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Feasibility test for Defined Contribution pension schemes

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HELEEN SCHUTTE

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"Feasibility test for Defined Contribution pension schemes"

HELEEN SCHUTTE

Delft University of Technology

Daily supervisors

Dr. P. Cirillo

Drs. A. van Ool

Other thesis comittee members

Dr. C. Kraaikamp

Dr. Ir. F. van der Meulen

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Delft, the Netherlands

Abstract

In the Netherlands Dutch pension funds perform the feasibility test. This test is originally designed for DB pension schemes. Nowadays DC pension schemes are getting a more prominent role in the Dutch pension system. Therefore an improved design of the feasibility test for DC pension schemes would be beneficial. In this research we search for a new definition for the pension result in DC pension schemes. The pension result is an important part of the feasibility test. We prefer to stay close to the concept of pension result in DB schemes and we conclude that a pension result based on indexed pension entitlements is an appropriate definition for DC pension schemes since it measures the maintenance of purchasing power similar to how it is done in the current feasibility test. We investigate how robust this new definition of the pension result is by considering an alternative premium policy, an alternative investment strategy, an alternative pension payment policy and an extension of the financial market model. All factors have an influence on the pension result. The pension result is calculated using economic scenarios which are based on the financial market model from Koijen, Nijman and Werker (KNW model). The pension result is highly sensitive to the interest rate and inflation rate development during the pension accrual period and the retirement period. Historical data give a motivation for the use of a jump diffusion model for both the interest rate and inflation rate. These variables are modeled as diffusion only in the KNW model. An extension of the KNW model is proposed in which jumps are added to the interest rate process and the inflation rate process. The addition of jumps influences the pension result in DC schemes.

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Contents

Chapter 1

Introduction

The Dutch government has started a discussion about renewing the pension system and is considering to revise the current use of pension schemes. The currently most common contracts are Defined Benefit contracts (DB), in these contracts the pension participant has the right to receive a predetermined pension benefit; the pension goal. The revision would be to assign a more prominent role to Defined Contribution schemes (DC). In DC schemes the social partners only make an agreement about the premium. They usually do not formulate a pension goal which means that the pension payments in the retirement period are uncertain. DC pension schemes are getting more popular since employers prefer to be exposed to less risk associated with pensions. Moreover, DC pension schemes allow for more customization and freedom of choice for pension participants, which in most cases does not hold for DB pension schemes.

With the introduction of the nFTK (nieuw Financieel Toetsingskader)[1] in 2015 Dutch pension funds are required to yearly perform the feasibility test. This feasibility test is an important instrument for pension funds, as it gives insight in the financial design of the pension fund, in the expectations about the pension payments and the associated risks.
The feasibility test cannot be applied to most DC pension schemes because, from a legal point of view, there are no 'pension rights' or pension entitlements (the right that one receives a certain amount pension capital during retirement) in a DC scheme during the accrual period that can serve as a norm in the same way as in a DB scheme. The existence of pension entitlements in a pension scheme are crucial for the application of the feasibility test. The feasibility test which pension funds use for DB schemes can be used for some DC schemes, but it can only be applied to DC schemes in which the accumulated capital will be converted into DB pension benefits at retirement. As a result, the feasibility test is only applied to the retirement period. Therefore it ignores the exposure to risk during the pension accrual period.

From the point of view of the pension participants however it is useful to know what risks one's pension capital gets exposed to when using a DC pension scheme. A design for the feasibility test for DC schemes is beneficial to give insight in the pension development for social partners, the board of the pension fund and for the supervisor (DNB). The feasibility test for DC schemes, which are converted into DB pension benefits at retirement, would get a more powerful meaning if we involve the accrual period of DC schemes as well. For other DC schemes we need to introduce a form of a feasibility test for both the accrual and retirement benefit period. The main research question we will address in this research for De Nederlandsche Bank is:

[1] new Financial Assessment Framework

How can the feasibility test for DC pension schemes be improved?

This question can be further refined using the following subquestions:

1. How should we define the pension result in DC pension schemes?

An important element in the current feasibility test is the calculation of the pension result. We can give content to our main research question by addressing whether we can give the feasibility test in DC schemes a similar set up with a meaningful definition of the pension result in it. From here this first subquestion rises; what is a proper definition for a pension result in DC schemes? In this research will mainly focus on the pension result and we analyze definitions for a pension result in variable annuity DC schemes[2]. The definitions of pension result will be tested and we draw conclusions about the quality and appropriateness of the definitions.

2. What is the relation between the risk attitude and the life-cycle investment strategy?

Pension providers must determine their risk attitude[3]. For DB schemes there exists a link between the risk attitude and the feasibility test given a certain investment strategy. For DC schemes there does not exist a link since the feasibility test is not required for and applicable to all DC schemes. We will investigate how we can make a link between the risk attitude for DC schemes and the feasibility test. After designing the feasibility test for DC schemes we analyze what values of the pension result based on a certain investment strategy we accept given the risk attitude of the pension funds.

3. Can we define the pension result in a general way such that it applies to all types of pension schemes?

In order to make the set up of the feasibility test easy and practical to use it would be beneficial to have one general form of the pension result in each type of pension scheme (i.e. DB and all types of DC schemes).

4. How robust is the pension result?

We will do several sensitivity analyses for the pension result. We will first analyze how sensitive the pension result is for several changes in the DC pension set up in the accrual period. We test the sensitivity for the way in which we set the premium payments and the investment strategy. Afterwards we look at the sensitivity for changes in the retirement period. We analyze how sensitive the test is to a fixed decrease in pension payments in a variable annuity. As a final analysis we look at both the accrual and the retirement period. We check if we see significant changes in the pension result when we make a different set of financial market scenario's; the scenario set. We change the way in which the financial market data is generated by extending

[2]Apart from variable annuities there exist DC pension schemes which are converted into DB benefits at retirement. Pension participants can also buy a fixed annuities at an insurance company at retirement. We do not include this forms of DC pension schemes in this research since the new definitions of pension result can be applied easily to these schemes as well.

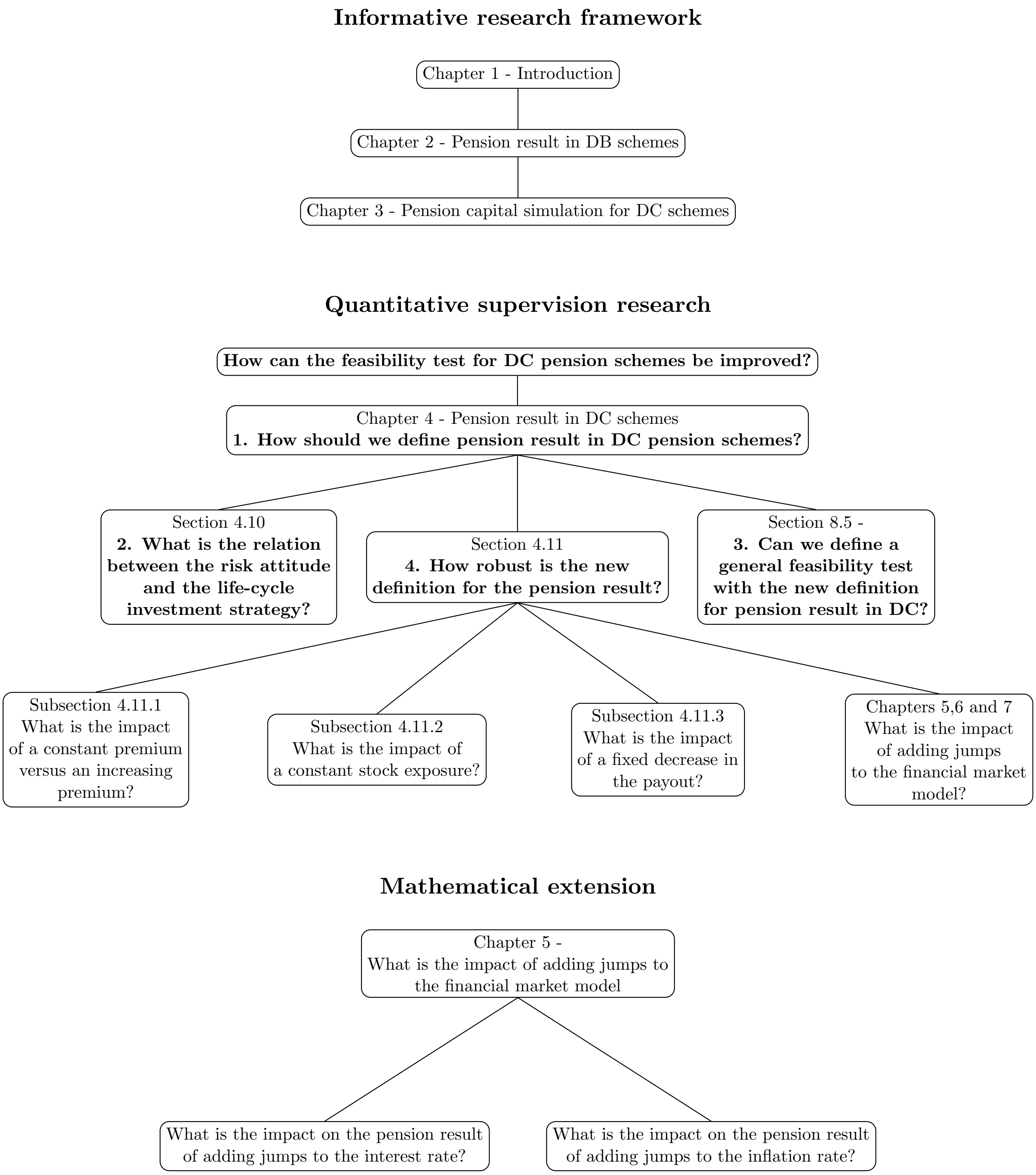
[3]The risk attitude is the extent to which the group of participants is willing to take investment risk to realize the goals of the pension fund and the extend to which they are able to take investment risk given the characteristics of the pension fund. The pension provider determines the risk attitude. The risk attitude is reflected in the maximum acceptable deviation of the expected pension in the bad weather scenario relative to the expected pension in the expected scenario. The definition of the risk attitude is stated as 'Risicohouding' in article 1a, paragraph 1, 2 and 3, 'Besluit financieel toetsingskader pensioenfondsen'.

the underlying model for the financial market parameters. With this adapted model and the new scenario set that it generates we analyze whether the pension result is highly sensitive to the financial market model. In this research we will make two changes in the model for the financial market parameters. We add a jump process in the interest rate and we add a jump process tot the inflation rate. We analyze the influence of the two extensions on the pension result.

Outline

The main focus of this research is formulating a new definition for the pension result in DC schemes. Figure 1.1 illustrates the outline of this thesis and highlights the links between the different chapters. Three different layers are distinguished. In the first layer we will introduce a more detailed description of DB pension schemes in chapter 2. In chapter 2 also gives a motivation for why the feasibility test is used and it gives a description of the current feasibility test and the pension result in DB schemes. Chapter 3 gives a more detailed description of DC pension schemes and it gives the mathematical background for the simulation of a DC pension capital. We introduce the KNW model, which is the financial market model that is used in the feasibility test. We introduce the investment strategy, other assumptions that are made and we describe the process of the premium payments and the pension payments.
The second layer involves the quantitative research for DNB. It starts in chapter 4 were we discuss new definitions for pension result which are applicable for DC schemes. We state the results of the definitions, elaborate on which definition suits best in the feasibility test for DC schemes and we analyze the best definition with several robustness checks.
The third layer is the mathematical extension of the research. The mathematical extension is part of the robustness checks. We start the mathematical extension with chapter 5 in which theory about jump processes is provided. In chapter 6 we change the financial market model by adding jumps to the interest rate and the inflation rate. In chapter 7 we will discuss the parameter estimation methods that are used for the jump process and we will discuss the results with respect to the pension result. In chapter 8 we conclude the research and finish the thesis with some points of discussion.

Figure 1.1: **Graphic representation of the thesis outline**



Chapter 2

Pension result in DB schemes

In this chapter we address the feasibility test and the pension result in DB schemes. First we explain how DB pension schemes work. Afterwards we will describe what the feasibility test is and we look at the functions of the 'feasibility test' to give the motivation for why pension funds use it and why the test is useful for the supervision of DNB. We then explain how the feasibility is used in DB pension schemes and how pension result is defined in DB schemes.

2.1 Defined Benefit pension scheme

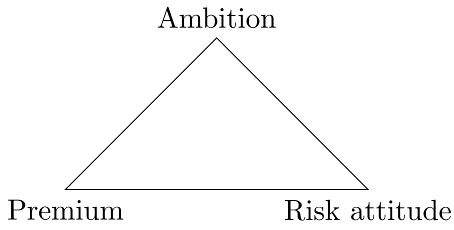
In a Defined Benefit (DB) pension scheme pension participants have a right on a pension benefit that is independent of the investment results of the pension fund. The pension benefit is determined by the number of years worked and the average salary of the beneficiary. In DB pension schemes the pension fund bears the investment risk, the longevity risk and the interest rate risk. Investment risk is the risk that the invested pension capital will be worth less than expected because of unfavorable developments in the financial markets. Longevity risk is the risk that pension participants get older than expected. We can distinguish between micro and macro longevity risk. Micro longevity risk results from non-systematic deviations from an individuals expected remaining lifetime, i.e. the risk that an individual gets older or dies earlier than expected. Macro-longevity risk results from the fact that survival probabilities change over time, i.e. the risk that the whole population gets older than expected [1]. Interest rate risk is the risk that changes in the interest rate will have an unfavorable effect on the value of the pension payments.

In order to pay out the promised pension benefits in the future pension funds have a few variables that they can change during different circumstances. These variables are the premium, the investment policy and the indexation policy. In most DB schemes the participants have a nominal claim which, if the financial position of the fund allows for it, will be adapted to the inflation. We call this indexation. Indexation can only be applied if the coverage ratio is high enough[1]. The coverage ratio is the ratio between the value of the assets and the value of the entitlements of the pension fund.

The indexation policy is part of the financial design of a DB pension scheme. The financial design can be explained by the pension triangle in Figure 2.1. The first vertex is the amount of premium the employers and the pension participants pay. The second vertex is the risk attitude of the pension fund and the third vertex of the pension triangle is the pension ambition. The

[1] Article 15, 'Besluit financieel toetsingskader pensioenfondsen'.

Figure 2.1: **Pension triangle**



ambition reflects the goals and the expectations that pension funds and pension participants have. The triangle represents that each aspect influences the other two. The key fact here is that a pension fund can never optimize all three vertexes. Improving with respect to one vertex means that the financial design will deteriorate with respect to the other vertexes. For instance, if the amount of premium is low and the participants are very risk averse, than the ambition cannot be high.

2.2 The feasibility test

The feasibility test is a tool used by pension funds to provide information about their performance to government institutions, pension participants and to themselves. This test is an important part of the monitoring of pension funds. DNB checks if the pension funds correctly apply the feasibility test. According to the Dutch pension law[2] it is mandatory for every pension fund to perform the feasibility test. In the first feasibility test (the commencement feasibility test) that a pension fund performs, the quantitative boundaries are set (see section 2.4). Afterwards the test should be performed on an annual basis with respect to the boundaries which are set in the commencement feasibility test. The feasibility test test whether the pension fund meets the set boundaries. The feasibility test should be performed in between the yearly tests in case of a significant adaption. We talk about a significant adaption if for instance the pension scheme is adjusted, the financial assessment framework is adjusted or if there has been an adjustment of the parameters for the underlying models. There will be a clear signal for the fund when, in bad economic conditions or in times of deficit, the fund can no longer meet the boundaries from the commencement feasibility test.

It is determined that each fund uses the same scenario set when performing the Feasibility test. This scenario set is based on the KNW [4] model by Kooijman, Nijman an Werker. The scenario set is delivered by DNB to the pension funds and consists of 2000 different scenarios for the development of the financial market over a time horizon of 60 years.

The feasibility test gives insight into the financial design of the pension fund. This design can be explained, as mentioned before, by the pension triangle in Figure 2.1. The feasibility test can be used to argue why the pension results and the possible policy choices are based on a balanced consideration of the interest of all the pension participants. For instance, in a downfall in the economy, pension funds should equally distribute the consequences of the downfall on all age groups. It also tests whether the goals of a pension fund can be realized. It works as a warning signal whenever pension funds do not meet their chosen boundaries. It also performs as a way of communication to inform the pension funds board and the individual participants about the financial position of the pensions.

[2] Article 22, 'Besluit financieel toetsingskader pensioenfondsen'.

2.3 Pension result

The main focus of the feasibility test is the calculation of the expected pension result. The expected pension result gives insight in how the expectation of the pension benefit, during its lifetime, quantifies itself in comparison to the maintenance of purchasing power. The feasibility test tests whether the expected pension result is in line with the set boundaries of the commencement feasibility test and the ambition of the fund. In the current feasibility test there is only a definition for the pension result that is applicable for all DB schemes. For DB schemes the feasibility test is performed during the period of accrual and the retirement period. For both periods the pension result is calculated. The pension result in DB schemes is by the Dutch law defined per scenario as a quotient in terms of a percentage. In the numerator we take the sum of all the benefits of all the retirement pensions and the survivor pensions in the future. In the denominator we take the sum of all the benefits of the retirement period and the survivor pension, both completely corrected for price inflation.

Definition 2.1. *Pension result in DB schemes*
Let Q_t^i be the nominal and let ${}^rQ_t^i$ be the real pension benefit in scenario i on time t according to the pension fund policies. The real benefit is equal to the nominal benefit corrected for inflation. Let ${}^r_dQ_t^i$ be the nominal and let ${}^r_dQ_t^i$ be the real benefit in scenario i in the case that indexation is always applied and benefits are never reduced. Let PD be the pension date; the first day of the retirement period. Let T the date of death and let $PD \leq T$. Then we can define the nominal and the real pension result at time s in a DB scheme as follows:

$$PR_t^i = 100\% \cdot \frac{\sum_{s=PD}^{T} Q_s^i}{\sum_{s=PD}^{T} {}^dQ_s^i} \mathbf{1}_{s \geq t} \tag{2.1}$$

$$^rPR_t^i = 100\% \cdot \frac{\sum_{s=PD}^{T} {}^rQ_s^i}{\sum_{s=PD}^{T} {}^r_dQ_s^i} \mathbf{1}_{s \geq t} \tag{2.2}$$

The denominator represents the ideal situation of the development of a pension since all pension benefits are completely compensated for inflation and face no benefit reduction. The pension result gives insight in the maintenance of purchasing power.

2.4 Risk attitude

Another function of the feasibility test is the quantification of the risk attitude[3] of the pension fund. The risk attitude is part of the agreement between social partners and the pension fund in order to execute the pension scheme. The pension fund and the social partners together should specify two quantitative criteria in the commencement feasibility test which will serve as boundaries for the following feasibility tests. These two quantitative criteria are a lower bound and a maximum deviation. The lower bound is represented by the 5th percentile of the pension result in the commencement feasibility test. The maximum deviation is defined as the difference between the median of the pension result and the 5th percentile of the pension result. Both boundaries, the lower bound and the maximum deviation, are set in the commencement feasibility test. Pension funds with for instance a large risk appetite choose a larger maximum deviation. The risk attitude is expressed as the combination of the lower bound and the maximum deviation. Based on the results of the annual feasibility tests conclusions can be drawn whether the pension fund still meets their expectations. If not the pension fund can decide to

[3] Article 1a, 'Besluit financieel toetsingskader pensioenfondsen'.

either change the premium, the investment strategy, cut the pension benefits [4] or the fund can decide to change the lower bounds and with that the expectations. The risk attitude forms the starting point for the board of the pension fund to discuss with other pension fund bodies how to set the investment strategy.

[4] Pension funds can only cut pension benefits as a last resort, whenever the financial position cannot be improved in any other way. Cuts in the pension benefits are applied based on the funding ratio.

Chapter 3

Pension capital simulation for DC pension schemes

In this chapter we will describe how we will set up the model for the simulation of DC pension capital and the computation of the associated pension results. We will start with the description of DC pension schemes, then we give an introduction of the model we use to generate the financial market scenarios. Afterwards we will explain how we chose the investment strategy. We will first shortly explain why in general pension capital is invested according to the life cycle principle and we will derive the life cycle which we will use in the simulation. In the final section we will give a detailed description of the remaining model assumptions.

3.1 Defined Contribution pension scheme

In a Defined Contribution (DC) scheme the pension participants bear most of the pension risks such as investment risk and macro longevity risk. The premium is fixed i.e. the premium does not change based on the financial situation of the pension fund, and the premium differs per age. Indexation does not exist in DC schemes because the participant does not have a pension right. An important advantage of DC pension schemes is that the investment strategy can be tailored to the individual preferences of the pension participant. Tailoring the investment strategy can for example be done by using a life cycle investment strategy. A life cycle investment strategy is an investment strategy in which the amount of investment risk depends on the age of the participant. The life cycle determines how we invest the accrued pension in risky assets. Studies have shown [11] that it is beneficial to take more investment risk at young ages than at older ages and that it is beneficial to let the investment risk decrease while the participants ages.

There are several types of DC schemes. In this thesis we will focus on the variable annuity DC. In a variable annuity the pension capital is not converted into pension benefits. The pension capital will be partly payed out and partly invested from the pension date on, which results in a variable payout during the retirement period due to variable investment returns and the survival probabilities. Restriction here is that the payouts must be a life long payout guarantee. This differs form the DB scheme since there the payouts are benefits and fixed from the pension date on.

3.2 Financial market model: the KNW model

In order to calculate the pension result we need to make a prediction for the development of pension capital in all sorts of representative scenarios of the financial market. For the feasibility test it is legally stated that pension funds must use the same scenario set generated and provided to the funds by DNB [6]. This scenario set consists of 2000 scenarios for 60 years. To generate these scenarios DNB makes use of the so-called KNW [4] model, since the Parameters committee [5] advises to use this model.

The KNW model is an affine factor model for the term structure were we assume a complete market. An affine factor model is a type of financial market model that relates zero-coupon bond prices to a spot rate model. The financial market model contains relations between key financial risk factors of pension funds. The simulated portfolio consists of a stock index S_t, long-term nominal and real [1] bonds which are respectively denoted as $P_t(N)$ and ${}^rP_t(N)$. The simulated portfolio also has a nominal money account. We state the equations that are used in the KNW model below (and with that the relations between the risk factors). All variables that are vectors or matrices will denoted in bold in this chapter.

- **Unobserved states**
  The number of states in this model is 2. The two state variables $\boldsymbol{X_t}$ have an influence on the real interest rate, the inflation rate, the prices of risk and the bond price. The time series for interest rate and inflation rate have a high first order auto correlation, so small changes now can have significant impact on the capital in the future i.e. for any t $\boldsymbol{X_t}$ is highly dependent on $\boldsymbol{X_{t-1}}$. So for the interest rate and the inflation rate the values in the near past are of great importance. The stochastic differential equation of the state variables is defined as:

  $$d\boldsymbol{X_t} = -\boldsymbol{K}\boldsymbol{X_t}dt + \boldsymbol{\Sigma_X}'\boldsymbol{dZ_t}.$$

  With $\boldsymbol{X_t} \in \mathbb{R}^2$, $\boldsymbol{K} \in \mathbb{R}^{2\times 2}$ is a lower triangular matrix and

  $$\boldsymbol{\Sigma_X}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

  $\boldsymbol{Z_t} \in \mathbb{R}^4$ is a vector of independent Brownian motions driving the uncertainty in the financial market. Four sources of uncertainty can be identified; uncertainty about the real interest rate, uncertainty about the instantaneous expected inflation, uncertainty about the unexpected inflation and uncertainty about the stock return.

- **Instantaneous real interest rate**
  The uncertainty and dynamics in the instantaneous real interest rate r_t are modeled using the two state variables. The interest rate is assumed to be affine in all factors.

  $$r_t = \delta_{0r} + \boldsymbol{\delta}'_{\boldsymbol{1r}}\boldsymbol{X_t}.$$

  With $\delta_{0r} \in \mathbb{R}$ and $\boldsymbol{\delta_{1r}} \in \mathbb{R}^{1\times 2}$.

- **Instantaneous expected inflation**
  The uncertainty and dynamics in the instantaneous expected inflation π_t are also modeled using the two state variables. The inflation is assumed to be affine in all factors.

  $$\pi_t = \delta_{0\pi} + \boldsymbol{\delta}'_{\boldsymbol{1\pi}}\boldsymbol{X_t}.$$

[1] With real we mean that the referred asset is compensated for inflation, so we translated the expected future value in terms of the value of the used currency today.

With $\delta_{0\pi} \in \mathbb{R}$ and $\boldsymbol{\delta_{1\pi}} \in \mathbb{R}^{1\times 2}$. Any correlation between the real interest and inflation rate is modeled using $\boldsymbol{\delta'_{1r}}$ and $\boldsymbol{\delta'_{1\pi}}$.

- **Price index**
For the relative return of the price index process (cumulative inflation) Π_t we assume a Geometric Brownian motion.

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \boldsymbol{\sigma_{\Pi} dZ_t}.$$

With $\boldsymbol{\sigma_{\Pi}} \in \mathbb{R}^4$ and $\Pi_0 = 1$. $\sigma_{\Pi(4)} = 0$ such that the price index is independent of the random shocks from the stock return.

- **Stock index**
For the stock index S_t we assume a Geometric Brownian motion. The relative stock returns are modeled as follows:

$$\frac{dS_t}{S_t} = ({}^0R_t + \eta_S)dt + \boldsymbol{\sigma'_S dZ_t}.$$

With $\boldsymbol{\sigma_S} \in \mathbb{R}^4$, $S_0 = 1$ and η_s the equity risk premium. We will derive the formula for ${}^0R_t$ in subsection 3.2.2.

- **Bond returns**
For the relative bond returns we assume the following SDE:

$$\frac{dP_t^B(N)}{P_t^B(N)} = ({}^0R_t + \boldsymbol{B(N)'\Lambda_t\Sigma'_X})dt + \boldsymbol{B(N)'\Sigma'_X dZ_t}.$$

The formula for $\boldsymbol{B(N)}$ will be derived in section 3.2.1 where we will derive the bond price of the KNW model.

- **Prices of risk**
The time-varying price of risk $\boldsymbol{\Lambda_t}$ is affine in state variables $\boldsymbol{X_t}$.

$$\boldsymbol{\Lambda_t} = \boldsymbol{\Lambda_0} + \boldsymbol{\Lambda_1 X_t}.$$

With $\boldsymbol{\Lambda_t}, \boldsymbol{\Lambda_0} \in \mathbb{R}^4$ and $\boldsymbol{\Lambda_1} \in \mathbb{R}^{4\times 2}$. The prices of risk will depend on the risk aversion of the investor. It is imposed here that the price of unexpected inflation risk, which cannot be identified using nominal bond data, equals zero i.e. the row in $\boldsymbol{\Lambda_1}$ representing the price of unexpected inflation will contain zeros. This assumption is imposed since inflation-linked bonds have been launched in the US only as of 1997, the data available is insufficient to estimate this price of risk accurately [4].

$$\boldsymbol{\Lambda_1} = \begin{bmatrix} \Lambda_{1(1,1)} & \Lambda_{1(1,2)} \\ \Lambda_{1(2,1)} & \Lambda_{1(2,2)} \\ 0 & 0 \\ \Lambda_{1(4,1)} & \Lambda_{1(4,2)} \end{bmatrix}$$

- **Nominal stochastic discount factor**
For the nominal stochastic discount factor $\frac{d\phi_t}{\phi_t}$ we assume a Geometric Brownian motion. The stochastic discount factor gives the marginal utility ratio between consumption today and in the future. So it displays how much we value to consume capital today or save capital for tomorrow. The marginal utility ratio is for everyone the same in case of complete markets (see theorem 3.2) .

$$\frac{d\phi_t}{\phi_t} = -{}^0R_t dt - \boldsymbol{\Lambda'_t dZ_t}.$$

3.2.1 The bond price in the KNW model

Bonds are fixed-income products. The theoretical fair value of a bond is the present value of the stream of cash flows it is expected to generate. Hence, the value of a bond is obtained by discounting the bond's expected cash flows to the present using an appropriate discount rate. If we assume that a bond terminates at maturity N, pays yearly discrete coupons or dividends D and has the face value M at maturity then the equation for the price of the bond at time t P_t is as follows:

$$\begin{aligned} P_t &= \frac{D}{1+R_{t+1}} + \frac{D}{(1+R_{t+2})^2} + ... + \frac{D}{(1+R_{t+N-1})^{N-1}} + \frac{M}{(1+R_{t+N})^N} \\ &= \sum_{n=1}^{N-1} \frac{D}{(1+R_{t+n})^n} + \frac{M}{(1+R_{t+N})^N} \end{aligned}$$

The simplest fixed-income instrument is a zero-coupon bond. A zero-coupon bond is an agreement to pay one dollar (a nominal bond) or one unit of the consumption good (a real bond) on a specified date (maturity). The pricing of a zero coupon bond with discrete compounding and maturity N is therefore:

$$P_t = \frac{1}{(1+R_{t+N})^N}$$

If we have a continuous discount factor we must divide the time intervals in n intervals with $\lim n \to \infty$. For the discount factor of the bond we then observe:

$$\lim_{n\to\infty} \left(1 + \frac{R}{n}\right)^{nt}$$

Now we make the change of variables $n = mR$ and we observe

$$\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{mRt} = \left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{m}\right)^{Rt} = e^{Rt}$$

So when we assume continuous compounding we compute the zero coupon bond price without maturity by:

$$P_t = e^{-Rt}$$

To get to the bond price of the KNW model we use the fact that the KNW model is an affine term structure model. In an affine term structure model bond prices are affine.

Definition 3.1. *Affine term structure models*
Affine term structure models are a special class of interest rate models in which it is assumed that the interest rate is affine and the term structure is affine. The interest rate is of the form:

$$dR_t = (\alpha_t - \beta_t R_t)dt + \sqrt{\delta_t + \gamma_t R_t} dZ_t$$

Where Z_t is a standard Brownian motion. In the KNW model it holds that:

$$dR_t = \boldsymbol{R_1' dX_t} = -\boldsymbol{R_1' K X_t} dt + \boldsymbol{R_1' \Sigma_X' dZ_t}.$$

Therefore the interest rate is affine in the state variables. The term structure i.e. the bond prices are of the following form:

$$P(t,T) = e^{A(N) + \boldsymbol{B(N)' X_t}}$$

where $A(N)$ and $\boldsymbol{B(N)}$ are deterministic functions.

In order to solve the bond price of the KNW model we use the basic pricing equation [16] (see theorem 10.2). To get to this equation we go back to non zero coupon bonds which pay dividends D. We now assume that the dividend payout is continuous. Let a generic bond security have price P_t and dividends D_t at each time t. The instantaneous total return at each time t is then the change in the bond price P_t plus the payed out dividend D_t times the change in time:

$$\frac{dP_t}{P_t} + \frac{D_t}{P_t}dt$$

Since we model risky assets as diffusion, our relative change in the bond price will be modeled according to the Bacheliers type:

$$\frac{dP_t}{P_t} = \mu dt + \sigma dZ_t$$

Where Z_t is a standard Brownian motion.

Theorem 3.2. *The basic pricing equation is:*

$$E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\right] - R_t dt = -E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\frac{d\phi_t}{\phi_t}\right]$$

Proof. Let ϕ_t be the stochastic discount factor at time t. If we buy a security today the payoff next period will be the security price plus the dividend. So we know that the price of the bond security will be accordingly:

$$P_t\phi_t = E_t^{\mathbb{Q}}\left[\int_0^\infty D_{t+s}\phi_{t+s}ds\right]$$

So the price at $P_{t+\Delta t}$ equals:

$$P_{t+\Delta t}\phi_{t+\Delta t} = E_{t+\Delta t}^{\mathbb{Q}}\left[\int_0^\infty D_{t+\Delta t+s}\phi_{t+\Delta t+s}ds\right]$$

Taking the difference of both we find:

$$\begin{aligned}
P_t\phi_t - P_{t+\Delta t}\phi_{t+\Delta t} &= E_t^{\mathbb{Q}}\left[\int_0^\infty D_{t+s}\phi_{t+s}ds\right] - E_{t+\Delta t}^{\mathbb{Q}}\left[\int_0^\infty D_{t+\Delta t+s}\phi_{t+\Delta t+s}ds\right] \Rightarrow \\
P_t\phi_t - P_{t+\Delta t}\phi_{t+\Delta t} &= E_t^{\mathbb{Q}}\left[\int_0^{\Delta_t} D_{t+s}\phi_{t+s}ds\right] + E_t^{\mathbb{Q}}\left[\int_{\Delta_t}^\infty D_{t+s}\phi_{t+s}ds\right] \\
&\quad - E_{t+\Delta t}^{\mathbb{Q}}\left[\int_0^\infty D_{t+\Delta t+s}\phi_{t+\Delta t+s}ds\right] \Rightarrow \\
P_t\phi_t - P_{t+\Delta t}\phi_{t+\Delta t} &= E_t^{\mathbb{Q}}\left[\int_0^{\Delta t} D_{t+s}\phi_{t+s}ds\right] + E_t^{\mathbb{Q}}\left[\int_0^\infty D_{t+\Delta t+s}\phi_{t+\Delta t+s}ds\right] \qquad (3.1) \\
&\quad - E_{t+\Delta t}^{\mathbb{Q}}\left[\int_0^\infty D_{t+\Delta_t+s}\phi_{t+\Delta_t+s}ds\right] \Rightarrow \\
P_t\phi_t &= E_t^{\mathbb{Q}}\left[\int_0^{\Delta t} D_{t+s}\phi_{t+s}ds\right] + E_t^{\mathbb{Q}}\left[P_{t+\Delta t}\phi_{t+\Delta t}\right] \Rightarrow \\
P_t\phi_t &\approx D_t\phi_t \cdot \Delta t + E_t^{\mathbb{Q}}\left[P_{t+\Delta t}\phi_{t+\Delta t}\right] \Rightarrow \\
P_t\phi_t &\approx D_t\phi_t \cdot \Delta t + E_t^{\mathbb{Q}}\left[P_t\phi_t - P_t\phi_t + P_{t+\Delta t}\phi_{t+\Delta t}\right] \Rightarrow \\
0 &\approx D_t\phi_t\Delta t + E_t^{\mathbb{Q}}\left[P_{t+\Delta t}\phi_{t+\Delta t} - P_t\phi_t\right] \text{ with } \Delta \to 0 \Rightarrow \\
0 &= D_t\phi_t dt + E_t^{\mathbb{Q}}\left[d(P_t\phi_t)\right] \qquad (3.2)
\end{aligned}$$

We rewrite $d(P_t\phi_t)$ using Ito's Lemma:

$$d(P_t\phi_t) \quad = \quad P_t d\phi_t + \phi_t dP_t + dP_t d\phi_t$$

then (3.2) becomes:

$$\begin{aligned} 0 &= D_t\phi dt + E_t^{\mathbb{Q}}\left[P_t d\phi_t + \phi_t dP_t + dP_t d\phi_t\right] \Rightarrow \\ 0 &= \frac{D_t}{P_t}dt + E_t^{\mathbb{Q}}\left[\frac{d\phi_t}{\phi_t} + \frac{dP_t}{P_t} + \frac{d\phi_t}{\phi_t}\frac{dP_t}{P_t}\right] \end{aligned} \tag{3.3}$$

Now we can interpreted 1 euro cash as a security that has a constant price equal to 1 and pays the interest rate as a dividend.

$$\frac{dP_t}{P_t} = R_t$$

Applying (3.3) to cash gives us:

$$0 \quad = \quad R_t dt + E_t^{\mathbb{Q}}\left[\frac{d\phi_t}{\phi_t}\right]$$

We can rewrite (3.3) as:

$$E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\right] + \frac{D_t}{P_t}dt = R_t dt - E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\frac{d\phi_t}{\phi_t}\right] \tag{3.4}$$

□

The holding period return of a bond with maturity N can be expressed as:

$$\begin{aligned} hpr &= \frac{P_{t+\Delta t}(N-\Delta) - P_t(N)}{P_t(N)} \\ &= \frac{P_{t+\Delta t}(N-\Delta) - P_{t+\Delta t}(N) + P_{t+\Delta t}(N) - P_t(N)}{P_t(N)}. \end{aligned}$$

Taking the limit we find:

$$hpr = \frac{dP_t(N)}{P_t} - \frac{1}{P_t}\frac{\partial P_t(N)}{\partial N}dt.$$

The fundamental pricing equation applied to the hpr, given maturity N, becomes:

$$E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\right] - \left(\frac{1}{P_t}\frac{\partial P(N,t)}{\partial N} + R_t\right)dt \quad = \quad -E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\frac{d\phi_t}{\phi_t}\right] \tag{3.5}$$

we assume that all the time dependence of the bond price comes through the state variables $\boldsymbol{X_t}$ of KNW. We use Ito's lemma to rewrite the terms $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\right]$ and $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\frac{d\phi_t}{\phi}\right]$ from the fundamental pricing equation (3.5). For expectation of the relative change in the bond price $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\right]$ we find according to Ito's lemma:

$$\begin{aligned} E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\right] &= E_t^{\mathbb{Q}}\left[\frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{dX_t} + \frac{1}{P}\frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}}(\boldsymbol{dX_t})^2\right] \\ E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\right] &= E_t^{\mathbb{Q}}\left[\frac{1}{P}\left(\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}}\boldsymbol{\sigma_X^2}\right)dt + \frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma_X dZ_t}\right] \\ &= \frac{1}{P}\left(\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}}\boldsymbol{\sigma_X^2}\right)dt. \end{aligned}$$

With $\boldsymbol{\mu_X} = (-\boldsymbol{KX_t})$ and $\boldsymbol{\sigma^2_X} = \boldsymbol{\Sigma'_X \Sigma_X}$.

For expectation of the relative change in the bond price times the relative change in the stochastic discount factor $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\frac{d\phi_t}{\phi}\right]$ we find:

$$\begin{aligned} E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\frac{d\phi_t}{\phi}\right] &= E_t^{\mathbb{Q}}\left[\frac{1}{P}\left(\left(\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}}\boldsymbol{\sigma^2_X}\right)dt + \frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma_X dZ_t}\right)(-R_t dt - (\boldsymbol{\Lambda_t})\boldsymbol{dZ_t})\right] \\ &= \frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma'_X \Lambda_t} dt. \end{aligned}$$

When we plug the new expressions for $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\right]$ and $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\frac{d\phi_t}{\phi}\right]$ into (3.5), we get a another expression for the basic pricing equation:

$$\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}}\boldsymbol{\sigma^2_X} - \frac{\partial P}{\partial N} - R_t P_t = \frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma'_X \Lambda_t}. \tag{3.6}$$

Recall that the log price of the bonds are linear in the affine KNW model:

$$P(N, X_t) = e^{A(N) + \boldsymbol{B(N) X_t}} \tag{3.7}$$

with the boundary condition: $P(0, \boldsymbol{X_t}) = 1$, so $A(0) - \boldsymbol{B(0)' X_t} = 0 \Rightarrow A(0) = 0$ and $\boldsymbol{B(0)} = \boldsymbol{0}$. We denote the derivative of $A(N)$ and $\boldsymbol{B(N)}$ with respect to the maturity N with a dot notation:

$$\begin{aligned} \dot{A}(N) &= \frac{\partial A(N)}{\partial N} \\ \boldsymbol{\dot{B}(N)} &= \frac{\boldsymbol{\partial B(N)}}{\partial N} \end{aligned}$$

with this we can deduce that:

$$\frac{1}{P_t}\frac{\partial P_t}{\boldsymbol{\partial X_t}} = \boldsymbol{B(N)} \tag{3.8}$$

$$\frac{1}{P_t}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}} = \boldsymbol{B(N)B(N)'} \tag{3.9}$$

$$\frac{1}{P_t}\frac{\partial P_t}{\partial N} = -\frac{1}{P_t}\frac{\partial P_t}{\partial N} = -\dot{A}(N) - \boldsymbol{\dot{B}(N)' X_t}. \tag{3.10}$$

Substituting this in (3.8), (3.9) and (3.10) into (3.6) we find together with the formulas for the economy parameters from KNW that:

$$\begin{aligned} &\boldsymbol{B(N)'}(-\boldsymbol{KX_t}) + \frac{1}{2}(\boldsymbol{\Sigma_X B(N) B(N)' \Sigma'_X}) - \dot{A}(N) - \boldsymbol{\dot{B}(N)' X_t} - (R_0 + \boldsymbol{R'_1 X_t}) = \\ &\boldsymbol{B(N)' \Sigma'_X}(\boldsymbol{\Lambda_1 X_t} + \boldsymbol{\Lambda_0}). \end{aligned}$$

Now we set all the terms with X_t terms equal:

$$\begin{aligned} &\boldsymbol{B(N)'}(-\boldsymbol{KX_t}) - \boldsymbol{\dot{B}(N)' X_t} - \boldsymbol{R'_1 X_t} = \boldsymbol{B(N)' \Sigma'_X \Lambda_1 X_t} \Rightarrow \\ &\boldsymbol{\dot{B}(N)} = -R_1 - (\boldsymbol{K} + \boldsymbol{\Lambda'_1 \Sigma_X})\boldsymbol{B(N)} \Rightarrow \\ &\boldsymbol{B(N)} = (\boldsymbol{K} + \boldsymbol{\Lambda'_1 \Sigma'_X})^{-1}\left[e^{-(\boldsymbol{K} + \boldsymbol{\Lambda'_1 \Sigma'_X})N} - \boldsymbol{I}_{2\times 2}\right]\boldsymbol{R_1} \\ &\boldsymbol{B(N)} = \boldsymbol{M}^{-1}\left[e^{-\boldsymbol{M}N} - \boldsymbol{I}_{2\times 2}\right]\boldsymbol{R_1} \end{aligned} \tag{3.11}$$

with $M = (\boldsymbol{K} + \boldsymbol{\Sigma_X'}\boldsymbol{\Lambda_1})$. We also set all the non X_t terms equal:

$$
\begin{aligned}
&- \dot{A}(N) + \frac{1}{2}\boldsymbol{B(N)'\Sigma_X'\Sigma_X B(N)} - R_0 = (\boldsymbol{\Lambda_0'\Sigma_X})\boldsymbol{B(N)} \Rightarrow \\
&\dot{A}(N) = -R_0 + \frac{1}{2}\boldsymbol{B(N)'\Sigma_X'\Sigma_X B(N)} - (\boldsymbol{\Lambda_0\Sigma_X'})\boldsymbol{B(N)} \Rightarrow \\
&A(N) = \int_0^N \dot{A}(s)ds
\end{aligned}
\tag{3.12}
$$

so the bond price in the KNW model can be written as:

$$P_t(N) = e^{A(N)+\boldsymbol{B(N)'X_t}} \tag{3.13}$$

where $A(N)$ and $\boldsymbol{B(N)}$ are respectively as equation (3.11) and (3.12).

3.2.2 Implications of nominal and inflation linked bonds

The stochastic discount factor can be used to determine the value of all discounted assets because the described markets are complete.

Definition 3.3. *Complete markets*
We say that a market is complete if:

- *there are no transaction costs and there is perfect information*
- *there exists a price for every asset in all possible states of the world*

In the theoretical representation of the KNW model the first theorem of fundamental asset pricing holds since the KNW market can be characterized by a risk-neutral measure [8].

Theorem 3.4. *First fundamental theorem of asset pricing*
If a market is characterized by at least one risk-neutral measure $\mathbb{Q}$ that is equivalent to the original probability measure $\mathbb{P}$ (i.e. they agree on which sets in filtrations $\mathscr{F}^n$ have probability zero), then it does not allow arbitrage [9]

This has a few implications. Since the discounted bond price is a martingale in a complete market under the risk neutral measure there does not exist arbitrage in the bond price [9], so the expected change in the discounted value of the price of a nominal bond does not change over time. The fundamental pricing equation for a nominal zero coupon bond is thus:

$$E\left[d\phi P\right] = 0.$$

From this it follows that also the discounted value of the real bond price corrected for inflation does not change over time, so for inflation linked bonds the following holds:

$$E\left[d\phi^r P\Pi\right] = 0$$

where ${}^r\phi = \phi\Pi$ is the real stochastic discount factor. Using the Ito Doeblin theorem we derive for the real stochastic discount factor:

$$
\begin{aligned}
\frac{d^r\phi}{{}^r\phi} &:= \frac{d(\phi\Pi)}{\phi\Pi} \\
&= \frac{d\phi}{\phi} + \frac{d\Pi}{\Pi} + \frac{d\phi}{\phi}\frac{d\Pi}{\Pi} \\
&= -(0_t^R - \pi_t + \boldsymbol{\sigma_\Pi'\Lambda_t})dt - (\boldsymbol{\Lambda_t'} - \boldsymbol{\sigma_\Pi})\boldsymbol{dZ_t} \\
&= -r_t - (\boldsymbol{\Lambda_t'} - \boldsymbol{\sigma_\Pi'})\boldsymbol{dZ_t}
\end{aligned}
$$

Using Ito's isometry we find that the nominal rate can thus be written as:

$$\begin{aligned} R_t &= r_t + \pi_t - \boldsymbol{\sigma'_\Pi \Lambda_t} \\ &= (\delta_{0r} + \delta_{0\phi} - \boldsymbol{\sigma'_\Pi \Lambda_0}) + (\delta'_{1r} + \boldsymbol{\delta_{1\phi}} - \boldsymbol{\sigma'_\Pi \Lambda_1})\boldsymbol{X_t} \\ &:= R_0 + \boldsymbol{R'_1 X_t} \end{aligned}$$

3.2.3 Implications for the equity risk premium

The fundamental valuation equation of the equity index implies that the expected value of the discounted stock price does not change over time [8]:

$$E\left[d\phi S\right] = 0.$$

This equation implies a restriction. Using the Ito-Doeblin theorem gives:

$$\begin{aligned} \frac{d\phi S}{\phi S} &= \frac{d\phi}{\phi} + \frac{dS}{S} + \frac{d\phi}{\phi} \cdot \frac{dS}{S} \\ &= -R_t dt - \boldsymbol{\Lambda'_t dZ_t} + (R_t + \eta_S)dt + \boldsymbol{\sigma'_s dZ_t} \\ &+ \left(-R_t dt - \boldsymbol{\Lambda'_t dZ_t}\right)\left((R_t + \eta_S)dt + \boldsymbol{\sigma'_s dZ_t}\right) \\ &= \left(\eta_S - \boldsymbol{\Lambda'_t \sigma_S}\right)dt - \left(\boldsymbol{\Lambda'_t} - \boldsymbol{\sigma'_S}\right)d\boldsymbol{Z_t}. \end{aligned} \tag{3.14}$$

Again using Ito's isometry we get:

$$\begin{aligned} E\left[\frac{d\phi S}{\phi S}\right] &= E\left[\left(\eta_S - \boldsymbol{\Lambda'_t \sigma_S}\right)dt - \left(\boldsymbol{\Lambda'_t} - \boldsymbol{\sigma'_S}\right)d\boldsymbol{Z_t}\right] \\ &= E\left[\left(\eta_S - \boldsymbol{\Lambda'_t \sigma_S}\right)dt\right] \\ &= \left(\eta_S - \boldsymbol{\Lambda'_t \sigma_S}\right)dt. \end{aligned}$$

Which implies $\boldsymbol{\sigma'_S \Lambda_0} = \eta_S$ and $\boldsymbol{\sigma'_S \Lambda_1} = 0$. This restrictions are imposed on the model.

3.2.4 Term structure

The formula for the interest term structure is given by coefficients $A(N)$ and $\boldsymbol{B(N)}$ respectively defined in equations (3.12) and (3.11). Where $\boldsymbol{A} \in \mathbb{R}^{1x75}$, $\boldsymbol{B} \in \mathbb{R}^{2x75}$ and maturity $N \leq 75$. The interest term structure with maturity N in scenario i equals:

$${}^{N}R_t^i = \exp\bigl(A(N) + B(1,N) \cdot X_t^i(1) + B(2,N) \cdot X_t^i(2)\bigr) - 1.$$

3.3 Investment strategy

The investment strategy determines how the pension capital is invested. This strategy generally changes over time as the participant ages. In general individuals face two main decisions in their financial planning over their life cycle, namely the saving decision and the investment decision [11]. Through the saving decision, individuals decide how to smooth consumption over time by setting the pension premium and the pension benefits. The next question then is how to optimally allocate savings between stocks and bonds; the investment decision. Here we will focus on the latter. The savings decision will not be optimized since we assume a set premium. The investment decision for pension capital is influenced by the risk aversion of pension participants.

3.3.1 CAPM model

We follow Campbell and Viceira [3] and use the CAPM model to determine the investment strategy. We consider an investment portfolio with holdings in a risky asset R_t and a risk free asset R_t^f. We assume that we invest a rate equal to ω_t in risky assets and $(1-\omega_t)$ in risk free-assets. The return of the portfolio will be as follow:

$$\begin{aligned} R_t^p &= \omega_t R_t + (1-\omega_t)R_t^f \\ &= R_t^f + \omega_t(R_t - R_t^f). \end{aligned} \tag{3.15}$$

Investors will prefer a high mean and a low variance of the portfolio returns. The mean of the portfolio is:

$$E_t[R_t^p] = R_t^f + \omega_t(E[R_t] - R_t^f).$$

The variance of the portfolio is:

$$(\sigma_t^p)^2 = \omega_t^2\sigma^2.$$

We assume that the investor trades off mean and variance in a linear fashion, so we will maximize a linear combination of mean and variance with a positive weight on the mean and negative weight on the variance. We get the following maximization problem:

$$max_{\{\omega_t\}}\left(R_t^f + \omega_t(E[R_t] - R_t^f) - \frac{1}{2}(1-\gamma)\omega_t^2\sigma^2\right).$$

We can subtract R_t^f without changing the maximization. We get:

$$max_{\{\omega_t\}}\left(\omega_t(E[R_t] - R_t^f) - \frac{1}{2}(1-\gamma)\omega_t^2\sigma^2\right).$$

We can solve this maximization problem when we relate the log portfolio returns to the log returns of the individual assets using Taylor series. The portfolio return of (3.15) can be written as:

$$\begin{aligned} \frac{1+R_t^p}{1+R_t^f} &= \frac{1+R_t^f + \omega_t(R_t + R_t^f)}{1+R_t^f} \\ &= 1 + \omega_t\frac{(1+R_t+R_t^f-1)}{1+R_t^f} \\ &= 1+\omega_t\left(\frac{1+R_t}{1+R_t^f}\right). \end{aligned} \tag{3.16}$$

We set:

$$\begin{aligned} \log(R_t^p) &= r_t^p \\ \log\left(R_t^f\right) &= r_t^f. \end{aligned}$$

Now taking the *log* of (3.16) we get:

$$\begin{aligned}\log\left(\frac{1+R_t}{1+R_t^f}\right) &= \log\left(1+\omega_t\left(\frac{1+R_t}{1+R_t^f}\right)\right) \Rightarrow \\ r_t^p - r_t^f &= \log\Big(1+\omega_t\left(e^{r_t - r_t^f}-1\right)\Big).\end{aligned}$$

This equation gives a non-linear relation between the log excess return on the risky asset $(r_t - r_t^f)$ and the log excess return on the portfolio $(r_t^p - r_t^f)$. This relation can be approximated using a second order Taylor expansion around the point $(r_t - r_t^f) = 0$. The function $f_t(r_t - r_t^f) = log\Big(1+\omega_t\left(e^{r_t - r_t^f}-1\right)\Big)$ is approximated as:

$$\begin{aligned}f_t(r_t - r_t^f) &\approx f_t(0) + f_t'(0)(r_t - r_t^f) + \frac{1}{2}f''(0)(r_t - r_t^f)^2 \\ &\text{with} \\ f_t(0) &= 0 \\ f_t'(0) &= \omega_t \\ f_t''(0) &= \omega_t(1-\omega_t).\end{aligned}$$

We replace $(r_t - r_t^f)^2$ by it's conditional expectation: σ^2. So for the Taylor expansion we get:

$$r_t^p - r_t^f \approx \omega_t(r_t - r_t^f) + \frac{1}{2}\omega_t(1-\omega_t)\sigma^2.$$

Now we substitute this into the maximization problem:

$$max_{\{\omega_t\}}\left(\omega_t(E[R_t] - R_t^f) + \frac{1}{2}\omega_t(1-\omega_t)\sigma^2 - \frac{1}{2}(1-\gamma)\omega_t^2\sigma^2\right)$$

and we solve for a_t:

$$\begin{aligned}E[R_t] - R_t^f + \frac{1}{2}\sigma_t^2 - \omega_t\sigma^2 + \omega_t(1-\gamma)\sigma^2 &= 0 \\ \omega_t((1-\gamma)\sigma^2 - \sigma^2) &= -E[R_t] + R_t^f - \frac{1}{2}\sigma^2 \\ \omega_t &= \frac{-E[R_t] + R_t^f - \frac{1}{2}\sigma^2}{((1-\gamma)\sigma^2 - \sigma_t^2)} \\ \omega_t &= \frac{E[R_t] - R_t^f + \frac{1}{2}\sigma^2}{\gamma\sigma^2} \\ \omega_t &= \frac{\mu + \frac{1}{2}\sigma^2}{\gamma\sigma^2}.\end{aligned}$$

With $\mu, \sigma, \gamma \in \mathbb{R}$ and γ coming from the utility function $U(W_t)$ with $W_t \in \mathbb{R}$ the wealth at time t. The derivation of $U(W)$ can be found in appendix 9.4. By calculating μ and σ we can create a life cycle ω_t for the equity exposure. The life cycle ω_t is diplayed in Figure 3.1. It depends on the risk preferences of the participant via the risk aversion parameter γ and on the financial market parameters. Obviously, the expected return is decreasing in γ since a high γ implies a low equity exposure. We will use ω_t as coefficient for the equity exposure in the simulation of the pension capital. The CAPM model, however, gives an optimal life cycle for the Merton model [12] and not for the KNW model. In the appendix we sketch an approach to deriving the optimal life cycle in the KNW model. We will not use this in the pension capital simulation since pension funds use more simple life cylces such as the Merton life cycle in practice.

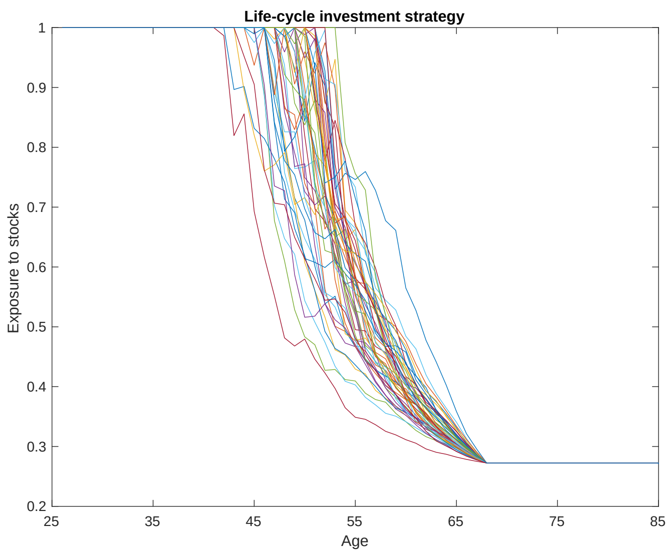
3.3.2 Human capital

If we assume that our pension participant receives labor income until retirement, from which he will pay a percentage as premium, the optimal allocation to risky assets will be related to this labor income and with that the amount of premium as well.
As mentioned before, the fraction of the agent's financial wealth invested in the risky asset should decrease as the age of our agent increases [2]. The first reason for this is that human capital, which is the discounted value of future labor incomes, is usually seen as a risk free asset. The value of human capital decreases as the investor ages. A second argument relates to the flexibility young investors have to alter their labor supply. This allows them to invest more aggressively in stocks compared to older agents.

The economic intuition is that early in life the fraction of human capital is high compared to the fraction of financial wealth. Young agents are less dependent on financial wealth for consumption since they have labor income as alternative income source. It is therefore affordable for them to take more risk with financial wealth then elderly agents who almost entirely depend on financial wealth for their consumption.

Although there are studies in the academic literature which assume that human capital is risky[2], most studies assume that human capital is risk free; human capital can be seen as an implicit holding in risk free assets. This implicit holding pays out dividends in the form of labor income. Hence, the agent's wealth consists of financial assets which can be traded and human capital which cannot be traded.

Figure 3.1: **Merton life cycle** ($\gamma = 5$)



[2]Benzoni et al. (2007), Baxter and Jerman (1997), Lettau and Ludvigson (2001) and Santos and Veronezi (2006).

3.4 Model description simulation pension development DC

To simulate the pension capital during the accrual period and the retirement benefit period we set up a model in which we use the scenario set from DNB generated by the KNW model. We will analyze the development of the pension capital in all scenarios. The scenario set contains 60 years so our simulation will cover a maximum of 60 years. We will assume that our participant starts to build up pension capital at an age of 25 retires at the age of 68 and dies at the age of 85. At $t = 0$ the pension participant is 25 years old, so the pension date will be at $s = 43$ and the time of death will be $T = 60$. We will only analyze a complete life cycle of a pension participant starting at the age of 25. We omit micro-longevity risk and macro-longevity risk since its influence on the pension result is negligible. For the simulation we also do not take into account investment costs or pension implementation costs.

3.4.1 Premium

- **Career pattern**
  We use a career pattern to simulate a realistic labor income development. In the first 10 years (age 25-34) we assume a labor income growth of 3%. For the next 10 years (age 35-44) we assume a yearly labor income growth of 2%. For the age group 45-54 the growth rate equals 1% and for the remaining labor years the growth rate equals 0%. This career pattern is also used to determine the pension premium in a DC pension scheme.[3]

- **Labor income**
  The initial yearly labor income equals $y^1 = 37000$ Euro, the average labor income in the Netherlands. We have to substract 70 percent of the 'Franchise' from this amount, which is the part of the pension capital that belongs to the AOW pension. The AOW pension is a basis pension payment of the Dutch government that every Dutch citizen or formal Dutch citizen is entitled to. We assume a unmarried pension participant and we assume a 'middelloonregeling' [10] which means that the pension aim is to get pension payments which are 70% of the average labor income during the career of the pension participant. The AOW in this case will be equal to 19518 and the AOW franchise will be equal to $0.70 \cdot 19518$ [10], this number is set by the Dutch government and based on our model assumptions. The 70% of the AOW for the franchise comes again from the aim to get pension payments which are 70% of the average labor income. We adapt the labor income for inflation. To correctly apply the inflation rate to labor income we must use the discrete price index Π_t^i:

  $$\Pi_t^i = (1 + \pi_1^i) \cdot (1 + \pi_2^i) \dots (1 + \pi_{t-1}^i) \cdot (1 + \pi_t^i).$$

  Let $y_t^1 \in \mathbb{R}^+$ for $t \in [0, s-1]$ be the labor income according to the career pattern at time t, then we compensate the labor income for the inflation by:

  $$y_t^i = \begin{cases} y_t^1 \cdot \Pi_t^i & t \in [0, s-1] \\ 0 & t \in [s, T]. \end{cases}$$

- **Premium**
  The pension premium is a percentage of the salary which is invested each year. This percentage c is determined by the 4% DC fiscal maximum premium ladder for the accrual of old-age pension from the 'Staffels' [10] and is dependent on the age of the participant

[3] Article 18a, paragraph 3b, 'Wet op de loonbelasting 1964'.

(See Appendix paragraph 2). The premium ladders are based on the fact that a pension participant will have 1.875% pension accrual every year. The aim is that after 40 years of pension payments the pension participant has accrued 70% of his average salary as pension payments. This 70% is translated into a premium percentage per age group. We will assume that the premium is payed at the end of the year and we will not take partner pensions into account. We calculate the premium as follows:

$$p_t^i = c \cdot y_t^i.$$

3.4.2 Interest rate strategy

For the simulation of pension accrual we assume that the bond portfolio consists of cash ${}^0R_t^i$, a one year maturity bond ${}^1R_t^i$ and in a five year maturity bond ${}^5R_t^i$. We will invest the remainder that is not invested in stocks in a combination of these three assets. Let $\theta_1 \in [0,1]$ and $\theta_2 \in [0,1]$

$$R_t^i = \theta_1 {}^0R_t^i + \theta_2 \cdot {}^1R_t^i + (1 - \theta_1 - \theta_2) \cdot {}^5R_t^i$$

We invest a weight of $(1-\omega_t)$ in R_t^i. We assume for simplicity that spread of the weight $(1-\omega_t)$ over cash and bonds are constant. We choose $\theta_1 = 0.1$ and $\theta_2 = 0.4$. This weights are chosen such that we invest the more in the longest maturity, which is the 5 year bond ${}^5R_t^i$, since pensions are a long term investment product and we want to hedge against long term interest rate risk.

3.4.3 Pension capital development

Let $W_t^i \in \mathbb{R}$ be pension capital at time t in scenario i, let $p_t^i \in \mathbb{R}$ be the premium at time t at scenario i, $\omega_t \in (0,1)$ is the investment strategy deduced in section 3.3 i.e. the fraction of pension capital invested in stocks S_t. Let Q_t^i be the pension payment at time t and in scenario i. We assume that the pension payment is payed at the end of the year. We will distinguish two definitions of pension capital. In the first case we do not take inflation into account; the nominal pension capital. We simulate the pension capital in the nominal case per scenario as follows:

$$W_{t+1}^i = \begin{cases} W_t^i(1 + (1-\omega_t)R_t^i + \omega_t dS_t^i) + p_t^i \text{ if } t \in [0, s-1] \\ W_t^i\left(1 + (1-\omega_t) \cdot R_t^i + \omega_t dS_t^i\right) - Q_t^i \text{ if } t \in [s, T]. \end{cases} \tag{3.17}$$

The second definition of pension capital is the real pension capital. The real pension capital is equal to the nominal pension capital corrected for inflation:

$${}^rW_t^i = \frac{W_t^i}{\Pi_t^i} \tag{3.18}$$

we assume that the portfolio is rebalanced after every year and with $W_1 = 0$ is the initial wealth value.

3.4.4 Pension payment

The pension payment Q_t^i is calculated using the price of an annuity. It is determined such that the participant receives pension payments until he dies. The discount factor determines how the pension capital is distributed over the retirement period. We have three options for the price of an annuity. We can use:

1. the bond portfolio R_t^i

2. the term structure ${}^{N}R_t^i$

3. the risk free rate ${}^{0}R_t^i$

The bond portfolio is specifically chosen for the investment portfolio and not for the reference measures that we will use to calculate pension result. Since we want the same annuity for all definitions of pension result, in order to make a fair comparison, we will chose a more general discounting factor; the risk free rate. In the case of a variable annuity DC pension scheme we have to update our annuity and with that the pension payout every year. The pension payments will be determined as in equation (3.19) in every scenario i and at every time $t \in [s, T]$, where s is the pension date and T the time of death.

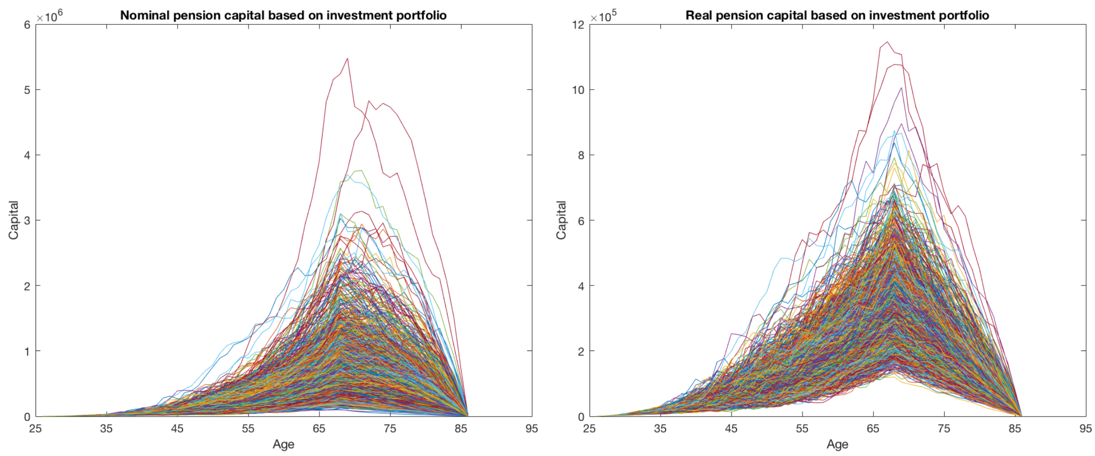
$$\begin{aligned} W_t^i &= \frac{Q_t^i}{(1+{}^{0}R_t^i)} + \frac{Q_t^i}{(1+{}^{0}R_t^i)^2} + ... + \frac{Q_t^i}{(1+{}^{0}R_t^i)^{T-t}} \quad t \in [s-1, T] \\ &= Q_t^i \sum_{k=1}^{T-t} \frac{1}{(1+{}^{0}R_t^i)^k} \quad t \in [s-1, T] \\ Q_t^i &= \frac{W_t^i}{\sum_{k=1}^{T-t} \frac{1}{(1+{}^{0}R_t^i)^k}} \quad t \in [s-1, T] \end{aligned} \tag{3.19}$$

The real pension payout is defined as the nominal pension payout corrected for inflation:

$${}^{r}Q_t^i = \frac{{}^{r}W_t^i}{\sum_{k=1}^{T-t} \frac{1}{(1+{}^{0}R_t^i)^k}} \quad t \in [s-1, T] \tag{3.20}$$

In Figure 3.2 respectively the real and the nominal DC pension capital development are displayed. We see that till the age of 68 the capital is increasing because of the payed premium and the investment returns. After the pension date the capital is decreasing due to the pension payouts. Note that the axes have a different scale since a similar scale would not show the real pension capital very well.

Figure 3.2: **DC pension capital development W_t^i and ${}^{r}W_t^i$**



Chapter 4

Pension result in DC schemes

In this chapter we define the pension result in DC schemes in a quantitative way. We will first introduce the way in which we want to formulate the pension result in DC schemes in a general way, and afterwards we will discuss alternatives which suit this general definition in the sections 4.3 up till and including 4.7.

4.1 Pension result

As already mentioned, the pension result is not sufficiently applicable to DC schemes in the existing form. In the current set up of the pension result applied to DC schemes we ignore the fact that every pension agreement has the aim to support a suited benefit for the post-active period. The pension result only takes into account the development of the pension capital from the pension date, the first day of the retirement period, on. In a DB scheme the participant accrues a pension entitlement by paying an amount of premium which leads to a certain pension level during retirement. These pension entitlements give a lot of insights for the participants about their pension benefit after retirement. In a DC scheme social partners only make an agreement about the amount of premium but the social partners do not formulate a pension goal, such as (indexed) pension entitlements. However we can still communicate about the expected result. We can do this by formulating the pension result for all DC schemes as well. If we add a pension result as a benchmark in the feasibility test which is applicable to DC pension schemes this must not form any guarantee; it only has an informative role.
The four most important factors[1] that influence the pension result are the premium, the investment returns, the interest rate (in particular the interest rate on the pension date and during the rest of the retirement period) and the inflation rate. We will start with the premium. Although the premium is an important factor that influences the pension result, we do however wish that the pension result is independent of the amount of premium but dependent on the spread of the premiums over the years. When the pension result would be dependent on the amount of premium the solution to obtain a high pension result would be to choose a very high premium. This is not a desirable result. The premium is determined by the social partners during the set up of the pension agreement, it is not a desired feature that the pension result tests the height of the premium and with that the specific financial situation of the pension participant, we rather want to measure other factors that influence the quality of pension. The pension result is also influenced by the investment returns. The investment returns depend on the investment strategy. The interest rate during the retirement period determines the height of the

[1] Other sources that influence the pension result are the chances of death which determine over how many years we have to spread the pension capital during retirement. In this research the chances of death are left out since this source of risk is not very important for the conclusions about the pension result.

pension payments during retirement. These pension payments are key factors in the definition of the pension result. Apart from this we also have some other individual factors such as the length of the pension accrual period and the specific salary development from the participant that influence the pension result. The pension result in general gives us insight to which extent purchasing power is maintained. From this it follows that also the inflation rate has an influence on the pension result.

To define a comparable definition for the pension result which is applicable in the DC scheme in general we need a reference value that follows the inflation rate. In order to stay close to the definition of the pension result in DB schemes we keep the numerator of the definition the same in the sense that it is calculated by summing the flow of expected pension payments. These pension payments are, however, determined in a different way when a DC pension scheme is applied instead of a DB pension scheme.

Definition 4.1. *Pension result in DC schemes*
Let Q_t^i be the nominal pension payment and let ${}^rQ_t^i$ be the real pension payment in scenario i at time t based on the accrued pension capital using respectively formula (3.19) and (3.20). Let aQ_t^i be the nominal pension payment and let ${}^r_aQ_t^i$ be the real pension payment, according to a reference capital accrual by investing in portfolio a:

$$\begin{aligned} {}_aQ_t^i &= \frac{{}_aW_t^i}{\sum_{k=1}^{T-t} \frac{1}{(1+{}^0R_t^i)^k}} \\ {}^r_aQ_t^i &= \frac{{}^r_aW_t^i}{\sum_{k=1}^{T-t} \frac{1}{(1+{}^0R_t^i)^k}} \end{aligned} \tag{4.1}$$

Let s be the pension date and let T be the time of death. The nominal and real pension result in DC schemes is defined as:

$$\begin{aligned} PR^i &= \frac{\sum_{t=s}^{T} Q_t^i}{\sum_{t=s}^{T} {}_aQ_t^i} \\ {}^rPR^i &= \frac{\sum_{t=s}^{T} {}^rQ_t^i}{\sum_{t=s}^{T} {}^r_aQ_t^i} \end{aligned} \tag{4.2}$$

Below we define possible definitions of the pension result, where we analyze different choices for the reference values ${}^r_aQ_t^i$. We do not discuss ${}_aQ_t^i$ since we will focus on the real pension payments and not on the nominal pension payments. We analyze the the results that every definition of pension result gives and we will draw a conclusion about which definition is most appropriate.

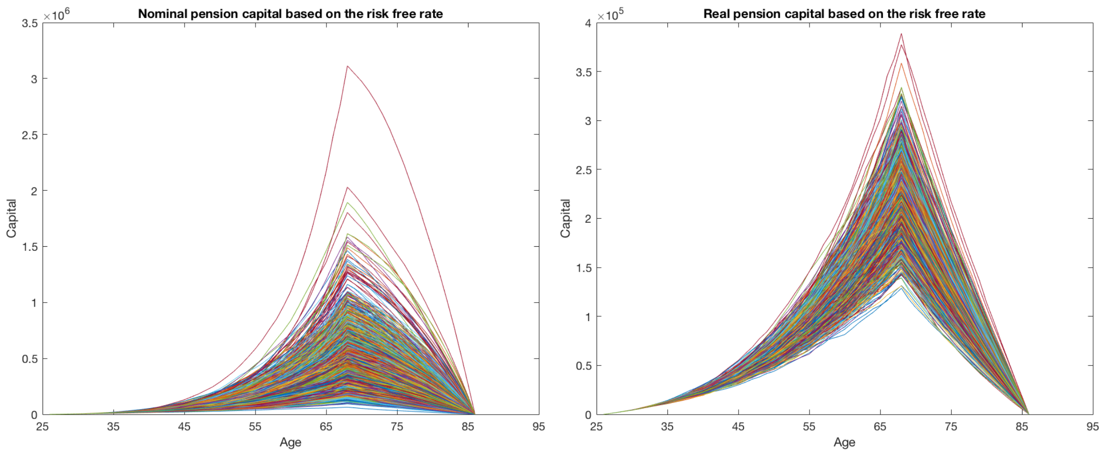
4.2 Pension result based on the risk free rate

For the first definition of the pension result we compare the development of pension capital according to the investment portfolio, described in the previous chapter, with a pension capital ${}_1W_t^i$ which has a return equal to the risk free interest rate. This is the interest rate we would get when saving capital on the bank. Pension capital for this first reference value develops as follows:

$$\begin{aligned} {}_1W_{t+1}^i &= \begin{cases} {}_1W_t^i(1+{}^0R_t^i) + p_t^i \text{ if } t \in [0, s-1] \\ {}_1W_t^i\left(1+{}^0R_t^i\right) - {}_1Q_t^i \text{ if } t \in [s, T]. \end{cases} \\ {}^r_1W_{t+1}^i &= \frac{{}_1W_{t+1}^i}{\Pi_t^i} \end{aligned}$$

To calculate the pension result we use ${}^r_1Q^i_t$ as pension payment, calculated as in (4.1). We determine the pension result as in equation (4.2). In Figure 4.1 respectively the nominal and real pension capital development based on the risk free rate are displayed. We see rather straight lines here since we do not invest in stocks.

Figure 4.1: **Pension capital development based on the risk free rate ${}_1W^i_t$ and ${}^r_1W^i_t$.**



4.3 Pension result based on a constant rate

We compare the development of pension capital according to the investment portfolio with a pension capital with a constant return rate α. We basically set a minimum for the investment return which we wish to reach. The pension accrual in respectively the nominal and real case will be :

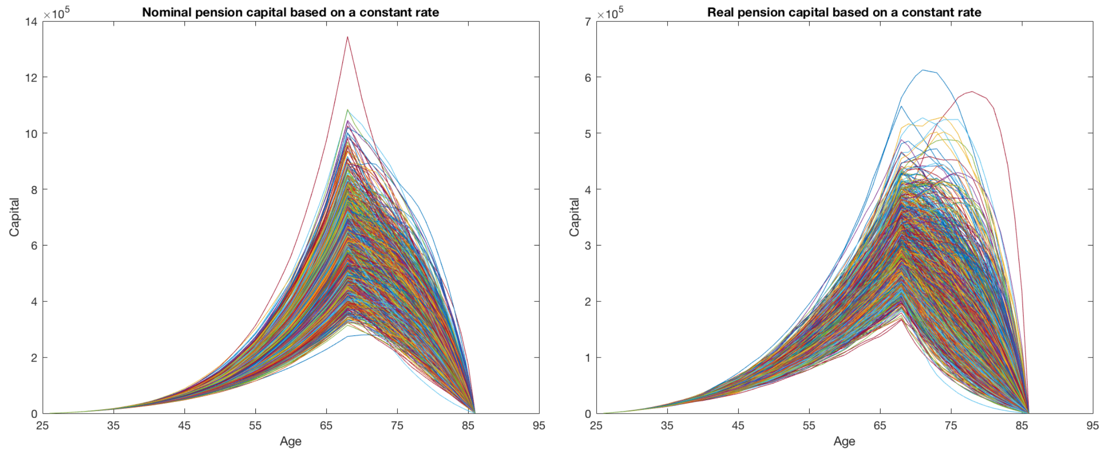
$$
\begin{aligned}
{}_2W^i_{t+1} &= \begin{cases} {}_2W^i_t(1+\alpha) + p^i_t \text{ if } t \in [0, s-1] \\ {}_2W^i_t\,(1+\alpha) - {}_2Q^i_t \text{ if } t \in [s, T]. \end{cases} \\
{}^r_2W^i_t &= \frac{{}_2W^i_t}{\Pi^i_t}
\end{aligned}
$$

To calculate the pension result we use ${}^r_2Q^i_t$ as pension payment, calculated as in (4.1). We determine the pension result as in equation (4.2). In Figure 4.2 respectively the nominal and real pension capital development based on a constant rate are displayed. We can see that in the accrual period the scenario lines are rather straight since the return is constant and the same for every year. In the retirement period we also have to deal with the interest rate to determine the pension payouts. Therefor we see less straight lines in the retirement period.

4.4 Pension result based on inflation rate

When we focus on measuring the maintenance of purchasing power it is desirable that the realized investment returns are at least equal to the inflation rate. We compare the development of pension capital according the investment portfolio with a pension capital which has a return equal to the inflation rate. If the return rate we use in equation (3.17) for the investment portfolio would be equal to the inflation rate, i.e. $(1 + (1-\omega_t)R^i_t + \omega S^i_t) = (1 + \pi^i_t)$ we would

Figure 4.2: **Pension capital development based on a constant rate $_2W_t^i$ and $_2^rW_t^i$.**

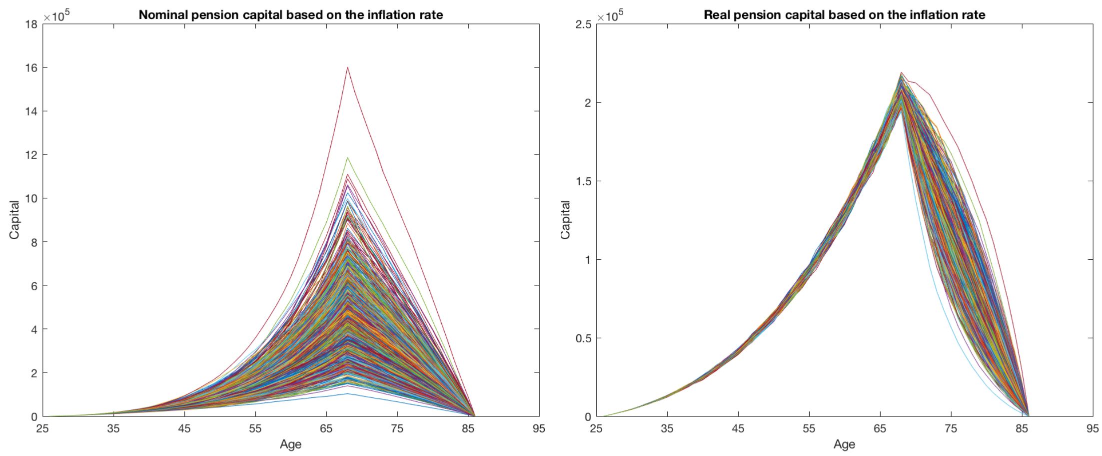


not lose purchasing power. The pension accrual in respectively the nominal and real case will be as follows:

$$
\begin{aligned}
{}_3W_{t+1}^i &= \begin{cases} {}_3W_t^i(1+\pi_t^i) + p_t^i \text{ if } t \in [0, s-1] \\ {}_3W_t^i\left(1+\pi_t^i\right) - {}_3Q_t^i \text{ if } t \in [s, T]. \end{cases} \\
{}_3^rW_t^i &= \frac{{}_3W_t^i}{\Pi_t^i}
\end{aligned}
$$

To calculate the pension result we use $_3^rQ_t^i$ as pension payment, calculated as in (4.1). We determine the pension result as in equation (4.2). In Figure 4.3 respectively the nominal and real pension capital development based on the inflation rate are displayed. The pension capital in the denominator is mainly dependent on the inflation rate, so in the real case we do not see big differences between the scenarios.

Figure 4.3: **Pension capital development based on the inflation rate $_3W_t^i$ and $_3^rW_t^i$.**



4.5 Pension result based on non indexed pension entitlements

There exist DC pension schemes in which the pension capital is converted into DB pension entitlements, in this type of pension scheme the pension fund buys pension entitlements on the pension date. From then on it will be a DB scheme on which we can apply the feasibility test in its existing state. In this case the pension participant also bears the pension conversion risk. The pension conversion risk is the risk one gets exposed to when the pension capital is converted into an annuity or in DB pension benefits. The level of pension payouts, ${}^{r}_{a}Q^{i}_{t}$ from equation (4.1), depends on a number of factors, including the interest rate (in this specific pension scheme mainly the interest rate on the pension date), the inflation rate and the expected age of death. The Pension conversion risk is the risk that the investments do not protect against the cost of buying an income at retirement. For this definition of pension result we use a similar structure as this specific DC pension scheme. The only difference is that the pension capital will already be converted into pension entitlements before the pension date. We analyze at every time t what pension benefit we can buy at that time with the premium paid in that year. This is in line with the accrual of nominal pension benefits in DB schemes. We compare the sum of these benefits with the pension payments generated by the pension capital according to the investment portfolio.

$$
\begin{aligned}
B^{i}_{t} &= \frac{p^{i}_{t}}{\sum_{k=1}^{T-s} \frac{1}{(1+{}_{f}R^{i}_{t})^{(s-t)+k}}} \\
BS^{i}_{t} &= \sum_{l=1}^{t} B^{i}_{l} \\
{}^{r}BS^{i}_{t} &= \frac{BS^{i}_{t}}{\Pi^{i}_{t}}
\end{aligned}
\tag{4.3}
$$

We calculate the pension result PR per scenario i as follows in respectively the nominal and real case:

$$
\begin{aligned}
{}_{4}PR^{i} &= \frac{\sum_{t=s}^{T} Q^{i}_{t}}{(T-s)\cdot BS^{i}_{t}} \\
{}^{r}_{4}PR^{i} &= \frac{\sum_{t=s}^{T} {}^{r}Q^{i}_{t}}{\sum_{s}^{T} \cdot {}^{r}BS^{i}_{t}}
\end{aligned}
$$

For the nominal case we use $(T-s)\cdot BS^{i}_{t}$ in the denominator since BS^{i}_{t} is the same for all $t \in [s,T]$, so BS^{i}_{t} is not really dependent on t anymore. In the real case we use $\sum_{s}^{T} \cdot {}^{r}BS^{i}_{t}$ in the denominator since the multiplication of BS^{i}_{t} by the cumulative inflation Π^{i}_{t} results in time dependence, so ${}^{r}BS^{i}_{t}$ does depend on t.

4.6 Pension result based on instant indexed pension entitlements

We will analyze at every time t what pension benefit we can buy at that time with the premium payed in that year, which will be indexed at all times $t \in [0, s-1]$. We will compare the sum of these pension benefits with the pension payments generated by the pension capital according to the investment portfolio. We calculate the pension result PR per scenario i as follows in respectively the nominal and real case:

$$
\begin{aligned}
{}_5PR^i &= \frac{\sum_{t=s}^{T} Q_t^i}{\sum_s^T \cdot BS_t^i \cdot \Pi_t^i} \\
{}_5^rPR^i &= \frac{\sum_{t=s}^{T} {}^rQ_t^i}{(T-s) \cdot {}^rBS_t^i \cdot \Pi_t^i}
\end{aligned}
$$

4.7 Replacement ratio

Besides the different definitions of pension result it is also interesting to look at the replacement ratio.

Definition 4.2. *Real Replacement ratio*
We define the real replacement ratio RR_t^i at every time t and in each scenario i as:

$$RR_t^i = \frac{{}^rQ_t^i}{\frac{1}{s}\sum_{t=0}^{s} \frac{y_t^i}{\Pi_t^i}}. \tag{4.4}$$

The replacement ratio is a reference of the pension payments in comparison to the earned salary. This gives an idea of what the pension participant can expect in comparison to his labor period.

4.8 Results

In table 4.1 we displayed the results of all the definition of the pension result. In table 4.2 we displayed some basic information about the financial market. We will analyze the different definitions for the pension result in DC schemes and we will conclude about the appropriateness of the formulated definitions.

We start with addressing some general properties of the formulated definitions of the pension result. The pension result depends on the scenario number for every definition. The nominator dependents on the investment policy for every definition but the denominator is independent of the investment policy. Another property is that all the definitions of the pension result do not give a conclusion about the absolute amount op pension, but about the relative amount of pension capital compared to a benchmark. We can also see that the definitions of pension result are independent of the amount of premium since the amount of premium is taken into account in both the nominator and denominator. So we do not get a higher pension result if we would consume less during the accrual period. This is a desired feature since we do not want the result to be the best if we would invest as much premium as we can. Paying more premium is an obvious way of improving the pension capital, but the pension result does not aim to measure the ratio between consumption and premium. The pension result aims to measure the maintenance of purchasing power given the premium. If a higher premium would improve the pension result, then it could not sufficiently measure the maintenance of purchasing power. In that case it would also be difficult to analyze the influence of the other policy assumption, since a higher premium would always be the solution to a higher pension result.

It can be seen in Table 4.1 that the pension result based on the risk free rate is higher than 1 in every percentile, so the investment portfolio always gets a higher return than the risk free rate. The pension result based on a constant rate asses a comparable value in the 95th percentile as the pension result based on the risk free rate. In the 5th percentile the pension result based on a constant rate is lower than the pension result based on the risk free rate. The pension result based on the inflation rate is equal to 3.344 in the 95th percentile which is higher than the pension result based on the risk free rate and the pension result based on a constant rate in the 95th percentile. In the lowest 5 percentiles the pension result based on the inflation rate gets slightly lower than 1. The pension result based on non indexed pension entitlements asses the highest value of 3.344 of all the definitions in the 95th percentile. In the 5th percentile it is equal to 0.687 which lies between the values of the 5th percentile of the pension result based on the inflation rate and the pension result based on a constant rate. These are respectively equal to 0.911 and 0.383. The pension result based on instant indexed pension entitlements asses the lowest value in both the 95th percentile and the 5th percentile. The highest difference between the median and the 5th percentile is seen in the case of the pension result based on non indexed pension entitlements where it is equal to 0.866. The lowest difference between the median and the 5th percentile is seen in the case of the pension result based on instant indexed pension entitlements where it is equal to 0.347.

We can see with these observations from Table 4.1 that there exists a clear order for the 'strictness' of the definitions. One denominator gives a higher pension result than the other. Here we look at which definition has the most values of the pension result lower than 1. The order of the definitions is as follows:

1. ${}_5PR^i$ based on instant indexed pension entitlements

2. ${}_2PR^i$ based on a constant rate with constant rate equal 4%

3. ${}_4PR^i$ based on non indexed pension entitlements

4. ${}_3PR^i$ based on inflation rate

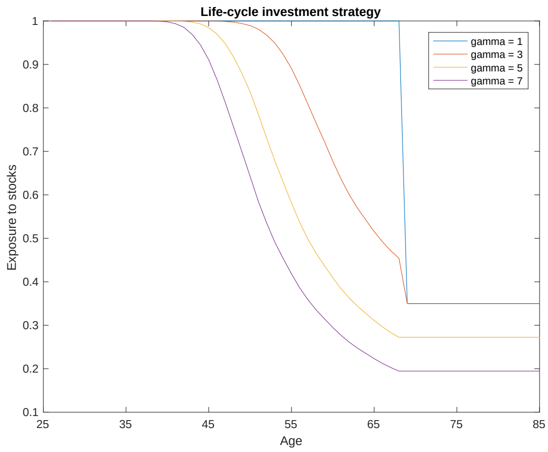
5. ${}_1PR^i$ based on the risk free rate

The definition based on the risk free rate is the least strict definition. The investment portfolio outperforms the risk free bank saving in 96% of the scenarios. The definition based on instant indexed pension entitlements and the definition based on the inflation rate are the only ones that explicitly measure the maintenance of purchasing power.

4.9 Pension result and the risk attitude

The risk attitude is the extent to which the group of participants is willing to and is able to take investment risk to realize the pension goals. The risk attitude is reflected in the maximum acceptable deviation between the median and the 5th percentile of the expected pension result. In general it holds that the more risk a participant is willing to take the higher the maximum deviation gets. We can see this in Table 4.3. We can clearly see that the 95th percentile gets higher as we take more risk. i.e. let γ decrease, but we also see that the maximum deviation indeed gets bigger and the 5th percentile gets lower as γ declines. So we see that there exists a link between the life cycle investment strategy which is based on γ and the risk attitude. The KNW scenario set is, however, a very positive scenario set. The excess return on stocks has a high mean of 0.0452. We are also considering a long time horizon of 60 years. Therefore a low γ i.e. a low risk aversion, still gives relatively high pension results in the 5th percentile.

Figure 4.4: **Life Cycles for $\gamma = 7$, $\gamma = 5$, $\gamma = 3$ and $\gamma = 1$**



In figure 4.4 the median of the life cycles for different values of γ are shown. The blue line represents the life cycle when $\gamma = 1$. The red line represents the life cycle when $\gamma = 3$, there is a discontinuous part right before retirement at the age of 68 since the amount that one is allowed to invest in stocks after retirement has a maximum of 0.35. The yellow line represents the life cycle when $\gamma = 5$ and the purple line represents the life cycle when $\gamma = 7$. We see that when γ is lower the graph decreases later in time.

4.10 Sensitivity Analysis

In this section we determine which definition for the pension result would be the best fit for the feasibility test and afterwards we do some robustness tests. We first discuss the appropriateness of the definition for the pension result.
The first definition based on the risk free rate is less suitable to use in the definition of the pension result in the feasibility test for DC schemes since it only suits for short term pensions, so for relatively older people. As soon as we look at long term pension capital development, then in general it holds that partly investing in stocks is always better than a risk free investment. Unless, however, the ratio invested in stock gets excessively high and/or the participant has a very high risk aversion. The definition for the pension result based on a constant rate and the pension result based on non indexed pension entitlements mainly measure the performance of an investment product, the focus lies on the height of the stock returns. The pension result based on non indexed pension entitlements also measures the interest rate risk. Definition three and five, respectively the pension result based on the inflation rate and the pension result based on indexed pension entitlements, explicitly measure the maintenance of purchasing power. Since the feasibility test focuses on measuring the maintenance of purchasing power we would consider these definitions to fit the best. Another favorable feature of definition five is that it resembles the definition of pension result in DB schemes the most. There for we will choose the definition based on instant indexed pension entitlements as the best definition for the pension result in DC schemes. One has to be aware that a definition for the pension result in DC schemes has the goal to inform about the quality of the pension. In DB schemes we compare the realized benefits with fully indexed pension entitlements. In DC schemes, we do not want the denominator of the definition of pension result to be interpreted as an ambition or an entitlement[2]. It should be noted that the interpretation of the denomiator as an entitlement cannot be used in DC schemes.

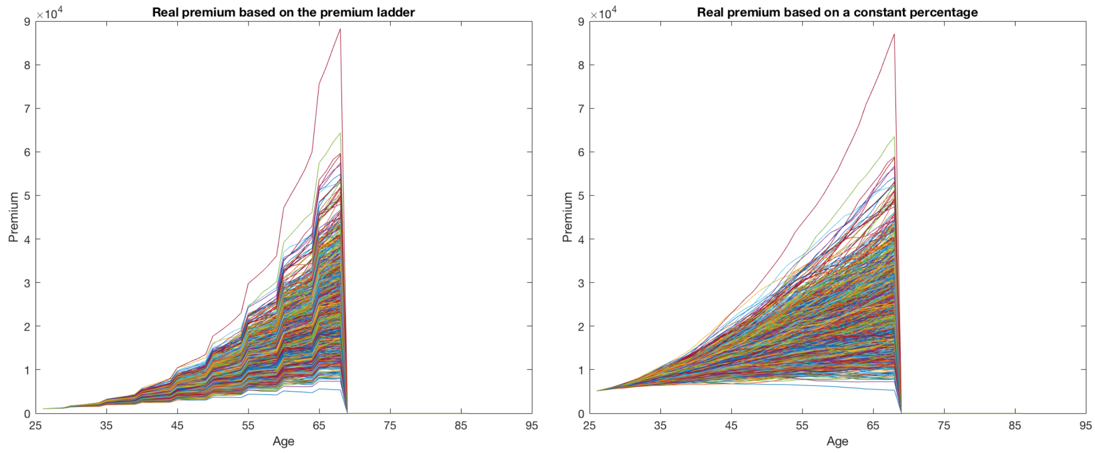
[2] Since in a DC scheme the participant does not accrue pension entitlements.

We will chose the definition based on indexed pension entitlements as the best definition and we will focus from now on on this definition in the sensitivity analysis.

4.10.1 Sensitivity Analysis 1: constant premium

As stated before in section 4.8 about the observations of the definitions of the pension result, all definitions are independent of the amount of premium. However the way in which premium payments are distributed over the years does influence the pension result. We will look at the sensitivity of the pension result to the premium development. We compare an increasing premium based on the premium ladder with a constant premium percentage. When we use a constant percentage it does not matter how high the percentage is as long as it remains constant over the time horizon[3]. In figure 4.5 the development of the premium is displayed and in figure 4.6 the development of the premium percentage is shown. We can see that when a constant premium percentage is used the premium development shows a smooth line which is not the case when using the 4 % premium ladder.

Figure 4.5: **Premium development**

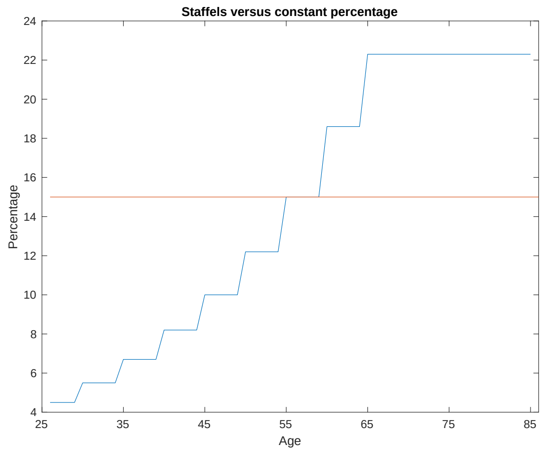


In Figure 4.5 we chose a constant premium percentage of 22 percent. It is chosen such that the total amount of payed premium resembles the total amount of payed premium when the 4 percent premium ladder is used.

We can see in table 4.4 that in general the pension result is higher when we use a constant premium instead of the premium ladder. The maximum deviation between the median and the 5th percentile, however, is bigger when using a constant premium. The higher pension result in the case of a constant premium percentage is the result of the increase of the premium percentage in the premium ladder. A low percentage is payed at young ages and the percentages increase with age. The premium payed at young ages can be invested for a longer period than the premium payed at older ages. So in general one can make a higher investment return on the premium payed at young ages compared to the premium payed at older ages. If we change the spread of the premium percentages into a constant which is the same for every age group we will see a higher pension result since the return made on premium payed at young ages is relatively higher compared to using a premium ladder. We can see this effect since we only incorporate the investment return in the numerator of the quotient for pension result.

[3] For the real replacement ratio however, the height of the constant premium percentage does matter, but we will not focus on that in this research.

Figure 4.6: **Premium percentage development**



In figure **??** the development of the premium percentage is show when using the 4 % premium ladder displayed in blue and when using a constant percentage displayed in red.

4.10.2 Sensitivity Analysis 2: constant stock exposure

In our model we used the Merton life cycle for the exposure to stocks. This life cycle is optimal in the Merton model, but it is not optimal in the KNW model. The Merton life cycle can be used as an approximation. The question is whether it is a good choice to use this approximation. To answer this question we will first show if it is better to invest according to a life cycle than according to a constant stock exposure by making a comparison between the Merton life cycle and a constant mix strategy. The constant-mix strategy invests 38% in stocks. This proportion is determined in such a way that the total median equity exposure (weighted for the available financial wealth) is the same as the total median equity exposure of the Merton strategy in order to properly compare the constant-mix strategy with the other strategy. We call the constant mix ω_{cm} and we calculate the constant mix ω_{cm} according to the formula in (4.5). Here M denotes the scenario number of the median.

$$\omega_{cm} = \frac{\sum_{t=1}^{T} \left(W_t^M \cdot \omega_t^M \right)}{\sum_{k=1}^{T} W_k^M} \tag{4.5}$$

When we compare the pension results in Table 4.1 with Table 4.5 we can see that investing according to a life cycle gives a higher pension result in all the percentiles. This is a desired result since investing according to a constant mix does not have the feature that it takes more risk for pension participants at a young age and less risk for pension participants at an older age. It is favorable to take more investment risk at young ages since the negative shocks that can occur at young ages can still be compensated during the rest of the pension payment period. At an older age there exists a need of protection of the pension capital which results in taking less investment risk.

4.10.3 Sensitivity Analysis 3: fixed decrease

Pension participants in the Netherlands who buy a variable annuity have the possibility to choose how they divide their pension capital over the various pension payments during retirement and with that they have the possibility to choose the height of the pension payments. This is called the pension payment policy. One could for instance choose to begin with a higher payout in the first years of the retirement period, based on the assumption that the investment returns can compensate for this in the upcoming years. We compute the pension payments as in equation (4.1) but with a different discount rate which takes into account the expected investment returns. We call this different discount rate the assumed interest rate ${}_aR_t^i$,

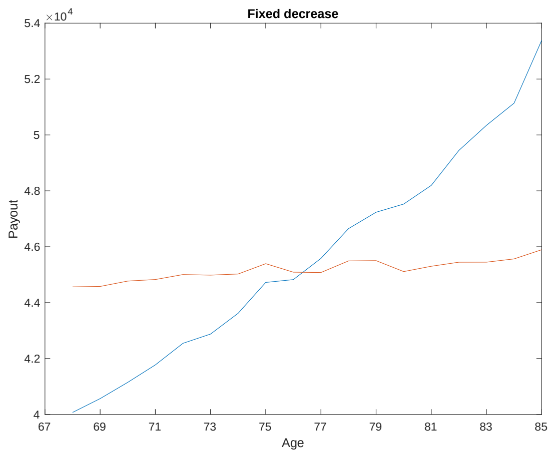
$$ {}_aR_t^i = {}_fR_t^i + \omega_t^i \eta_S \quad \text{for } t \geq s \tag{4.6} $$

The parameters $\omega_t^i \cdot \eta_S$ determine the maximum fixed decrease pension participants can use. Here ω_t^i represents the equity exposure (see section 3.3.1). The investment strategy ω_t^i depends on the risk preferences of the participant via the risk aversion parameter γ and on the financial market parameters. Obviously, the expected return is decreasing in γ since a high γ implies a low equity exposure. A low ${}_aR_t^i$ implies a relatively low benefit level at retirement. A high ${}_aR_t^i$ implies a high pension payment level at retirement but also a higher probability of a decrease in the pension payment level during retirement.

To measure the sensitivity of the pension result to a fixed decrease in the payout during the retirement period, we implement a fixed decrease in both the numerator and the denominator of the definition of pension result. This because we assume that the choice to have a fixed decrease is based on a pattern of consumption and not on the maintenance of purchasing power. In the pension result we want to measure the maintenance of purchasing power and not the choice of a consumption pattern. So in the nominator and the denominator we use the same structure to determine the pension payouts using the annuity and the pension capital. In this way we also stay consistent with the accrual period where the same premium is invested in the numerator and the denominator.

We observe in table 4.6 that the pension result is similar when we apply a fixed decrease to the pension payouts, the maximum deviation is also similar.

Figure 4.7: **Payout development**



In Figure 4.7 we displayed the median of the payouts using no fixed decrease in the retirement period, the blue line, and we displayed the median of payouts using a fixed decrease in the retirement period, the red line. only the median is shown because this displays the effect of the fixed decrease the best. In case of a fixed decrease we see a stable payout and in the case of no fixed decrease we see an increasing payout.

4.10.4 Sensitivity Analysis 4: different scenario set

An important point of criticism on the feasibility test is the underlying scenario set. The scenario set is a prediction of the development of certain economic variables during the up coming 60 years. As most predictions this scenario set does not give us exact reality which naturally makes it a point of discussion. The scenario set is, as mentioned before, based on the KNW model. So the parameter and model risk of the KNW model is expressed in the scenario set.
In the KNW model the state variables, which resemble an interest rate factor and an inflation factor, are modeled with a diffusion part and they are mean reverting. From a statistical point of view, there exist three features for the interest rate process and the inflation rate process that consistently exist in bond markets:

1. Autocorrelation and mean reversion
2. Volatility behavior
3. Significant skewness and kurtosis behavior

The first two features are present in the interest rate and inflation rate formulas of the KNW model, but the third feature is not. It is well known that any normal distribution has zero skewness and a kurtosis of 3. Therefore the skewness and kurtosis of the returns of a GBM are zero [23]. Changes in interest rate demonstrate considerable skewness and kurtosis [14]. Kurtosis in the interest rate is often present in the form of Leptokurtosis. Leptokurtosis or 'fat tailed risk' occurs when the shape of a distribution is more peaked than the shape of a normal or 'bellcurve' distribution. In such a distribution, small changes are less frequent than in a normal distribution, but large price moves are more likely to happen and are potentially larger than in a normal distribution. The central peak is more narrow, but the tails are significantly longer and fatter. Backus et al. (1997) state that jumps better explain the high degree of curvature (more peeks) in yield curves. Surprises i.e. jumps or shocks occur with significant magnitude and regularity and have substantial impact on the yields, bond prices and bid-ask spreads. So adding jumps to the interest rate process in the KNW model might lead to more realistic interest rate development.

The presence of jumps in the inflation rate has not been considered so much in existing literature. We will analyze the historical data of the inflation rate and we will check whether is could be modeled by a jump diffusion process as well.

The absence of kurtosis in the interest rate process in the KNW model is a point of criticism and discussion. In the next chapters we will address this discussion point of the scenario set by extending the KNW model. We will add jumps to the interest rate and the inflation rate of the KNW model. We will define the formulas of the extension of the KNW model with jumps in the interest rate and the inflation rate. We will conclude with an analysis about the impact of a new scenario set, based on the extended KNW models, to the definitions of pension result. With this we want to analyze if adding jumps is an important improvement of the KNW model in order to give a better prediction of the economy parameters.

4.10.5 Goal of the feasibility test

In practice it is sometimes difficult to work with the feasibility test. The definition of pension result does not give us any information about the height of the payout during the retirement benefit period. The pension result is said to be an easy definition to communicate with about the quality of a pension, but in practice this is not always the case.

Instead of giving an alternative definition for the pension result in DC schemes we can also consider an alternative goal for the feasibility test. We could shift from the goal to measure the maintenance of purchasing power to the goal of measuring what pension payout we could actually expect in comparison with the average salary the pension participant received before retirement. The benchmark for the quality of a pension will then be the replacement ratio. The question is now how the replacement ratio and the pension result relate to each other. Can we transform one into another via a formula, such that when we define a new definition for the pension result in DC schemes we can also make a link to the replacement ratio? We know that the pension result based on instant indexed pension entitlements is defined as follows:

$$ {}_5^rPR^i \quad = \quad \frac{\sum_{t=s}^{T} {}^rQ_t^i}{(T-s) \cdot {}^rBS_t^i \cdot \Pi_t^i} $$

From this it follows that the average real replacement ratio can be written using ${}_5^rPR^i$,

$$ \begin{aligned} \sum_{t=s}^{T} {}^rRR_t^i &= \sum_{t=s}^{T} \frac{{}^rQ_t^i}{E_i[{}^ry_t^i]} \\ &= \sum_{t=s}^{T} \frac{{}^rBS_t^i \cdot \Pi_t^i}{E_i[{}^ry_t^i]} \cdot \frac{{}^rQ_t^i}{{}^rBS_t^i \cdot \Pi_t^i} \\ &= \sum_{t=s}^{T} \frac{{}^rBS_t^i \cdot \Pi_t^i}{E_i[{}^ry_t^i]} \cdot \frac{1}{(T-s)} \cdot {}_5^rPR^i \end{aligned} \tag{4.7} $$

Since $y_t^i = 0$ for $t \in [s, T]$ the expectation $E[y_t^i]$ will be evaluated over the accrual period only. We can see in equation (4.7) that we cannot make a direct link between the pension result and the replacement ratio, since the term ${}^rBS_t^i \cdot \Pi_t^i$ has an inverse effect on the pension result and the replacement ratio. When ${}^rBS_t^i \cdot \Pi_t^i$ is high the pension result gets low but the sum of ${}^rRR_t^i$ over the retirement period gets higher. This results in a higher mean of the replacement ratio's. This effect can also be seen in table 4.1, where we do not generally see a high replacement ratio if we find a high pension result. The use of the replacement ratio has been a point of discussion. Since the replacement ratio is generally easier to understand the replacement ratio seems to be a favored alternative. The Dutch pension law, however, does not act on the agreements between the employer and the employees about the amount of premium and the height of the salary. This is the reason why the existing definition of the pension result does not measure the actual height of the pension payments and why the pension result is chosen as a measure in the feasibility test to inform about the quality of pensions.

Table 4.1: **Pension result definitions** ($\gamma = 5$)

Pension result based on the risk free rate

Percentile	Annuity	${}^rW_s^i$	Payout	Pension Result	${}^r_1W_s^i$	Payout	RR
95	20.164	471241	27852	2.671	222932	10426	0.502
50	13.436	395205	26308	1.606	264507	16385	0.474
4.1	14.738	257499	15924	1.000	259397	15929	0.287
5	8.633	245991	17257	1.025	216836	16834	0.311

Pension result based on a constant rate

Percentile	Annuity	${}^rW_s^i$	Payout	Pension Result	${}^r_2W_s^i$	Payout	RR
95	10.255	534455	37730	2.693	205507	14012	0.680
50	13.039	337344	18148	1.007	260566	18021	0.327
49.5	24.132	412517	24853	1.000	310763	24854	0.448
5	22.031	197698	11279	0.383	372541	29488	0.203

Pension result based on the inflation rate

Percentile	Annuity	${}^rW_s^i$	Payout	Pension Result	${}^r_3W_s^i$	Payout	RR
95	17.665	455182	39028	3.344	207096	11672	0.703
50	17.945	400927	19617	1.681	207507	11672	0.353
8.8	21.004	159973	11669	1.000	209145	11672	0.210
5	17.730	200218	10629	0.911	208904	11672	0.191

Pension result based on non indexed pension entitlements

Percentile	Annuity	${}^rW_s^i$	Payout	Pension Result	${}^r_4W_s^i$	Payout	RR
95	12.683	556474	30959	3.353	0	13705	0.558
50	14.625	264614	17466	1.553	0	14652	0.315
17.2	25.370	317690	14850	1.000	0	13571	0.268
5	15.538	172405	9585	0.687	0	14948	0.173

Pension result based on instant indexed pension entitlements

Percentile	Annuity	${}^rW_s^i$	Payout	Pension Result	${}^r_5W_s^i$	Payout	RR
95	23.280	257736	16535	1.329	0	12446	0.298
85.4	16.080	314221	22264	1.000	0	22264	0.401
50	24.119	241986	13042	0.571	0	22854	0.235
5	18.817	191134	10661	0.224	0	47572	0.192

In Table 4.1 the first column displays the 95th percentile, the median, the percentile where the pension result is approximately equal to 1 and the 5th percentile. In the second column the average annuity price during the retirement period is displayed. In the third column the total accrued real DC pension capital at the pension date s is displayed. In the fourth column the average real pension payout during retirement is displayed, this is calculated according to equation (3.20). In the fifth column the pension result is displayed. The sixth column displays the real pension capital according to the reference value ${}^r_aW_t^i$ used in the denominator of the definition for the pension result in DC schemes. The seventh column displays the real pension payout ${}^r_aQ_t^i$ from equation (4.1) in case of the pension result based on the risk free rate, the pension result based on a constant rate and the pension result based on the inflation rate. The seventh column displays ${}^rBS_t^i$ from equation (4.3) in case of the pension result based on non indexed pension entitlements. The seventh column displays ${}^rBS_t^i \cdot \Pi_t^i$ in case of the pension result based on instant indexed pension entitlements. The last column displays the average associated replacement ratio. For all the values in this table a risk aversion of $\gamma = 5$ is assumed.

Table 4.2: **Basic information financial market** ($\gamma = 5$)

Variable	${}^1R^i_t$	R^i_t	${}^0R^i_t$	π^i_t	dS^i_t
μ	0.0247	0.0267	0.0229	0.0198	0.0682
σ	0.0324	0.0306	0.0331	0.0162	0.1822

In Table 4.2 the basic information about the financial market is displayed assuming that the parameter of risk aversion $\gamma = 5$. In the first row respectively the mean of a one year maturity bond ${}^1R^i_t$ is displayed, the mean of the return of interest rate strategy from section 3.4.2 R^i_t, the mean of the short rate ${}^0R^i_t$, the mean of the inflation rate π^i_t and the mean of the stock return dS^i_t are displayed. In the second row the variances of the same parameters are displayed.

Table 4.3: **Risk attitude and the life-cycle investment strategy**

Pension result $\gamma = 7$

Percentile	Annuity	${}^rW^i_s$	Payout	Pension Result	${}^r_5W^i_s$	Payout	RR
95	15.927	256878	15679	1.172	0	13380	0.282
90.8	9,571	632810	49384	1.000	0	49402	0.890
50	19.691	285874	15189	0.523	0	29027	0.274
5	16.123	221725	13994	0.221	0	63189	0.252
1	13.328	305750	18914	0.148	0	127620	0.341

Pension result $\gamma = 3$

Percentile	Annuity	${}^rW^i_s$	Payout	Pension Result	${}^r_5W^i_s$	Payout	RR
95	18.114	590773	31926	1.754	0	18206	0.575
75	12.400	348979	21910	1.000	0	21919	0.395
50	16.028	483651	41654	0.656	0	63518	0.750
5	15.355	371533	19171	0.238	0	80512	0.345
1	9.321	270880	18758	0.147	0	127690	0.338

Pension result $\gamma = 1$

Percentile	Annuity	${}^rW^i_s$	Payout	Pension Result	${}^r_5W^i_s$	Payout	RR
95	17.133	493417	47783	3.017	0	15840	0.861
60.05	12.404	237516	38682	1.000	0	38687	0.697
50	18.857	402904	16312	0.798	0	20448	0.294
5	11.417	571960	50432	0.213	0	236911	0.909
1	16.193	282790	10303	0.132	0	78339	0.186

In Table 4.3 the results of the pension result based on instant indexed pension entitlements are displayed using different values for the risk aversion parameter γ. It can be compared with the last tabular of Table 4.1 i.e. the pension result based on instant indexed pension entitlements. In Table 4.3 the first tabular displays the results when the risk aversion parameter $\gamma = 7$ is applied, in the second tabular $\gamma = 3$ is applied. In the last tabular $\gamma = 1$ is applied. In each tabular the same set up is used as in Table 4.1 regarding the columns. We only added the first percentile here in the last rows to stress the behavior of the life cycle in bad weather scenarios.

Table 4.4: **Pension result constant percentage premium** ($\gamma = 5$)

Percentile	Annuity	${}^{r}W_s^i$	Payout	Pension Result	${}_5^rW_s^i$	Payout	RR
95	13.350	248757	19223	1.541	0	12476	0.346
50	15.757	294604	15251	0.639	0	23871	0.275
78.7	23.141	291994	19470	1.000	0	19467	0.351
5	22.485	261628	14557	0.240	0	60552	0.262

In Table 4.4 the results of the pension result based on instant indexed pension entitlements are displayed when using a constant premium percentage instead of the used premium ladder in Table 4.1. We chose a constant premium percentage of 15 percent. It is chosen such that the total amount of payed premium resembles the total amount of payed premium when the 4 percent premium ladder is used. Table 4.4 can be compared with the last tabular of Table 4.1 i.e. the pension result based on instant indexed pension entitlements. The set up of Table 4.4 is the same as the set up of Table 4.1 regarding the columns.

Table 4.5: **Pension result using a constant stock exposure** ($\gamma = 5$)

Percentile	Annuity	${}^{r}W_s^i$	Payout	Pension Result	${}_5^rW_s^i$	Payout	RR
95	23.010	165979	11680	1.216	0	9602	0.208
50	12.013	274041	18478	0.527	0	35054	0.220
89.35	9.571	576551	49443	1.001	0	49402	0.676
5	15.710	243639	12355	0.214	0	57631	0.146

In Table 4.5 the results of the pension result based on instant indexed pension entitlements are displayed when using a constant mix instead of the Merton life cycle. Here we assume that the parameter of risk aversion $\gamma = 5$. It can be compared with the last tabular of Table 4.1 i.e. the pension result based on instant indexed pension entitlements. The set up of Table 4.4 is the same as the set up of Table 4.1 regarding the columns.

Table 4.6: **Pension result fixed decrease payout during retirement period** ($\gamma = 5$)

Percentile	Annuity	${}^{r}W_s^i$	Payout	PR	${}_5^rW_s^i$	Payout	RR
95	13.460	247153	15188	1.332	0	11403	0.274
85.5	10.350	318180	22811	1.000	0	22815	0.411
50	19.916	238195	16084	0.572	0	28099	0.290
5	13.219	384491	24072	0.224	0	107492	0.434

In Table 4.6 the results for the pension result based on instant indexed pension entitlements are displayed when using a fixed decrease payout during the retirement period. Here we assume that the parameter of risk aversion $\gamma = 5$. The results in Table 4.6 can be compared with the last tabular of Table 4.1 i.e. the pension result based on instant indexed pension entitlements. The set up of Table 4.4 is the same as the set up of Table 4.1 regarding the columns.

Chapter 5

Jump diffusion processes

For the modeling of asset price fluctuations diffusion is the most common stochastic process that is used. The random component of diffusion is the Brownian motion process, this is a random process with continuous sample paths. But the property of being continuous is often not found in real life prices. Lots of assets show returns which are widely dispersed in their amplitude and manifest frequent large peaks corresponding to discontinuous 'jumps' in the price. This high variability is an often observed feature of financial asset returns. In statistical terms this results in heavy tails in the empirical distribution of returns; the tail of the distribution decays slowly at infinity and large moves have a significant probability of occurring. This leads to a poor representation of the distribution of financial asset returns by a normal distribution [15].

Jump diffusion models generically lead to highly variable returns with realistic tail behavior [15]. The strongest argument for using discontinuous models with jumps is the presence of sudden significant moves in the price. While diffusion models, in some specific cases, can generate heavy tails in the returns, they cannot generate sudden, discontinuous moves in prices. In a diffusion model tail events are the result of the accumulation of many small moves. So if one aims to build a model that captures the perception of risks with large unpredictable movements jump diffusion models are helpful. This chapter is rather theoretical and serves as an introduction on jump processes. The theorems and derivations will be applied in the next chapters.

5.1 Jump Processes

As mentioned before Brownian motions have continuous sample paths and therefor large jumps are not allowed. Jump processes have discontinuous sample paths and therefore they allow for large sudden moves in the underlying price process. We assume that the shocks in the prices process arrive randomly. Let $\tau_1 < \tau_2 < ...$ be the arrival times of the shocks. We assume that $\lim_{n\to\infty} \tau_n = +\infty$, meaning that the number of shocks in a finite time interval is finite, so we do not observe extreme explosions. Define the inter arrival times of the shocks by:

$$T_n = \tau_n - \tau_{n-1},\ n \geq 1$$

with the assumption that $S_0 := 0$.

5.1.1 Poisson Process

If the sequence of inter arrival times $(T_n)_{n\geq 1}$ is an i.i.d sequence of exponential distributed random variables with mean $\frac{1}{\lambda}$, then the stochastic process $\{N_t \text{ with } t \geq 0\}$, which is defined as:

$$N_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_n \leq t\}}$$

for $t \geq 0$, is called a homogeneous Poisson process with intensity λ. The prototype of jump processes is the Poisson process $\{N_t \text{ with } t \geq 0\}$.

Definition 5.1. *Poisson Process*
The Poisson process has stationary and independent increments. It starts at $t_0 = 0$ with $N(t_0) = 0$. It has Poisson distributed increments. We denote the intensity rate by λ with $\lambda \in (0, \infty)$. Then the probability that $N(t) = n$ is given by:

$$\mathbb{P}(N(t) = n) = \tfrac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, 2, ...$$

and

$$E[N_t] = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t) = \lambda t. \tag{5.1}$$

From equation (5.1) we deduce that:

$$E[N_t - \lambda t] = 0$$

since $N_t - \lambda t$ has independent increments, we get the following proposition:

Proposition 5.2. *let $\mathscr{F} := \sigma(N_s : s \in [0, t])$, $t \in \mathbb{R}^+$ denote the filtration generated by the Poisson process $(N_t)_{t\geq 0}$. The compensated Poisson process*

$$(N_t - \lambda t)$$

is a Martingale with respect to $\mathscr{F}_s \subset \mathscr{F}$.

5.1.2 Compound Poisson Process

The Poisson process defined above is a starting point, but it can only jump up by unity when we use the Poisson process. This is not a suitable model for simulating the nominal interest rate with random jumps. First we would like to have a range of possible jump amplitudes. We can capture this by using a compound Poisson process.

To define the Compound Poisson process we assume that $T = (T_n)_{n\geq 1}$ is a given i.i.d sequence of exponential distributed random variables with mean $\frac{1}{\lambda}$. Then the cumulative sum

$$\tau_n = \sum_{j=1}^{n} T_j \; n \geq 1$$

are the arrival times of the shocks in a homogeneous Poisson process.

Definition 5.3. *Compound Poisson processes*
A compound Poisson process with intensity $\lambda > 0$ is a stochastic process J_t which we can define in two equivalent representations

1. *Infinite sum representation:*

$$J_t = \sum_{n=1}^{\infty} Y_n \mathbb{1}_{\{\tau_n \leq t\}}.$$

2. *Stochastic sum representation:*

$$J_t = \sum_{i=1}^{N_t} Y_i \tag{5.2}$$

where jumps sizes Y_i are i.i.d according to a given distribution with mean μ_Y and variance σ_Y^2. $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ independent from $(Y_i)_{i\geq 1}$. The convention is always that if $N_t = 0$, i.e. if the sum is empty, then $J_t = 0$.

In particular we note that the increment of the jump size

$$\Delta J_t := J_t - J_{t-}, \quad t \geq 0$$

of $(J_t)_{t\geq 0}$, at time t, is given by the relation:

$$\Delta J_t = Y \Delta N_t, \quad t \geq 0$$

where

$$\Delta N_t := N_t - N_{t-} \in \{0, 1\}, \quad t \geq 0$$

denotes the jump size of the standard Poisson process $(N_t)_{t\geq 0}$, and N_{t-} is the left limit

$$N_{t-} := \lim_{s \to t} N_s, \quad t > 0.$$

We consider the process J_t, $t \geq 0$, with $J_0 := 0$ and we derive the expectation of J_t with respect to the risk neutral measure as follows:

$$\begin{aligned}
E_{t_0}^{\mathbb{Q}}[J_t] &= E_{t_0}^{\mathbb{Q}}\left[\sum_{n=1}^{\infty} Y_n \mathbb{1}_{\{\tau_n \leq t\}}\right] \\
&= \sum_{n=1}^{\infty} E^{\mathbb{Q}}\left[Y_n \mathbb{1}_{\{\tau_n \leq t\}}\right] \\
&= \sum_{n=1}^{\infty} E^{\mathbb{Q}}\left[Y_n\right] \mathbb{P}(\tau_n \leq t) \\
&= \lambda t \mu_Y.
\end{aligned}$$

Since the compensated compound Poisson process also has independent increments we have the following proposition:

Proposition 5.4. *The compensated compound Poisson process*

$$M_t := \{\sum_{i=1}^{N(t)} Y_i - \lambda t \mu_Y\}, \quad t \geq 0 \tag{5.3}$$

is a martingale. In differential notation this is written as:

$$E_{t_0}^{\mathbb{Q}}[Y dN(t) - \lambda \mu_Y dt] = 0$$

or equivalently:

$$E_{t_0}^{\mathbb{Q}}[Y dN(t)] = \lambda \mu_Y dt. \tag{5.4}$$

Proof.

$$E[M_t|M_s] = E[\sum_{i=1}^{N(t)} Y_i - \lambda t \mu_Y |M_s] = E[\sum_{i=1}^{N(t)} Y_i|M_s] - \lambda s \mu_Y = \sum_{i=1}^{N(s)} Y_i - \lambda s \mu_Y = M_s.$$

□

5.1.3 Stochastic calculus with Jump processes

Based on the relation

$$dJ_t = Y dN_t$$

and given that $\{N_t = n\}$, i.e. the n jump sizes of $(J_t)_{t\geq 0}$ on $[0, T]$, are independent random variables which are distributed on $\mathbb{R}$ according to $\nu(dx)$. The next proposition allows us to compute the moment generating function of the increment $J_T - J_t$ [17]:

Proposition 5.5. *For any $t \in [0, T]$ we have:*

$$E[e^{a(J_T - J_t)}] = \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(e^{ax} - 1)\nu(dx)\right)$$

with $a \in \mathbb{R}$.

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Y_i)_{i\geq 1}$ $\forall a \in \mathbb{R}$ we have, by conditioning on the value of $N_T - N_t = n$,

$$\begin{aligned}
E[\exp(a(J_T - J_t))] &= E\left[\exp\left(a \sum_{i=N_t+1}^{N_T} Y_i\right)\right] \\
&= E\left[\exp\left(a \sum_{i=1}^{N_T - N_t} Y_i\right)\right] \\
&= \sum_{n=0}^{\infty} E\left[\exp\left(a \sum_{i=1}^{n} Y_i\right) |(N_T - N_t) = n\right] \mathbb{P}((N_T - N_t) = n) \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(T-t)^n E\left[\exp\left(a\sum_{i=1}^{n} Y_i\right)\right] \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(T-t)^n \prod_{i=1}^{n} E\left[e^{aY}\right] \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(T-t)^n (E\left[e^{aY}\right])^n \\
&= \exp\left(\lambda(T-t))(E\left[e^{aY}\right] - 1\right) \\
&= \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(e^{ax} - 1)\nu(dx)\right).
\end{aligned}$$

Since the probability distribution of $\nu(dx)$ of Y satisfies:

$$E[\exp(aY)] = \int_{-\infty}^{\infty} e^{ax}\nu(dx)$$

and

$$\int_{-\infty}^{\infty} \nu(dx) = 1.$$

$\square$

From this moment generating function we can compute the variance of J_t for fixed t.

$$\begin{aligned} E[J_t^2] &= \frac{\partial^2}{\partial a^2} E\left[e^{aJ_t}\right] |_{a=0} \\ &= \lambda t \int_{-\infty}^{\infty} x^2\nu(dx) + (\lambda(T-t))^2 \left(\int_{-\infty}^{\infty} x\nu(dx)\right)^2 \\ &= \lambda t E[|Y|^2] + (\lambda t E[|Y|])^2 \end{aligned}$$

which yields:

$$var(J_t) = \lambda t E[|Y^2|]$$

so it follows that

$$\begin{aligned} E[(dJ_t)^2] &= \lambda E[|Y|^2]dt \\ &= \lambda(var(Y) + (E[Y])^2)dt \\ &= \lambda(\sigma_Y + \mu_Y^2)dt. \end{aligned} \tag{5.5}$$

A common approach to stochastic differential equation which include jump processes is the approach of Ahn and Thompson [21] and Baz and Das [20]. Here a linearization technique is used for the moment generating function of the jump process J_t which includes a two-term Taylor-series approximation. We call this the standard linearization,

$$E[\exp(J_t)] \approx E_t\left[1 + J_t + \frac{1}{2}J_t^2\right] = 1 + \lambda t E[Y] + \frac{1}{2}\left(\lambda t E[|Y|^2] + \lambda^2 t^2 \left(E[|Y|]\right)^2\right). \tag{5.6}$$

The last term $\lambda^2 t^2 \left(E[|Y|]\right)^2$ is negligible. This linearization technique gives a closed-form approximation of the bond price.

Chapter 6

Extending the financial market model with a jump process

In this chapter we extend the KNW model by adding jumps to the bond prices and with that to the interest rate process. We derive the bond prices with a added jump process. However affine term structure models with a jump process in the bond prices do not always provide an analytical solution. Solutions to the partial difference differential equation (PDDE) for bond prices implied by jump-diffusion processes must often be found by solving an approximate PDDE. Here we use the standard linearization technique of the bond PDDE. Affine jump-diffusion models make the same functional form assumptions on the drift and the volatility as in the case without jumps. In addition, functional form assumptions are needed for the jump intensities and the distribution of the jump sizes conditional on information 'right before' the jump. The first assumption that we make is about the intensity of the Poisson process. We assume that the (stochastic) intensity λ of the Poisson process is affine. We will take $\lambda(\boldsymbol{X_t}) = \lambda$. We can note that the standard linearization does not require explicit assumptions about the distribution of jump sizes. We do assume that the jump sizes $\boldsymbol{Y_i}$ are iid variables with mean $\boldsymbol{\mu_Y}$ and variance $\boldsymbol{\sigma_Y}$.

6.1 Solving the bond price for extended KNW

In order to add jumps in the nominal interest rate of KNW we have to define the bond price as follows:

$$P_{t,N}(\boldsymbol{X_t} + \boldsymbol{J_t}) = e^{A(N)+\boldsymbol{B(N)}(\boldsymbol{X_t}+\boldsymbol{J_t})}. \tag{6.1}$$

Where $\boldsymbol{J_t}$ is the added jump.

In general, using the fundamental pricing equation given maturity N and assuming that all the time dependence of the bond price comes through the state variables X_t and J_t, We can rewrite the fundamental pricing equation using Ito's lemma. Recall that the fundamental pricing equation is given as in (6.2):

$$E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\right] - \left(\frac{1}{P_t}\frac{\partial P(N,t)}{\partial N} + R_t\right) dt \quad = \quad -E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\frac{d\phi_t}{\phi_t}\right] \tag{6.2}$$

When using the standard linearization technique following Ahn and Thompson [21] and Baz and Das [20] we can follow the approach we used to determine the bond price in the original KNW model. This technique can produce an exact solution for the bond prices with respect to an approximate PDDE using a two-term Taylor-series approximation. We rewrite the components $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\right]$ and $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\frac{d\phi_t}{\phi_t}\right]$ of equation (6.2) with added jumps.

For the component $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P_t}\right]$ of equation (6.2) we find:

$$
\begin{aligned}
&E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\right]\\
&= E_t^{\mathbb{Q}}\left[\frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{dX_t} + \frac{1}{P}\frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t^2}}(\boldsymbol{dX_t})^2 + \frac{1}{P}\left(P(\boldsymbol{X_t}+\boldsymbol{Y}) - P(\boldsymbol{X_t})\right)dN_t\right]\\
&= E_t^{\mathbb{Q}}\left[\frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{dX_t} + \frac{1}{P}\frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2}(\boldsymbol{dX_t})^2 + \left(e^{\boldsymbol{B(N)'Y}} - 1\right)dN_t\right]\\
&= E_t^{\mathbb{Q}}\left[\frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{1}{P}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2}\boldsymbol{\sigma_X}^2 dt + \frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma_X}\boldsymbol{dZ_t} + \left(e^{\boldsymbol{B(N)'Y}} - 1\right)dN_t\right]\\
&= \frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{1}{P}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2}\boldsymbol{\sigma_X}^2 dt + E_t^{\mathbb{Q}}\left[\left(e^{\boldsymbol{B(N)'Y}} - 1\right)dN_t\right]
\end{aligned} \tag{6.3}
$$

For the component $E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\frac{d\phi_t}{\phi}\right]$ of equation (6.2) we find:

$$
\begin{aligned}
&E_t^{\mathbb{Q}}\left[\frac{dP_t}{P}\frac{d\phi_t}{\phi}\right]\\
&= E_t^{\mathbb{Q}}\left[\frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{1}{P}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2}(\boldsymbol{\sigma_X}^2)dt + \frac{1}{P}\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma_X}\boldsymbol{dZ_t} + \left(e^{\boldsymbol{B(N)'Y}} - 1\right)dN_t\right]\\
&\cdot E_t^{\mathbb{Q}}\left[(-R_t dt - (\boldsymbol{\Lambda_t})\boldsymbol{dZ_t})\right]\\
&= \frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma'_X}\boldsymbol{\Lambda_t} dt
\end{aligned} \tag{6.4}
$$

Let M be the compensated Poisson process $dM_t = dN_t - \lambda dt$ as in equation (5.3). Intuitively, the compensated Poisson process is a centered version of the Poisson process because we are taking out the conditional mean change λdt. This leaves us with a mean 0 shock process dM, similar to the Brownian motion $\boldsymbol{dZ_t}$. Using the second order Taylor approximation we get:

$$
\begin{aligned}
&E_t^{\mathbb{Q}}[\left(e^{(\boldsymbol{B(N)'Y})} - 1\right)dN_t]\\
&\approx E_t^{\mathbb{Q}}\left[-\boldsymbol{B(N)'Y}dN_t + \frac{1}{2}(\boldsymbol{B(N)'Y})^2 dN_t\right]\\
&= E_t^{\mathbb{Q}}\left[\left(-\boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)}\right)dN_t\right]\\
&= E_t^{\mathbb{Q}}\left[\lambda\left(-\boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)}\right)dt + \left(-\boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)}\right)dM_t\right]\\
&= \lambda\left(-\boldsymbol{B(N)'E[Y]} + \frac{1}{2}\boldsymbol{B(N)'E[Y]E[Y]'B(N)}\right)dt
\end{aligned} \tag{6.5}
$$

Plugging equation (6.5) into (6.3) and plugging (6.3) and (6.4) into the fundamental pricing equation (6.2) we get, when we divide both sides by dt,:

$$
\begin{aligned}
&\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\mu_X} + \frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2}\boldsymbol{\sigma_X}^2 + \lambda\left(-\boldsymbol{B(N)'\mu_Y} + \frac{1}{2}\boldsymbol{B(N)'}\lambda(\boldsymbol{\sigma'_Y\sigma_Y} + \boldsymbol{\mu'_Y\mu_Y}))'\boldsymbol{B(N)}\right) - \frac{\partial P}{\partial N} - R_t P_t\\
&= \frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma'_X}\boldsymbol{\Lambda_t}
\end{aligned} \tag{6.6}
$$

We can note here that one can easily relax the assumptions that jump risk can be diversified away without considerably complicating the problem[19]. Following Das and Foresi [22], the pricing kernel with systematic jump risk follows:

$$\frac{d\phi_t}{\phi_t} \quad = \quad -R_t dt - \boldsymbol{\Lambda_t' dZ_t} + h\lambda dt - h dN_t.$$

Where the market price of (systematic) jump risk is constant and denoted by h. Now one can easily account for the systematic jump risk in the two-term Taylor approximation outlined in this section by substituting $\lambda(1-h)$ for λ. However relaxation of the assumption that jump risk is diversifiable becomes more costly in the context of estimating parameters.
We continue to rewrite the fundamental pricing equation (6.2) with jumps. Again we use for equation (6.6) that:

$$\begin{aligned}
\frac{1}{P_t}\frac{\partial P_t}{\boldsymbol{\partial X_t}} &= \boldsymbol{B(N)} \\
\frac{1}{P_t}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2} &= \boldsymbol{B(N)B(N)'} \\
\frac{1}{P_t}\frac{\partial P_t}{\partial N} &= -\dot{A}(N) - \boldsymbol{\dot{B}(N)'X_t}
\end{aligned}$$

where we use a dot notation for the derivative of $A(N)$ and $\boldsymbol{B(N)}$ with respect to the maturity N. With this, equation (6.6) and together with the formulas for the economic parameters from KNW we find the following equality:

$$\begin{aligned}
&\boldsymbol{B(N)'}\left(-\boldsymbol{KX_t} - \boldsymbol{\mu_Y}\right) + \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y'\mu_Y}))\boldsymbol{B(N)} - \dot{A}(N) \\
&-\boldsymbol{\dot{B}(N)'X_t} - (R_0 + \boldsymbol{R_1'X_t}) = \boldsymbol{B(N)'\Sigma_X'}(\boldsymbol{\Lambda_0} + \boldsymbol{\Lambda_1 X_t})
\end{aligned}$$

Now we set all the terms with X_t equal:

$$\begin{aligned}
&-\boldsymbol{B(N)'}\left(-\boldsymbol{KX_t}\right) - \boldsymbol{\dot{B}(N)'X_t} - \boldsymbol{R_1'X_t} = \boldsymbol{B(N)'\Sigma_X'\Lambda_1 X_t} \Rightarrow \\
&\boldsymbol{\dot{B}(N)} = -\boldsymbol{R_1} - \boldsymbol{B(N)'}(\boldsymbol{K} + \boldsymbol{\Sigma_X'\Lambda_1}) \Rightarrow \\
&\boldsymbol{B(N)} = \boldsymbol{M}^{-1}\left[e^{-\boldsymbol{M}N} - \boldsymbol{I}_{2\times 2}\right]\boldsymbol{R_1}
\end{aligned}$$

with $\boldsymbol{M} = (\boldsymbol{K} + \boldsymbol{\Sigma_X'\Lambda_1})$

As a next step we set all the non X_t terms equal:

$$\begin{aligned}
&\boldsymbol{B(N)'}\lambda\boldsymbol{E[Y]} - \dot{A}(N) + \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y'\mu_Y}))\boldsymbol{B(N)} \\
&- R_0 = \boldsymbol{B(N)'\Sigma_X'\Lambda_0} \Rightarrow \\
&\dot{A}(N) = -R_0 + \frac{1}{2}\boldsymbol{B(N)'}\left(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y'\mu_Y})\right)\boldsymbol{B(N)} \\
&- \boldsymbol{B(N)'\Sigma_X'\Lambda_0} - \boldsymbol{B(N)'}\lambda\boldsymbol{E[Y]} \Rightarrow \\
&A(N) = \int_0^N \dot{A}(s)ds. \qquad (6.7)
\end{aligned}$$

So for the bond price we find

$$P_{t,N}(\boldsymbol{X_t} + \boldsymbol{J_t}) = e^{A(N) + \boldsymbol{B(N)}(\boldsymbol{X_t} + \boldsymbol{J_t})}. \qquad (6.8)$$

With $A(N) = \int_0^N \dot{A}(s)ds$ and $\boldsymbol{B(N)} = \boldsymbol{M}^{-1}\left[e^{-\boldsymbol{M}N} - \boldsymbol{I}_{2\times 2}\right]\boldsymbol{R_1}$. The nominal zero coupon bond with duration $N = 0$ and payout 1 has a price $P_{t,T}(\boldsymbol{X_t}, \boldsymbol{J_t}) = 1$. This implies that $A(0) = 0$ and $\boldsymbol{B(0)} = \boldsymbol{0}_{1\times 2}$. The instantaneous nominal yield of a bond with duration zeros (cash) is defined as the derivative with respect to the maturity N of the bond price. Since $\boldsymbol{X_t}$ and $\boldsymbol{J_t}$ are not functions of the maturity N we get:

$$\begin{aligned} -d\ln P_{t,N}(\boldsymbol{X_t} + \boldsymbol{J_t}) &= -(\dot{A}(0) + \dot{\boldsymbol{B}}\boldsymbol{(0)}'(\boldsymbol{X_t} + \boldsymbol{J_t})) \\ &:= R_0 + \boldsymbol{R_1'}(\boldsymbol{X_t} + \boldsymbol{J_t}) \end{aligned}$$

so the instantaneous nominal yield of a bond with duration N is:

$$-d\ln(P + t, N(\boldsymbol{X_t} + \boldsymbol{J_t})) = -(\dot{A}(N) + \dot{\boldsymbol{B}}\boldsymbol{(N)'}(\boldsymbol{X_t} + \boldsymbol{J_t}))$$

6.1.1 Bond funds implementing constant duration

The KNW model is estimated using yields of bonds with different duration. The introduction of bond funds which implement constant duration is convenient to calculate these yields. In this subsection we will follow the approach of Draper[8] but we adapted by taking the added jumps into account. The development of the bond index can be derived by applying the Ito-Doeblin lemma to:[12]

$$P^{F_N}(\boldsymbol{X_t}) = P(\boldsymbol{X_t}, t, N) = e^{A(N) + \boldsymbol{B(N)'}(\boldsymbol{X_t} + \boldsymbol{J_t})}$$

holding N constant leads to:

$$dP^{F_N} = P^{F_N}\boldsymbol{B(N)'dX} + \frac{1}{2}P^{F_N}\boldsymbol{dX'B(N)B(N)'dX}P^{F_N} + P^{F_N}\left(P(\boldsymbol{X_t} + \boldsymbol{Y}) - P(\boldsymbol{X_t})\right)dN_t$$

we get for the relative bond return:

$$\begin{aligned} \frac{dP^{F_N}}{P^{F_N}} &= \boldsymbol{B(N)'}\left(-\boldsymbol{K}\boldsymbol{X_t}\right)dt + \left(\frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X})\right)dt \\ &+ \boldsymbol{B(N)'\Sigma_X' dZ_t} + \left(\boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)}\right)dN_t \end{aligned} \tag{6.9}$$

This equation together with stochastic discount factor $\frac{d\phi_t}{\phi_t} = -R_t dt + \boldsymbol{\Lambda_t' dZ_t}$ are consistent with the fundamental asset valuation equation if:

$$E^{\mathbb{Q}}\left[\frac{dP^{F_N}}{P^{F_N}} + \frac{d\phi_t}{\phi_t} + \frac{dP^{F_N}}{P^{F_N}}\frac{d\phi_t}{\phi_t}\right] = 0$$

Which yields to the restriction:

$$\begin{aligned} &\boldsymbol{B(N)'}\left(-\boldsymbol{K}\boldsymbol{X_t} + \lambda\boldsymbol{\mu_Y}\right) \\ &+ \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y'\mu_Y}))\boldsymbol{B(N)} \\ &- R_t - \boldsymbol{B(N)'\Sigma_X'\Lambda_t} = 0 \Rightarrow \\ &\boldsymbol{B(N)'}\left(-\boldsymbol{K}\boldsymbol{X_t} + \lambda\boldsymbol{\mu_Y}\right) \\ &+ \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y^2} + \boldsymbol{\mu_Y^2}))\boldsymbol{B(N)} = \\ &- R_t - \boldsymbol{B(N)'\Sigma_X'\Lambda_t} \end{aligned} \tag{6.10}$$

[2]Note that the fund's value index can not be determined using the instantaneous return of a bond with constant maturity. The instantaneous return does not take into account changes in the state variables.

If we substitute (6.10) into (6.9), this leads to the relative funds price dynamics equation:

$$\frac{dP^{F_N}}{P^{F_N}} = \left(\left(R_t + \boldsymbol{B(N)'\Sigma_X'\Lambda_t}\right) dt + \boldsymbol{B(N)'\Sigma_X' dZ_t} + \left(\boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)} \right) dN_t \right)$$

6.1.2 Exact Discretization of KNW

Exact discretization is possible by writing the whole model as a multivariate Ornstein-Uhlenbeck process. We need this exact discretization in order to simulate the model in Matlab.

$$\boldsymbol{d\Psi_t} = (\boldsymbol{\Theta_0} + \boldsymbol{\Theta_1\Psi_t})dt + \boldsymbol{\sigma_\Psi dZ_t} + \boldsymbol{\sigma_N} dN_t$$

with

$$\Psi' = \begin{bmatrix} \boldsymbol{X_t} \\ \ln(\Pi_t) \\ \ln(S_t) \\ \ln\left(P^{F_0}\right) \\ \ln\left(P^{F_N}\right) \end{bmatrix}.$$

We use the Ito-Doeblin theorem for the log inflation:

$$\begin{aligned} d\ln(\Pi) &= \frac{\partial \ln(\Pi)}{\partial \Pi} d\Pi + \frac{1}{2}\left(\frac{\partial^2 \ln(\Pi)}{\partial \Pi^2}\right)(d\Pi)^2 \\ &= (\pi_t dt + \boldsymbol{\sigma_\Pi' dZ_t}) - \frac{1}{2}\left(\pi_t dt + \boldsymbol{\sigma_\Pi' dZ_t}\right)^2 \\ &= (\pi_t - \frac{1}{2}\boldsymbol{\sigma_\Pi'\sigma_\Pi})dt + \boldsymbol{\sigma_\Pi' dZ_t}. \end{aligned}$$

We also use the Ito-Doeblin theorem for the log equity:

$$\begin{aligned} d\ln(S) &= \frac{\partial \ln(S)}{\partial S} dS + \frac{1}{2}\left(\frac{\partial^2 \ln(S)}{\partial S^2}\right)(dS)^2 \\ &= (R_t + \eta_S)dt + \frac{1}{2}\left((R_t + \eta_S)dt + \boldsymbol{\sigma_S' dZ_t}\right)^2 \\ &= \left(R_0 + \boldsymbol{R_1' X_t} + \eta_S - \frac{1}{2}\boldsymbol{\sigma_S'\sigma_S}\right) dt + \boldsymbol{\sigma_S' dZ_t}. \end{aligned}$$

The log wealth invested in a constant duration fund develops according to:

$$\begin{aligned} d\ln\left(P^{F_N}\right) &= \frac{\partial \ln\left(P^{F_N}\right)}{\partial P^{F_N}} dP^{F_N} + \frac{1}{2}\left(\frac{\partial^2 \ln\left(P^{F_N}\right)}{(P^{F_N})^2}\right)(dP^{F_N})^2 + \left(P(X_t + Y) - P(X_t)\right) dN_t \\ &= \left(R_t + \boldsymbol{B(N)'\Sigma_X'\Lambda_t} - \frac{1}{2}\boldsymbol{B(N)'(\Sigma_X'\Sigma_X)B(N)}\right) dt \\ &\quad + \boldsymbol{B(N)'\Sigma_X' dZ_t} + \left(-\boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)}\right) dN_t \end{aligned}$$

This implies that the discretized multivariate Ornstein-Uhlenbeck process, which we use to simulate the financial market model, is as follows:

$$\boldsymbol{d\Psi_t} = (\boldsymbol{\Theta_0} + \boldsymbol{\Theta_1\Psi_t})dt + \boldsymbol{\sigma_\Psi dZ_t} + \boldsymbol{\sigma_N} dN_t$$

with:

$$
\begin{aligned}
\boldsymbol{\Psi_t} &= \begin{bmatrix} \boldsymbol{X_t} \\ \ln(\Pi_t) \\ \ln(S_t) \\ \ln\left(P^{F_0}\right) \\ \ln\left(P^{F_N}\right) \end{bmatrix} \\
\boldsymbol{\Theta_0} &= \begin{bmatrix} \mathbf{0}_{2x1} \\ \delta_{0\pi} - \frac{1}{2}\boldsymbol{\sigma'_\Pi \sigma_\Pi} \\ R_0 + \eta_S - \frac{1}{2}\boldsymbol{\sigma'_S \sigma_S} \\ R_0 \\ R_0 + \boldsymbol{B(N)'\Sigma_X \Lambda_0} - \frac{1}{2}\boldsymbol{B(N)'(\Sigma'_X \Sigma_X) B(N)} \end{bmatrix} \\
\boldsymbol{\Theta_1} &= \begin{bmatrix} -\boldsymbol{K} & \mathbf{0}_{2\times 4} \\ \boldsymbol{\delta'_{1\pi}} & \mathbf{0}_{1\times 4} \\ \boldsymbol{R'_1} & \mathbf{0}_{1\times 4} \\ \boldsymbol{R'_1} & \mathbf{0}_{1\times 4} \\ \boldsymbol{R'_1} + \boldsymbol{B(N)'\Sigma'_X \Lambda_1} & \mathbf{0}_{1\times 4} \end{bmatrix} \\
\boldsymbol{\sigma_\Psi} &= \begin{bmatrix} \boldsymbol{\Sigma'_X} \\ \boldsymbol{\sigma'_\Pi} \\ \boldsymbol{\sigma'_S} \\ \mathbf{0}_{1\times 4} \\ \boldsymbol{B(N)'\Sigma'_X} \end{bmatrix} \\
\boldsymbol{\sigma_N} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \boldsymbol{B(N)'Y} + \frac{1}{2}\boldsymbol{B(N)'YY'B(N)} \end{bmatrix}
\end{aligned}
$$

6.2 Alternative approximation for bond PDDE

In the previous section we applied the standard linearization technique of the PDDE for the bond price in KNW following Baz and Das [20]. Although this approximations technique has proven to be close to numerical estimates, they suggest that the degree of accuracy of this approach is still an open question. Durham [19] provides an alternative closed-form approximation for the PDDE of the bond price. This alternative approximation incorporates the specific distributional assumptions for the jump process J more explicitly and employs two-term Taylor-series approximations after the expectation in the bond pricing equation has been taken.
Durham's research shows that, when using Gaussian assumptions for the jump size, the alternative approximation is clearly more accurate than the standard linearization according to Ahn and Thompson [21] and Baz and Das[20]. The higher accuracy of the alternative approximation is independent of how much the jumps contribute to the variance in the interest rate compared to the diffusion. In this section we will derive the PDDE of the bond price using the alternative approximation from Durham [19].

In the standard approximation we approximated $e^{(\boldsymbol{J_t})}$ by a two-term Taylor approximation. The alternative method of Durham [19] approximates the moment generating function of Y. Given the Gaussian assumptions of Y and the independence of the Poisson process and the jump size we can write:

$$E_t^{\mathbb{Q}}[\exp\big(\boldsymbol{B(N)'Y}-1\big)dN_t]=\lambda dt\exp\bigg(\boldsymbol{B(N)'\mu_Y}+\frac{1}{2}\boldsymbol{B(N)'(\sigma_Y'\sigma_Y)B(N)}-1\bigg).$$

Taking the two-term Taylor expansion we get:

$$\begin{aligned}
&\exp\bigg(\boldsymbol{B(N)'\mu_Y}+\frac{1}{2}\boldsymbol{B(N)'(\sigma_Y'\sigma_Y)B(N)}\bigg)=\\
&1-\boldsymbol{B(N)'\mu_Y}+\frac{1}{2}\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)}\\
&+\frac{1}{2}\left(\boldsymbol{B(N)'\mu_Y'\mu_Y B(N)}-\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)B(N)'\mu_Y}\right)\\
&+\frac{1}{2}\left(\frac{1}{4}(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})\right).
\end{aligned}$$

With the fundamental pricing equation we get:

$$\begin{aligned}
&\frac{\partial P_t}{\boldsymbol{\partial X_t}}(-\boldsymbol{KX_t}-\lambda\boldsymbol{\mu_Y})+\frac{1}{2}\frac{\partial^2 P_t}{\boldsymbol{\partial X_t}^2}(\boldsymbol{\sigma_X}^2+\lambda(\boldsymbol{\sigma_Y}^2+\boldsymbol{\mu_Y}^2))-\frac{1}{2}\frac{\partial^3 P_t}{\boldsymbol{\partial X_t}^3}(\boldsymbol{\mu_Y\sigma_Y}^2)\\
&+\frac{1}{8}\frac{\partial^4 P_t}{\boldsymbol{\partial X_t}^4}\boldsymbol{\sigma_Y}^4-\frac{\partial P}{\partial N}-R_tP_t=\frac{\partial P_t}{\boldsymbol{\partial X_t}}\boldsymbol{\sigma_X\Lambda_t}\Rightarrow\\
&\boldsymbol{B(N)'}(-\boldsymbol{KX_t}-\lambda\boldsymbol{\mu_Y})+\frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\sigma_X}^2+\lambda(\boldsymbol{\sigma_Y}^2+\boldsymbol{\mu_Y}^2))\boldsymbol{B(N)}-\\
&\frac{1}{2}\boldsymbol{B(N)'\sigma_Y^2B(N)B(N)'\mu_Y}+\frac{1}{8}\boldsymbol{B(N)'\sigma_Y^2B(N)B(N)'\sigma_Y^2B(N)}\\
&-(\dot{A}(N)+\boldsymbol{\dot{B}(N)X_t})-R_t\boldsymbol{X_t}=\boldsymbol{B(N)\sigma_X}(\boldsymbol{\Lambda_0}+\boldsymbol{\Lambda_1X_t}).
\end{aligned}$$

For $\boldsymbol{B(N)}$ we get again: $\boldsymbol{B(N)}=\boldsymbol{M}^{-1}\left[e^{-\boldsymbol{M}N}-\boldsymbol{I}_{2\times2}\right]\boldsymbol{R_1}$, with $\boldsymbol{M}=(\boldsymbol{K}+\boldsymbol{\Sigma_X'\Lambda_1})$. For $A(N)$ we get:

$$\begin{aligned}
\dot{A}(N)&=-R_0+\frac{1}{2}\left(\boldsymbol{\Sigma_X'\Sigma_X}+\boldsymbol{B(N)'}\lambda(\boldsymbol{\sigma_Y'\sigma_Y}+\boldsymbol{\mu_Y'\mu_Y})\right)\boldsymbol{B(N)}-\boldsymbol{B(N)'\Sigma_X'\Lambda_0}-\boldsymbol{B(N)'}\lambda\boldsymbol{\mu_Y}\\
&\quad-\frac{1}{2}\boldsymbol{B(N)'\sigma_Y^2B(N)B(N)'\mu_Y}+\frac{1}{8}\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)B(N)'\sigma_Y'\sigma_Y B(N)}\Rightarrow\\
A(N)&=\int_0^N\dot{A}(s)ds.
\end{aligned}$$

6.2.1 Bond funds implementing constant duration

Again we will follow the approach of Draper[8] but we adapted it for the added jump process. The development of the bond index can be derived by applying the Ito-Doeblin lemma to:

$$P^{F_N}(\boldsymbol{X_t})=P(\boldsymbol{X_t},t,N)=e^{A(N)+\boldsymbol{B(N)'X_t}+\boldsymbol{J_t}}$$

holding N constant leads to:

$$dP^{F_N}=P^{F_N}\boldsymbol{B(N)'dX}+\frac{1}{2}P^{F_N}\boldsymbol{dX'B(N)B(N)'dX}P^{F_N}+P^{F_N}(P(\boldsymbol{X_t}+\boldsymbol{Y})-P(\boldsymbol{X_t}))dN_t.$$

We get for the relative bond return:

$$
\begin{aligned}
\frac{dP^{F_N}}{P^{F_N}} &= \left(\boldsymbol{B(N)'}(-\boldsymbol{KX_t})\right) dt \\
&+ \left(\frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X})\right) dt + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y'\mu_Y}))\boldsymbol{B(N)}dt \\
&- \left(\frac{1}{2}\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)B(N)'\mu_Y}\right)\lambda dt \\
&+ \left(\frac{1}{8}(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})\right)\lambda dt \\
&+ \boldsymbol{B(N)'\Sigma_X' dZ_t} - \boldsymbol{B(N)'Y}\lambda dt. \qquad (6.11)
\end{aligned}
$$

This equation together with stochastic discount factor $\frac{d\phi_t}{\phi_t} = -R_t dt + \boldsymbol{\Lambda_t' dZ_t}$ are consistent with the fundamental asset valuation equation if:

$$
E^{\mathbb{Q}}\left[\frac{dP^{F_N}}{P^{F_N}} + \frac{d\phi_t}{\phi_t} + \frac{dP^{F_N}}{P^{F_N}}\frac{d\phi_t}{\phi_t}\right] = 0
$$

which yields to the restriction:

$$
\begin{aligned}
&\boldsymbol{B(N)'}(-\boldsymbol{KX_t} + \lambda\boldsymbol{\mu_Y}) \\
&+ \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y\mu_Y}))\boldsymbol{B(N)} \\
&+ \frac{1}{2}\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)B(N)'\mu_Y} + \frac{1}{8}(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)}) \\
&= -R_t - \boldsymbol{B(N)'\Sigma_X\Lambda_t}. \qquad (6.12)
\end{aligned}
$$

If we substitute (6.12) into (6.11) leads to the relative funds price dynamics equation:

$$
\frac{dP^{F_N}}{P^{F_N}} = \frac{1}{P^{F_N}}\left(\left(R_t + \boldsymbol{B(N)'\Sigma_X'\Lambda_t}\right)dt + \boldsymbol{B(N)'\Sigma_X' dZ_t} + (P(\boldsymbol{X_t} + \boldsymbol{Y}) - P(\boldsymbol{X_t}))dN_t\right).
$$

6.2.2 Exact discretization of KNW

The log wealth invested in a constant duration fund develops according to:

$$
\begin{aligned}
d\ln\left(P^{F_N}\right) &= \frac{\partial\ln\left(P^{F_N}\right)}{\partial P^{F_N}}dP^{F_N} + \frac{1}{2}\left(\frac{\partial^2\ln\left(P^{F_N}\right)}{(P^{F_N})^2}\right)(dP^{F_N})^2 + (P(\boldsymbol{X_t} + \boldsymbol{Y}) - P(\boldsymbol{X_t}))dN_t \\
&= \left(R_t + \boldsymbol{B(N)'\Sigma_X'\Lambda_t} - \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} + \lambda(\boldsymbol{\sigma_Y'\sigma_Y} + \boldsymbol{\mu_Y'\mu_Y}))\boldsymbol{B(N)}\right)dt \\
&+ \left(-\frac{1}{2}\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)B(N)'\mu_Y} + \frac{1}{8}(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})(\boldsymbol{B(N)'\sigma_Y'\sigma_Y B(N)})\right)dt \\
&+ \boldsymbol{B(N)'\Sigma_X' dZ_t} - \boldsymbol{B(N)'Y}dN_t.
\end{aligned}
$$

This implies that the discretized multivariate Ornstein-Uhlenbeck process, which we use to simulate the financial market model, is as follows:

$$
\boldsymbol{d\Psi_t} = (\boldsymbol{\Theta_0} + \boldsymbol{\Theta_1\Psi_t})dt + \boldsymbol{\sigma_\Psi dZ_t}
$$

with:

$$
\begin{aligned}
\boldsymbol{\Psi_t} &= \begin{bmatrix} \boldsymbol{X_t} \\ \ln(\Pi_t) \\ \ln(S_t) \\ \ln\left(P^{F_0}\right) \\ \ln\left(P^{F_N}\right) \end{bmatrix} \\
\boldsymbol{\Theta_0} &= \begin{bmatrix} \mathbf{0}_{2\times 1} \\ \delta_{0\pi} - \frac{1}{2}\boldsymbol{\sigma_\Pi'}\boldsymbol{\sigma_\Pi} \\ R_0 + \eta_S - \frac{1}{2}\boldsymbol{\sigma_S'}\boldsymbol{\sigma_S} \\ R_0 \\ R_0 + \boldsymbol{B(N)'\Sigma_X\Lambda_0} - \boldsymbol{B(N)'\mu_Y} - \frac{1}{2}\boldsymbol{B(N)'}(\boldsymbol{\Sigma_X'\Sigma_X} - \lambda(\boldsymbol{\sigma_Y}^2 + \boldsymbol{\mu_Y}^2))\boldsymbol{B(N)} \\ +\frac{1}{3}\boldsymbol{B(N)}^3\boldsymbol{\mu_Y\sigma_Y}^2 + \frac{1}{8}(\boldsymbol{B(N)\sigma_Y})^4 \end{bmatrix} \\
\Theta_1 &= \begin{bmatrix} -\boldsymbol{K} & \mathbf{0}_{2\times 4} \\ \boldsymbol{\delta_{1\pi}'} & \mathbf{0}_{1\times 4} \\ \boldsymbol{R_1'} & \mathbf{0}_{1\times 4} \\ \boldsymbol{R_1'} & \mathbf{0}_{1\times 4} \\ \boldsymbol{R_1'} + \boldsymbol{B(N)'\Sigma_X'\Lambda_1} & \mathbf{0}_{1\times 4} \end{bmatrix} \\
\boldsymbol{\sigma_\Psi} &= \begin{bmatrix} \boldsymbol{\Sigma_X'} \\ \boldsymbol{\sigma_\Pi'} \\ \boldsymbol{\sigma_S'} \\ \mathbf{0_{1\times 4}} \\ \boldsymbol{B(N)'\Sigma_X'} \end{bmatrix}
\end{aligned}
$$

6.3 Jump size distributions

So far we did not discuss the choice of the distribution for the jump size in detail. Assumptions regarding the distribution of the jump size in general are critical when it comes to closed-form solutions, numerical estimates and closed-form approximations of PDE's of the bond price. To simulate a realistic behavior of interest rate we want the probability of big jumps to be less likely than the probability of small jumps. We also want to allow for negative jump sizes. Regularly used distributions for the jump sizes are:

1. Gaussian distribution
2. Exponential/Bernoulli distribution (Das and Foresi [22])
3. Uniform distribution
4. Bimodal Gaussian mixture distribution (Durham [19])

A common assumption in the literature is that jump J has an expected value of zero and follows a normal distribution. This is, however, not the most logical choice since it does not provide meaningful movements compared to diffusion, i.e. it does not provide significant jumps different from diffusion only models. This standard normal assumption is often just made for simplicity.

Durham[19] elaborates on an exponential distribution were the jump sign follows a Bernoulli distribution. This assumption produces a analytical closed-form solution for bond prices. Therefore, in contrast to Baz and Das [20], one can directly assess the relative accuracy of the numerical and closed-form approximations of the bond price with respect to an analytical solution. In other words, with this assumption one can test whether the numerical solutions to the exact

PDDE are more accurate than the closed-form solutions to an approximated PDDE. This is a nice feature to conclude about the use of the numerical solutions as a benchmark for accuracy. The exponential assumption for the jump size distribution is, however, more driven by the fact that it produces an analytical bond price solution rather than that is has theoretical notions about the true distribution of jump sizes in the interest rate.
Durham [19] concludes that both closed-form approximations, the standard and the alternative, give a higher accuracy with respect to the numerical solution when assuming an exponential distribution for the jump sizes instead of Gaussian assumptions on the jump size distribution. Durham [19] shows that the alternative closed-form approximation is clearly more accurate than the standard linearization technique on average and for each maturity point. Moreover he shows that on average the alternative closed-form approximation is more precise than the numerical solution independent of how much the jumps contribute to the variance in the short rate compared to the diffusion. This result is notably driven by greater accuracy for bonds with a maturity between one and 10, for higher maturities the numerical approximation is superior. This result generally questions the use of numerical solutions as a benchmark. That is, one cannot in principle be sure whether approximate solutions of the exact PDDE are more precise than exact solutions of an approximate PDDE.

The assumption in Das and foresi [22] that the absolute value of the jump size follows an exponential distribution and that the jump sign follows a Bernoulli distribution is a considerable improvement compared to the Gaussian assumption with respect to significant jumps. However, another desirable feature of any possible distribution for jump sizes would be that the modes are of considerable distance from the mean to allow for significantly higher interest rate movements compared to diffusion. Jumps can also have a symmetrical distribution around the means, which is also not feature of exponential distributions. Therefore, a useful alternative distribution would be the Bimodal distribution with jumps symmetrically distributed around modes of sufficient distance from the origin.This assumption for the jump size distribution is also elaborated by Durham. Here the average jump size is zero. The Bimodal distribution has the feature that positive and negative values cancel out when taking the average or calculating expectations.

Durham [19] concludes about the Bimodal jump size assumption that the alternative approximation consistently gives a slightly more precise result in estimating the term structure but does not provide significantly more accuracy than the standard linearization. The precision of both closed-form approximations seem to decrease with maturity. When jumps contribute more to the total variance of the short rate compared to the diffusion both closed-form approximations become somewhat less accurate but still the absolute degree of precision is roughly comparable. Therefor we will only derive the bond price in KNW using standard linearization of the bond PDDE here when assuming a Bimodal distribution for the jump sizes.

6.3.1 The Bimodal distribution

A multi-modal distribution is a continuous probability distribution with two or more modes. These modes appear as distinct peaks (local maxima) in the probability density function. A multi-modal distribution most commonly arises as a convex mixture of different unimodal distributions (i.e. distributions having only one mode). Note here that there is no one to one connection between the number of components in a mixture and the number of modes of the resulting density. The Gaussian distribution is a unimodal distribution. The multi-modal distribution based on Gaussian distributions is defined the following mixture:

$$f(y) = \sum_{i=1}^{n} w_i \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(y-\mu_i)^2}{2\sigma_i^2}\right) \tag{6.13}$$

Where $f(y)$ is the total density of the mixture, w_i is the weight assigned to the ith component Gaussian distribution. It holds that: $\sum_{i=1}^{n} w_i = 1$. The number of component distributions in the mixture is n, μ_i is the mean of the ith component distribution, and σ_i^2 is the variance of the ith component distribution.

For the distribution of the jump size we will use the specific case of a multi-modal distribution with two different modes ($n = 2$); the bimodal distribution. The probability distribution is defined as follows:

$$f(y) = \frac{w}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(y-\mu_1)^2}{2\sigma_1^2}\right) + \frac{(1-w)}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right) \tag{6.14}$$

We can deduce that:

$$\begin{aligned} E[Y] &= w\mu_1 + (1-w)\mu_2 \\ var(Y) &= \frac{w(\mu_1^2 + \sigma_1^2) + (1-w)(\mu_2^2 + \sigma_2^2)}{2} \end{aligned}$$

Since the standard linearization method does not require an explicit assumption about the jump size distribution, we can simply fill in $E[Y]$ and $var(Y)$ for respectively μ_Y and σ_Y^2. We can also make the intuitive assumption that the component distributions of Y are symmetric reflexions across the origin, with equal but opposite means and equal variances. Under this formulation $\mu_1 = -\mu_2$ and $\sigma_1 = \sigma_2$. The total expectation of the Bimodal variable will be equal to zero.

6.4 Jumps in the inflation rate

The KNW model is an affine two factor model. The two factors, or state variables influence both the interest rate and the inflation rate. The pension result is also dependent on both the interest rate and the inflation rate. The maintenance of purchasing power is highly influenced by the inflation rate, so the way in which the inflation rate is modeled has a big influence on the pension result. Therefor we will analyze what the impact is of adding jumps to the inflation rate as well. Let L_t be the jump for the inflation rate, let Y_2 be the variable for the jump size and let λ_2 be the expectation of the Poisson process for the jumps in the inflation rate. We adapt the formula for price index in the KNW model as follows:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \boldsymbol{\sigma_\Pi dZ_t} + \boldsymbol{Y_2} dN_t \tag{6.15}$$

Where $\boldsymbol{\delta_{1\pi}}, \boldsymbol{X_t}, \boldsymbol{L_t} \in \mathbb{R}^{\not{k}}$. We will only observe a change in the exact discretization part:

$$\begin{aligned} d\ln(\Pi) &= \frac{\partial \ln(\Pi)}{\partial \Pi} d\Pi + \frac{1}{2}\left(\frac{\partial^2 \ln(\Pi)}{\partial \Pi^2}\right)(d\Pi)^2 \\ &= ((\pi_t + \lambda_2)dt + \boldsymbol{\sigma'_\Pi dZ_t} + \boldsymbol{\delta'_{1\pi} Y_2} dM_t) - \frac{1}{2}\left((\pi_t + \lambda_2)dt + \boldsymbol{\sigma'_\Pi dZ_t} + \boldsymbol{\delta'_{1\pi} Y_2} dM_t\right)^2 \\ &= (\pi_t + \lambda_2 - \frac{1}{2}\boldsymbol{\sigma'_\Pi \sigma_\Pi})dt + \boldsymbol{\sigma'_\Pi dZ_t} + \frac{1}{2}\lambda_2 \boldsymbol{\delta'_{1\pi}}(\boldsymbol{\mu^2_{Y_2}} + \boldsymbol{\sigma^2_{Y_2}})dt + \boldsymbol{\delta'_{1\pi} \mu_{Y_2}} dM_t \end{aligned}$$

Chapter 7

Parameter estimation

In this chapter we investigate the parameter estimation of the jump size and the jump intensity. The investigation of the parameter estimation will be based on both historical data of the interest rate and the inflation rate. For the historical data of the interest rate we use the Euro one monthly OIS middel rate, see Figure 7.1. For the historical data of the inflation rate we use the monthly all-items HICP of NADJ, see Figure 7.2. In chapter 6 we have seen that when we add jumps to the bond price we get:

$$P_{t,N}(\boldsymbol{X_t} + \boldsymbol{J_t}) = e^{A(N)+\boldsymbol{B(N)}(\boldsymbol{X_t}+\boldsymbol{J_t})}.$$

The instantaneous nominal yield of a bond with duration zeros (cash) is defined as the derivative with respect to the maturity N of the bond price. Since $\boldsymbol{X_t}$ and $\boldsymbol{J_t}$ are not functions of the maturity N we get:

$$\begin{aligned} -d\ln P_{t,N}(\boldsymbol{X_t} + \boldsymbol{J_t}) &= -(\dot{A}(0) + \dot{\boldsymbol{B}}\boldsymbol{(0)}'(\boldsymbol{X_t} + \boldsymbol{J_t})) \\ &:= R_0 + \boldsymbol{R_1'}(\boldsymbol{X_t} + \boldsymbol{J_t}) \end{aligned}$$

so the equation for the interest rate will be as follows when we add a jump process:

$$-d\ln P_t(0) = R_0 + \boldsymbol{R_1'}(\boldsymbol{X_t} + \boldsymbol{J_t})$$

For the equation of the price index with jumps we found:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt + \boldsymbol{\sigma_\Pi dZ_t} + \boldsymbol{Y_2}dN_t$$

A method for estimating parameters for discrete, binned assets log returns is the multinomial maximum likelihood estimation. The performance of the multinomial maximum likelihood is superior to the method of the least squares [24]. We will estimate the parameters of the jump size and the jump frequency for both a Gaussian jump size assumption and a Bimodal jump size assumption. In case of the estimation of the interest rate we can easily extend to the two dimensions for J_t for which Y and N_t are assumed independent. Let μ^* be the estimated mean of the jumps then we get using a parameter estimation:

$$\begin{aligned} E\left[\boldsymbol{R_1'}\boldsymbol{J_t}\right] &= \lambda\mu^* t \Rightarrow \\ E\left[\boldsymbol{R_1^{(1)}}\boldsymbol{J_t^{(1)}} + \boldsymbol{R_1^{(2)}}\boldsymbol{J_t^{(2)}}\right] &= \lambda\mu^* t \Rightarrow \\ E\left[\boldsymbol{R_1^{(1)}}\boldsymbol{J_t^{(1)}}\right] + E\left[\boldsymbol{R_1^{(2)}}\boldsymbol{J_t^{(2)}}\right] &= \lambda\mu^* t \Rightarrow \\ E\left[\boldsymbol{R_1^{(1)}}\boldsymbol{J_t^{(1)}}\right] = E\left[\boldsymbol{R_1^{(2)}}\boldsymbol{J_t^{(2)}}\right] &= \frac{1}{2}\lambda\mu^* t \end{aligned}$$

here we assume that the entries of the jumps are symmetric. For the variance and the frequency of the Poisson process we have the same.

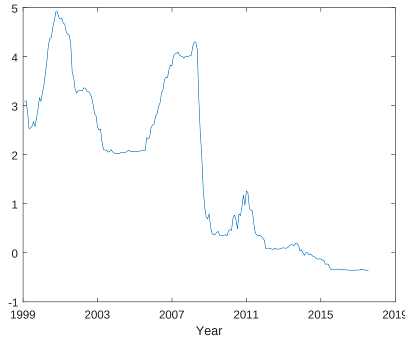


Figure 7.1: Euro monthly OIS middel rate

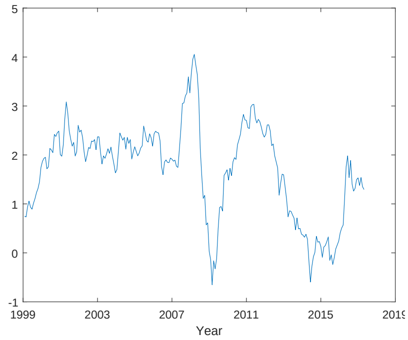


Figure 7.2: Monthly all-items HICP of NADJ

7.1 Probability distribution of jump diffusion with normal jump size

When we assume a Gaussian distribution for the jump size we can follow Floyed Hanson and Zongwu Zu [24]:

$$dr_t = (a - br_t)dt + \sigma dZ_t. \tag{7.1}$$

where $a, \sigma \geq 0$ and $b \in \mathbb{R}$. Using the Ito-Doeblin formula, we can immediately verify that:

$$r_t = r_0 e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) + \sigma \int_0^t e^{-b(t-s)} dZ_t \tag{7.2}$$

To get to the probability distribution of r_t we use the relationship between the Brownian motion and the drift with the right change of measure according to the first theorem of Girsanov[9].

Theorem 7.1. *Girsanov 1*
Let $L(t) \in \mathbb{R}^n$ be a stochastic process on the space $(\Omega, \mathscr{F}^n, \mathbb{P})$ such that:

$$dL(t) = a(t, \omega)dt + dZ_t.$$

with $L(0) = 0$, $\omega \in \Omega$, $0 \leq t \leq T < \infty$ and independent elements $Z_t^1, Z_t^2, ..., Z_t^n$. Define the process:

$$D(t) = \exp\left(-\int_0^t a(s,\omega)dZ_s - \tfrac{1}{2}\int_0^t a^2(s,\omega)ds\right) \quad \forall t \leq T.$$

Assume that $a(t, \omega)$ is adapted and the following condition (the Novikov condition) holds:

$$E_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^t a^2(s,\omega)ds\right)\right] < \infty.$$

Define a new measure $\mathbb{Q}$ on (Ω, F_t^n) equivalent to measure $\mathbb{P}$ (i.e. they agree on which sets in $\mathscr{F}^n$ have probability zero) , by setting:

$$dQ(\omega) = D_T(\omega)dP(\omega).$$

Then under the new measure $\mathbb{Q}$, the process $L(t)$ is a standard Brownian motion.

From here it follows directly that:

$$r_t \sim N\left(r_0 e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right), \frac{\sigma^2}{2b}\left(1 - e^{-2bt}\right)\right).$$

We call:

$$\begin{aligned}\mu_r &= r_0 e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right)\\ \sigma_r &= \frac{\sigma^2}{2b}\left(1 - e^{-2bt}\right).\end{aligned}$$

Knowing the distribution of r_t we can now use the theorem of Floyed Hanson and Zongwu Zu [24] to get the probability density of the interest rate process with jumps.

Theorem 7.2. *Let $\Delta J_t = \sum_{i=1}^{\Delta N_t} Y_i$ be a stochastic jump with $Y_i \sim N(\mu_Y, \sigma_Y)$ and N_t a Poisson random variable; $P(\Delta N_t = k) = \frac{e^{-\lambda \Delta t}(\lambda \Delta t)^k}{k!}$. For a return on a fixed income asset r_t the probability density for the jump-diffusion returns of r_t is:*

$$f(\Delta r_t) = \sum_{k=0}^{\infty} p_k(\lambda \Delta t)\phi\left(\Delta r_t; (r_0 e^{-b\Delta t} + \frac{a}{b}\left(1 - e^{-b\Delta t}\right) + \mu_Y \cdot k, \frac{\sigma^2}{2b}\left(1 - e^{-2b\Delta_t}\right) + \sigma_Y^2 \cdot k\right)$$

where $\Delta r_t \in \mathbb{R}, p_k(\lambda \Delta t) = \mathbb{P}(\Delta N_t = k)$ and ϕ is the normal density:

$$\phi(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

with μ the mean and σ^2 the variance.

Proof. Since

$$\Delta r_t = r_0 e^{-b\Delta t} + \frac{a}{b}\left(1 - e^{-b\Delta t}\right) + \sigma \int_0^{\Delta t} e^{-b(s)} dZs + J_t$$

And

$$\left(r_0 e^{-b\Delta t} + \frac{a}{b}\left(1 - e^{-b\Delta t}\right) + \sigma \int_0^{\Delta t} e^{-b(s)} dZ_t\right) \sim N\left(r_0 e^{-b\Delta t} + \frac{a}{b}\left(1 - e^{-b\Delta t}\right), \frac{\sigma^2}{2b}\left(1 - e^{-2b\Delta t}\right)\right)$$

we get for the distribution of Δr_t given that $\Delta N_t = k$:

$$\Delta r_t | \Delta N_t = k \sim N\left(r_0 e^{-b\Delta t} + \frac{a}{b}\left(1 - e^{-b\Delta t}\right), \frac{\sigma^2}{2b}\left(1 - e^{-2b\Delta t}\right)\right) + \sum^k N\left(\mu_Y, \sigma_Y^2\right).$$

We will first focus on $\sum^k N\left(\mu_Y, \sigma_Y^2\right)$. Since Y_i are normal i.i.d and independent of N_t we get that

$$\begin{aligned}P(\Delta J_t = j | \Delta N_t = k) &= P\left(\sum_{i=0}^k Y_i = j\right)\\ &= \sum_{i=1}^k \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \mu_Y)^2}{2\sigma_Y^2}\right).\end{aligned}$$

We use the fact that the sum of k independent normal random variables also has a normal distribution with the mean equal to $\mu_Y \cdot k$ and the variance equal to $\sigma_Y^2 \cdot k$. It follows that:

$$P(\Delta J_t = j|\Delta N_t = k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \mu_Y \cdot k)^2}{2\sigma_Y^2 \cdot k}\right).$$

Now we use the fact that the diffusion term and the jump term are independent normal random variables and we use again that the sum of l independent normal random variables also has a normal distribution where we sum the means and the variances of the separate normals. We then get for Δr_t given that $\Delta N_t = k$:

$$\Delta r_t|\Delta N_t = k \sim N\left(\Delta r_t; (r_0 e^{-b\Delta t} + \frac{a}{b}\left(1 - e^{-b\Delta t}\right) + \mu_Y \cdot k, \frac{\sigma_Y^2}{2b}\left(1 - e^{-2b\Delta_t}\right) + \sigma_Y^2 \cdot k\right)$$

The last step is to drop the assumption that $N_t = k$ and to multiply by the distribution of N_t. □

7.2 Probability distribution of jump diffusion with Bimodal jump size

The Bimodal distribution consists of the weighted sum of two normals. If we would sum k identical Bimodal random variables we can use the fact that we are summing k i.i.d random variables and get:

$$\begin{aligned} E[J|N_t = k] &= E\left[\sum^k Y_i\right] = kw\mu_{1Y} + k(1-w)\mu_{2Y} \\ var(J|N_t = k) &= var\left(\sum^k Y_i\right) = kw(\mu_{1Y}^2 + \sigma_{1Y}^2) + k(1-w)(\mu_{2Y}^2 + \sigma_{2Y}^2) \end{aligned}$$

The Bimodal distribution is not closed under convolution To get to the distribution of the sum of the distribution we need to use convolution. This means that the sum of Bimodal distributions is not a Bimodal distribution. using the convolution of Bimodal distributions (see appendix 9.5) we get:

$$\begin{aligned} f_Z(z) = &\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_{r_t}^2}} \exp\left(-\frac{(x - \mu_{r_t})^2}{2\sigma_{r_t}}\right) \\ &\cdot \frac{e^{-\lambda\Delta t}(\lambda\Delta t)^k}{k!} \left(\frac{w}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(z-x) - \mu_1)^2}{2\sigma_1^2}\right) + \frac{(1-w)}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{((z-x) - \mu_2)^2}{2\sigma_2^2}\right)\right) dx \end{aligned} \tag{7.3}$$

7.3 Multinomial maximum likelihood estimation

The multinomial maximum likelihood estimation of model parameters is justified for binned financial data [25]. This estimation method with discrete bins is used since the jumps diffusion process is a discontinuous process. Let n be the number of data points we wish to estimate. The main idea for this method is the following:

1. Step 1: sort the historical data into nb bins (b_i, b_{i+1}) for $i = 1 : b$ of the same length Δb. With that a histogram is constructed. We get the sample frequency $f_b^{(s)}$, for $1 : nb$.

2. Step 2: Get the theoretical jump-diffusion frequency with parameter vector x. Here $x = (\mu_y, \sigma_Y, \lambda)$ in the case of a Gaussian jump size assumption and $x = (\mu_{1Y}, \sigma_{1Y}, \mu_{2Y}, \sigma_{2Y}, \lambda)$ in the case of a Bimodal jump size assumption. We get:

$$f_b^{(jd)}(x) := (n-1)\int_{b_i}^{b_{i+1}} \phi^{(jd)}(\eta, x)d\eta$$

where $\phi^{(jd)}(\eta; x)$ is some jump-diffusion density in η.

3. Step 3: minimize the objective function $y(x)$:

$$y(x) := -\sum_{b=1}^{nb}\left[f_b^{(s)}\ln\Big(f_b^{(jd)}(x)\Big)\right]$$

The minimization MATLAB function $fmincon$ is used to get the optimal parameters x^*. We want to maximize the chance of r_t being inside a bin where we find many observation from the historical data. So whenever $f_b^{(s)}$ is high for bin b we want $f_b^{(jd)}(x)$ to be high as well. We use the following constraint:

$$\sigma_Y^2 \geq 0$$

We base this parameter estimation on the method of Hanson and Zhu [24] but we adapt it slightly. In the paper of Hanson and Zhu the multinomial maximum likelihood is used for the stock returns. Instead of the sample frequency of the returns $f_b^{(s)}$ they use sample frequency of the returns of the log prices. Since we are dealing here with interest rates which can have negative values we cannot use the log returns. We perform the multinomial maximum likelihood for both the interest rate and the inflation rate. The data contains 232 data points for the monthly interest rate from January 1999 up till and including may 2018. For the inflation the data set contains 229 monthly data points from January 1998 up till and including January 2018.

7.4 Results

We follow Synowiec [28] and set the mean μ_r of the diffusion part in $dr_t = \mu_r r_t dt + \sigma_r dZ_t + dJ_t$ equal to the mean of the historical data. We also set the mean and μ_π) of the diffusion part in $d\pi_t = \mu_\pi \pi_t dt + \sigma_{pi} dZ_t + dL_t$ equal to the mean of the historical data. The mean is divided by 100 since we want percentages. We set the variance of the diffusion (σ_r^2 and σ_π^2) equal to the variance of the historical data divided by 10000. We make this assumption since the KNW model is already calibrated using historical data, so the mean and the variance of the historical data are already been taken into account for the estimation of the KNW parameters. Therefore we estimate the jump parameters given the mean and the variance of the historical data to get parameter values that will match more with the KNW model. The mean and variance of respectively the historical data of the interest rate and the inflation rate are:

$$\mu_r = 0.017079397$$
$$\sigma_r^2 = 0.00027646$$
$$\mu_\pi = 0.017098$$
$$\sigma_\pi^2 = 0.000091$$

7.4.1 Results parameter estimation

In Table 7.1 we stated the results of the multinomial maximum likelihood estimation using a Gaussian assumption for the jump sizes. The first we do not limit the size of λ. We then find that the jump frequency λ is bigger than 50. This high number is similar to the results of Synowiec [28]. The jump sizes Y, however, do not really differ from the variance of the diffusion in this case. When we limit the amount of jumps per year we observe bigger jump sizes. Durham [19] uses estimates where the amount of jumps are either 10 or 16 jumps per year assuming a normal distribution for jump sizes, so we display the results for the likelihood as well for $\lambda \leq 10$. This amount of jumps is about 5 times smaller than the amount of jumps estimated assuming no upper bound for λ. The focus in the KNW model is on yearly data and the used time step is one year. When we observe the historical data in Figure 7.1 and 7.2, we would rather have a low jump frequency and a high jump mean since the data shows that a significant jump does not happen every year and if the event of a jump takes place the jump size is significant compared to the diffusion behavior. We could implement this by limiting the amount of jumps to less than 1 significant jump per year. We will chose $\lambda = \frac{2}{5}$, which corresponds to 2 big jumps once in 5 years. This choice is arbitrary but it is 25 times smaller than the previous upper bound and it serves the goal of testing whether jumps influence the pension result. The results of the multinomial maximum likelihood estimation are stated in the Table 7.1. In this table we can see that in general it holds that the lower λ the bigger the jump size is.

For the Bimodal jump size distribution the optimization program is very heavy since the program numerically computes two integrals, one for the convolution and one with respect to the bins. We can only get an answers if we set the bin size very low in this case. But this makes the estimation method very inaccurate. Here we can conclude that the multinomial maximum likelihood is not a suitable estimation method for a Bimodal assumption for jump sizes when using $fmincon$ in Matlab.

7.4.2 Results for the estimation error

When using $fmincon$ for negative log likelihood functions the Hessian matrix can be used for a confidence interval for the estimation. The square root of the diagonal of the inverse Hessian matrix gives an estimate for the error term. Diffusion models with jumps, however, have continuous-time dynamics, but the sampling in continuous time is not feasible. Local asymptotic normality is also an ongoing area of research for stochastic processes with jumps. No results for general jump-diffusions have been established so far [27]. Therefore we cannot use the Hessian method in this case. The multinomial maximum likelihood function that we have introduced here is an estimation based on discrete-time observations and is a pseudo-likelihood function.

We do, however, want to know the accuracy of the estimation. We will use a very basic verification in which we simulate a jump diffusion process where we chose the mean and the variance of the diffusion and drift part equal to the mean and the variance of the historical data. For the mean and the variance of the jump size and the jump frequency λ we will chose some arbitrary value. We run the multinomial maximum likelihood for a 1000 times to see if the program gives back similar values for the parameters of the jump as we had chosen. We set $\mu_Y = 0.02$, $\sigma_Y = 0.001$ and $\lambda = \frac{2}{5}$. We generate a data set based on a jump diffusion model with this chosen parameter values. When we run the multinomial maximum likelihood with respect to this generated data we find that the maximum likelihood method would always give us a jump frequency between 50 and 53 with a small mean of the jump sizes between 0.001 and 0.0008. The variances of the jump sizes which are of the order 10^{-8}. From this it seems that the multinomial maximum likelihood estimation is not so accurate for estimation jump diffusion parameters in a interest rate or inflation rate process since the data is based on a jump diffusion process where the jump frequency is approximately 125 times smaller and the mean jump size is at least 20 bigger. Also the variance was chosen much larger than the number that the multinomial maximum likelihood returns.

Another verification of the accuracy of the multinomial maximum likelihood estimation can be done by checking the estimated values with a grid search. We chose a search domain for all three variables of the jump process, so we chose $\lambda \in [\lambda_{lb}, \lambda_{ub}]$, $\mu_Y \in [\mu_{Ylb}, \mu_{Yup}]$ and $\sigma_Y \in [\sigma_{Ylb}, \sigma_{Yub}]$. We then construct a for loop which starts at respectively λ_{lb}, μ_{Ylb} and σ_{Ylb}. For each loop we add a step size Δ_λ, Δ_μ and Δ_σ to the start values of λ, μ and σ, such that we asses values within the chosen domains. So forinstance λ, will attain the grid points $\lambda_{lb} + \Delta_\lambda, \lambda_{lb} + 2\Delta_\lambda, ..., \lambda_{lb} + (n-1)\Delta_\lambda, \lambda_{lb} + n\Delta_\lambda, \lambda_{ub}$. We fill in these values in the likelihood function $f(\Delta r_t)$. Again we set the mean and the variance of the drift and diffusion part equal to the mean and the variance of the historical data. We calculate the likelihood estimator for all the grid points:

$$\hat{l}(\lambda, \mu_Y, \sigma_Y; x_i) = \frac{1}{n}\sum_{i=1}^{n} \ln(f(x_i|\lambda, \mu_Y, \sigma_Y))$$

Where n is the number of data point and x_i are the observations in the historical data. We check for which parameter values of λ, μ and σ the maximum likelihood estimator is the largest. For this grid search we find the same result as for the multinomial maximum likelihood method. The method always selects high jump frequencies in combination with small average jump sizes μ_Y.

Although we can conclude that the estimation method seems not reliable, our goal is to check whether the presence of any reasonable jump in the KNW model has an influence on the pension result. When we look at the historical data the values that the estimation method gives for Gaussian jump size assumptions, which are stated in Table 7.1, are not odd. We could still use them to draw a conclusion about the influence of adding jumps in the KNW model on the pension result. For the Bimodal jump size assumptions we will use values from literature. We follow Durham [19] who uses a jump frequency of 10. This will be discussed in the next section. Another motivation to use the values that we found with the multinomial maximum likelihood estimation is that jump size mean that Durham [19] uses coincides with the values that we found for the Gaussian jump size assumption in the case that we set an upper bound of 10 for the jump frequency λ in the multinomial maximum likelihood estimation.

7.4.3 The pension result based on the standard linearization and Gaussian jumps in the interest rate

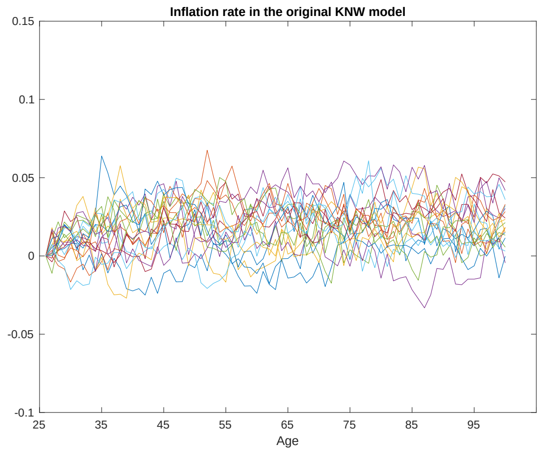
We implement the values of the parameter estimation in the extended KNW model for Gaussian jump size assumptions. We calculate the pension result in the case jumps are added to the interest rate and the inflation rate of the financial market model. We state the results in Tables 7.2 and 7.3. We will first look at the values of the pension result in the case that we add a jump process to the interest rate. In chapter 4 we can see in Table 4.1 that the pension result in the 95th percentile is equal to 1.329. In Table 7.2 we see, for $\lambda \leq \infty$, that the pension result in the 95th percentile is equal to 1.349. For $\lambda = 10$ we find that the pension result in the 95th percentile is equal to 1.400 and for $\lambda = \frac{2}{5}$ we see that the pension result is the 95th percentile is equal to 1.3934. So we see that adding a positive jump mean in the interest rate gives a slightly higher pension result in the 95th percentile. In Table 4.1 in chapter 4 we see that the 5th percentile is equal to 0.224 in Table 7.2 we see that the 5th percentile for all values of the jump frequency λ is not significantly different. The maximum deviation is also not significantly different compared to the values in Table 4.1 and the pension result is higher than 1 in the more or less the same percentile compared with Table 4.1.

We also analyzed the case in which the mean of the jump sizes are negative. The results are also stated in Table 7.2. We do not observe a significant difference in the values for the pension result. It seems that the influence of λ is higher than the influence of the mean of the jump size. When the jump sizes are normally distributed with mean zero we only see an impact on the pension result when we assume extreme variances. This is not likely according to the historical data. This coincide with Durhams [19] conclusion that jumps sizes which are normally distributed with mean zero do not contribute much with respect to diffusion, except when their variance is extremely high. When we use the alternative linearization instead of the standard linearization we do not see significant difference in the pension result compared to the standard linearization for all combinations of parameter values.

In general we see an effect of jumps in the interest rate process but this effect is not very significant compared to a scenario set with no jumps in the interest rate. When we look at the interest rate in the original KNW model, see Figure 7.3 then we see that a movement of for instance size 0.03 in the interest rate is not so uncommon. According to the data these are quite big jumps. The estimated mean of the jump size is of comparable size of these movements in the KNW model. So if we want to add a jump to the nominal interest rate which distinguishes from the diffusion in the KNW model we need to chose jumps with an average size bigger than this, but that would not be very likely according to the historical data. Movements of the same order of size as 0.03 are less common in the inflation rate in the original KNW model, see Figure 7.4. Therefore we see a bigger impact compared with the added jumps in the interest rate on the pension result when we use the values of the multinomial maximum likelihood for the jumps in the inflation.

Figure 7.3: **Interest rate in the original KNW model ($^0\boldsymbol{R}_t^i$)**

Figure 7.4: **Inflation rate in the original KNW model (π_t^i)**



7.4.4 The pension result based on the standard linearization and Gaussian jumps in the inflation rate

We add jumps to the inflation rate process in the KNW model and the results are stated in Table 7.3. In Table 4.1 in chapter 4 the pension result surpasses 1 in the upper 15% of the scenarios. In Table 7.3 we see, when use $\lambda = 49$ with the corresponding mean $\mu_y = 0.0008$, that the pension result does not even reach 1 in any scenario. For $\lambda = 10$ and $\mu_y = 0.0045$ the pension result is even lower and does not reach 0.5 in any scenario. For $\lambda = \frac{2}{5}$ and $\mu_y = 0.02$ we see in Table 7.3 that the pension result is equal to 1 in the 97.3 percentile. So in general positive values for μ_Y make the pension result lower. This is due to the fact the the inflation rate gets higher on average when we add positive jumps to it. A higher inflation rate means that a bigger investment return is needed to compensate for the loss of purchasing power that occurs with high inflation rates. When we look at negative values for μ_Y the pension result gets higher. In Table 7.3 we see that when we implement $\lambda = 49$ and $\mu_Y = -0.0008$ the pension result gets higher than 1 in 99% of the scenarios. For $\lambda = 10$ and $\mu_Y = -0.0045$ the pension result is even higher. For $\lambda = \frac{2}{5}$ we see that the pension result is higher than 1 in 44 percent of the scenarios. So in Table 7.3 we can see that adding jumps to the inflation rate has a significant influence on the pension result and highly depends on the choice of the parameter values. We can also conclude that adding Gaussian jumps to the inflation rate has a bigger influence on the pension result than when we add Gaussian jumps to the interest rate.

7.4.5 The pension based on the standard linearization and Bimodal jumps

In Table 7.4 we stated the results for adding jumps with Bimodal jump sizes to the interest rate and the inflation rate. We did not find estimates of the jump parameters, therefor we only used parameters from Durham [19]. When applying the assumption of Bimodal jump sizes with the restriction that the total expectation of the jump sizes is zero, see the approach in section 6.7.2, we see in Table 7.4 that the jumps have a significant influence on the pension result. In Table 4.1 in chapter 4 we see that the difference between the pension result in the 95th percentile and the 50th percentile is equal to 0.758. In Table 7.4 we see that the difference between the 95th percentile and the 50th percentile is equal to 1.817 for the interest rate and 1.829 for the inflation rate. This is more than twice as big compared to the difference in Table 4.1. The difference between the 50th percentile and the 5 in Table 4.1 is 0.347. In Table 7.4 we see that the difference between the 50th percentile and the 5 is 0.434 for the interest rate and 0.442 for the inflation rate. The 95 percentile is higher and the 5th percentile is lower when we add jumps with a Bimodal jump size to KNW. So overall we see that the Bimodal jumps have a significant influence on the pension result. The pension result gets less stable when we add jumps to KNW, i.e. the values are more spread. We also see that a Bimodal jump size assumption has a bigger influence on the pension result for the interest rate and a lower influence for the inflation rate than the Gaussian jumps size assumption. When we use the alternative linearization instead of the standard linearization we do not see significant difference in the pension result compared to the standard linearization.

7.4.6 Concluding remarks

We can conclude that in the current setting of the KNW model the addition of jumps has an influence the pension result. This influence is highly dependent on what parameter values we assume for the jump process. We also see that Bimodal jump sizes are conceptually more realistic than Gaussian jumps. Since Gaussian jumps can be simulated with a diffusion only model and Bimodal jumps allow for an average jump size of zero and it allows for positive and negative jumps. The impact of Bimodal jumps on the interest rate and the pension result is bigger assuming the same average jump size and comparable amounts of jumps per year, than the impact of a Gaussian jump size assumption. The impact of a Bimodal jump size assumption for the inflation rate influences the pension result less than a Gaussian jump size assumption. However, we see that the added jumps do not deviate so much form the movements of the diffusion in the original KNW model. The price movements of the interest rate and the inflation rate are very high compared to historical data in the original KNW model. This could be a result of the fact that the KNW model is estimated based on historical data already. The historical data show diffusion and jumps and the KNW parameter estimation incorporates this behavior in diffusion only. In fact, the total distribution of the combination of diffusion with Gaussian jumps is again Gaussian. So the combination of diffusion and Gaussian jumps can always be approximated by a diffusion only model with a higher volatility to cover for the jumps. What we then see is the result of the KNW estimation; higher volatility when compared to market data. While what we aim to see is small movements at most time points and whenever a jump occurred, big movements. A Bimodal jump size assumption for the interest rate an the inflation rate seems the most reasonable assumption. This assumption has a significant influence on the pension result. The only issue here is to find a suitable parameter estimation technique that can be applied to a Bimodal jump size assumption.

Table 7.1: **Results parameter estimation with Gaussian jump size**

Process	μ_Y	σ_Y	$\lambda \leq \infty$	number of bins
R_t	0.00096179	1,11E-09	50.66	56
Π_t	0.000817538	1,06E-07	49.74	56

Process	μ_Y	σ_Y	$\lambda \leq 10$	number of bins
R_t	0.004542874	1.10E-08	10	56
Π_t	0.004511779	1.92E-08	10	56

Process	μ_Y	σ_Y	$\lambda \leq \frac{2}{5}$	number of bins
R_t	0.022824471	3.06E-09	$\frac{2}{5}$	56
Π_t	0.022557848	5.47E-09	$\frac{2}{5}$	56

In this table we state the results of the parameter estimation of the jumps using the multinomial maximum likelihood estimation method and assuming Gaussian jump sizes. In the first tabular the results are displayed with no restriction on λ. In the second tabular the results are displayed when an upper bound of 10 for λ is set. In the last tabular the results are displayed when an upper bound of $\frac{2}{5}$ is set.

Table 7.2: **Results for the Pension result with Gaussian jumps in the interest rate**

$\lambda < \infty$		$\lambda \leq 10$		$\lambda \leq \frac{2}{5}$	$\lambda < \infty$		$\lambda \leq 10$		$\lambda \leq \frac{2}{5}$		
%	PR	%	PR	%	PR	%	PR	%	PR	%	PR
95	1.349	95	1.400	95	1.393	95	1.332	95	1.382	95	1.390
84.75	1.000	85,55	1.000	85,7	1.000	85,55	1.000	86,2	1.001	85.75	1.001
50	0.573	50	0.573	50	0.580	50	0.566	50	0.567	50	0.579
5	0.227	5	0.218	5	0.219	5	0.225	5	0.215	5	0.219

In this Table we state the results for the pension result based on a extension of the KNW model where a jump process with a Gaussian jump size assumption is added to the interest rate process. In the first 6 columns we state the percentile in the % column and the pension result in the PR column that we obtain when using μ_Y and σ_Y from the multinomial maximum likelihood estimation values for three constraints for λ. In the last 6 columns we state the percentile in the % column and the pension result in the PR column that we obtain when using $-\mu_Y$ and σ_Y from the multinomial maximum likelihood estimation values for three constraints for λ.

Table 7.3: **Results for the Pension result with Gaussian jumps in the inflation rate**

$\lambda < \infty$		$\lambda \leq 10$		$\lambda \leq \frac{2}{5}$	$\lambda < \infty$		$\lambda \leq 10$		$\lambda \leq \frac{2}{5}$		
%	PR	%	PR	%	PR	%	PR	%	PR	%	PR
100	0.731	100	0,416	97,3	1.001	95	12.643	95	16.518	95	2.273
95	0.159	95	0.127	95	0.868	50	4.996	50	6.332	55.75	1.000
50	0.067	50	0.054	50	0.354	5	1.805	5	2.334	50	0.928
5	0.028	5	0.022	5	0.139	0.65	0.976	0.4	1.010	5	0.361

In this Table we state the results for the pension result based on a extension of the KNW model where we added a jump process with a Gaussian jump size assumption to the inflation rate process. In the first 6 columns we state the percentile in the % column and the pension result in the PR column that we obtain when using μ_Y and σ_Y from the multinomial maximum likelihood estimation values for three constraints for λ. In the last 6 columns we state the percentile in the % column and the pension result in the PR column that we obtain when using $-\mu_Y$ and σ_Y from the multinomial maximum likelihood estimation values for three constraints for λ.

Table 7.4: **Results for the Pension result with Bimodal jumps in the interest rate and inflation rate**

R_t	%	PR	π_t	%	PR
	95	2.403		95	2.421
	73.05	1.000		72.1	0.998
	50	0.586		50	0.592
	5	0.152		5	0.150

In this Table we state the results for the pension result based on a extension of the KNW model where a jump process with a Bimodal jump size assumption is added to the interest rate and to the inflation rate process. The parameters for the jump process are based on Durham [19]: $\lambda = 10$, $\mu_{Y1} = 0.006$, $\sigma_{Y1} = 0.0015$, $\mu_{Y2} = -0.004$, $\sigma_{Y2} = 0.001$ and $w = 0.4$. In the first two columns we state the results for adding a jump process to the interest rate process. The % column represents the percentile and the PR column states the number for the pension result. In column 3 and 4 the result for adding a jump process to the inflation rate process is displayed.

Chapter 8

Conclusion and recommendations

The aim of this thesis is to propose an improvement of the design of the feasibility test for DC pension schemes. For this purpose we analyzed definitions of the pension result in DC schemes. The pension result currently used for DC schemes is the same as the definition for the pension result in DB schemes. The pension result in DB pension schemes, however, is not applicable to most DC pension schemes.

8.1 How should we define the pension result in DC pension schemes?

In chapter 4, sections 4.3 up to and including 4.7, five possible new definitions for the pension result in DC schemes have been proposed. The design of the definition for the pension result is in line with the pension result in DB schemes: the concept of a quotient, with expected pension payments in the nominator according to the investment portfolio of the pension fund and a denominator with pension payments according to a norm, is maintained. The aim to measure the maintenance of purchasing power with the pension result is also maintained. In DB pension schemes the norm is the height of the indexed pension entitlements without cut backs. In DB schemes this is also the pension goal. For DC pension schemes we cannot use the existing norm for DB pension schemes since there are no pension entitlements, so the search for a new definition of pension result in DC schemes comes down to choosing a new norm. This new norm, however, cannot serve as a pension goal. We suggest five norms for the denominator of the pension result quotient for DC pension schemes. It follows that comparing the expected pension payouts with pension payouts which are generated from instant indexed pension entitlements is the best norm among the five suggested norms. This definition measures the maintenance of purchasing power in DC pension schemes and can be used as a new definition for the pension result in DC schemes.

8.1.1 Discussion about the pension result in DC schemes

In chapter 4 we justified the choice for a new definition for the pension result in DC schemes which is based on indexed pension entitlements. This definition measures the maintenance of purchasing power. This has been the goal of the pension result in DB schemes as well. We can ask ourselves, however, if maintenance of purchasing power is the best term to define the quality a pension. The pension result depends on the height of the pension payments in both the denominator and the numerator. When the pension payments in the denominator are lower than the pension payments in the numerator the pension result will be relatively higher than the case in which the pension payments in the denominator are bigger than the pension payments in the nominator. The height of the pension result depends on three types of risk[1]. The interest rate risk (or pension conversion risk), i.e. the risk that the interest rate will be low during retirement, which will result in a low pension payment. The second type of risk is the inflation rate risk. This is the risk that the inflation rate will be high which results in a lower real value of the pension payment. The third type of risk is the investment risk. This is the risk that the investment returns can be low which can result in losing capital. So the question is which risk we mainly want to measure to define the quality of the pension payment. The definitions based on the risk free rate, the constant rate and the inflation rate show the performance of an investment product. We only measure investment return here. The pension result based on the inflation rate also measures the maintenance of purchasing power. The definition based on non indexed pension entitlements and indexed entitlements show the exposure to interest risk and investment risk. So the conversion risk is only shown in the latter two definitions. The definition of the pension result based on instant indexed pension entitlements also shows the exposure to inflation rate.

One can also debate what the focus of the feasibility test should be. Do we want to stress the stability of the median or do we want to stress the minimization of the deviation between the median and the 5th percentile?

8.2 What is the relation between the risk attitude and the life-cycle investment strategy?

The impact of the life-cycle investment strategy on the pension result is analyzed. We see that when we implement a defensive life-cycle investment strategy that then the deviation between the median an the 5th percentile gets smaller, so the risk attitude is more preserved in this case. When we implement offensive life-cycle investment strategy, then we see the opposite result and therefore the risk attitude is less preserved in this case. With this we can conclude that we see the same result as for DB pension schemes; the more risk averse the pension participant is, the lower the maximum deviation.

8.3 How robust is the new definition for pension result in DC schemes?

Several sensitivity analyses have been done in order to test the proposed new definition for the pension result in DC schemes. The influence of the development of the premium payments over the accrual period on the pension result has been analyzed as well as the influence of a constant life cycle, a fixed decrease and an extension of the financial market model.

8.3.1 Sensitivity Analysis 1: constant premium

In chapter 4 section 4.11.1 we analyze how the development of the premium payments over the years influences the pension result. Pension funds use the premium ladder to determine the percentage of premium during each year in the pension accrual period. We have seen that using a constant percentage gives a better pension result. The pension result is independent of the exact height of this constant percentage. The conclusion here is that one could determine the constant premium percentage that results in a pension capital, after 40 accrual years, which

[1]Note that we do not take longevity and chances of death into account.

pays a pension payment of 70% of the average salary. With this we preserve the premium ladder aim, but the pension result will be higher.

8.3.2 Sensitivity Analysis 2: constant stock exposure

The influence of a constant investment mix versus the used Merton life-cycle is analyzed in chapter 4 section 4.11.2. Using a life cycle that incorporates the risk attitude of a pension participant and for which the risk averseness decreases with age always gives a higher pension result than using a constant mix which does not take the age of the pension participant into account. When a life cycle is used in DC pension schemes the optimal stock expose decreases because of the decrease of the human capital. The conclusion here is that a use of life-cycle is always necessary to achieve a better pension result.

8.3.3 Sensitivity Analysis 3: fixed decrease

The influence of using a fixed decrease, when determining the pension payments, on the pension result is analyzed in chapter 4 section 4.11.3. We have seen that the pension result is similar when we apply a fixed decrease to when we do not apply a fixed decrease. We have also seen that the maximum deviation is similar and the 5th percentile is similar. We can conclude here that using a fixed decrease does not have an influence on the pension result. If the pension participant prefers more capital in the beginning of the retirement period or if ones believe is that the stock returns will be positive during the retirement benefit period, then a fixed decrease can be a preferred option.

8.3.4 Sensitivity Analysis 4: different scenario set

The financial market model, the KNW model is extended with a added jump process to the interest rate and the inflation rate since historical data obviously show a jump diffusion process which is not present in the original KNW model. We see that adding jumps influences the pension result. The way in which the pension result changes when jumps are added is highly dependent on the parameter values of the jumps and the assumption we make about the distribution of the jump size. We see that Bimodal jump sizes are conceptually more realistic than Gaussian jumps. We also see that a Bimodal jump size assumption, in the case of the interest rate process, has a more significant impact on the pension result. A Bimodal jump size assumption in the inflation rate has less influence on the pension result than a Gaussian jump size assumption in the inflation rate, but the influence is still significant.

8.3.5 Calibration KNW

We see that the jump sizes, which in the multinomial maximum likelihood estimation where recognized as jumps and which were significantly higher than the diffusion volatility, are very common moves in the KNW model. We can therefore suspect that the diffusion volatility in the KNW model is too high. This can be a result of the fact that the KNW model is estimated based on historical data. The historical data seem to be a jump diffusion model while the KNW model uses a diffusion only model. Therefore the volatility in the KNW model has to compensate for the rare big jumps in the historical data, which may cause a volatility which is too high to measure the impact of jumps estimated according to historical data. Lowering the KNW volatility and adding jumps will give a more realistic interest rate behavior. We have estimated the variables of the jump intensity and the jump size apart, without incorporating the rest of the KNW model. A better analysis could be given if the KNW model as a whole were calibrated with the jumps included.

8.3.6 Jumps in mean reversion factor

Whether a diffusion only or a jump-diffusion process better captures interest rate movements is an open empirical question[19]. As a consequence the computation of the nominal interest rate in the KNW model is still a point of discussion. The KNW model uses a mean reverting process, but if we check the historical data figure 7.1 then we see a different behavior. We can see that the mean reversion is obviously present, but we see that the long time mean shows jumps. So An another extension of the KNW model, with a clear motivation based on real life date, would be to not implement the jump process in the bond price as done in this research, but to implement the jump process in the mean of the nominal interest rate. This is an interesting topic for further research.

8.4 Goal of the feasibility test

An important part of the feasibility test is the calculation of pension result, which has the aim to measure the maintenance of purchasing power. We can, however, change the way in which we qualify pension payments. We could also look at the real replacement ratio. This is a measure which is conceptually easier to understand. The real replacement ratio gives more information about the height of pension payment and it also takes inflation into account. We see that for a pension payment where the maintenance of purchasing power is very high the pension payments can still be low. A high real replacement ratio indicates high pension payments and a correction for inflation. But since the feasibility test does not aim to be dependent on the amount of premium, the real replacement ratio does not fit so well in the feasibility test framework. The Dutch pension law does not act on the agreements between the employer and the employees about the amount of premium and the height of the salary. This is the reason why the existing definition of the pension result does not measure the actual height of the pension payments and why the pension result is chosen as a measure in the feasibility test to inform about the quality of pensions. The real replacement ratio does not have a direct link with the pension result in the feasibility test but it can be used for communicational purposes about the actual hight of the pension payments and it can be used to determine the height of the premium payments.

8.5 Can we design the feasibility test in a general way such that it applies to all types of pension schemes?

The improvement for the pension result in DC pension schemes that we have suggested in this research is chosen in such a way that the calculation of pension result serves the same goal as the calculation of pension result in DB schemes. We also choose the same set up for the definition of pension result with a similar norm in the denominator as for DB schemes but the numerator differs conceptually since it are two different pension schemes. In this way we stay close tot the current set up of the feasibility test and we adapted the test such that it can be applied to all types of pension schemes.

Chapter 9

Appendix 1: Parameters, details and background knowledge

9.1 Derivation of the risk aversion

9.1.1 Utility theory

In order to define the life cycle for investment strategy the level of risk averseness is used. In order to define the level of risk aversion we need some theory about utility functions. Utility is a measure of preference over some set of goods and a utility function is a function that quantifies the utility. The expected utility is measure of preference over some set of choices with uncertain outcomes. To define this in more detail we asses this uncertain outcomes by obtaining lotteries. A lottery is characterized by an ordered set of probabilities $p = \{p_1, ..., p_n\}$ where $\sum_{i=1}^{n} p_i = 1$ with $p_i \geq 0$. Let $\succ, \prec, \succeq, \preceq$ and $\sim$ denote preference and indifferent relationships.

- If an individual prefers lottery P^* to P this is denoted by $P^* \succ P$ or $P \prec P^*$.
- When an individual is indifferent between two lotteries this is denoted as $P^* \sim P$.
- If an individual prefers lottery P^* to P or is indifferent between P^* and P this is written as $P^* \succeq P$ or $P \preceq P^*$.

Utility axioms:

Theorem 9.1. *Axioms*

1. *Completeness,*
   for any two lotteries P and P^* either $P^* \succ P$, $P^* \prec P$ or $P^* \sim P$.

2. *Transitivity,*
   if $P^{**} \succeq P^*$ and $P^* \succeq P \Rightarrow P^{**} \succeq P$.

3. *Continuity,*
   if $P^{**} \succeq P^* \succeq P \Rightarrow \exists\ \lambda \in [0, 1]$ such that $P^* \sim \lambda P^{**} + (1 - \lambda)P$ where $\lambda P^{**} + (1 - \lambda)P$ denotes a compound lottery. With probability λ one receives lottery P^{**} and with probability $(1 - \lambda)$ one receives lottery P.

4. *Independence,*
   for any two lotteries P^* and P it holds that $P^* \succ P \Leftrightarrow \forall \lambda \in (0, 1]$ and $\forall P^{**}$ it holds that:

$$\lambda P^* + (1 - \lambda)P^{**} \succ \lambda P + (1 - \lambda)P^{**}$$

moreover for any two lotteries P and P^+ it holds that $P \sim P^+ \Leftrightarrow \forall \lambda \in (0,1]$ and $\forall P^{**}$ it holds that:

$$\lambda P + (1-\lambda)P^{**} \sim \lambda P^+ + (1-\lambda)P^{**}.$$

5. *Dominance,*
   let P^1 be the compounded lottery $\lambda_1 P^* + (1-\lambda_1)P^+$ and let P^2 be the compounded lottery $\lambda_2 P^* + (1-\lambda_2)P^+$. If $P^* \succ P^+ \Rightarrow P_1 \succ P_2 \Leftrightarrow \lambda_1 > \lambda_2$.

9.1.2 Deriving expected utility

As mentioned before, the expected utility is measure of preference over some set of choices with uncertain outcomes. To display this uncertainty we define lottery $e_i = \{p_1, ..., p_n\} = \{0, 0, ..., 1, ..., 0, 0\}$ with $p_i = 1$, $p_j = 0\ \forall j \neq i$ and with the payoff set: $\{x_1, ..., x_n\}$. We observe outcome x_i with probability 1 and outcome $x_j\ \forall j \neq i$ with probability 0. Without loss of generality we assume that the outcomes are ordered such that $e_n \geq e_{n-1} \geq ... \geq e_1$. This follows from the completeness axiom for this case of n elementary lotteries. From the continuous axiom we can derive that $\forall\ e_i\ \exists\ U_i$ in $[0,1]$ such that:

$$e_i \sim U_i e_n + (1-U_i)e_1 \tag{9.1}$$

so for $i = 1 \Rightarrow U_1 = 0$ and for $i = n \Rightarrow U_n = 1$. A given arbitrary lottery can be viewed as a compound lottery over the n elementary lotteries where elementary lottery e_i is obtained with probability p_i.

$$p \sim p_1 e_1 + ... + p_n e_n$$

. By the independence axiom and by equation (9.1) the individual is indifferent between lottery p and the following lottery:

$$p_1 e_1 + ... + p_n e_n \sim p_1 e_1 + ... + p_{i-1} e_{i-1} + p_i[U_i e_n + (1-U_i)e_1] + p_{i+1} e_{i+1} + ... + p_n e_n.$$

Repeating this substitution gives us:

$$p_1 e_1 + ... + p_n e_n \sim \left(\sum_{i=1}^{n} p_i U_i\right) e_n + \left(1 - \sum_{i=1}^{n} p_i U_i\right) e_1.$$

Now we set $\Lambda = \sum_{i=1}^{n} p_i U_i$ then

$$p \sim \Lambda e_n + (1-\Lambda)e_1.$$

Similarly we can show that any other arbitrary lottery

$$p^* = \{p_1^*, ..., p_n^*\} \sim \Lambda^* e_n + (1-\Lambda^*)e_1$$

where $\Lambda^* = \sum_{i=1}^{n} p_i^* U_i$. We know from the dominance axiom that:

$$p^* > p \Leftrightarrow \Lambda^* > \Lambda \Rightarrow \sum_{i=1}^{n} p_i^* U_i > \sum_{i=1}^{n} p_i U_i$$

9.1.3 Merton's coefficient of risk aversion

Expected utility is not unique under linear transformation. This is why in the following part of this chapter we will derive a different measure to display the extent of risk averseness. The intuition for why expected utility is unique up to a linear transformation comes from equation (9.1). Here we express elementary lottery i in term of the least and the most preferred elementary lottery. However other bases for ranking a given lottery are possible.

Risk averse individuals reject every form of risk, even fair risk. We now introduce a different measure for the level of risk averseness, the risk premium $\rho_u(\tilde{x})$, using the expected utility. This measure is instead of simply the expected utility, unique under nonlinear transformations.

Definition 9.2. *The risk premium $\rho_u(\tilde{x})$ of a lottery in Merton's model with payoff $\tilde{x}$ for an agent characterized by a utility function u is the maximum amount of money which the agent is willing to pay to receive instead of $\tilde{x}$ its expected value with certainty: $\rho_u(\tilde{x})$ is such that $u(E[\tilde{x}] - \rho_u(\tilde{x})) = U(\tilde{x})$*

Merton's risk premium is the amount that an individual is willing to pay to avoid risk. Let $\rho_u(\tilde{x})$ denote the risk premium of an individual for the lottery with payoff $\tilde{\epsilon}$:

$$U(W - \rho_u(\tilde{x})) = E[u(W + \tilde{\epsilon})] < u(E[W + \tilde{\epsilon}]) = u(W + 0) = u(W) \tag{9.2}$$

Here $W - \rho_u(\tilde{x})$ is the certainty equivalent associated with $\tilde{\epsilon}$. We see that for concave utility Jensens inequality implies that $\rho_u(\tilde{x}) > 0$ when $\tilde{\epsilon}$ is fair, i.e. the individual would accept wealth lower than his expected wealth to avoid the lottery. For small $\tilde{\epsilon}$ we can take the Taylor expansion of equation (9.2) around $\tilde{\epsilon} = 0$ and $\rho_u(\tilde{x}) = 0$. Expanding the left hand side about $\rho_u(\tilde{x}) = 0$ gives:

$$\begin{aligned} U(W - \rho_u(\tilde{x})) &= E[u(W - \rho_u(\tilde{x}))] \\ &\approx u(W) - \rho_u(\tilde{x})u'(W) \end{aligned}$$

and the expansion of the right hand side about $\tilde{\epsilon} = 0$ gives:

$$\begin{aligned} U(\tilde{x}) &= E[u(W + \tilde{\epsilon})] \\ &\approx E[u(W) + \tilde{\epsilon}u'(W) + \frac{1}{2}\tilde{\epsilon}^2 u''(W)] \\ &= u(W) + 0 + \frac{1}{2}\tilde{\epsilon}^2 u''(W) \end{aligned} \tag{9.3}$$

with $\sigma^2 = E[\tilde{\epsilon}^2]$ the variance of the lottery. Setting the result of (9.2) and (9.3) equal we get:

$$\pi = -\frac{1}{2}\sigma^2 \frac{u''(W)}{u'(W)} = \frac{1}{2}\sigma^2 R(W) \tag{9.4}$$

Where $R(W)$ is the Pratt-Arrow measure of absolute risk aversion in correspondence of the wealth W. Absolute risk aversion is the rate of decay for marginal utility $u'(W)$. More particularly, absolute risk aversion measures the rate at which marginal utility decreases when wealth is increased by one unit of the used currency. This means that absolute risk aversion is currency dependent. It is often preferred to use a unit-free measurements of sensitivity. To this end, we define the index of relative risk aversion $\rho_u^r(\tilde{x})$ as the rate at which marginal utility $u'(W)$ decreases. Relative risk gives us a lottery with relative payoff $W(1 + \tilde{\epsilon})$. We get the following relative risk aversion coefficient:

$$\begin{aligned} \rho_u^r(\tilde{x}) &= \frac{\mathrm{d}u'(W)}{\mathrm{d}W} \cdot \frac{W}{u'(W)} \\ &= \frac{Wu''(W)}{u'(W)} \end{aligned}$$

9.1.4 Utility for pension accrual

Describing a preference relation in the context of individual wealth is founded by Merton [12]. In this model the constant relative risk aversion (CRRA) is used as a measure of risk aversion to represent the preference of individuals for the utility function. Intuitively an individual would get less risk averse if his wealth W_t grows. This means that Merton's risk premium ρ_u for utility function u, which is proportional to the term $\frac{U''(W_t)}{U'(W_t)}$, decreases when W_t increases. Here U is the expected utility function of the pension participant. So in order to describe a natural attitude towards risk exposure we would want to obtain a function which is inversely proportional to the wealth W_t. We also need a coefficient to determine to which extend the risk aversion gets less as W_t grows which will be defined as γ. The function of the attitude towards risk exposure is defined in such a way that if γ grows then our risk aversion is higher compared to capital W_t. We get:

$$\frac{U''(W_t)}{U'(W_t)} = \frac{-1}{\frac{1}{\gamma}W_t}.$$

We can solve this differential equation in the following way:

$$\begin{aligned}
\frac{U''(W_t)}{U'(W_t)} &= \frac{-1}{\frac{1}{\gamma}W_t} \\
-U''(W_t) &= \frac{U'(W_t)}{\frac{1}{\gamma}W_t} \\
-\frac{1}{\gamma}W_t U''(W_t) &= U'(W_t) \\
&\Rightarrow \quad U(W_t) = \frac{W_t^{1-\gamma}}{1-\gamma}.
\end{aligned}$$

When $\gamma = 1$ we solve:

$$-U''(W_t) = \frac{U'(W_t)}{W_t}$$

From this it follows that $U(W_t) = \ln(W_t)$. The time separable utility function of an individual featuring CRRA preferences thus equals:

$$U(W_t) = \begin{cases} \frac{W_t^{1-\gamma}}{1-\gamma} \text{ if } \gamma \neq 1 \\ \ln(W_t) \text{ if } \gamma = 1 \end{cases}. \tag{9.5}$$

This utility function describes the process in which money is favored. Since the function is a concave utility function for $\gamma > 1$ it holds that the less capital you have, the more you favor. Whenever people have more capital the favor tends to weaken. If $\gamma < 1$ then the utility function is convex.

9.2 Definition for convolutions

Definition 9.3. *Convolution [26]*
Suppose X and Y are two continuous random variables with f_X the distribution function of X and f_Y the distribution function of Y. Let $f(x,y)$ be the joint distribution function of X and Y and let $Z = X + Y$. To find the distribution function of Z we will first find the cumulative distribution function of Z and then differentiate this. To find the cumulative distribution function of Z we calculate:

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x,y)dydx.$$

In the inner integral we make the change of variables $y = v - x$ to obtain:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z} f(x, v-x)dvdx \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f(x, v-x)dvdx. \end{aligned}$$

Differentiating with respect to v, we have, if $\int_{-\infty}^{\infty} f(x, z-x)dx$ is continuous at z,

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx.$$

If X and Y are independent, we have:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx.$$

This integral is called the convolution of the functions f_X and f_Y.

9.3 List of parameters

$t \in [25; 85]$	time in years
N	maturity in years
$i \in \mathbb{N}$	scenario number
$L_t \in [25, 85]$	age of the participant
$\boldsymbol{X_t} \in \mathbb{R}^2$	unobserved state variable
$r_t \in \mathbb{R}$	nominal interest
$\pi_t \in \mathbb{R}$	instantaneous expected inflation
$\Pi_t \in \mathbb{R}$	cumulative inflation
$S_t \in \mathbb{R}$	investment return
$P_t(N) \in \mathbb{R}$	bond price
$\boldsymbol{\Lambda_t} \in \mathbb{R}^4$	prices of risk
$\phi_t \in \mathbb{R}$	stochastic discount factor
$R_t \in \mathbb{R}$	nominal interest at time
${}^N R_t \in \mathbb{R}$	term structure
${}_0 R_t \in \mathbb{R}$	risk free interest rate
$p_t^i \in \mathbb{R}$	premium
$\omega_t \in (0, 1)$	the life cycle
${}^r y_t^i \in \mathbb{R}$	indexed salary
$W_t^i \in \mathbb{R}$	pension capital at time
${}^r W_t^i \in \mathbb{R}$	real pension capital
${}_1 W_t^i \in \mathbb{R}$	pension capital based on risk free rate
${}_1^r W_t^i \in \mathbb{R}$	real pension capital based on risk free rate
${}_2 W_t^i \in \mathbb{R}$	pension capital based on constant rate
${}_2^r W_t^i \in \mathbb{R}$	real pension capital based on constant rate
${}_3 W_t^i \in \mathbb{R}$	pension capital based on constant rate
${}_3^r W_t^i \in \mathbb{R}$	real pension capital based on constant rate
${}_4 W_t^i \in \mathbb{R}$	pension capital based on non indexed pension entitlements
${}_4^r W_t^i \in \mathbb{R}$	real pension capital based on non indexed pension entitlements
${}_5 W_t^i \in \mathbb{R}$	pension capital based on indexed pension entitlements
${}_5^r W_t^i \in \mathbb{R}$	real pension capital based on indexed pension entitlements
$Q^i \in \mathbb{R}$	pension payment
${}_1 Q_t^i \in \mathbb{R}$	pension payment based on the risk free rate
${}_1^r Q_t^i \in \mathbb{R}$	real pension payment based on the risk free rate
${}_2 Q_t^i \in \mathbb{R}$	pension payment based on a constant rate
${}_2^r Q_t^i \in \mathbb{R}$	real pension payment in scenario based on a constant rate
${}_3 Q_t^i \in \mathbb{R}$	pension payment based on a inflation rate
${}_3^r Q_t^i \in \mathbb{R}$	real pension payment based on a inflation rate
${}_4 Q_t^i \in \mathbb{R}$	pension payment based on a on non indexed pension entitlements
${}_4^r Q_t^i \in \mathbb{R}$	real pension payment based on a on non indexed pension entitlements
${}_5 Q_t^i \in \mathbb{R}$	pension payment based on indexed pension entitlements
${}_5^r Q_t^i \in \mathbb{R}$	real pension payment based on indexed pension entitlements
$PR_t^i \in \mathbb{R}$	pension result based on the risk free rate
${}^r PR_t^i \in \mathbb{R}$	real pension result based on the risk free rate
${}_2 PR_t^i \in \mathbb{R}$	pension result based on a constant rate
${}_2^r PR_t^i \in \mathbb{R}$	real pension result based on a constant rate
${}_3 PR_t^i \in \mathbb{R}$	pension result based on inflation rate
${}_3^r PR_t^i \in \mathbb{R}$	real pension result based on inflation rate

${}_4PR_t^i \in \mathbb{R}$	pension result based on non indexed pension entitlements
${}_4^rPR_t^i \in \mathbb{R}$	real pension result based on non indexed pension entitlements
${}_5PR_t^i \in \mathbb{R}$	pension result based on indexed pension entitlements
${}_5^rPR_t^i \in \mathbb{R}$	real pension result based on indexed pension entitlements
${}^rRR_t^i \in \mathbb{R}$	real replacement ratio
$u \in \mathbb{R}$	utility
$U \in \mathbb{R}$	expected utility
$D_t \in \mathbb{R}$	dividend
$N_t \in \mathbb{N}$	Poisson process
$\lambda \in \mathbb{N}$	mean Poisson process
$\boldsymbol{J_t} \in \mathbb{R}^2$	jump i.e. compound Poisson process
$\boldsymbol{Y_t} \in \mathbb{R}^2$	jump size
$\boldsymbol{\mu_Y} \in \mathbb{R}^2$	mean jump size
$\boldsymbol{\sigma_Y} \in \mathbb{R}^2$	mean jump size
s	time of retirement
T	time of death

9.4 Premium ladder ('Staffels')

Table 9.1: **4% premium Staffels [10]**

Age cohort	Percentage of pension base
18-19	3.3%
20-24	3.7%
25-29	4.5%
30-34	5.5%
35-39	6.7%
40-44	8.2%
42-49	10%
50-54	12.2%
55-59	15%
60-64	18.6%
65-67	22.3%

9.5 Parameter estimation KNW

Table 9.2: **Estimation results Parameters KNW**[8]

Parameter	Estimate	Standard deviation
$\delta_{0\pi}$	1.81 %	2.79 %
$\delta_{0\pi(1)}$	-0.63 %	0.10 %
$\delta_{0\pi(2)}$	0.14 %	0.24 %
R_0	2.40 %	6.06 %
$R_{1(1)}$	-1.48 %	0.22 %
$R_{1(2)}$	0.53 %	0.56 %
$K_{(1,1)}$	0.08	0.11
$K_{(2,2)}$	0.35	0.18
$K_{(2,1)}$	-0.19	0.08
$K_{(1,2)}$	0	0
$\sigma_{\Pi(1)}$	0.02 %	0.07 %
$\sigma_{\Pi(2)}$	-0.01 %	0.06 %
$\sigma_{\Pi(3)}$	0.61 %	0.04 %
$\sigma_{\Pi(4)}$	0 %	0 %
η_S	4.52 %	3.73 %
$\sigma_{S(1)}$	-0.53 %	1.44%
$\sigma_{S(2)}$	-0.76 %	1.54 %
$\sigma_{S(3)}$	-2.11 %	1.51 %
$\sigma_{S(4)}$	16.59 %	0.96 %
$\Lambda_{0(1)}$	0.403	0.333
$\Lambda_{0(2)}$	0.039	0.270
$\Lambda_{0(3)}$	0	0
$\Lambda_{0(4)}$	0	0
$\Lambda_{1(1,1)}$	0.149	0.156
$\Lambda_{1(1,2)}$	-0.381	0.039
$\Lambda_{1(1,3)}$	0	0
$\Lambda_{1(1,4)}$	0	0
$\Lambda_{1(2,1)}$	0.089	0.075
$\Lambda_{1(2,2)}$	-0.083	0.129
$\Lambda_{1(2,3)}$	0	0
$\Lambda_{1(2,4)}$	0	0

Chapter 10

Appendix 2: Additional results on Portfolio allocation

In the model for pension capital simulation we used the Merton life cycle to determine the exposure to stocks. The Merton life cycle is, however, optimal in the Merton model but not in the KNW model. The Merton model assumes that there are only stocks and risk free bonds to invest in, moreover the prices of risk are constant in the Merton model. Ideally we want to apply a life cycle which gives an optimal investment policy for both the stock exposure and the bond exposure. Therefore we will try to derive the optimal life cycle. We want to take into account the risk preferences of pension participants. We use the same utility function as the Merton model. So to determine the optimal investment strategy we consider a pension participant with an isoelastic utility function:

$$\frac{1}{1-\gamma}c^{1-\gamma}. \tag{10.1}$$

Optimal investment strategy problems, however, do not allow for analytical solutions in general. Simple models as the Merton model [12] and the Brennan Xia model [13] do have analytical solutions. Koijen, Nijman and Werker [4] use numerical techniques to determine the optimal investment strategy. This is necessary since the dynamic of the evolution of the participants wealth W_t is endogenously determined. As a result, the solution is recursive and must be solved using dynamic programming [30]. In this chapter we will explore how to deduce the closest analytical solution for the KNW model.

We change the approach we used for Merton's life cycle. Instead of maximizing the wealth on the pension date, we maximize the flow of pension payments during the retirement period using the martingale approach [29]. Since we need an investment strategy during the retirement period as well.

Let c_t^i be the nominal pension payment at time t and in scenario i, Π_t^i be the cumulative inflation rate, W_t^i the nominal financial wealth, H_t^i the nominal human capital and ϕ_t^i the real stochastic discount factor, which is the nominal stochastic discount factor from chapter 3 compensated for inflation. The investor's optimal portfolio choice problem can be formulated in the following way:

$$max_{c_t} E_t^{\mathbb{P}} \left[\int_{43}^{60} \frac{1}{1-\gamma} \left(\frac{c_l^i}{\Pi_l^i} \right)^{1-\gamma} dl \right] \tag{10.2}$$

$$\text{s.t.} \quad \frac{W_t^i + H_t^i}{\Pi_t^i} = E_t^{\mathbb{P}} \left[\int_{43}^{60} \frac{c_l^i}{\Pi_l^i} \frac{\phi_l^i}{\phi_t^i} dl \right]. \tag{10.3}$$

The constraint comes from the fact that we want the expectation of all our pension payments together to be equal to the accrued pension capital at time t, so we are maximizing the expected real pension payments during the whole pension period. We get the following lagrangian:

$$\mathcal{L} = E_t^{\mathbb{P}} \left[\int_{43}^{60} \frac{1}{1-\gamma} \left(\frac{c_l^i}{\Pi_l^i} \right)^{1-\gamma} dl \right] - \lambda \left(E_t^{\mathbb{P}} \left[\int_{43}^{60} \frac{c_l^i}{\Pi_l^i} \frac{\phi_l^i}{\phi_t^i} dl \right] - \frac{W_t^i + H_t^i}{\Pi_t^i} \right).$$

All the expectations in the next sections will be with respect to the physical measure $\mathbb{P}$. We will leave out the notation of it for simplicity. We will also leave out the bold notation for matrices.

10.1 Optimal benefit

Taking the derivative with respect to $\frac{c_l^i}{\Pi_l^i}$ $\forall l, i$ we get:

$$\frac{\partial \mathcal{L}}{\partial \frac{c_l^i}{\Pi_l^i}} = \int_{43}^{60} \left(\frac{c_l^i}{\Pi_l^i} \right)^{-\gamma} dl - \lambda \int_{43}^{60} \frac{\phi_l^i}{\phi_t^i} dl$$

we set $\frac{\partial \mathcal{L}}{\partial \frac{c_l^i}{\Pi_l^i}} = 0$ $\forall t$ such that:

$$\frac{W_t^i + H_t^i}{\Pi_t^i} = E_t \left[\int_{43}^{60} \frac{c_l^i}{\Pi_l^i} \frac{\phi_l^i}{\phi_t^i} dl \right].$$

We get:

$$\int_{43}^{60} \left(\frac{c_l^i}{\Pi_l^i} \right)^{-\gamma} dl = \lambda \int_{43}^{60} \frac{\phi_l^i}{\phi_t^i} dl.$$

This results in:

$$\int_{43}^{60} \left(\frac{c_l^i}{\Pi_l^i} \right)^{-\gamma} dl = \lambda \int_{43}^{60} \frac{\phi_l^i}{\phi_t^i} dl \Rightarrow$$

$$\left(\frac{c_l^i}{\Pi_l^i} \right)^{-\gamma} = \lambda \frac{\phi_l^i}{\phi_t^i} \Rightarrow$$

$$\frac{c_l^i}{\Pi_l^i} = \left(\lambda \frac{\phi_l^i}{\phi_t^i} \right)^{-\frac{1}{\gamma}} \Rightarrow \tag{10.4}$$

$$c_l^i = \left(\lambda \frac{\phi_l^i}{\phi_t^i} \right)^{-\frac{1}{\gamma}} \Pi_l^i. \tag{10.5}$$

We substitute (5.4) into (5.2) such and solve for λ

$$\frac{W_t^i + H_t^i}{\Pi_t^i} = E_t\left[\int_{43}^{60}\left(\lambda\frac{\phi_l^i}{\phi_t^i}\right)^{-\frac{1}{\gamma}}\frac{\phi_l^i}{\phi_t^i}dl\right] \Rightarrow$$

$$\frac{W_t^i + H_t^i}{\Pi_t^i} = \lambda^{-\frac{1}{\gamma}}E_t\left[\int_{43}^{60}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}dl\right] \Rightarrow$$

$$\lambda^{-\frac{1}{\gamma}} = \frac{\frac{W_t^i + H_t^i}{\Pi_t^i}}{E_t\left[\int_{43}^{60}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}dl\right]} \Rightarrow$$

$$\lambda = \frac{\left(\frac{W_t^i + H_t^i}{\Pi_t^i}\right)^{-\gamma}}{\left(E_t\left[\int_{43}^{60}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}dl\right]\right)^{-\gamma}}. \tag{10.6}$$

Now we fill in λ (5.6) in the optimal pension benefit (5.5):

$$c_l^i = \frac{\left(\frac{W_t^i + H_t^i}{\Pi_t^i}\right)}{E_t\left[\int_{l=43}^{60}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}dl\right]}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{-\frac{1}{\gamma}}\Pi_l^i.$$

10.2 Portfolio return

We have the following static variational problem stated in in equation (5.2) and (5.3). Let $\frac{W_t^i + H_t^i}{\Pi_t^i} = w_i^t$, so w_i^t is the real total wealth in scenario i and at time t. From there we can deduce for the real return of the total wealth i.e. (dw_i^t).

$$\frac{W_t^i + H_t^i}{\Pi_t^i} = E_t\left[\int_{43}^{60}\frac{c_l^i}{\Pi_l^i}\frac{\phi_l^i}{\phi_t^i}dl\right] \Rightarrow$$

$$w_t = E_t\left[\int_{43}^{60}\frac{c_l^i}{\Pi_l^i}\frac{\phi_l^i}{\phi_t^i}dl\right] \Rightarrow$$

$$d\log\big(w_t^i\big) = d\log\left(c_t^* E_t\left[\int_{43}^{60}\frac{c_l^i}{c_t^*\Pi_l^i}\frac{\phi_l^i}{\phi_t^i}dl\right]'\right) \Rightarrow$$

$$d\log\big(w_t^i\big) = d\log(c_t^*) + d\log\big(K_t^i\big) \tag{10.7}$$

To simplify writing we define the following parameters:

$$w_t^i = \frac{W_t^i + H_t^i}{\Pi_t^i}$$

$$c_0^* = \frac{w_0^i}{E_t\left[\int_{43}^{60}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}dl\right]}$$

$$c_t^* = \frac{w_t^i}{E_t\left[\int_{43}^{60}\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}dl\right]}\left(\frac{\phi_t^i}{\phi_t^i}\right)^{\frac{-1}{\gamma}} \tag{10.8}$$

$$K_t^i = E_t\left[\int_{43}^{60}\frac{c_l^i}{c_t^*\Pi_l^i}\frac{\phi_l^i}{\phi_t^i}dl\right]. \tag{10.9}$$

Note that c_0^* and c_t^* are not actual benefits but convenient parameters we will use later. From equation (5.8) it follows that:

$$d\log(c_t^*) = \frac{-1}{\gamma} d\log\left(\phi_t^i\right). \tag{10.10}$$

For K_t^i we find:

$$\begin{aligned} K_t^i &= \frac{1}{c_t^*} E_t\left[\int_{43}^{60} \frac{c_l^i}{\Pi_l^i} \frac{\phi_l^i}{\phi_t^i} dl\right] \\ &= \frac{1}{c_t^*} \int_{43}^{60} E_t\left[\frac{w_t^i}{E_t\left[\int_{43}^{60} \left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}} dl\right]} \left(\frac{\phi_l^i}{\phi_t^i}\right)^{\frac{-1}{\gamma}} \frac{\phi_l^i}{\phi_t^i}\right] dl \\ &= \int_{43}^{60} E_t\left[\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}\right] dl. \end{aligned} \tag{10.11}$$

10.3 Expectation of the pricing kernel

We are looking for the value of $E_t\left[\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}\right]$ to determine the value $d\log\left(K_t^i\right)$ in the portfolio return. Recall that X_t is an Ornstein Uhlenbeck process, which has the following solution:

$$X_t = e^{-(t-s)K} X_s + \int_s^t e^{-(t-u)K} \Sigma_x' dZ(u).$$

For the prices of risk we then find that:

$$\Lambda_t = \Lambda_0 + \Lambda_1 e^{-(t-s)K} X_s + \Lambda_1 \int_s^t e^{-(t-u)K} \Sigma_x' dZ(u).$$

Also recall that $K \in \mathbb{R}^{2\times 2}$,

$$\Sigma_X' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$dZ, \lambda_0 \in \mathbb{R}^{4\times 1}$ and $\Lambda_1 \in \mathbb{R}^{4\times 2}$. We can write the pricing kernel as:

$$\frac{\phi_l}{\phi_t} = exp\left\{\int_t^l \left(-r_u - \frac{1}{2}\Lambda_u' \rho \Lambda_u\right) du + \int_t^l \Lambda_u' dZ(u)\right\}$$

since the Brownian motions are independent $\rho = I$. We know that $\left(\frac{\phi_l^i}{\phi_t^i}\right)$ has a log normal distribution. The mean of $\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}$ is therefore equal to $e^{\left(1-\frac{1}{\gamma}\right)\mu + \left(1-\frac{1}{\gamma}\right)^2 \frac{1}{2}\sigma^2}$ with μ and σ^2 respectively the mean and variance of:

$$\left(\int_t^l \left(-r_u - \frac{1}{2}\Lambda_u' \Lambda_u\right) du + \int_t^l \Lambda_u' dZ(u)\right).$$

So we are looking for:

$$\begin{aligned} \mu &= E_t\left[\int_t^l \left(-R_u^0 - \frac{1}{2}\Lambda_u' \Lambda_u\right) du + \int_t^l \Lambda_u' dZ(u)\right] \\ \sigma^2 &= var\left(\int_t^l \left(-R_u^0 - \frac{1}{2}\Lambda_u' \Lambda_u\right) du + \int_t^l \Lambda_u' dZ(u)\right). \end{aligned}$$

In order to solve $E_t\left[\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}\right]$ we will make some assumption to simplify the problem. We deduce an analytical solution of a simplified version of the KNW model following the Brennan and Xia model[13]. The Brennan and Xia model differs from KNW assuming that prices of risk are constant. Here we will not take the prices of risk completely constant but we discretize the prices of risk. So instead of using the stochastic differential equation of Λ_t we assume that Λ_t is a constant for each t such that the derivation is still manageable. To solve our expectation we will set: $\Lambda_t = \Lambda_{ct}$. We get the following:

$$E_t\left[\left(\frac{\phi_l^i}{\phi_t^i}\right)^{-\frac{1}{\gamma}}\right] = E_t\left[\left(\exp\left\{\int_t^l\left(-r_u-\frac{1}{2}\Lambda_{ct}'\Lambda_{ct}\right)du+\int_t^l\Lambda_{ct}'dZ(u)\right\}\right)\left(1-\frac{-1}{\gamma}\right)\right].$$

So we are looking for:

$$\begin{aligned}\mu &= E_t\left[\int_t^l\left(-r_u-\frac{1}{2}\Lambda_{ct}'\Lambda_{ct}\right)du+\int_t^l\Lambda_{ct}'dZ(u)\right]\\ &= -E_t\left[\int_t^l r_u du\right]-\frac{1}{2}E_t\left[\int_t^l\Lambda_{ct}'\Lambda_{ct}du\right]+E_t\left[\int_t^l\Lambda_{ct}'dZ(u)\right]\\ &= -\int_t^l E_t\left[r_u\right]du-\frac{1}{2}\int_t^l E_t\left[\Lambda_{ct}'\Lambda_{ct}\right]du+E_t\left[\int_t^l\Lambda_{ct}'dZ(u)\right].\end{aligned}$$

We compute the three expectations separately:
1.

$$\begin{aligned}-\int_t^l E_t\left[r_u\right]du &= -\int_t^l E_t\left[\delta_{0R}+\delta_{1R}'X_u\right]du\\ &= -\int_t^l E_t\left[\delta_{0R}\right]+E_t\left[\delta_{1R}'e^{-(u-t)K}X_t\right]\\ &\quad +E_t\left[\delta_{1R}'\int_t^u e^{-(u-v)K}\Sigma_x'dZ(v)\right]du\\ &= -\int_t^l\delta_{0R}du+\int_t^l\delta_{1R}'e^{-(u-t)K}X_tdu\\ &= -\delta_{0R}(l-t)+\int_t^l\delta_{1R}'e^{-uK}e^{tK}X_tdu\\ &= -\delta_{0R}(l-t)+\int_t^l\delta_{1R}'\sum_{m=0}^{\infty}\frac{1}{m!}(-uk)^m e^{tK}X_tdu\\ &= -\delta_{0R}(l-t)-\left[\delta_{1R}'K^{-1}\sum_{m=0}^{\infty}\left(\frac{1}{m!}(-uk)^m-I\right)e^{tK}X_t\right]_t^l\\ &= -\delta_{0R}(l-t)-\delta_{1R}'K^{-1}\left(e^{-lK}-e^{-tK}\right)e^{tK}X_t.\end{aligned}$$

2.

$$\begin{aligned}-\frac{1}{2}\int_t^l E_t\left[\Lambda_{ct}'\Lambda_{ct}\right]du &= -\frac{1}{2}\left[\Lambda_{ct}'\Lambda_{ct}u\right]_t^l\\ &= -\frac{1}{2}\Lambda_{ct}'\Lambda_{ct}(l-t).\end{aligned}$$

3.

$$E_t\left[\int_t^l \Lambda'_{ct} dZ(u)\right] = 0.$$

For σ we find:

$$\begin{aligned}
\sigma^2 &= var\left(\int_t^l \left(-r_u - \frac{1}{2}\Lambda'_{ct}\Lambda_{ct}\right) du + \int_t^l \Lambda'_{ct} dZ(u)\right) \\
&= var(\int_t^l \Lambda'_{ct} dZ(u)) \\
&= E\left[\left(\int_t^l \Lambda'_{ct} dZ(u)\right)^2\right] \\
&= E\left[\int_t^l \Lambda'_{ct}\Lambda_{ct} du\right] \\
&= \int_t^l \Lambda'_{ct}\Lambda_{ct} du \\
&= \Lambda'_{ct}\Lambda_{ct}(l-t).
\end{aligned}$$

So:

$$\mu = -\delta_{0R}(l-t) - \delta'_{1R}K^{-1}\left(e^{-tK} - e^{-lK}\right)e^{tK}X_t - \frac{1}{2}\Lambda'_{ct}\Lambda_{ct}(l-t) \quad (10.12)$$

$$\sigma^2 = \Lambda'_{ct}\Lambda_{ct}(l-t). \quad (10.13)$$

10.4 Optimal portfolio allocation

We have seen that the log return of the portfolio dw_t^i is defined as follows:

$$d\log(w_t^i) = d\log(c_t^*) + d\log(K_t^i).$$

With the conversion factor K_t^i:

$$\begin{aligned}
K_t^i &= \int_{43}^{60} E_t\left[\left(\frac{\phi_l^i}{\phi_t^i}\right)^{1-\frac{1}{\gamma}}\right] dl \\
&= \int_{43}^{60} e^{\left(1-\frac{1}{\gamma}\right)\mu + \left(1-\frac{1}{\gamma}\right)^2 \frac{1}{2}\sigma^2} dl \\
&= \int_{43}^{60} k^i(l,t) dl
\end{aligned}$$

with

$$dK_t^i = \frac{\partial K_t^i}{\partial t}dt + \frac{\partial K_t^i}{\partial Z_t}dZ_t + \frac{1}{2}\frac{\partial^2 K_t^i}{\partial Z_t^2}dt$$

such that for $d\log(K_t^i)$ we get:

$$\begin{aligned}
d\log(K_t^i) &= \frac{1}{K_t^i}dK_t^i - \frac{1}{2}\frac{1}{(K_t^i)^2}(dK_t^i)^2 \\
&= \frac{1}{K_t^i}\left(\frac{\partial K_t^i}{\partial t}dt + \frac{\partial K_t^i}{\partial Z_t}dZ_t + \frac{1}{2}\frac{\partial^2 K_t^i}{\partial Z_t^2}dt\right) - \frac{1}{2}\frac{1}{(K_t^i)^2}(dK_t^i)^2.
\end{aligned}$$

We only focus on the dZ_t terms.

$$
\begin{aligned}
& \frac{1}{K_t^i}\frac{\partial K_t^i}{\partial Z_t}dZ_t \\
= \ & \frac{1}{K_t^i}\frac{\partial \int_{43}^{60} k^i(l,t)dl}{\partial Z_t}dZ_t.
\end{aligned}
$$

Now we use the Leibniz integration rule. We can use this since $k^i(l,t) = e^{(1-\frac{1}{\gamma})\mu+(1-\frac{1}{\gamma})^2\frac{1}{2}\sigma^2}$ is continuous differentiable.

$$
\begin{aligned}
= \ & \frac{1}{K_t^i}\frac{\int_{43}^{60} \partial k^i(l,t)dl}{\partial Z_t}dZ_t \\
= \ & \frac{1}{K_t^i}\int_{43}^{60}\left(-(1-\frac{1}{\gamma})\delta_{1R}'K^{-1}\left(e^{-lK}-e^{-tK}\right)e^{tK}\Sigma_X\right) \\
& \cdot e^{\left(1-\frac{1}{\gamma}\right)\mu+\left(1-\frac{1}{\gamma}\right)^2\frac{1}{2}\sigma^2}dldZ_t \\
= \ & \int_{43}^{60}\frac{\left(-(1-\frac{1}{\gamma})\delta_{1R}'K^{-1}\left(e^{-lK}-e^{-tK}\right)e^{tK}\Sigma_X\right)e^{\left(1-\frac{1}{\gamma}\right)\mu+\left(1-\frac{1}{\gamma}\right)^2\frac{1}{2}\sigma^2}}{\int_{43}^{60} e^{\left(1-\frac{1}{\gamma}\right)\mu+\left(1-\frac{1}{\gamma}\right)^2\frac{1}{2}\sigma^2}dl}dldZ_t.
\end{aligned}
$$

For the determination of the optimal portfolio allocation we define the duration of the conversion factor Dk_t as follows:

$$
Dk_t^i = \int_{43}^{60}\frac{\left(-(1-\frac{1}{\gamma})\delta_{1R}'K^{-1}\left(e^{-lK}-e^{-tK}\right)e^{tK}\Sigma_X\right)e^{\left(1-\frac{1}{\gamma}\right)\mu+\left(1-\frac{1}{\gamma}\right)^2\frac{1}{2}\sigma^2}}{\int_{43}^{60} e^{\left(1-\frac{1}{\gamma}\right)\mu+\left(1-\frac{1}{\gamma}\right)^2\frac{1}{2}\sigma^2}dl}dl \tag{10.14}
$$

we then get,

$$
\begin{aligned}
d\log\left(w_t^i\right) &= d\log(c_t^*) + d\log\left(K_t^i\right) \\
&= (...)dt - \frac{1}{\gamma}d\log\left(\phi_t^i\right) + Dk_t^i dZ_t \\
&= (...)dt - \frac{1}{\gamma}(-R_t dt - \Lambda_{ct}' dZ_t) + Dk_t^i dZ_t \\
&= (...)dt + \frac{1}{\gamma}\Lambda_{ct}' dZ_t + Dk_t^i dZ_t.
\end{aligned}
$$

Note that Dk_t^i only influences the uncertainty about the real interest rate and the instantaneous expected inflation.

10.4.1 Stochastic change in the wealth process

For the stochastic change in the wealth process of equation (3.8) and (3.9), we find using the Ito-Doeblin formula:

$$
\begin{aligned}
d\log\left(\frac{W_t^i + H_t^i}{\Pi_t^i}\right) &= (1-\omega_t)dR_t^i + \omega_t dS_t^i - d\Pi_t^i \\
&= (1-\omega_t)R_1'(-KX_t dt + \Sigma_X' dZ_t) + \omega_t((R_t+\eta_s)dt + \sigma_s' dZ_t) \\
&\quad -\pi_t dt - \sigma_\Pi' dZ_t \\
&= \left((1-\omega_t)R_1'(-KX_t) + \omega_t((R_t+\eta_s) - \pi_t\right) dt \\
&\quad + \left((1-\omega_t)\Sigma_X' + \omega_t\sigma_s' - \sigma_\Pi'\right) dZ_t.
\end{aligned}
$$

For the change in the log wealth we have to take into account the change in cash, the one year maturity bond, the 5 year maturity bond, stock return, inflation rate and the payed premium. We can see premium as a bond that pays a fixed dividend over 43 years. The present value of the premium is:

$$
\begin{aligned}
PV_{premium} &= \int_0^{43} p_t^i \cdot P(N)dN \Rightarrow \\
dPV_{premium} &= \int_0^{43} p_t^i \cdot dP(N)dN
\end{aligned}
$$

where $P(N)$ is the bond price for maturity N. It follows that:

$$
\frac{dPV_{premium}}{PV_{premium}} = \frac{\int_0^{43} p_t^i \left((R_t(0) + B(N)\lambda_t)\, dt + B(N)\Sigma_X dZ_t\right) dN}{\int_0^{43} p_t^i \cdot P(N)dN} \tag{10.15}
$$

so it follows that:

$$
d\log(PV_{premium}) = \frac{\int_0^{43} p_t^i \left((R_t(0) + B(N)\lambda_t)\, dt + B(N)\Sigma_X dZ_t\right) dN}{\int_0^{43} p_t^i \cdot P(N)dN}
$$

We can rewrite the equation $d\ln\left(\frac{W_t^i+H_t^i}{\Pi_t^i}\right) = (1-\omega_t)dR_t^i + \omega_t dS_t^i - d\Pi_t^i + dln(PV_{premium})$ since the interest rate term R_t^i consist of three factors, the short rate ${}^0R_t^i$, a one year maturity bond ${}^1R_t^i$ and a five year maturity bond ${}^5R_t^i$. Every factor of the term R_t^i gets an separate weight. For the one year maturity bond we assume that we invest a weight of ω_1, for the five year maturity bond we assume that we invest a weight of ω_2. The weight invested in stocks will be ω_3. We get

$$
\begin{aligned}
d\ln\left(\frac{W_t^i+H_t^i}{\Pi_t^i}\right) &= \begin{bmatrix} 1-\omega_1-\omega_2-\omega_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ 1 \\ 1 \end{bmatrix}' \begin{bmatrix} d^0R_t^i \\ d^1R_t^i \\ d^5R_t^i \\ dS_t^i \\ -d\Pi_t^i \\ d\log(PV_{premium}) \end{bmatrix} \\
&= \begin{bmatrix} 1-\omega_1-\omega_2-\omega_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ 1 \\ 1 \end{bmatrix}' \begin{bmatrix} R_1'(-KX_tdt+\Sigma_X'dZ_t) \\ (r_t+B(1)\Sigma_x\Lambda_t)\,dt+B(1)'\Sigma_X'dZ_t \\ (r_t+B(5)\Sigma_x\Lambda_t)\,dt+B(5)'\Sigma_X'dZ_t \\ (R_t+\eta_s)dt+\sigma_s'dZ_t \\ -\pi_tdt-\sigma_\Pi'dZ_t \\ d\log(PV_{premium}) \end{bmatrix} \\
&= (...)dt+\begin{bmatrix} 1-\omega_1-\omega_2-\omega_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ 1 \\ 1 \end{bmatrix}' \begin{bmatrix} R_1'\Sigma_X'dZ_t \\ B(1)'\Sigma_XdZ_t \\ B(5)'\Sigma_XdZ_t \\ \sigma_s'dZ_t \\ \sigma_\Pi'dZ_t \\ \frac{\int_0^{43}B(N)'\Sigma_XdN}{\int_0^{43}P(N)dN} \end{bmatrix} \Rightarrow
\end{aligned}
$$

We get the following system of equations:

1.

$$
\left((1-\omega_1-\omega_2-\omega_3)R_1^{(1)}+\omega_1B(1)^{(1)}+\omega_2B(5)^{(1)}+\omega_3\sigma_s^{(1)}+\sigma_\Pi^{(1)}\right)dZ_t^{(1)}
$$
$$
-\left(\frac{\int_0^{43}B(N)'\Sigma_XdN}{\int_0^{43}P(N)dN}\right)^{(1)}dZ_t^{(1)}=\left(\frac{1}{\gamma}\Lambda_{ct}'+Dk_t^i\right)^{(1)}dZ_t^{(1)}
$$

2.

$$
\left((1-\omega_1-\omega_2-\omega_3)R_1^{(2)}+\omega_1B(1)^{(2)}+\omega_2B(5)^{(2)}+\omega_3\sigma_s^{(2)}+\sigma_\Pi^{(2)}\right)dZ_t^{(2)}
$$
$$
-\left(\frac{\int_0^{43}B(N)'\Sigma_XdN}{\int_0^{43}P(N)dN}\right)^{(2)}dZ_t^{(2)}=\left(\frac{1}{\gamma}\Lambda_{ct}'+Dk_t^i\right)^{(2)}dZ_t^{(2)}
$$

3.

$$
\left(\omega_3\sigma_s^{(3)}+\sigma_\Pi^{(3)}\right)dZ_t^{(3)}=\frac{1}{\gamma}\Lambda_{ct}^{(3)}dZ_t^{(3)}
$$

4.

$$
\omega_3\sigma_s^{(4)}dZ_t^{(4)}=\frac{1}{\gamma}\Lambda_{ct}^{(4)}dZ_t^{(4)}.
$$

This does not give an unique analytical solution since there is no other asset apart from S_t^i that influences the third shock dZ_t^3. A solution would be to add the restriction $\Lambda_{ct}^3=0$ and $\sigma_s^3=-\frac{1}{\omega_3}\sigma_\Pi^3$.

We can then rewrite our problem in the following form:

$$\omega' H dZ_t = \left(\frac{1}{\gamma}\Lambda'_{ct} + Dk_t^i\right) dZ_t$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}' \begin{bmatrix} B(1)^{(1)} - R_1^{(1)} & B(1)^{(2)} - R_1^{(2)} & 0 \\ B(5)^{(1)} - R_1^{(1)} & B(5)^{(2)} - R_1^{(2)} & 0 \\ \sigma_s^{(1)} - R_1^{(1)} & \sigma_s^{(2)} - R_1^{(2)} & \sigma_s^{(4)} \end{bmatrix} \begin{bmatrix} dZ_t^{(1)} \\ dZ_t^{(2)} \\ dZ_t^{(4)} \end{bmatrix} =$$

$$\begin{bmatrix} \left(\frac{1}{\gamma}\Lambda'_{ct} + Dk_t^i\right)^{(1)} - \sigma_\Pi^{(1)} - {}^0R_1^{(1)} - \left(\frac{\int_0^{43} B(N)'\Sigma_X dN}{\int_0^{43} P(N)dN}\right)^{(1)} \\ \left(\frac{1}{\gamma}\Lambda'_{ct} + Dk_t^i\right)^{(2)} - \sigma_\Pi^{(2)} - {}^0R_1^{(2)} - \left(\frac{\int_0^{43} B(N)'\Sigma_X dN}{\int_0^{43} P(N)dN}\right)^{(2)} \\ \left(\frac{1}{\gamma}\Lambda'_{ct}\right)^{(4)} \end{bmatrix} \begin{bmatrix} dZ_t^{(1)} \\ dZ_t^{(2)} \\ dZ_t^{(4)} \end{bmatrix}.$$

It then follows that we also have the restriction $\Lambda_{ct}^4 = -\frac{\sigma_s^3\sigma_s^4\gamma}{\sigma_\Pi^3}$. We call ω_3 the speculative demand for stocks. Intuitively, it is the quotient of how well risk taking in stocks is rewarded i.e. the prices of risk $\Lambda_{ct}^{(4)}$ divided by how risky stocks really are (how volatile they are, which is incorporated in the parameter $\sigma_S^{(4)}$). This quotient is multiplied with the attitude towards risk which is symbolized by γ; the parameter for risk aversion. The higher γ the more risk averse we are and the lower ω_3 is, resulting in a low investment rate in stocks.

$$\omega_3 = \frac{\Lambda_{ct}^{(4)}}{\gamma\sigma_S^{(4)}}$$

ω_2 contains a speculative demand for a one year maturity bond and a hedging demand. The hedging demand is equivalent with the duration. The same holds for ω_1.
When we determine the optimal investment strategy using a utility function we take into account the risk preferences of a pension participant. If we would only maximize the sum of the expected pension payouts and not apply a utility function we would get a high $50th$ percentile in the pension result but the left tail, i.e. the 5th percentile can be low due to risky investments.
In this chapter we used the same utility function as in the Merton model, see equation (5.1). The Merton model, however, assumes that there are only stocks and risk-free bonds to invest in. There is no interest rate risk in the Merton model. In the optimal life cycle that we derived in this chapter we do take interest rate risk into account. Let $u(c) = \int_s^T \frac{1}{1-\gamma}\left(\frac{c_l^i}{\Pi_l^i}\right)^{1-\gamma} dl$ and let c^* be our optimal pension payout. It holds that:

$$E_t[u(c^*)] \geq E_t[u(c)]$$

but on the other hand, since we only maximized the life cycle of the pension payments, we cannot say whether the optimal c^* is expected to be the maximum flow of pension payments in general:

$$E\left[\int_s^T \frac{c_l^*}{\Pi_l^i} dl\right] \lesseqgtr E\left[\int_s^T \frac{c_l}{\Pi_l^i} dl\right]$$

With this we can see that the real pension payouts are not maximized and therefor the pension result will not be optimal using the derived life cycle in this section. So in general we expect to find a lower $50th$ percentile, i.e. a lower pension result. Since we avoid more risk. On the other hand we also expect to find a lower maximum deviation, so a lower difference between the $50th$ and the $5th$ percentile since we governed all risky assets in the optimal life cycle.

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