omega} (\partial_t u)h_2 dS d\tau\right| \le \delta||\nabla u||^2_{L^2(J;L^2(\Omega))} + C(u_0,f,g,h_2,\delta). \tag{71}$$

Now notice that for $q \ge 7/5$ the Sobolev embedding $B^{(2-1/q)(1-1/p)}_{q,p}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$. Furthermore, if $q \ge 8/7$ then $B^{3-1/q-2/p}_{q,p}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$. Then we can estimate using Hölder's inequality,

$$||u_0(\cdot)h_2(0,\cdot)||_{L^1(\partial\Omega)} \le ||u_0||_{L^2(\partial\Omega)}||h_2(0,\cdot)||_{L^2(\partial\Omega)} \lesssim ||u_0||_{X^{\mathrm{Tr}}}||h_2||_{Y^{\mathrm{Tr}}}.$$

Similarly, we obtain using Poincaré inequality,

$$||u(t,\cdot)h_2(t,\cdot)||_{L^1(\partial\Omega)} \le ||u(t,\cdot)||_{L^2(\partial\Omega)}||h_2(t,\cdot)||_{L^2(\partial\Omega)} \lesssim ||u(t,\cdot)||_{H^{1,2}(\Omega)}||h_2(t,\cdot)||_{Y^{\mathrm{Tr}}}.$$

Now notice that $H^{1,2}(\Omega) \hookrightarrow H^{1-1/2,2}(\partial\Omega) \hookrightarrow L^{q^*}(\partial\Omega)$ provided that $q \ge 3/2$ and $1 = 1/q + 1/q^*$, from which we obtain

$$\begin{aligned}\left|\int_0^t\int_{\partial\Omega} u\partial_t h_2 dS d\tau\right| &\le \int_0^t ||u||_{L^{q^*}(\partial\Omega)}||\partial_t h_2||_{L^q(\partial\Omega)}d\tau \\ &\le \int_0^t ||u||_{H^{1,2}(\partial\Omega)}||\partial_t h_2||_{L^q(\partial\Omega)}d\tau\end{aligned}$$

From this point on, we can estimate as before. Therefore, for the case $q \in [3/2, 2]$ we also obtain an estimate of the form (71).

(iv) Suppose $q \in [2, \infty)$, then we see that

$$\left|\int_\Omega (\partial_t u) g dx\right| \leq ||\partial_t u||_{L^2(\Omega)} ||g||_{L^2(\Omega)} \lesssim \delta ||\partial_t u||^2_{L^2(\Omega)} + \frac{1}{4\delta} ||g||_{L^q(\Omega)}. \tag{72}$$

If $q \in [3/2, 2]$, then $H^{1,2}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ with $1 = 1/q + 1/q^*$, from which we see that

$$||(\partial_t u) g||_{L^1(\Omega)} \leq ||\partial_t u||_{L^q(\Omega)} ||g||_{L^{q^*}(\Omega)} \lesssim ||\partial_t u||_{L^2(\Omega)} ||g||_{H^{1,q}(\Omega)} \leq \delta ||\partial_t u||^2_{L^2(\Omega)} + \frac{1}{4\delta} ||g||^2_{H^{1,q}(\Omega)}.$$

So, now we see that for $p \in [2, \infty)$ and $q \in [3/2, \infty)$ we have that

$$E(u) + \varepsilon(||\partial_t u||^2_{L^2(J;L^2(\Omega))} + ||\nabla \mu||^2_{L^2(J;L^2(\Omega))}) \leq \delta(||\partial_t u||^2_{L^2(J;L^2(\Omega))} + ||\nabla \mu||^2_{L^2(J;L^2(\Omega))}) + C_0 \int_0^T E(u(t, \cdot)) + C(u_0, f, g, h_1, h_2, \delta, \theta). \tag{73}$$

Now by choosing δ sufficiently small, we can absorb the $||\partial_t u||$ and $||\nabla u||$ terms, which yields

$$\begin{aligned} E(u(t, \cdot)) &\leq E(u(t, \cdot)) + C_1(||\partial_t u||^2_{L^2(J;L^2(\Omega))} + ||\nabla \mu||^2_{L^2(J;L^2(\Omega))}) \\ &\leq C_0 \int_0^T E(u(t, \cdot)) + C(u_0, f, g, h_1, h_2, \delta, \theta). \end{aligned}$$

Notice that by the first assumption on the non-linearity Φ we see that $||u||_{H^{1,2}(\Omega)} \lesssim E(u)$, and hence $E(u)$ is non-negative. Therefore, we may apply Grönwall's Lemma to see that $E(u(t, \cdot))$ is bounded for $t \in J_{\max}$. From $||u(t, \cdot)||_{H^{1,2}(\Omega)} \lesssim E(u(t, \cdot))$ we then see that $u \in L^\infty(J_{\max}; H^{1,2}(\Omega))$. □

Proposition 5.4 (Gagliardo-Nirenberg interpolation inequality). *Suppose $n, j, m \in \mathbb{N}$ and $p, q, r \in [1, \infty]$ and $a \in [0, 1]$ such that*

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right) a + \frac{1-a}{q} \quad \text{and} \quad \frac{j}{m} \leq a \leq 1.$$

Consider a function $u : \Omega \to \mathbb{R}$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ such that $u \in L^q(\Omega)$ and its m-th weak derivative lies in $L^r(\Omega)$, i.e. $D^m u \in L^r(\Omega)$. Then there exists constants $C_1, C_2 > 0$ depending on m, n, j, q, r, a such that

$$||D^j u||_{L^p(\Omega)} \leq C_1 ||D^m u||^a_{L^r(\Omega)} ||u||^{1-a}_{L^q(\Omega)} + C_2 ||u||_{L^s(\Omega)}.$$

Moreover, if $m - n/r \geq -n/q$, then

$$||D^j u||_{L^p(\Omega)} \leq C ||u||^a_{H^{m,r}(\Omega)} ||u||^{1-a}_{L^q(\Omega)}.$$

Proof. For the first statement, see [Nir59] Lecture 2. For the second statement, set $s = q$ and notice that by hypothesis we have $H^{m,r}(\Omega) \hookrightarrow L^q(\Omega)$ as $m - n/r \geq -n/q$.

$$\begin{aligned} ||D^j u||_{L^p(\Omega)} &\leq C_1 ||D^m u||^a_{L^r(\Omega)} ||u||^{1-a}_{L^q(\Omega)} + C_2 ||u||^a_{L^q(\Omega)} ||u||^{1-a}_{L^q(\Omega)} \\ &\leq C_1 ||u||^a_{H^{m,r}(\Omega)} ||u||^{1-a}_{L^q(\Omega)} + \tilde{C}_2 ||u||^a_{H^{m,r}(\Omega)} ||u||^{1-a}_{L^q(\Omega)} \\ &\leq (C_1 + \tilde{C}_2) ||u||^a_{H^{m,r}(\Omega)} ||u||^{1-a}_{L^q(\Omega)} \end{aligned}$$

□

Lemma 5.5. *Let $p, q \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\gamma \in [1, \infty)$ if $n \in \{1, 2\}$ and $\gamma \in [1, 4)$ if $n = 3$. Suppose there exists a constant $C_0 > 0$ such that*

$$|\Phi''(s)| \leq C_0(1 + |s|^\gamma), \quad s \in \mathbb{R}. \tag{74}$$

Then there exists constants $\kappa \in (0, 1)$, $m > 0$ such that the bound

$$||\Phi'(u)||_{L^p(J;H^{1,q}(\Omega))} \leq C(T) ||u||^\kappa_{L^p(J;H^{1,q}(\Omega))} ||u||^m_{L^\infty(0,T_{max};H^{1,2}(\Omega))} \tag{75}$$

holds for all $T \in [0, T_{max})$ and where $\sup_{T \in [0,T_{max})} C(T) < \infty$.

Proof. We start with estimating the term $\nabla\Phi'(u) = \Phi''(u)\nabla u$ in $L^q(\Omega)$. As $\frac{1}{q} = \frac{1}{3q} + \frac{2}{3q}$ it follows by Hölder's inequality that

$$\begin{aligned}||\Phi''(u)\nabla u||_{L^q(\Omega)} &\leq ||\Phi''(u)||_{L^{3q/2}(\Omega)}||\nabla u||_{L^{3q}(\Omega)} \\ &\leq C\left(1 + ||u||^{\gamma}_{L^{3\gamma q/2}(\Omega)}\right)||\nabla u||_{L^{3q}(\Omega)}.\end{aligned}$$

Now we apply the Gagliardo-Nirenberg interpolation inequality, see Proposition 5.4, which gives

$$||u||_{L^{3\gamma q/2}(\Omega)} \lesssim ||u||^{a}_{H^{3,q}(\Omega)}||u||^{1-a}_{L^r(\Omega)}. \tag{76}$$

provided that there exists $a \in [0, 1]$ such that

$$\frac{2}{3\gamma q} = \left[\frac{1}{q} - \frac{3}{n}\right]a + \frac{1-a}{r} \text{ and } 3 - \frac{n}{q} \geq -\frac{n}{r}. \tag{77}$$

For the moment, we will assume that this requirement holds. Similarly, we have that

$$||\nabla u||_{L^{3q}(\Omega)} \lesssim ||u||^{b}_{H^{3,q}(\Omega)}||u||^{1-b}_{L^r(\Omega)}, \tag{78}$$

provided that there exists a $b \in [0, 1]$ such that

$$\frac{1}{3q} = \frac{1}{n} + \left[\frac{1}{q} - \frac{3}{n}\right]b + \frac{1-b}{r} \text{ and } 3 - \frac{n}{q} \geq -\frac{n}{r}. \tag{79}$$

Again, under the assumption that such b exists, by combining (76) and (78) we obtain

$$||\Phi''(u)\nabla u||_{L^q(\Omega)} \lesssim ||u||^{\gamma a+b}_{H^{3,q}(\Omega)}||u||^{\gamma(1-a)+(1-b)}_{L^r(\Omega)} + ||u||^{b}_{H^{3,q}(\Omega)}||u||^{1-b}_{L^r(\Omega)}. \tag{80}$$

Now, in order to gain something from this, we suppose that we can choose r in such a way that $H^{1,2}(\Omega) \hookrightarrow L^r(\Omega)$, i.e.

$$1 - \frac{n}{2} \geq -\frac{n}{r}. \tag{81}$$

Now we show that such a, b and r exists.

- If $n \in \{1, 2\}$, then (81) holds for all $r \in (1, \infty)$. Also the inequality $3 - n/q \geq -n/r$ holds for all $q, r \in (1, \infty)$ in these dimensions. Now notice that as $n < 3q$, we can choose r such that

  $$\frac{\gamma n}{2} < r < \frac{3\gamma q}{2}.$$

  We claim that a as implicitly defined by (77) lies in the interval $[0, 1]$. By simplifying we see that

  $$a = \underbrace{\frac{n}{3\gamma}}_{0<\cdot<1}\frac{3\gamma q - 2r}{(q-r)n + 3qr}.$$

  We see that by our choice of r we have that the numerator is positive, i.e. $3\gamma q - 2r > 0$. The denominator is also positive, which can be seen by noticing that $3q - n > 0$, which implies that

  $$(q-r)n + 3qr = qn + (3q-n)r > 0.$$

  This shows that $a \geq 0$. Now we show that $a \leq 1$. For the enumerator we have $0 \leq 3\gamma q - 2r < 3\gamma q - \gamma n$, using this we see that

  $$a < \frac{n}{3\gamma}\frac{3\gamma q - \gamma n}{qn + (3q-n)r} = \underbrace{\frac{n}{3}}_{<1}\underbrace{\frac{1}{c+r}}_{<1} \leq 1, \text{ where } c := \frac{qn}{3q-n} > 0.$$

  Similarly, we claim that b as implicitly defined by (79) also lies in the interval $[0, 1]$. By rewriting we see that

  $$b = \frac{1}{3}\frac{(3q-n)r + 3qn}{(3q-n)r + qn}.$$

  We see that both the enumerator and the denominator are positive, as $3q - n > 0$, hence $b \geq 0$. Now we show that $b \leq 1$. Notice that

  $$b = \frac{1}{3}\frac{(3q-n)r}{(3q-n)r + qn} + \frac{1}{3}\frac{3qn}{(3q-n)r + qn} = \frac{1}{3}\frac{1}{1+c_0} + \frac{1}{1+c_0^{-1}}, \text{ with } c_0 := \frac{c}{r} > 0.$$

Using an argument with contradiction, suppose that $b > 1$, then

$$\frac{1}{3}\frac{1+c_0^{-1}}{1+c_0} + 1 > 1 + c_0^{-1} \longrightarrow \frac{1}{3}\frac{1+c_0}{1+c_0} > 1,$$

which is a contradiction. So this shows that indeed $b \le 1$.

- If $n = 3$, then (81) holds for $r \in (1, 6)$. If $\gamma \in [1, 4)$, then $\gamma n/2 < 6$. Using this, we see that we choose r such that
$$\frac{\gamma n}{2} < r < \min\left\{6, \frac{3\gamma q}{2}\right\}.$$
With this choice of r, we can again show that $a \in [0, 1]$ and $b \in [0, 1]$.

Now that we know (80) is valid, set $\kappa = \gamma a + b$. It can be seen that $\kappa \in (0, 1)$. Firstly we see that $\kappa > 0$ as γ, a, and b are all positive. Next, notice that by definition of (77) and (79) we have

$$\kappa\left[\frac{3}{n} + \frac{1}{r} - \frac{1}{q}\right] = \gamma\left[\frac{1}{r} - \frac{2}{3\gamma q}\right] + \left[\frac{1}{r} + \frac{1}{n} - \frac{1}{3q}\right] = \frac{1}{n} + \frac{\gamma+1}{r} - \frac{1}{q}.$$

Now by the assumption on r we have that $\frac{\gamma}{r} < \frac{2}{n}$, which shows the existence of $\kappa \in (0, 1)$. Hence (80) yields

$$||\Phi''(u)\nabla u||_{L^q(\Omega)} \lesssim ||u||^{\kappa}_{H^{3,q}(\Omega)} ||u||^{(1-a)\gamma+(1-b)}_{H^{1,2}(\Omega)}. \tag{82}$$

We also claim that

$$||\Phi'(u)||_{L^q(\Omega)} \lesssim ||u||^{\kappa}_{H^{3,q}(\Omega)} ||u||^{m}_{H^{1,2}(\Omega)}. \tag{83}$$

This can be seen from the inequality

$$|\Phi'(s)| \le \int_0^s |\Phi''(r)| dr = \left(1 + \frac{|s|^\gamma}{\gamma+1}\right)|s|.$$

Then by Hölder's inequality we again have

$$||\Phi'(u)||_{L^p(\Omega)} \lesssim (1 + ||u||^{\gamma}_{L^{3\gamma q/2}(\Omega)})||u||_{L^{3q}(\Omega)},$$

and from this point we can estimate as before. Thus combining the estimates (82) and (83), and applying Hölder's inequality yields

$$\begin{aligned}
||\Phi'(u)||^p_{L^p(J,H^{1,q}(\Omega))} &= \int_0^T ||\Phi'(u(t,\cdot))||^p_{H^{1,q}(\Omega)} dt \lesssim \int_0^T ||u||^{\kappa p}_{H^{3,q}(\Omega)} ||u||^{mp}_{H^{1,2}(\Omega)} dt \\
&\le ||u||^{\kappa p}_{L^{\kappa p}(J;H^{3,q}(\Omega))} ||u||^{mp}_{L^\infty(J;H^{1,2}(\Omega))} \\
&\le ||1||^{\kappa p}_{L^{\kappa p/(1-\kappa)}(J)} ||u||^{\kappa p}_{L^p(J;H^{3,q}(\Omega))} ||u||^{mp}_{L^\infty(J;H^{1,2}(\Omega))}
\end{aligned}$$

Now taking p-th roots on both sides yields the desired result. □

Theorem 5.6 (Global well-posedness)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^3$ and denote $J = [0, T]$. Suppose there exists constants $c_0, ...c_3 > 0$, $\theta \in (0, 1)$ and $\gamma \in [1, 4)$ such that the non-linearity $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies the following three bounds for all $s \in \mathbb{R}$,*

(i) $\Phi(s) \ge -\frac{\eta}{2}s^2 - c_0$, with $0 < \eta < \lambda_1$ and where λ_1 is the first non-trivial eigenvalue of the Neumann Laplacian,

(ii) $|\Phi'(s)| \le (c_1\Phi(s) + c_2 s^2 + c_3)^\theta$, and

(iii) $|\Phi''(s)| \le C_0(1 + |s|^\gamma)$.

Suppose $p \in [2, \infty)$, $q \in [3/2, \infty)$ and $\alpha \in [0, p-1)$ such that

$$\frac{2}{3}\frac{1+\alpha}{p} + \frac{1}{q} < 1.$$

Suppose (β, a, c, b) satisfy (61). Then there exists a global solution of the semilinear system (57)

$$\begin{aligned}
&u \in Z^1 := H^{1,p}(J, w_\alpha; H^{1,q}(\Omega)) \cap L^p(J, w_\alpha; H^{3,q}(\Omega)), \text{ and} \\
&\mu \in Z^2 := L^p(J, w_\alpha; H^{2,q}(\Omega)),
\end{aligned}$$

if the data are subject to

$$
\begin{aligned}
f \in X^1 &:= L^p(J, w_\alpha; L^q(\Omega)),\\
g \in X^2 &:= L^p(J, w_\alpha; H^{1,q}(\Omega)),\\
h_1 \in Y^1 &:= L^p(J, w_\alpha; W^{1-1/q,q}(\partial\Omega)),\\
h_2 \in Y^2 &:= F^{1+\varepsilon}_{p,q}(J; L^q(\partial\Omega)) \cap L^p(J; W^{2-1/q,q}(\partial\Omega)),\\
u_0 \in X^{Tr} &:= B^{3-2(\alpha+1)/p}_{qp}(\Omega), \ \textit{and}\\
\partial_\nu u_0 &= h_2|_{t=0} \ \textit{if} \ 1 - \frac{1}{2q} > \frac{1+\alpha}{p},
\end{aligned}
$$

where $\varepsilon > 0$.

Proof. Follows from Lemma 5.3 and Lemma 5.5, and the argument at the beginning of this section. □

Remark 5.7. Notice that the non-linearity $\Phi(u) = \frac{1}{4}(u^2-1)^2$ satisfies all 3 criteria in the above Theorem. Figure 8 shows the condition of parameters for the global well-posedness.

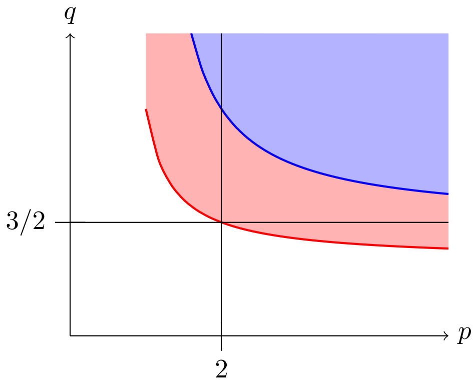


Figure 8: Comparison of the conditions on p and q in the semilinear setting with $\Phi(u) = (u^2-1)^2$, $\alpha = 0$, and $n = 3$. The area shaded in red is associated to the critical spaces setting and the area shaded in blue is associated to the classical setting. For the global well-posedness argument proved in this thesis, we are constrained to $p \in [2, \infty)$ and $q \in [3/2, \infty)$.

6 Conclusion

In this thesis, we first set out to develop linear theory for the Cahn-Hilliard-Gurtin equations on $\mathbb{R}^n$ in the setting of weighted L^pL^q-spaces. By connecting the weighted anisotropic Mikhlin multiplier theorem with the method of Newton polygons in Section 2, this goal was achieved for the linear Cahn-Hilliard-Gurtin equations on $\mathbb{R}^n$ in Section 3. Thanks to this result, we may be able to also treat other mixed-order systems that fit the Newton polygon approach on weighted L^pL^q-spaces, which could be an interesting starting point for future work. Due to limited time, we were not able to consider the linear theory for the half-space and domains. Therefore we postulated maximal regularity results in these settings together with a localization result. We hope to prove these postulations rigorously in future work. Note that they are likely to hold true, as Wilke considered the half-space, domains and a localization argument already in the L^p-setting. Furthermore, one direction of Maximal regularity results are always easy to prove, which likewise gives us confidence that the postulations will indeed be shown to hold true.

In Section 4.1 we considered the local well-posedness of the quasilinear equation in a classical setting by adapting Wilke's proof. In this approach, the integrability parameters p and q are assumed to be large, such that the trace space X^{Tr} embeds into C^2. As we are working on time-weighted L^pL^q-spaces, this already led to new results, which enable us to treat rough initial conditions.

Then, in Section 4.2 we considered the local well-posedness for the semilinear equation in the setting of Critical Spaces. This allows us to consider lower values for p, q and α. Specifically, for the double well-potential $\Phi(u) = (u^2 - 1)^2$ we require that $\frac{2}{3}\frac{1+\alpha}{p} + \frac{n}{3q} < 1$, which is much more flexible than the classical condition $3 - 2\frac{1+\alpha}{p} > \frac{n}{q} + 2$, as can be seen in Figure 7.

In Section 5 we adapted the global well-posedness result from Wilke to be compatible with the local well-posedness result from the critical space setting. By instantaneous regularization, it is enough to consider unweighted L^pL^q-spaces. We were not able to recover global well-posedness for all the possible parameters p, q and α obtained from the local well-posedness. This his due to the fact that the 'starting point' (62) for the a priori energy estimate is an L^2-type estimate. For the spatial regularity, we are able to go down to $q = 3/2$ for $n = 3$, by utilizing the spatial regularity of u, μ and the data using Sobolev embeddings. As there is little time regularity available, we cannot hope to go below $p = 2$ using this L^2-type estimate. In future work, by trying to construct a different starting point for an L^p- or L^q-type of energy estimate, we may be able to recover more parameters p, q and α for the global well-posedness. Another aspect that can be improved is relaxing the extra requirement on h_2, by treating the $||(\partial_t u)h_2||_{L^1(\partial\Omega)}$-term using fractional derivatives.

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