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A Quantum Annealing Approach for Solving Hard Variants of the Stable Marriage Problem

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Abstract. The Stable Marriage Problem (SMP) describes the problem, of finding a stable matching between two equally sized sets of elements (e.g., males and females) given an ordering of preferences for each element. A matching is stable, when there does not exist any match of a male and female which both prefer each other to their current partner under the matching. Finding such a matching of maximum cardinality, when ties and incomplete preference lists are allowed, is called MAX-SMTI and is an NP-hard variation of the SMP.

In this work a Quadratic Unconstrained Binary Optimization (QUBO) formulation for MAX-SMTI is introduced and solved both with D-Wave Systems quantum annealing hardware and by their classical meta-heuristic QBSolv. Both approaches are reviewed against existing state-of-the-art approximation algorithms for MAX-SMTI. Additionally, the proposed QUBO problem can also be used to count stable matchings in SMP instances, which is proven to be a #P-complete problem. The results show, that the proposed (quantum) methods can compete with the classical ones regarding the solution quality and might be a relevant alternative, when quantum hardware scales with respect to the number of qubits and their connectivity.

Keywords: Quantum annealing · Stable marriage problem · Optimization · D-wave systems · Heuristic · MAX-SMTI

1 Introduction

The Stable Marriage Problem (SMP) was first defined by Gale and Shapley in 1962 [6]. The problem consists of two sets with n males and n females, which rank the opposite gender in strictly ordered and complete preference lists. The goal is then to find stable matchings for those males m and females w. Stable means, that there is no pair (m_i, w_j) such that both would prefer each other to their current match.

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SMP and its generalized variants have large applications in industry and science. For instance, Maggs et al. used the Stable Marriage Problem in order to explain a content delivery network (CDN) in which clients are mapped to server clusters of the CDN [17]. Other well known applications are the assignment of graduating medicine students to their first hospital appointments [22] or the design of the clearinghouse adopted by the National Residency Matching Program [23].

Since SMP has a large field of applications a lot of research has been made over the years. A survey, regarding SMP and its variants is given in [12]. In our work an NP-hard variation of the SMP, where the goal is to find the maximum cardinality with ties and incomplete preference lists, the MAX-SMTI, will be investigated. Podhradsky et al. compared the state-of-the-art approximation algorithms for MAX-SMTI [21], while Delorme et al. reviewed the mathematical models for MAX-SMTI [5].

With Quantum Computing (QC) technology emerging, there is now the possibility to solve such kind of problems in a completely different way. One field of QC is Quantum Annealing (QA), a meta-heuristic, which is implemented in hardware by D-Wave Systems. In order to perform quantum computations on such a machine, it is necessary to cast the problem into a certain mathematical form, the Quadratic Unconstrained Binary Optimization (QUBO) or the equivalent Ising formulation. In [8] and [16], many QUBO and Ising formulations for well known NP-hard problems are introduced.

In the following, SMP and the more complex variation MAX-SMTI are defined in Sect. 2. After giving some fundamentals of Quantum Annealing, our QUBO formulation for MAX-SMTI is presented in Sect. 3. In Sect. 4 the experimental setup is given and the results of the MAX-SMTI QUBO formulation by using D-Wave Systems 2000Q Quantum Annealer and the classical software tool QBSolv are compared against two state-of-the-art approximation algorithms, SHIFTBRK and Kiraly2 [21]. The optimal solutions (ground truth) of the small test instances are determined by a linear programming solver [24]. The algorithms are compared regarding their solution quality and a theoretical outlook w.r.t the computational time is given.

2 Background

2.1 Stable Marriage Problem

The Stable Marriage Problem describes the problem of finding a stable matching between two sets, males $\mathcal{M} = \{m_1, m_2, ..., m_n\}$ and females $\mathcal{W} = \{w_1, w_2, ..., w_n\}$. Elements of both, $\mathcal{M}$ and $\mathcal{W}$, rank the other sex in strictly ordered and complete preference lists. Therefore every preference list has the length of n.

Let $M(m|w)$ describe the notation for $m|w$'s matching partner and let $w_2 \succ_{m_2} w_3$, describe the notation that m_2 (strictly) prefers w_2 over w_3. A pair $p = (m, w)$ is considered as blocking towards M, if $w \succ_m M(m)$ and $m \succ_w M(w)$.

A matching $M = \{(m_i, w_j), ...\}$ of size n is considered as stable, if there exists no blocking pair p.

Note, that this doesn't mean everybody is "happy" in terms of being matched to the most desired possibility available. Furthermore this can be expressed, that everybody is matched in a way he or she cannot complain.

It is known, that SMP can be solved in $O(n^2)$ by the Gale-Shapley (GS) algorithm [18]. The basic GS algorithm only works for complete and strict preference lists, nevertheless this often does not suit reality well. Therefore, mainly three variants of the SMP were developed. The Stable Marriage Problem with ties in its preference lists (SMT), the Stable Marriage Problem with incomplete preference lists (SMI), and the combination of both, the Stable Marriage Problem with ties and incomplete preference lists (SMTI).

SMT - Stable Marriage Problem with Ties. In the basic SMP, the preference lists must be ordered in a strict way. Relaxing this rule, so that indifferences are allowed in preference lists, results in two different ways of describing preferences and in three definitions of stability.

- **Strict Preference:** This is true for (m_1, w_2, w_3), if m_1 prefers w_2 over w_3 and does not tie w_2 and w_3. It is denoted as $w_2 \succ_{m_1} w_3$.
- **(Weak) Preference:** This is true for (m_1, w_2, w_3), if m_1 prefers w_2 over w_3 and both are tied on m_2's preference list. The notation here is $w_2 \succcurlyeq_{m_1} w_3$.

Consequently, three definitions for stability of a matching can be made [18]:

- **Weakly Stable:** There is no pair that would strictly prefer each other over their partner in M.
- **Strongly Stable:** There is no pair (m, w), so that $w \succ_m M(m)$ and $m \succcurlyeq_w M(w)$.
- **Super Stable:** There is no pair that would (weakly) prefer each other over their current match.

Such a pair (described in the manner of weakly, strongly and super stability), like in SMP, is a blocking pair if it exists.

SMI - Stable Marriage Problem with Incomplete Preference Lists. Incomplete preference lists do refer to the term that not every person of the opposite sex needs to be listed on a persons preference list. This results in the fact that not every male can be matched to every female.[1] Therefore an acceptable pair can be defined as followed.

Acceptable Pair: A pair (m, w) is acceptable if m has listed w on his preference list and w has listed m. In SMI, a matching is stable, if it is stable in terms of SMP and no unacceptable pair was matched.

[1] Consider the situation, in which m has w on his preference list, but w does not mention m. A match (m, w) is therefore impossible.

SMTI - Stable Marriage Problem with Ties and Incomplete Preference Lists. Combining both generalizations from before, we obtain the Stable Marriage Problem with ties and incomplete preference lists (SMTI). Finding a weakly stable match for a given SMTI problem, by breaking down ties arbitrarily (like in SMP), will affect the size of the resulting match, as explained in Example 1. This effect will be interesting for MAX-SMTI and therefore, in the scope of this work, weakly stable SMTI will be reviewed as SMTI. So a pair (m, w) is considered as blocking towards a matching M, when the following takes account:

$$(free(m) \vee w \succ_m M(m)) \wedge (free(w) \vee m \succ_w M(w)) \tag{1}$$

where $free(m|w)$ denotes if $m|w$ are matched to anyone in M. Note, that either $w \succ_m M(m))$ or $free(m)$ is sufficient to create an unblocking pair towards the matching M.

Example 1. Consider the example from Table 1, with $n = |\mathcal{M}| = |\mathcal{W}|$. It is easy to review that $M_1 = \{(m_2, w_3), (m_3, w_1)\}$ is weakly stable (has no blocking pairs). Also $M_2 = \{(m_2, w_2), (m_3, w_1), (m_1, w_3)\}$ can be reviewed as weakly stable. Note that the cardinality of the two stable matchings differ in size. This effect of possible indifference in the cardinality of the matchings does only occur in the weakly stability criteria.

Table 1. Arbitrary SMTI instance: the persons in the brackets are tied.

Males				Females			
m_1:	w_1	w_3		w_1:	$(m_2$	$m_3)$	
m_2:	$(w_2$	$w_3)$	w_1	w_2:	$(m_1$	m_2	$m_3)$
m_3:	w_1	w_3	w_2	w_3:	$(m_3$	$m_2)$	m_1

MAX-SMTI. In the previous section it was discussed, that breaking ties arbitrarily in SMTI will affect the size of the resulting match. Trying to find the matching with the largest cardinality is an NP-hard optimization problem, referenced in the following by MAX-SMTI [11].

2.2 Quantum Annealing

Quantum Annealing is a meta-heuristic, which was formulated in its general and present form by T. Kadowaki and H. Nishimori in 1998 [14] and is commonly used to solve problems with a discrete search space with many local minima. D-Wave Systems implemented the meta-heuristic in their hardware, in order to find low energy states of a spin glass system, described by an Ising Hamiltonian,

$$\mathcal{H}(s) = \sum_i h_i x_i + \sum_{i<j} J_{ij} x_i x_j \tag{2}$$

where h_i is the energy, which directly affects qubit i. J_{ij} represent the energies interacting between the two qubits i and j and x_i is the spin $(-1, +1)$ of the i-th qubit. Many optimization problems can be formulated as an Ising Hamiltonian and therefore be executed on D-Wave's quantum annealing hardware [16], in attempt to find good or acceptable solutions.

Within the quantum annealing process an initial Hamiltonian H_I with an easy to prepare minimal energy configuration (or ground state) is physically interpolated to a problem Hamiltonian H_P, whose minimal energy configuration is sought and corresponds to the best solution of the defined problem, see (3). This transition is described by an adiabatic evolution path which is mathematically represented as function $s(t)$ and decreases from 1 to 0 [19].

$$H(t) = s(t)H_I + (1 - s(t))H_P \tag{3}$$

If this transition is executed sufficiently slow, the probability to find the ground state of the problem Hamiltonian is close to 1 [1]. Thus, by mapping the MAX-SMTI problem onto a spin glass system, quantum annealing in theory is able to find the solution of it.

For completeness, we map our optimization problem to a QUBO problem, which is an alternative formulation of the Ising spin glass system. It is mathematically equivalent and uses 0 and 1 for the spin variables [2,25]. The quantum annealer is also capable of minimizing the functional form of the QUBO problem,

$$\min x^t Q x \qquad \text{with } x \in \{0, 1\}^n \tag{4}$$

with x being a binary vector of size n, and Q being an $n \times n$ real-valued matrix describing the relationship between the variables. Given matrix $Q : n \times n$, the annealing process tries to find binary variable assignments $x \in \{0, 1\}^n$ to minimize the objective function in (4).

3 QUBO Formulation for MAX-SMTI

In the following section, the QUBO formulation for MAX-SMTI gets introduced.

3.1 Encoding

When reformulating MAX-SMTI towards a QUBO problem, every bit variable of the bit-vector x corresponds to an acceptable pair in the SMTI problem instance. Therefore, the size of the bit-vector and the QUBO matrix is determined by the number of acceptable pairs in the SMTI instance. The index of a bit-variable x_i is therefore a unique identifier for one possible match in SMTI. Hence, the notation x_{m_i,w_j} references the unique index for the acceptable pair (m_i, w_j).[2] The reversed encoding (deduce the male/female from the current index of x_i) is written as m_{x_i} or w_{x_i}. In case of $x_i = 1$, it can be concluded that (m_{x_i}, w_{x_i}) are matched.

[2] This kind of notation implies, that the referenced pair is always acceptable.

3.2 Objective Function and Constraints

In MAX-SMTI it must be ensured that no one is matched twice and that the matching is stable. Additionally, the matching with the maximum cardinality is searched. In the following sections, those two constraints and the objective function will be explained in detail.

Constraint - No One is Matched Twice. Since QUBO is unconstrained, the quantum annealer could try to match a person twice, in order to gain a smaller final solution energy.

This would not result in a stable matching nor any valid response. So, the constraint needs to be enforced by (5).

$$p_3 \cdot \sum_{i,j} x_i x_j \cdot [i \neq j]([m_{x_i} == m_{x_j}] + [w_{x_i} == w_{x_j}]) \tag{5}$$

Note, that the square brackets indicate boolean formulations, which values are cast to their integer counterparts 0 and 1 for false and true, respectively. So, if either one of the males or females in that formula equals each other, both bit variables are 1 and they are matched twice. Therefore, the penalty p_3 is added to the solution energy and states it as invalid.

Constraint - The Matching is Stable. In order to assure the stability of the match, (1) was reformulated into the following constraint:

$$\begin{aligned} p_2 \cdot (&\sum_{i,j}^{n} x_{m,w_j} x_{m_i,w} \cdot [\neg w \succ_m w_j \wedge \neg m \succ_w m_i] \\ -&\sum_{i}^{n} x_{m_i,w}[\neg m \succ_w m_i] - \sum_{i}^{n} x_{m,w_i}[\neg w \succ_m w_i] + 1) \end{aligned} \tag{6}$$

Note that (6) must hold for all $(m, w) \in A$, with A being the set of acceptable pairs.

The first term ensures, that two matched pairs that would create a blocking pair are less energy effective. The second and third terms enforce a couple (w, m_i), where $\neg m \succ_w m_i$, regardless of whether (w, m_i) gets matched or not. So while the first part adds up a penalty in case a pair tends to be blocking, the second part promotes pairs, which are more likely to be stable by adding negative values on the diagonal of the QUBO matrix.

Objective Function - The Maximum Cardinality is Found. To enforce the stable matching with the maximum cardinality, $-p_1$ needs to be assigned onto every element of the diagonal. This has the effect, that solutions with more matched pairs have a lower energy than solutions with less matched pairs.

$$-p_1 \sum_i x_i \quad \forall i \in |A| \tag{7}$$

3.3 Penalty Assignment

By giving the constraints in the sections above, three penalties (p_1, p_2 and p_3) were introduced. p_1 was set to 1 by default, since it is just used for the objective function to define, respectively count, the maximum cardinality. However, it is more important to ensure that a match is stable than finding a match of maximum cardinality. Therefore, p_2 was set to n, with n being the size of the sets of males/females. The last penalty is $p_3 = n \cdot p_2 + p_1$. Since p_3 references the hard constraint that no one is matched twice, it needs to be larger than every value assigned to the diagonal of the QUBO matrix. The minimum value, that can be assigned onto the diagonal is, $n \cdot p_2 + p_1$.

3.4 Resulting Energy

For a simple verification of the solution, the resulting energy e can be used. Considering (6), it can be seen, that per acceptable pair, the equation equals $-p_2$. Since this constraint needs to hold for all acceptable pairs, the energy for a stable solution has to be $e \leq -p_2 \cdot |A|$. In (7), $-p_1$ is assigned to every element of the diagonal and added up per couple in the resulting match. Combining the two observations, the resulting energy of a stable solution is defined in (8). Here n_r is the size of the resulting match with $n_r > 0$.

$$e = -(p_2 \cdot |A| + p_1 \cdot n_r) \tag{8}$$

So in general, with having the energy to a solution, the stability criteria can be verified easily. However, without knowing the maximum cardinality, it is not possible to determine the energy of the optimal solution for MAX-SMTI.

4 Experimental Setup

4.1 Datasets

For the computational experiments, two kinds of datasets were created. One consists of random SMTI-Instances, while the other one consists of random SMP-Instances, further called SMTI-Dataset and SMP-Dataset, respectively.

SMTI-Dataset. For the SMTI-Dataset, 50 instances per size in the range of $[3; 30]$ were created by an algorithm proposed by Gent et al. [7]. This algorithm takes two parameters g_1 and g_2. g_1 describes the probability of one element being added to a preference list and g_2 describes the probability of the occurrence of a tie in a preference list. For each sample, both parameters were randomly drawn to promote uniformly distributed ties and preference lists over all samples per size.

SMP-Dataset. For the creation of the SMP-Dataset, the same approach as for the SMTI-Dataset has been used just with 20 samples per size. For the generation of SMP instances it is sufficient to shuffle each preference list to create a new SMP instance.

4.2 Methods

Linear Programming Solver. Regarding MAX-SMTI, the optimal solution can be found by computing every matching for every possible arrangement of tie breaks and return the matching with the maximum cardinality. Since this brute force approach is quite inefficient, another approach to find the optimal solution was developed by Roth et al. using integer linear programming (LP) [24]. For our experiments, we used COIN-OR‘s branch and cut method to solve the LP model of Roth et al. [13]. In this work this LP approach will be reviewed as MAX-SMTI-LP.

Kiraly2. Kiraly et al. introduced two simple linear approximation algorithms for MAX-SMTI [15]. The first, a linear 3/2-approximation algorithm, works for SMTI instances with men strict preference lists. This was followed by a second algorithm for general SMTI problems with an approximation ratio of 5/3, based on the former mentioned algorithm. See [21] for a detailed description and implementation.

SHIFTBRK. Halldòrsson et al. introduced SHIFTBRK, an approximation algorithm with a ratio of $2/(1 + L^2)$, where L is the length of the longest tie among all ties in the preference lists [9]. So the approximation ratio decreases within the increase of the length of the longest tie. The algorithm is based on iteratively breaking and shifting ties and can solve the resulting SMI instances via the well known GS algorithm [6].

D-Wave Systems Quantum Annealing. As already mentioned in Sec. 2.2, D-Wave Systems Quantum Annealer takes a QUBO problem as input and takes care of the annealing process. However, since the hardware is still quite restricted in its resources, w.r.t. the number of qubits and their connectivity, we could only use it for relatively small problem instances up to a size of $n = 7$. In this work this approach is reviewed as QUBO-MAX-SMTI (QA).

QBSolv. QBSolv is a software tool, which splits a QUBO problem into smaller components (subQUBOs) of a predefined subproblem size, which are then solved independently of each other. This process is executed iteratively as long as there is an improvement and it can be defined using the QBSolv parameter "num_repeats". This parameter determines the number of times to repeat the splitting of the QUBO problem matrix after finding a better sample. With doing so, the QUBO matrix is split into different components using a classical tabu search heuristic in each iteration. QBSolv can be used in a completely classical way to solve the subQUBOs or as a quantum-classic hybrid method by solving the single subQUBOs on the quantum annealer.

Besides embedding and splitting the QUBO into subQUBOs, QBSolv also takes care of unembedding and merging of the subproblems' solutions. See [20] for more details on QBSolv. In this work this approach is reviewed as QUBO-MAX-SMTI (QBSolv).

5 Results and Discussion

5.1 Solution Quality

In Fig. 1 we evaluated the solution quality of our QUBO-MAX-SMTI (QA/QBSolv) methods against two state-of-the-art approximation algorithms for MAX-SMTI, Kiraly2 and SHIFTBRK. The ground truth was calculated using the MAX-SMTI-LP solver. For problem sizes ranging from 3 to 30, 50 randomly sampled instances per size were used. The approximation ratio was calculated by the number of optimal solutions found divided by the number of samples per size.

One can see, that the QUBO-MAX-SMTI (QBSolv) could find every optimal stable matching of the problem instances till the size of $n = 13$ and even outperforms the other approximation methods in some cases ($n = 10, 11, 12, 15$). Regarding the larger problem instances, QUBO-MAX-SMTI (QBSolv) decreases in solution quality to a minimum of 35% in problem size $n = 30$, while SHIFTBRK, for example, stays in the approximation ratio of around 90%.

QUBO-MAX-SMTI (QA) was run on the D-Wave 2000Q quantum chip for only the relatively small problem instances, due to the reasons mentioned in Sect. 4.2. The number of measurements per problem was set to 100. The results show, that the proposed approach can keep up with QUBO-MAX-SMTI (QBSolv) and SHIFTBRK and even outperforms Kiraly2 till the problem size of $n = 5$. However, afterwards the approximation ratio decreases to around 60%. This might be due to large physical qubit chains, which occur, when the logical qubits of the QUBO problem don't fit directly to the physical qubits and their connectivity of the quantum architecture. However, one could improve performance by adjusting the hyperparameter *chain_strength* of those qubit chains [4]. Additionally, we ran the two proposed methods against an exact solver, the MAX-SMTI-LP, for the same problem instances as above. In Fig. 2 the solution energies of QUBO-MAX-SMTI (QA/QBSolv) are compared against the energy of the global optimum found by the MAX-SMTI-LP and the solution energies describing a stable matching, which can be determined as described in Sect. 3.4. The energy of the global optimum can be calculated by encoding the exact MAX-SMTI-LP solution on the binary vector x of the QUBO formulation and computing the matrix multiplication $x^t Qx$.

In Fig. 2a the QUBO-MAX-SMTI (QA) method found for the two smallest problem sizes ($n \leq 4$) the optimal solution (determined by the resulting energy) in every problem instance. For the larger problem sizes ($5 \leq n$), the percentage

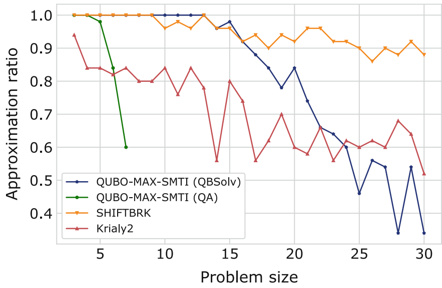
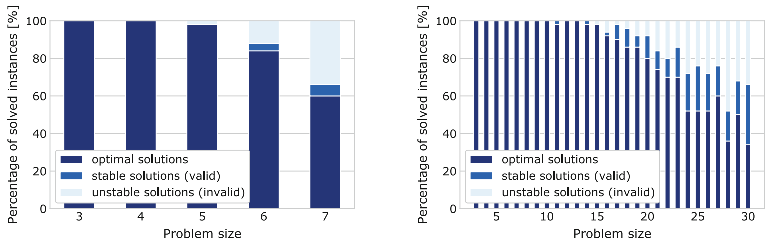


Fig. 1. Solution qualities of the approximation algorithms for different sized MAX-SMTI problem instances. For each problem size, 50 random sampled instances were used. The approximation ratio was calculated by the number of optimal solutions found, divided by the number of samples per size.

of optimal solved instances decreases to a minimum of 60%. Additionally one can see, that for problem size $n = 6$ and $n = 7$, the MAX-SMTI-LP found a better solution, i.e. it has a lower solution energy, for 5% respectively 8% of the instances. In some cases, the QUBO-MAX-SMTI (QA) method could not find stable matches, as shown in light blue.

The QUBO-MAX-SMTI (QBSolv) method found for a problem size of $n \leq 13$ every optimal solution of the used problem instances, except for $n = 11$, as shown in Fig. 2b. From then on, the percentage of optimal solved instances decreases to a minimum of 35% in problem size 30. Moreover, it can be seen, that for the largest problem size ($n = 30$), the percentage of optimal solved instances roughly equals the percentage of non optimal solved instances and invalid, i.e. unstable solutions. Besides that, another finding is, that QUBO-MAX-SMTI (QA/QBSolv) can be used to find and count the number of stable solutions for SMP instances, which in general is #P-complete [10]. In Fig. 3 the percentage of all stable matchings per problem size found by the proposed methods, are shown. For each problem size, 20 instances were used. The baseline was delivered by the backtracking algorithm proposed in [26] and was run as long as QBSolv. The results show, that QUBO-MAX-SMTI (QBSolv) found every stable matching till the size of $n \leq 11$ for every problem instance. From then on, the percentage of finding all stable matchings decreases to a minimum of 9% in problem size 18. However, for larger problem instances ($n \geq 15$), QUBO-MAX-SMTI (QBSolv) found more stable matches in the same amount of time as the backtracking algorithm. The QA approach was only able to find every stable matching for $n \leq 4$. With $n = 5$ only 30% of the stable matchings were found, while for $n \geq 6$ QUBO-MAX-SMTI (QA) was not able to find any stable machting for those SMP instances. The reason why this QA method finds stable solutions for the SMTI instances for the same problem size lies in the size of the actual QUBO matrix, which is proportional to the number of acceptable pairs. For SMTI, the number of acceptable pairs is less or equal to n^2, due to the incomplete preference



(a) *Solution energy of QUBO-MAX-SMTI (QA)* **(b)** *Solution energy of QUBO-MAX-SMTI (QBSolv)*

Fig. 2. The percentage of optimal (dark blue), stable but not optimal (medium blue) and unstable (light blue), i.e. invalid, solutions of the problem instances. The problem size varies from 3 to 30. For each problem size 50 problem instances were used. (Color figure online)

lists, while the number of acceptable pairs for SMP is equal to n^2. As a result, the size of the QUBO matrix for SMP instances increases much faster than for the SMTI instances and therefore the corresponding solution space.

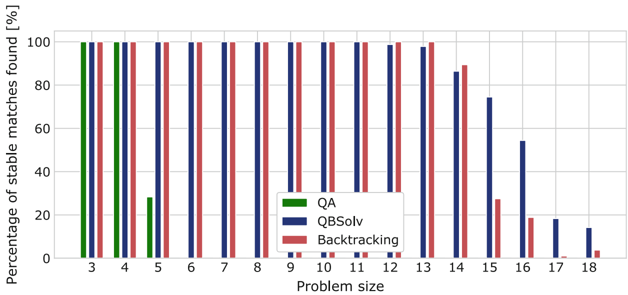


Fig. 3. The percentage of all stable matchings per problem size found by the proposed methods. For each problem size 20 instances were used. The baseline was delivered by the backtracking algorithm proposed in [26]. The backtracking algorithm was run as long as QBSolv and then stopped.

5.2 Computational Results

Regarding the computational results, it is hard to draw a fair practical comparison between the state-of-the-art classical methods for MAX-SMTI and the quantum annealing approach presented in this paper. Since the interaction of classical and quantum hardware in our approach leads to additional overhead (cloud access time and job queuing time at D-Wave Systems) we didn't do a time to solution comparison but rather give a theoretical outlook of the computation times.

The preprocessing time of MAX-SMTI-QUBO (QA/QBSolv) is in $O(n^4)$ and contains the computation of acceptable pairs and the creation of the QUBO matrix. MAX-SMTI-LP has a slightly better preprocessing time of $O(n^3)$, when setting up the integer linear program. However, when it comes to solving the models (QUBO and ILP) we expect the former being computational superior. The QA algorithm implemented in D-Wave Systems QPU basically runs in constant time $O(1)$, while the runtime of the mixed-integer-programming solver (COIN-OR's branch and cut) is definitely larger than $O(1)$ and since it is a heuristic its runtime complexity is hard to determine exactly. We expect, that with quantum annealing hardware getting larger in the number of qubits and their connectivity, it could outperform the state-of-the-art methods w.r.t. bigger problem size instances ($n > 7$).

6 Conclusion

We introduced QUBO-MAX-SMTI (QA/QBSolv), the first QUBO formulation for solving MAX-SMTI instances with classical and also quantum annealing hardware. The QUBO formulation for MAX-SMTI requries maximal n^2 variables with n being the number of acceptable pairs and therefore does not need any additional slack variables. Our experiments show the current state of applicability and provide a comparison to some state-of-the-art classical algorithms.

Regarding the solution quality, the approach of using the QUBO formulation with QA and QBSolv can keep up with the state-of-the-art algorithms and even outperform them in some cases. Additionally, the QUBO formulation can be used to find and count multiple stable solutions, which is a #P-complete task. However, since the quantum hardware is still in its infancy the QA method was only applicable for $n \leq 7$ and is therefore at the moment not competitive with the classical methods for the relevant problem sizes. Nevertheless, the experiments give hope, that the quantum approach gains in importance when the corresponding hardware increases w.r.t the number of qubits and their connectivity.

The addressed problem of the immaturity of the quantum hardware makes it likewise hard to draw a final conclusion regarding the hoped computational quantum advantage. Since at the moment the classical methods outclass the quantum method from a practical point of view (time to solution), the theoretical outlook predicts, that using quantum annealing hardware for solving QUBO fomulation might be comparatively more efficient.

Regarding future work it would be interesting to see, if the computational times and the solution quality improve when using D-Wave Systems new annealer called Advantage [3]. The new hardware chip has up to 5000 qubits and a better connectivity. This enables to solve larger fully connected QUBO problems and a shorter total computation time might be achieved.

Concluding, the proposed QUBO approach might be a relevant alternative to the classical methods, when quantum hardware scales.

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