Delft University of Technology
Faculty of Electrical Engineering, Mathematics & Computer Science
Delft Institute of Applied Mathematics

The existence of travelling waves of the discrete Nagumo equation

Dutch title: Het bestaan van golf oplossingen van de discrete Nagumo vergelijking

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"The existence of travelling waves of the discrete Nagumo equation"

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Abstract

This thesis considers solutions to the discrete Nagumo equation

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), n \in \mathbb{Z}.$$

For sufficiently large d, the solutions are of the form $u_n(t) = U(n + ct)$ with $c > 0$. This thesis contains the proof of existence of traveling wave solutions of the discrete Nagumo equations and originates from Bertram Zinner's article "Existence of Traveling Wavefront Solutions for the Discrete Nagumo equation" [Zin90]. In the first chapter, all the prerequisite knowledge needed to understand the proof, such as Brouwer's fixed point theorem, is presented. The proof starts by considering $f(u_n)$ as a linear function and thus simplifying the problem. The simplified problem is converted into a fixed point problem by considering a Poincaré map which can be solved using Brouwer's fixed point theorem. Finally, the proof ends by confirming that the solutions of the approximated, simplified problem have a limit point which corresponds to the traveling wave solutions of the discrete Nagumo equation.

Contents

Chapter 1

Introduction

In this thesis solutions to the discrete Nagumo equation

$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n) \tag{1.1}$$

are considered. Zinner's article [Zin90] discusses the proof of existence of solutions of the form $u_n(t) = U(n + ct)$ with $c > 0$ of the discrete Nagumo equation. This proof will be discussed in detail in this thesis.
The thesis starts with chapter 2 which describes how electric impulses being transmitted from one membrane of a nerve cell to another can be modelled. The discrete Nagumo equation (1.1) models myelinated axons between nervous cells where u_n represents the membrane potential at cell n.

Chapter 3 gives the needed prior knowledge to understand Zinner's article. In section 3.1.1 the Picard-Lindelöf theorem will be stated and proven. This theorem states the existence of a unique local solution of an initial value problem. The section continues by describing how the solution of the initial value problem is dependent on the initial value. Brouwer's fixed point theorem 3.2.5 will be discussed in section 3.2. This theorem states that every continuous mapping in $\mathbb{R}^n$ from a closed ball to itself contains at least one fixed point. The "hairy ball" theorem 3.2.4, stated in section 3.2.1, proves that there is no continuous tangent vector field on even-dimensional n-spheres and will be used to prove Brouwer's fixed point theorem 3.2.5. The final section 3.3 of chapter 3 contains the general homotopy theorem (3.3.4). This theorem states that the fixed point index, which "counts" the number of fixed points of a map h_λ from an open subset of a closed subspace to that space, is independent of λ of the chosen homotopy h_λ.

The article by Zinner [Zin90] will be discussed in chapter 4. In section 4.4 the discrete Nagumo equation (1.1) will be simplified by considering the linear function $h_0(x) = x - \frac{1}{4}$ instead of $f(u_n)$ and by restricting u_n to take values between 0 and 1. In this section it will also be proven in theorem 4.4.2 that the number of solutions to the simplified equation (4.2) is finite. In section 4.5 it will be concluded by the Picard-Lindelöf theorem 3.1.2 that the simplified equation (4.2) with initial values $u_n(0) = x_n$ has a unique solution $u(x;t)$ which depends continuously on the initial value x. Then the shifted Poincaré map

$$(Tx)_n = \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x;\tau) & \text{for } n = 1, .., N \end{cases}$$

is defined and it follows by Brouwer's fixed point theorem 3.2.5 that the map T_0 has a fixed point which corresponds to the traveling wave solution $u_n(t)$. So far, the construction of T_0

depended on the linear function h_0. In section 4.7 a homotopy h_λ will be defined which deforms h_0 into a Lipschitz continuous function h_1. The general homotopy invariance theorem 3.3.4 will be used to conclude that the number of fixed points of T_{h_1} is independent of λ. In the final section 4.7 f will be approximated by h^k. It will be proven that if the traveling wave u_n^k is a solution to

$$\dot{v}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h^k(u_n)$$

then u_n^k is a traveling wave solution of the discrete Nagumo equation (1.1). This concludes the proof of theorem 4.1.1.

Chapter 2

Application

Neurons in animal nerve systems are connected by “wires”, called axons. Axons carry electrical signals from one membrane of a neuron to another. Myelin is a material that surrounds parts of nerve cell axons. It increases the rate at which the nervous cells send out electrical impulses, also called action potentials [Bea07]. Myelin does not form a single long cover for an axon but rather multiple, small covers. Therefore the action potential “jumps” from one cover to another [MW 07]. When the signal reaches the final axon, this electrical signal starts a chemical reaction. Myelin is crucial for the nerve system to work accurately.

When waves fail to transmit the action potential, it can lead to failure of the nerve system. For example, when the cardiac action potential fails, it can lead to fatalities. Therefore, studying models of action potentials of animal cells is highly relevant for neurophysiology.

The goal is to model the action potential between animal neuron cells [Kee87]. Suppose that u_n represents the membrane potential at cell n, where $n \in \mathbb{N}$. In figure 2.1 a visualisation of the structure of a simplified neuron is visible.

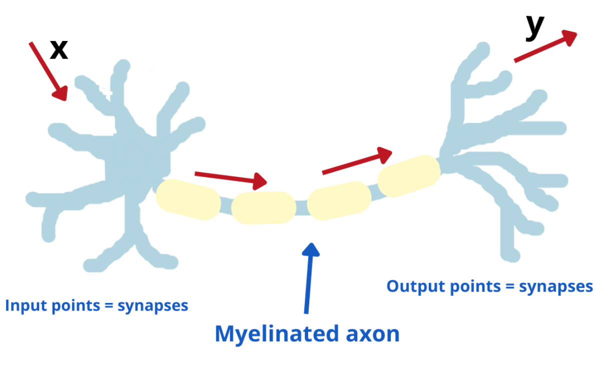


Figure 2.1: Structure of simplified neuron

Kirkchhoff’s laws deal with potential difference in electrical circuits. Currents between cells satisfy Kirchhoff's laws and it is assumed that all n cells are coupled. Furthermore, it is assumed that the potential is equal for every cell. This gives the following equation

$$\dot{u}_n = d(u_{n+1} - 2u_n + u_{n+1}) + f(u_n).$$

Here d is the coupling coefficient, $d = \frac{1}{R}$, where R is the intercellular resistance. Thus d represents the "amount" of lost transmitted electrical signal.

Chapter 3

Prerequisite knowledge

3.1 Ordinary differential equations

The simplified discrete Nagumo equation (4.3) with initial condition $v_n(0) = x_n$ with $0 \leq x_n \leq 1$ for $n = 0, ..., N$ is an initial value problem. The Picard-Lindelöf theorem proves the existence of a unique local solution of an initial value problem. This section starts with the Contraction Principle which will be used to prove the Picard-Lindelöf theorem. Furthermore, in this section the dependence of the initial value on the solution of the problem will be described. Theorems stated in this section originate from "Ordinary differential equations and Dynamical Systems" by Gerald Teschl[Tes12].

3.1.1 Picard-Lindelöf

A fixed point of a mapping $K : C \subset X \to C$ is an element $x \in C$ such that $K(x) = x$. K is a *contraction* if there exists a constant $\theta \in [0, 1)$ such that $\|K(x) - K(y)\| \leq \theta \|x - y\|$ for $x, y \in C$. The following theorem is called the Contraction principle or Banach's fixed point theorem.

Theorem 3.1.1 (Contraction principle)**.** *Let C be a nonempty, closed subset of a Banach space X and let $K : C \to C$ be a contraction. Then K has a unique fixed point $\bar{x} \in C$ such that $\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta} \|K(x) - x\|$ for $x \in C$.*

Proof. Suppose $x_0 \in C$ and consider the sequence $x_n = K^n(x_0)$. It follows that

$$\|x_{n+1} - x_n\| \leq \theta \|x_n - x_{n-1}\| \leq ... \leq \theta^n \|x_1 - x_0\|.$$

Then for $n > m$ by the triangle inequality

$$\begin{aligned}
\|x_n - x_m\| &\leq \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \\
&\leq \theta^m \sum_{j=0}^{n-m-1} \theta^j \|x_1 - x_0\| \\
&= \theta^m \frac{1 - \theta^{n-m}}{1 - \theta} \|x_1 - x_0\| \\
&= \frac{1 - \theta^n}{1 - \theta} \|x_1 - x_0\| \\
&\leq \frac{\theta^m}{1 - \theta} \|x_1 - x_0\|.
\end{aligned}$$

Thus x_n is a Cauchy sequence and converges to a limit $\bar{x}$. Since

$$\|K(\bar{x}) - \bar{x}\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

it follows that $\bar{x}$ is a fixed point. Furthermore,

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta}\|K^1(x) - K^0(x)\| = \frac{\theta^n}{1-\theta}\|K(x) - x\|.$$

Now suppose that K has two fixed points, namely $\bar{x}$ and $\tilde{x}$. Then

$$\|\bar{x} - \tilde{x}\| = \|K(\bar{x}) - K(\tilde{x})\| \leq \theta\|\bar{x} - \tilde{x}\|$$

which implies that $\bar{x} = \tilde{x}$. Therefore, it follows that K has a unique fixed point $\bar{x} \in C$ such that $\|K(x) - K(y)\| \leq \theta\|x - y\|$ for $x, y \in C$. □

The contraction principle will be used to show existence and uniqueness of solutions of the general initial value problem

$$\dot{x} = f(t, x), x(t_0) = x_0. \tag{3.1}$$

The following theorem states this result.

Theorem 3.1.2 (Picard-Lindelöf). *Suppose $f \in C(U, \mathbb{R}^n)$, U is an open subset of $\mathbb{R}^{n+1}$ and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$, and M denotes the maximum of $|f|$ on V, then there exists a unique local solution $\bar{x}(t) \in C^1([t_0, t_0 + T_0])$ and the solution remains in $\overline{B_\delta(x_0)}$ of the initial value problem 3.1, where $T_0 = min\{T, \frac{\delta}{M}\}$.*

Proof. Integrating both sides of the general initial value problem (3.1) with respect to t gives the following result

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Since $x_0(t) = x_0$ is an approximating solution for small t, it can be substituted in the integral equation

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s)).$$

Iterating this procedure, gives the following sequence of approximating solutions

$$x_m(t) = K^m(x_0)(t),\ K(x)(t) = x_0 + \int_{t_0}^{t} f(x, x(s))ds.$$

Now without loss of generality t_0 is set to 0 and $t \geq 0$. Furthermore, X is defined as $X = C([0, T], \mathbb{R}^n)$ for some suitable $T > 0$ Since $(0, x_0) \in U$ and U is open, V is defined as $V = [0, 0 + T] \times \overline{B_\delta(x_0)} \subset U$ where $B_\delta(x_0) = \{x \in \mathbb{R}^n | |x - x_0| < \delta\}$. The maximum M of $|f|$ exists by continuity of f and compactness of V. When $\{(t, x(t))|t \in [0, T]\} \subset V$, it follows that

$$|K(x)(t) - x_0| \leq \int_0^t |f(s, x(s))|ds \leq tM.$$

Hence, for $t \leq T_0$ it follows that $T_0 M \leq \delta$ and the graph of $K(x)$ restricted to $[0, T_0]$ is again in V. If $M = 0$, then this implies that $\frac{\delta}{M} = \infty$ such that $T_0 = \min\{T, \infty\} = T$. Since $[0, T_0] \subset [0, T]$, the same constant M will also bound $|f|$ on $V_0 = [0, T_0] \times \overline{B_\delta(x_0)} \subset V$.

Since f is Lipschitz continuous, it holds for every compact set $V_0 \subset U$ that

$$L = \sup_{(t,x)\neq(t,y)\in V_0} \frac{|f(t,x)-f(t,y)|}{|x-y|}$$

is finite. If both graphs of $x(t)$ and $y(t)$ lie in V_0, then

$$\begin{aligned}
\|K(x)-K(y)\| &\leq \int_0^t |f(s,x(s))-f(s,y(s))|ds \\
&\leq L\int_0^t |x(s)-y(s)|ds \\
&\leq Lt \sup_{0\leq s\leq t} |x(s)-y(s)| \\
&\leq LT_0\|x-y\|
\end{aligned}$$

for $x, y \in C$. Now choosing $T_0 < L^{-1}$, it follows that K is a contraction. Then by the contraction principle 3.1.1 it follows that there exists a unique solution to the initial value problem (3.1). □

3.1.2 Dependence on initial condition

Lemma 3.1.3 (Gronwall's inequality). *Suppose $\psi(t)$ satisfies $\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)ds$ for $t \in [0,T]$ with $\alpha \in \mathbb{R}$ and $\beta(t) \geq 0$. Then $\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{(\int_t^s \beta(r)dr)}ds$ for $t \in [0,T]$. Moreover, if in addition $\alpha(s) \leq \alpha(t)$ for $s \leq t$, then $\psi(t) \leq \alpha(t)e^{\int_0^t \beta(s)ds}$ for $t \in [0,T]$.*

Proof. Define $\phi(t) = e^{-\int_0^t \beta(s)ds}$. Then since it was assumed that $\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)ds$ it follows that

$$\frac{d}{dt}(\phi(t)\int_0^t \beta(s)\psi(s)ds) = \beta(t)\phi(t)\Big(\psi(t) - \int_0^t \beta(s)\psi(s)ds\Big) \leq \alpha(t)\beta(t)\phi(t).$$

Integrating this inequality with respect to t and dividing the resulting equation by $\phi(t)$ gives

$$\int_0^t \beta(s)\psi(s)ds \leq \int_0^t \alpha(s)\beta(s)\frac{\phi(s)}{\phi(t)}ds.$$

Adding $\alpha(t)$ to both sides gives

$$\int_0^t \beta(s)\psi(s)ds + \alpha(t) \leq \int_0^t \alpha(s)\beta(s)\frac{\phi(s)}{\phi(t)}ds + \alpha(t).$$

Since $\phi(t)$ was defined as $\phi(t) = e^{-\int_0^t \beta(s)ds}$ it follows that

$$\begin{aligned}
\frac{\phi(s)}{\phi(t)} &= \frac{e^{-\int_0^s \beta(r)dr}}{e^{-\int_0^t \beta(r)dr}} \\
&= e^{-\int_0^s \beta(r)dr+\int_0^t \beta(r)dr} \\
&= e^{(\int_s^t \beta(r)dr)}.
\end{aligned}$$

Since $\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)ds$ was assumed to be true, it follows that

$$\psi(t) \leq \int_0^t \alpha(s)\beta(s)\frac{\phi(s)}{\phi(t)}ds + \alpha(t) = \int_0^t \alpha(s)\beta(s)e^{(\int_s^t \beta(r)dr)}ds + \alpha(t).$$

If in addition the function α is non-decreasing, the fundamental theorem of calculus implies that

$$\begin{aligned}\psi(t) &\leq \alpha(t) + (-\alpha(t)e^{(\int_s^t \beta(r)dr)})|_{s=0}^{s=t} \\ &= \alpha(t)e^{(\int_0^t \beta(r)dr)}.\end{aligned}$$

This concludes the proof. □

Corollary 3.1.4. *If $\psi(t) \leq \alpha + \int_0^t (\beta\psi(s) + \gamma)ds$ for $t \in [0,T]$ for given constants $\alpha \in \mathbb{R}, \beta \geq 0$ and $\gamma \in \mathbb{R}$ and α is non-decreasing, then $\psi(t) \leq \alpha e^{|\beta t|} + \frac{\gamma}{\beta}(e^{(\beta t)} - 1)$ for $t \in [0,T]$.*

Proof. Suppose $\beta \neq 0$. Then $\psi(t) + \frac{\gamma}{\beta} \leq \alpha + \frac{\gamma}{\beta} + \int_0^t (\beta\psi(s) + \frac{\gamma}{\beta})ds$. By Gronwall's inequality 3.1.3, it follows that

$$\begin{aligned}\psi(t) + \frac{\gamma}{\beta} &\leq (\alpha + \frac{\gamma}{\beta})e^{\int_0^t \beta(s)ds} \\ &= (\alpha + \frac{\gamma}{\beta})e^{\beta t}.\end{aligned}$$

Rewriting gives

$$\begin{aligned}\psi(t) &\leq (\alpha + \frac{\gamma}{\beta})e^{\beta t} - \frac{\gamma}{\beta} \\ &= \alpha e^{\beta t} + \frac{\gamma}{\beta}(e^{\beta t} - 1).\end{aligned}$$

If $\beta = 0$, then it follows that $\psi(t) \leq \alpha + \gamma t$. This concludes the proof. □

Suppose $x(t)$ is a solution of the initial value problem

$$\dot{x} = f(t,x), x(t_0) = x_0$$

and $y(t)$ is a solution of the initial value problem

$$\dot{y} = g(t,y), y(t_0) = y_0.$$

The following theorem states the dependence of the solution of the initial value problems on the initial values.

Theorem 3.1.5. *Suppose $f, g \in C(U, \mathbb{R}^n)$ and let f be locally Lipschitz continuous in the second argument, uniformly with respect to the first. Then*

$$|x(t) - y(t)| \leq |x_0 - y_0|e^{L|t-t_0|} + \frac{M}{L}(e^{L|t-t_0|} - 1)$$

where

$$L = \sup_{(t,x)\neq(t,y)\in V} \frac{|f(t,x) - f(t,y)|}{|x-y|},\ M = \sup_{(t,x)\in V} |f(t,x) - g(t,x)|$$

, with $V \subset U$ some set containing the graphs of $x(t)$ and $y(t)$.

Proof. Without loss of generality t_0 is set to 0. The triangle inequality gives

$$\begin{aligned}|f(s,x(s)) - g(s,y(s))| &\leq |f(s,x(s)) - f(s,y(s))| + |f(s,y(s)) - g(s,y(s))| \\ &\leq L|x(s) - y(s)| + M.\end{aligned}$$

If for $t \in [0,T]$, $\psi(t) \leq \alpha + \int_0^t (\beta\psi(s) + \gamma)ds$ then for given constants $\alpha \in \mathbb{R}, \beta \geq 0$ and $\gamma \in \mathbb{R}$, it follows by corollary 3.1.4 that for $t \in [0,T]$, $\psi(t) \leq \alpha e^{\beta t} + \frac{\gamma}{\beta}(e^{\beta t} - 1)$. Combining these results gives

$$\begin{aligned} |x(t) - y(t)| &\leq |x_0 - y_0| + \int_0^t |f(s, x(s)) - g(s, y(s))| ds \\ &\leq |x_0 - y_0| + \int_0^t L|x(s) - y(s)| + M \\ &\leq |x_0 - y_0| e^{L|t-t_0|} + \frac{M}{L}(e^{L|t-t_0|} - 1), \end{aligned}$$

and the conclusion follows. □

3.2 Brouwer's fixed point theorem

In section 4.5 the simplified initial value problem (4.2) will be converted into a fixed point problem. To conclude that the shifted Poincaré map (4.9) has a fixed point, Brouwer's fixed point theorem is used. This theorem states that every continuous mapping in $\mathbb{R}^n$ from a closed ball to itself contains at least one fixed point. Brouwer's fixed point theorem will be proven using the "hairy ball" theorem, which states that there is no continuous tangent vector fields on even-dimensional n-spheres. This section starts with the proof of the "hairy ball" theorem and continues with the proof of Brouwer's fixed point theorem. Every lemma, theorem and their proofs stated in this section originate from the article "Analytic proofs of the 'hairy ball theorem' and the Brouwer fixed point theorem" by John Milnor [Mil78].

3.2.1 "Hairy ball" theorem

The sphere S^{n-1} is defined as the set of all vectors $u = (u_1, ..., u_n)$ in the Euclidean space $\mathbb{R}^n$ such that the Euclidean length $\|u\|$ equals 1. A vector $v(u)$ in $\mathbb{R}^n$ is *tangent* to S^{n-1} at u if the inner product $u \cdot v(u)$ is equal to zero.

Suppose A is a compact region in $\mathbb{R}^n$, $x \mapsto v(x)$ is a continuously differentiable vector field which is defined throughout a neighborhood of A and $v(x)$ are arbitrary vectors in $\mathbb{R}^n$.

Lemma 3.2.1. *If the real number t is sufficiently small, then the mapping $f_t(x) : A \to f_t(A)$ by $\mathbf{f}_t(x) = x + tv(x)$ is one-to-one and the volume of region $\mathbf{f}_t(A)$ can be expressed as a polynomial function of t.*

Proof. Suppose A is a cube with edges parallel to the coordinate axes and suppose $x, y \in A$. Since $v(x)$ is continuously differentiable on A, the Mean Value theorem states that there exists a point $c \in (x, y)$ such that

$$\begin{aligned} v_i(x) - v_i(y) &= \frac{d}{dt} v_i(x + t(y - x))|_{t=\theta} \cdot 1 \\ &= (\nabla v_i(x + \theta(y - x)) \cdot (y - x) \\ &= \nabla v_i(c) \cdot (y - x) \\ &= \sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j}(c) \cdot (y_j - x_j) \end{aligned}$$

where $c = x + \theta(y - x)$ and $\theta \in (0, 1)$. This implies

$$|v_i(x) - v_i(y)| \leq \sum_j^n \sup_A |\frac{\partial v_i}{\partial x_j}||y_j - x_j|.$$

Therefore,

$$\|v(x) - v(y)\| \leq \sum_i |v_i(x) - v_i(y)| \leq \sum_{i,j} \sup_A |\frac{\partial v_i}{\partial x_j}||y_j - x_j| \leq \sum_{i,j} \sup_A |\frac{\partial v_i}{\partial x_j}|\|x - y\| \leq n^2 \sup_A |\frac{\partial v_i}{\partial x_j}|.$$

Hence there exists a Lipschitz constant d such that

$$\|v(x) - v(y)\| \leq d\|x - y\|$$

for all x and y in A, namely $d = n^2 \sup_A |\frac{\partial v_i}{\partial x_j}|$.
An arbitrary compact set A in $\mathbb{R}^n$ can be covered by finitely many open cubes I_i where $i \in \{1, n\}$ such that the following holds

$$\|v(x) - v(y)\| \leq \begin{cases} \sup_A |\frac{\partial v_i}{\partial x_j}|\|x - y\| & \text{if } x, y \in I_i \\ e\|x - y\| & \text{if } x \in I_i \text{ and } y \in I_i \text{ with } i \neq j \end{cases}$$

where $e \in \mathbb{Z}$ and $e \neq 0$. Now define

$$g = \begin{cases} n^2 \sup_A |\frac{\partial v_i}{\partial x_j}| & \text{if } x, y \in I_i \\ e & \text{if } x \in I_i, y \in I_j \text{ and } i \neq j. \end{cases}$$

Then for every $x, y \in A$, it follows that there exists a Lipschitz constant g so that $\|v(x) - v(y)\| \leq g\|x - y\|$.
Let t be arbitrary with $|t| < g^{-1}$. If $f_t(x) = f_t(y)$ then $x + tv(x) = y + tv(y)$ and thus $x - y = t(v(y) - v(x))$. Due to the Lipschitz condition it then follows that

$$\|x - y\| = \|t(v(y) - v(x))\| \leq |t||g|\|x - y\|$$

which implies that $x = y$. Since $x = y$ whenever $f_t(x) = f_t(y)$, it can be concluded that the function f_t is one-to-one.
The matrix of first derivatives of f_t can be written as $I + t[\frac{\partial v_i}{\partial x_j}]$, where I is equal to the identity matrix. The determinant of this matrix is a polynomial function of t, of the form $1 + ta_1(x) + ... + t^n a_n(x)$. By integrating over A, the volume of the image region is as follows

$$\text{vol } f_t(A) = b_0 + b_1 t + ... + b_n t^n$$

where $b_k = \int_A a_k dx_1 ... dx_n$. Hence it follows that the volume of f_t can be expressed as a polynomial function of t and the conclusion follows. □

Now suppose the sphere S^{n-1} has a continuously differentiable field $u \mapsto v(u)$ of unit tangent vectors. Define ω_n as the unit sphere in $\mathbb{R}^n$.

Lemma 3.2.2. *If the parameter t is sufficiently small, then the transformation $u \mapsto u + tv(u)$ maps ω_n onto the sphere of radius $\sqrt{1 + t^2}$*

Proof. For any real number t the vector $u + tv(u)$ has length $\sqrt{1+t^2}$. The matrix of first derivatives of f_t was defined in the proof of Lemma **??**. If t is sufficiently small, the determinant $1 + ta_1(x) + ... + t^n a_n(x)$ will be close to 1 and therefore the matrix is non-singular throughout the compact region A. The Inverse Function theorem states that if a function f is continuously differentiable on some open set containing a point a and $\det Jf(a) \neq 0$, then there is some open set V containing a and an open W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable for all $y \in W$. From this theorem it follows that f_t maps open sets contained in the interior of A to open sets. Thus the image $f_t(S^{n-1})$ is an open subset of the sphere of radius $\sqrt{1+t^2}$. But the image $f_t(S^{n-1})$ is also compact and thus also closed. Since S^{n-1} is connected, an open and closed subset must be the entire sphere. And therefore the transformation $u \mapsto v(u)$ maps ω_n onto the sphere of radius $\sqrt{1+t^2}$. □

Theorem 3.2.3. *An even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors.*

Proof. Define region A as $A = \{x \in \mathbb{R}^n | a \leq \|x\| \leq b\}$ where $a, b \in \mathbb{N}$. The vector field v is extended throughout this region by setting $v(ru) = rv(u)$ for $a \leq r \leq b$. By Lemma 3.2.1 and Lemma 3.2.2, for sufficiently small t the mapping $f_t(x) = x + tv(x)$ is defined throughout the region A, and maps the sphere of radius r onto the sphere of radius $r\sqrt{1+t^2}$. Since

$$f_t(ru) = ru + tv(ru) = ru + trv(u) = r(u + tv(u)) = rf_t(u),$$

f_t maps A onto the region $\{x \in \mathbb{R}^n | a\sqrt{1+t^2} \leq \|x\| \leq b\sqrt{1+t^2}\}$. It then follows that

$$\begin{aligned} \text{vol}(f_t(A)) &= \text{ vol } (\omega_n)(b^n(\sqrt{1+t^2})^n - a^n(\sqrt{1+t^2})^n) \\ &= (\sqrt{1+t^2})^n \text{ vol}(\omega_n)(b^n - a^n) \\ &= (\sqrt{1+t^2})^n \text{ vol}(A). \end{aligned}$$

If n is odd, this volume is not a polynomial function of t. This is a contradiction to Lemma 3.2.1 and therefore it can be concluded that an even-dimensional sphere does not possess any continuously differentiable field of unit tangent vectors. □

Theorem 3.2.4 (”Hairy ball” theorem)**.** *An even dimensional sphere does not admit any continuous field of non-zero tangent vectors.*

Proof. Suppose that the sphere S^{n-1} possesses a continuous field of non-zero tangent vectors $v(u)$. Define $m = \min_{u \in S^{n-1}} \|v(u)\|$. The Weierstrass Approximation theorem states that there exists a polynomial mapping p from S^{n-1} to $\mathbb{R}^n$ satisfying

$$\|p(u) - v(u)\| < \frac{m}{2}$$

for all u. The differentiable vector field $w(u)$ is defined by the formula $w = p - (p \cdot u)u$ for every u. Since u is a unit vector, $u \cdot u = 1$ and it thus follows that

$$\begin{aligned} w \cdot u &= (p - (p \cdot u)u) \cdot u \\ &= p \cdot u - (p \cdot u)(u \cdot u) \\ &= p \cdot u - p \cdot u \\ &= 0. \end{aligned}$$

Therefore, $w(u)$ is tangent to S^{n-1} at u. Again since u is a unit vector, it follows that

$$\begin{aligned}
\|w - p\| &= \|p - (p \cdot u)u - p\| \\
&= \|(p \cdot u)u\| \\
&= |p \cdot u| \\
&= |p \cdot u - v(u) \cdot u| \\
&= |(p - v(u)) \cdot u| \\
&\leq \|p - v(u)\| \cdot \|u\| \\
&= \|p - v(u)\| < \frac{m}{2},
\end{aligned}$$

where the Cauchy-Schwarz inequality was applied. By the triangle inequality

$$\begin{aligned}
\|w - v(u)\| &\leq \|w - p\| + \|p - v(u)\| \\
&< \frac{m}{2} + \frac{m}{2} = m
\end{aligned}$$

and by the reverse triangle inequality

$$\begin{aligned}
\|w\| &\geq \|v(w)\| - \|w - v(u)\| \\
&\geq m - \|w - v(u)\|.
\end{aligned}$$

Since it has been shown that $\|w - v(u)\| < m$ this implies that $w \neq 0$. Therefore, the quotient $\frac{w(u)}{\|w(u)\|}$ is an infinitely differentiable field of unit tangent vectors on the sphere S^{n-1}. If $n-1$ is even, this is impossible by Theorem 3.2.3. Thus an even-dimensional sphere does not admit any continuous field of non-zero tangent vectors. □

3.2.2 Brouwer's fixed point theorem

Now that the "hairy ball" theorem has been proven, it can be used to prove Brouwer's fixed point theorem. The disk D^n is defined as $D^n = \{x \in \mathbb{R}^n | \|x\| \leq 1\}$.

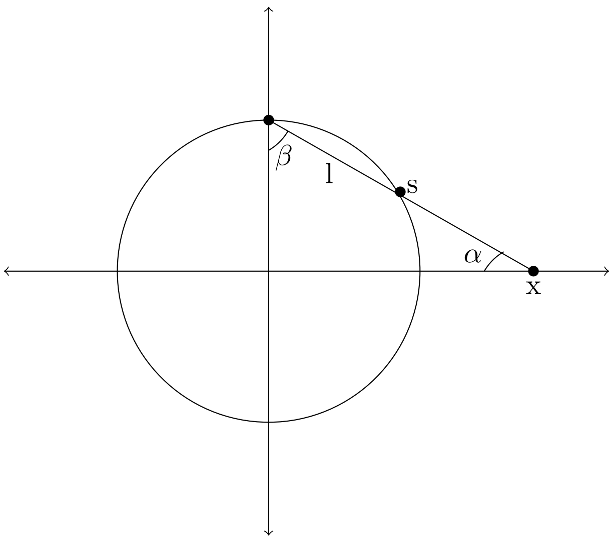
Theorem 3.2.5. *Every continuous mapping f from the disk D^n to itself possesses at least one fixed point.*

Proof. Define $w(x) = x - \frac{y(1-x\cdot x)}{(1-x\cdot y)}$ where $y = f(x) \neq x$. Whenever $x \cdot x = 1$, it follows that $w(x) = x$. When x and y are linearly independent, it follows that $w(x) \neq 0$. When x and y are linearly dependent, then $(x \cdot x)y = (x \cdot y)x$ implies that

$$w(x) = x - \frac{y(1 - x \cdot x)}{(1 - x \cdot y)} = \frac{x(1 - x \cdot y)}{(1 - x \cdot y)} - \frac{y(1 - x \cdot x)}{(1 - x \cdot y)} = \frac{x - x(x \cdot y) - y + y(x \cdot x)}{1 - x \cdot y} = \frac{x - y}{1 - x \cdot y} \neq 0.$$

Hence w is a non-zero vector field on D^n which points directly outward everywhere on the boundary.
$\mathbb{R}^n$ is identified as S^n with the hyperplane $x_{n+1} = 0$ which passes through the equator. The goal is to find a projection $s(x) : D^n \to U_s$ where $U_s = \{u \in S^n | u_{n+1} < 0\}$ is the southern hemisphere of the unit sphere S^n in $\mathbb{R}^{n+1}$.
The construction of $s(x)$ will be demonstrated using a 2D-plot. Define s as the intersection the projection makes with the sphere, define β as the angel the projection makes with the sphere and define l as the length of point s to the north pole. An illustration of this construction is visible in figure 3.1

Figure 3.1



Then

$$
\begin{aligned}
\tan\beta &= x \\
\cos\beta = \frac{l}{2} \frac{1-(\cos\beta)^2}{(\cos\beta)^2} &= x^2 \\
1-(\cos\beta)^2 &= x^2(\cos\beta)^2 \\
\frac{1}{1+x^2} &= (\cos\beta)^2 \\
(\cos\beta)^2 &= \frac{l^2}{4} \\
\frac{1}{1+x^2} &= \frac{l^2}{4}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(\sin\beta)^2 &= 1-(\cos\beta)^2 \\
&= 1-\frac{1}{1+x^2} \\
&= \frac{x}{1+x^2}.
\end{aligned}
$$

$s(x)$ is equal for x_j with $j \in \{1, n\}$, namely

$$
\begin{aligned}
s(x) &= \sin\beta \cdot l \\
&= \sqrt{\frac{x}{1+x^2}} \cdot \sqrt{\frac{4}{1+x^2}} \\
&= \frac{x}{\sqrt{1+x^2}} \cdot \frac{2}{\sqrt{1+x^2}} \\
&= \frac{2x}{1+x^2}
\end{aligned}
$$

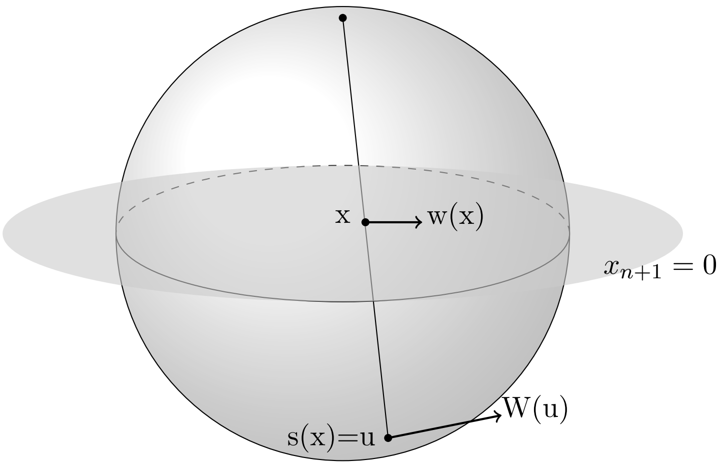
and for x_j with $j = n + 1$

$$\begin{aligned} s(x) &= 1 - \cos\beta \cdot l \\ &= 1 - \frac{1}{1+x^2} \cdot \frac{4}{1+x^2} \\ &= 1 - \frac{2}{1+x^2} \\ &= \frac{x^2-1}{1+x^2}. \end{aligned}$$

Therefore the precise formula for $s(x)$ is

$$s(x) = \frac{(2x_1, ..., 2x_n, x \cdot x - 1)}{(x \cdot x + 1)}.$$

Figure 3.2



The tangent vector $W(x)$ is defined as $W(x) = \frac{ds(x+tw(x))}{dt}$. Therefore, $W(x) \in U_s$ and $W(x) \neq 0$. $W(u)$ is then $W(x)$ at the image point $s(x) = u$ at $t = 0$. An illustration of vectors $s(x)$ and $W(x)$ are visible in Figure 3.2.

At every point of the equator of S^n, so when $u = s(u)$, it follows that$w(u) = u$. By the definition of $w(x)$, the vector $w(u) = u$ points directly outward on the boundary. Then the corresponding tangent vector $W(u) = \frac{ds(x)}{dt} = (0, ..., 0, 1)$ points away from U_s. Now define $U_n = \{u \in S^n | u_{n+1} > 0\}$ as the northern hemisphere of the unit sphere S^n in $\mathbb{R}^{n+1}$. Repeating these steps to the vector field $-w(x) : D^n \to U_n$, then gives a vector field with $W(u)$ pointing away from U_n. Combining $w(x)$ and $-w(x)$ gives a non-zero tangent vector field W which is defined continuously everywhere on S^n.

If n is even, then this is a contradiction to Theorem 3.2.4. Therefore, it can be concluded that it cannot be possible that $f(x) \neq x$ for all x in D^n.

Now suppose $n = 2k - 1$, then for any map f from D^{2k-1} to itself with $f(x) \neq x$ for $x \in D^{2k-1}$, this map can be transformed to $F(x_1, ..., x_{2k}) = (f(x_1, ..., x_{2k-1}), 0)$. This is a map from D^{2k} to itself, and since $n = 2k$ is even, this is again in contradiction to Theorem 3.2.4. Hence, the claim is true for all $n \in \mathbb{N}$ and it can be concluded that every continuous mapping f from the disk D^n to itself possesses at least one fixed point.

□

3.3 Index theory

When simplifying the discrete Nagumo equation (1.1) h_1 will be substituted by a linear function h_0. In section 4.6 a homotopy $h_\lambda : [0,1] \to \mathbb{R}$ will be constructed that deforms $h_0(x)$ into $h_1(x)$ continuously and it will be proven using the general homotopy invariance theorem 3.3.4 that this construction is independent of λ. The fixed point index "counts" the number of fixed points of a map from an open subset of a closed subspace to that space. The general homotopy invariance theorem 3.3.4 states that the fixed point index is well-defined and independent of λ.
The proof of the general homotopy theorem 3.3.4 uses the Leray-Schauder degree, which "counts" the number of fixed points of a map from an open subset of a normed linear space to that space. The Leray-Schauder degree is constructed by extending the Brouwer degree, however this construction exceeds the content of this thesis. In this section the Leray-Schauder degree will be defined and the general homotopy theorem 3.3.4 will be stated.
For this section, X is a metric space, $I = [0,1]$, $A \subset X \times I$, and for $\lambda \in I$, the "slice" at λ is defined as $A_\lambda = \{x \in X : (x,\lambda) \in A\}$.

3.3.1 Leray-Schauder degree

The following definition of the Leray-Schauder degree originates from the article "Leray-Schauder degree: a half century of extensions and applications" by Jean Mawhin [Maw99]. If $U \subset X$ is an open bounded set, $f : \overline{U} \to X$ is compact, and $z \notin (I-f)(\partial U)$, the *Leray-Schauder degree* is notated by $\deg_{LS}[I-f, U, z]$ of $I-f$ in U over z. This degree "algebraically counts" the number of fixed points of $f(\cdot) - z$ in U.

Theorem 3.3.1. *The Leray-Schauder degree satisfies the following properties.*

1. *(Additivity) If $U = U_1 \cup U_2$, where U_1 and U_2 are open and disjoint, and if $z \notin (I-f)(\partial U_1) \cup (I-f)(\partial U_2)$, then $deg_{LS}[I-f, U, z] = deg_{LS}[I-f, U_1, z] + deg_{LS}[I-f, U_2, z]$.*

2. *(Existence) If $deg_{LS}[I-f, U, z] \neq 0$, then $z \in (I-f)(U)$.*

3. *(Homotopy invariance) Let $\sigma \subset X \times I$ be a bounded open set, and let $F : \overline{\sigma} \to X$ be compact. If $x - F(x,\lambda) \neq z$ for each $(x,\lambda) \in \partial\sigma$, then $deg_{LS}[I - F(\cdot,\lambda), \sigma_\lambda, z]$ is independent of λ.*

3.3.2 Fixed point index

The following presentation of the fixed point index is based on the article "fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces" by Herbert Amann [Ama76]. A nonempty subset A of a metric space X is called a *retract* of X if there exists a continuous map $r : X \to A$ such that $r|A = id_A$. Every retract is a closed subspace of X. The following theorem describes the properties of the fixed point index.

Theorem 3.3.2. *Let X be a retract of some Banach space E. For every open subset U of X and every compact map $f : \overline{U} \to X$ which has no fixed points on ∂U, there exists an integer $i(f, U, X)$ satisfying the following conditions:*

1. *(Normalization) For every constant map f mapping $\overline{U}$ into U, $i(f, U, X) = 1$;*

2. *(Additivity) For every pair of disjoint open subsets U_1, U_2 of U such that f has no fixed points on $\overline{U}\backslash(U_1 \cup U_2)$,*

$$i(f,U,X) = i(f,U_1,X) + i(f,U_2,X),$$

   where $i(f,U_k,X) = i(f|\overline{U}_k,U_k,X)$, $k = 1,2$;

3. *(Homotopy invariance) For every compact interval $\Lambda \subset \mathbb{R}$, and every compact map $h : \Lambda \times \overline{U} \to X$ such that $h(\lambda,x) \neq x$ for $(\lambda,x) \in \Lambda \times \delta U$,*

$$i(h(\lambda,\cdot),U,X) \tag{3.2}$$

   is well-defined and independent of $\lambda \in \Lambda$.

4. *(Permanence) If Y is a retract of X and $f(\overline{U}) \subset Y$, then $i(f,U,X) = i(f,U \cap Y,Y)$, where $i(f,U \cap Y,Y) = i(f|\overline{U \cap Y},U \cap Y,Y)$*

The family $\{i(f,U,X)|X$ retract of E, U open in $X, f : \overline{U} \to X$ compact without fixed points on $\partial U\}$ is uniquely determined by properties (1.)-(4.), and $i(f,U,X)$ is called the fixed point index of f over U with respect to X.

This theorem can be proven using the properties of the Leray-Schauder degree. However, this exceeds the scope of this thesis.

Corollary 3.3.3. *The fixed point index has the following further properties:*

- *(Excision) For every open subset $V \subset U$ such that f has no fixed point in $\overline{U}\backslash V$, $i(f,U,X) = i(f,V,X)$;*
- *(Solution property) If $i(f,U,X) \neq 0$, then f has at least one fixed point in U.*

Now let $\Lambda \subset \mathbb{R}$ be an arbitrary interval and let A be a subset of $\Lambda \times X$. The "slice" at λ A_λ is open in X if A is open in $\Lambda \times X$.

Theorem 3.3.4 (General homotopy theorem)**.** *Let Λ be a nonempty compact interval, let X be a retract of some Banach space E, and let U be an open subset of $\Lambda \times X$. Suppose $h : \overline{U} \to X$ is a compact map such that $h(\lambda,x) \neq x$ for every $(\lambda,x) \in \partial U$. Then $i(h(\lambda,\cdot),U_\lambda,X)$, $\lambda \in \Lambda$ is well-defined and independent of $\lambda \in \Lambda$.*

Chapter 4

Article ”Existence of Traveling Wavefront Solutions for the Discrete Nagumo Equation” by Bertram Zinner

4.1 Introduction

The discrete Nagumo equation is defined for $n \in \mathbb{Z}$ as

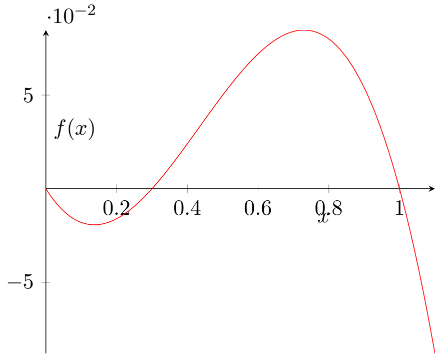
$$\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n). \tag{4.1}$$

Here d is a positive real number and f denotes a Lipschitz continuous function satisfying the following properties

- $f(0) = f(a) = f(1)$,
- $f(x) < 0$ for $0 < x < a$,
- $f(x) > 0$ for $a < x < 1$,
- $\int_0^1 f(x)dx > 0$,

where $a \in (0, 1)$. A typical example of a function satisfying these properties is the cubic polynomial $f(x) = x(x - a)(1 - x)$ where $0 < a < \frac{1}{2}$.

Figure 4.1: Example of cubic polynomial



In Figure 4.1 an example of a cubic polynomial is visible where $a = 0.3$. A result known for this discrete Nagumo equation is that traveling wave solutions of equation 1.1 exist. The following theorem states this result.

Theorem 4.1.1. *Suppose f is a Lipschitz continuous function satisfying $f(0) = f(a) = f(1)$, $f(x) < 0$ for $0 < x < a$, $f(x) > 0$ for $a < x < 1$ and $\int_0^1 f(x)dx > 0$. Then there exists some $d^* > 0$ such that for $d > d^*$ the discrete Nagumo equation 1.1 admits a solution $u_n(t) = U(n + ct)$, where $c > 0$, $U \in C^1(\mathbb{R}, (0, 1))$, $U(-\infty) = 0$, $U(\infty) = 1$, and $U'(x) > 0$ for all $x \in \mathbb{R}$.*

The previous chapter 3 contained all the prerequisite knowledge needed to understand the proof of Theorem 4.1.1 and thus the existence of traveling wavefront solutions for the Discrete Nagumo equation by Bertram Zinner [Zin90]. In this chapter the content of this proof will be discussed and expanded. The chapter starts with a general strategy of the proof.

4.2 General strategy

The proof of the discrete Nagumo equation given in this chapter can be broken into 5 steps.
Step 1: Providing a basis for the proof
The unique solution of the fixed point problem (4.3) $u(t)$ is defined as $\{u_n(t)\}_{n=0}^N$. In section 4.3, several lemmata will be stated and proven describing certain properties of initial values that are invariant under the flow of $u(t)$. The first section provides a basis on which the rest of the proof will be built.
Step 2: Simplifying the problem
Instead of the discrete Nagumo equation, in this step the following simplified equation is considered for $n \in \mathbb{Z}$

$$\dot{v}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) \tag{4.2}$$

where $u_n = P(v_n)$ and $h(u_n) = u_n - \frac{1}{4}$. Here $P(v_n)$ is defined as

$$P(v_n) = \begin{cases} 0 & \text{for } v_n < 0, \\ v_n & \text{for } 0 \leq v_n \leq 1, \\ 1 & \text{for } 1 < v_n. \end{cases}$$

The simplified problem is to find a monotone traveling wave solution of equation 4.2 on an interval $[0, \tau]$ satisfying the following conditions

1. $u_n(\tau) = u_{n+1}(0)$,
2. $v_n(0) \leq v_{n+1}(0)$,
3. $d(u_{n-1}(0) - 2u_n(0) + u_{n+1}(0) + h(u_n(0)) > 0$ if $u_n(0) > 0$,
4. $\lim_{n\to-\infty} v_n(0) = 0$ and $\lim_{n\to\infty} v_n(0) = 1$.

It turns out that the conditions imply the existence of a function $U : \mathbb{R} \to \mathbb{R}$ with $U(-\infty) = 0$, $U(\infty) = 1$ and $0 \leq U \leq 1$ such that $u_n(t) = U(n + ct)$ and $\tau = \frac{1}{c}$ for all $n \in \mathbb{Z}, t \in [0, \tau]$. In section 4.4 it will also be shown that it suffices to consider only finitely many solutions of the simplified equation 4.2.

Step 3: Converting the simplified problem into a fixed point problem
In section 4.5 it will be concluded that by the Picard-Lindelöf theorem the initial value problem for $u_{-1} = 0$ and $u_{n+1} = 1$

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n), \\ u_n &= P(v_n), \\ v_n(0) &= x_n \text{ with } 0 \le x_n \le 1 \text{ for } n = 0, ..., N. \end{aligned} \tag{4.3}$$

has a unique solution $u(x;t) = \{u_n(x;t)\}_{n=0}^N$ which depends continuously on the initial value $x = \{x_n\}_{n=0}^N$. The set X is defined as

$$\begin{aligned} X = \Big\{ &\{x_n\}_{n=0}^N \in \mathbb{R}^{N+1} : x_0 = 0, x_1 = \frac{-h(0)}{d}, x_n \le x_{n+1}, \\ &d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0, \\ &x_n \ge \frac{n}{N}, \text{for } n = 1, .., N, \text{where } x_{N+1} = 1 \Big\} \end{aligned}$$

and the following shifted Poincaré map $T : \bar{X} \to \mathbb{R}^{N+1}$ is considered

$$(Tx)_n = \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x;\tau) & \text{for } n = 1, .., N \end{cases} \tag{4.4}$$

where τ is defined by $u_0(x;\tau) = x_1$. The map T_0 has a fixed point by Brouwer's fixed point theorem 3.2.5 if the map satisfies the following four properties:

- $\overline{C_0 \cap O_0}$ is a closed, bounded, and convex subset of $\mathbb{R}^{N+1}$,
- $T_0(\overline{C_0 \cap O_0}) \subset C_0 \cap O_0$,
- T_0 is continuous,
- $C_0 \cap O_0$ is nonempty.

This will be proven in section 4.5. Finally, this fixed point x corresponds to the traveling wave $\{u_n(t)\}_{-\infty}^{\infty}$.
Step 4: Deforming h_0 into h_1
The construction of T_0 depended on the linear function h_0 so far. The function h_0 can be deformed continuously into h_1 using a homotopy h_λ where $h_1 \in B_{app}$. The set B_{app} is defined as

$$\begin{aligned} B_{app} = \Big\{ &h : [0,1] \to \mathbb{R} : h \text{ is Lipschitz continuous}, h(0) < 0, h(1) > 0, \\ &h \text{ has a unique zero in } (0,1), \text{ and } \int_0^1 h(s)ds > 0 \Big\}. \end{aligned}$$

The fixed points of T_{h_0} are then continued into the fixed points of T_{h_1}. The general homotopy invariance theorem 3.3.4 will be used to conclude that the number of fixed points of T_{h_1} is independent of λ. It then again follows that the fixed point of T_1 corresponds to a traveling wave solution.
Step 5: Convergence of the approximate solutions
In the final step f is approximated by h^k. Suppose $\{u_n^k\}$ is a traveling wave of

$$\begin{aligned} \dot{v}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h^k(u_n), \\ u_n = P(v_n) \end{aligned} \tag{4.5}$$

for $n \in \mathbb{Z}$. Assuming that $\|f - h^k\|_2 \to 0$ as $k \to \infty$, it will be proven in section 4.7 that the traveling wave solution u_n^k to (4.5) is a traveling wave solution to the discrete Nagumo equation (1.1). This concludes the proof of Theorem 4.1.1.

4.3 Invariance results

The fixed point problem (4.3) has a unique solution $u(t) = \{u_n(t)\}_{n=0}^N$ by the Picard-Lindelöf theorem 3.1.2. Here h is an arbitrary element of B_{app}. In this section it will be shown that certain properties of initial values $x = \{x_n\}_{n=0}^N \in \mathbb{R}^{N+1}$ are invariant under the flow of $u(t)$.

Lemma 4.3.1. *Suppose $x_0 = 0$, $x_1 = \frac{-h(0)}{d}$ and $d > -h(0)$ and suppose $x = \{x_n\}_{n=0}^N \in \mathbb{R}^{N+1}$ satisfies the following properties*

1. $0 \leq x_0 \leq ... \leq x_N \leq 1$,
2. $x_n < x_{n+1}$ *whenever* $0 < x_n < 1$,
3. $d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0$ *whenever* $0 < x_n$, *where* $x_{N+1} = 1$ *and* $x_{-1} = 0$.

Then $u(t)$ satisfies properties 1,2 and 3 for all $t \geq 0$.

Proof. Define t_1 as $t_1 = \sup\{t \geq 0 : u(s)$ satisfies properties 1.,2. and 3. for all $0 \leq s \leq t\}$. Because $u(0) = x$ satisfies properties 1,2 and 3, t_1 is well defined. To prove Lemma 4.3.1, it suffices to show that $t_1 = \infty$. Since $x_1 = \frac{-h(0)}{d}$ and $d > -h(0)$ it follows that $x_1 > 0$. By property 1 it then follows that $x_n > 0$ for $n = 1, 2, ..., N$ and thus by property 3 that $\dot{v}_n(0) = d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0$ for $n = 1, 2, ..., N$. $u_0(t)$ is differentiable at $t = 0$ if the limit $\lim_{t\to 0} |\frac{u_0(t)-u_0(0)}{t}|$ exists. Since

$$\dot{v}_0(0) = d(x_{-1} - 2x_0 + x_1) + h(x_0) = dx_1 + h(0) = 0,$$

by the Mean Value theorem and $u_0(0) = 0$ it follows that for $\theta \in (0, 1)$

$$\begin{aligned}
\lim_{t\to 0} |\frac{u_0(t) - u_0(0)}{t}| &= \lim_{t\to 0} |\frac{u_0(t)}{t}| \\
&= \lim_{t\to 0} |\frac{P(v_0(t))}{t}| \\
&= \lim_{t\to 0} |\frac{P(v_0(0) + \dot{v}_0(\theta)t)}{t}| \\
&= \lim_{t\to 0} |\frac{P(\dot{v}_0(\theta)t)}{t}| \\
&\leq \lim_{t\to 0} |\frac{\dot{v}_0(\theta)t}{t}| \\
&= \lim_{t\to 0} |\dot{v}_0(\theta)| \\
&= \lim_{t\to 0} |\dot{v}_0(0)| \\
&= 0.
\end{aligned}$$

Thus $u_0(t)$ is differentiable at $t = 0$ and $\dot{u}_0(0) = 0$. Since h is Lipschitz continuous, there exists a real constant L such that for all u_0, $|h(u_0(t)) - h(u_0(0))| \leq L|u_0(t) - u_0(0)|$. Therefore,

$$\begin{aligned}
\lim_{t\to 0} |\frac{h(u_0(t)) - h(u_0(0))}{t}| &\leq \lim_{t\to 0} L|\frac{u_0(t) - u_0(0)}{t}| \\
&= \lim_{t\to 0} L|\dot{u}_0(0)| \\
&= L \cdot 0 \\
&= 0
\end{aligned}$$

and it follows that $h(u_0(t))$ is differentiable at $t = 0$ and this derivative is equal to 0. Since

$$\begin{aligned}
\ddot{v}_0(0) &= d(\dot{u}_{-1}(0) - 2\dot{u}_0(0) + \dot{u}_1(0)) + \frac{d}{dt}h(u_0(0)) \\
&= d(0 - 2 \cdot 0 + \dot{u}_1(0)) + 0 \\
&= d^2(u_0(0) - 2u_1(0) + u_2(0)) + h(u_1(0)) \\
&> 0,
\end{aligned}$$

it can be concluded that $\dot{v}_0(t) > 0$ for all sufficiently small $t > 0$. And since $\dot{v}_n$ and u_n are continuous, it follows that $t_1 > 0$. Now suppose that $t_1 < \infty$. Then either

1. $u_n(t_1) = u_{n+1}(t_1)$ for some $0 < u_n(t_1) < 1$
2. $\dot{v}_n(t_1) = 0$ for some $u_n(t_1) > 0$.

Suppose 1 is true. Because $u_{N+1} = 1$, n is chosen such that $u_n(t_1) = u_{n+1}(t_1) < u_{n+2}(t_1)$. It then follows that

$$\begin{aligned}
\frac{d}{dt}(v_{n+1} - v_n)(t_1) &= d(u_n(t_1) - 2u_{n+1}(t_1) + u_{n+2}(t_1) - u_{n-1}(t_1) + 2u_n(t_1) - u_{n+1}(t_1)) \\
&\quad + h(u_{n+1}(t_1)) - h(u_n(t_1)) \\
&= d(3u_n(t_1) - 3u_{n+1}(t_1) + u_{n+2}(t_1) - u_{n-1}(t_1)) + h(u_{n+1}(t_1)) - h(u_n(t_1)) \\
&= d(u_{n+2}(t_1) - u_{n-1}(t_1)) \\
&> 0
\end{aligned}$$

and therefore $u_{n+1}(t_1 - \epsilon) = v_{n+1}(t_1 - \epsilon) < v_n(t_1 - \epsilon) = u_n(t_1 - \epsilon)$ for sufficiently small ϵ. This is a contradiction, since $u(t)$ satisfies $u_n < u_{n+1}$ whenever $0 < u_n < 1$. Thus there does not exist an $u_n(t)$ such that $0 < u_n(t_1) < 1$ for which $u_n(t_1) = u_{n+1}(t_1)$.
Now suppose 2 is true. Consider $u_n(t)$ on the interval $[0, t_1]$. If there exists a $t_0 \in (0, t_1)$ such that $u_n(t_0) = 1$, then because $v_n(t)$ is non-decreasing on $[0, t_1]$, the following three cases may occur for all $t \in [t_0, t_1]$,

1. $u_n(t) = v_n(t)$ for all $t \in [0, t_1]$,
2. $u_n(t) = \begin{cases} v_n(t) \text{ for all } t \in [0, t_0] \\ 1 \text{ for all } t \in (t_0, t_1] \end{cases}$,
3. $u_n(t) = 1$ for all $t \in [0, t_1]$.

Let $a \in (0, t_1]$ be arbitrary.

$$\lim_{t\to 0^-} |\frac{u_n(t) - u_n(a)}{t}| = \lim_{t\to 0^-} |\frac{P(v_n(t) - P(v_n)(a)}{t}|.$$

so for the first case where $u_n(t) = v_n(t)$ for all $t \in [0, t_1]$ it follows that

$$\lim_{t \to 0^-} |\frac{P(v_n(t) - P(v_n)(a)}{t}| = \lim_{t \to 0^-} |\frac{v_n(t) - v_n(a)}{t}| = \dot{v}_n(a).$$

For the second case it holds that

$$\lim_{t \to 0^-} |\frac{P(v_n(t) - P(v_n)(a)}{t}| \begin{cases} = \lim_{t \to 0^-} |\frac{v_n(t) - v_n(a)}{t}| = \dot{v}_n(a) \text{ for all } t \in [0, t_1] \\ = \lim_{t \to 0^-} \frac{1-1}{t} = 0 \text{ for all } t \in (t_0, t_1], \end{cases}$$

For the final case where $u_n(t) = 1$, for all $t \in [0, t_1]$ it holds that

$$\lim_{t \to 0^-} |\frac{P(v_n(t) - P(v_n)(a)}{t}| = \lim_{t \to 0^-} |\frac{1 - 1}{t}| = 0.$$

Since the left derivative of $u_n(t)$ exists for every case, it exists for every $t \in (0, t_1]$ and is denoted by $\dot{u}_n(t-)$. Since 2 was assumed to be true, it follows that $\dot{u}_n(t_1) = 0$ since $u_n(t) = 1$ for all $t \in [t_0, t_1]$. Then

$$\ddot{v}_n(t_1-) = d(\dot{u}_{n-1}(t_1-) - 2\dot{u}_n(t_1-) + \dot{u}_{n+1}(t_1-)) + h(\dot{u}_n(t_1-)) = d(\dot{u}_{n-1}(t_1-) + \dot{u}_{n+1}(t_1-)) \geq 0$$

where $\dot{u}_{n-1}(t_1-) \geq 0$ and $\dot{u}_{n+1}(t_1-) \geq 0$.
If $\ddot{v}_n(t_1-) > 0$, this would imply that $\dot{v}_n(t_1 - \epsilon) < 0$ for some small positive ϵ, which is impossible. Therefore $\ddot{v}_n(t_1-) = 0$ and thus

$$\dot{u}_{n-1}(t_1-) = 0 \text{ and } \dot{u}_{n+1}(t_1-) = 0.$$

The inequality $v_{n-1}(t_1) > 1$ would give $\dot{v}_n(t_1) = h(1) > 0$ which is not possible. Therefore it holds that $\dot{v}_{n-1}(t_1) = \dot{u}_{n-1}(t_1-) = 0$. By repeating this argument for $j \in \{0, n-1\}$ it follows that

$$\dot{u}_j(t_1-) = 0 \text{ for } j = 0, 1, ..., n-1. \tag{4.6}$$

If $v_{n+1} > 1$ then $u_{n+1}(t) = 0$ for t sufficiently close to t_1 and since $0 \leq u_0 \leq ... \leq u_N \leq 1$, it follows that $\dot{u}_{n+1}(t_1) = ... = \dot{u}_N(t_1-) = 0$. If $v_{n+1} \leq 1$, then $\dot{v}_{n+1} \leq 1$, then $\dot{v}_{n+1}(t_1) = \dot{u}_{n+1}(t_1-) = 0$. So together with equation (4.6) it can be concluded that

$$\dot{u_n}(t_1-) = 0 \text{ for } n = 0, 1, ..., N.$$

Let n_1 be the maximal $n \in \{0, 1, ..., N\}$ for which $u_n(t) < 1$ for all $0 \leq t < t_1$. Then there exists $t_0 \in [0, t_1)$ such that $\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n)$ for $n = 0, 1, ..., n_1$, where $u_{n_1+1} = 1$ for $t \in [t_0, t_1]$. Since $\dot{u}_n(t_1-) = 0$ for $n = 0, 1, ..., N$, one has $\dot{u}_n(t_1) = 0$ for $n = 0, 1, ..., n_1$ and by the uniqueness of the initial value problem $u_n(t) = u_n(t_1)$ for $t \in [t_0, t_1], n = 0, 1, ..., n_1$. Therefore $\dot{v}_n(t) = 0$ for all $t \in (t_0, t_1)$ and $n \in \{0, 1, ..., n_1\}$ in contradiction to the choice of t_1. Therefore, it can be concluded that $t_1 = \infty$ and therefore the unique solution $u(t)$ of the initial value problem satisfies properties 1, 2, and 3 for all $t \geq 0$. □

For $d > -h(0)$ and $N \in \mathbb{N}$, the sets $C(h, d, N)$ and $O(h, d, N)$ are defined as

$$C(h, d, N) = \Big\{ x \in \mathbb{R}^{N+1} : x_0 = 1, x_1 = \frac{-h(0)}{d} \leq x_2 \leq ... \leq x_N \leq 1, \text{and } x_n \geq \frac{n}{N}, \\ \text{for } n = 1, 2, .., N \Big\}$$

$$O(h,d,N) = \Big\{x \in \mathbb{R}^{N+1} : d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0,$$
$$\text{for } n = 1, 2, ..., N, \text{where } x_{N+1} = 1\Big\}.$$

Note that every element in $C \cap O$ satisfies Lemma 4.3.1. Furthermore, the function $t^* : \overline{C \cap O} \to (0, \infty]$ is defined by $t^*(x) = \sup\{t : u_0(x;t) < \frac{-h(0)}{d}\}$ and the map $T : \{x \in \overline{C \cap O} : t^*(x) < \infty\} \to \mathbb{R}^{N+1}$ is defined by

$$(Tx)_n = \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x;t^*) & \text{for } n = 1, ..., N. \end{cases}$$

Lemma 4.3.2. $T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset O$

Proof. Suppose $x \in C \cap O$. If x satisfies $x_n < x_{n+1}$ whenever $0 < x_n < 1$ then by Lemma 4.3.1 it follows that $Tx \in O$. If x does not necessarily satisfy $x_n < x_{n+1}$ whenever $0 < x_n < 1$, it will be proven that it still follows that $Tx \in O$. Define n_1 as the largest $n \in \{0, 1, ..., N\}$ for which $x_n < 1$ and define for $\epsilon > 0$

$$x_n^\epsilon = \begin{cases} x_n + n\epsilon & \text{for } n = 0, ..., n_1 \\ x_n & \text{for } n = n_1 + 1, ..., N. \end{cases}$$

For ϵ small it follows that $x^\epsilon \in C \cap O$ and $x_n^\epsilon < x_{n+1}^\epsilon$ whenever $0 < x_n^\epsilon < 1$. By Lemma 4.3.1 $u(x^\epsilon, t^*(x))$ satisfies $d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) > 0$ whenever $0 < u_n$, where $u_{N+1} = 1$ and $u_{-1} = 0$. Now since $x^\epsilon \to x$ as $\epsilon \to 0$ and u is continuous, it follows that $y = u(x;t^*)$ satisfies $d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) \geq 0$ whenever $0 < y_n$. In the proof of Lemma 4.3.1 the contradiction that $d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) = 0$ was derived. This contradiction thus implies that $d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) > 0$ whenever $0 < y_n$. Therefore again by Lemma 4.3.1 $Tx \in O$. Now suppose $x \in \overline{C \cap O}$. Then there exists a sequence $\{x^k\}$ in $C \cap O$ such that $x^k \to x$ as $k \to \infty$. Since $u(x^k, t^*(x))$ also satisfies $d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) > 0$ whenever $0 < u_n$ for each k, using the same argument as before, the same result applies for $u(x, t^*(x))$. Therefore $Tx \in O$ and thus for every $x \in \{x \in \overline{C \cap O} : t^*(x) < \infty\}$, $x \in O$ and it can be concluded that $T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset O$. □

Lemma 4.3.3. *For every $h \in B_{app}$ there exists a positive δ such that $T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset C$ for $N \geq \frac{1}{\delta}$.*

Proof. Define $y = Tx$. Since $t^*(x)$ is defined as $t^*(x) = \sup\{t : u_0(x;t) < \frac{-h(0)}{d}\}$, it follows that $y_0 = 0$ and $y_1 = \frac{-h(0)}{d}$. By the same argument as in the proof of Lemma 4.3.2 the result that $y_1 \leq y_2 ... \leq y_N \leq 1$ follows. To prove that for $N \geq \frac{1}{\delta}$, $T\{x \in \overline{C \cap O} : t^*(x) < \infty\} \subset C$ it remains to prove that there exists a $\delta > 0$ such that $y_n \geq \frac{n}{N}$ for $n = 1, ..., N$, whenever $N \geq \frac{1}{\delta}$. Define $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$, the values of δ_i for $i = 1, ..., 6$ will be constructed along the way. Define a, b, m_1 and m_2 such that

- $a \in (0, 1)$ is the unique zero of h,
- $b(x) = \max_{a \leq s \leq x} h(s)$ for $x \in [a, 1]$,
- $m_1 = d + \sup_{0 \leq s \leq 1} |h(s)|$,
- $m_2 = m_1(4d + \sup_{s \neq t} |\frac{h(s)-h(t)}{(s-t)}|)$.

Now suppose that $N \geq \frac{1}{\delta}$ and suppose that $y_n \leq a$. Because $y \in O(h,d,N)$ it follows that

$$\begin{aligned} d(y_{n-1} - 2y_n + y_{n+1}) + h(y_n) &> 0 \\ d(y_{n-1} - 2y_n + y_{n+1}) &> -h(y_n) \\ y_{n-1} - 2y_n + y_{n+1} &> \frac{-h(y_n)}{d} \\ y_{n-1} - y_n &> y_n - y_{n-1} - \frac{h(y_n)}{d}. \end{aligned}$$

Since $y_n \leq a$ and thus $h(y_n) \leq 0$,it follows that

$$y_{n-1} - y_n > y_n - y_{n-1}. \tag{4.7}$$

By induction it will be proven that for $i = 1, ..., n+1$, $y_{i+1} - y_i > y_1 - y_0$. When $i = 1$, $y_2 - y_1 > y_1 - y_0$ follows from equation (4.7). Hence, the equation fulfills the basis step. Now assume that the following induction hypothesis is true: for $i = k, y_{k+1} - y_k > y_1 - y_0$. Then for $i = k+1$

$$y_{k+1} - y_{k+1} > y_{k+1} - y_k > y_1 - y_0$$

again by equation (4.7). Hence by induction, $y_{i+1} - y_i > y_1 - y_0$ for $i = 1, ..., n+1$. Define $\delta_1 = \frac{-h(0)}{d}$. Then $y_1 - y_0 = \frac{-h(0)}{d} = \delta_1$, and it follows that $y_i - y_0 = \sum_{j=1}^{i} y_j - y_{j-1}$ and thus $y_i - y_0 > i(\frac{-h(0)}{d}) = i\delta_1$ and thus $y_i > i\delta + y_0 \geq i\delta_1 \geq i\delta = \frac{i}{N}$ for $i = 1, ..., n+1$. If $y_{N-1} \leq a$, then the conclusion follows. Otherwise there is an index n_0 such that $y_{n_0-1} \leq a$ and $y_{n_0} > a$. Hence to prove the lemma, it remains to check that $y_n \geq \frac{n}{N}$ for $n = n_0+1, ..., N$. Now define $\delta_2 = \max\{x - a : a < x \leq 1 \text{ and } b(x) \leq (\frac{d}{2})\delta_1\}$. Either $y_n \leq a + \delta_2$ in which case

$$y_{n_0+1} - y_{n_0} > (y_{n_0} - y_{n_0-1}) - (\frac{1}{d})h(y_{n_0}) \geq \delta_1 - \frac{1}{2}\delta_1 = \frac{1}{2}\delta_1$$

or $y_{n_0} > a + \delta_2$. Now define $\delta_3 = \min\{\frac{1}{2}\delta_1, \delta_2\}$. Then it suffices to show that $y_n \geq \frac{n}{N}$ for $y_n \geq a + \delta_3$. This will be proven by contradiction. Let n_1 be the smallest index for which $y_{n_1} \geq a + \delta_3$ and $y_{n_1} < \frac{n_1}{N}$. Then

$$\frac{n_1 - 2}{N} \leq x_{n_1-2} \leq x_{n_1-1} \leq y_{n_1} < \frac{n_1}{N}$$

and $x_{n_1-1} \geq a + \frac{\delta_3}{2}$. Now define $\delta_4 = \frac{\delta_3}{2}$ and $\delta_5 = \min\{\frac{h(s)}{4d} : a + \frac{\delta_3}{2} \leq s \leq 1\}$. Since $\frac{1}{N} \leq \delta_5 \leq \frac{h(x_{n_1-1})}{4d}$, it follows that

$$x_{n_1-1} - x_{n_1-2} < \frac{n_1}{N} - \frac{n_1 - 2}{N} = \frac{2}{N} \leq \frac{h(x_{n_1} - 1)}{2d}.$$

Suppose $0 < u_n(0) < 1$ and $0 < u_n(t) < 1$. The goal is to find an estimation for $|\dot{u}_n(t) - \dot{u}_n(0)|$. Firstly,

$$\begin{aligned} |\dot{u}_n(t) - \dot{u}_n(0)| &= |d(u_{n-1}(t) - 2u_n(t) + u_{n+1}(t)) + h(u_n(t)) \\ &\quad - d(u_{n-1}(0) - 2u_n(0) + u_{n+1}(0)) - h(u_n(0))| \\ &= |d(u_{n-1}(t) - u_{n-1}(0) - 2u_n(t) + 2u_n(0) \\ &\quad + u_{n+1}(t) - u_{n+1}(0)) + h(u_n(t)) - h(u_n(0))| \\ &\leq |d||(u_{n-1}(t) - u_{n-1}(0) - 2u_n(t) + 2u_n(0) + u_{n+1}(t) - u_{n+1}(0))| \\ &\quad + |h(u_n(t)) - h(u_n(0))| \\ &\leq d \int_0^t |\dot{u}_{n-1}(s) - 2\dot{u}_n(s) + \dot{u}_{n+1}(s)| ds + \frac{|h(u_n(t) - h(u_n(0))|}{|u_n(t) - u_n(0)|} |u_n(t) - u_n(0)|. \end{aligned}$$

Filling in the formula again then gives

$$
\begin{aligned}
|\dot{u}_n(t) - \dot{u}_n(0)| \leq d \int_0^t & |d(u_{n-2}(s) - 2u_{n-1}(s) + u_n(s) - 2u_{n-1}(s) + 4u_n(s) - 2u_{n+1}(s) \\
& + u_n(s) - 2u_{n+1}(s) + u_{n+2}(s) + h(u_{n-1}(s)) - 2h(u_n(s)) + h(u_{n+1}(s))|ds \\
& + \frac{|h(u_n(t) - h(u_n(0))|}{|u_n(t) - u_n(0)|} \int_0^t |d(u_{n-1}(s) - 2u_n(s) + u_{n+1}(s)) + h(u_n(s))|ds \\
& \leq d^2 \int_0^t |u_{n-2}(s) - 4u_{n-1}(s) + 6u_n(s) - 4u_{n+1}(s) + u_{n+2}|ds \\
& + d \int_0^t |h(u_{n-1}(s)) - 2h(u_n(s)) + h(u_{n+1}(s))|ds \\
& + \frac{|h(u_n(t) - h(u_n(0))|}{|u_n(t) - u_n(0)|} d \int_0^t |u_{n-1}(s) - 2u_n(s) + u_{n+1}(s)|ds \\
& + \frac{|h(u_n(t) - h(u_n(0))|}{|u_n(t) - u_n(0)|} \int_0^t |h(u_n(s))|ds.
\end{aligned}
$$

Let's first examine the first part of this equation. It can be rewritten as

$$
d^2 \int_0^t |u_{n-2}(s) - 4u_{n-1}(s) + 6u_n(s) - 4u_{n+1}(s) + u_{n+2}|ds =
$$

$$
d^2 \int_0^t | - (u_{n-1} - u_{n-2}) + 3(u_n - u_{n-1}) - 3(u_{n+1} - u_n) + (u_{n+2} - u_{n+1})|ds.
$$

Since it was assumed that $0 < u_n(0) < 1$ and $0 < u_n(t) < 1$ and the u_n's follow the ordering property, it follows that $0 \leq u_j - u_{j-1} \leq 1$ for $j = 1, ..., N$. Suppose $(u_n - u_{n-1}) = 1$. Then $u_n = 1$, $u_{n-1} = 0$ and by the ordering property it follows that $u_{n+1} = u_{n+2} = 1$ and $u_{n-1} = u_{n-2} = 0$. Then $| - (u_{n-1} - u_{n-2}) + 3(u_n - u_{n-1}) - 3(u_{n+1} - u_n) + (u_{n+2} - u_{n+1})| = 3$. The same conclusion follows if $(u_{n+1} - u_n) = 1$. Therefore, the maximum value of $| - (u_{n-1} - u_{n-2}) + 3(u_n - u_{n-1}) - 3(u_{n+1} - u_n) + (u_{n+2} - u_{n+1})|$ is 3. Thus it follows that

$$
\begin{aligned}
d^2 \int_0^t |u_{n-2}(s) - 4u_{n-1}(s) + 6u_n(s) - 4u_{n+1}(s) + u_{n+2}|ds & \leq d^2 \int_0^t 3ds \\
& = d^2 3t.
\end{aligned}
$$

For computation purposes, define the following

$$
M = \sup_{0 \leq s \leq 1} |h(s)|,
$$

$$
L = \sup_{s \neq t} \frac{|h(s) - h(t)|}{|s - t|}.
$$

Then it follows that

$$
\begin{aligned}
d \int_0^t |h(u_{n-1}(s)) - 2h(u_n(s)) + h(u_{n+1}(s))|ds & \leq d \int_0^t 4Mds \\
& = d4tM.
\end{aligned}
$$

By the same argument as before

$$\begin{aligned}\frac{|h(u_n(t)-h(u_n(0))|}{|u_n(t)-u_n(0)|}d\int_0^t |u_{n-1}(s)-2u_n(s)+u_{n+1}(s)|ds &\leq dL\int_0^t |u_{n-1}(s)-2u_n(s)+u_{n+1}(s)|ds\\ &\leq dL\int_0^t |(u_{n+1}-u_n)-(u_n-u_{n-1})|ds\\ &= dL\int_0^t 1ds\\ &= dLt.\end{aligned}$$

Finally it follows that

$$\begin{aligned}\frac{|h(u_n(t)-h(u_n(0))|}{|u_n(t)-u_n(0)|}\int_0^t |h(u_n(s))|ds &\leq LM\int_0^t 1ds\\ &= LMt.\end{aligned}$$

Recall that $m_1 = d+\sup_{0\leq s\leq 1}|h(s)|$ and $m_2 = m_1(4d+\sup_{s\neq t}|$. Combining these results then gives

$$\begin{aligned}|\dot{u}_n(t)-\dot{u}_n(0)| &\leq d^2\int_0^t |u_{n-2}(s)-4u_{n-1}(s)+6u_n(s)-4u_{n+1}(s)+u_{n+2}|ds\\ &\quad + d\int_0^t |h(u_{n-1}(s))-2h(u_n(s))+h(u_{n+1}(s))|ds\\ &\quad + \frac{|h(u_n(t)-h(u_n(0))|}{|u_n(t)-u_n(0)|}d\int_0^t |u_{n-1}(s)-2u_n(s)+u_{n+1}(s)|ds\\ &\quad + \frac{|h(u_n(t)-h(u_n(0))|}{|u_n(t)-u_n(0)|}\int_0^t |h(u_n(s))|ds\\ &\leq d^2 3t + d4tM + dLt + LMt\\ &= t(3d^2+d4M+dL+LM)\\ &\leq t(4d^2+d4M+dL+LM)\\ &= t(d+M)(4d+L)\\ &= t(d+\sup_{0\leq s\leq 1}|h(s)|)(4d+\sup_{s\neq t}\frac{|h(s)-h(t)|}{|s-t|})\\ &= t(m_1(4d+\sup_{s\neq t}\frac{|h(s)-h(t)|}{|s-t|}))\\ &= tm_2.\end{aligned}$$

Thus an estimation for $|\dot{u}_n(t)-\dot{u}_n(0)|$ has been found, namely

$$|\dot{u}_n(t)-\dot{u}_n(0)| \leq tm_2.$$

This implies that $\dot{u}_n(t) \geq \dot{u}_n(0) - tm_2$ and thus $\dot{u}_n(t) \geq \frac{1}{2}\dot{u}_n(0)$ for $0 \leq t \leq \frac{\dot{u}_n(0)}{2m_2}$. This gives

$$u_n(t)-u_n(0) = \int_0^t \dot{u}_n(s)ds \geq t\frac{1}{2}\dot{u}_n(0)$$

for $0 \leq t \leq \frac{\dot{u}_n(0)}{2m_2}$. Then deriving

$$\begin{aligned}
\dot{u}_{n_1-1} &= d(u_{n_1-2}(0) - 2u_{n_1-1}(0) + u_{n_1}(0)) + h(u_{n_1-1}(0)) \\
&\geq h(u_{n_1-1}(0)) - d(u_{n_1-1}(0) - u_{n_1-2}(0)) \\
&= h(x_{n_1-1}) - d(x_{n_1-1} - x_{n_1-2}) \\
&\geq \frac{h(x_{n_1-1})}{2}.
\end{aligned}$$

and using this for $n = n_1 - 1$, gives

$$u_{n_1-1}(t) - x_{n_1-1} \geq t\frac{1}{2}\dot{u}_{n_1-1}(0) = t\frac{1}{2}\frac{h(x_{n_1-1})}{2} = t\frac{h(x_{n_1-1})}{4}$$

for $0 \leq t \leq \frac{h(x_{n_1-1})}{4m_2}$. The definition $t^*(x) = \sup\{t : u_0(x;t) < \frac{-h(0)}{d}\}$ can be used to find that

$$\frac{-h(0)}{d} = u_0(t^*) - u_0(0) = \int_0^{t^*} \dot{u}_0(s)ds \leq t^* \text{sup}_s \dot{u}_0(s)$$

and since $\dot{u}_0(t) = d(u_1(t) - 2u_0(t)) + h(u_0(t))$, it follows that

$$\begin{aligned}
t^* \text{sup}_s \dot{u}_0(s) &\geq \frac{-h(0)}{d} \\
t^* &\geq \frac{-h(0)}{d}\frac{1}{\text{sup}_s \dot{u}_0(s)}.
\end{aligned}$$

Since $\text{sup}_s(u_1(s) - 2u_0(s)) = 1$ it follows that

$$\begin{aligned}
\sup_s \dot{u}_0(s) &= \sup_s d(u_1 - 2u_0)(s) + h(u_0)(s) \\
&, \geq d + \sup_{0\leq s\leq 1} |h(s)|.
\end{aligned}$$

Thus it can be concluded that $t^* \geq \frac{-h(0)}{dm_1}$. Finally, define $\delta_6 = \min\{\frac{d\delta_5}{m_2}, \frac{-h(0)}{dm_1}\}d\delta_5$. Tt then follows that

$$u_{n_1-1}(t^*) - x_{n_1-1} \geq \delta_6$$

and thus

$$u_{n_1-1}(t^*) \geq \delta_6 + x_{n_1-1} \geq \frac{n_1}{N}.$$

This is a contradiction to $y_{n_1} < \frac{n_1}{N}$. Hence, it can be concluded that there exists a $\delta > 0$, namely $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$, such that $y_n \geq \frac{n}{N}$ whenever $N \geq \frac{1}{\delta}$. □

4.4 Simplified problem

In this section, the simplified equation (4.2) is discussed. Since $u_n = P(v_n)$, the variable u_n takes values in $[0, 1]$. Suppose u_n is a solution of this simplified equation on an interval $[0, \tau]$ satisfying the following conditions

1. $u_n(\tau) = u_{n+1}(0)$,
2. $v_n(0) \leq v_{n+1}(0)$,
3. $d(u_{n-1}(0) - 2u_n(0) + u_{n+1}(0) + h(u_n(0)) > 0$ if $u_n(0) > 0$,

4. $\lim_{n\to-\infty} v_n(0) = 0$ and $\lim_{n\to\infty} v_n(0) = 1$.

By the Picard-Lindelöf theorem the solution to (4.2) exists on $t \in [0, \tau]$ and is unique. By Lemma 4.3.1 the initial values x are invariant under the flow of $u(t)$. By theorem 3.1.5 the solution $u(t)$ is independent of t. Then $u_n(\tau) = u_{n+1}(0)$ implies that $u_{n+1}(t) = u_n(t + \tau)$. Iterating n times then gives

$$\begin{aligned} u_n(t) &= u_0(t + n\tau) \\ &= u_0(\tau(n + \frac{t}{\tau})). \end{aligned}$$

Furthermore, $u_n(-\infty) = \lim_{n\to-\infty} v_n(0) = 0$ and $u_n(\infty) = \lim_{n\to\infty} v_n(0) = 1$. Now define $U(s) = u_0(\tau s)$. Then the conditions imply the existence of a function $U : \mathbb{R} \to \mathbb{R}$ with $U(-\infty) = 0$, $U(\infty) = 1$ and $0 \leq U \leq 1$ such that $u_n(t) = U(n + ct)$ and $\tau = \frac{1}{c}$ for all $n \in \mathbb{Z}, t \in [0, \tau]$. Therefore, the solution to (4.2) is a traveling wave solution.

Suppose a is the unique zero of $h \in B_{app}$ in (0,1).

Lemma 4.4.1. *Suppose x is a fixed point of T, and let $\tau = t^*(x)$. Let $e(h) = \min_{\frac{a}{4} \leq s \leq \frac{a}{2}} \frac{-h(s)}{d}$, $m(h) = \min\left\{\frac{a(h)}{4}, e(h)\right\}$ and $M(h) = \max_{0 \leq s \leq 1}\{2d + h(s)\}$. Then $\tau \geq \tau_0(h) = \frac{m(h)}{M(h)} > 0$.*

Proof. Suppose there exists an integer n such that $\frac{a}{4} \leq x_n \leq \frac{a}{2}$. $Tx \in O$ by Lemma 4.3.2, so we have

$$\begin{aligned} d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) &> 0 \\ d(x_{n-1} - 2x_n + x_{n+1}) &> -h(x_n) \\ x_{n-1} - 2x_n + x_{n+1} &> \frac{-h(x_n)}{d} \\ -2x_n + x_{n+1} &> \frac{-h(x_n)}{d} - x_{n-1} \\ x_{n+1} - x_n &> \frac{-h(x_n)}{d} + x_n - x_{n-1} \\ x_{n+1} - x_n &> e(h) + x_n - x_{n-1} \\ x_{n+1} - x_n &> e(h). \end{aligned}$$

Suppose there does not exist an integer n such that $\frac{a}{4} \leq x_n \leq \frac{a}{2}$. Then this inequality also holds for $n + 1$. Since $x_{n+1} \geq x_n$ it follows that $x_{n+1} - x_n > \frac{a}{4}$. Either way, there exists an integer n such that $x_n \leq \frac{a}{2}$ and $x_{n-1} - x_n \geq m(h)$. Since $\dot{u}_n(t) \leq M(h)$ for all $t \geq 0$, the following result follows

$$m(h) \leq u_{n+1}(0) - u_n(0) = u_n(\tau) - u_n(0) = \int_0^\tau \dot{u}_n(t)dt \leq \tau M(h)$$

from which it follows that $\tau \geq \frac{m(h)}{M(h)}$ ☐

Lemma 4.4.1 will be used to prove the following lemma that shows that for a monotone traveling wave only finitely many of the values $x_n = u_n(0)$ are different from 0 and 1.

Lemma 4.4.2. *Let $\epsilon_0 = \frac{1}{2}min\{a,1\text{-}a\}$ and for $x = \{x_0\}_0^N$ let $\#(x, e)$ be the number of $x_n's$ in x such that $\epsilon < x_n < 1 - \epsilon$. Then for all $\epsilon \in [0, \epsilon_0]$ there exists a bound $S(e, h)$ independent of N such that $\#(x, \epsilon) \leq S(\epsilon, h)$ for all $x \in \{x \in \bigcup_N \overline{C \cap O} : Tx = x\}$.*

Proof. Let $h \in B_{app}$ and $\epsilon \in [0, \epsilon_0]$ be arbitrary and assume x is a fixed point of T. Define $p(\epsilon, h)$ as $p(\epsilon, h) = \max_{\epsilon \leq s \leq \frac{a}{2}} \frac{d}{-h(s)}$ and suppose $\epsilon < x_i \leq x_{i+1} \leq ... \leq x_j \leq \frac{a}{2}$. It follows that $j + 1 - i \leq p(\epsilon, h)$ as

$$\begin{aligned}
1 &\geq x_{j+1} - x_i \\
&= \sum_{n=i}^{j} (x_{n+1} - x_n) \\
&\geq \sum_{n=i}^{j} (x_n - x_{n-1}) - \frac{1}{d} \sum_{n=i}^{j} h(x_n) \\
&\geq -\frac{1}{d} \sum_{n=i}^{j} h(x_n) \\
&\geq \frac{j+1-i}{p(\epsilon, h)}.
\end{aligned}$$

Thus the number of $x_n's$ with $\epsilon < x_n \leq \frac{a}{2}$ is bounded above by $p(\epsilon, h)$. In the proof of Lemma 4.4.1 it is shown that there exists an integer n_0 such that $x_{n_0} \leq \frac{a}{2}$ and $x_{n_0+1} - x_{n_0} \geq m(h)$ where $m(h)$ is defined as in Lemma 4.4.1. Since for all $0 \leq x_n \leq a$

$$\begin{aligned}
x_{n+1} - x_n &> x_n - x_{n-1} - \frac{1}{d} h(x_n) \\
&\geq x_n - x_{n-1}
\end{aligned}$$

it follows that for all $\frac{a}{2} \leq x_n \leq a$, $x_{n+1} - x_n \geq m(h)$. Therefore, the number of x_n's with $\frac{a}{2} \leq x_n \leq a$ is bounded above by $1 + \frac{a}{2} m(h)$. Now define

- $b(x, h) = \max_{a \leq s \leq x} \frac{h(s)}{d}$,
- $\sigma_1(h) = \max\{x - a : a \leq x \leq 1, b(x, h) \leq \frac{m(h)}{2}\}$,
- $\sigma_2(h) = \min\{\sigma_1(h), \frac{m(h)}{2}\}$.

Since $\sigma_1 > 0$ and $m(h) > 0$, it follows that $\sigma_2(h) > 0$. Either $x_{n_0} \leq a + \sigma_1(h)$ or $x_{n_0} > a + \sigma_1(h)$. If $x_{n_0} \leq a + \sigma_1(h)$, then

$$\begin{aligned}
x_{n_0+1} - x_{n_0} &> (x_{n_0} - x_{n_0-1}) - \frac{1}{d} h(x_{n_0}) \\
&\geq m(h) - \frac{1}{2} m(h) \\
&= \frac{m(h)}{2}.
\end{aligned}$$

In both cases, there is at most one x_n such that $a < x_n \leq a + \sigma_2(h)$. Suppose $a + \sigma_2(h) \leq x_n \leq x_{n+1} < 1 - \epsilon$. For $\tau = t^*(x)$ and $0 \leq s \leq \tau$, $\dot{u}_n(s)$ is estimated as follows

$$\begin{aligned}
\dot{u}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n) \\
&\geq h(u_n) - d(u_n - u_{n-1}) \\
&\geq \min_{x_n \leq s \leq x_{n+1}} h(s) - d(x_{n+1} - x_{n-1}).
\end{aligned}$$

Then

$$\begin{aligned}
x_{n+1} - x_{n-1} &> x_{n+1} - x_n \\
&= u_n(\tau) - u_n(0) \\
&= \int_0^\tau \dot{u}_n(s)ds \\
&\geq \int_0^\tau \min_{x_n \leq s \leq x_{n+1}} h(s) - d(x_{n+1} - x_{n-1})ds \\
&= \tau(\min_{x_n \leq s \leq x_{n+1}} h(s) - d(x_{n+1} - x_{n-1})) \\
&= \tau(\min_{x_n \leq s \leq x_{n+1}} h(s)) - d\tau(x_{n+1} - x_{n-1})).
\end{aligned}$$

It follows that

$$\begin{aligned}
x_{n+1} - x_{n-1} + d\tau(x_{n+1} - x_{n-1}) &\geq \tau(\min_{x_n \leq s \leq x_{n+1}} h(s)) \\
(1 + d\tau)(x_{n+1} - x_{n-1}) &\geq \tau(\min_{x_n \leq s \leq x_{n+1}} h(s)) \\
x_{n+1} - x_{n-1} &\geq \frac{\tau}{d\tau + 1}(\min_{x_n \leq s \leq x_{n+1}} h(s)).
\end{aligned}$$

Since $\frac{\tau}{d\tau+1} \geq \frac{\tau_0}{\tau_0+1}$ for $\tau \geq \tau_0$, one concludes by Lemma 4.4.1 that

$$x_{n+1} - x_{n-1} \geq \tau_3 = \frac{\tau_0(h)}{d\tau_0(h) + 1}(\min_{a+\tau_2 \leq s \leq 1-\epsilon} h(s)).$$

So since it was assumed that $a + \tau_2 \leq x_i \leq ... \leq x_{j+1} < 1 - \epsilon$, then $2 \geq \sum_{n=i}^{j}(x_{n+1} - x_{n-1}) \geq (j + 1 - i)\tau_3(h)$ and therefore that $j - i \leq \frac{2}{\tau_3(h)} + 1$. Therefore, this shows that the number of x_n's such that $a + \tau_2 \leq x_n < 1 - \epsilon$ is bounded above by $\frac{2}{\tau_3(h)} + 1$. The following conclusions have thus been drawn:

- The number of x_n's with $\epsilon < x_n \leq \frac{a}{2}$ is bounded above by $p(\epsilon, h)$,
- The number of x_n's with $\frac{a}{2} \leq x_n \leq a$ is bounded above by $1 + \frac{a(h)}{2m(h)}$,
- There is at most one x_n such that $a < x_n \leq a + \tau_2(h)$,
- The number of x_n's such that $a + \tau_2 \leq x_n < 1 - \epsilon$ is bounded above by $\frac{2}{\tau_3(h)} + 1$.

Summarizing these conclusions gives

$$\#(x, \epsilon) \leq p(\epsilon, h) + (1 + \frac{a(h)}{2m(h)} + 1 + (\frac{2}{\tau_3(h)} + 1) = S(\epsilon, h)$$

for $\epsilon < x_n < 1 - \epsilon$. □

Therefore it suffices to consider only finitely many solutions of the simplified problem 4.2.

4.5 Fixed point problem

For this section the following initial value problem is considered

$$\begin{aligned}
\dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n), \\
u_n &= P(v_n), \\
v_n(0) &= x_n \text{ with } 0 \leq x_n \leq 1 \text{ for } n = 0, ..., N.
\end{aligned} \tag{4.8}$$

Then for $u_{-1} = 0$ and $u_{N+1} = 1$ the initial value problem has a unique solution by the Picard Lindelöf theorem 3.1.2. By Theorem 3.1.5 it follows that this solution depends continuously on the initial value $x = \{x_n\}_{n=0}^N$. For computation purposes, the set X is defined as

$$
\begin{aligned}
X = \Big\{ &\{x_n\}_{n=0}^N \in \mathbb{R}^{N+1} : x_0 = 0, x_1 = \frac{-h(0)}{d}, x_n \leq x_{n+1}, \\
&d(x_{n-1} - 2x_n + x_{n+1}) + h(x_n) > 0, \\
&x_n \geq \frac{n}{N}, \text{for } n = 1, .., N, \text{where } x_{N+1} = 1 \Big\}.
\end{aligned}
$$

Note that $X = C \cap O$ and thus every element of X satisfies Lemma 4.3.1. Then the following shifted Poincaré map $T : \bar{X} \to \mathbb{R}^{N+1}$ is considered

$$
(Tx)_n = \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x;\tau) & \text{for } n = 1, .., N \end{cases} \tag{4.9}
$$

where τ is defined by $u_0(x;\tau) = x_1$. Further details on Poincaré maps can be examined in the book of Teschl [Tes12].
The following two lemmata will be used to show that for d sufficiently large $t^*(x) < \infty$ for all $x \in \overline{C \cap O}$.

Lemma 4.5.1. *Let $x \in C(h, d, N) \cap O(h, d, N)$, $\dot{u}(t) = \dot{u}(x, t)$, and $D > 0$. Suppose that for all $n \in \{1, 2, ..., N\}$, $0 < u_n(t) < 1$ implies $\dot{u}_n(t) < D$. Then for all $k \in \mathbb{N}$ for which $d \geq k^2(D + \sup|h|)$, $u_n(t) - u_{n-1}(t) \leq \frac{2}{k}$ for $n = 1, 2, ..., N$.*

Proof. To simplify the notation, let $\triangle_n = u_n(t) - u_{n-1}(t)$. For $0 < u_n(t) < 1$ one has $\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n)$ and therefore

$$
\begin{aligned}
\triangle_{n+1} - \triangle_n &= u_{n+1}(t) - u_n(t) - (u_n(t) - u_{n-1}(t)) \\
&= u_{n+1}(t) - 2u_n(t) - u_{n-1}(t) \\
&= \frac{1}{d}(\dot{u}_n(t) - h(u_n(t)).
\end{aligned}
$$

Now using the assumption that $\dot{u}_n(t) < D$ and $d \geq k^2(D + \sup|h|)$, it follows that

$$
\begin{aligned}
|\triangle_{n+1} - \triangle_n| &= |\frac{1}{d}(\dot{u}_n(t) - h(u_n(t))| \\
&< |\frac{1}{d}(D - h(u_n(t))| \\
&\leq \frac{1}{k^2}
\end{aligned}
$$

if $0 < u_n < 1$. Now suppose that $u_n(t) = 0$, then $n = 0$, $t = 0$, and $u_1(0) = \frac{-h(0)}{d}$. Since $d(\triangle_{n+1} - \triangle_n) + h(u_n) > 0$, it follows that $|\triangle_{n+1} - \triangle_n| = \frac{-h(0)}{d} < \frac{1}{k^2}$. If $u_n(t) = 1$, then $\triangle_{n+1} = 0$. Again since $d(\triangle_{n+1} - \triangle_n) + h(u_n) > 0$, it follows that $-d\triangle_n > -h(u_n)$ and thus $|\triangle_{n+1} - \triangle_n| = \frac{h(1)}{d} < \frac{1}{k^2}$. In both cases, one has $|\triangle_{n+1} - \triangle_n| \leq \frac{1}{k^2}$. Now contradiction will be used to prove that $u_n(t) - u_{n-1}(t) \leq \frac{2}{k}$ for $n = 1, 2, ..., N$. Suppose by contradiction that there exists an n_0 such that $\triangle_{n_0} > \frac{2}{k}$. Then

$$
\triangle_{n_0+m} = \triangle_{n_0} - \sum_{i=1}^{m}(\triangle_{n_0+i-1} - \triangle_{n_0+i}) > \frac{2}{k} - \frac{m}{k^2}
$$

and thus for $m = 0, 1, ..., k$ for $\triangle_{n_0+m} > \frac{1}{k}$. Since $\triangle_{N+1} = 0$, this implies that $n_0 + k \leq N$. Therefore

$$1 \geq u_{n_0+k}(t) - u_{n_0-1}(t) = \sum_{m=0}^{k} \triangle_{n_0+m} > \sum_{m=0}^{k} \frac{1}{k} = \frac{k+1}{k}$$

which leads to a contradiction. Thus for all $k \in \mathbb{N}$, for which $d \geq k^2(D + \sup|h|)$ it follows that $u_n(t) - u_{n-1}(t) \leq \frac{2}{k}$ for $n = 1, 2, ..., N$. □

Lemma 4.5.2. *There exists a number d_1 which depends only on $\sup|h|$, $\sup_{s\neq t} \frac{h(s)-h(t)}{(s-t)}$ and $\int_0^1 h(s)ds$, such that for all $x \in C(h,d,N) \cap O(h,d,N)$, $t \in [0, t^*)$, $d > d_1$, the following holds*

$$\sup_n \dot{u}_n(t) \geq \frac{1}{2}\int_0^1 h(s)ds.$$

The supremum here is taken over all n for which $0 < u_n < 1$.

Proof. Let $n_1 > 0$ be such that $u_{n_1}(t) < 1$. For this proof assume that t is fixed. The definition of $\dot{u}_n = d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n)$ for $n = 0, ..., n_1$ implies that

$$\begin{aligned}\dot{u}_n(u_{n+1} - u_n) &= (d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n))(u_{n+1} - u_n)\\ &= d(u_{n+1}u_{n-1} - u_n u_{n-1} - 2u_n u_{n+1} + 2u_n^2 + u_{n+1}^2 - u_n u_{n+1}) + h(u_n)(u_{n+1} - u_n)\end{aligned}$$

and

$$\begin{aligned}\dot{u}_n(u_n - u_{n-1}) &= (d(u_{n-1} - 2u_n + u_{n+1}) + h(u_n))(u_n - u_{n-1})\\ &= d(u_{n-1}u_n - u_{n-1}^2 - u_n^2 + 2u_n u_{n-1} + u_{n+1}u_n - u_{n+1}u_{n-1}) + h(u_n)(u_n - u_{n-1}).\end{aligned}$$

Adding the two results gives the following

$$\begin{aligned}\dot{u}_n(u_{n+1} - u_n) + \dot{u}_n(u_n - u_{n-1}) &= h(u_n)(u_{n+1} - u_n + u_n - u_{n-1}) +\\ &\quad d(u_{n+1}u_{n-1} - 2u_n u_{n+1} + u_{n+1}^2 - u_{n-1}u_n + 2u_n^2 - u_n u_{n+1}\\ &\quad + u_{n-1}u_n - 2u_n^2 + u_n u_{n+1} - u_{n-1}^2 + 2u_{n-1}u_n - u_{n-1}u_{n+1})\\ &= h(u_n)(u_{n+1} - u_{n-1}) + d(-2u_n u_{n+1} + u_{n+1}^2 + 2u_{n-1}u_n - u_{n-1}^2)\\ &= (u_{n+1} - u_{n-1})(h(u_n) + d(u_{n-1} - 2u_n + u_{n+1}))\\ &= (u_{n+1} - u_{n-1})\dot{u}_n\\ &\leq (\max_{0\leq n\leq n_1} \dot{u}_n)(u_{n+1} - u_{n-1}).\end{aligned}$$

Adding this result over n from 0 to n_1 gives

$$\begin{aligned}&\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + h(u_n)(u_{n+1} - u_n) + d(u_{n+1}^2 - u_{n-1}^2 + 2u_{n-1}u_n - 2u_n u_{n+1}) =\\ &\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) + \sum_{n=0}^{n_1} d(u_{n+1}^2 - u_{n-1}^2 + 2u_{n-1}u_n - 2u_n u_{n+1}) =\\ &\sum_{n=0}^{n_1} h(u_n)(u_{n+1} - u_n) + \sum_{n=0}^{n_1} h(u_n)(u_n - u_{n-1}) + d((u_{n_1+1} - u_{n_1})^2 - u_0^2) \leq\\ &(\max_{0\leq n\leq n_1} \dot{u}_n)(u_{n_1+1} + u_{n_1} - u_0)\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{n=0}^{n_1} h(u_n)(u_{n+1}-u_n)+\sum_{n=0}^{n_1} h(u_n)(u_n-u_{n-1}) &\le (\max_{0\le n\le n_1}\dot{u}_n)(u_{n+1}+u_{n_1}-u_0)\\ &\quad -(d((u_{n_1+1}-u_{n_1})^2-u_0^2))\\ &\le du_0^2+2\max_{0\le n\le n_1}\dot{u}_n.\end{aligned}$$

Since $u_0 \le u_1 = \frac{-h(0)}{d}$, it follows that

$$\sum_{n=0}^{n_1} h(u_n)(u_{n+1}-u_n)+\sum_{n=0}^{n_1} h(u_n)(u_n-u_{n-1}) \le \frac{h(0)^2}{d}+2\max_{0\le n\le n_1}\dot{u}_n.$$

The conclusion will be proven by contradiction. Suppose $\dot{u}_n < D = \frac{1}{2}\int_0^1 h(s)ds$ for $n = 0, 1, ..., n_1$. The variable $n_1(d)$ is defined as the largest $n \in \{0, ..., N\}$ such that $u_n < 1$. By Lemma 4.5.1 there exists a number d_1 such that $d_1 > \frac{2(h(0))^2}{\int_0^1 h(s)ds}$. Then for $d > d_1$, it follows that

$$|\sum_{n=0}^{n_1(d)} h(u_n)(u_{n+1}-u_n)-\int_0^1 h(s)ds| < \frac{1}{4}\int_0^1 h(s)ds$$

and

$$|\sum_{n=0}^{n_1(d)} h(u_n)(u_n-u_{n-1})-\int_0^1 h(s)ds| < \frac{1}{4}\int_0^1 h(s)ds.$$

Since $d > d_1$

$$\frac{h(0)^2}{d} < \frac{h(0)^2}{d_1} = h(0)^2\frac{\int_0^1 h(s)ds}{2h(0)^2} = \frac{1}{2}\int_0^1 h(s)ds.$$

Then

$$\begin{aligned}\sum_{n=0}^{n_1(d)} h(u_n)(u_{n+1}-u_n)+\sum_{n=0}^{n_1(d)} h(u_n)(u_n-u_{n-1}) &> 2\int_0^1 h(s)ds-2*\frac{1}{4}\int_0^1 h(s)ds-\frac{h(0)^2}{d}\\ &= \frac{3}{2}\int_0^1 h(s)ds-\frac{h(0)^2}{d}\\ &> \int_0^1 h(s)ds.\end{aligned}$$

But this leads to a contradiction by the assumption

$$\sum_{n=0}^{n_1} h(u_n)(u_{n+1}-u_n)+\sum_{n=0}^{n_1} h(u_n)(u_n-u_{n-1})-\frac{h(0)^2}{d} \le 2\max_{0\le n\le n_1}\dot{u}_n < 2\frac{1}{2}\int_0^1 h(s)ds = \int_0^1 h(s)ds$$

and thus the conclusion follows. Note that d_1 only depends on $\sup|h|$, $\sup_{s\neq t}\frac{h(s)-h(t)}{(s-t)}$ and $\int_0^1 h(s)ds$. □

Now define $d_2 = \max\{8, d_1\}$, where d_1 is chosen according to Lemma 4.5.2. δ is defined as $\min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ such that the conclusion of Lemma 4.3.3 holds. Now assume that $d > d_2$ and $N \ge \frac{1}{\delta}$. Recall that h was assumed to be a linear function, namely $h_0(x) = x - \frac{1}{4}$.

For shorter notation, $C_0 = C(h_0, d, N)$ and $D_0 = D(h_0, d, N)$. By Lemma 4.3.1 and the proof of Lemma 4.3.2 it follows that all u_n's are nondecreasing. Then for $t \in (0, t^*)$, by Lemma 4.5.2

$$N \geq \sum_{n=0}^{N} u_n(t) - \sum_{n=0}^{N} u_n(0) = \int_0^t \sum_{n=0}^{N} \dot{u}_n(v-)dv \geq \int_0^t \sup_n \dot{u}_n(v-)dv \geq \int_0^t \frac{1}{2}dv \int_0^1 h_0(s)ds = t\frac{1}{2}\int_0^1 h_0(s)ds.$$

Therefore

$$t^*(x) \leq \frac{2N}{\int_0^1 h_0(s)ds} = M^*.$$

Now define $T_0 = T(h_0)$. T was defined as $T : \{x \in \overline{C \cap O} : t^*(x) < \infty\} \to \mathbb{R}^{N+1}$ by

$$(Tx)_n = \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x;\tau) & \text{for } n = 1, .., N \end{cases} \tag{4.10}$$

where τ is defined implicitly by $u_0(x;\tau) = x_1$. The map T_0 is well-defined on $\overline{C_0 \cap O_0}$ as $t^*(x) \leq M^*$. The map T_0 has a fixed point by Brouwer's fixed point theorem if the map satisfies the following four properties

- $\overline{C_0 \cap O_0}$ is a closed, bounded, and convex subset of $\mathbb{R}^{N+1}$,
- $T_0(\overline{C_0 \cap O_0}) \subset C_0 \cap O_0$,
- T_0 is continuous,
- $C_0 \cap O_0$ is nonempty.

Lemma 4.5.3. *$\overline{C_0 \cap O_0}$ is a closed, bounded, and convex subset of $\mathbb{R}^{N+1}$.*

Proof. By definition of the closure of a set, it follows that $\overline{C_0 \cap O_0}$ is a closed set. Since for every x_n, it holds that $0 \leq x_n \leq 1$ the set has a lower and upper bound. Hence $\overline{C_0 \cap O_0}$ is bounded. It remains to prove that the set is convex. Suppose $\{a\}_{n=0}^N$ and $\{b\}_{n=0}^N$ are in $\mathbb{R}^{N+1}$ and that $\{a\}_{n=0}^N$, $\{b\}_{n=0}^N \in \overline{C_0 \cap O_0}$. Define θ as an arbitrary constant in $[0, 1]$. Since $a_0 = b_0 = 0$, $\theta a_0 + (1-\theta)b_0 = 0$. Furthermore, $a_1 = b_1 = \frac{-h_0(0)}{d}$ and thus

$$\theta a_1 + (1-\theta)b_1 = \theta\frac{-h_0(0)}{d} + (1-\theta)\frac{-h_0(0)}{d} = \theta\frac{-h_0(0)}{d} + \frac{-h_0(0)}{d} + \theta\frac{h_0(0)}{d} = \frac{-h_0(0)}{d}.$$

Since $a_n \leq a_{n+1}$ and $b_n \leq b_{n+1}$

$$\theta a_n + (1-\theta)b_n \leq \theta a_{n+1} + (1-\theta)b_{n+1}.$$

Since θ is a nonnegative constant, if $d(a_{n-1} - 2a_n + a_{n+1}) + h_0(a_n) > 0$, then also $\theta d(a_{n-1} - 2a_n + a_{n+1}) + \theta h_0(a_n) > 0$. In a similar way, since $(1-\theta)$ is a nonnegative constant, this implies that $(1-\theta)d(b_{n-1} - 2b_n + b_{n+1}) + (1-\theta)h_0(b_n) > 0$. So combining these results gives

$$\theta d(a_{n-1} - 2a_n + a_{n+1}) + \theta h_0(a_n) + (1-\theta)d(b_{n-1} - 2b_n + b_{n+1}) + (1-\theta)h_0(b_n) > 0.$$

Finally, $a_n \geq \frac{n}{N}$ and $b_n \geq \frac{n}{N}$ for $n = 1, ..., N$ where $a_{N+1} = b_{N+1} = 1$ implies that

$$\theta a_n + (1-\theta)b_n \geq \theta\frac{n}{N} + (1-\theta)\frac{n}{N} = \frac{n}{N}$$

for $n = 1, ..., N$ where $\theta a_{N+1} + (1-\theta)b_{N+1} = 1$. Therefore, it can be concluded that $\theta a_n + (1-\theta)b_n \in \overline{C_0 \cap O_0}$. Hence by definition the set is convex. In conclusion, $\overline{C_0 \cap O_0}$ is a closed, bounded and convex subset of $\mathbb{R}^{N+1}$. □

Lemma 4.5.4. $T_0(\overline{C_0 \cap O_0}) \subset C_0 \cap O_0$.

Proof. By Lemma 4.3.2 it follows that $T_0(\overline{C_0 \cap O_0}) \subset O_0$. By Lemma 4.3.3 it follows that for $N \geq \frac{1}{\delta}$, $T_0(\overline{C_0 \cap O_0}) \subset C_0$. Thus it can be concluded that $T_0(\overline{C_0 \cap O_0}) \subset C_0 \cap O_0$. □

Lemma 4.5.5. *T_0 is continuous.*

Proof. Define $r : \overline{C_0 \cap O_0} \times [0, M^*] \to \mathbb{R}$ by $r(x,t) = u_0(x;t)$, where $M^* = \frac{2N}{\int_0^1 h_0(s)ds}$. Since $u_0(x;t) = x_1$, this function is continuous. Furthermore, define $s : \overline{C_0 \cap O_0}$ X $[0, M^*] \to \mathbb{R}$ by $s(x,t) = \frac{-h(0)}{d}$ which is also a continuous function. Now let
$G(t^*) = \{(x,t) : x \in \overline{C_0 \cap O_0}, t = t^*(x)\}$ be the graph of t^*. Since $u_0(x;t) = x_1$, then
$G(t^*) = \{(x,t) : r(x,t) = s(x,t)\}$. Since $G(t^*)$ is the pre-image of a closed set of a continuous function, $G(t^*)$ is closed in $\overline{C_0 \cap O_0} \times [0, M^*]$. As $\overline{C_0 \cap O_0} \times [0, M^*]$ is compact, it follows that $G(t^*)$ is compact. Now suppose that t^* is not continuous at a point $x \in \overline{C_0 \cap O_0} \times [0, M^*]$.
This implies that there exists $\epsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\overline{C_0 \cap O_0}$ such that $x_n \to x$ as $n \to \infty$ and

$$|t^*(x_n) - t^*(x)| \geq \epsilon \tag{4.11}$$

for all $n \in \mathbb{N}$. Since $t^*(x_n) \in [0, M^*]$ for all $n \in \mathbb{N}$ and is thus bounded, it follows by the Bolzano-Weierstrass theorem that $\{x_n\}_n$ has a subsequence $\{x_{n_k}\}_n$ such that $t(x_{n_k}) \to \bar{t}$ where $\bar{t} \in \mathbb{R}$. By equation (4.11) it follows that $|\bar{t} - t^*(x)| \geq \epsilon$. For any $x \in \overline{C_0 \cap O_0}$ and $t \in \mathbb{R}$, $(x,t) \in G(t^*)$ implies that $t = t^*(x)$ by the definition of the graph. It then follows that $(x, \bar{t}) \notin G(t^*)$. Hence, if $(x_{n_k}, t^*(x_{n_k})) \in G(t^*)$ then as $n \to \infty$, $(x, \bar{t}) \notin G(t^*)$. Then $G(t^*)$ is not closed and therefore not compact. Hence, a contradiction has been reached. Thus it can be concluded that t^* is continuous. Then T_0 is continuous by its definition. This concludes the proof. □

Lemma 4.5.6. *$C_0 \cap O_0$ is nonempty.*

Proof. To show that $C_0 \cap O_0$ is nonempty, it suffices to show that there exists at least one x such that $x \in C_0 \cap O_0$. Recall that $h_0(x) = x - a$. Let $x_0 = 0$ and define inductively
$x_{n+1} = x_n + (n+1)\frac{a}{d}$ for $n = 0, 1, ..., n_0$ where n_0 is such that $x_{n_0} \leq a$ and $x_{n_0+1} > a$. Then for $n = 1, ..., n_0$

$$\begin{aligned}
d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) &= d((x_{n+1} - x_n) - (x_n - x_{n-1})) + h_0(x_n) \\
&= d(n+1)\frac{a}{d} - n\frac{a}{d}) + x_n - a \\
&= (n+1)a - na + x_n - a \\
&= a + x_n - a \\
&= x_n \\
&> 0.
\end{aligned}$$

Because of how x_{n+1} was defined inductively, it follows that $x_{n+1} - x_n = \frac{(n+1)a}{d}$ and $x_n - x_{n-1} = \frac{na}{d}$. Now x_n can be rewritten to

$$\begin{aligned}
x_n &= (x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1} \\
&= \frac{a}{d}(1 + 2 + ... + n) \\
&= \frac{a}{d}\frac{n}{2}(n+1)
\end{aligned}$$

where the well-known result $\sum_{n=1}^{n} k = \frac{n(n+1)}{2}$ is used. Then since $x_{n_0} \leq a < x_{n_0+1}$, it follows that $\frac{a}{d}\frac{n_0(n_0+1)}{2} \leq a < \frac{a}{d}\frac{(n_0+1)(n_0+2)}{2}$. This then implies that $2d < (n_0+1)(n_0+2)$. So since $(n_0+1)(n_0+2) - 2d > 0$, it follows that $n_0^2 + 3n_0 + 2 - 2d > 0$ and moreover $n_0^2 + 4n_0 + 4 - 2d > 0$. Solving this inequality using the abc-formula gives

$$\begin{aligned} n_0 &> \frac{-4 + \sqrt{4^2 - 4(4-2d)}}{2} \\ &= \frac{-4 + \sqrt{16 - 16 + 8d}}{2} \\ &= \frac{-4 + \sqrt{8d}}{2} \\ &= \frac{-4 + 2\sqrt{2d}}{2} \\ &= \sqrt{2d} - 2. \end{aligned}$$

Since $d > 8$, it follows that $n_0 > \sqrt{2d} - 2 > 2$. Hence

$$x_{n_0+1} = x_{n_0} + (n_0+1)\frac{a}{d} \leq a + \frac{2a}{n_0} < a + \frac{2a}{2} = 2a.$$

Since $a < x_{n_0+1} < 2a$, it follows that $x_{n_0+1} < \frac{1}{2}$. Now define $x_{n+1} = x_n + (n_0+1)\frac{a}{d}$ for $n = n_0+1, ..., n_1$ where n_1 is such that $x_{n_1} \leq \frac{1}{2}$ and $x_{n_1+1} > \frac{1}{2}$. If $n_0 = n_1$, then no new x_n's are constructed. Hence, suppose that $n_1 \geq n_0 + 1$. Then for $n = n_0+1, ..., n_1$

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - a > 0.$$

Next let $x_{n+1} = x_n + (n_1 + n_0 + 1 - n)\frac{a}{d}$ for $n = n_1+1, ..., n_0+n_1$. Then for $n = n_1+1, ..., n_0+n_1$, it follows that

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - 2a > 0.$$

Since

$$x_{n_0+n_1+1} = x_{n_1+1} + (\sum_{n=1}^{n_0} n)\frac{a}{d} = x_{n_1+1} + x_{n_0}$$

and

$$x_{n_1+1} = x_{n_1} + (n_0+1)\frac{a}{d} \leq \frac{1}{2} + x_{n_0+1} - x_{n_0} < 1 - x_{n_0}$$

it can be concluded that

$$x_{n_0+n_1+1} = x_{n_1+1} + x_{n_0} < 1 - x_{n_0} + x_{n_0} = 1.$$

Define $x_{n+1} = x_n + \frac{a}{d}$ for $n = n_0+n_1+1, ..., n_2$ where n_2 is such that $1 - \frac{a}{d} \leq x_{n_2+1} < 1$. Then for $n = n_0+n_1+1, ..., n_2$, it follows that

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) = x_n - a > 0.$$

Finally let $x_n = 1$ for all $n = n_2+2, ..., N$. Then for $n = n_2+1, ..., N$, it follows that

$$d(x_{n-1} - 2x_n + x_{n+1}) + h_0(x_n) > x_n - 2a > 0.$$

Therefore, there exists an $x \in C_0 \cap O_0$ and thus it can be concluded that $C_0 \cap O_0$ is nonempty. □

Thus by lemmata 4.5.3, 4.5.4, 4.5.5 and 4.5.6 it follows that the Poincaré map satisfies the following properties

- $\overline{C_0 \cap O_0}$ is a closed, bounded, and convex subset of $\mathbb{R}^{N+1}$,
- $T_0(\overline{C_0 \cap O_0}) \subset C_0 \cap O_0$,
- T_0 is continuous,
- $C_0 \cap O_0$ is nonempty.

Then by Brouwer's fixed point theorem 3.2.5, it follows that the map T_0 has a fixed point. Since the initial values x are invariant under the flow of $u(t)$ by Lemma 4.3.1 and the solution $u(t)$ is independent of t by Lemma 3.1.5, it follows that the fixed point x corresponds to the traveling wave $\{u_n(t)\}_{-\infty}^{\infty}$ of the simplified equation 4.2 where

$$u_n(t) = \begin{cases} 0 & \text{for } n \leq -1 \\ u_n(x;t) & \text{for } 0 \leq n \leq N \\ 1 & \text{for } n \geq N+1 \end{cases}$$

and $t \in [0, \tau]$, where τ is defined implicitly by $u_0(x;\tau) = x_1$.

4.6 Solution for general h

So far, the construction of $T_0 : \overline{X} \to \mathbb{R}^{N+1}$ depends on h_0. It was shown that, for certain values of d, a traveling wave solution of the simplified discrete Nagumo equation (4.2) with $h_0(x) = x - \frac{1}{4}$ was found. The goal is now to deform continuously h_0 into h_1, where $h_1 \in B_{app}$ such that the fixed points of T_{h_0} are continued into the fixed points of T_{h_1}. Let $h_1 \in B_{app}$. The unique zero of h_0 is defined as a_0 and similarly, a_1 as the unique zero of h_1. Define the following

$$h(\lambda, x) = \begin{cases} (1 - \frac{\lambda}{a_0})h_0(x) + \frac{\lambda}{a_0}g(a_0, x) & \text{for } 0 \leq \lambda \leq a_0 \\ g(\lambda, x) & \text{for } a_0 \leq \lambda \leq a_1 \\ (\frac{1-\lambda}{1-a_1})g(a_1, x) + \frac{\lambda - a_1}{1-a_1}h_1(x) & \text{for } a_1 \leq \lambda \leq 1 \end{cases}$$

where $g(\lambda, x) : [a_0, a_1] \times [0, 1] \to \mathbb{R}$ is defined by

$$g(\lambda, x) = \max\{-\tfrac{(1-\lambda)^2}{4\lambda}, x - \lambda\}.$$

Therefore

$$\begin{aligned} \int_0^1 g(\lambda, x) &\geq \int_0^\lambda -\frac{(1-\lambda)^2}{4\lambda} + \int_\lambda^1 x - \lambda \\ &= \frac{-(1-\lambda)^2}{4} + \frac{1}{2} - \lambda - \frac{1}{2}\lambda^2 + \lambda^2 \\ &= \frac{-(1-\lambda)^2}{4} + \frac{(\lambda-1)^2}{2} \\ &= \frac{(\lambda-1)^2}{4} \\ &> 0 \end{aligned}$$

for $a_0 \le \lambda \le a_1$. Then since $h_\lambda = h(\lambda, \cdot) \in B_{app}$ it follows that $h_\lambda = h(\lambda, \cdot)$ is a homotopy that deforms $h_0(x)$ continuously into $h_1(x)$ in the set B_{app}. Define the following

$$\begin{aligned} C_\lambda &= C(h_\lambda, d, N) \\ O_\lambda &= O(h_\lambda, d, N) \\ \mathfrak{C} &= \{(\lambda, x) : \lambda \in [0,1], x \in C_\lambda\} \\ \mathfrak{O} &= \{(\lambda, x) : \lambda \in [0,1], x \in O_\lambda\}. \end{aligned}$$

By Lemma 4.5.2 there exists a constant $d_3 > d_2$ such that $\sup_n \dot{u}_n(t) \ge \frac{1}{2}\int_0^1 h_\lambda(s)ds$ for all $(\lambda, x) \in \overline{\mathfrak{C} \cap \mathfrak{O}}$. For the rest of this section let $d > d_3$. The definition of δ depends continuously on h. Therefore a δ can be chosen such that the conclusion of Lemma 4.3.2 holds for all h_λ. For the rest of this section let N be some integer greater than $\frac{1}{\delta}$. Let $t^* : \overline{\mathfrak{C} \cap \mathfrak{O}}$ be defined by

$$t^*(\lambda, x) = \sup\{t : u_0(x, h_\lambda; t) < \tfrac{-h_\lambda(0)}{d}\}.$$

Then for $t \in (0, t^*)$, by Lemma 4.5.2 it follows that

$$N \ge \sum_{n=0}^{N} u_n(t) - \sum_{n=0}^{N} u_n(0) = \int_0^t \sum_{n=0}^{N} \dot{u}_n(v-)dv \ge \int_0^t \sup_n \dot{u}_n(v-)dv \ge \int_0^t \frac{1}{2}dv \int_0^1 h_\lambda(s)ds = t\frac{1}{2}\int_0^1 h_\lambda(s)ds.$$

Therefore

$$t^*(x) \le \frac{2N}{\int_0^1 h_\lambda(s)ds} \le \frac{2N}{(\min_{0\le\lambda\le1}\int_0^1 h_\lambda(s)ds)} = M^*.$$

In section 4.5 it was proven that $t^*(x)$ is continuous. Let $T(\lambda, x)$ be defined by

$$(T(\lambda, x))_n = \begin{cases} 0 & \text{for } n = 0 \\ u_{n-1}(x, h_\lambda; t^*(\lambda, x)) & \text{for } n = 1, ..., N. \end{cases}$$

Then by the definition of $(T(\lambda, x))_n$ and $t^*(\lambda, x)$, it follows that T is continuous. Define $\phi : [0,1] \times \mathbb{R}^{N+1} \to [0,1] \times \mathbb{R}^{N+1}$ with $\phi(\lambda, x) = (\lambda, y)$, where $y = \{y_n\}_{n=0}^N$ is given by

$$y_n = \begin{cases} \frac{h_\lambda(0)}{h_0(0)} x_n & \text{for } n = 1 \\ x_n & \text{for } n \ne 1 \end{cases} \tag{4.12}$$

Then $\phi : [0,1] \times \mathbb{R}^{N+1} \to [0,1] \times \mathbb{R}^{N+1}$ is a homeomorphism. Let $U = \phi^{-1}(\mathfrak{C} \cap \mathfrak{O})$ and $A = C_0$. Since C_0 is closed and convex, there exists a function $r : \mathbb{R}^{N+1} \to A$ such that $r(a) = a$ for all $a \in A$. Therefore, A is a retract of $\mathbb{R}^{N+1}$. The restriction of $F : X \to \mathbb{R}^{N+1}$ to the slice X_λ is denoted by $F_\lambda : X_\lambda \to \mathbb{R}^{N+1}$. Define

$$F : \overline{U} \to A \text{ by } F(\lambda, x) = \phi_\lambda^{-1}(T). \tag{4.13}$$

By lemmata 4.3.2 and 4.3.3 for $(\lambda, x) \in \mathfrak{C} \cap \mathfrak{O}$ it follows that $T(\lambda, x) \in C_\lambda \cap O_\lambda$. Then by the same arguments as in the proof of the second property of Lemma 4.3.1, for $(\lambda, x) \in \overline{\mathfrak{C} \cap \mathfrak{O}}$ it follows that $T(\lambda, x) \in C_\lambda \cap O_\lambda$. Therefore, $F(\overline{U}) \subset A$. Since $\mathfrak{O}$ is an open set, $U = ([0,1] \times A) \cap \phi^{-1}(\mathfrak{O})$ is an open subset of $[0,1] \times A$. Now the general homotopy invariance theorem 3.3.4 can be used to conclude that $i(F_\lambda, U_\lambda, A)$ is well defined and independent of $\lambda \in [0,1]$. Since

$$\phi_0(x) = \phi(0, x) = (0, y) = (0, x) \tag{4.14}$$

it follows that $F_0 = T_0$ and $U_0 = C_0 \cap O_0$. Since C_0 is convex, it follows that U_0 is also convex. Therefore, the map $F_0 : \overline{U} \to U_0$ is constant and by the normalization property of Theorem 3.3.2 it follows that

$$i(F_0, U_0, A) = 1. \tag{4.15}$$

Since $i(F_\lambda, U_\lambda, A)$ is independent of $\lambda \in [0, 1]$, it can be concluded that

$$i(F_1, U_1, A) = 1. \tag{4.16}$$

Then by the solution property of the index of corollary 3.3.3, there exists an $x \in U_1$, such that $F_1 x = x$. Therefore $\phi_1(x)$ is a fixed point of T_1. This fixed point of T_1 corresponds to a traveling wave solution of (4.3).

4.7 Convergence of approximate solutions

In the simplified problem, f was substituted by $h(u_n)$. Let $\{h_k\}$ be a sequence in B_{app} that converges to f in the norm defined by

$$\|h\| = \sup |h| + \sup_{s \neq t} |\frac{h(s) - h(t)}{s - t}|$$

where $s, t \in [0, 1]$. By the previous chapters, there exists a number $d_3(h_k)$ such that for every $h_k \in B_{app}$

$$\begin{aligned} \dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + h_k(u_n), \\ u_n &= P(v_n) \end{aligned}$$

for $n \in \mathbb{Z}$ has a traveling wave solution for $d > d_3(h_k)$. According to Lemma 4.5.2, d_3 depends only on $\sup |h_k|$, $\sup_{s \neq t} |\frac{h_k(s) - h_k(t)}{s - t}|$ and $\int_0^1 h_k(s) ds$. Therefore there exists a number $d^* < \infty$ such that equation (4.5) has a traveling wave solution for all $d > d^*$. Hence for the rest of this section, suppose d is any number greater than d^*. Then for every $k \in \mathbb{N}$ there exists a traveling wave solution v^k to (4.5). Let $x^k = \{x_n^k\}_{n=-\infty}^{\infty} \in \ell^\infty$ be defined by $x_n^k = v_n^k(0)$. Then there exists an integer n_k such that

$$x_{n_k}^k \leq \frac{1}{2} < x_{n_k+1}^k.$$

Then x^k is shifted to the sequence y^k such that $y_0^k \leq \frac{1}{2} < y_1^k$. So let $y^k = \{y_n^k\}_{-\infty}^{\infty}$ be defined by $y_n^k = x_{n+n_k}^k$ for $n \in \mathbb{Z}$. Recall from Lemma 4.4.2 that for all $\epsilon \in [0, \epsilon_0]$ there exists a bound $S(\epsilon, h)$ independent of N for the number of y_n's in y such that $\epsilon < y_n < 1 - \epsilon$. This bound was defined as

$$S(\epsilon, h) = p(\epsilon, h) + (1 + \frac{a(h)}{2m(h)}) + 1 + (\frac{2}{\sigma_3(h)} + 1).$$

Since $\{h_k\}$ converges it follows that

$$\begin{aligned} \limsup_{k \to \infty} S(\epsilon, h^k) &= \limsup_{k \to \infty} (p(\epsilon, h^k) + (1 + \frac{a(h^k)}{2m(h^k)}) + 1 + (\frac{2}{\sigma_3(h^k)} + 1)) \\ &< \infty. \end{aligned}$$

Therefore, the number of fixed points of y_n is finite. Since the sequence $\{y^k\}$ is bounded, the sequence has a convergent subsequence in ℓ^∞. Therefore,

$$\lim_{k \to \infty} y^k = y \text{ for some } y \in \ell^\infty \text{ with } \lim_{n \to -\infty} y_n = 0 \text{ and } \lim_{n \to \infty} y_n = 1.$$

Define $\tau_k = t^*(x^k)$, where $t^* = t^*(x) = \sup\{t : u_0(x;t) < \frac{-h(0)}{d}\}$. Since τ_k is bounded, it follows that

$$\lim_{k\to\infty} \tau_k = \tau \text{ for some } 0 < \tau < \infty.$$

By Lemma 4.4.1, it follows that $0 < \tau_0(h_k) \leq \tau$. Furthermore since

$$\begin{aligned}
1 &= \sum_n (u_{n+1}(0) - u_n(0)) \\
&= \sum_n (u_n(\tau) - u_n(0)) \\
&= \sum_n \int_0^\tau \dot{u}_n(s)ds \\
&= \int_0^\tau \sum_n \dot{u}_n(s)ds \\
&\geq \tau \inf_{0\leq t\leq \tau} (\sup_n \dot{u}_n(t)),
\end{aligned}$$

then by Lemma 4.5.2 it can be concluded that

$$\begin{aligned}
\tau_k &\leq \frac{1}{\inf_{0\leq t\leq\tau}(\sup_n \dot{u}_n(t))} \\
&\leq \frac{1}{\sup_n \dot{u}_n(t)} \\
&\leq \frac{1}{\frac{1}{2}\int_0^1 h_k(s)} \\
&= \frac{2}{\int_0^1 h_k(s)} \\
&< \infty.
\end{aligned}$$

Now consider the following initial value problem

$$\begin{aligned}
\dot{u}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n) \\
v_n(0) &= y_n
\end{aligned}$$

for $n \in \mathbb{Z}$. It will now be shown that he solution $\{u_n\}$ of this initial value problem is a traveling wave with velocity $c = \frac{1}{\tau}$. $\{u_n\}$ satisfies $0 \leq u_n(t) \leq 1$ for all $n \in \mathbb{Z}$ and all $t \geq 0$, since $\underline{u}_n = 0$ is a lower solution and $\overline{u}_n = 1$ is an upper solution. Therefore $\{v_n\}$ where $v_n = u_n$ is the unique solution of

$$\begin{aligned}
\dot{v}_n &= d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n) \\
u_n &= P(v_n) \\
v_n(0) &= y_n
\end{aligned}$$

for $n \in \mathbb{Z}$. Since the solution of this initial value problem depends continuously on its initial condition y and on the function f, it can be concluded that $u_n(t)$ is nondecreasing and $u_n(\tau) = u_{n+1}(0)$. For all $j \in \mathbb{Z}$ and all $s \in \mathbb{R}$, it holds that $\dot{u}_j(s) \geq 0$. Suppose that $\dot{u}_n(t) = 0$ for some $n \in \mathbb{Z}$ and some $t \in \mathbb{R}$. Then $\ddot{u}_n(t)$ exists and

$$\ddot{u}_n(t) = d(\dot{u}_{n-1}(t) + \dot{u}_{n+1}(t)).$$

Since $\dot{u}_{n-1}(t) \geq 0$ and $\dot{u}_{n+1}(t) \geq 0$, it follows that $\ddot{u}_n(t) \geq 0$. Now $\ddot{u}_n(t) > 0$ would imply that $\dot{u}_n(t-\epsilon) < 0$ for sufficiently small $\epsilon > 0$, which leads to a contradiction. Therefore, $\ddot{u}_n(t) = 0$ and this implies that $\dot{u}_{n-1}(t) = \dot{u}_{n+1}(t) = 0$. Since this is true for every $n \in \mathbb{Z}$, it follows that $\dot{u}_n(t) = 0$ for all $n \in \mathbb{Z}$. But this implies that the wave $\{u_n\}$ has zero speed in contradiction to $c = \frac{1}{\tau} > 0$. Therefore, it can be concluded that $\dot{u}_n(t) > 0$.
In this final step it has been shown that there exists some $d^* > 0$ such that for $d > d^*$ the discrete Nagumo equation 1.1 admits a continuous solution $u_n(t)$ with $0 \leq u_n(t) \leq 1$ for all $n \in \mathbb{Z}$ and all $t \geq 0$ and $\dot{u}_n(t) > 0$. This therefore completes the proof of Theorem 4.1.1.

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