DELFT UNIVERSITY OF TECHNOLOGY

MASTER THESIS

The Internal Language of Comprehension Categories

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Abstract

Denotational semantics of type theories provide a framework for understanding and reasoning about type theories and the behaviour of programs and proofs. In particular, it is important to study what can and can not be proved within Martin-Löf Type Theory (MLTT) as it is the basis of proof assistants like Agda, Lean and Coq. Many models, including a certain class of comprehension categories, *full and split comprehension categories*, have been studied for the semantics of dependent type theories. The motivation for this work comes from the fact that not all comprehension categories are full and split, and one expects that type theories more general than MLTT can be interpreted in a comprehension category which is not full and split.

In this thesis, we first study how MLTT is interpreted in full split comprehension categories through concrete examples. Next, we investigate type theories that can be interpreted in comprehension categories which are not necessarily full and split. For this, we propose a candidate type theory for the internal language of comprehension categories by extracting a type theory from the semantics given by a general comprehension category which is not full and split. We also give an interpretation of this type theory in every comprehension category.

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Chapter 1

Introduction

Background and Motivation The significance of denotational semantics in the study of programming languages has been well-recognised since Dana Scott's foundational work [Sco70]. Denotational semantics of type theories provide a framework for understanding and reasoning about type theories and the behaviour of programs and proofs. This can ensure that the rules of the type theory are interpreted in a consistent way that reflects the intended behaviour of the type theory. This consistency and meaningfulness are critical for both theoretical validation and practical application.

For example, one can show that a result can not be proved in the type theory by showing that the result does not hold for a model. This is relevant due to the inherent difficulty in proving non-derivability in a type theory. As a particular case for this, one can show that a type theory is consistent by showing that the type corresponding to `False` is interpreted as the empty set. This ensures that there are no terms inhabiting the type `False`, thereby demonstrating the consistency of the type theory. This is particularly important when using type theory as a logical framework in a proof assistant.

An example of showing the consistency of a type theory using denotational semantics can be found in the work of Kapulkin and Lumsdaine [KL21], where the consistency of univalent type theory is discussed. The authors provide a model for dependent type theories in simplicial sets and show that the translation of the univalence axiom in this model is a non-empty Kan complex. This implies that Martin-Löf type theory plus the univalence axiom is at least as consistent as the classical foundations used. Another example can be found in the lecture notes of Gratzer [Gra23], where the author gives an interpretation of functional extensionality in categories with families [Dyb96] equipped with intentional identity type and shows that this interpretation is inhabited in the model. This implies that functional extensionality is a consistent addition to the rules of the type theory.

Martin-Löf Type Theory (MLTT) [Mar84] is used as a logical framework and is the basis of proof assistants like Agda[1], Lean[2] and Coq[3]; hence, it is worthwhile to study what can and can not be proved within this type theory. Various categorical structures have been studied as a framework for denotational semantics of dependent type theories, which include the work of Seely [See84] on locally cartesian closed categories, Cartmell [Car86] on categories with attributes, Taylor [Tay87] on categories with display maps and Jacobs [Jac93] on comprehension categories. In this report, we focus on the use of comprehension categories as denotational semantics for dependent-type theories. An overview of how some of the other structures compare to comprehension categories is provided in Chapter 2.

Comprehension categories, introduced by Jacobs [Jac93], are inspired by hyperdoctrines in categorical logic, introduced by Lawvere [Law70], and use Grothendieck fibrations to give a general framework for type dependency semantics. As discussed by Lumsdaine and Warren [LW15], a certain class of comprehension categories, full split comprehension categories, serve as models for dependent type

[1] https://wiki.portal.chalmers.se/agda/pmwiki.php
[2] https://lean-lang.org
[3] https://coq.inria.fr

theories that adhere to the structural rules of MLTT. Comprehension categories and the interpretation of MLTT in full split comprehension categories are discussed in Chapters 4 and 5 respectively.

Although full split comprehension categories are models for MLTT, all instances of comprehension categories are not full and split. One expects that dependent type theories that are more general than MLTT can be interpreted in comprehension categories that are not full and split. For example, type theories with a weaker notion of substitution where substitution is “functorial” only up to isomorphism, as opposed to MLTT, which has strictly functorial substitution, are interpreted in non-split comprehension categories (see Section 5.1.5). In much of the literature, however, comprehension categories are taken to be full. This arises due to how terms in MLTT are interpreted in a comprehension category as certain morphisms in the category of contexts. The main objective of this thesis is to study the type theories that are interpreted in general comprehension categories without the requirements of fullness and splitness.

Research Problem and Contributions Usually, one proposes semantics for a type theory by considering the requirements that the category should have such that it is possible to interpret the components of the type theory in it. Examples of this can be found in the work of Seely [See84], Cartmell [Car86], Taylor [Tay87], and Jacobs [Jac93]. Conversely, one can study the general type theories that can be interpreted in a certain semantic framework by starting from the semantics and deriving the syntax of a type theory such that all the structure of the semantic framework is reflected in the type theory. If soundness and completeness of the rules of the type theory with respect to the class of models are proved, then this type theory is called the internal language of the semantic framework. Given a class of models and their internal language, one can show a certain property of the type theory by showing the counterpart of the property in semantics and vice versa. This approach is taken by Ahrens et al. [ANW23], where they derive the syntax of a directed type theory called Bicategorical Type Theory (BTT) from comprehension bicategories, the bicategorical generalisation of comprehension categories.

The first contribution of this thesis is to conduct an in-depth discussion of the interpretation of MLTT in full split comprehension categories by discussing extensive examples of interpretations in specific full split comprehension categories and providing more detailed proofs and explanations for the general case. This is done in Chapter 5, through two running examples of interpretations of certain type theories in specific comprehension categories.

The second contribution of this thesis is to investigate type theories that are interpreted in comprehension categories which are not necessarily full. For this, in Section 6.1, we introduce a type theory with rules extracted from the structure of comprehension categories such that all categorical structures are reflected in the type theory, similar to the approach taken by Ahrens et al. [ANW23]. By comparing the rules of this type theory to MLTT, we aim to obtain a better understanding of comprehension categories and the type theories that are interpreted in them. We also show the soundness of the rules of the derived type theory in Section 6.2 by providing an interpretation of this type theory in comprehension categories.

The third contribution of this thesis is to further investigate the differences between BTT and other proposed directed type theories by focusing on the one-dimensional, non-directed case. For example, one of the differences between BTT and other type theories is the way terms are interpreted as morphisms in the category of types instead of certain morphisms in the category of contexts. As we take a similar approach to Ahrens et al. [ANW23] when extracting the rules of our type theory, this difference also appears between our derived type theory and MLTT. To study this difference, in Sections 6.3 and 6.4, we propose how a unit type can be added to the syntax of the type theory and investigate the requirements for having a semantic one-to-one correspondence between the terms of this type theory and those of MLTT. We expect this result to be in line with how the mismatch between terms of BTT and other directed-type theories with terms interpreted similarly to MLTT could be reconciled.

Outline

- In Chapter 2, we give an overview of some of the categorical semantics for dependent type theory, focusing on how they are related to comprehension categories. We then discuss the work of Lindgren [Lin21] on the semantics of type dependency in non-full comprehension categories. We conclude the related work by discussing the type theory derived from comprehension bicategories by Ahrens et al. [ANW23].
- Chapter 3 contains the category theory and type theory preliminaries.
- In Chapter 4, we discuss the definition of comprehension categories and some relevant results from the literature.
- In Chapter 5, we discuss the interpretation of MLTT in full split comprehension categories. The contribution of this report in this chapter is to discuss extensive examples of interpretations in specific full split comprehension categories as well as providing more detailed proofs and explanations.
- In Chapter 6, we introduce our type theory extracted from the structure of a comprehension category and compare the syntax of this type theory to MLTT. We then show the soundness of the rules by giving an interpretation in comprehension categories. We also add a unit type to the type theory and discuss the requirements for having a semantic one-to-one correspondence between the terms of this type theory and those of MLTT.
- Chapter 7 contains a discussion of the results of Chapter 6, a comparison between our type theory and MLTT and a comparison between the results of this report and those of Ahrens et al. [ANW23]. We also mention some limitations and possible directions for future work.

Chapter 2

Related Work

Comprehension Categories for Semantics of Type Dependency Cartesian closed categories are known to be models for simply typed lambda calculus [LS88]. Locally cartesian closed categories have been shown to be models of dependent type theories [See84; CD11; CGH14]. Several other categorical semantics have been studied for interpreting dependent type theories with Π- and Σ-types, e.g. Martin-Löf type theory. These include categories with display maps [Tay87], contextual categories [Car86; Str91], categories with attributes [Car86; Mog91], categories with families [Dyb96; Hof97] and comprehension categories [Jac93].

Comprehension categories, as introduced by Jacobs [Jac93], are based on the work on hyperdoctrines in categorical logic, particularly the contributions of Lawvere [Law70] and Seely [See83], as well as on Ehrhard's [Ehr88] work on D-categories. Among the models for MLTT, comprehension categories are more general and less close to syntax and allow for certain type formers to be interpreted more elegantly as (fibred) adjunctions [Hof97].

A certain class of comprehension categories, full split comprehension categories, are known to be equivalent to categories with attributes [Car86; Mog91] and categories with families [Dyb96; Hof97]. The equivalence of full split comprehension categories and categories with attributes was shown by Blanco [Bla91] and some back-and-forth constructions of these equivalences are discussed in Jacob's work [Jac93; Jac99]. Contextual categories, however, are closer to syntax and are not directly equivalent to full split comprehension categories.

Non-full Comprehension Categories Lindgren [Lin21] discusses what type-formers for unit and Π-types correspond to in comprehension categories that are not full. In Chapter 6, we use the definition of a (non-full) comprehension category with unit from this work to derive the rules concerning the unit type for the type theory presented in this thesis which is extracted from a comprehension category.

Deriving a Type Theory from Semantics Ahrens et al. [ANW23] derive the syntax of their directed type theory (BTT) from comprehension bicategories such that all categorical structures are reflected in the type theory. This is similar to what is done in Chapter 6, in that we also extract the rules of the type theory from semantics such that all the structure is reflected in the type theory.

As comprehension bicategories are a two-dimensional generalisation of comprehension categories, we expect our type theory to be the 1-dimensional restriction of BTT. The bicategorical nature of comprehension bicategories gives rise to BTT being a directed type theory. The type theory presented in Chapter 6 is extracted from comprehension categories which do not feature bicategorical structure; hence, this type theory is different from BTT in that it is not a directed type theory.

The terms in BTT are interpreted as 1-cells in the bicategory of types. This is different from the usual interpretation of terms in other directed type theories as certain context morphisms, particularly sections of projections from extended contexts to original contexts. Similarly, the terms of the type theory introduced in Chapter 6 are interpreted as morphisms in the category of types, similar to terms of BTT. This is different from terms in MLTT which are interpreted as sections of projections from extended contexts to original contexts. Because of this, one expects the terms of our type theory to be different from MLTT in ways similar to how terms in BTT differ from the other proposed directed type theories in which terms are interpreted as certain context morphisms.

Additionally, both BTT and our type theory feature explicit substitution in the syntax. Since we derive the rules of our type theory from comprehension categories that are not full and split, the substitution is not strictly functorial and terms are interpreted differently than in MLTT. This is also the case for BTT, as BTT is derived from comprehension bicategories that do not have a requirement on fullness and splitness.

We derive our type theory from the syntax of a comprehension category where the base category has a terminal object. The terminal object of the base category corresponds to the empty context in the type theory. Hence, our type theory features rules regarding the well-formedness of the empty context, whereas BTT does not.

Chapter 3

Preliminaries

Category theory preliminaries are discussed in Section 3.1 and type theory preliminaries are discussed in Section 3.2.

3.1 Category Theory Preliminaries

In this section we review the category theory preliminaries needed for the rest of this report. The basics of category theory is presumed to be known by the reader and is therefore not covered. The reader is referred to [Mac71] for basics of category theory. Nevertheless, as the arrow category is repeatedly used throughout this thesis, we discuss the background knowledge regarding arrow categories in Section 3.1.1. The content presented in this section is based on [Mac71]. In section Section 3.1.2, we review the required background knowledge about Grothendiek Fibrations. Much of the content of this section is based on [Str18; AL19; JY21].

3.1.1 Arrow Category

Definition 3.1. Given a category $\mathcal{C}$, the *arrow category* $\mathcal{C}^{\rightarrow}$ consists of

1. An object a of $\mathcal{C}^{\rightarrow}$ is a morphism of $\mathcal{C}$,
2. Given $a : \mathcal{C}(a_0, a_1)$ and $b : \mathcal{C}(b_0, b_1)$, a morphism $f : \mathcal{C}^{\rightarrow}(a, b)$ is a pair $(f_0 : a_0 \to b_0, f_1 : a_1 \to b_1)$ such that $\begin{array}{ccc} a_0 & \xrightarrow{f_0} & b_0 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ a_1 & \xrightarrow[f_1]{} & b_1 \end{array}$ commutes,
3. for all $a_0, a_1 : \mathcal{C}$ and $a : \mathcal{C}(a_0, a_1)$, $1_a = (1_{a_0}, 1_{a_1})$,
$$\begin{array}{ccc} a_0 & \xrightarrow{1_{a_0}} & a_0 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ a_1 & \xrightarrow[1_{a_1}]{} & a_1 \end{array}$$
4. for $a : \mathcal{C}(a_0, a_1)$, $b : \mathcal{C}(b_0, b_1)$, $c : \mathcal{C}(c_0, c_1)$ composition is defined by
$$\begin{aligned} \mathcal{C}^{\rightarrow}(a, b) \times \mathcal{C}^{\rightarrow}(b, c) &\to \mathcal{C}^{\rightarrow}(a, c) \\ ((f_0, f_1), (g_0, g_1)) &\mapsto (g_0 \circ f_0, g_1 \circ f_1). \end{aligned}$$

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[r, "g_0"] \arrow[d, "b"] & c_0 \arrow[d, "c"] \\ a_1 \arrow[r, "f_1"'] & b_1 \arrow[r, "g_1"'] & c_1 \end{tikzcd}$$

Definition 3.2. The *codomain functor* $\mathsf{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ is given on objects by the codomain of the morphisms in $\mathcal{C}$, which means that given $\alpha : \mathcal{C}(a_0, a_1)$,

$$\mathsf{cod} : \alpha \mapsto a_1.$$

On morphisms, it acts as follows:

$$\mathsf{cod} : \left(\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\ a_1 \arrow[r, "f_1"'] & b_1 \end{tikzcd} \right) \mapsto (f_1 : a_1 \to b_1).$$

Definition 3.3. The *domain functor* $\mathsf{dom} : \mathcal{C}^{\to} \to \mathcal{C}$ is given on objects by the domain of the morphisms in $\mathcal{C}$, which means that given $\alpha : \mathcal{C}(a_0, a_1)$,

$$\mathsf{cod} : \alpha \mapsto a_0.$$

On morphisms, it acts as follows:

$$\mathsf{dom} : \left(\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\ a_1 \arrow[r, "f_1"'] & b_1 \end{tikzcd} \right) \mapsto (f_0 : a_0 \to b_0).$$

Remark 3.4. A functor into the arrow category $\chi : \mathcal{T} \to \mathcal{C}^{\to}$, can be thought of as two functors, $\mathsf{cod} \circ \chi : \mathcal{T} \to \mathcal{C}$ and $\mathsf{dom} \circ \chi : \mathcal{T} \to \mathcal{C}$, and a natural transformation $\eta : \mathsf{dom} \circ \chi \Rightarrow \mathsf{cod} \circ \chi$.

For each $A \in \mathcal{T}$, $\eta_A := \chi A$. For η to be a natural transformation we need the following commuting diagram for each $f \in \mathcal{T}(A, B)$ with $\chi A = \alpha \in \mathcal{C}(a_1, a_2)$ and $\chi B = \beta \in \mathcal{C}(b_1, b_2)$:

$$\begin{tikzcd} (\mathsf{dom} \circ \chi)A \arrow[r, "(\mathsf{dom}\circ\chi)f"] \arrow[d, "\chi A"'] & (\mathsf{dom} \circ \chi)B \arrow[d, "\chi B"] \\ (\mathsf{cod} \circ \chi)A \arrow[r, "(\mathsf{cod}\circ\chi)f"'] & (\mathsf{cod} \circ \chi)B. \end{tikzcd}$$

This is exactly the commuting diagram corresponding to χf in $\mathcal{C}^{\to}$:

$$\begin{tikzcd} a_1 \arrow[r, "\gamma_1"] \arrow[d, "\alpha"'] & b_1 \arrow[d, "\beta"] \\ a_2 \arrow[r, "\gamma_2"'] & b_2. \end{tikzcd}$$

3.1.2 Grothendieck Fibration

Definition 3.5 ([Str18, Definition 2.1]). Let $p : \mathcal{D} \to \mathcal{C}$ be a functor. A morphism $\varphi : Y \to X$ in $\mathcal{D}$ above $u := p(\varphi)$ is called *cartesian* if and only if for all $v : \Theta \to \Delta$ in $\mathcal{C}$ and $\theta : Z \to X$ with $p(\theta) = u \circ v$ there exists a unique morphism $\psi : Z \to Y$ with $p(\psi) = v$ and $\theta = \varphi \circ \psi$:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\psi"'] \arrow[drr, "\theta"] & & & \\
& Y \arrow[r, "\varphi"'] & X & \mathcal{D} \arrow[dd, "p"] \\
\Theta \arrow[dr, "v"'] \arrow[drr, "u \circ v"] & & & \\
& \Delta \arrow[r, "u"'] & \Gamma & \mathcal{C}.
\end{tikzcd}$$

A morphism $\alpha : Y \to X$ is called *vertical* if and only if $p(\alpha)$ is an identity morphism in $\mathcal{C}$. For $\Gamma \in \mathcal{C}$, we write $\mathcal{D}_\Gamma$ for the subcategory of $\mathcal{D}$ consisting of those morphisms α with $p(\alpha) = \mathsf{id}_\Gamma$. $\mathcal{D}_\Gamma$ is called the *fibre* of p over Γ. For $f : A \to B$ in $\mathcal{C}$, $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$, we denote the collection of morphisms above f with domain X and codomain Y as:

$$\mathcal{D}_f(X, Y) = \{u : X \to Y \mid pf = u\}.$$

Lemma 3.6. *Cartesian morphisms are closed under composition.*

Proof. Let $p : \mathcal{D} \to \mathcal{C}$ be a functor, $s : \Delta \to \Gamma$, $u : \Theta \to \Delta$ and $v : \Omega \to \Theta$ morphisms in $\mathcal{C}$, $\delta : Y \to X$ a cartesian morphism in $\mathcal{D}$ above s and $\varphi : Z \to Y$ a cartesian morphism above u:

$$\begin{tikzcd}
H \arrow[dr, "\psi"'] \arrow[drr, "\xi"] \arrow[drrr, "\theta"] & & & & \\
& Z \arrow[r, "\varphi"'] & Y \arrow[r, "\delta"'] & X & \mathcal{D} \arrow[d, "p"] \\
\Omega \arrow[r, "v"'] & \Theta \arrow[r, "u"'] & \Delta \arrow[r, "s"'] & \Gamma & \mathcal{C}.
\end{tikzcd}$$

To show that $\delta \circ \varphi$ is cartesian, we have to show that for all $v : \Omega \to \Theta$ and $\theta : H \to X$ with $p(\theta) = s \circ u \circ v$, there exists a unique $\psi : H \to Z$ with $p(\psi) = v$ and $\theta = (\delta \circ \varphi) \circ \psi$. From δ being cartesian, we have that there exists a unique $\xi : H \to Y$ with $p(\xi) = u \circ v$ and $\theta = \delta \circ \xi$. From v being cartesian we get that there exists a unique $\psi' : H \to Z$ with $p(\psi') = v$ and $\xi = \varphi \circ \psi'$. Hence, $\theta = \delta \circ \varphi \circ \psi'$ and $\psi' : H \to Z$ is the unique morphism that we needed. $\square$

Lemma 3.7. *Let $p : \mathcal{D} \to \mathcal{C}$ be a functor, $u : \Delta \to \Gamma$ a morphism in $\mathcal{C}$, and $\varphi : Y \to X$ and $\theta : Z \to X$ cartesian morphisms in $\mathcal{D}$ above u. We have $Z \cong Y$ in $\mathcal{D}$.*

$$\begin{tikzcd} Z \arrow[d, "i"', "\cong"] \arrow[dr, "\theta"] & & \\ Y \arrow[r, "\varphi"'] & X & \mathcal{D} \arrow[d, "p"] \\ \Delta \arrow[r, "u"'] & \Gamma & \mathcal{C} \end{tikzcd}$$

Proof. From φ being cartesian, we get a unique morphism $i : Z \to Y$ such that $\varphi \circ i = \theta$. From θ being cartesian, we get a unique morphism $i^{-1} : Y \to Z$ such that $\theta \circ i^{-1} = \varphi$. From these we get $\varphi \circ i \circ i^{-1} = \varphi$. Since φ is cartesian, there is a unique morphism of the form $Y \to Y$ such that $\varphi \circ 1_Y = \varphi$, the identity morphism 1_Y. Hence, $i \circ i^{-1} = 1_Y$. Similarly, one can show that $i^{-1} \circ i = 1_X$ and we get $i : Z \cong Y$. □

Lemma 3.8. *Let $p : \mathcal{D} \to \mathcal{C}$ be a functor, $u : \Delta \to \Gamma$ a morphism in $\mathcal{C}$, $\varphi : Y \to X$ and $\varphi' : Y' \to X$ in $\mathcal{D}$ above u, and an isomorphism $i : Y' \cong Y$ such that $\varphi' = \varphi \circ i$. If φ' is cartesian, then so is φ.*

$$\begin{tikzcd} Y' \arrow[d, "i"', "\cong"] \arrow[dr, "\varphi'"] & & \\ Y \arrow[r, "\varphi"'] & X & \mathcal{D} \arrow[d, "p"] \\ \Delta \arrow[r, "u"'] & \Gamma & \mathcal{C} \end{tikzcd}$$

Proof. We need to show for all $v : \Theta \to \Delta$ in $\mathcal{C}$ and $\theta : Z \to X$ with $p(\theta) = u \circ v$ there exists a unique morphism $\psi : Z \to Y$ with $p(\psi) = v$ and $\theta = \varphi \circ \psi$. Let $v : \Theta \to \Delta$ in $\mathcal{C}$ and $\theta : Z \to X$ with $p(\theta) = u \circ v$. From φ being cartesian, we have a unique morphism $\psi' : Z \to Y'$ with $p(\psi') = v$ and $\theta = \varphi' \circ \psi'$. We take ψ to be $i \circ \psi'$. Since $p(i) = 1_\Delta$, we have $p(\psi) = v$. We have $\theta = \varphi \circ \psi$ from $\theta = \varphi' \circ \psi'$ and $\varphi' = \varphi \circ i$.

Now to show uniqueness, let $\psi'' : Z \to Y$ with $p(\psi'') = v$ and $\theta = \varphi \circ \psi''$. We have to show $\psi = \psi''$. From $\theta = \varphi \circ \psi''$, we get $\theta = \varphi \circ i \circ i^{-1} \circ \psi'' = \varphi' \circ i^{-1} \circ \psi''$. Since ψ' is the unique morphism such that $p(\psi') = v$ and $\theta = \varphi' \circ \psi'$, we get $i^{-1} \circ \psi'' = \psi'$. Hence, $\psi'' = i \circ \psi'$ and $\psi'' = \psi$. □

Lemma 3.9. *Cartesian morphisms in $\mathcal{C}^{\to}$ are pullback squares in $\mathcal{C}$.*

Proof. Let $a_1, a_2, b_1, b_2 : \mathcal{C}, \alpha : \mathcal{C}(a_1, a_2)$ and $\beta : \mathcal{C}(b_1, b_2)$. We have to show that if $(\gamma_1, \gamma_2) : \mathcal{C}^{\to}(\alpha, \beta)$ is a cartesian morphism in $\mathcal{C}^{\to}$, then the following commuting diagram is a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} a_1 \arrow[r, "\gamma_1"] \arrow[d, "\alpha_1"'] & b_1 \arrow[d, "\beta"] \\ a_2 \arrow[r, "\gamma_2"'] & b_2. \end{tikzcd}$$

To show that the diagram is a pullback square, we have to show that for any commuting

$$\begin{tikzcd} a_1' \arrow[r, "\gamma_1'"] \arrow[d, "\alpha_1'"'] & b_1 \arrow[d, "\beta"] \\ a_2 \arrow[r, "\gamma_2"'] & b_2, \end{tikzcd}$$

there exists a unique $h : \mathcal{C}(a_1', a_1)$ such that $\alpha \circ h = \alpha'$ and $\gamma_1 \circ h = \gamma_1'$. From the second diagram commuting, we have $(\gamma_1', \gamma_2) : \mathcal{C}^{\to}(\alpha', \beta)$. Since (γ_1, γ_2) is cartesian, for all $k : \mathcal{C}(a_2, a_2)$ such that $\mathsf{cod}(\gamma_1, \gamma_2) \circ k = \mathsf{cod}(\gamma_1', \gamma_2)$, there exists a unique $(h', k) : \mathcal{C}^{\to}(\alpha', \alpha)$ such that $(\gamma_1, \gamma_2) \circ (h', k) = (\gamma_1', \gamma_2)$. Now, we show that this h' is the unique h that we need. From $\mathsf{cod}(\gamma_1, \gamma_2) \circ k = \mathsf{cod}(\gamma_1', \gamma_2)$, we get $\gamma_2 \circ k = \gamma_2$. Hence, $k = 1_{a_2}$. From $(\gamma_1, \gamma_2) \circ (h', 1_{a_2}) = (\gamma_1', \gamma_2)$ we have that h' is the unique morphism for which $\gamma_1 \circ h' = \gamma_1'$. From $(h', 1_{a_2})$ being a morphism in $\mathcal{C}^{\to}(\alpha', \alpha)$ we get $\alpha \circ h' = \alpha'$. Thus, cartesian morphisms in $\mathcal{C}^{\to}$ are pullback squares in $\mathcal{C}$. □

Definition 3.10 ([Str18, Definition 2.2]). A functor $p : \mathcal{D} \to \mathcal{C}$ is called a *(Grothendieck) fibration* if and only if for all $u : \Delta \to \Gamma$ in $\mathcal{C}$ and $X \in \mathcal{D}_\Gamma$ there exists a cartesian arrow $\varphi : Y \to X$ above u called a *cartesian lifting* of u to X.

Definition 3.11 ([AL19, Definition 5.3]). A *cleaving* for a fibration $p : \mathcal{D} \to \mathcal{C}$ is a function giving, for each $f : c' \to c$ in $\mathcal{C}$ and $d : \mathcal{D}_c$, a cartesian lift of f into d. A *cloven* fibration over $\mathcal{C}$ is a fibration equipped with a cleaving.

Definition 3.12 ([AL19, Definition 5.6]). A fibration $p : \mathcal{D} \to \mathcal{C}$ is *split* if and only if

1. the chosen lifts of identities are identities; and
2. the chosen lift of any composite is the composite of the individual lifts.

Example 3.13. For each category $\mathcal{C}$, the domain functor $\mathsf{dom} : \mathcal{C}^{\to} \to \mathcal{C}$, defined in Definition 3.3, is a split fibration. If $\mathcal{C}$ has all pullbacks, the codomain functor $\mathsf{cod} : \mathcal{C}^{\to} \to \mathcal{C}$, defined in Definition 3.2, is also a split fibration and a fibre over $\Gamma \in \mathcal{C}$ is the slice category $\mathcal{C}/\Gamma$.

We show these two statement in the following two lemmas.

Lemma 3.14. $\mathsf{dom} : \mathcal{C}^{\to} \to \mathcal{C}$ *is a split fibration.*

Proof. We need to show that for each $\Delta, \Gamma : \mathcal{C}$, $f : \mathcal{C}(\Delta, \Gamma)$ and $\beta : \Gamma/\mathcal{C}$, there exists a cartesian f_β in $C^{\to}$ into β. We take f_β to be the morphism corresponding to the following commuting diagram:

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "\beta\circ f"'] & \Gamma \arrow[d, "\beta"] \\ X \arrow[r, equal] & X. \end{tikzcd}$$

Now we need to show that f_β is cartesian, which is for each $\gamma : \Theta/\mathcal{C}$, $g : \gamma \to \beta$ and $h_1 : \Theta \to \Delta$, where $\gamma : \Theta \to Y$, with $\mathsf{dom} \circ g = g_1$ and $f \circ h_1 = g_1$, there exists a unique h such that $\mathsf{dom} \circ h = h_1$

and $f_\beta \circ h = g$:

$$\begin{tikzcd}
\gamma \arrow[dr, "h"'] \arrow[drr, "g"] & & & & \mathcal{C}^{\rightarrow} \arrow[ddd, "\mathsf{dom}"] \\
& \beta \circ f \arrow[r, "f_\beta"'] & \beta & & \\
\Theta \arrow[dr, "h_1"'] \arrow[drr, "g_1"] & & & & \\
& \Delta \arrow[r, "f"'] & \Gamma & & \mathcal{C}
\end{tikzcd}$$

We show that the unique h is the map corresponding to the following commuting diagram:

$$\begin{tikzcd}
\Theta \arrow[r, "h_1"] \arrow[d, "\gamma"'] & \Delta \arrow[d, "\beta \circ f"] \\
Y \arrow[r, "\mathsf{cod} \circ g"'] & X.
\end{tikzcd}$$

By definition $\mathsf{dom} \circ h = h_1$. To show that $f_\beta \circ h = g$, we need to show that the composition corresponding to

$$\begin{tikzcd}
\Theta \arrow[r, "h_1"] \arrow[d, "\gamma"'] & \Delta \arrow[r, "f"] \arrow[d, "\beta \circ f"] & \Gamma \arrow[d, "\beta"] \\
Y \arrow[r, "\mathsf{cod} \circ g"'] & X \arrow[r, equal] & X
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
\Theta \arrow[r, "g_1"] \arrow[d, "\gamma"'] & \Gamma \arrow[d, "\beta"] \\
Y \arrow[r, "\mathsf{cod} \circ g"'] & X,
\end{tikzcd}$$

which follows from $g_1 = f \circ h_1$ and the composition of morphisms in the arrow category. Now, to prove the uniqueness, we need to show that if the diagram

$$\begin{tikzcd}
\Theta \arrow[r, "h_1"] \arrow[d, "\gamma"'] & \Delta \arrow[d, "\beta \circ f"] \\
Y \arrow[r, "k"'] & X
\end{tikzcd}$$

commutes, and the composition corresponding to

$$\begin{tikzcd}
\Theta \arrow[r, "h_1"] \arrow[d, "\gamma"'] & \Delta \arrow[r, "f"] \arrow[d, "\beta \circ f"] & \Gamma \arrow[d, "\beta"] \\
Y \arrow[r, "k"'] & X \arrow[r, equal] & X
\end{tikzcd}$$

is equal to

$$\begin{tikzcd} \Theta \arrow[r, "g_1"] \arrow[d, "\gamma"'] & \Gamma \arrow[d, "\beta"] \\ Y \arrow[r, "\mathsf{cod} \circ g"'] & X, \end{tikzcd}$$

then $k = \mathsf{cod} \circ g$. From $f_\beta \circ h = g$, we have $f \circ h_1 = g_1$ and $k = \mathsf{cod} \circ g$; hence, the morphism is unique.

By the definition of identity morphism and composition in $\mathcal{C}^\rightarrow$, dom is split. $\square$

Lemma 3.15. *If $\mathcal{C}$ has all pullbacks,* $\mathsf{cod} : \mathcal{C}^\rightarrow \rightarrow \mathcal{C}$ *is a split fibration and a fibre over $\Gamma \in \mathcal{C}$ is the slice category $\mathcal{C}/\Gamma$.*

Proof. From Lemma 3.9 we know that cartesian morphisms in $\mathcal{C}^\rightarrow$ are pullback squares in $\mathcal{C}$. Since $\mathcal{C}$ has all pullbacks, for all $u : \Delta \rightarrow \Gamma$ in $\mathcal{C}$ and $\alpha \in \mathcal{C}^\rightarrow_\Gamma$ we have the following pullback square in $\mathcal{C}$, which corresponds to a cartesian morphism in $\mathcal{C}^\rightarrow$ into α:

$$\begin{tikzcd} \Delta \times_\Gamma \mathsf{dom}(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{dom}(\alpha) \arrow[d, "\alpha"] \\ \Delta \arrow[r, "u"'] & \Gamma. \end{tikzcd}$$

This means that $\mathsf{cod} : \mathcal{C}^\rightarrow \rightarrow \mathcal{C}$ is a (cloven) fibration. By the definition of cod, a fibre over $\Gamma \in \mathcal{C}$ is the slice category $\mathcal{C}/\Gamma$.

Now, we show that cod is a split fibration. For each $\Gamma \in \mathcal{C}$ and $\alpha : \mathcal{C}/\Gamma$, we have:

$$\begin{tikzcd} \mathsf{id}_\Gamma^* \alpha \arrow[r, "\mathsf{id}_{\Gamma_\alpha}"] & \alpha & \mathcal{C}^\rightarrow \arrow[d, "\mathsf{cod}"] \\ \Gamma \arrow[r, "\mathsf{id}_\Gamma"'] & \Gamma & \mathcal{C}, \end{tikzcd}$$

where the cartesian lift $\mathsf{id}_{\Gamma_\alpha}$ corresponds to the following pullback square in $\mathcal{C}$:

$$\begin{tikzcd} \mathsf{dom}(\alpha) \arrow[r, "\mathsf{id}_{\mathsf{dom}(\alpha)}"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{dom}(\alpha) \arrow[d, "\alpha"] \\ \Gamma \arrow[r, "\mathsf{id}_\Gamma"'] & \Gamma. \end{tikzcd}$$

This means that $\mathsf{id}_\Gamma^* \alpha = \alpha$. For each $\Gamma, \Gamma', \Gamma'' \in \mathcal{C}$, $s : \Gamma \rightarrow \Gamma'$, $s' : \Gamma' \rightarrow \Gamma''$ and $\alpha : \mathcal{C}/\Gamma''$, we have:

$$\begin{tikzcd} s^*(s'^*\alpha) \arrow[r, "s_{s'^*\alpha}"] & s'^*\alpha \arrow[r, "s'_\alpha"] & \alpha & \mathcal{C}^\rightarrow \arrow[d, "\mathsf{cod}"] \\ \Gamma \arrow[r, "s"'] & \Gamma' \arrow[r, "s'"'] & \Gamma'' & \mathcal{C}, \end{tikzcd}$$

where the cartesian morphisms s'_α and $s_{s'^*\alpha}$ correspond to the following pullback squares in $\mathcal{C}$:

$$\begin{tikzcd}
\mathsf{dom}(s^*(s'^*\alpha)) \arrow[r] \arrow[d, "s^*(s'^*\alpha)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{dom}(s'^*\alpha) \arrow[r] \arrow[d, "s'^*\alpha"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{dom}(\alpha) \arrow[d, "\alpha"] \\
\Gamma \arrow[r, "s"'] & \Gamma' \arrow[r, "s'"'] & \Gamma''
\end{tikzcd}$$

The pullback of α along $s' \circ s$ is $s^*(s'^*\alpha)$, which means that $(s' \circ s)^*\alpha = s^*(s'^*\alpha)$. Hence, cod is a split fibration.

□

Definition 3.16 ([Str18, Section 3]). Let $p : \mathcal{D} \to \mathcal{C}$ be a cloven fibraton. Every morphism $s : \Gamma \to \Delta$ induces a *reindexing functor* $s^* : \mathcal{D}_\Delta \to \mathcal{D}_\Gamma$ with the following action on objects and morphisms:

1. for each $A \in \mathcal{D}_\Delta$, s^*A is the domain of the chosen life s_A.

$$\begin{tikzcd}
s^*A \arrow[r, "s_A"] & A & \mathcal{D} \arrow[d, "p"] \\
\Gamma \arrow[r, "s"] & \Delta & \mathcal{C}
\end{tikzcd}$$

2. for each $A, B \in \mathcal{D}_\Delta$, and $\alpha \in \mathcal{D}_\Delta(A, B)$, $s^*\alpha \in \mathcal{D}_\Gamma(s^*A, s^*B)$ is the unique morphism that makes the following diagram commute given by s_B being cartesian:

$$\begin{tikzcd}
s^*A \arrow[r, "s_A"] \arrow[d, "s^*\alpha"'] & A \arrow[d, "\alpha"] \\
s^*B \arrow[r, "s_B"'] & B.
\end{tikzcd}$$

Remark 3.17 ([Str18, Section 3]). Let $p : \mathcal{D} \to \mathcal{C}$ be a fibration. If p is not split, for composable u and v in $\mathcal{C}$, $v^* \circ u^*$ is not equal to $(u \circ v)^*$ in general, but the functors are canonically isomorphic.

$$\begin{tikzcd}
v^*u^*A \arrow[dr, "v_{u^*A}"] \arrow[dd, "i_A"', "\cong"] & & & \\
& u^*A \arrow[dr, "u_A"] & & \\
(uv)^*A \arrow[rr, "(uv)_A"'] & & A & \mathcal{D} \arrow[d, "p"] \\
\Gamma \arrow[rr, "u \circ v"] & & \Delta & \mathcal{C}
\end{tikzcd}$$

For each $A \in \mathcal{D}_\Delta$, we know from Lemma 3.6, that the composition $u_A \circ v_{u^*A}$ is cartesian. From Lemma 3.7 we get the isomorphism $i_A : v^*u^*A \cong (uv)^*A$. If p is split, for composable u and v in $\mathcal{C}$ we have $v^* \circ u^* = (u \circ v)^*$. This is because from requirement 2 of Definition 3.12 we know that the lift of the composition is the composition of lifts and we have $uv_A = u_A \circ v_{u^*A}$. Similarly for the identity morphism, if p is not split, for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{D}_\Gamma$ we have $A \cong \mathsf{id}^* A$:

$$\begin{tikzcd}
A \arrow[d, "\cong"'] \arrow[dr, equal] & & \\
\mathsf{id}^* A \arrow[r, "\mathsf{id}^*_A"'] & A & \mathcal{D} \arrow[d, "p"] \\
\Gamma \arrow[r, "\mathsf{id}_A"'] & \Gamma & \mathcal{C}.
\end{tikzcd}$$

Remark 3.18. Each Grothendieck fibration $p : \mathcal{D} \to \mathcal{C}$ can equivalently be thought of as a pseudofunctor of the form $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$, where $\mathcal{C}$ is a small category. The Grothendieck construction associates to each pseudofunctor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$ a cloven fibration [JY21, Section 10.1]. This cloven fibration is split if and only if the pseudofunctor is a strict functor [JY21, Proposition 10.1.11]. This construction defines a 2-equivalence of 2-categories [JY21, Theorem 10.6.16].

3.1.3 Fibred Functor and Adjunction

Definition 3.19 ([Jac93, Definition 2.4]). Let $p : \mathcal{C} \to \mathcal{B}$ and $q : \mathcal{D} \to \mathcal{B}$ be fibrations with the same basis $\mathcal{B}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is called a *fibred functor* (also called a *cartesian functor*) if it preserves cartesian morphisms and $q \circ F = p$.

Proposition 3.20 ([Jac93, Lemma 2.5]). *Let $p : \mathcal{C} \to \mathcal{B}$ and $q : \mathcal{D} \to \mathcal{B}$ be fibrations and let $F : p \to q$ be a fibred functor over $\mathcal{B}$. For every $A \in \mathcal{C}$, one obtains a fibrewise functor $F_A : \mathcal{C}_A \to \mathcal{D}_A$ by restriction. Then F is full (faithful) if and only if every F_A is full (faithful).*

Definition 3.21 ([Jac99, 1.8.6 Definition]). Let $p : \mathcal{C} \to \mathcal{B}$ and $q : \mathcal{D} \to \mathcal{B}$ be fibrations with the same base category $\mathcal{B}$. A *fibred adjunction* over $\mathcal{B}$ is given by fibred functors F, G in:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "F"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[ll, bend left, "G"] \arrow[dl, "q"] \\
& \mathcal{B}, &
\end{tikzcd}$$

such that there is an ordinary adjunction $\mathcal{D} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{C}$ with F and G as functors, and natural transformations $\eta : \mathsf{id}_\mathcal{C} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathsf{id}_\mathcal{D}$ given by this adjunction are vertical.

Proposition 3.22. *An adjunction $\mathcal{D} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{C}$ of fibred functors F and G in:*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "F"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[ll, bend left, "G"] \arrow[dl, "q"] \\
& \mathcal{B}, &
\end{tikzcd}$$

is a fibred adjunction if and only if

$$\mathcal{D}_f(FX, Y) \cong \mathcal{C}_f(X, GY),$$

for each $f : A \to B$ in $\mathcal{B}$ and objects X in $\mathcal{C}_A$ and Y in $\mathcal{D}_B$.

Proof. For the proof, see for example [Lin21, Lemma 2.2.0.11, Lemma 2.2.0.12]. □

3.2 Type Theory Preliminaries

In this section we discuss the restriction of Martin-Löf type theory that is used in this report. When we talk about MLTT, we mean a type theory with judgements and structural rules of MLTT, as discussed in [Mar84], with unit, Π- and Σ- types. We do not consider identity types, W−types and universes. Therefore, these topics are not included in this section. Throughout this report we consider the (definitional) equality of contexts, types and terms up to renaming of bound variables.

An overview on the judgements and structural rules of the type theory is given in Section 3.2.1. In Section 3.2.2 we discuss the notion of context morphisms which are mappings between contexts, used in categorical semantics for interpreting how substitution translates one context into the other by replacing variables with terms of certain types. Unit, Π- and Σ- types in MLTT are then discussed in Sections 3.2.4, 3.2.5 and 5.2, respectively.

The content of this section is based on [Hof97] and Appendix A.2 of [Uni13].

3.2.1 Judgements and Structural Rules

The Judgements of the type theory are as follows:

1. Γ ctx, which is read as "Γ is a (well-formed) context";
2. $\Gamma \vdash A$ type, which is read as "A is type in context Γ";
3. $\Gamma \vdash a : A$, which is read as "a is a term of type A in context Γ";
4. $\Gamma \equiv \Delta$ ctx, which is read as "Γ and Δ are (definitionally) equal contexts";
5. $\Gamma \vdash A \equiv B$ type, which is read as "A and B are (definitionally) equal types in context Γ";
6. $\Gamma \vdash a \equiv a' : A$, which is read as "$a$ and a' are (definitionally) equal terms of type A in context Γ".

Contexts can be thought of as lists of the form $[x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n]$ saying distinct terms $x_1, \ldots, x_n$ have assumed types $A_1 \ldots A_n$ respectively. Here, terms $x_1, \ldots, x_i$ may occur free in A_i, which expresses type dependency in the type theory. Well-formedness of context Γ is given by the judgement Γ ctx. Empty list corresponds to the empty context denoted as $\diamond$ ctx.

The type theory has the following structural rules.

1. Rules for context formation:

$$\frac{}{\diamond\ \mathsf{ctx}}\ \textsf{C-Emp} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.x : A\ \mathsf{ctx}}\ \textsf{C-Ext}$$

$$\frac{\Gamma \equiv \Delta\ \mathsf{ctx} \quad \Gamma \vdash A \equiv B\ \mathsf{type}}{\Gamma.x : A \equiv \Delta.y : B\ \mathsf{ctx}}\ \textsf{C-Ext-Eq}$$

2. Variable rule:

$$\frac{\Gamma.x : A.\Delta \text{ ctx}}{\Gamma \vdash x : A}\,\textsf{Var}$$

3. Rules expressing definitional equalities are equivalence relations:

$$\frac{\Gamma \text{ ctx}}{\Gamma \equiv \Gamma \text{ ctx}}\,\textsf{C-Eq-R} \qquad \frac{\Gamma \equiv \Delta \text{ ctx}}{\Delta \equiv \Gamma \text{ ctx}}\,\textsf{C-Eq-S}$$

$$\frac{\Gamma \equiv \Delta \text{ ctx} \quad \Delta \equiv \Theta \text{ ctx}}{\Gamma \equiv \Theta \text{ ctx}}\,\textsf{C-Eq-T}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}}\,\textsf{Ty-Eq-R} \qquad \frac{\Gamma \text{ ctx} \quad \Gamma \vdash B \equiv A \text{ type}}{\Gamma \vdash A \equiv B \text{ type}}\,\textsf{Ty-Eq-S}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \equiv B \text{ type} \quad \Gamma \vdash B \equiv C \text{ type}}{\Gamma \vdash A \equiv C \text{ type}}\,\textsf{Ty-Eq-T}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type} \quad \Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A}\,\textsf{Tm-Eq-R}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type} \quad \Gamma \vdash b \equiv a : A}{\Gamma \vdash a \equiv b : A}\,\textsf{Tm-Eq-S}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type} \quad \Gamma \vdash a \equiv b : A \quad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}\,\textsf{Tm-Eq-T}$$

4. Rules relating typing and definitional equality:

$$\frac{\Gamma \text{ ctx} \quad \Gamma : A \text{ type} \quad \Gamma \vdash a : A \quad \Gamma \equiv \Delta \text{ ctx} \quad \Gamma \vdash A \equiv B \text{ type}}{\Delta \vdash a : B}\,\textsf{Tm-Conv}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \equiv \Delta \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Delta \vdash A \text{ type}}\,\textsf{Ty-Conv}$$

5. We also have the following weakening, substitution and contraction rules which can be derived from the other rules. We explicitly mention these here as they are frequently used in the rest of the report. In these rules, $\mathcal{J}$ ranges over one of the judgements $a : A$, A type, $a \equiv b : A$, $A \equiv B$ type.

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \quad \Gamma \vdash C \text{ type}}{\Gamma.x : C.\Delta \vdash \mathcal{J}}\,\textsf{Weak}$$

$$\frac{\Gamma.x : C \vdash \mathcal{J} \quad \Gamma \vdash c : C}{\Gamma.\Delta[c/x] \vdash \mathcal{J}[c/x]}\,\textsf{Subst}$$

$$\frac{\Gamma.x : C.y : C, \Delta \vdash \mathcal{J}}{\Gamma.x : C.\Delta[x/y] \vdash \mathcal{J}[x/y]}\,\textsf{Contr}$$

Here, $\mathcal{J}[c/x]$ (and $\Delta[c/x]$) is the capture-free substitution of c for all occurrences of x in $\mathcal{J}$, which means that no free variables in c will become bound in $\mathcal{J}[c/x]$ [Hof97].

3.2.2 Context Morphisms

Context morphisms are mappings between contexts, used in categorical semantics for interpreting how substitution translates one context into the other by replacing variables with terms of certain types. The notion of context morphism can be used in formulating a typing rule for generalised substitution, that is the simultaneous substitution of all the variables in the target context up to renaming of variables.

In the rest of this report, we sometimes use the word "substitution" when referring to the notion of context morphism discussed in this section.

Definition 3.23 ([Hof97, Definition 2.11]). Let Γ and Δ be well-formed contexts with Δ of the form $[x_1 : A_1, \ldots, x_n : A_n]$. A *context morphism* from Γ to Δ is an $n-$tuple of terms $(M_1, \ldots, M_n)$ such that for each $i \leq n$ the following judgement holds:

$$\Gamma \vdash M_i : A_i[M_1/x_1, \ldots, M_{i-1}/x_{i-1}].$$

Notation 3.24. Given contexts Γ and Δ, a context morphism s from Γ to Δ is denoted as $s : \Gamma \to \Delta$ or $\Gamma \vdash s : \Delta$.

Remark 3.25 ([Hof97, Proposition 2.12]). One can derive the following rule for generalised substitution by induction on the length of Δ and using the Rules Weak and Subst.

$$\frac{\Gamma \vdash s : \Delta \quad \Delta.\Theta \vdash \mathcal{J}}{\Gamma.\Theta[f/\Delta] \vdash \mathcal{J}[f/\Delta]}\ \textsf{Gen-Subst}$$

Here by $\Theta[f/\Delta]$ (and similarly $\mathcal{J}[f/\Delta]$), we mean the simultaneous substitution of all variables in Δ with the terms that make up the context morphism f.

Example 3.26. For any context Γ, there is a unique context morphism $() : \Gamma \to \diamond$, the empty context morphism.

Example 3.27. Let $\Gamma = [n : \mathbb{N}, v : \mathsf{Vec}(n)]$, where $\mathsf{Vec}(n)$ is the type of vectors of natural numbers of length n. $(2, [0, 1])$ is a context morphism from $\diamond$ to Γ as $\diamond \vdash 2 : \mathbb{N}$ and $\diamond \vdash [0, 1] : \mathsf{Vec}[2/n]$.

Example 3.28 ([Hof97, Section 2.4]). Let $\Gamma = [n : \mathbb{N}, p : \mathsf{id}_{\mathbb{N}}(0, n)]$, where $\mathsf{id}_{\mathbb{N}}$ is the identity type of natural numbers. $(0, \mathsf{refl}_{\mathbb{N}}(0))$ is a context morphism from $\diamond$ to Γ as $\diamond \vdash 0 : \mathbb{N}$ and $\diamond \vdash \mathsf{refl}_{\mathbb{N}}(0) : (\mathsf{id}_{\mathbb{N}}(0, n))[0/n]$.

Example 3.29. Let $\Gamma = [b : \mathsf{Bool}]$ and $\Delta = [b_1 : \mathsf{Bool}, b_2 : \mathsf{Bool}, c : \mathsf{Conj}(b_1, b_2)]$, where $\mathsf{Conj}(b_1, b_2)$ is the boolean result of $b_1 \wedge b_2$. $(\mathsf{False}, b, \mathsf{False})$ is a context morphism from Γ to Δ as $\Gamma \vdash \mathsf{False} : \mathsf{Bool}$, $\Gamma \vdash b : \mathsf{Bool}$, $\Gamma \vdash \mathsf{False} : (\mathsf{Conj}(b_1, b_2))[False/b_1, b/b_2]$.

Example 3.30. For any context $\Gamma = [x_1 : A_1, \ldots, x_n : A_n]$, we have a context morphism $(x_1, \ldots, x_n)$ from Γ to Γ. This is called the identity context morphism.

Notation 3.31. For any context Γ, we denote the identity context morphism defined in Example 3.30 as id_Γ or 1_Γ.

Example 3.32 ([Hof97, Proposition 2.13]). Given context morphisms $f : \Gamma \to \Delta$ and $g : \Delta \to \Theta$, where $g = (M_1, \ldots, M_n)$, we have a context morphism $(M_1[f], \ldots, M_n[f])$ from Γ to Θ.

Notation 3.33. Given context morphisms $f : \Gamma \to \Delta$ and $g : \Delta \to \Theta$, the composition context morphism defined in Example 3.32 is denoted as $g \circ f$.

Example 3.34. For any $\Gamma = [x_1 : A_1, \ldots, x_n : A_n]$, a context morphism of the form $\Gamma \to \Gamma.x : A$ is $(x_1, \ldots, x_n, x)$. A context morphism of the form $\Gamma.x : A \to \Gamma$ is given by $\pi_A := (x_1, \ldots, x_n)$, the projection from the extended context to the non-extended one. Note that by this definition, an extension context morphism of the form $\Gamma \to \Gamma.x : A$ composed with a projection context morphism of the form $\Gamma.x : A \to \Gamma$ is the identity context morphism on the context Γ.

Notation 3.35. Given a context morphism s from Γ to Δ, by $(s, x : A)$ we mean a context morphism s' which is made of all the terms in s plus a term x of type A. For example, we denote the context morphism from Γ to $\Gamma.x : A$ defined in Example 3.34, which is of the form $(x_1, \ldots, x_n, x)$, as (id_Γ, x) or $(\mathsf{id}_\Gamma, x : A)$.

Notation 3.36. Given a context $\Gamma = [x_1 : A_1, \ldots, x_n : A_n]$. We denote the projection context morphism from $\Gamma.x : A$ to Γ defined in Example 3.34 as π_A.

Example 3.37. If type A is not dependent on any variables from context Γ, a context morphism $s' : \Gamma \to \Delta.x : A$ is equivalent to a morphisms $s : \Gamma \to \Delta$ and a term $x : A$; hence, the context morphism s' is of the form $(s, x : A)$, where we use Notation 3.35. If A contains variables from Δ, a context morphism $s' : \Gamma \to \Delta.x : A$ is equivalent to a morphism $s : \Gamma \to \Delta$ and a term $x : A[s/\Delta]$; hence, the context morphism s' is of the form $(s, x : A[s/\Delta])$.

Proposition 3.38 ([Hof97, Proposition 2.13]). *Given* $f : \Gamma \to \Delta, g : \Delta \to \Theta, \Theta \vdash A$ type *and* $\Theta \vdash t : A$, *we have:*

1. $\Gamma \vdash A[g \circ f] \equiv A[g][f]$;
2. $\Gamma \vdash t[g \circ f] \equiv t[g][f]$.

Proposition 3.39 ([Hof97, Proposition 2.13]). *The composition of substitutions is associative.*

Proposition 3.40 ([Hof97, Exercise 2.14]). *For* $s : \Gamma \to \Delta$, $\mathsf{id}_\Delta \circ s \equiv s \equiv s \circ \mathsf{id}_\Gamma$.

3.2.3 Unit Type

Unit type is either defined as a type with a unique term in each context or as an inductive type with elimination and computation rules.

The following rules define unit type as a type with a unique term in each context.

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma : \ \mathsf{type}} \ \textsf{Unit-Ty}$$

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma} \ \textsf{Unit-Tm}$$

$$\frac{\Gamma \vdash x : \mathbb{1}_\Gamma}{\Gamma \vdash x \equiv \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma} \ \textsf{Unit-Unique}$$

The following rules define unit type as an inductive type with elimination and computation rules.

$$\frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma \ \mathsf{type}} \ \textsf{Unit-Form}$$

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \textsf{Unit-Intro}$$

$$\frac{\Gamma.x : \mathbb{1}_\Gamma \vdash A\ \mathsf{type} \quad \Gamma \vdash a : A[\mathsf{tt}_\Gamma/x]}{\Gamma.x : \mathbb{1}_\Gamma \vdash \mathsf{urec}_{A,a} : A}\ \textsf{Unit-Elim}$$

$$\frac{\Gamma.x : \mathbb{1}_\Gamma \vdash A\ \mathsf{type} \quad \Gamma \vdash a : A[\mathsf{tt}_\Gamma/x]}{\Gamma \vdash \mathsf{urec}_{A,a}[\mathsf{tt}_\Gamma/x] \equiv a : A[\mathsf{tt}_\Gamma/x]}\ \textsf{Unit-Comp}$$

The elimination and computation rules state that for a context Γ and type A in Γ, to have a term of type A in context $\Gamma.x : 1$, it suffices to have a term of type $A[tt/x]$ in context Γ. This can also be thought of as meaning that for defining a function of the form $\mathbb{1} \to A$ it suffices to define the function for tt.

3.2.4 Π-Types

Π-types correspond to cartesian product over a family of sets in set theory; given a family of sets $(B_i)_{i\in I}$, one can form the set $\Pi_{i\in I} B_i$ that has as elements functions mapping an index i to an element of the corresponding set B_i. Similarly, in type theory, given a type B dependent on A, which corresponds to the judgement $\Gamma.x : A \vdash B$ type, we get a type $\prod_{x:A} B$ in context Γ. A term of $\prod_{x:A} B$ corresponds to a dependent function that takes a parameter a of type A and has a result of type $B[a/x]$. This idea is captured in the following formation, introduction, elimination and computation rules:

$$\frac{\Gamma \vdash A\ \mathsf{type} \quad \Gamma.x : A \vdash B\ \mathsf{type}}{\Gamma \vdash \Pi_{(x:A)} B\ \mathsf{type}}\ \Pi\textsf{-Form}$$

$$\frac{\Gamma.x : A \vdash b : B}{\Gamma \vdash \lambda(x : A).b : \Pi_{(x:A)} B}\ \Pi\textsf{-Intro}$$

$$\frac{\Gamma \vdash f : \Pi_{(x:A)} B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B[a/x]}\ \Pi\textsf{-Elim}$$

$$\frac{\Gamma, x : A \vdash b : B \quad \Gamma \vdash a : A}{\Gamma \vdash (\lambda(x : A).b)(a) \equiv b[a/x] : B[a/x]}\ \Pi\textsf{-Comp}$$

The elimination and computation rules reflect the idea of how the function is applied to its parameter. In addition to these, we also have the rules that state definitional equality is preserved for the formation and introduction rules.

As a special case for Π-Types, if B does not depend on A, which corresponds to the judgements $\Gamma \vdash A$ type and $\Gamma \vdash B$ type, then the type $\prod_{x:A} B$ is the definition of a (non-dependent) function type $A \to B$.

3.2.5 Σ-Types

Σ-types correspond to disjoint union in set theory; given a family of sets $(B_i)_{i\in I}$, one can form the set $\sum_{i\in I} B_i := \{(i,b) | i \in I \land b \in B_i\}$, where the elements are pairs of an index i and an element of B_i. Similarly, in type theory, given a type B dependent on A, which corresponds to the judgement $\Gamma.x : A \vdash B$ type, we get a type $\sum_{x:A} B$ in context Γ. The terms of $\sum_{x:A} B$ are pairs where the first element is a term a of type A and the second element is of type $B[a/x]$. This is stated in the following formation and introduction rules:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.x : A \vdash B \text{ type}}{\Gamma \vdash \Sigma_{(x:A)} B \text{ type}} \ \Sigma\text{-Form}$$

$$\frac{\Gamma.x : A \vdash B \text{ type} \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x]}{\Gamma \vdash (a, b) : \Sigma_{(x:A)} B} \ \Sigma\text{-Intro}$$

The elimination and computation rules reflect the idea that to define a function out of $\sum_{x:A} B$, it is enough to specify the function on the pairs:

$$\frac{\Gamma.z : \Sigma_{(x:A)} B \vdash C \text{ type} \quad \Gamma.x : A.y : B \vdash g : C[(x,y)/z] \quad \Gamma \vdash p : \Sigma_{(x:A)} B}{\Gamma \vdash \mathsf{rec}_{z:\Sigma_{(x:A)} B}(C, g, p) : C[p/z]} \ \Sigma\text{-Elim}$$

$$\frac{\Gamma.z : \Sigma_{(x:A)} B \vdash C \text{ type} \quad \Gamma.x : A.y : B \vdash g : C[(x,y)/z] \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \mathsf{rec}_{z:\Sigma_{(x:A)} B}(C, g, (a,b)) \equiv g[a, b/x, y] : C[(a,b)/z]} \ \Sigma\text{-Comp}$$

The computation rule states that a function out of $\sum_{x:A} B$ defined using $\mathsf{rec}_{\sum_{x:A} B}$ from the elimination rule, acts on the pairs as specified by the term g.

Two examples of functions defined out of $\sum_{x:A} B$ are the first and second projections π_1 and π_2. These projections are defined as follows, assuming $\Gamma \vdash A$ type, $\Gamma.x : A \vdash B$ type and $\Gamma \vdash p : \sum_{x:A} B$:

$$\pi_1(p) :\equiv \mathsf{rec}_{z:\sum_{a:A} B}(A, x, p) : A$$

$$\pi_2(p) :\equiv \mathsf{rec}_{z:\sum_{a:A} B}(B[\pi_1(z)/x], y, p) : B[\pi_1(p)/x]$$

There is a weak version of the elimination rule, called weak Σ-elimination. In this case, the elimination rule is restricted to the cases where C does not depend on $\sum_{x:A} B$, i.e. having the following as the elimination and computation rules:

$$\frac{\Gamma \vdash C \text{ type} \quad \Gamma.x : A.y : B \vdash g : C \quad \Gamma \vdash p : \sum_{(x:A)} B}{\Gamma \vdash \mathsf{rec}_{\sum_{(x:A)} B}(C, g, p) : C} \ \text{Weak } \Sigma\text{-Elim}$$

$$\frac{\Gamma \vdash C \text{ type} \quad \Gamma.x : A.y : B \vdash g : C \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \mathsf{rec}_{\sum_{(x:A)} B}(C, g, (a,b)) \equiv g[a, b/x, y] : C} \ \text{Weak } \Sigma\text{-Comp}$$

Here, it is still possible to define the first projection, similar to the previous case, but not the second one. Under proposition-as-types, one can consider the strong and weak Σ- types as counterparts of the constructive existential quantification and the existential quantification in classical mathematics, respectively.

As a special case, if B does not depend on A, which corresponds to the judgements $\Gamma \vdash A$ type and $\Gamma \vdash B$ type, then the type $\sum_{x:A} B$ is the definition of an ordinary product type $A \times B$.

Chapter 4

Comprehension Categories

In this section we go over the definition of a comprehension category as introduced in [Jac93] and discuss examples of comprehension categories. We also mention and prove some relevant results from the literature that will be used in the rest of this report.

Comprehension categories feature a cloven Grothendieck fibration $p : \mathcal{T} \to \mathcal{C}$, where $\mathcal{C}$ corresponds to the category of contexts and for each $\Gamma \in \mathcal{C}$ the fibre $\mathcal{T}_\Gamma$ corresponds to the category of types in context Γ. The reindexing functors corresponding to the lifts provided by the cleaving of the fibration give the semantics for substitution between contexts. Context extension is captured by a functor $\chi_0 : \mathcal{T} \to \mathcal{C}$ which sends a type A in context Γ to the extended context $\Gamma.A$, and a natural transformation $\pi : \chi_0 \to p$ that provides the coherence condition needed for projections from extended contexts to the original contexts. These two functors and natural transformation can be thought of as one functor $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ into the arrow category, which is called the comprehension. Lastly, to capture how context morphisms from Γ to Δ are built of terms in context Γ, an additional requirement is added that χ preserves cartesian morphisms. This ensures that cartesian morphisms in $\mathcal{T}$ correspond to pullback squares in $\mathcal{C}$.

Definition 4.1 ([Jac93, Definition 4.1]). A *comprehension category* consists of a category $\mathcal{C}$, a (cloven) fibration $p : \mathcal{T} \to \mathcal{C}$, and a functor $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ preserving cartesian arrows, such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C}. & \end{tikzcd}$$

Here, χ is called the comprehension.

We will see later that the objects of $\mathcal{C}$ correspond to contexts, and the objects of the fibre $\mathcal{T}_\Gamma$ over $\Gamma \in \mathcal{C}$ correspond to (dependent) types in context Γ. Usually, $\mathcal{C}$ is required to have a terminal object. The terminal object of $\mathcal{C}$ corresponds to the empty context in the type theory.

Notation 4.2. In the rest of this report, unless specified otherwise, by a comprehension category we mean comprehension category $(\mathcal{C}, \mathcal{T}, \chi, p)$ which means that the base is denoted as $\mathcal{C}$, the corresponding fibration as $p : \mathcal{T} \to \mathcal{C}$ and the comprehension as χ.

Definition 4.3 ([Jac93, Definition 4.1]). A comprehension category is called *full* if $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ is fully faithful and is called *split* if $p : \mathcal{T} \to \mathcal{C}$ is a split fibration.

Remark 4.4. $\mathcal{C}$ is not required to have all pullbacks. In other words, we do not require $\mathsf{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ to be a fibration.

Remark 4.5. χ is a functor into the arrow category $\mathcal{C}^{\rightarrow}$, so, as discussed in Remark 3.4, it can be thought of as two functors $\mathsf{dom} \circ \chi : \mathcal{T} \to \mathcal{C}$ and $\mathsf{cod} \circ \chi : \mathcal{T} \to \mathcal{C}$ and a natural transformation $\pi : \mathsf{dom} \circ \chi \Rightarrow \mathsf{cod} \circ \chi$.

We denote $\mathsf{dom} \circ \chi$ as χ_0 due to [Jac93, Notation 4.2]. For each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, $\chi_0 A$ is denoted as $\Gamma.A$. We will see later that this corresponds to the extension of context Γ with a term of type A. χA, which by Remark 4.5 can also be thought of as $\pi_A : \mathcal{C}(\Gamma.A, \Gamma)$, is called the projection from $\Gamma.A$ to Γ. We will see that this corresponds to the projection from an extended context to the original context.

Example 4.6 ([Jac93, Example 4.5]). Recall from Lemma 3.15 that if $\mathcal{C}$ has all pullbacks, $\mathsf{cod} : \mathcal{C}^{\rightarrow} \to \mathcal{C}$ is a split fibration. Hence, if $\mathcal{C}$ has all pullbacks, using the identity functor $1 : \mathcal{C}^{\rightarrow} \to \mathcal{C}^{\rightarrow}$ we get the *identity* comprehension category:

$$\begin{tikzcd} \mathcal{C}^{\rightarrow} \arrow[rr, "1"] \arrow[dr, "\mathsf{cod}"'] & & \mathcal{C}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C}. & \end{tikzcd}$$

The identity comprehension category is full and split.

Example 4.7. The category Set has pullbacks. Let $X, Y, Z \in \mathsf{Set}$ be sets, and $f : X \to Z$ and $g : Y \to Z$ be functions. We have the following pullback square in Set:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z, \end{tikzcd}$$

where $X \times_Z Y := \{(x, y) \in X \times Y | fx = gy\}$. Hence, $\mathsf{cod} : \mathsf{Set}^{\rightarrow} \to \mathsf{Set}$ is a fibration and we have the following full split comprehension category:

$$\begin{tikzcd} \mathsf{Set}^{\rightarrow} \arrow[rr, "1"] \arrow[dr, "\mathsf{cod}"'] & & \mathsf{Set}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\ & \mathsf{Set}. & \end{tikzcd}$$

This is a special case of the comprehension category explained in Example 4.6.

4.1 Extended Example: Syntactic Category

Given a dependent type theory with judgements and structural rules of Martin-Löf type theory, one can construct a full split comprehension category from the syntax. In this subsection we explain the construction of this syntactic category as it is described in [Jac93, Example 4.3], and why it makes a full split comprehension category. We use $\|-\|$ to denote the semantic counterpart of a type-theoretic entity. The construction is as follows.

1. Category $\mathcal{C}$ has contexts as its objects, up to definitional equality of contexts, and context morphisms as its morphisms. Using Propositions 3.39 and 3.40 we can see that this makes a category.

Recall from Definition 3.23 that for $\Delta = [x_1 : A_1, \ldots, x_n : A_n]$, a context morphism $s : \Gamma \to \Delta$ is an n-tuple of terms (up to definitional equality) $(M_1, \ldots, M_n)$ satisfying:

$$\Gamma \vdash M_i : A_i[M_1, \ldots, M_{i-1}/x_1, \ldots, x_{i-1}].$$

Note that the category $\mathcal{C}$ constructed in this way, has a terminal object, namely the empty context $\diamond$. This is justified by Example 3.26.

2. Objects of $\mathcal{T}_{\|\Gamma\|}$ are of the form $\|A\|$ for $\Gamma \vdash A$ type, up to definitional equality. This means that for a type A in context Γ, $p : \mathcal{T} \to \mathcal{C}$ sends $\|A\|$ to its context $\|\Gamma\|$. For types A and B in context Γ, morphisms from $\|A\|$ to $\|B\|$ correspond to terms t of type B in context $\Gamma.A$, which is $\Gamma.x : A \vdash t : B$.

   Now, we show that this makes a category.

   (a) For each context Γ and type A in Γ the identity morphism $\mathsf{id}_{\|A\|}$ is given by the variable rule $\Gamma.x : A \vdash x : A$.

   (b) The composition of morphisms is given by the following substitution. Let $\Gamma.x : A \vdash t_1 : B$ and $\Gamma.y : B \vdash t_2 : C$. We have $\|t_1\| \in \mathcal{T}(\|A\|, \|B\|)$ and $\|t_2\| \in \mathcal{T}(\|B\|, \|C\|)$. The composition defined as $\|t_2\| \circ \|t_1\| := \|t_2[t_1/y]\|$ is a morphism in $\mathcal{T}(\|A\|, \|C\|)$, since $\Gamma.x : A \vdash t_2[t_1/y] : C$.

   (c) Let $\Gamma.y : A \vdash t : B$. From the definition of composition we have $\|t\| \circ \mathsf{id}_{\|A\|} = \|t[y/x]\|$ and $\mathsf{id}_B \circ \|t\| = \|x[t/x]\|$. From the type theory we know that $\Gamma.x : A \vdash t[y/x] \equiv t : B$ and $\Gamma.y : A \vdash x[t/x] \equiv t : B$ which means $\|t\| \circ \mathsf{id}_{\|A\|} = \|t\| = \mathsf{id}_{\|B\|} \circ \|t\|$.

   (d) Let $\Gamma.x : A \vdash t_1 : B$, $\Gamma.y : B \vdash t_2 : C$ and $\Gamma.z : C \vdash t_3 : D$. We have $\|t_1\| \in \mathcal{T}(\|A\|, \|B\|)$, $\|t_2\| \in \mathcal{T}(\|B\|, \|C\|)$ and $\|t_3\| \in \mathcal{T}(\|C\|, \|D\|)$. From the definition of composition we get $\|t_3\| \circ (\|t_2\| \circ \|t_1\|) = \|t_3[(t_2[t_1/y])/z]\|$ and $(\|t_3\| \circ \|t_2\|) \circ \|t_1\| = \|t_3[t_2/z])[t_1/y]\|$. From the type theory we know that $\Gamma.x : A \vdash t_3[(t_2[t_1/y])/z] \equiv (t_3[t_2/z])[t_1/y] : D$ which means $\|t_3\| \circ (\|t_2\| \circ \|t_1\|) = (\|t_3\| \circ \|t_2\|) \circ \|t_1\|$.

3. χ preserves cartesian arrows, and sends an object $\|A\|$ in $\mathcal{T}_{\|\Gamma\|}$ to $\pi_{\|A\|} \in \mathcal{C}(\|\Gamma.A\|, \|\Gamma\|)$. This gives $p = \mathsf{cod} \circ \chi$. For each $A, B : \mathcal{T}_\Gamma$ and a morphism $\alpha : \mathcal{T}_\Gamma(A, B)$, χ sends α to a morphism β that makes the following diagram commute:

$$\begin{tikzcd}
\Gamma.A \arrow[rr, "\beta"] \arrow[dr, "\pi_A"'] & & \Gamma.B \arrow[dl, "\pi_B"] \\
& \Gamma. &
\end{tikzcd}$$

   The context morphism β is of the form $(\mathsf{id}_{\|\Gamma\|}, \|t\|)$, where $\mathsf{id}_{\|\Gamma\|}$ is the identity morphism on $\|\Gamma\|$ and we have $\Gamma.A \vdash t : B$. Hence, for each $\Gamma \in \mathcal{C}$ and $A, B \in \mathcal{T}_\Gamma$ there is a bijection between $\mathcal{T}_\Gamma(A, B)$ and $\mathcal{C}^{\to}(\pi_A, \pi_B)$, which means that χ is fibrewise fully faithful; hence, fully faithful.

4. It remains to show that p is a cloven split fibration to show that we have constructed a full split comprehension category.

   (a) Let $\Delta = [x_1 : A_1, \cdots, x_n : A_n]$ and $s = (M_1, \cdots, M_n) : \Gamma \to \Delta$. For each type A_i in Δ the cartesian lift of $\|s\|$ into $\|A_i\|$ is given by $\|s\|_{\|A_i\|} := \|M_i\| : \mathcal{T}(\|A_i[s]\|, \|A_i\|)$, as $\Gamma.x : A_i[s] \vdash M_i : A_i$. We will see why this morphism is cartesian in Sections 5.1.2 to 5.1.4.

(b) For $\Gamma \vdash A$ type we have $\Gamma \vdash A[\mathrm{id}_\Gamma] \equiv A$ and from part 1 of Proposition 3.38 we know that $\Gamma \vdash A[g \circ f] \equiv A[g][f]$ for $f : \Gamma \to \Delta$, $g : \Delta \to \Theta$, $\Theta \vdash A$ type. This means that p is a split fibration.

This concludes the extended example.

Remark 4.8. The initiality of a full split comprehension category given by the construction in Section 4.1 is discussed in [LW15]. This means that full split comprehension categories can be considered models for dependent type theories with judgements and structural rules of Martin-Löf type theory.

Lemma 4.9 ([Jac93, Lemma 4.4], [Jac99, Lemma 10.3.1]). *In a comprehension category, for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $s : \Gamma \to \Delta$, the pullback of π_A along s exists and is itself a projection, particularly π_{s^*A}. This means that for an arbitrary $s : \Gamma \to \Delta$ and $A \in \Delta$, we always have a pullback square of the following form:*

$$\begin{tikzcd} \Gamma.s^*A \arrow[r, "s.A"] \arrow[d, "\pi_{s^*A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\ \Gamma \arrow[r, "s"'] & \Delta, \end{tikzcd}$$

where $s.A$ is χs_A.

Proof. For each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $s : \Gamma \to \Delta$, the cleaving of the fibration gives a cartesian morphism $s_A : s^*A \to A$. χs_A corresponds to:

$$\begin{tikzcd} \Gamma.s^*A \arrow[r, "\chi s_A"] \arrow[d, "\pi_{s^*A}"'] & \Delta.A \arrow[d, "\pi_A"] \\ \Gamma \arrow[r, "s"'] & \Delta, \end{tikzcd}$$

From Lemma 3.9 we know that cartesian morphisms in $\mathcal{C}^{\to}$ are pullback squares in $\mathcal{C}$ and that χ preserves cartesian morphisms; hence, we always have a pullback square of the following form in $\mathcal{C}$:

$$\begin{tikzcd} \Gamma.s^*A \arrow[r, "s.A"] \arrow[d, "\pi_{s^*A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\ \Gamma \arrow[r, "s"'] & \Delta, \end{tikzcd}$$

where $s.A = \chi s_A$. □

Example 4.10. The full subcategory of $\mathcal{C}^{\to}$ with all the projections as objects is denoted as $\mathcal{D}^{\to}$ due to [Jac99]. By Lemma 4.9, the pullback of all the projections in $\mathcal{D}^{\to}$ exists in $\mathcal{C}$. Hence, $\mathrm{cod} : \mathcal{D}^{\to} \to \mathcal{C}$ is a fibration. This means that the following is a comprehension category:

$$\begin{tikzcd} \mathcal{D}^{\to} \arrow[rr, "\iota"] \arrow[dr, "\mathrm{cod}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C}, & \end{tikzcd}$$

where ι is the inclusion. This comprehension category is split, since cartesian morphisms in $\mathcal{D}^{\to}$ are pullback squares in $\mathcal{C}$.

Lemma 4.11 ([Jac93, Lemma 4.4]). *In a comprehension category, for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $s : \Gamma \to \Delta$, we have the following bijection:*

$$\mathcal{C}/\Delta(s, \pi_A) \cong \{t \in \mathcal{C}(\Gamma, \Gamma.s^*A) | t \text{ is a section of } \pi_{s^*A}\}.$$

In other words, there is a bijection between morphisms $s' : \Gamma \to \Delta.A$ in $\mathcal{C}$ that make the following diagram commute:

$$\begin{tikzcd}
\Gamma \arrow[rr, "s'"] \arrow[dr, "s"'] & & \Delta.A \arrow[dl, "\pi_A"] \\
& \Delta, &
\end{tikzcd}$$

and the sections $t_{s^*A} : \Gamma \to \Gamma.s^*A$ of the projection $\pi_{s^*A} : \Gamma.s^*A \to \Gamma$ in $\mathcal{C}$.

Proof. The bijection is given by factorising s' through $s.A$, in the following pullback square from Lemma 4.9:

$$\begin{tikzcd}
\Gamma \arrow[dr, "t_{s^*A}"] \arrow[drr, bend left, "s'"] \arrow[ddr, equal, bend right] & & \\
& \Gamma.s^*A \arrow[r, "s.A"] \arrow[d, "\pi_{s^*A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

□

Lemma 4.12 ([Jac93, Section 5.5]). *In a full comprehension category, we have the following bijection for each $\Gamma \in \mathcal{C}$ and $A, B \in \mathcal{T}_\Gamma$:*

$$\mathcal{T}_\Gamma(A, B) \cong \{t \in \mathcal{C}(\Gamma, \Gamma.s^*A) | t \text{ is a section of } \pi_{\pi_A^* B}\}$$

Proof. From χ being fully faithful we have, $\mathcal{T}_{\Gamma(A,B)} \cong \mathcal{C}/\Gamma(\pi_A, \pi_B)$ which means that there is a bijection between elements of $\mathcal{T}_{\Gamma(A,B)}$ and morphisms $s : \Gamma.A \to \Gamma.B$ in $\mathcal{C}$ that make the following diagram commute:

$$\begin{tikzcd}
\Gamma.A \arrow[rr, "s"] \arrow[dr, "\pi_A"'] & & \Gamma.B \arrow[dl, "\pi_B"] \\
& \Gamma. &
\end{tikzcd}$$

For each such $s : \Gamma.A \to \Gamma.B$, we get a unique section of $\pi_{\pi_A^* B}$ from the following pullback square:

$$\begin{tikzcd}
\Gamma.A \arrow[dr, "t"] \arrow[drr, bend left, "s"] \arrow[ddr, equal, bend right] & & \\
& \Gamma.A.\pi_A^* B \arrow[r] \arrow[d, "\pi_{\pi_A^* B}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.B \arrow[d, "\pi_B"] \\
& \Gamma.A \arrow[r, "\pi_A"'] & \Gamma.
\end{tikzcd}$$

Hence, in a full comprehension category, there is a bijection between elements of $\mathcal{T}_\Gamma(A, B)$ and sections of $\pi_{\pi_A^* B}$. $\square$

Remark 4.13. As discussed in [Jac93, Lemma 4.9], the inclusion functor $\iota : \mathsf{Comp}_{\mathsf{full}}(\mathcal{C}) \to \mathsf{Comp}(\mathcal{C})$, where $\mathsf{Comp}(\mathcal{C})$ is the category of comprehension categories with base $\mathcal{C}$ and suitable cartesian functors as morphism, has a left adjoint. This means that every comprehension category can be turned into an equivalent full one. [CGH14; LW15; Str18] explain two ways to turn a comprehension category into an equivalent split one using left and right adjoints of the inclusion functor $\iota : \mathsf{Comp}_{\mathsf{split}}(\mathcal{C}) \to \mathsf{Comp}(\mathcal{C})$, where $\mathsf{Comp}_{\mathsf{split}}$ is the category of split comprehension categories with base $\mathcal{C}$ and suitable cartesian functors that preserve the splitting as morphisms.

Chapter 5

Interpretation of Dependent Type Theories

In Chapter 4, we have briefly pointed to how each component of a comprehension category corresponds to the components of a type theory. In this section, we make this more precise and discuss the interpretation of a dependent type theory with judgements and structural rules of Martin-Löf type theory in a full split comprehension category. Whenever the assumptions of the comprehension category being full or split are used, it is explicitly stated. Then, we discuss the interpretation unit type, Π- and Σ-types in comprehension categories.

Assuming a Martin-Löf type theory with judgements and structural rules, the high-level idea behind the interpretation in a comprehension category is as follows.

1. The objects of $\mathcal{C}$ can be thought of as contexts, and the morphisms can be thought of as context morphisms. Identity context morphism, composition and their properties are as described in Section 3.2.1. The terminal object of $\mathcal{C}$ can be thought of as the empty context, with a unique context morphism, the empty context morphism, from each context to it.

2. The objects of a fibre $\mathcal{T}_\Gamma$ over Γ can be thought of as (dependent) types in context Γ. Hence, $p : \mathcal{T} \to \mathcal{C}$ can be thought of as sending a type to its context.

3. For each $A \in \mathcal{T}_\Gamma$ the comprehension χA, denoted as $\pi_A : \Gamma.A \to \Gamma$, can be thought of as a projection projection from the extended context $\Gamma.A$ back to Γ. This means that χ_0 can be thought of as context extension.

4. For each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, the sections of the projection π_A, morphisms $t : \Gamma \to \Gamma.A$ in $\mathcal{C}$ such that $\pi_A \circ t = 1_\Gamma$, can be thought of as terms of type A.

5. Substitution is given by the universal property of the cartesian morphisms. In particular, for $\Delta = [x_1 : A_1, \cdots x_n : A_n]$ and a context morphism $s = (M_1, \cdots M_n) : \Gamma \to \Delta$, as defined in Definition 3.23, the term M_i is identified by the lift $s_{A_i} \in \mathcal{T}_\Gamma(s^* A_i, A_i)$ given by the cleaving.

Throughout this section, we discuss two running examples of dependent type theories that can be interpreted in two certain comprehension categories. After discussing the interpretation of each component of dependent type theory in comprehension categories, we go back to the examples and discuss the interpretations in those certain instances.

The first example concerns the type theory that can be interpreted in the comprehension category made from the category Set, which is discussed in Example 4.7. This example corresponds to how dependent type theories can be interpreted in set theory. The interpretations of the judgements and structural rules of the type theory in this category are discussed in Section 5.1.8. Whether unit, Σ- and Π- types can be interpreted in this comprehension category and the interpretations are discussed in Example 5.16 and Section 5.3.4.

The second example is about a comprehension category in which a type theory with no type dependency can be interpreted. We discuss the type theory that can be interpreted in the comprehension category defined in Example 5.5 and compare this with our expectations of what the interpretation should be in the presence of no type dependency. In this sense, this example serves as a sanity check of the interpretations discussed in this section for the special case of not having type dependency. The interpretations of the judgements and structural rules of the type theory in this category are discussed in Section 5.1.9. Whether unit, Σ- and Π- types can be interpreted in this comprehension category and the interpretations are discussed in Example 5.17 and Section 5.3.5.

5.1 Judgements and Structural Rules

In this section, we describe the interpretation of a type theory with judgements and structural rules of Martin-Löf type theory in a comprehension category. We use $\|-\|$ to denote the semantic counterpart of a type-theoretic entity. The interpretation of the judgements in a comprehension category is as follows:

1. Γ ctx is interpreted as an object $\|\Gamma\|$ in $\mathcal{C}$ (up to definitional equality of contexts);
2. $\Gamma \vdash A$ type is interpreted as an object $\|A\|$ in the fibre $\mathcal{T}_{\|\Gamma\|}$ (up to definitional equality of types);
3. $\Gamma \vdash t : A$ is interpreted as a morphism $\|t\|$ in $\mathcal{C}$ such that $\|t\|$ is a section of the projection $\pi_{\|A\|}$, i.e. $\|t\| : \|\Gamma\| \to \chi_0\|\Gamma\|$ in $\mathcal{C}$ and $\pi_{\|A\|} \circ \|t\| = 1_{\|\Gamma\|}$.

Context extension is interpreted as the action of χ_0 on the objects of $\mathcal{T}$. For a type A in context Γ, $\chi_0\|A\| := \|\Gamma.A\|$ is the interpretation of the context Γ extended with a term of type A. Substitution is interpreted as the reindexing functors induced by the fibration p. s being a context morphism (substitution), as defined is Definition 3.23, from context Γ to Δ is interpreted as a morphism $\|s\| : \|\Gamma\| \to \|\Delta\|$ in $\mathcal{C}$. Weakening and contraction are also interpreted as certain reindexing functors. In particular, weakening corresponds to reindexing functors of the form $\pi_A^* : \mathcal{T}_\Gamma \to \mathcal{T}_{\Gamma.A}$ where $\pi_A : \Gamma.A \to \Gamma$ and contraction corresponds to reindexing functors of the form $\delta_A^* : \mathcal{T}_{\Gamma.A.A} \to \mathcal{T}_{\Gamma.A}$ where $\delta_A : \Gamma.A \to \Gamma.A.A$. This is further explained in Sections 5.1.1 to 5.1.7.

5.1.1 Context Extension

Given a context Γ and a type A in context Γ, the context extension rule states that the extended context $\Gamma.A$ is well-formed:

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A \text{ type}}{\Gamma.A \text{ ctx}}$$

Recall from Remark 4.5 that $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ can be thought of as $\mathsf{dom} \circ \chi : \mathcal{T} \to \mathcal{C}$, denoted as χ_0, $\mathsf{cod} \circ \chi : \mathcal{T} \to \mathcal{C}$ which is equal to p, and $\pi : \chi_0 \Rightarrow p$. Context extension is interpreted as the action of χ_0 of the objects of $\mathcal{T}$, which is $\|\Gamma.A\| := \chi_0\|A\|$.

5.1.2 Substitution : Types

Recall from Definition 3.16 that for each morphism $s : \Gamma \to \Delta$ in $\mathcal{C}$, the reindexing functor $s^* : \mathcal{T}_\Delta \to \mathcal{T}_\Gamma$ takes $A \in \mathcal{T}_\Delta$ to $s^*A \in \mathcal{T}_\Gamma$, where s^*A is the domain of s_A, the cartesian cartesian lift of s in A

given by the cleaving of the fibration:

$$\begin{tikzcd} s^*A \arrow[r, "s_A"] & A & \mathcal{T} \arrow[d, "p"] \\ \Gamma \arrow[r, "s"] & \Delta & \mathcal{C}. \end{tikzcd}$$

As mentioned above, s being a context morphism (substitution) from context Γ to Δ is interpreted as a morphism $\|s\| : \|\Gamma\| \to \|\Delta\|$ in $\mathcal{C}$. Given a context morphism $s : \Gamma \to \Delta$ and a type A in context Δ corresponding to the judgement $\Delta \vdash A$ type, substitution gives a type $A[s]$ in context Γ, corresponding to the judgement $\Gamma \vdash A[s]$ type. In the comprehension category, this is interpreted as the action of the reindexing functor $\|s\|^* : \mathcal{T}_{\|\Delta\|} \to \mathcal{T}_{\|\Gamma\|}$ on objects. This means that the result of the substitution s applied to type A in context Δ, which is $A[s/\Delta]$, is interpreted as $\|s\|^*\|A\|$.

Notation 5.1. For $\Gamma, \Delta \in \mathcal{C}$, $s : \Gamma \to \Delta$ and $A \in \mathcal{T}_\Delta$, the comprehension $\chi_0(s_A)$ is denoted by $s.A : \Gamma.s^*A \to \Delta.A$.

Example 5.2. As an example we consider the case of substitution of one variable of type A with a term of type A in a type B in context $\Gamma.x : A$. This is given by the following substitution rule:

$$\frac{\Gamma.x : A \vdash B \ \mathsf{type} \quad \Gamma \vdash a : A}{\Gamma \vdash B[a/x] \ \mathsf{type}}$$

As mentioned in Example 3.34, the corresponding context morphism to this substitution is $(\mathsf{id}_\Gamma, a : A)$, which is a section of the projection $\pi_{\|A\|}$. In other words, the corresponding context morphism is $\|a\| : \|\Gamma\| \to \|\Gamma.A\|$. The type $B[a/x]$ in context Γ is interpreted as $\|B[a/x]\| := \|a\|^*\|B\|$:

$$\begin{tikzcd} \|B[a/x]\| \arrow[r, "\|a\|_{\|B\|}"] & \|B\| & \mathcal{T} \arrow[d, "p"] \\ \|\Gamma\| \arrow[r, "\|a\|"] & \|\Gamma.A\| & \mathcal{C}. \end{tikzcd}$$

5.1.3 Substitution is Pullback in $\mathcal{C}$

We have now seen the interpretation of substitution for types in a comprehension category. We now see how context morphisms being $n-$tuples of terms is interpreted in comprehension categories. Recall from Definition 3.23 that in the syntax, for $\Delta = [x_1 : A_1, \ldots, x_n : A_n]$, a context morphism $s : \Gamma \to \Delta$ is an $n-$tuple of terms (up to definitional equality) $(M_1, \ldots, M_n)$ satisfying $\Gamma \vdash M_i : A_i[M_1/x_1, \ldots, M_{i-1}/x_{i-1}]$.

We saw in Example 3.37 that a context morphism $s' : \Gamma \to \Delta.x : A$ is equivalent to a morphism $s : \Gamma \to \Delta$ and a term $t : A[s/\Delta]$. To interpret this in a comprehension category, we need to have the following bijection:

$$\frac{\text{a section } t \text{ of } \pi_{\|s\|^*\|A\|}}{\overline{s' : \|\Gamma\| \to \|\Delta.A\| \text{ in } \mathcal{C}}} \tag{5.1}$$

for each contexts Γ, Δ, type A in context Δ and context morphism $s : \Gamma \to \Delta$. We know from Lemma 4.9 that for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $s : \Gamma \to \Delta$ we have the following pullback square:

$$\begin{tikzcd}
\Gamma.s^*A \arrow[r, "s.A"] \arrow[d, "\pi_{s^*A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\
\Gamma \arrow[r, "s"'] & \Delta,
\end{tikzcd}$$

and from Lemma 4.11 we get the required bijection in (5.1) by factorising s' through $\|s\|.\|A\|$ in:

$$\begin{tikzcd}
\|\Gamma\| \arrow[dr, "t"] \arrow[drr, bend left, "s'"] \arrow[ddr, equal, bend right] & & \\
& \|\Gamma.A[s]\| \arrow[r, "\|s\|.\|A\|"] \arrow[d, "\pi_{\|s\|^*\|A\|}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \|\Delta.A\| \arrow[d, "\pi_{\|A\|}"] \\
& \|\Gamma\| \arrow[r, "\|s\|"'] & \|\Delta\|.
\end{tikzcd}$$

This justifies why χ should preserve cartesian morphisms in a comprehension category. We see in the next section that the morphism t is the interpretation of $A[s/\Delta]$.

5.1.4 Substitution: Terms

Given a context morphism $s : \Gamma \to \Delta$, a type A in context Δ and a term $\Delta \vdash t : A$, substitution gives a term $t[s]$ of type $A[s]$ in context Γ, corresponding to the judgement $\Gamma \vdash t[s] : A[s]$. In the comprehension category, $t[s]$ is interpreted as the unique morphism given by the universal property of the following pullback:

$$\begin{tikzcd}
\|\Gamma\| \arrow[rr, "\|s\|"] \arrow[dr, "\|t[s]\|"] \arrow[ddr, equal, bend right] & & \|\Delta\| \arrow[dr, "\|t\|"] & \\
& \|\Gamma.A[s]\| \arrow[rr, "\|s\|.\|A\|"] \arrow[d, "\pi_{\|A[s]\|}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \|\Delta.A\| \arrow[d, "\pi_{\|A\|}"] \\
& \|\Gamma\| \arrow[rr, "\|s\|"] & & \|\Delta\|,
\end{tikzcd}$$

where $\|A[s]\| = \|s\|^*\|A\|$. Note that the outer diagram commutes as $\|t\|$ is a section of $\pi_{\|A\|}$.

Example 5.3. As an example we consider the case of substitution of one variable given by the following substitution rule:

$$\frac{\Gamma.x : A \vdash t : B \quad \Gamma \vdash a : A}{\Gamma \vdash t[a/x] : B[a/x]}$$

Similar to what we saw in Example 5.2, the corresponding context morphism to this substitution is $\|a\| : \|\Gamma\| \to \|\Gamma.A\|$. The term $t[a/x]$ of type $B[a/x]$ in context Γ is interpreted as the morphism given by the universal property of the following pullback:

$$
\begin{tikzcd}
\|\Gamma\| \arrow[rr, "\|a\|"] \arrow[dr, "\|t[a/x]\|"'] \arrow[ddr, equal] & & \|\Gamma.A\| \arrow[dr, "\|t\|"] & \\
& \|\Gamma.B[a/x]\| \arrow[rr, "\|a\|.\|B\|"] \arrow[d, "\pi_{\|B[a/x]\|}"] \arrow[dr, phantom, "\lrcorner", very near start] & & \|\Gamma.A.B\| \arrow[d, "\pi_{\|B\|}"] \\
& \|\Gamma\| \arrow[rr, "\|a\|"] & & \|\Gamma.A\|.
\end{tikzcd}
$$

5.1.5 Functoriality of Substitution

If p is not a split fibration, the interpretation of substitution defined in Section 5.1.2 is functorial only up to isomorphism. If p is a split fibration, the substitution can be interpreted up to equality. We know from Propositions 3.39 and 3.40 that substitution in Martin-Löf type theory is strictly functorial; hence, to interpret Martin-Löf type theory the comprehension category should be split.

Let $\Gamma, \Gamma', \Gamma'' \in \mathcal{C}$, $s : \Gamma \to \Gamma'$ and $s' : \Gamma' \to \Gamma''$. We know from Remark 3.17 that $(s' \circ s)^* \cong s^* \circ s'^*$, which means that for each $A \in \mathcal{T}_{\Gamma''}$ we have the following isomorphism:

$$
\begin{tikzcd}
s^* s'^* A \arrow[dd, "\cong"'] \arrow[dr, "s_{s'^* A}"] & & & \\
& s'^* A \arrow[dr, "s'_A"] & & \mathcal{T} \arrow[ddd, "p"] \\
(s's)^* A \arrow[rr, "(s's)_A"] & & A & \\
& & & \\
\Gamma \arrow[r, "s"] & \Gamma' \arrow[r, "s'"] & \Gamma'' & \mathcal{C}.
\end{tikzcd}
$$

In the type theory, this corresponds to substitution being associative only up to isomorphism. Similarly for each identity morphism $\mathsf{id}_\Gamma : \Gamma \to \Gamma$ in $\mathcal{C}$, and each $A \in \mathcal{T}_\Gamma$ we have the following isomorphism:

$$
\begin{tikzcd}
A \arrow[d, "\cong"'] \arrow[dr, equal] & & \\
\mathsf{id}_\Gamma^* A \arrow[r, "\mathsf{id}_A^*"'] & A & \mathcal{T} \arrow[d, "p"] \\
\Gamma \arrow[r, "\mathsf{id}_\Gamma"] & \Gamma & \mathcal{C},
\end{tikzcd}
$$

which corresponds to $A[\mathsf{id}_\Gamma/\Gamma] \cong A$ in the type theory.

This means that comprehension categories with a non-split p can model type theories with substitution that is functorial only up to isomorphism, and comprehension categories with a split p can model type theories with strictly functorial substitution. Note that as discussed in Remark 4.13, a fibration $p : \mathcal{T} \to \mathcal{C}$ can be turned into an equivalent split one with the same base.

5.1.6 Weakening

In a comprehension category, for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, the projection $\pi_A : \Gamma.A \to \Gamma$ in $\mathcal{C}$ induces a reindexing functor $\pi_A^* : \mathcal{T}_\Gamma \to \mathcal{T}_{\Gamma.A}$ which corresponds to weakening:

$$\begin{tikzcd} \pi_A^* B \arrow[r, "\pi_{A_B}"] & B & \mathcal{T} \arrow[d, "p"] \\ \Gamma.A \arrow[r, "\pi_A"] & \Gamma & \mathcal{C}. \end{tikzcd}$$

Given a context Γ and a type A in context Γ, the weakening rule states that for a type B in context Γ, B is a type in context $\Gamma.A$.

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma.x : A \vdash B \text{ type}}$$

In the comprehension category, this is interpreted as the action of the reindexing functor $\pi_{\|A\|}^* : \mathcal{T}_{\|\Gamma\|} \to \mathcal{T}_{\|\Gamma.A\|}$ on objects. This means that given $\Gamma \vdash B$ type, which is interpreted as $\|B\| \in \mathcal{T}_{\|\Gamma\|}$, $\|B\|$ is mapped to $\pi_{\|A\|}^* \|B\| \in \mathcal{T}_{\|\Gamma.A\|}$ which corresponds to the judgement $\Gamma.A \vdash B$ type.

For terms the weakening rule is as follows:

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \vdash t : B}{\Gamma.x : A \vdash t : B}$$

Term t of type B in the extended context $\Gamma.A$ is interpreted as the morphism given by the universal property of the following pullback:

$$\begin{tikzcd} \|\Gamma.A\| \arrow[rr, "\pi_{\|A\|}"] \arrow[dr, dashed, "\|t_{B[\pi_A]}\|"] \arrow[ddr, equal, bend right] & & \|\Gamma\| \arrow[dr, dashed, "\|t\|"] & \\ & \|\Gamma.A.B[\pi_A]\| \arrow[rr, "\pi_{\|A\|}.\|B\|"] \arrow[d, "\pi_{\|B[\pi_A]\|}"] \arrow[drr, phantom, "\lrcorner", very near start] & & \|\Gamma.B\| \arrow[d, "\pi_{\|B\|}"] \\ & \|\Gamma.A\| \arrow[rr, "\pi_{\|A\|}"] & & \|\Gamma\|, \end{tikzcd}$$

where $B[\pi_A]$ corresponds to $\pi_{\|A\|}^* \|B\|$.

The reindexing functors of the form π_A^* are called weakening functors. We will see in section Section 5.3 that Π- and Σ-types can be interpreted in a comprehension category as right and left adjoints to the weakening functors.

5.1.7 Contraction

In a comprehension category, for each $\Gamma : \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, a morphism $\delta_A : \Gamma.A \to \Gamma.A.A$ in $\mathcal{C}$ induces a reindexing functor $\delta_A^* : \mathcal{T}_{\Gamma.A.A} \to \mathcal{T}_{\Gamma.A}$ which corresponds to contraction:

$$\begin{tikzcd}
\delta_A^* B \arrow[r, "\delta_{A_B}"] & B & \mathcal{T} \arrow[d, "p"] \\
\Gamma.A \arrow[r, "\delta_A"] & \Gamma.A.A & \mathcal{C}.
\end{tikzcd}$$

Given a context $\Gamma.x : A.y : A$, and a type B in context $\Gamma.x : A.y : A$, the contraction rule states that $B[x/y]$ is a type in context $\Gamma.x : A$:

$$\frac{\Gamma, x : A, y : A \vdash B \ \mathsf{type}}{\Gamma, x : A \vdash B[x/y] \ \mathsf{type}}$$

In the comprehension category, this is interpreted as the action of a reindexing functor $\delta_{\|A\|}^* : \mathcal{T}_{\|\Gamma.A.A\|} \to \mathcal{T}_{\|\Gamma.A\|}$ on objects, where $\delta_{\|A\|}$ is the interpretation of a context morphism of the form $(\mathsf{id}_\Gamma, x : A, y : A)$. This means that $B[x/y]$ is interpreted as $\delta_{\|A\|}^* \|B\|$.

For terms the weakening rule is as follows:

$$\frac{\Gamma, x : A, y : A \vdash t : B}{\Gamma, x : A \vdash t[x/y] : B[x/y]}$$

Term t of type B in the extended context $\Gamma.A$ is interpreted as the morphism given by the universal property of the following pullback:

$$\begin{tikzcd}
\|\Gamma.A\| \arrow[rr, "\delta_{\|A\|}"] \arrow[dr, "\|t[x/y]\|"] \arrow[ddr, equal, bend right] & & \|\Gamma.A.A\| \arrow[dr, "\|t\|"] & \\
& \|\Gamma.A.B[x/y]\| \arrow[rr, "\delta_{\|A\|}.\|B\|"] \arrow[d, "\pi_{\|B[x/y]\|}"] \arrow[drr, phantom, "\lrcorner", very near start] & & \|\Gamma.A.A.B\| \arrow[d, "\pi_{\|B\|}"] \\
& \|\Gamma.A\| \arrow[rr, "\delta_{\|A\|}"] & & \|\Gamma.A.A\|.
\end{tikzcd}$$

5.1.8 Extended Example: $(\mathsf{Set}, \mathsf{Set}^\to, 1, \mathsf{cod})$

We have seen in Example 4.7 that $(\mathsf{Set}, \mathsf{Set}^\to, 1, \mathsf{cod})$ is a full split comprehension category:

$$\begin{tikzcd}
\mathsf{Set}^\to \arrow[rr, "1"] \arrow[dr, "\mathsf{cod}"'] & & \mathsf{Set}^\to \arrow[dl, "\mathsf{cod}"] \\
& \mathsf{Set}. &
\end{tikzcd}$$

By Remark 4.8, we can interpret the judgements and structural rules of a dependent type theory in this comprehension category. The interpretation is as follows.

1. The judgement Γ ctx is interpreted as an object $\|\Gamma\|$ in Set, which means $\|\Gamma\|$ is a set.

2. We know from Lemma 3.15 that for the fibrations $\mathsf{cod} : \mathsf{Set}^{\to} \to \mathsf{Set}$, a fibre over $\Gamma \in \mathsf{Set}$ is the slice category Set/Γ. The judgement $\Gamma \vdash A$ type is interpreted as a morphism $\|A\|$ in Set into $\|\Gamma\|$, which is a function with $\|\Gamma\|$ as its codomain.

3. Context extension is interpreted as 1_0 which is $\mathsf{dom} : \mathsf{Set}^{\to} \to \mathsf{Set}$. For $\Gamma \vdash A$ type, the context Γ extended with type A is interpreted as $\|\Gamma.A\| := \mathsf{dom}(\|A\|)$. This means that $\|A\|$ coincides with the projection $\pi_{\|A\|} : \|\Gamma.A\| \to \|\Gamma\|$.

4. Given $\Gamma \vdash t : A$, the term t is interpreted as $\|t\| : \|\Gamma\| \to \|\Gamma.A\|$ in Set such that $\|A\| \circ \|t\| = 1_{\|\Gamma\|}$. This means that terms of type A are interpreted as right inverses of the function $\|A\| : \|\Gamma.A\| \to \|\Gamma\|$.

5. Given contexts Γ and Δ, a type A in context Δ and a context morphism s from Γ to Δ, the result of applying the substitution to type A, i.e. $A[s]$, is interpreted as the first projection of the pullback of $\|A\|$ along $\|s\|$:
$$\begin{tikzcd} \|\Gamma\| \times_{\|\Delta\|} \|\Delta.A\| \arrow[r] \arrow[d, "\|A[s]\|"'] \arrow[dr, phantom, "\lrcorner", very near start] & \|\Delta.A\| \arrow[d, "\|A\|"] \\ \|\Gamma\| \arrow[r, "\|s\|"'] & \|\Delta\|. \end{tikzcd}$$
This means that $\|\Gamma.A[s]\| := \|\Gamma\| \times_{\|\Delta\|} \|\Delta.A\|$.

6. For a context Γ and a type A in Γ, weakening from context Γ to $\Gamma.A$ is interpreted as substitution along $\pi_{\|A\|}$. Here $\pi_{\|A\|}$ coincides with $\|A\|$; hence, weakening is interpreted as the first projection of the pullback along $\|A\|$:
$$\begin{tikzcd} \|\Gamma.A\| \times_{\|\Gamma\|} \|\Gamma.B\| \arrow[r] \arrow[d, "\pi^*_{\|A\|}\|B\|"'] \arrow[dr, phantom, "\lrcorner", very near start] & \|\Gamma.B\| \arrow[d, "\|B\|"] \\ \|\Gamma.A\| \arrow[r, "\|A\|"'] & \|\Gamma\|. \end{tikzcd}$$

5.1.9 Extended Example: No Type Dependency

In a type theory with no type dependency, extended context $\Gamma.x : A$ can be thought of as the cartesian product $\Gamma \times [x : A]$. This is the motivation behind defining the following comprehension category.

Example 5.4 ([Jac93, Example 4.11]). Let $\mathcal{C}$ be a category with products. A category $\bar{\mathcal{C}}$ is defined with pairs of objects (Γ, A) from $\mathcal{C}$ as objects. Morphisms $(\Gamma, A) \to (\Delta, B)$ in $\bar{\mathcal{C}}$ are given by two maps $u : \Gamma \to \Delta$ and $f : \Gamma \times A \to B$ in $\mathcal{C}$. Composition is described by $(v, g) \circ (u, f) = (v \circ u, g \circ \langle u \circ \pi_1, f\rangle)$ and identities by (id, π_2), where π_1 and π_2 are the projections of the binary product. The first projection $\pi^1 : \bar{\mathcal{C}} \to \mathcal{C}$ is then a split fibration.

For $s : \Gamma \to \Delta$ in $\mathcal{C}$ and $(\Delta, A) : \bar{\mathcal{C}}_\Delta$, the cartesian lift of s into (Δ, A) is given by $(s, 1_A) : (\Gamma, A) \to (\Delta, A)$. For $(s, 1_A)$ to be cartesian, for each $u : \Theta \to \Gamma$ in $\mathcal{C}$ and $(s \circ u : \Theta \to \Delta, f : \Theta \times B \to A) : (\Theta, B) \to (\Delta, A)$ in $\bar{\mathcal{C}}$, there should exist a unique $g : \Theta \times B \to A$ in $\mathcal{C}$ such that $(s \circ u, f) = (s, 1_A) \circ (u, g) = (s \circ u, 1_A \circ \langle u \circ \pi_1, g\rangle)$. The unique g is f.

$$\begin{tikzcd}
(\Theta, B) \arrow[dr, dashed, "(u,g)"'] \arrow[drr, "(s\circ u, f)"] & & & \\
& (\Gamma, s^*A) \arrow[r, "(s, s_A)"'] & (\Delta, A) & \bar{\mathcal{C}} \arrow[dd, "\pi^1"] \\
\Theta \arrow[dr, "u"'] \arrow[drr, "s\circ u"] & & & \\
& \Gamma \arrow[r, "s"'] & \Delta & \mathcal{C}
\end{tikzcd}$$

This fibration gives the following full split comprehension category:

$$\begin{tikzcd}
\bar{\mathcal{C}} \arrow[rr, "\mathcal{P}_{\mathcal{C}}"] \arrow[dr, "\pi^1"'] & & \mathcal{C}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where $\mathcal{P}_{\mathcal{C}} : \bar{\mathcal{C}} \to \mathcal{C}^{\rightarrow}$ maps a pair (Γ, A) to the first projection of the binary product $\pi_1 : \Gamma \times A \to \Gamma$.

Example 5.5. Set has products; hence, by Example 5.4 $(\mathsf{Set}, \overline{\mathsf{Set}}, \mathcal{P}_{\mathsf{Set}}, \pi^1)$ is a full split comprehension category:

$$\begin{tikzcd}
\overline{\mathsf{Set}} \arrow[rr, "\mathcal{P}_{\mathsf{Set}}"] \arrow[dr, "\pi^1"'] & & \mathsf{Set}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\
& \mathsf{Set}. &
\end{tikzcd}$$

Now, we discuss the interpretation of the judgements and structural rules of a type theory with no type dependency in the comprehension category $(\mathsf{Set}, \overline{\mathsf{Set}}, \mathcal{P}_{\mathsf{Set}}, \pi^1)$ from Example 5.5. Here, contexts and types can both be thought of as sets, and the extended context $\Gamma.A$ can be thought of as $\Gamma \times A$. This is in line with there being no type dependency in the type theory.

The interpretation is as follows.

1. The judgement Γ ctx is interpreted as a set $\|\Gamma\|$.

2. The judgement $\Gamma \vdash A$ type is interpreted as a set $\|A\|$, and an object $(\|\Gamma\|, \|A\|)$ in $\overline{\mathsf{Set}}$ over Γ.

3. Context extension is given by $\mathsf{dom} \circ \mathcal{P}_{\mathsf{Set}}$. For $\Gamma \vdash A$ type, the context Γ extended with type A is interpreted as $\|\Gamma.A\| := \|\Gamma\| \times \|A\|$. This is in line with there being no type dependency in the type theory.

4. Given $\Gamma \vdash t : A$, the term t is interpreted as $\|t\| : \|\Gamma\| \to \|\Gamma\| \times \|A\|$ in Set such that $\pi_1 \circ \|t\| = 1_{\|\Gamma\|}$, where π_1 is the first projection of the product. This means that terms of type A are interpreted as right inverses of the first projections of products.

5. Let Γ and Δ be contexts, A a type in context Δ and s a context morphism from Γ to Δ. We know from Example 5.4 that the cartesian lift of $\|s\|$ into $(\|\Delta\|, \|A\|)$ is $(\|s\|, 1_{\|A\|})$. This means that the interpretation $\|A[s]\|$ coincides with $\|A\|$. This is in line with there being no type dependency in the type theory.

6. Weakening from context Γ to $\Gamma.A$ is interpreted as substitution along $\pi_{\|A\|}$. Here $\pi_{\|A\|}$ coincides with π_1; hence, weakening is interpreted as substitution along π_1. Similar to part 5, for a type B in context $\Gamma.A$, the interpretation of this substitution applied to B coincides with $\|B\|$.

5.2 Unit Type

In this section, by unit type, we mean a type inhabited with exactly one term. This corresponds to the first definition of unit type in Section 3.2.3. In Remark 5.18 we discuss how unit type defined as an inductive type (see the second definition in Section 3.2.3) can be interpreted in a comprehension category.

Intuitively, to interpret the unit type of the type theory discussed in Section 4.1 in a full split comprehension category, each fibre $\mathcal{T}_\Gamma$ should have an object $\mathbb{1}_\Gamma$, such that for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, there is a unique morphism $(1_\Gamma, \mathrm{tt}) : \Gamma.A \to \Gamma.\mathbb{1}_\Gamma$.

$$\begin{tikzcd}
\Gamma.A \arrow[r, "(1_\Gamma{,}\mathrm{tt})"] \arrow[d, "\pi_A"'] & \Gamma.\mathbb{1}_\Gamma \arrow[d, "\pi_{\mathbb{1}_\Gamma}"] \\
\Gamma \arrow[r, equal] & \Gamma
\end{tikzcd}$$

From χ being fully faithful, we have a bijection between $\mathcal{C}(\Gamma.A, \Gamma.\mathbb{1})$ and $\mathcal{T}(A, \mathbb{1})$, hence for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, there should be a unique morphism from each A to $\mathbb{1}_\Gamma$, i.e. each fibre should have a terminal object $\mathbb{1}_\Gamma$. In addition, for each $s : \Gamma \to \Delta$, the reindexing functor $s^* : \mathcal{T}_\Delta \to \mathcal{T}_\Gamma$ should preserve terminal objects. Hence, we expect that a type theory with unit types can be interpreted in full a comprehension category with fibrewise terminals [Jac99, Definition 1.8.1].

Definition 5.6 ([Jac99, Definition 1.8.1]). A comprehension category has *fibrewise terminals* if:

1. for each $\Gamma \in \mathcal{C}$, $\mathcal{T}_\Gamma$ has a terminal object;

2. for each $s : \Gamma \to \Delta$ the reindexing functor $s^* : \mathcal{T}_\Delta \to \mathcal{T}_\Gamma$ preserves terminal objects.

Definition 5.7. In a comprehension category with fibrewise terminals, we can define a *terminal object functor*:

$$\begin{aligned} \mathbb{1} : \mathcal{C} &\to \mathcal{T} \\ \Gamma &\mapsto \mathbb{1}_\Gamma, \end{aligned}$$

where $\mathbb{1}_\Gamma$ is a terminal object in the fibre $\mathcal{T}_\Gamma$. $\mathbb{1}$ takes a morphism $s : \Gamma \to \Delta$ in $\mathcal{C}$ to $s_{\mathbb{1}_\Delta} \circ i : \mathbb{1}_\Gamma \to \mathbb{1}_\Delta$ in $\mathcal{T}$:

$$\begin{tikzcd}
\mathbb{1}_\Gamma \arrow[d, "i"', "\cong"] \arrow[dr] & & \\
s^*\mathbb{1}_\Delta \arrow[r, "s_{\mathbb{1}_\Delta}"'] & \mathbb{1}_\Delta & \mathcal{T} \arrow[dd, "p"] \\
& & \\
\Gamma \arrow[r, "s"'] & \Delta & \mathcal{C},
\end{tikzcd}$$

where i is the isomorphism from requirement 2 of Definition 5.6. Now we show that this defines a functor. For each $\Gamma, \Delta \in \mathcal{C}$ and $s : \Gamma \to \Delta$ in $\mathcal{C}$, there is a unique morphism of the form $\mathbb{1}_\Gamma \to \mathbb{1}_\Delta$

above s, since $s_{\mathbb{1}_\Delta}$ is cartesian and $s^*\mathbb{1}_\Delta$ is terminal in $\mathcal{T}_\Gamma$. In particular, for each $\Gamma \in \mathcal{C}$, we have $\mathbb{1}(1_\Gamma) = 1_{\mathbb{1}_\Gamma}$. Additionally, for each $s : \Gamma \to \Gamma'$ and $s' : \Gamma' \to \Gamma''$ in $\mathcal{C}$, we have $\mathbb{1}_{s' \circ s} = \mathbb{1}_{s'} \circ \mathbb{1}_s$ since $\mathbb{1}_{s' \circ s}$ and $\mathbb{1}_{s'} \circ \mathbb{1}_s$ are both of the form $\mathbb{1}_\Gamma \to \mathbb{1}''_\Gamma$ in $\mathcal{T}$ above $s' \circ s$.

Lemma 5.8. *In a comprehension category with fibrewise terminals, $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ is fully faithful.*

Proof. We need to show that for each $\Gamma, \Delta \in \mathcal{C}$, $\mathcal{C}(\Gamma, \Delta) \cong \mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Delta)$. Let $\Gamma, \Delta \in \mathcal{C}$ and $s : \Gamma \to \Delta$. Given $s : \mathcal{C}(\Gamma, \Delta)$ we get $s' : \mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Delta) := s_{\mathbb{1}_\Delta} \circ i$, where i is the isomorphism $i : \mathbb{1}_\Gamma \cong s^*\mathbb{1}_\Delta$ from requirement 2 of Definition 5.6:

$$\begin{tikzcd}
\mathbb{1}_\Gamma \arrow[d, "i"', "\cong"] \arrow[dr, "s'"] & & \\
s^*\mathbb{1}_\Delta \arrow[r, "s_{\mathbb{1}_\Delta}"'] & \mathbb{1}_\Delta & \mathcal{T} \arrow[d, "p"] \\
\Gamma \arrow[r, "s"'] & \Delta & \mathcal{C}.
\end{tikzcd}$$

Given $s : \mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Delta)$ we get $\bar{s} : \mathcal{C}(\Gamma, \Delta) := p(s)$. For each $s : \mathcal{C}(\Gamma, \Delta)$ we have $\overline{(s_{\mathbb{1}_\Delta} \circ i)} = p(s_{\mathbb{1}_\Delta} \circ i) = s$, and for each $s : \mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Delta)$ we have $\bar{s}_{\mathbb{1}_\Delta} \circ i = s$; hence $\mathcal{C}(\Gamma, \Delta) \cong \mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Delta)$ and $\mathbb{1}$ is fully faithful. ☐

Definition 5.9 ([Jac93, Definition 4.12]). A *comprehension category with unit* is given by a fibration $p : \mathcal{T} \to \mathcal{C}$ provided with a terminal object functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$, which has a right adjoint $\chi_0 : \mathcal{T} \to \mathcal{C}$. The comprehension $\chi : \mathcal{T} \to \mathcal{C}^\to$ is then given by $A \mapsto p(\epsilon_A)$ where $\epsilon : \mathbb{1} \circ \chi_0 \Rightarrow \mathsf{id}_\mathcal{T}$ is the counit of the adjunction. See [Jac99, Lemma 1.8.9] for proof that this makes a comprehension category.

Example 5.10. Recall from Example 4.6 that for $\mathcal{C}$ with pullbacks, the identity functor $1 : \mathcal{C}^\to \to \mathcal{C}^\to$ gives a full split comprehension category, the identity comprehension category:

$$\begin{tikzcd}
\mathcal{C}^\to \arrow[rr, "1"] \arrow[dr, "\mathsf{cod}"'] & & \mathcal{C}^\to \arrow[dl, "\mathsf{cod}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

The identity comprehension category has unit. For each $\Gamma \in \mathcal{C}$ and $\alpha \in \mathcal{C}^\to$, we have the bijection $\mathcal{C}^\to(\mathsf{id}_\Gamma, \alpha) \cong \mathcal{C}(\Gamma, \mathsf{dom}(\alpha))$. This corresponds to the adjunction $\begin{tikzcd} \mathcal{C}^\to \arrow[r, shift right, "\mathsf{dom}"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "\mathsf{id}_{(-)}"'] \end{tikzcd}$, where $\mathsf{id}_{(-)}$ maps each $\Gamma \in \mathcal{C}$ to id_Γ.

Lemma 5.11 ([Jac93, Lemma 4.13]). *In a comprehension category with unit,*

1. *for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, we have a bijection between the sections $t_A : \Gamma \to \Gamma.A$ of the projection $\pi_A : \Gamma.A \to \Gamma$ and the morphisms in $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A)$;*

2. *for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $s : \Gamma \to \Delta$, we have a bijection between $s' : \Gamma \to \Delta.A$ in $\mathcal{C}$ such that $\pi_A \circ s' = s$ and morphisms in $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, s^*A)$;*

The first statement indicates that the terms of type A in context Γ in the syntax correspond to (type) morphisms $t : \mathcal{T}_\Gamma(1_\Gamma, A)$ in the comprehension category.

Proof. 1. From the adjunction in Definition 5.9, for each $\Gamma, \Delta \in \mathcal{C}$ and $A \in \mathcal{T}_\Delta$ we have $\mathcal{T}(\mathbb{1}_\Gamma, A) \cong \mathcal{C}(\Gamma, \Delta.A)$. As a special case, we have $\mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Gamma) \cong \mathcal{C}(\Gamma, \Gamma)$ for each $\Gamma \in \mathcal{C}$. Since $\mathbb{1}_\Gamma$ is the terminal object in the fibre $\mathcal{T}_\Gamma$, this means that there is a unique morphism in $\mathcal{C}(\Gamma, \Gamma)$ for each $\Gamma \in \mathcal{C}$, the identity morphism 1_Γ. As a consequence, for $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$ all morphisms in $\mathcal{C}(\Gamma, \Gamma.A)$ are sections of the projection $\pi_A : \Gamma.A \to \Gamma$. From the adjunction we also have $\mathcal{T}(\mathbb{1}_\Gamma, A) \cong \mathcal{C}(\Gamma, \Gamma.A)$ for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$. Since morphisms in $\mathcal{C}(\Gamma, \Gamma.A)$ are all sections of π_A, we get a bijection between sections $t_A : \Gamma \to \Gamma.A$ of the projection π_A and the morphisms in $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A)$.

2. From Lemma 4.11 we have a bijection between such morphisms $s' : \Gamma \to \Delta.A$ and sections of π_{s^*A}. Using part 1 of Lemma 5.11 we get a bijection between such morphisms $s' : \Gamma \to \Delta.A$ in $\mathcal{C}$ and morphisms in $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, s^*A)$.

□

Lemma 5.12. *In a comprehension category with unit, for each* $\Gamma \in \mathcal{C}$, $\pi_{\mathbb{1}_\Gamma}$ *has a unique section* $\mathrm{tt}_\Gamma : \Gamma \to \Gamma.\mathbb{1}_\Gamma$.

Proof. From part 1 of Lemma 5.11, we have that for each $\Gamma \in \mathcal{C}$, there is a bijection between sections of $\pi_{\mathbb{1}_\Gamma}$ and $\mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Gamma)$. $\mathbb{1}_\Gamma$ is the terminal object of the fibre $\mathcal{T}_\Gamma$; hence, $\pi_{\mathbb{1}_\Gamma}$ has a unique section. □

Remark 5.13. Lemma 5.12 justifies why a type theory with unit type can be interpreted in a comprehension category with unit.

Lemma 5.14. *In a comprehension category with unit, for each* $\Gamma \in \mathcal{C}$ *we have* $\Gamma \cong \Gamma.\mathbb{1}_\Gamma$.

Proof. From Lemma 5.8 we get $\mathcal{C}(\Gamma.\mathbb{1}_\Gamma, \Gamma.\mathbb{1}_\Gamma) \cong \mathcal{T}(\mathbb{1}_{\Gamma.\mathbb{1}_\Gamma}, \mathbb{1}_{\Gamma.\mathbb{1}_\Gamma})$. $\mathbb{1}_{\Gamma.\mathbb{1}_\Gamma}$ is terminal in $\mathcal{T}_{\Gamma.\mathbb{1}_\Gamma}$; hence, there is a unique morphism in $\mathcal{C}(\Gamma.\mathbb{1}_\Gamma, \Gamma.\mathbb{1}_\Gamma)$, the identity morphism $1_{\Gamma.\mathbb{1}_\Gamma}$. From Lemma 5.12, we know that $\pi_{\mathbb{1}_\Gamma}$ has a unique section $\mathrm{tt}_\Gamma : \Gamma \to \Gamma.\mathbb{1}_\Gamma$. We have $\mathrm{tt} \circ \pi_{\mathbb{1}_\Gamma} = 1_{\Gamma.\mathbb{1}_\Gamma}$ since there is a unique morphism in $\mathcal{C}(\Gamma.\mathbb{1}_\Gamma, \Gamma.\mathbb{1}_\Gamma)$. Hence, $\mathrm{tt}_\Gamma : \Gamma \cong \Gamma.\mathbb{1}_\Gamma$. □

Remark 5.15. As a result of Lemma 5.14, in a full comprehension category with unit the singleton context $[\mathrm{tt} : \mathbb{1}]$ is isomorphic to the terminal empty context $\diamond$.

5.2.1 Extended Example : Syntactic Category

In case the construction explained in Section 4.1 is applied to a type theory that has a unit type, the resulting comprehension category has unit. In a full comprehension category constructed from a type theory with unit type $\Gamma \vdash \mathbb{1}_\Gamma$ type that has a unique term $\Gamma \vdash \mathrm{tt} : \mathbb{1}_\Gamma$, we can define a functor

$$\begin{aligned} \mathbb{1} : \mathcal{C} &\to \mathcal{T} \\ \|\Gamma\| &\mapsto \|\mathbb{1}_\Gamma\|. \end{aligned}$$

To show that the full comprehension category has unit, we need to show that:

1. $\mathcal{T}$ has fibrewise terminals and $\mathbb{1}$ is a terminal object functor;

2. we have the adjunction $\mathcal{T} \underset{\chi_0}{\overset{\mathbb{1}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$.

Each context morphism $s : \Gamma.A \to \Gamma.\mathbb{1}_\Gamma$ in the syntax is of the form $(\mathsf{id}_\Gamma, \mathsf{tt}_\Gamma)$ where tt_Γ is the unique term of $\mathbb{1}_\Gamma$; hence, for each context Γ and type A in context Γ there is a unique context morphism $s : \Gamma.A \to \Gamma.\mathbb{1}_\Gamma$. This means that in the comprehension category, there is a unique morphism in $\mathcal{C}(\|\Gamma.A\|, \|\Gamma.\mathbb{1}_\Gamma\|)$. From χ being fully faithful we have $\mathcal{C}(\|\Gamma.A\|, \|\Gamma.\mathbb{1}_\Gamma\|) \cong \mathcal{T}(\|A\|, \|\mathbb{1}_\Gamma\|)$; hence, $\|\mathbb{1}_\Gamma\|$ is a terminal object in the fibre $\mathcal{T}_\Gamma$.

For each context morphism $s : \Gamma \to \Delta$, there is a unique extension of s to $(s, \mathsf{tt}_\Delta) : \Gamma \to \Delta.\mathbb{1}_\Delta$ in the syntax. This means that in the comprehension category, for each $\|s\| : \|\Gamma\| \to \|\Delta\|$ in $\mathcal{C}$ we have a unique $s' : \|\Gamma\| \to \|\Delta\|.\|\mathbb{1}_\Delta\|$ such that $\pi_{\|\mathbb{1}_\Delta\|} \circ s' = \|s\|$. From Lemma 4.9, we know that there is a bijection between such $s' : \|\Gamma\| \to \|\Delta\|.\|\mathbb{1}_\Delta\|$ and sections $t_{\|s\|^*\|1_\Delta\|}$ of $\pi_{\|s\|^*\|1_\Delta\|}$; hence, there is a unique section of $\pi_{\|s\|^*\|1_\Delta\|}$ in $\mathcal{C}$.

From the syntax we know that for each type A in context Γ, there is a bijection between context morphisms of the form $\Gamma.A \to \Gamma.B$ and context morphisms of the form $\Gamma \to \Gamma.B$; hence, we have the bijection $\mathcal{C}(\|\Gamma.A\|, \|\Gamma.s\|^*\|\mathbb{1}_\Delta\|) \cong \mathcal{C}(\|\Gamma\|, \|\Gamma.s\|^*\|\mathbb{1}_\Delta\|)$. This means that there is a unique morphism in $\mathcal{C}(\|\Gamma.A\|, \|\Gamma.s\|^*\|\mathbb{1}_\Delta\|)$. Using χ being fully faithful, we get that there is a unique morphism in $\mathcal{T}(\|A\|, \|s\|^*\|\mathbb{1}_\Delta\|)$. This means that $\|\mathbb{1}_\Delta\|$ is a terminal object in $\mathcal{T}_\Gamma$ and that reindexing preserves terminal objects. This concludes showing $\mathcal{T}$ has fibrewise terminals and $\mathbb{1}$ is a terminal object functor.

Let Γ, Δ ctx, $\Delta \vdash A$ type. From the syntax we know that for each type A in Γ and type B in context Δ, there is a bijection between context morphisms of the form $\Gamma.A \to \Gamma.B$ and context morphisms of the form $\Gamma \to \Gamma.B$; hence, we have the bijection $\mathcal{C}(\|\Gamma\|, \|\Delta\|.\|A\|) \cong \mathcal{C}(\|\Gamma.\mathbb{1}_\Gamma\|, \|\Delta\|.\|A\|)$. From χ being fully faithful we have $\mathcal{C}(\|\Gamma.\mathbb{1}_\Gamma\|, \|\Delta\|.\|A\|) \cong \mathcal{T}(\|\mathbb{1}_\Gamma\|, \|A\|)$; hence, $\mathcal{T}(\|\mathbb{1}_\Gamma\|, \|A\|) \cong \mathcal{C}(\|\Gamma\|, \|\Delta\|.\|A\|)$. For naturality in $\|\Gamma\|$ and $\|A\|$, we need to show that for each contexts $\Gamma_1, \Gamma_2, \Delta_1$ and Δ_2, types A_1 in Δ_1 and A_2 in Δ_2, context morphism $g : \Gamma_2 \to \Gamma_1$ and a pair of context morphisms $h = (h_1 : \Delta_1 \to \Delta_2, h_2 : \Delta_1.A_1 \to \Delta_2.A_2)$ such that $\pi_{A_2} \circ h_2 = h_1 \circ \pi_{A_1}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T}(\|\mathbb{1}_{\Gamma_1}\|, \|A_1\|) \arrow[r, "\cong"] \arrow[d, "\mathsf{hom}(\mathbb{1}(\|g\|), \|h\|)"'] & \mathcal{C}(\|\Gamma_1\|, \|\Delta_1.A_1\|) \arrow[d, "\mathsf{hom}(\|g\|, \chi_0\|h\|)"] \\
\mathcal{T}(\|\mathbb{1}_{\Gamma_2}\|, \|A_2\|) \arrow[r, "\cong"] & \mathcal{C}(\|\Gamma_2\|, \|\Delta_2.A_2\|).
\end{tikzcd}$$

Hence, we have the adjunction $\mathcal{T} \underset{\chi_0}{\overset{\mathbb{1}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$.

This concludes the extended example.

Example 5.16. Recall from Example 4.7 that the comprehension category $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ is a special case of the identity comprehension category:

$$\begin{tikzcd}
\mathsf{Set}^{\to} \arrow[rr, "1"] \arrow[dr, "\mathsf{cod}"'] & & \mathsf{Set}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathsf{Set}. &
\end{tikzcd}$$

By the same reasoning as in Example 5.10, the comprehension category $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ has unit. The adjunction corresponding to unit is $\mathsf{Set}^{\to} \underset{\mathsf{dom}}{\overset{\mathsf{id}_{(-)}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Set}$.

Continuing the interpretation given in Section 5.1.8, we discuss the interpretation of a type theory with unit type in the comprehension category $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$. From the adjunction, we know that the interpretation of unit type $\mathbb{1}$ in context Γ is the identity morphism 1_Γ is Set, i.e. $\|\mathbb{1}_\Gamma\| := 1_{\|\Gamma\|}$. This means that the interpretation of the extended context $\Gamma.\mathbb{1}$ coincides with the interpretation of Γ. As a special case, the interpretation of the context $[\mathsf{tt} : \mathbb{1}]$ coincides with the interpretation of the empty context. Since $\|\Gamma.\mathbb{1}_\Gamma\|$ and $\|\Gamma\|$ coincide for each context Γ, we have $\|\mathsf{tt}_\Gamma\| = 1_{\|\Gamma\|} = \|\mathbb{1}_\Gamma\|$.

Example 5.17. The comprehension category discussed in Example 5.4 has unit if $\mathcal{C}$ has a terminal object denoted as $1_\mathcal{C}$:

$$\begin{tikzcd}
\bar{\mathcal{C}} \arrow[rr, "\mathcal{P}_\mathcal{C}"] \arrow[dr, "\pi^1"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

For each Γ in $\mathcal{C}$ and (Δ, A) in $\bar{\mathcal{C}}$ we have the bijection $\bar{\mathcal{C}}((\Gamma, 1_\mathcal{C}), (\Delta, A)) \cong \mathcal{C}(\Gamma, \Delta \times A)$ as $\Gamma \times 1_\mathcal{C} = \Gamma$ which is natural in Γ and (Δ, A). This corresponds to the adjunction $\bar{\mathcal{C}}\ \underset{\mathsf{dom}\circ\mathcal{P}_{\mathcal{C}}}{\overset{\mathbb{1}_{\mathcal{C}}}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}}}\ \mathcal{C}$, where $\mathbb{1}_\mathcal{C}$ sends each Γ in $\mathcal{C}$ to $(\Gamma, 1_\mathcal{C})$.

Set has a terminal object; hence, the following comprehension category discussed has unit:

$$\begin{tikzcd}
\bar{\mathsf{Set}} \arrow[rr, "\mathcal{P}_\mathsf{Set}"] \arrow[dr, "\pi^1"'] & & \mathsf{Set}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathsf{Set}. &
\end{tikzcd}$$

with $\overline{\mathsf{Set}}\ \underset{\mathsf{dom}\circ\mathcal{P}_{\mathsf{Set}}}{\overset{\mathbb{1}_{\mathsf{Set}}}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}}}\ \mathsf{Set}$, as the adjunction, where $\mathbb{1}_\mathsf{Set}$ sends each set Γ to $(\Gamma, \{*\})$. Continuing the interpretation given in Section 5.1.9, we now discuss the interpretation of unit type in a type theory with no type dependency in this category. From the adjunction, we know that the interpretation of the unit type in the singleton set, i.e. $\|\mathbb{1}\| := \{*\}$. For each context Γ, the interpretation of the extended context $\Gamma.\mathbb{1}_\Gamma$ is $\|\Gamma\| \times \{*\}$ which is $\|\Gamma\|$. As a special case, the interpretation of the context $[\mathsf{tt} : \mathbb{1}]$ coincides with the interpretation of the empty context. Since $\|\Gamma.\mathbb{1}_\Gamma\|$ and $\|\Gamma\|$ coincide for each context Γ, we have $\|\mathsf{tt}_\Gamma\| = 1_{\|\Gamma\|}$.

Remark 5.18. We have now seen how unit type defined as a type with a unique term can be interpreted in a comprehension category. We know from Section 3.2.3, that in a type theory, unit type can also be defined as an inductive type with formation, introduction, elimination and computations rules.

A type theory with unit type defined like this can be interpreted in a full comprehension category where for each $\Gamma \in \mathcal{C}$, we have:

1. a type $\mathbb{1}_\Gamma \in \mathcal{T}_\Gamma$;

2. a section $\mathsf{tt}_\Gamma : \Gamma \to \Gamma.\mathbb{1}_\Gamma$;

3. for any type $A \in \mathcal{T}_{\Gamma.\mathbb{1}_\Gamma}$ and section a of $\pi_{A[\mathsf{tt}]}$, a section $\mathsf{urec}_{A,a}$ of π_A, such that $\mathsf{urec}_{A,a} \circ \mathsf{tt}_\Gamma = \mathsf{tt}_\Gamma.A \circ a$:

$$\begin{tikzcd}
\Gamma \arrow[rr, "\mathsf{tt}_\Gamma"] \arrow[dr, "a"'] \arrow[ddr, equal, bend right] & & \Gamma.\mathbb{1}_\Gamma \arrow[dr, "\mathsf{urec}_{A,a}"] & \\
& \Gamma.A[\mathsf{tt}_\Gamma] \arrow[rr, "\mathsf{tt}_\Gamma.A"] \arrow[d, "\pi_{A[\mathsf{tt}_\Gamma]}"] \arrow[drr, phantom, "\lrcorner", very near start] & & \Gamma.\mathbb{1}_\Gamma.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[rr, "\mathsf{tt}_\Gamma"] & & \Gamma.\mathbb{1}_\Gamma.
\end{tikzcd}$$

This is the corrected version of the definition of a unit type from [LW15, Definition 3.4.4.5]. There was a mistake in the original paper, which was confirmed over email by the first author: In the paper, the third requirement is stated as $\mathsf{urec}_{A,a} \circ \mathsf{tt}_\Gamma = a$.

5.3 Π- and Σ- Types

Dependent product and sum types can be expressed as certain fibred adjunctions or equivalently as fibrewise adjunctions plus an additional condition on the fibred adjunctions being compatible with reindexing: the "Beck-Chevalley" condition. Here, we focus on the latter approach. The equivalent definition of a comprehension category with products and sums using fibred adjunctions is stated in [Jac93, Section 5].

Dependent product types correspond to right adjoints to weakening functors, and dependent sum types correspond to left adjoints to weakening functors. In the following definition, this is expressed using fibrewise adjunctions plus a Beck-Chevalley condition, which states that the adjunctions commute with reindexing. Recall from Section 5.1.6 that weakening functors are reindexing functors of form π_A^* where $A \in \mathcal{T}$.

Definition 5.19 ([Jac93, Section 5]). A comprehension category has *products (sums)* if

1. for every $A \in \mathcal{T}_\Gamma$, every weakening functor $\pi_A^* : \mathcal{T}_\Gamma \to \mathcal{T}_{\Gamma.A}$ has a right adjoint $\prod_A : \mathcal{T}_{\Gamma.A} \to \mathcal{T}_\Gamma$ (left adjoint $\sum_A$):

$$\begin{tikzcd}
\mathcal{T}_{\Gamma.A} \arrow[r, shift right=1.5ex, "\prod_A"'] \arrow[r, phantom, "\perp"] & \mathcal{T}_\Gamma \arrow[l, shift right=1.5ex, "\pi_A^*"']
\end{tikzcd}
\qquad
\left(\begin{tikzcd}
\mathcal{T}_\Gamma \arrow[r, shift right=1.5ex, "\pi_A^*"'] \arrow[r, phantom, "\perp"] & \mathcal{T}_{\Gamma.A} \arrow[l, shift right=1.5ex, "\sum_A"']
\end{tikzcd}\right);$$

2. the Beck-Chevalley condition holds, which means that for each $s : \Gamma \to \Delta$ in $\mathcal{C}$ and $A \in \mathcal{T}_\Delta$, the natural transformation $s^* \prod_A \Rightarrow \prod_{s^*A}(s.A)^*$ ($\sum_{s^*A}(s.A)^* \Rightarrow s^* \sum_A$) is an isomorphisms. Note that $s.A : \Gamma.s^*A \to \Delta.A$ is $\chi_0 s_A$.

Remark 5.20. The natural transformation $s^* \prod_A \Rightarrow \prod_{s^*A}(s.A)^*$ mentioned in Definition 5.19 comes from:

$$\begin{tikzcd}
& \mathcal{T}_{\Delta.A} \arrow[r, "(s.A)^*"] & \mathcal{T}_{\Gamma.s^*A} \arrow[r, "\prod_{s^*A}"] & \mathcal{T}_\Gamma \\
\mathcal{T}_{\Delta.A} \arrow[ur, equal] \arrow[r, "\prod_A"'] \arrow[ur, phantom, shift left=4ex, "\Uparrow\epsilon"] & \mathcal{T}_\Delta \arrow[u, "\pi_A^*"] \arrow[r, "s^*"'] & \mathcal{T}_\Gamma \arrow[u, "\pi_{s^*A}^*"'] \arrow[ul, Rightarrow, "\cong"] \arrow[ur, equal] \arrow[ur, phantom, shift left=4ex, "\Uparrow\eta"] &
\end{tikzcd}$$

where η and ϵ are the unit and counit of the corresponding adjunctions, and the middle isomorphism is weakening commuting with substitution up to isomorphism, i.e. $\pi_{s^*A}s^* \cong (s.A)^*\pi_A^*$. From Remark 3.17, we have $(s.A)^* \circ \pi_A^* \cong (\pi_A \circ s_A)^*$, and we get the isomorphism using $\pi_A \circ s.A = s \circ \pi_{s^*A}$ from the following commuting square corresponding to χs_A:

$$\begin{tikzcd}
\Gamma.s^*A \arrow[r, "s.A"] \arrow[d, "\pi_{s^*A}"'] & \Delta.A \arrow[d, "\pi_A"] \\
\Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

5.3.1 Interpretation of $\prod$-Types

Type theoretic dependent product types are interpreted in a comprehension category with products by interpreting the formation, introduction, elimination, computation and $\eta-$reduction rules. The formation rule is interpreted as the action of $\prod_A$ on the objects of $\mathcal{T}_{\Gamma.A}$ for each $A \in \mathcal{T}_\Gamma$. Each $B \in \mathcal{T}_{\Gamma.A}$, which corresponds to the judgement $\Gamma.x : A \vdash B$ type, can be seen as a type B that contains a variable of type A.

To be able to interpret the other four rules, it suffices to have a bijection between the terms f of type $\prod_A B$ and terms b of type B in context $\Gamma.A$, where a is a term of type A in context Γ. Hence, in a comprehension category with products we want to have a bijection between pairs of sections f of $\pi_{\prod_A B} : \Gamma.\prod_A B \to \Gamma$ and sections b of $\pi_B : \Gamma.B \to \Gamma$:

$$\frac{\text{section } f \text{ of } \pi_{\prod_A B}}{\text{section } b \text{ of } \pi_B}$$

First, we obtain a map from sections of $\pi_{\prod_A B}$ to sections of π_B using Lemma 4.9 and the adjunction. Given a section f of $\pi_{\prod_A B}$, we can obtain a section $t_{\pi_A^*(\prod_A B)}$ of $\pi_{\pi_A^*(\prod_A B)}$ using Lemma 4.9. For $A \in \mathcal{T}_\Gamma$, $B \in \mathcal{T}_{\Gamma.A}$ and each section $f : \Gamma \to \Gamma.\prod_A B$ of $\pi_{\prod_A B}$, we get a unique section of $\pi_{\pi_A^*(\prod_A B)}$ from the following diagram, where we have the pullback square from Lemma 4.9 and the outer diagram commutes as f is a section of $\pi_{\prod_A B}$:

$$\begin{tikzcd}
\Gamma.A \arrow[rr, "\pi_A"] \arrow[dr, "t_{\pi_A^*(\prod_A B)}"] \arrow[ddr, bend right, equal] & & \Gamma \arrow[dr, "f"] & \\
& \Gamma.A.\pi_A^*(\prod_A B) \arrow[rr] \arrow[d, "\pi_{\pi_A^*(\prod_A B)}"] \arrow[drr, phantom, "\lrcorner", very near start] & & \Gamma.\prod_A B \arrow[d, "\pi_{\prod_A B}"] \\
& \Gamma.A \arrow[rr, "\pi_A"'] & & \Gamma.
\end{tikzcd}$$

The desired map maps f to $(\chi_0\epsilon_B) \circ t$, where ϵ is the counit of the adjunction, $\epsilon_B : \pi_A^*(\prod_A B) \to B$ and $\chi_0\epsilon_B : \Gamma.A.\pi_A^*(\prod_A B) \to \Gamma.A.B$.

This map, together with the substitution rule, is reflected in the type theory as the elimination rule of the dependent product type, i.e. the application of a term $f : \prod_A B$ to a variable of type A. To show that this map is an isomorphism, however, doesn't come for free in comprehension categories with product. In the following lemma, we discuss that this map is an isomorphism if and only if χ preserves products, which is if and only if we have $\mathcal{C}/\Delta(s, \pi_{\prod_A B}) \cong \mathcal{C}/\Delta.A(s.A, \pi_B)$ for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $B \in \mathcal{T}_{\Delta.A}$ and $s : \Gamma \to \Delta$. This means that the type-theoretic dependent product type can be interpreted in a comprehension category with products where χ preserves products. We also show that this condition holds in a comprehension category with unit.

Definition 5.21 ([Jac93, Section 5.1]). In a comprehension category, χ *preserves products* if and only if

$$\mathcal{C}/\Delta(s, \pi_{\prod_A B}) \cong \mathcal{C}/\Delta.A(s.A, \pi_B)$$

for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $B \in \mathcal{T}_{\Delta.A}$ and $s : \Gamma \to \Delta$. This means that χ preserves products if and only if there is a bijection between commuting diagrams of the form

$$\begin{tikzcd} \Gamma \arrow[rr] \arrow[dr, "s"'] & & \Delta.\prod_A B \arrow[dl, "\pi_{\prod_A B}"] \\ & \Delta & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Gamma.A \arrow[rr] \arrow[dr, "s.A"'] & & \Delta.A.B \arrow[dl, "\pi_B"] \\ & \Delta.A & \end{tikzcd}$$

in $\mathcal{C}$.

Lemma 5.22 ([Jac93, Lemma 5.2]). *In a comprehension category with products, the following two statements are equivalent for each $A \in \mathcal{T}_\Gamma$ and $B \in \mathcal{T}_{\Gamma.A}$:*

1. *there is a bijection between sections of $\pi_{\prod_A B}$ and sections of π_B;*
2. *χ preserves products.*

Proof. Let $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $B \in \mathcal{T}_{\Delta.A}$ and $s : \Gamma \to \Delta$. For brevity, we write $\{\text{sections of } \pi_A\}$ for $\{t_A : \Gamma \to \Gamma.A | \pi_A \circ t_A = 1_\Gamma\}$. First we show that $\{\text{sections of } \pi_{\prod_A B}\} \cong \{\text{sections of } \pi_B\}$ then $\mathcal{C}/\Delta(s, \pi_{\prod_A B}) \cong \mathcal{C}/\Delta.A(s.A, \pi_B)$:

$$\begin{aligned} \mathcal{C}/\Delta(s, \pi_{\prod_A B}) &\cong \mathcal{C}/\Gamma(\mathsf{id}_\Gamma, \pi_{s^*(\prod_A B)}) && \text{(by Lemma 4.11)} \\ &= \{\text{sections of } \pi_{s^* \prod_A B}\} \\ &\cong \{\text{sections of } \pi_{\prod_{s^*A}((s.A)^*B)}\} && \text{(by Beck-Chevalley)} \\ &\cong \{\text{sections of } \pi_{(s.A)^*B}\} && \text{(by assumption)} \\ &= \mathcal{C}/\Gamma.A(\mathsf{id}_{\Gamma.A}, \pi_{(s.A)^*B}) \\ &\cong \mathcal{C}/\Delta.A(s.A, \pi_B). && \text{(by Lemma 4.11)} \end{aligned}$$

Conversely, if $\mathcal{C}/\Delta(s, \pi_{\prod_A B}) \cong \mathcal{C}/\Delta.A(s.A, \pi_B)$ then $\{\text{sections of } \pi_{\prod_A B}\} \cong \{\text{sections of } \pi_B\}$:

$$\begin{aligned} \{\text{sections of } \pi_{\prod_A B}\} &= \mathcal{C}/\Delta(\mathsf{id}_\Delta, \pi_{\prod_A B}) \\ &\cong \mathcal{C}/\Delta.A(\mathsf{id}_\Delta.A, \pi_B) && \text{(by assumption)} \\ &\cong \mathcal{C}/\Delta.A(\mathsf{id}_\Delta, \pi_B) && \text{(by Remark 3.17)} \\ &= \{\text{sections of } \pi_B\}. \end{aligned}$$

□

Lemma 5.23 ([Jac93, Lemma 5.3]). *A comprehension category with unit preserves products.*

Proof. Let $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$ and $B \in \mathcal{T}_{\Delta.A}$ and $s : \Gamma \to \Delta$. For brevity, we write $\{\text{sections of } \pi_A\}$ for $\{t_A : \Gamma \to \Gamma.A | \pi_A \circ t_A = 1_\Gamma\}$. Using Lemma 5.22, we need to show that $\{\text{sections of } \pi_{\prod_A B}\} \cong \{\text{sections of } \pi_B\}$:

$$\begin{aligned}
\{\text{sections of } \pi_{\prod_A B}\} &\cong \mathcal{T}_\Gamma(\mathbb{1}_\Gamma, \prod_A B) && \text{(by part 1 of Lemma 5.11)}\\
&\cong \mathcal{T}_{\Gamma.A}(\pi_A^* \mathbb{1}_\Gamma, B) && \text{(by the adjunction in Definition 5.19)}\\
&\cong \mathcal{T}_{\Gamma.A}(\mathbb{1}_{\Gamma.A}, B) && \text{(by requirement 2 of Definition 5.6)}\\
&\cong \{\text{sections of } \pi_B\}. && \text{(by part 1 of Lemma 5.11)}
\end{aligned}$$

□

5.3.2 Interpretation of Σ-Types

Similar to the Π-types, the interpretation of the formation rule of Σ-types for $\Gamma \vdash A$ type and $\Gamma.x : A \vdash B$ type is the action of $\sum_{\|A\|} \|B\|$, left adjoint to $\pi^*_{\|A\|}$, on the objects of $\mathcal{T}_{\|\Gamma.A\|}$. To be able to interpret the introduction, elimination and computation rules, it suffices to have the following bijection in a comprehension category with sums:

$$\{\text{sections of } \pi_{\sum_{\|A\|} \|B\|}\} \cong \{(a, b) | a \text{ is a section of } \pi_{\|A\|} \text{ and } b \text{ is a section of } \pi_{\|B\|[a]}\}$$

We get this bijection if the contexts $\|\Gamma.A\|.\|B\|$ and $\|\Gamma\|.\sum_{\|A\|} \|B\|$ are isomorphic. This means that we can interpret Σ-types in a comprehension category with sums, if for each $\Gamma \in \mathcal{C}$, $A \in \mathcal{T}_\Gamma$ and $B \in \mathcal{T}_{\Gamma.A}$, $\chi_0(\pi_{A_{\sum_A B}} \circ \eta_B) : \Gamma.A.B \to \Gamma.\sum_A B$ is an isomorphism, where $\pi_{A_{\sum_A B}}$ is the lift of π_A into $\sum_A B$ and η is the unit of the adjunction. Hence, we have the following definition of a comprehension category with strong sums:

Definition 5.24 ([Jac93, Definition 5.8]). A comprehension category has *strong sums* if it has sums in such a way that for each $\Gamma \in \mathcal{C}$, $A \in \mathcal{T}_\Gamma$ and $B \in \mathcal{T}_{\Gamma.A}$, the following map is an isomorphism:

$$\chi_0(\pi_{A_{\sum_A B}} \circ \eta_B) : \Gamma.A.B \to \Gamma.\sum_A B.$$

Σ-types are interpreted in a comprehension category with strong sums.

Example 5.25 ([Jac93, Example 5.14 (iv)]). The construction discussed in Section 4.1 applied to a type theory with unit type, Π-types and Σ-types yields a full comprehension category with unit, product and strong sum.

5.3.3 Interpretation of Weak Σ-Types

For Σ-types with a weak elimination rule, as discussed in Section 3.2.5, we require the following bijection in a comprehension category with sums for each $\Gamma \in \mathcal{C}$, $A, C \in \mathcal{T}_\Gamma$ and $B \in \mathcal{T}_{\Gamma.A}$:

$$\{\text{sections of } \pi_{\pi_B^*(\pi_A^* C)}\} \cong \{\text{sections of } \pi_{\pi^*_{(\sum_A B)} C}\},$$

which corresponds to the following bijection in the type theory:

$$\frac{\Gamma.z : \sum_{x:A} B \vdash t : C}{\Gamma.x : A.y : B \vdash t' : C}$$

Lemma 5.26 ([Jac93, Section 5.5]). *In a full comprehension category with sums, we have the following bijection:*

$$\{\text{sections of } \pi_{\pi_B^*(\pi_A^* C)}\} \cong \{\text{sections of } \pi_{\pi^*_{(\Sigma_A B)} C}\}.$$

Proof.

$$\begin{aligned}
\{\text{sections of } \pi_{\pi_B^*(\pi_A^* C)}\} &\cong \mathcal{T}_\Gamma(\textstyle\sum_A B, C) && \text{by Lemma 4.12} \\
&\cong \mathcal{T}_{\Gamma.A}(B, \pi_A^* C) && \text{(by the adjunction} \\
& && \text{in Definition 5.19)} \\
&\cong \{\text{sections of } \pi_{\pi^*_{(\Sigma_A B)} C}\} && \text{(by Lemma 4.12}
\end{aligned}$$

□

Hence, weak Σ-types can be interpreted in a full comprehension category with sums.

5.3.4 Extended Example : $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$

We saw in Section 5.1.8 and Example 5.16 that $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ is a full split comprehension category with unit. We also saw how a type theory with unit type is interpreted in this comprehension category. Now, we discuss whether $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ has strong sums (products), and whether a type theory with Σ-types (Π- types) can be interpreted in this comprehension category.

Proposition 5.27 ([Jac93, Examples 5.14(i)]). *If $\mathcal{C}$ has finite limits, the identity comprehension category from Example 4.6 is full with unit and strong sums.*

Set has finite limits; hence, by Proposition 5.27 the comprehension category $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ is full with unit and strong sums. This means that we can interpret a type theory with unit and Σ-types in $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ as explained in Section 5.3.2. We have already seen the interpretation of unit type in this comprehension category in Example 5.16. Now we discuss Σ-types.

To interpret Σ-types in this comprehension category we need an adjunction and a Beck-Chevalley condition as explained in Definition 5.19. We also need the isomorphism corresponding to strong sums given in Definition 5.24. For the adjunction we need a left adjoint to the pullback functor π_A^*. A candidate for this left adjoint is $\pi_A \circ -$:

$$\begin{tikzcd} \mathsf{Set}/\Gamma \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "\pi_A^*"'] & \mathsf{Set}/\Gamma.A. \arrow[l, shift right=1.5ex, "\pi_A \circ -"'] \end{tikzcd}$$

To show that this is an adjunction, we need to show the following bijection for each $\pi_A, \pi_C \in \mathsf{Set}/\Gamma$ and $\pi_B \in \mathsf{Set}/\Gamma.A$:

$$\mathsf{Set}/\Gamma(\pi_A \circ \pi_B, \pi_C) \cong \mathsf{Set}/\Gamma.A(\pi_B, \pi_{\Gamma.A \times_\Gamma \Gamma.C}),$$

where $\pi_{\Gamma.A \times_\Gamma \Gamma.C}$ is the first projection of the pullback, and that this is natural in π_B and π_C. The construction is as follows.

1. For each $f \in \mathsf{Set}/\Gamma(\pi_A \circ \pi_B, \pi_C)$ we can get a $f' \in \mathsf{Set}/\Gamma.A(\pi_B, \pi_{\Gamma.A \times_\Gamma \Gamma.C})$ by $b \mapsto (\pi_B b, f b)$ for each $b \in \Gamma.A.B$. This works because $\pi_A(\pi_B(b)) = \pi_c(f(b))$ from the definition of f.

2. For each $f \in \mathsf{Set}/\Gamma.A(\pi_B, \pi_{\Gamma.A\times_\Gamma\Gamma.C})$ we can get a $\bar{f} \in \mathsf{Set}/\Gamma(\pi_A \circ \pi_B, \pi_C)$ by $\bar{f} := \pi_2 \circ f$. This works because from the definition of f we have $\pi_1 \circ f = \pi_B$; hence, $\pi_A \circ \pi_1 \circ f = \pi_A \circ \pi_B$, which gives $\pi_C \circ \pi_2 \circ f = \pi_A \circ \pi_B$ using the commutativity of the following pullback square:

$$\begin{tikzcd} \Gamma.A \times_\Gamma \Gamma.C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.C \arrow[d, "\pi_c"] \\ \Gamma.A \arrow[r, "\pi_A"'] & \Gamma. \end{tikzcd}$$

3. Now we show that $\overline{f'} = f$ for each $f \in \mathsf{Set}/\Gamma(\pi_A \circ \pi_B, \pi_C)$. $\overline{f'}$ sends $b \in \Gamma.A.B$ to $\pi_2(\pi_B b, fb)$ which is fb; hence, $\overline{f'} = f$ by functional extensionality.

4. We also show that $(\bar{f})' = f$ for each $f \in \mathsf{Set}/\Gamma.A(\pi_B, \pi_{\Gamma.A\times_\Gamma\Gamma.C})$. $(\bar{f})' = f$ sends $b \in \Gamma.A.B$ to $(\pi_B b, \pi_2(fb)$. We know from the definition of f that $\pi_B b = \pi_1(fb)$; hence, $(\pi_B b, \pi_2(fb) = (\pi_1(fb), \pi_2(fb)) = fb$ and $(\bar{f})' = f$ by functional extensionality.

For naturality, we need to show that for each $g : \mathsf{Set}/\Gamma.A(\pi_{B_2}, \pi_{B_1})$ and $h : \mathsf{Set}_\Gamma(\pi_{C_1} \to \pi_{C_2})$, we have the following commuting square:

$$\begin{tikzcd} \mathsf{Set}/\Gamma(\pi_A \circ \pi_{B_1}, \pi_{C_1}) \arrow[r, "\cong"] \arrow[d, "\mathsf{hom}((\pi_A\circ -)g, h)"'] & \mathsf{Set}/\Gamma.A(\pi_{B_1}, \pi_{\Gamma.A\times_\Gamma\Gamma.C_1}) \arrow[d, "\mathsf{hom}(g, \pi_A^* h)"] \\ \mathsf{Set}/\Gamma(\pi_A \circ \pi_{B_2}, \pi_{C_2}) \arrow[r, "\cong"'] & \mathsf{Set}/\Gamma.A(\pi_{B_2}, \pi_{\Gamma.A\times_\Gamma\Gamma.C_2}). \end{tikzcd}$$

This concludes showing $\pi_A \circ -$ is left adjoint to π_A^*.

For the Beck-Chevalley condition to hold, for each $s : \Gamma \to \Delta$ in Set, $\pi_A \in \mathsf{Set}/\Delta$ and $\pi_B \in \mathsf{Set}/\Gamma.A$ we need:

$$\pi_{s^*\pi_A} \circ (s_A^* \pi_B) \cong s^*(\pi_A \circ \pi_B).$$

We have $\pi_{s^*\pi_A} \circ (s_A^* \pi_B) = s^*(\pi_A \circ \pi_B)$ from the following pullback square:

$$\begin{tikzcd} P_2 \arrow[r] \arrow[d, "s_A^*\pi_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A.B \arrow[d, "\pi_B"] \\ P_1 \arrow[r, "s_A"] \arrow[d, "s^*\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\ \Gamma \arrow[r, "s"] & \Delta, \end{tikzcd}$$

where P_1 is the pullback of π_A along s and P_2 is the pullback of π_B along s_A. Lastly, we show that sums defined this way are strong sums. In this comprehension category, for each $\pi_B \in \mathsf{Set}/\Gamma.A$ we have $\mathsf{dom}(\pi_A \circ \pi_B) = \Gamma.A.B$; hence, the isomorphism corresponding to strong sums given in Definition 5.24 is $\Gamma.A.B \cong \Gamma.A.B$ which holds.

This means that for each context Γ, type A in context Γ and type B in context $\Gamma.A$, we can interpret $\sum_A B$ as $\|A\| \circ \|B\|$.

Now, we see whether Π-types can be interpreted in this comprehension category as discussed in Section 5.3.1.

Definition 5.28. A category $\mathcal{C}$ is *locally cartesian closed* if and only if all of its slice categories $\mathcal{C}/X$ are cartesian closed, i.e. they have all finite products and all exponentials.

Proposition 5.29 ([Jac93, Examples 5.14(i)]). *The identity comprehension category* $(\mathcal{C}, \mathcal{C}^{\to}, 1, \mathsf{cod})$ *is a full comprehension category with unit, products and strong sums if and only if* $\mathcal{C}$ *is locally cartesian closed.*

Set is locally cartesian closed; hence, from Proposition 5.29 we have that $(\mathsf{Set}, \mathsf{Set}^{\to}, 1, \mathsf{cod})$ is a full comprehension category with unit, products and strong sums. This means that we can interpret Π- types in this comprehension category as well.

For Π- types, we need an adjunction and a Beck-Chevalley condition of the forms given in Definition 5.19. In addition to this, we need $1 : \mathsf{Set}^{\to} \to \mathsf{Set}^{\to}$ to preserve products.

It is explained in [Jac99, Exercise 1.3.3] and [Jac99, Proposition 1.9.8] that for $\pi_A \in \mathsf{Set}/\Gamma$, the adjunction corresponding to Π-types is given by:

$$\begin{tikzcd} \mathsf{Set}/\Gamma.A \arrow[r, shift right, "\Pi_{\pi_A}"'] \arrow[r, phantom, "\perp"] & \mathsf{Set}/\Gamma, \arrow[l, shift right, "\pi_A^*"'] \end{tikzcd}$$

where for each $\pi_B \in \mathsf{Set}/\Gamma.A$, $\Pi_{\pi_A}\pi_B$ is the domain of the following equaliser in Set/Γ:

$$\begin{tikzcd} \Pi_{\pi_A}\pi_B \arrow[r, "e"'] & (\pi_A \circ \pi_B)^{\pi_A} \arrow[r, shift left, "\pi_B \circ -"] \arrow[r, shift right, "1_{\pi_A}"'] & \pi_A^{\pi_A}. \end{tikzcd}$$

In other words, $\Pi_{\pi_A}\pi_B$ is the set of all sections $s : \Gamma.A \to \Gamma.A.B$ of π_B such that $\pi_A \circ \pi_B \circ s = \pi_A$. For this to be an adjunction, we need to show:

$$\mathsf{Set}/\Gamma.A(\pi_A^*(\pi_C), \pi_B) \cong \mathsf{Set}/\Gamma(\pi_C, \Pi_{\pi_A}\pi_B),$$

for each $\pi_C \in \mathsf{Set}/\Gamma$ and $\pi_B \in \mathsf{Set}/\Gamma.A$ and a naturality condition. This is given by the following natural isomorphism from $(\pi_A \circ \pi_B)$ being an exponential object:

$$\mathsf{Set}/\Gamma(\pi_C, (\pi_A \circ \pi_B)^{\pi_A}) \cong \mathsf{Set}/\Gamma(\pi_C \times \pi_A, \pi_A \circ \pi_B),$$

and $\pi_C \times \pi_A$ in Set/Γ being $\pi_A \circ \pi_A^*(\pi_C)$ in Set.

[Jac99, Lemma 1.9.7] states that in fibration for which each reindexing functor has both a left and a right adjoint, the Beck-Chevalley holds for sums if and only if it holds for products. We have already shown that the Beck-Chevalley condition holds for sums; hence, it also holds for products. Lastly, 1 preserves products by the definition of $\Pi_{\pi_A}\pi_B$ and Lemma 5.22.

This means that for each context Γ, type A in context Γ and type B in context $\Gamma.A$, we can interpret $\prod_A B$ to be $\Pi_{\|A\|}\|B\|$ in Set/Γ, which is the set of all sections $s : \|\Gamma.A\| \to \|\Gamma.A.B\|$ of $\|B\|$ such that $\|A\| \circ \|B\| \circ s = \|A\|$.

5.3.5 Extended Example : No Type Dependency

We know from Examples 5.5 and 5.17 that $(\mathsf{Set}, \overline{\mathsf{Set}}, \mathcal{P}_{\mathsf{Set}}, \pi^1)$ is a full split comprehension category with unit. Continuing the interpretation given in Section 5.1.9 and Example 5.17, we now discuss whether this category has strong sums (products), and whether a type theory with Σ-types (Π- types) can be interpreted in this comprehension category. Since there is no type dependency in the type theory, we expect the interpretation of Σ-types to coincide with that of dependent product type and the interpretation of Π-types to coincide to that of (non-dependent) function types as explained in Sections 3.2.4 and 3.2.5.

Proposition 5.30 ([Jac93, Examples 5.14(ii)]). *If $\mathcal{C}$ has finite products, $(\mathcal{C}, \bar{\mathcal{C}}, \mathcal{P}_{\mathcal{C}}, \pi^1)$ is full with unit and strong sums. This comprehension category is full with unit, products and strong sums if and only if $\mathcal{C}$ is cartesian closed.*

Set is cartesian closed; hence, by Proposition 5.30 the comprehension category $(\mathsf{Set}, \overline{\mathsf{Set}}, \mathcal{P}_{\mathsf{Set}}, \pi^1)$ is full with unit, strong sums and products. This means that we can interpret Σ-types in a type theory with no type dependency in $(\mathsf{Set}, \overline{\mathsf{Set}}, \mathcal{P}_{\mathsf{Set}}, \pi^1)$ as explained in Section 5.3.2. To check if we can interpret Π-types in this comprehension category as explained in Section 5.3.1 we need to check whether $\mathcal{P}_{\mathsf{Set}}$ preserves products or not.

We start with Σ-types. To interpret Σ-types in this comprehension category we need an adjunction of the form given in part 1 of Definition 5.19, a Beck-Chevalley condition as explained in part 2 of Definition 5.19, and the isomorphism corresponding to strong sums given in Definition 5.24. The adjunction for Σ-types is:

$$\begin{tikzcd} \overline{\mathsf{Set}}_\Gamma \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\pi_A^*"'] & \overline{\mathsf{Set}}_{\Gamma\times A} \arrow[l, bend right, "(\pi_1, A\times -)"'] \end{tikzcd}.$$

This is an adjunction since for each $\Gamma, A, B, C : \mathsf{Set}$ and $(\Gamma \times A, B), (\Gamma, A), (\Gamma, C) : \overline{\mathsf{Set}}$ we have the following bijection:

$$\overline{\mathsf{Set}}_\Gamma((\Gamma, A \times B), (\Gamma, C)) \cong \overline{\mathsf{Set}}_{\Gamma\times A}((\Gamma \times A, B), (\Gamma \times A, C)),$$

using $\Gamma \times (A \times B) = (\Gamma \times A) \times B$, natural in (Γ, C) and $(\Gamma \times A, B)$.

For the Beck-Chevalley condition to hold $s^*A \times (s.A)^* \Rightarrow s^*(A \times -)$ should be an isomorphism for each $s : \Gamma \to \Delta$ in Set and $(\Delta, A) \in \overline{\mathsf{Set}}_\Delta$. From Section 5.1.9 we know that in this comprehension category $s_A = 1_A$ and $s^*A = A$, so the natural transformation of the Beck-Chevalley condition is $A \times - \Rightarrow A \times -$ which is an isomorphism. For the comprehension category to have strong sums $\Gamma.A.B$ should be isomorphic to $\Gamma. \sum_A B$ for each $\Gamma : \mathsf{Set}$, $(\Gamma, A) \in \overline{\mathsf{Set}}_\Gamma$ and $(\Gamma \times A, B) \in \overline{\mathsf{Set}}_{\Gamma\times A}$ (See Definition 5.24), but in this comprehension category these two sets coincide.

This means that for each context Γ and types A and B in context Γ, the Σ-type $\sum_A B$ is interpreted as an object $(\|\Gamma\|, \|A\| \times \|B\|)$ in $\overline{\mathsf{Set}}_\Gamma$. This coincides with the interpretation of product type $A \times B$ defined as follows:

$$\frac{\Gamma \vdash A, B \ \mathsf{type}}{\Gamma \vdash A \times B \ \mathsf{type}} \times\text{-form}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma \vdash x : A \quad \Gamma \vdash y : B}{\Gamma \vdash (x, y) : A \times B} \times\text{-intro}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma \vdash z : A \times B}{\begin{array}{c}\Gamma \vdash p_1(z) : A \\ \Gamma \vdash p_2(z) : B\end{array}} \times\text{-elim}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma \vdash x : A \quad \Gamma \vdash y : B}{\begin{array}{c}\Gamma \vdash p_1((x, y)) \equiv x : A \\ \Gamma \vdash p_2((x, y)) \equiv y : B\end{array}} \times\text{-comp}$$

The projections are called p_1 and p_2 (instead of the usual π_1 and π_2 notation), to distinguish between these and the projections of a categorical product.

The interpretation of the given product type in $(\mathsf{Set}, \overline{\mathsf{Set}}, \mathcal{P}_{\mathsf{Set}}, \pi^1)$ is as follows.

1. For context Γ and types A and B in context Γ, the type $A \times B$ is interpreted as an object $(\|\Gamma\|, \|A\| \times \|B\|)$ in $\overline{\mathsf{Set}}_{\|\Gamma\|}$.

2. For the introduction rule, we have $\|x\| : \|\Gamma\| \to \|\Gamma\| \times \|A\|$ such that $\pi_1 \circ \|x\| = 1_{\|\Gamma\|}$ and $\|y\| : \|\Gamma\| \to \|\Gamma\| \times \|B\|$ such that $\pi_1 \circ \|y\| = 1_{\|\Gamma\|}$ (see Section 5.1.9). We define $\|(x, y)\|$ as $\langle 1_\Gamma, \langle \|x\|, \|y\| \rangle \rangle : \|\Gamma\| \to \|\Gamma\| \times (\|A\| \times \|B\|)$, which satisfies $\pi_1 \circ \|(x, y)\| = 1_{\|\Gamma\|}$.

3. For the elimination rule, we have $\|z\| : \|\Gamma\| \to \|\Gamma\| \times (\|A\| \times \|B\|)$ such that $\pi_1 \circ \|z\| = 1_{\|\Gamma\|}$. We define $\|p_1(z)\|$ as $\langle 1_\Gamma, \pi_{1_{\|A\|\times\|B\|}} \circ \pi_{2_{\|\Gamma\|\times(\|A\|\times\|B\|)}} \circ \|z\| \rangle : \|\Gamma\| \to \|\Gamma\| \times \|A\|$, which satisfies $\pi_1 \circ \|p_1(z)\| = 1_{\|\Gamma\|}$. Similarly, we can define $\|p_2(z)\|$ as $\langle 1_\Gamma, \pi_{2_{\|A\|\times\|B\|}} \circ \pi_{2_{\|\Gamma\|\times(\|A\|\times\|B\|)}} \circ \|z\| \rangle$: $\|\Gamma\| \to \|\Gamma\| \times \|B\|$, which satisfies $\pi_1 \circ \|p_2(z)\| = 1_{\|\Gamma\|}$

4. For the computation rule, we have $\|x\| : \|\Gamma\| \to \|\Gamma\| \times \|A\|$ such that $\pi_1 \circ \|x\| = 1_{\|\Gamma\|}$ and $\|y\| : \|\Gamma\| \to \|\Gamma\| \times \|B\|$ such that $\pi_1 \circ \|y\| = 1_{\|\Gamma\|}$. We need to check whether $\|p_1((x, y))\| = \|x\|$, and $\|p_2((x, y))\| = \|y\|$. From the interpretation of the introduction and elimination rule we have:

$$\begin{aligned}
\|p_1((x,y))\| &= \langle 1_\Gamma, \pi_{1_{\|A\|\times\|B\|}} \circ \pi_{2_{\|\Gamma\|\times(\|A\|\times\|B\|)}} \circ \|(x,y)\| \rangle \\
&= \langle 1_\Gamma, \pi_{1_{\|A\|\times\|B\|}} \circ \pi_{2_{\|\Gamma\|\times(\|A\|\times\|B\|)}} \circ \langle 1_\Gamma, \langle \|x\|, \|y\| \rangle \rangle \rangle \\
&= \langle 1_\Gamma, \pi_{1_{\|A\|\times\|B\|}} \circ \langle \|x\|, \|y\| \rangle \rangle \\
&= \langle 1_\Gamma, \|x\| \rangle \\
&= \|x\|.
\end{aligned}$$

   Similarly, one can show $\|p_2((x, y))\| = \|y\|$.

Next, we move to Π-types. Similar to the previous case, we need an adjunction and a Beck-Chevalley condition of the forms given in Definition 5.19. In addition to this, we need $\mathcal{P}_{\mathsf{Set}}$ to preserve products. The adjunction corresponding to Π-types is:

$$\overline{\mathsf{Set}}_{\Gamma\times A} \underset{(\pi_1, (-)^A)}{\overset{\pi_A^*}{\underset{\perp}{\rightleftarrows}}} \overline{\mathsf{Set}}_\Gamma.$$

Here $(-)^A$ sends B to the exponential B^A, which in Set is the set of all functions from A to B. This is an adjunction since for each Γ, A, B, C : Set and $(\Gamma \times A, B), (\Gamma, A), (\Gamma, C) : \overline{\mathsf{Set}}$, we have the following bijection:

$$\overline{\mathsf{Set}}_{\Gamma\times A}((\Gamma \times A, C), (\Gamma \times A, B)) \cong \overline{\mathsf{Set}}_\Gamma((\Gamma, C), (\Gamma, B^A)),$$

natural in $(\Gamma \times A, B)$ and (Γ, B^A). This is because there is a unique morphism of the form $\Gamma \times C \to B^A$ for each morphism of the form $\Gamma \times C \times A \to B$ in Set, by the definition of exponentials.

For the Beck-Chevalley condition to hold $s^*(-)^A \Rightarrow ((s.A)^*(-))^{s^*A}$ should be an isomorphism for each $s : \Gamma \to \Delta$ in Set and $(\Delta, A) \in \overline{\mathsf{Set}}_\Delta$. From Section 5.1.9 we know that in this comprehension category $s_A = 1_A$ and $s^*A = A$, so the natural transformation of the Beck-Chevalley condition is $(-)^A \Rightarrow (-)^A$, which is an isomorphism.

It remains to check whether $\mathcal{P}_{\mathsf{Set}}$ preserves products defined this way. $\mathcal{P}_{\mathsf{Set}}$ preserves products if and only if there is a bijection between the commuting diagrams of the form

$$\begin{tikzcd} \Gamma \arrow[rr] \arrow[dr, "s"'] & & \Delta \times B^A \arrow[dl, "\pi_1"] \\ & \Delta & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Gamma \times A \arrow[rr] \arrow[dr, "s\times 1_A"'] & & \Delta \times A \times B \arrow[dl, "\pi_1"] \\ & \Delta \times A & \end{tikzcd}$$

for each $\Gamma, \Delta \in \mathsf{Set}$, $(\Delta, A) \in \overline{\mathsf{Set}}_\Delta$, $(\Delta \times A, B) \in \overline{\mathsf{Set}}_{\Delta\times A}$ and $s : \Gamma \to \Delta$ in Set (see Definition 5.21). Morphisms that make the left diagram commute are of the form $\langle s, f\rangle$ where $f : \Gamma \to B^A$, and morphisms that make the right diagram commute are of the form $\langle \pi_1 \circ (s \times 1_A), \langle \pi_2, g\rangle\rangle$ where $g : \Gamma \times A \to B$. By the definition of exponentials there is a bijection between $\mathsf{Set}(\Gamma, B^A)$ and $\mathsf{Set}(\Gamma \times A, B)$; hence, $\mathcal{P}_{\mathsf{Set}}$ preserves products.

This means that for each context Γ, type A in context Γ and type B in context $\Gamma.A$, the Π-type $\prod_A B$ is interpreted as $(\|\Gamma\|, \|B\|^{\|A\|})$ in $\overline{\mathsf{Set}}_{\|\Gamma\|}$, where $\|B\|^{\|A\|}$ is the set of all functions from $\|A\|$ to $\|B\|$. This coincides with the interpretation of (non-dependent) function type $A \to B$ defined as follows:

$$\frac{\Gamma \vdash A, B \ \mathsf{type}}{\Gamma \vdash A \to B \ \mathsf{type}}\ \mathsf{func\text{-}form}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma.x : A \vdash b : B}{\Gamma \vdash (\lambda x : A.b) : A \to B}\ \mathsf{func\text{-}intro}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma \vdash f : A \to B}{\Gamma.x : A \vdash f@x : B}\ \mathsf{func\text{-}elim}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma.x : A \vdash b : B}{\Gamma.x : A \vdash (\lambda x : A.b)@x \equiv b : B}\ \mathsf{func\text{-}beta}$$

$$\frac{\Gamma \vdash A, B \ \mathsf{type} \quad \Gamma \vdash f : A \to B}{\Gamma \vdash (\lambda x : A.f@x) \equiv f : A \to B}\ \mathsf{func\text{-}eta}$$

For context Γ and types A and B in context Γ, the type $A \to B$ is interpreted as an object $(\|\Gamma\|, \|B\|^{\|A\|})$ in $\overline{\mathsf{Set}}_{\|\Gamma\|}$. For the introduction rule, we have $\|b\| : \|\Gamma\| \times \|A\| \to (\|\Gamma\| \times \|A\|) \times \|B\|$ such that $\pi_1 \circ \|b\| = 1_{\|\Gamma\|\times\|A\|}$. For an exponential object B^A, we have an evaluation map $\mathsf{ev}_{B^A} : B^A \times A \to B$, which is universal in the sense that for each $e : C \times A \to B$, there exists a unique $u : C \to B^A$ such that $\mathsf{ev} \circ (u \times 1_A) = e$. Hence, for the exponential object $\|B\|^{\|A\|}$ and morphism $\pi_2 \circ \|b\| : \|\Gamma\| \times \|A\| \to \|B\|$, there exists a unique $u_{\pi_2\circ\|b\|} : \|\Gamma\| \to \|B\|^{\|A\|}$ such that:

$$\mathsf{ev}_{\|B\|^{\|A\|}} \circ (u_{\pi_2\circ\|b\|} \times 1_{\|A\|}) = \pi_2 \circ \|b\|. \tag{5.2}$$

The interpretation of $(\lambda x : A.b)$ is defined as follows:

$$\|(\lambda x : A.b)\| : \|\Gamma\| \to \|\Gamma\| \times \|B\|^{\|A\|} := \langle 1_{\|\Gamma\|}, u_{\pi_2\circ\|b\|}\rangle.$$

For the elimination rule, we have $\|f\| : \|\Gamma\| \to \|\Gamma\| \times \|B\|^{\|A\|}$ such that $\pi_1 \circ \|f\| = 1_{\|\Gamma\|}$. The interpretation of $f@x$ is defined as follows:

$$\|f@x\| : \|\Gamma\| \times \|A\| \to (\|\Gamma\| \times \|A\|) \times \|B\| \coloneqq \langle 1_{\|\Gamma\| \times \|A\|}, \mathsf{ev}_{\|B\|^{\|A\|}} \circ (\pi_2 \circ \|f\| \times 1_{\|A\|}) \rangle.$$

For the $\beta-$conversion rule we need to show:

$$\|(\lambda x : A.b)@x\| = \langle 1_{\|\Gamma\| \times \|A\|}, \mathsf{ev}_{\|B\|^{\|A\|}} \circ (u_{\pi_2 \circ \|b\|} \times 1_{\|A\|}) \rangle = \|b\|,$$

which is give by Eq. (5.2). For the $\eta-$conversion rule we need to show:

$$\|(\lambda x : A.f@x)\| = \langle 1_{\|\Gamma\|}, u_{\pi_2 \circ \|f@x\|} \rangle = f,$$

where $\pi_2 \circ \|f@x\| = \mathsf{ev}_{\|B\|^{\|A\|}} \circ (\pi_2 \circ \|f\| \times 1_{\|A\|})$. This is given by $u_{\pi_2 \circ \|f@x\|}$ being the unique morphism that satisfies $\pi_2 \circ \|f@x\| = \mathsf{ev}_{\|B\|^{\|A\|}} \circ (u_{\pi_2 \circ \|f@x\|} \times 1_{\|A\|})$.

Chapter 6

Type Theory Extracted from the Semantics

We have seen in Chapter 5 how MLTT can be interpreted in a full split comprehension category. All comprehension categories, however, are not full and split. One expects that more general dependent-type theories can be interpreted in comprehension categories that are not full and split. For example, a type theory with a weaker notion of substitution where substitution is "functorial" only up to isomorphism, as opposed to MLTT which has strictly functorial substitution, can be interpreted in a non-split comprehension category, as discussed in Section 5.1.5. In much of the literature, however, comprehension categories are taken to be full [LW15]. This arises due to how terms of MLTT are interpreted in a comprehension category as sections of the projection context morphisms in the category of contexts.

Usually, one proposes semantics for a type theory by considering the requirements that the category should have such that it is possible to interpret the components of the type theory in it. One can study the general type theories that can be interpreted in a certain semantic framework by starting from the semantics and deriving the syntax of a type theory such that all the structure of the semantic framework is reflected in the type theory. If soundness and completeness of the rules of the type theory with respect to the class of models is proven, then this type theory is called the internal language of the semantic framework.

Soundness means that if a judgement can be derived in the type theory from certain premises, then this result also holds for every model of the type theory given the premises. One can prove soundness by giving an interpretation of the type theory in every model. The converse of the soundness theorem is the completeness theorem, which states that if a statement holds for every model of the type theory, it can also be derived from the type theory. Given a class of models and their internal language one can show a certain property of the type theory by showing the semantic counterpart in the models and vice versa.

Some examples of internal languages are as follows. Simply typed lambda calculus is the internal language of cartesian closed categories [LS88]. Extensional dependent type theories are the internal language of locally cartesian closed categories [See84; CD11; CGH14]. In categorical logic, first order logic is the internal language of hyperdoctrines [See84].

In this section we investigate type theories that can be interpreted in comprehension categories which are not necessarily full. For this, in section Section 6.1 we introduce a type theory with rules extracted from the structure of comprehension categories such that all categorical structures are reflected in the type theory. We also show the soundness of the rules of the derived type theory by providing an interpretation of this type theory in comprehension categories in Section 6.2. Showing completeness, and showing that this type theory is the internal language of comprehension categories, is left for future work.

The type theory is then compared to MLTT to highlight the differences of a type theory that is interpreted in comprehension categories and MLTT which has full split comprehension categories as a model. The main differences between this type theory and MLTT is that the terms in this type theory are interpreted as morphisms in the category $\mathcal{T}$ of types, and that substitution in this type theory is functorial only up to isomorphism. This is contrary to MLTT where terms are interpreted as certain morphisms in $\mathcal{C}$, particularly sections of the projection context morphisms, and substitution is strictly functorial. These two differences result from there being no requirement on fullness and splitness, respectively.

We then propose how unit type can be added to this type theory and investigate the requirements for having a semantic one-to-one correspondence between the terms of this type theory and the terms of MLTT in Section 6.4.

6.1 Judgements and Structural Rules

In this section, we derive the judgements and structural rules of a type theory with explicit substitution from a comprehension category $(\mathcal{C}, \mathcal{T}, \chi, p)$, where $\mathcal{C}$ has a terminal object, without requiring that χ preserves cartesian morphisms. This is a weaker notion of a comprehension category where cartesian morphisms in $\mathcal{T}$ do not correspond to pullback squares in $\mathcal{C}$. In addition to this, we do not impose any requirements of fullness and splitness on the comprehension category.

6.1.1 Judgements

The judgements of the type theory are as follows:

1. Γ ctx, which is read as "Γ is a context";
2. $\Gamma \vdash s : \Delta$, which is read as "s is a substitution from Γ to Δ", where Γ, Δ ctx;
3. $\Gamma \vdash s \equiv s' : \Delta$, which is read as "$s$ is equal to s'", where $\Gamma \vdash s, s' : \Delta$;
4. $\Gamma \vdash A$ type, which is read as "A is a type in context Γ", where Γ ctx;
5. $\Gamma | A \vdash t : B$, which is read as "t is a term of type B depending on A in context Γ", where $\Gamma \vdash A, B$ type;
6. $\Gamma | A \vdash t \equiv t' : B$, which reads as "$t$ and t' are equal", where $\Gamma | A \vdash t, t' : B$.

The judgement $\Gamma \vdash s : \Delta$ can also be read as "s is a context morphism from Γ to Δ", since this judgement is interpreted as $\|s\|$ being a morphism in the category of contexts $\mathcal{C}$ from $\|\Gamma\|$ to $\|\Delta\|$ (see Section 6.2). Similarly, the judgement $\Gamma | A \vdash t : B$ can also be read as "t is a type morphism from A to B". Judgements 2 and 5 are the same as the corresponding judgements in the type theory derived from comprehension bicategories in [ANW23].

Unlike Martin-Löf type theory, this type theory has explicit substitution in the syntax in the sense discussed in [Aba+91]. Another difference between this type theory and Martin-Löf type theory is the terms. As we will see in Section 6.2, the terms of this type theory are interpreted as morphisms in $\mathcal{T}$, whereas as discussed in Section 5.1, the terms of Martin-Löf type theory are interpreted as certain morphisms in $\mathcal{C}$, particularly the sections of the projection context morphisms from extended contexts to the original contexts. We will refer to the particular context morphisms which are interpreted as a section of a projection context morphism as MLTT terms.

Notation 6.1. Similar to [ANW23, Section 8], we define the following notations that stand for four judgements, which use the composition and identities introduced in Rules ctx-mor-id, ctx-mor-comp, ty-mor-id and ty-mor-comp.

1. $\Gamma \overset{\sim}{\vdash} s : \Delta$ stands for the following four judgements:
   - $\Gamma \vdash s : \Delta$
   - $\Delta \vdash s^{-1} : \Gamma$
   - $\Delta \vdash s \circ s^{-1} \equiv 1_\Delta : \Delta$
   - $\Gamma \vdash s^{-1} \circ s \equiv 1_\Gamma : \Gamma$
2. $\Gamma|A \overset{\sim}{\vdash} t : B$ stands for the following four judgements:
   - $\Gamma|A \vdash t : B$
   - $\Gamma|B \vdash t^{-1} : A$
   - $\Gamma|B \vdash t \circ t^{-1} \equiv 1_B : B$
   - $\Gamma|A \vdash t^{-1} \circ t \equiv 1_A : A$

Notation 6.2. We use the notation $\Gamma \vdash t : A$ to refer to MLTT style terms. This means that we use the following notations for MLTT style terms, which stand for multiple judgements.

1. $\Gamma \vdash t : A$ stands for the following two judgements:
   - $\Gamma \vdash t : \Gamma.A$
   - $\Gamma \vdash \pi_A \circ t \equiv 1_\Gamma : \Gamma$
2. $\Gamma \vdash t \equiv t' : A$ stands for the following five judgements:
   - $\Gamma \vdash t : \Gamma.A$
   - $\Gamma \vdash \pi_A \circ t \equiv 1_\Gamma : \Gamma$
   - $\Gamma \vdash t' : \Gamma.A$
   - $\Gamma \vdash \pi_A \circ t' \equiv 1_\Gamma : \Gamma$
   - $\Gamma \vdash t \equiv t' : \Gamma.A$

Note that MLTT style terms are not part of the syntax of this type theory. This notation is only used for brevity, as sections of projections are frequently used.

When referring to a rule that uses one of these notations, it is clear from the context which judgement is used.

6.1.2 Rules for Context and Type Morphisms

From the category $\mathcal{C}$ of context we derive the following rules.

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash 1_\Gamma : \Gamma}\ \textsf{ctx-mor-id}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta}{\Gamma \vdash s' \circ s : \Theta}\ \textsf{ctx-mor-comp}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash s \circ 1_\Gamma \equiv s : \Delta}\ \textsf{ctx-id-unit-r}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash 1_\Delta \circ s \equiv s : \Delta}\ \text{ctx-id-unit-l}$$

$$b\ \frac{\Gamma, \Delta, \Theta, \Phi\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash s'' : \Phi}{\Gamma \vdash s'' \circ (s' \circ s) \equiv (s'' \circ s') \circ s : \Phi}\ \text{ctx-comp-assoc}$$

Rule ctx-mor-id is derived from the identity morphisms in $\mathcal{C}$ and Rule ctx-mor-comp is derived from the composition of morphisms in $\mathcal{C}$. Such definitions of identity context morphism and the composition of context morphisms are in line with the definitions given in Examples 3.30 and 3.32 for identity and composition of context morphisms in Martin-Löf type theory. Rules ctx-id-unit-r and ctx-id-unit-l are derived from the left and right unit laws of identity in category $\mathcal{C}$, and are in line with Proposition 3.40 about Martin-Löf type theory. Lastly, Rule ctx-comp-assoc is derived from the associativity of composition in $\mathcal{C}$, which is in line with Proposition 3.39 about Martin-Löf type theory.

From category $\mathcal{C}$ having a terminal object we derive the following rules.

$$\frac{}{\diamond\ \mathsf{ctx}}\ \text{empty-ctx}$$

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \langle\rangle_\Gamma : \diamond}\ \text{empty-ctx-mor}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash s : \diamond}{\Gamma \vdash s \equiv \langle\rangle_\Gamma : \diamond}\ \text{empty-ctx-mor-unique}$$

Rule empty-ctx is derived from $\mathcal{C}$ having a terminal object. Rules empty-ctx-mor and empty-ctx-mor-unique are derived from the universal property of the terminal object. This means that empty context is a well-formed context in the type theory and there is a unique substitution from each context Γ to the empty context, which is also the case for Martin-Löf type theory as discussed in Example 3.26.

From the category $\mathcal{T}$ of types we derive the following rules.

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma | A \vdash 1_A : A}\ \text{ty-mor-id}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C\ \mathsf{type} \quad \Gamma | A \vdash t : B \quad \Gamma | B \vdash t' : C}{\Gamma | A \vdash t' \circ t : C}\ \text{ty-mor-comp}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma | A \vdash t : B}{\Gamma | A \vdash t \circ 1_A \equiv t : B}\ \text{ty-id-unit-r}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma | A \vdash t : B}{\Gamma | A \vdash 1_B \circ t \equiv t : B}\ \text{ty-id-unit-l}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C, D\ \mathsf{type} \quad \Gamma | A \vdash t : B \quad \Gamma | B \vdash t' : C \quad \Gamma | C \vdash t'' : D}{\Gamma | A \vdash t'' \circ (t' \circ t) \equiv (t'' \circ t') \circ t : D}\ \text{ty-comp-assoc}$$

Similar to the rules derived from the category of contexts, Rule ty-mor-id is derived from the identity morphisms in $\mathcal{T}$ and Rule ty-mor-comp is derived from the composition of morphisms in $\mathcal{T}$. Rules ty-id-unit-r and ty-id-unit-l are derived from the left and right unit laws of identity in category $\mathcal{T}$ and Rule ty-comp-assoc is derived from associativity of composition in $\mathcal{T}$.

6.1.3 Rules for Context Extension

The rules regarding context extension are derived from functor χ, the comprehension. Recall from Remark 3.4 that a functor $\chi : \mathcal{T} \to \mathcal{C}^{\to}$ into the arrow category can be thought of as two functors $\chi_0 := \mathsf{dom} \circ \chi$, $\mathsf{cod} \circ \chi$ and a natural transformation $\pi : \chi_0 \Rightarrow \mathsf{cod} \circ \chi$. From the definition of a comprehension category, we know that $\mathsf{cod} \circ \chi = p$. The rules regarding context extension are derived from $\chi_0 : \mathcal{T} \to \mathcal{C}$ and $\pi : \chi_0 \Rightarrow p$. The fibration p is used in Section 6.1.4 to derive the rules regarding substitution.

The rules for context extension are as follows.

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A\ \mathsf{ctx}}\ \mathsf{ext\text{-}ty}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma | A \vdash t : B}{\Gamma.A \vdash \Gamma.t : \Gamma.B}\ \mathsf{ext\text{-}tm}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \Gamma.1_A \equiv 1_{\Gamma.A} : \Gamma.A}\ \mathsf{ext\text{-}id}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C\ \mathsf{type} \quad \Gamma | A \vdash t : B \quad \Gamma | B \vdash t' : C}{\Gamma.A \vdash \Gamma.(t' \circ t) \equiv \Gamma.t' \circ \Gamma.t : \Gamma.B}\ \mathsf{ext\text{-}comp}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \pi_A : \Gamma}\ \mathsf{ext\text{-}proj}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma | A \vdash t : B}{\Gamma.A \vdash \pi_B \circ \Gamma.t \equiv \pi_A : \Gamma}\ \mathsf{ext\text{-}coh}$$

Rule ext-ty is derived from the action of χ_{0_Γ} on the objects of $\mathcal{T}_\Gamma$, where χ_{0_Γ} is the restriction of χ_0 to the fibre $\mathcal{T}_\Gamma$. Rule ext-tm is derived from the action of χ_{0_Γ} on the morphisms in $\mathcal{T}_\Gamma$. Rules ext-comp and ext-id are derived from functoriality of χ_{0_Γ}. Rules ext-proj and ext-coh are derived from the morphisms in $\mathcal{C}$ and coherence conditions given by $\pi : \chi_0 \Rightarrow p$.

Rule ext-ty is the usual context extension rule. Rule ext-proj gives the context morphism corresponding to projection from an extended context to the original context, which aligns with the context morphism described in Example 3.34 for Martin-Löf type theory. Rule ext-tm states that for each type morphism, a term t in type B dependent on type A, there is a corresponding context morphism from the extended context $\Gamma.A$ to the extended context $\Gamma.B$, and Rule ext-coh gives the coherence condition for this correspondence. Lastly, Rules ext-id and ext-comp state that composition and identity are preserved under this correspondence. Note that here we do not have a one-to-one correspondence between such type morphisms and context morphisms, as there is no assumption of fullness on the comprehension category, i.e. χ is not necessarily fully faithful.

6.1.4 Rules for Substitution

Unlike Martin-Löf type theory, this type theory features explicit substitution in the syntax. From Remark 3.18, we know that the fibration $p : \mathcal{T} \to \mathcal{C}$ can be thought of as a pseudofunctor of the form $\mathcal{C}^{\mathsf{op}} \to \mathsf{Cat}$. The rules regarding substitution are derived from this pseudofunctor. Under the equivalence mentioned in Remark 3.18, the fibration $p : \mathcal{T} \to \mathcal{C}$ can be thought of as reindexing functors of the form $s^* : \mathcal{T}_\Delta \to \mathcal{T}_\Gamma$ for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, and two natural isomorphism corresponding to composition of reindexing functors and reindexing along identity morphisms. Namely, for each $s : \Gamma \to \Delta$ and $s' : \Delta \to \Theta$ in $\mathcal{C}$, we have a natural isomorphism $i_{\mathsf{comp}} : (s' \circ s)^* \cong s^* \circ s'^*$, and for each $A \in \mathcal{T}_\Gamma$ we have an isomorphism $i_{\mathsf{id}_A} : 1_\Gamma^* A \cong A$.

The rules for substitution are as follows.

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A \ \mathsf{type}}{\Gamma \vdash A[s] \ \mathsf{type}} \ \mathsf{sub\text{-}ty}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta | A \vdash t : B}{\Gamma | A[s] \vdash t[s] : B[s]} \ \mathsf{sub\text{-}tm}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A \ \mathsf{type}}{\Gamma | A \vdash 1_A[s] \equiv 1_{A[s]} : A} \ \mathsf{sub\text{-}pres\text{-}id}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A, B, C \ \mathsf{type} \quad \Delta | A \vdash t : B \quad \Delta | B \vdash t' : C}{\Gamma | A \vdash (t' \circ t)[s] \equiv t'[s] \circ t[s] : C} \ \mathsf{sub\text{-}pres\text{-}comp}$$

$$\frac{\Gamma \ \mathsf{ctx} \quad \Gamma \vdash A \ \mathsf{type}}{\Gamma | A[1_\Gamma] \overset{\sim}{\vdash} i_{\mathsf{id}_A} : A} \ \mathsf{sub\text{-}id}$$

$$\frac{\Gamma, \Delta, \Theta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash A \ \mathsf{type}}{\Gamma | A[s' \circ s] \overset{\sim}{\vdash} i_{\mathsf{comp}_{A,s,s'}} : (A[s'])[s]} \ \mathsf{sub\text{-}comp}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A \ \mathsf{type}}{\Gamma.A[s] \vdash s.A : \Delta.A} \ \mathsf{lift\text{-}compreh}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A \ \mathsf{type}}{\Gamma.A[s] \vdash \pi_A \circ s.A \equiv s \circ \pi_{A[s]} : \Delta} \ \mathsf{lift\text{-}coh}$$

$$\frac{\Gamma \ \mathsf{ctx} \quad \Gamma \vdash A \ \mathsf{type}}{\Gamma.A[1_\Gamma] \vdash 1_{\Gamma.A} \circ \Gamma.i_{\mathsf{id}_A} \equiv 1_\Gamma.A : \Gamma.A} \ \mathsf{iid\text{-}coh}$$

$$\frac{\Gamma, \Delta, \Theta \ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash A \ \mathsf{type}}{\Gamma.A[s' \circ s] \vdash s'.A \circ s.A[s'] \circ \Gamma.i_{\mathsf{comp}_{A,s,s'}} \equiv (s' \circ s).A : \Theta.A} \ \mathsf{icomp\text{-}coh}$$

Rules sub-ty and sub-tm are derived from the action of the reindexing functor $s^* : \mathcal{T}_\Delta \to \mathcal{T}_\Gamma$ on objects and morphisms respectively. Rules sub-pres-id and sub-pres-comp are derived from the functoriality of the reindexing functor. Rule sub-id is derived from the isomorphism $i_{\mathsf{id}_A} : 1_\Gamma^* A \cong A$ and Rule sub-comp from the isomorphism $i_{\mathsf{comp}_{A,s,s'}} : (s' \circ s)^* A \cong s^*(s'^* A)$.

Rules sub-ty and sub-tm are the usual substitution rules. Rules sub-id and sub-comp state that substitution is functorial up to isomorphism, unlike Martin-Löf type theory which has strictly functorial substitution as stated in Propositions 3.39 and 3.40. As discussed in Section 5.1.5, this is in line with there being no assumption on the splitness of the comprehension category. Rules sub-pres-id and sub-pres-comp state that identity and composition of type morphisms are preserved under substitution.

Rules lift-compreh and lift-coh are derived from $\chi s_A : \pi_{A[s]} \to \pi_A$ in $\mathcal{C}^\to$, where s_A is the cartesian lift of s into A. Rules iid-coh and icomp-coh concern the coherence conditions regarding the comprehension of i_{id_A} and $i_{\mathsf{comp}_{A,s,s'}}$, from Rules sub-id and sub-comp, and the comprehension of certain cartesian lifts.

Another difference between this type theory and Martin-Löf type theory is that in Martin-Löf type theory context morphisms are built from terms, whereas here, there is no such correspondence between context morphisms and terms (type morphisms). In addition to this, because there is no requirement that χ preserves cartesian morphisms, there is also no correspondence between the context morphisms and sections of the projections in $\mathcal{C}$.

The rules of the type theory are summarised in Fig. 6.1. In addition to these, we also have the rules related to $\equiv$ being a congruence for all the judgements, which are listed in Fig. 6.2.

Proposition 6.3. *From the rules given in Figs. 6.1 and 6.2, we can derive the following weakening rules:*

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type}}{\Gamma.A \vdash B[\pi_A]\ \mathsf{type}}\ \textit{weak-ty}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C\ \mathsf{type} \quad \Gamma|B \vdash t : C}{\Gamma.A|B[\pi_A] \vdash t[\pi_A] : C[\pi_A]}\ \textit{weak-tm}$$

Proof. For weakening of types, the derivation is as follows.

$$\frac{\Gamma\ \mathsf{ctx} \quad \dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A\ \mathsf{ctx}}\ \mathsf{ext\text{-}ty} \quad \dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \pi_A : \Gamma}\ \mathsf{ext\text{-}proj} \quad \Gamma \vdash B\ \mathsf{type}}{\Gamma.A \vdash B[\pi_A]\ \mathsf{type}}\ \mathsf{sub\text{-}ty}$$

For terms, we obtain the weakening rule as follows.

$$\frac{\Gamma\ \mathsf{ctx} \quad \dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A\ \mathsf{ctx}}\ \mathsf{ext\text{-}ty} \quad \dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \pi_A : \Gamma}\ \mathsf{ext\text{-}proj} \quad \Gamma \vdash B, C\ \mathsf{type} \quad \Gamma.B \vdash t : C}{\Gamma.A|B[\pi_A] \vdash t[\pi_A] : C[\pi_A]}\ \mathsf{sub\text{-}tm}$$

□

6.2 Interpretation in a Comprehension Category

In this section, we show soundness of the rules of the type theory by giving an interpretation of the type theory in every comprehension categories where $\mathcal{C}$ has a terminal object and where χ doesn't preserve cartesian morphisms. Note that there is no assumption of fullness and splitness on the comprehension category.

The judgements are interpreted as follows:

1. Γ ctx is interpreted as an object $\|\Gamma\|$ in $\mathcal{C}$;

2. $\Gamma \vdash s : \Delta$ is interpreted as a morpshim $\|s\| : \|\Gamma\| \to \|\Delta\|$ in $\mathcal{C}$;

3. $\Gamma \vdash s \equiv s' : \Delta$ is interpreted as $\|s\| = \|s'\|$;

4. $\Gamma \vdash A$ type is interpreted as an object $\|A\|$ in $\mathcal{T}_{\|\Gamma\|}$;

5. $\Gamma|A \vdash t : B$ is interpreted as a morphism $\|t\| : \|A\| \to \|B\|$ in $\mathcal{T}_{\|\Gamma\|}$;

6. $\Gamma|A \vdash t \equiv t' : B$ is interpreted as $\|t\| = \|t'\|$.

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash 1_\Gamma : \Gamma}\ \textsf{ctx-mor-id} \qquad \frac{\Gamma, \Delta, \Theta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta}{\Gamma \vdash s' \circ s : \Theta}\ \textsf{ctx-mor-comp}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash s \circ 1_\Gamma \equiv s : \Delta}\ \textsf{ctx-id-unit-r} \qquad \frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash 1_\Delta \circ s \equiv s : \Delta}\ \textsf{ctx-id-unit-l}$$

$$\frac{\Gamma, \Delta, \Theta, \Phi\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash s'' : \Phi}{\Gamma \vdash s'' \circ (s' \circ s) \equiv (s'' \circ s') \circ s : \Phi}\ \textsf{ctx-comp-assoc}$$

$$\frac{}{\diamond\ \mathsf{ctx}}\ \textsf{empty-ctx} \qquad \frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \langle\rangle_\Gamma : \diamond}\ \textsf{empty-ctx-mor} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash s : \diamond}{\Gamma \vdash s \equiv \langle\rangle_\Gamma : \diamond}\ \textsf{empty-ctx-mor-unique}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma|A \vdash 1_A : A}\ \textsf{ty-mor-id} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C\ \mathsf{type} \quad \Gamma|A \vdash t : B \quad \Gamma|B \vdash t' : C}{\Gamma|A \vdash t' \circ t : C}\ \textsf{ty-mor-comp}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma|A \vdash t : B}{\Gamma|A \vdash t \circ 1_A \equiv t : B}\ \textsf{ty-id-unit-r} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma|A \vdash t : B}{\Gamma|A \vdash 1_B \circ t \equiv t : B}\ \textsf{ty-id-unit-l}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C, D\ \mathsf{type} \quad \Gamma|A \vdash t : B \quad \Gamma|B \vdash t' : C \quad \Gamma|C \vdash t'' : D}{\Gamma|A \vdash t'' \circ (t' \circ t) \equiv (t'' \circ t') \circ t : D}\ \textsf{ty-comp-assoc}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A\ \mathsf{ctx}}\ \textsf{ext-ty} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma|A \vdash t : B}{\Gamma.A \vdash \Gamma.t : \Gamma.B}\ \textsf{ext-tm}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \Gamma.1_A \equiv 1_{\Gamma.A} : \Gamma.A}\ \textsf{ext-id} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B, C\ \mathsf{type} \quad \Gamma|A \vdash t : B \quad \Gamma|B \vdash t' : C}{\Gamma.A \vdash \Gamma.(t' \circ t) \equiv \Gamma.t' \circ \Gamma.t : \Gamma.B}\ \textsf{ext-comp}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A \vdash \pi_A : \Gamma}\ \textsf{ext-proj} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma|A \vdash t : B}{\Gamma.A \vdash \pi_B \circ \Gamma.t \equiv \pi_A : \Gamma}\ \textsf{ext-coh}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A\ \mathsf{type}}{\Gamma \vdash A[s]\ \mathsf{type}}\ \textsf{sub-ty} \qquad \frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta|A \vdash t : B}{\Gamma|A[s] \vdash t[s] : B[s]}\ \textsf{sub-tm}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A\ \mathsf{type}}{\Gamma|A \vdash 1_A[s] \equiv 1_{A[s]} : A}\ \textsf{sub-pres-id}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A, B, C\ \mathsf{type} \quad \Delta|A \vdash t : B \quad \Delta|B \vdash t' : C}{\Gamma|A \vdash (t' \circ t)[s] \equiv t'[s] \circ t[s] : C}\ \textsf{sub-pres-comp}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma|A[1_\Gamma] \mathrel{\tilde{\vdash}} i_{\mathsf{id}_A} : A}\ \textsf{sub-id} \qquad \frac{\Gamma, \Delta, \Theta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash A\ \mathsf{type}}{\Gamma|A[s' \circ s] \mathrel{\tilde{\vdash}} i_{\mathsf{comp}_{A,s,s'}} : (A[s'])[s]}\ \textsf{sub-comp}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A\ \mathsf{type}}{\Gamma.A[s] \vdash s.A : \Delta.A}\ \textsf{lift-compreh} \qquad \frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash A\ \mathsf{type}}{\Gamma.A[s] \vdash \pi_A \circ s.A \equiv s \circ \pi_{A[s]} : \Delta}\ \textsf{lift-coh}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type}}{\Gamma.A[1_\Gamma] \vdash 1_{\Gamma.A} \circ \Gamma.i_{\mathsf{id}_A} \equiv 1_\Gamma.A : \Gamma.A}\ \textsf{iid-coh} \qquad \frac{\Gamma, \Delta, \Theta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash A\ \mathsf{type}}{\Gamma.A[s' \circ s] \vdash s'.A \circ s.A[s'] \circ \Gamma.i_{\mathsf{comp}_{A,s,s'}} \equiv (s' \circ s).A : \Theta.A}\ \textsf{icomp-coh}$$

FIGURE 6.1: Rules of the type theory.

The rules introduced in Section 6.1.2 regarding context and type morphisms are interpreted as follows.

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma \vdash s \equiv s : \Delta}\ \textsf{ctx-eq-refl}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s_1, s_2 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta}{\Gamma \vdash s_2 \equiv s_1 : \Delta}\ \textsf{ctx-eq-sym}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s_1, s_2, s_3 : \Delta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta \quad \Gamma \vdash s_2 \equiv s_3 : \Delta}{\Gamma \vdash s_1 \equiv s_3 : \Delta}\ \textsf{ctx-eq-trans}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s_1, s_2 : \Delta \quad \Delta \vdash t : \Theta \quad \Gamma \vdash s_1 \equiv s_2 : \Delta}{\Gamma \vdash t \circ s_1 \equiv t \circ s_2 : \Theta}\ \textsf{ctx-comp-cong-1}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash t : \Delta \quad \Delta \vdash s_1, s_2 : \Theta \quad \Delta \vdash s_1 \equiv s_2 : \Theta}{\Gamma \vdash s_1 \circ t \equiv s_2 \circ t : \Theta}\ \textsf{ctx-comp-cong-2}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma | A \vdash t : B}{\Gamma | A \vdash t \equiv t : B}\ \textsf{ty-eq-refl}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma | A \vdash t_1, t_2 : B \quad \Gamma | A \vdash t_1 \equiv t_2 : B}{\Gamma | A \vdash t_2 \equiv t_1 : B}\ \textsf{ty-eq-sym}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma | A \vdash t_1, t_2, t_3 : B \quad \Gamma | A \vdash t_1 \equiv t_2 : B \quad \Gamma | A \vdash t_2 \equiv t_3 : B}{\Gamma | A \vdash t_1 \equiv t_3 : B}\ \textsf{ty-eq-trans}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B, C \text{ type} \quad \Gamma | A \vdash u_1, u_2 : B \quad \Gamma | B \vdash v : C \quad \Gamma | A \vdash u_1 \equiv u_2 : B}{\Gamma | A \vdash v \circ u_1 \equiv v \circ u_2 : C}\ \textsf{ty-comp-cong-1}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B, C \text{ type} \quad \Gamma | A \vdash v : B \quad \Gamma | B \vdash u_1, u_2 : C \quad \Gamma | B \vdash u_1 \equiv u_2 : C}{\Gamma | A \vdash u_1 \circ v \equiv u_2 \circ v : C}\ \textsf{ty-comp-cong-2}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma | A \vdash t_1, t_2 : B \quad \Gamma | A \vdash t_1 \equiv t_2 : B}{\Gamma . A \vdash \Gamma . t_1 \equiv \Gamma . t_2 : \Gamma . B}\ \textsf{ext-cong}$$

$$\frac{\Gamma \vdash s : \Delta \quad \Delta | A \vdash t_1, t_2 : B \quad \Delta | A \vdash t_1 \equiv t_2 : B}{\Gamma | A[s] \vdash t_1[s] \equiv t_2[s] : B[s]}\ \textsf{sub-cong}$$

FIGURE 6.2: Rules of the type theory regarding $\equiv$ being a congruence.

1. Rule ctx-mor-id is interpreted as the identity morphisms in $\mathcal{C}$. This means $\|1_\Gamma\| := 1_{\|\Gamma\|}$, for each context Γ.
2. Rule ctx-mor-comp is interpreted as the composition of morphisms in $\mathcal{C}$. This means $\|s' \circ s\| := \|s'\| \circ \|s\|$ for each context Γ, Δ and Θ and context morphisms s from Γ to Δ and s' from Δ to Θ.
3. Rules ctx-id-unit-r and ctx-id-unit-l are interpreted as the unit laws of identity in $\mathcal{C}$.
4. Rule ctx-comp-assoc is interpreted as the associativity of composition in $\mathcal{C}$.
5. Rule empty-ctx is interpreted as the terminal object in $\mathcal{C}$.

6. Rules empty-ctx-mor and empty-ctx-mor-unique are interpreted as the universal property of the terminal object $\|\diamond\|$. This means that $\|\langle\rangle_\Gamma\|$ is the unique morphism from Γ to the terimanl object in $\mathcal{C}$.

7. Rule ty-mor-id is interpreted as the identity morphisms in $\mathcal{T}$. This means $\|1_A\| := 1_{\|A\|}$, where A is a type in context Γ.

8. Rule ty-mor-comp is interpreted as the composition of morphisms in $\mathcal{T}$. This means $\|t' \circ t\| := \|t'\| \circ \|t\|$ for types A, B and C in context Γ, term t of type A dependent on B and term t' of type B dependent on C.

9. Rules ty-id-unit-r and ty-id-unit-l are interpreted as the unit laws of identity in $\mathcal{T}$.

10. Rule ty-comp-assoc is interpreted as the associativity of composition in $\mathcal{T}$.

This means that $\Gamma \mathrel{\widetilde{\vdash}} s : \Delta$ is interpreted as $\|s\| : \|\Gamma\| \cong \|\Delta\|$ in $\mathcal{C}$ with the inverse $\|s^{-1}\|$. Similarly, $\Gamma | A \mathrel{\widetilde{\vdash}} t : B$ is interpreted as $\|t\| : \|A\| \cong \|B\|$ in $\mathcal{T}_{\|\Gamma\|}$ with the inverse $\|t^{-1}\|$.

The rules introduced in Section 6.1.3 regarding comprehension are interpreted as follows.

1. Rule ext-ty is interpreted as the action of χ_0 on the objects of $\mathcal{T}_\Gamma$. This means $\|\Gamma.A\| := \mathsf{dom}(\chi(\|A\|))$ for a type A in context Γ.

2. Rule ext-tm is interpreted as the action of χ_0 on the morphisms of $\mathcal{T}_\Gamma$. This means $\|\Gamma.t\| := \mathsf{dom}(\chi(\|t\|))$ for a term t of type B dependent on A in context Γ.

3. Rule ext-id is interpreted as χ_0 preserving identity.

4. Rule ext-comp is interpreted as χ_0 preserving composition.

5. Rule ext-proj is interpreted as the action of χ on the objects of $\mathcal{T}$. This means $\|\pi_A\| := \chi(\|A\|)$ for a type A in context Γ.

6. Rule ext-coh is interpreted as the following commuting diagram corresponding to $\chi(\|t\|)$ for a term t of type B dependent on A in context Γ:

$$\begin{tikzcd}
\|\Gamma.A\| \arrow[rr, "\|\Gamma.t\|"] \arrow[dr, "\|\pi_A\|"'] & & \|\Gamma.B\| \arrow[dl, "\|\pi_B\|"] \\
& \|\Gamma\|. &
\end{tikzcd}$$

The rules introduced in Section 6.1.4 regarding substitution are interpreted as follows.

1. Rules sub-ty and sub-tm are interpreted as the action of the reindexing functor $\|s\|^* : \mathcal{T}_{\|\Delta\|} \to \mathcal{T}_{\|\Gamma\|}$ on objects and morphisms respectively. This means $\|A[s]\| := \|s\|^*\|A\|$ and $\|t[s] := \|s\|^*\|t\|\|$, for contexts Γ and Δ, a context morphism s from Γ to Δ, types A and B in Δ and a term t of type A dependent on B in context Δ.

2. Rules sub-pres-id and sub-pres-comp are interpreted as the reindexing functor $\|s\|^* : \mathcal{T}_{\|\Delta\|} \to \mathcal{T}_{\|\Gamma\|}$ preserving identity and composition respectively, for contexts Γ and Δ and a context morphism s from Γ to Δ.

3. Rule sub-id is interpreted as the isomorphism $\|A[1_\Gamma]\| \cong \|A\|$, which is $1^*_{\|\Gamma\|}\|A\| \cong \|A\|$, for a type A in context Γ. This isomorphism is discussed in Remark 3.17.

4. Rule sub-comp is interpreted as the isomorphism $\|A[s' \circ s]\| \cong \|(A[s'])[s]\|$, which is $(\|s'\| \circ \|s\|)^*\|A\| \cong \|s\|^*(\|s'\|^*\|A\|)$, for a type A in context Θ and context morphisms s from Γ to Δ and s' from Δ to Θ. This is discussed in Remark 3.17.

5. Rule lift-compreh is interpreted as $\|s.A\| := \mathsf{dom}(\chi(\|s\|_{\|A\|}))$, where $\|s\| : \|\Gamma\| \to \|\Delta\|$ in $\mathcal{C}$, $\|A\| \in \mathcal{T}_{\|\Delta\|}$ and $\|s\|_{\|A\|}$ is the cartesian lift of $\|s\|$ into $\|A\|$.

6. Rule lift-coh is interpreted as the following commuting square corresponding to $\chi(\|s\|_{\|A\|}) : \|\pi_{A[s]}\| \to \|\pi_A\|$ in $\mathcal{C}^{\to}$:

$$\begin{tikzcd}
\|\Gamma.A[s]\| \arrow[r, "\|s.A\|"] \arrow[d, "\|\pi_{A[s]}\|"'] & \|\Delta.A\| \arrow[d, "\|\pi_A\|"] \\
\|\Gamma\| \arrow[r, "\|s\|"'] & \|\Delta\|.
\end{tikzcd}$$

7. Rule iid-coh is interpreted as the following commuting diagram in $\mathcal{C}$:

$$\begin{tikzcd}
\|\Gamma.A\| \arrow[dr, "\|1_{\Gamma.A}\|"] & \\
\|\Gamma.A[1_\Gamma]\| \arrow[u, "\|\Gamma.i_{\mathsf{id}_A}\|"] \arrow[r, "\|1_\Gamma.A\|"'] & \|\Gamma.A\|
\end{tikzcd}$$

This follows from the following commuting diagram in $\mathcal{T}$ that we have from Remark 3.17:

$$\begin{tikzcd}
\|A\| \arrow[dr, "\|1_\Gamma\|"] & & \\
\|A[1_\Gamma]\| \arrow[u, "\|i_{\mathsf{id}_A}\|"] \arrow[r, "\|1_\Gamma\|_{\|A\|}"'] & \|A\| & \mathcal{T} \arrow[d, "p"] \\
\|\Gamma\| \arrow[r, "\|1_\Gamma\|"'] & \|\Gamma\| & \mathcal{C},
\end{tikzcd}$$

where $\|1_\Gamma\|_{\|A\|}$ is a cartesian lift.

8. Rule icomp-coh is interpreted as the following commuting diagram in $\mathcal{C}$:

$$\begin{tikzcd}
\|\Gamma.A[s' \circ s]\| \arrow[d, dashed, "\|\Gamma.i_{\mathsf{comp}_{A,s,s'}}\|"'] \arrow[drr, "\|(s' \circ s).A\|"] & & \\
\|\Gamma.A[s'][s]\| \arrow[r, "\|s.A[s']\|"'] & \|\Delta.A[s']\| \arrow[r, "\|s'.A\|"'] & \|\Theta.A\|.
\end{tikzcd}$$

This follows from the following commuting diagram in $\mathcal{T}$ that we have from Lemma 3.7 and Remark 3.17:

$$\begin{tikzcd}
\|A[s' \circ s]\| \arrow[d, dashed, "\|i_{\mathsf{comp}_{A,s,s'}}\|"'] \arrow[drr, "\|(s' \circ s)\|_{\|A\|}"] & & & \mathcal{T} \arrow[dd, "p"] \\
\|A[s'][s]\| \arrow[r, "\|s\|_{\|A[s']\|}"'] & \|A[s']\| \arrow[r, "\|s'\|_{\|A\|}"'] & \|A\| & \\
\|\Gamma\| \arrow[r, "s"'] & \|\Delta\| \arrow[r, "s'"'] & \|\Theta\| & \mathcal{C},
\end{tikzcd}$$

where $\|s'\|_{\|A\|}$, $\|s\|_{\|A[s']\|}$ and $\|(s' \circ s)\|_{\|A\|}$ are cartesian lifts.

6.3 Comprehension Preserving Cartesian Morphisms

As mentioned in Section 6.1.4, one of the differences between this type theory and MLTT is that here, context morphisms are not built out of terms. By context morphisms being built out of terms we mean that given $\Gamma \vdash s : \Delta$ and $\Delta \vdash A$ type, there is a bijection between terms of the form $\Gamma | \mathbb{1}_\Gamma \vdash t : A[s]$ and context morphisms of the form $\Gamma \vdash s' : \Delta.A$ that satisfy $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$, where $\mathbb{1}_\Gamma$ is unit type.

In this section we discuss the rules that need to be added to the type theory to have context morphism be built out of certain sections of the projection context morphisms (MLTT style terms), and what this means semantically. We will see that semantically this corresponds to χ preserving cartesian morphisms. In Section 6.4 we discuss how unit type can be added to the type theory, and the requirements for having a bijection between terms of the form $\Gamma | \mathbb{1}_\Gamma \vdash t : A$ and MLTT style terms of the form $\Gamma \vdash t : A$. Adding this to the type theory, results in having context morphisms that are built out of terms, in the sense described above.

Having context morphisms built out of MLTT style terms means that given $\Gamma \vdash s : \Delta$ and $\Delta \vdash A$ type, we have a bijection between MLTT style terms of the form $\Gamma \vdash t_{s,s'} : A[s]$, and context morphisms of the form $\Gamma \vdash s' : \Delta.A$ that satisfy $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$.

First, in Lemma 6.4 we show that with the rules of the type theory given in Figs. 6.1 and 6.2, given a section t of $\pi_{A[s]}$, $s.A \circ t$ is a context morphism from Γ to $\Delta.A$ that satisfies $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$. Next, in Lemma 6.5 we show that if the two following rules are added to the rules in Figs. 6.1 and 6.2, we get context morphisms that are made of MLTT style terms

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Delta \vdash A \ \mathsf{type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash s' : \Delta.A \quad \Gamma \vdash s \equiv \pi_A \circ s' : \Delta}{\begin{array}{c}\Gamma \vdash t_{s,s'} : A[s] \\ \Gamma \vdash s.A \circ t_{s,s'} \equiv s' : \Delta.A\end{array}} \ \mathsf{ctx\text{-}mor\text{-}to\text{-}ml}$$

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Delta \vdash A \ \mathsf{type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : A[s]}{\Gamma \vdash t_{s,s.A \circ t} \equiv t : A[s]} \ \mathsf{ctx\text{-}mor\text{-}to\text{-}ml\text{-}unique}$$

Lastly, in Lemma 6.6 we show that semantically, this is equivalent to χ preserving cartesian morphisms.

Lemma 6.4. *From the rules of the type theory given in Figs. 6.1 and 6.2, we can derive the following two rules:*

$$\frac{\Gamma, \Delta \ \mathsf{ctx} \quad \Delta \vdash A \ \mathsf{type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : A[s]}{\begin{array}{c}\Gamma \vdash s.A \circ t : \Delta.A \\ \Gamma \vdash \pi_A \circ s.A \circ t \equiv s : \Delta\end{array}}$$

Proof. The first conclusion can be derived as follows:

$$\dfrac{\dfrac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Delta \vdash A \text{ type}}{\Gamma.A[s] \vdash s.A : \Delta.A}\,\textsf{lift-compreh} \qquad \Gamma \vdash t : A[s]}{\Gamma \vdash s.A \circ t : \Delta.A}\,\textsf{ctx-mor-comp}$$

For the second conclusion we first show that $\Gamma \vdash s \circ \pi_{A[s]} \circ t \equiv s : \Delta$ using the assumption that t is a section of $\pi_{A[s]}$:

$$\dfrac{\dfrac{\Gamma \vdash \pi_{A[s]} \circ t \equiv 1_\Gamma : \Gamma \quad \Gamma \vdash s : \Delta}{\Gamma \vdash s \circ \pi_{A[s]} \circ t \equiv s \circ 1_\Gamma : \Delta}\,\textsf{ctx-comp-cong-1} \qquad \dfrac{\Gamma \vdash s : \Delta}{\Gamma \vdash s \circ 1_\Gamma \equiv s : \Delta}\,\textsf{ctx-id-unit-r}}{\Gamma \vdash s \circ \pi_{A[s]} \circ t \equiv s : \Delta}\,\textsf{ctx-eq-trans}$$

Now, using Rule lift-coh we can get the second conclusion as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash s : \Delta \quad \Delta \vdash A \text{ type}}{\Gamma.A[s] \vdash \pi_A \circ s.A \equiv s \circ \pi_{A[s]} : \Delta}\,\textsf{lift-coh} \qquad \Gamma \vdash t : \Gamma.A[s]}{\Gamma \vdash \pi_A \circ s.A \circ t \equiv s \circ \pi_{A[s]} \circ t : \Delta}\,\textsf{ctx-comp-cong-2} \qquad \Gamma \vdash s \circ \pi_{A[s]} \circ t \equiv s : \Delta}{\Gamma \vdash \pi_A \circ s.A \circ t \equiv s : \Delta}\,\textsf{ctx-eq-trans}$$

Semantically, this is showing that the following outer diagram commutes:

$$\begin{tikzcd}
\Gamma \arrow[dr, "t"] \arrow[drr, bend left, "s.A \circ t"] \arrow[ddr, bend right, equal] & & \\
& \Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"'] & \Delta.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

□

Lemma 6.5. *If the following two rules are added to the rules given in Figs. 6.1 and 6.2, the context morphisms of the resulting type theory are built out of MLTT style terms:*

$$\dfrac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash s' : \Delta.A \quad \Gamma \vdash s \equiv \pi_A \circ s' : \Delta}{\begin{array}{c}\Gamma \vdash t_{s,s'} : A[s] \\ \Gamma \vdash s.A \circ t_{s,s'} \equiv s' : \Delta.A\end{array}}\,\textit{ctx-mor-to-ml}$$

$$\dfrac{\Gamma, \Delta \text{ ctx} \quad \Delta \vdash A \text{ type} \quad \Gamma \vdash s : \Delta \quad \Gamma \vdash t : A[s]}{\Gamma \vdash t_{s,s.A \circ t} \equiv t : A[s]}\,\textit{ctx-mor-to-ml-unique}$$

Proof. Let A be a type in context Δ, and s a context morphism from Γ to Δ. We need to show that if Rules ctx-mor-to-ml and ctx-mor-to-ml-unique are added to the rules in Figs. 6.1 and 6.2, there is a bijection between MLTT style terms of the form $\Gamma \vdash t : A[s]$ and context morphisms of the form $\Gamma \vdash s' : \Delta.A$ that satisfy $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$.

We know from Lemma 6.4 that for each $\Gamma \vdash t : A[s]$, $s.A \circ t$ is a context morphism from Γ to $\Delta.A$ that satisfies $\Gamma \vdash \pi_A \circ s.A \circ t \equiv s : \Delta$. We know from the first conclusion of Rule ctx-mor-to-ml that for each context morphism of the form $\Gamma \vdash s' : \Delta.A$ satisfying $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$, we have an MLTT style term $\Gamma \vdash t_{s,s'} : A[s]$.

We need to show that for each $\Gamma \vdash t : A[s]$, $t_{s,s.A\circ t}$ given by Rule ctx-mor-to-ml is equal to t. This is given by Rule ctx-mor-to-ml-unique. We also need to show that for each $\Gamma \vdash s' : \Delta.A$ satisfying $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$, the corresponding context morphism from Γ to Δ, which is $s.A \circ t_{s,s'}$, is equal to s'. This is given by the second conclusion of Rule ctx-mor-to-ml. ☐

Semantically, Rules ctx-mor-to-ml and ctx-mor-to-ml-unique correspond to having a unique $u : \Gamma \to \Gamma.A[s]$ in $\mathcal{C}$ such that $\pi_{A[s]} \circ u = 1_\Gamma$ and $s.A \circ u = s'$ for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$, $s : \Gamma \to \Delta$ and $s' : \Gamma \to \Delta.A$ with $\pi_A \circ s' = s$:

$$\begin{tikzcd}
\Gamma \arrow[dr, "u"] \arrow[drr, bend left, "s'"] \arrow[ddr, equal, bend right] & & \\
& \Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"'] & \Delta.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

In the following lemma, we show that this condition holds in a (weak) comprehension category, if and only if χ preserves cartesian morphisms.

Lemma 6.6. *In a (weak) comprehension category $(\mathcal{C}, \mathcal{T}, \chi, p)$, χ preserves cartesian morphisms if and only if for each $\Gamma, \Delta \in \mathcal{C}$, $A \in \mathcal{T}_\Delta$, $s : \Gamma \to \Delta$ and $s' : \Gamma \to \Delta.A$ with $\pi_A \circ s' = s$, there exists a unique $u : \Gamma \to \Gamma.A[s]$ such that $\pi_{A[s]} \circ u = 1_\Gamma$ and $s.A \circ u = s'$. In other words, all commuting squares of the following form in $\mathcal{C}$ are pullback squares:*

$$\begin{tikzcd}
\Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\
\Gamma \arrow[r, "s"'] & \Delta,
\end{tikzcd}$$

if and only if for all commuting squares of such form and $s' : \Gamma \to \Gamma.A[s]$ with $\pi_A \circ s' = s$, there exists a unique $u : \Gamma \to \Gamma.A[s]$ such that $\pi_{A[s]} \circ u = 1_\Gamma$ and $s.A \circ u = s'$:

$$\begin{tikzcd}
\Gamma \arrow[dr, "u"] \arrow[drr, bend left, "s'"] \arrow[ddr, equal, bend right] & & \\
& \Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"'] & \Delta.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

Proof. Let $(\mathcal{C}, \mathcal{T}, \chi, p)$ be a (weak) comprehension category where for all $s : \Gamma \to \Delta$ and $s' : \Gamma \to \Delta.A$ in $\mathcal{C}$ such that $\pi_A \circ s' = s$, there exists a unique $u : \Gamma \to \Gamma.A[s]$ such that $\pi_{A[s]} \circ u = 1_\Gamma$ and $s.A \circ u = s'$. We show that χ preserves cartesian morphisms. Let $s : \Gamma \to \Delta$, $s_\Theta : \Theta \to \Gamma$ and $s'_\Theta : \Theta \to \Delta.A$ in $\mathcal{C}$ such that $\pi_A \circ s'_\Theta = s \circ s_\Theta$. We need to show that there exists a unique

$u_\Theta : \Theta \to \Gamma.A[s]$ such that $\pi_{A[s]} \circ u_\Theta = s_\Theta$ and $s.A \circ u_\Theta = s'_\Theta$:

$$\begin{tikzcd}
\Theta \arrow[dr, "u_\Theta"] \arrow[drr, bend left, "s'_\Theta"] \arrow[ddr, bend right, "s_\Theta"'] & & \\
& \Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

Since $\pi_A \circ s'_\Theta = s \circ s_\Theta$, using the assumption, there exists a unique $u_1 : \Theta \to \Theta.A[s][s_\Theta]$ such that

$$s.A \circ s_\Theta.A[s] \circ u_1 = s'_\Theta, \tag{6.1}$$

and

$$\pi_{A[s][s_\Theta]} \circ u_1 = 1_\Theta, \tag{6.2}$$

in the following diagram:

$$\begin{tikzcd}
\Theta \arrow[dr, "u_1"] \arrow[drrr, bend left, "s'_\Theta"] \arrow[ddr, bend right, equal] & & & \\
& \Theta.A[s][s_\Theta] \arrow[r, "s_\Theta.A[s]"] \arrow[d, "\pi_{A[s][s_\Theta]}"] & \Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"] & \Delta.A \arrow[d, "\pi_A"] \\
& \Theta \arrow[r, "s_\Theta"] & \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

We define $u_\Theta := s_\Theta.A[s] \circ u_1$ as a candidate for the unique morphism of the form $\Theta \to \Gamma.A[s]$ that satisfies $s.A \circ u_\Theta = s'_\Theta$ and $\pi_{A[s]} \circ u_\Theta = s_\Theta$. We have the first requirement from:

$$\begin{aligned}
s.A \circ u_\Theta &= s.A \circ s_\Theta.A[s] \circ u_1 && \text{(by definition of } u_\Theta) && (6.3)\\
&= s'_\Theta, && \text{(by (6.1))} && (6.4)
\end{aligned}$$

and the second from:

$$\begin{aligned}
\pi_{A[s]} \circ u_\Theta &= \pi_{A[s]} \circ s_\Theta.A[s] \circ u_1 && \text{(by definition of } u_\Theta) \\
&= s_\Theta \circ \pi_{A[s][s_\Theta]} && \text{(by commuting of the left inner square)} \\
&= s_\Theta \circ 1_\Theta && \text{(by (6.2))} \\
&= s_\Theta.
\end{aligned}$$

Now, we need to show that u_Θ is a unique morphism that satisfies these requirements. Let $t : \Theta \to$

$\Gamma.A[s]$ such that $s.A \circ t = s'_\Theta$ and $\pi_{A[s][s_\Theta]} \circ t = s_\Theta$. Since $\pi_{A[s]} \circ u_\Theta = s_\Theta$, using the assumption we have that there exists a unique $u : \Theta \to \Theta.A[s][s_\Theta]$ such that $s_\Theta.A[s] \circ u = u_\Theta$ and $\pi_{A[s][s_\Theta]} \circ u = 1_\Theta$. We know from the definition of u_Θ and (6.2), that u_1 satisfies these requirements; hence, we have:

$$s_\Theta.A[s] \circ u_1 = u_\Theta. \tag{6.5}$$

Similarly, since $\pi_{A[s]} \circ t = s_\Theta$, using the assumption we have that there exists a unique $u_t : \Theta \to \Theta.A[s][s_\Theta]$ such that $s_\Theta.A[s] \circ u_t = t$ and $\pi_{A[s][s_\Theta]} \circ u_t = 1_\Theta$. From $s_\Theta.A[s] \circ u_t = t$ and $s.A \circ t = s'_\Theta$, we get $s.A \circ s_\Theta.A[s] \circ u_t = s'_\Theta$. But we know that u_1 is the unique morphism that satisfies (6.1) and (6.2), which means $u_t = u_1$. Hence, we have:

$$s_\Theta.A[s] \circ u_1 = t. \tag{6.6}$$

From (6.5) and (6.6), we get $u_\Theta = t$. This concludes showing that the following is a pullback square in $\mathcal{C}$:

$$\begin{tikzcd}
\Theta \arrow[dr, "u_\Theta"] \arrow[drr, bend left, "s'_\Theta"] \arrow[ddr, bend right, "s_\Theta"'] & & \\
& \Gamma.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_{A[s]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.A \arrow[d, "\pi_A"] \\
& \Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

The converse holds as a special case of the definition of pullback. $\square$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \qquad \Delta \vdash A\ \mathsf{type} \qquad \Gamma \vdash s : \Delta \qquad \Gamma \vdash s' : \Delta.A \qquad \Gamma \vdash s \equiv \pi_A \circ s' : \Delta}{\begin{array}{c}\Gamma \vdash t_{s,s'} : A[s] \\ \Gamma \vdash s.A \circ t_{s,s'} \equiv s' : \Delta.A\end{array}}\ \textsf{ctx-mor-to-ml}$$

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \qquad \Delta \vdash A\ \mathsf{type} \qquad \Gamma \vdash s : \Delta \qquad \Gamma \vdash t : A[s]}{\Gamma \vdash t_{s,s.A \circ t} \equiv t : A[s]}\ \textsf{ctx-mor-to-ml-unique}$$

FIGURE 6.3: Rules of the type theory regarding χ preserving cartesian morphisms.

Remark 6.7. As a result of Lemma 6.6, if we start from a comprehension category and follow the approach used in Section 6.1 to obtain a type theory, the resulting type theory would have the rules from Figs. 6.1 to 6.3.

6.4 Unit Type

In Section 6.4.1, we discuss adding unit type to the type theory given in Figs. 6.1 to 6.3. Following the approach taken in Section 6.1, we start from an appropriate notion of comprehension category with unit and derive the rules such that all the structure is reflected in the type theory. We use the definition of unit type given in [Lin21] for comprehension categories that are not full. Next, in Section 6.4.2 we show soundness by giving an interpretation of the rules concerning unit type in any such comprehension category with unit. Finally, in Section 6.4.3, we discuss the requirement for having terms (type morphisms) on the form $\Gamma|\mathbb{1}_\Gamma \vdash t : A$ coincide with MLTT style terms of the form $\Gamma \vdash t : A$.

6.4.1 Unit Type from Semantics

Similar to Chapter 5, for each context Γ, we define unit type in context Γ, as a type $\mathbb{1}_\Gamma$ such that the projection context morphism $\pi_{\mathbb{1}_\Gamma}$ has a unique section. This is expressed as:

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma\ \mathsf{type}}\ \mathbb{1}\text{-form} \qquad \frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \mathbb{1}\text{-intro} \qquad \frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash t : \mathbb{1}_\Gamma}{\Gamma \vdash t \equiv \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \mathbb{1}\text{-unique}$$

where the last rule uses the notation introduced in Notation 6.2. Since our type theory has explicit substitution, we also have to include rules expressing how unit type behaves with respect to reindexing (substitution). The preservation of unit under substitution (up to isomorphism) can be expressed as:

$$\frac{\Gamma, \Delta\ \mathsf{ctx} \quad \Gamma \vdash s : \Delta}{\Gamma|\mathbb{1}_\Delta[s] \overset{\sim}{\vdash} i_{\mathbb{1},s} : \mathbb{1}_\Gamma}\ \mathsf{sub\text{-}pres\text{-}}\mathbb{1}$$

We add these rules to the type theory following the approach used in Section 6.1. We start from an appropriate notion of comprehension category with unit and extract the rules for adding unit type such that all the structure of the comprehension category is reflected in the type theory. Note that we are extending the type theory with the rules in Figs. 6.1 to 6.3, i.e. the rules corresponding to χ preserving cartesian morphisms are included; however, whenever the rules in Fig. 6.3 are used, we explicitly mention it.

The appropriate notion of a comprehension category with unit for Rules $\mathbb{1}$-form, $\mathbb{1}$-intro, $\mathbb{1}$-unique and sub-pres-$\mathbb{1}$ has for each object $\Gamma \in \mathcal{C}$:

1. an object $\mathbb{1}_\Gamma$ in $\mathcal{C}_\Gamma$;
2. a unique section $\mathsf{tt}_\Gamma : \Gamma \to \Gamma.\mathbb{1}_\Gamma$ of $\pi_{\mathbb{1}_\Gamma}$in $\mathcal{C}$;
3. such that for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, we have $s^*\mathbb{1}_\Delta \cong \mathbb{1}_\Gamma$, i.e. the reindexing functors preserve $\mathbb{1}$ (up to isomorphism).

We will first show that requirements 1 and 3, can be thought of as having a fibred functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ in :

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathbb{1}"] \arrow[dr, "\mathsf{id}_\mathcal{C}"'] & & \mathcal{T} \arrow[dl, "p"] \\ & \mathcal{C}. & \end{tikzcd}$$

For each $\Gamma \in \mathcal{C}$, we denote $\mathbb{1}(\Gamma)$ as $\mathbb{1}_\Gamma$. Then, we show that requirement 2, is equivalent to having

$$\mathcal{C}_s(\mathsf{cod}(s'), \Delta) \cong \mathcal{C}^{\to}_s(s', \pi_{\mathbb{1}_\Delta}),$$

for each $s : \Gamma \to \Delta$ in $\mathcal{C}$ and morphism s' in $\mathcal{C}$ with $\mathsf{cod}(s') = \Gamma$, which can be thought of as an adjunction $\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "\chi\circ\mathbb{1}"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\to} \arrow[l, shift right, "\mathsf{cod}"'] \end{tikzcd}$ that is “fibred” in some appropriate sense (see Remark 6.9). Finally, we will extract the rules for unit type from a comprehension category with unit defined as follows.

Definition 6.8. A comprehension category has unit types if there is a fibred functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ in :

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathbb{1}"] \arrow[dr, "\mathsf{id}_{\mathcal{C}}"'] & & \mathcal{T} \arrow[dl, "p"] \\ & \mathcal{C}, & \end{tikzcd}$$

and an adjunction $\mathcal{C} \underset{\chi\circ\mathbb{1}}{\overset{\mathsf{cod}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{C}^{\to}$ that is "fibred" in the sense that for each $s : \Gamma \to \Delta$ in $\mathcal{C}$ and morphism s' in $\mathcal{C}$ with $\mathsf{cod}(s') = \Gamma$, we have:

$$\mathcal{C}_s(\mathsf{cod}(s'), \Delta) \cong \mathcal{C}_s^{\to}(s', \pi_{\mathbb{1}_\Delta}),$$

as explained in Proposition 3.22.

Remark 6.9. As explained in [Lin21, Remark 3.1.0.5], the adjunction in Definition 6.8 is not a fibred adjunction in the proper sense (see Definition 3.21), as cod is not a fibration unless $\mathcal{C}$ has all pullbacks.

Remark 6.10 ([Lin21, Remark 3.1.0.6]). The isomorphism

$$\mathcal{C}_s(\mathsf{cod}(s'), \Delta) \cong \mathcal{C}_s^{\to}(s', \pi_{\mathbb{1}_\Delta}),$$

implies that for each $s : \Gamma \to \Delta$ and $s' : \Theta \to \Gamma$ in $\mathcal{C}$, there is a unique morphism$s : \Theta \to \Delta.\mathbb{1}_\Delta$ making the following diagram commute:

$$\begin{tikzcd} \Theta \arrow[r, "u"] \arrow[d, "s'"'] & \Delta.\mathbb{1}_\Delta \arrow[d, "\pi_{\mathbb{1}_\Delta}"] \\ \Gamma \arrow[r, "s"'] & \Delta. \end{tikzcd}$$

This is because $\mathcal{C}_s$ is fibred over $\mathcal{C}$ by $\mathsf{id}_{\mathcal{C}}$; hence the only morphism above s is s itself.

Remark 6.11. Definition 6.8 is equivalent to the definition of a comprehension category with pseudo-stable unit types defined in [Lin21, Definition 3.1.0.3], by Lemma 6.14. This is, however, different from the comprehension category with unit discussed in Definition 5.9, where we have the assumption of χ being fully faithful.

Lemma 6.12. *Given a functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ with $p \circ \mathbb{1} = \mathsf{id}_{\mathcal{C}}$:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathbb{1}"] \arrow[dr, "\mathsf{id}_{\mathcal{C}}"'] & & \mathcal{T} \arrow[dl, "p"] \\ & \mathcal{C}, & \end{tikzcd}$$

the following are equivalent:

1. *$\mathbb{1}$ is a fibred functor;*
2. *for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, $s^*\mathbb{1}_\Delta \cong \mathbb{1}_\Gamma$, where s^* is the reindexing functor corresponding to the fibration $p : \mathcal{T} \to \mathcal{C}$.*

Proof. First, we show if $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ preserves cartesian morphisms, then for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, $s^*\mathbb{1}_\Delta \cong \mathbb{1}_\Gamma$. Because $\mathcal{C}$ is fibred over $\mathcal{C}$ by $\mathsf{id}_{\mathcal{C}}$, then for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, s is cartesian in $\mathcal{C}$. Since

$\mathbb{1} : \mathcal{C} \to \mathcal{T}$ preserves cartesian morphisms, for each $s : \Gamma \to \Delta$, $\mathbb{1}(s)$ is cartesian in $\mathcal{T}$. We have $s^*\mathbb{1}_\Delta \cong \mathbb{1}_\Gamma$ from Lemma 3.7.

Next, we show if for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, $s^*\mathbb{1}_\Delta \cong \mathbb{1}_\Gamma$, then $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ preserves cartesian morphisms. Because $\mathcal{C}$ is fibred over $\mathcal{C}$ by $\mathsf{id}_\mathcal{C}$, then for each $s : \Gamma \to \Delta$ in $\mathcal{C}$, s is cartesian in $\mathcal{C}$. This means that we need to show $\mathbb{1}(s) : \mathbb{1}_\Gamma \to \mathbb{1}_\Delta$ is cartesian for each $s : \Gamma \to \Delta$. We have $\mathbb{1}(s)$ is cartesian from Lemma 3.8.

$$\begin{tikzcd}
s^*\mathbb{1}_\Delta \arrow[d, "\cong"'] \arrow[dr, "s_{\mathbb{1}_\Delta}"] & & \\
\mathbb{1}_\Gamma \arrow[r, "\mathbb{1}(s)"'] & \mathbb{1}_\Delta & \mathcal{T} \arrow[d, "p"] \\
\Gamma \arrow[r, "s"'] & \Delta & \mathcal{C}.
\end{tikzcd}$$

□

Lemma 6.13. *Given a fibred functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ in*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathbb{1}"] \arrow[dr, "\mathsf{id}_\mathcal{C}"'] & & \mathcal{T} \arrow[dl, "p"] \\
& \mathcal{C}, &
\end{tikzcd}$$

for each $s : \Gamma \to \Delta$, the following diagram is a pullback squares in $\mathcal{C}$:

$$\begin{tikzcd}
\Gamma.\mathbb{1}_\Gamma \arrow[r, "\chi_0(\mathbb{1}(s))"] \arrow[d, "\pi_{\mathbb{1}_\Gamma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.\mathbb{1}_\Delta \arrow[d, "\pi_{\mathbb{1}_\Delta}"] \\
\Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

Proof. Since both $\mathbb{1}$ and χ preserve cartesian morphisms and cartesian morphisms in $\mathcal{C}^\to$ are pullback squares in $\mathcal{C}$. □

Lemma 6.14 ([Lin21, Propositions 3.1.0.4 and 3.1.0.8]). *In a comprehension category equipped with a fibred functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$, the following are equivalent:*

1. *for each $\Gamma \in \mathcal{C}$, $\pi_{\mathbb{1}_\Gamma}$ has a unique section $\mathsf{tt}_\Gamma : \Gamma \to \Gamma.\mathbb{1}_\Gamma$;*
2. *for each $s : \Gamma \to \Delta$ in $\mathcal{C}$ and morphism s' in $\mathcal{C}$ with $\mathsf{cod}(s') = \Gamma$, we have:*

$$\mathcal{C}_s(\mathsf{cod}(s'), \Delta) \cong \mathcal{C}^\to_s(s', \pi_{\mathbb{1}_\Delta}).$$

Proof. We know from Remark 6.10, that the second statement is equivalent to having a unique morphisms u that makes the following diagram commute:

$$\begin{tikzcd}
\Theta \arrow[r, "u"] \arrow[d, "s'"'] & \Delta.\mathbb{1}_\Delta \arrow[d, "\pi_{\mathbb{1}_\Delta}"] \\
\Gamma \arrow[r, "s"'] & \Delta.
\end{tikzcd}$$

for each $s : \Gamma \to \Delta$ and $s' : \Theta \to \Gamma$ in $\mathcal{C}$.

First, we show $2 \Rightarrow 1$. Let $\Gamma \in \mathcal{C}$. By taking $s = s' = 1_\Gamma$, we have that there exists a unique morphisms $\mathsf{tt}_\Gamma : \Gamma \to \Gamma.\mathbb{1}_\Gamma$ that makes the following diagram commute:

$$\begin{tikzcd}
\Gamma \arrow[r, "\mathsf{tt}_\Gamma"] \arrow[d, "1_\Gamma"'] & \Gamma.\mathbb{1}_\Gamma \arrow[d, "\pi_{\mathbb{1}_\Gamma}"] \\
\Gamma \arrow[r, "1_\Gamma"'] & \Gamma.
\end{tikzcd}$$

This means that $\pi_{\mathbb{1}_\Gamma}$ has a unique section.

Next, we show $1 \Rightarrow 2$. Let $s : \Gamma \to \Delta$ and $s' : \Theta \to \Gamma$ in $\mathcal{C}$. We need to show that there exists a unique morphism $u : \Theta \to \Gamma.\mathbb{1}_\Gamma$ such that $\pi_{\mathbb{1}_\Gamma} \circ u = s \circ s'$. We take u to be $\chi s \circ \chi s' \circ \mathsf{tt}_\Theta$:

$$\begin{tikzcd}
\Theta \arrow[dr, dashed, "\mathsf{tt}_{\mathbb{1}_\Theta}"] \arrow[drrr, bend left, "u"] \arrow[ddr, equal] & & & \\
& \Theta.\mathbb{1}_\Theta \arrow[r, "\chi(s')"'] \arrow[d, "\pi_{\mathbb{1}_\Theta}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta.\mathbb{1}_\Delta \arrow[r, "\chi(s)"'] \arrow[d, "\pi_{\mathbb{1}_\Delta}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.\mathbb{1}_\Gamma \arrow[d, "\pi_{\mathbb{1}_\Gamma}"] \\
& \Theta \arrow[r, "s'"'] & \Delta \arrow[r, "s"'] & \Gamma.
\end{tikzcd}$$

This choice of u satisfies $\pi_{\mathbb{1}_\Gamma} \circ u = s \circ s'$ since $\mathsf{tt}_{\mathbb{1}_\Theta}$ is a section of $\pi_{\mathbb{1}_\Theta}$ and both of the two inner squares commute. To show uniqueness, let $u' : \Theta \to \Gamma.\mathbb{1}_\Gamma$ such that $\pi_{\mathbb{1}_\Gamma} \circ u' = s \circ s'$. We know from Lemma 6.13, that the outer square is a pullback square in $\mathcal{C}$; hence, there exists a unique $e : \Theta \to \Theta.\mathbb{1}_\Theta$ such that $\pi_{\mathbb{1}_\Theta} \circ e = 1_\Theta$ and $u' = \chi s \circ \chi s' \circ e$. Since, $\mathsf{tt}_{\mathbb{1}_\Theta}$ is the unique section of $\pi_{\mathbb{1}_\Theta}$, we have $u = \chi s \circ \chi s' \circ \mathsf{tt}_\Theta = \chi s \circ \chi s' \circ e = u'$. $\square$

Lemma 6.15. *In a comprehension category with unit as defined in Definition 6.8, for each* $\Gamma \in \mathcal{C}$*, we have* $\Gamma \cong \Gamma.\mathbb{1}_\Gamma$.

Proof. Let $\Gamma \in \mathcal{C}$. We know from Lemma 6.14 that $\pi_{\mathbb{1}_\Gamma}$ has a unique section tt. Hence, to prove $\Gamma \cong \Gamma.\mathbb{1}_\Gamma$, we need to show that $\mathsf{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} = 1_{\Gamma.\mathbb{1}_\Gamma}$. Using Remark 6.10 and taking $s = 1_\Gamma$ and $s' = \pi_{\mathbb{1}_\Gamma}$ we have there there is a unique morphism $u : \Gamma.\mathbb{1}_\Gamma \to \Gamma.\mathbb{1}_\Gamma$ that makes the following diagram commute:

$$\begin{tikzcd}
\Gamma.\mathbb{1}_\Gamma \arrow[r, "u"] \arrow[d, "\pi_{\mathbb{1}_\Gamma}"'] & \Gamma.\mathbb{1}_\Gamma \arrow[d, "\pi_{\mathbb{1}_\Gamma}"] \\
\Gamma \arrow[r, equal] & \Gamma.
\end{tikzcd}$$

Since $1_{\Gamma.\mathbb{1}_\Gamma}$ makes the diagram commute, we have that $1_{\Gamma.\mathbb{1}_\Gamma}$ is the unique morphism such that $\pi_{\mathbb{1}_\Gamma} \circ 1_{\Gamma.\mathbb{1}_\Gamma} = \pi_{\mathbb{1}_\Gamma}$. We know $\pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma = 1_\Gamma$ as tt_Γ is a section. This gives $\pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} = 1_\Gamma \circ \pi_{\mathbb{1}_\Gamma} = \pi_{\mathbb{1}_\Gamma}$. Since $1_{\Gamma.\mathbb{1}_\Gamma}$ is the unique morphism such that $\pi_{\mathbb{1}_\Gamma} \circ 1_{\Gamma.\mathbb{1}_\Gamma} = \pi_{\mathbb{1}_\Gamma}$, we get $\mathsf{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} = 1_{\Gamma.\mathbb{1}_\Gamma}$. $\square$

Now that we have shown Lemmas 6.12 and 6.14, we start from a comprehension category with unit, that is equipped with a fibred functor $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ with all the projections of units having unique sections, and extract the rules for adding unit type to the type theory. From $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ being a functor that satisfies $p \circ \mathbb{1} = \mathsf{id}_\mathcal{C}$, we gets the following rules:

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma\ \mathsf{type}}\ \mathbb{1}\text{-form}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma.\mathbb{1}_\Gamma \vdash \chi_0(\mathbb{1}(s)) : \Delta.\mathbb{1}_\Delta}\ \textsf{compreh-}\mathbb{1}\textsf{-mor}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_{\mathbb{1}_\Delta} \circ \chi_0(\mathbb{1}(s)) \equiv s \circ \pi_{\mathbb{1}_\Gamma} : \Delta}\ \textsf{compreh-}\mathbb{1}\textsf{-coh}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma.\mathbb{1}_\Gamma \vdash \chi_0(\mathbb{1}(1_\Gamma)) \equiv 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma}\ \textsf{compreh-}\mathbb{1}\textsf{-id}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta}{\Gamma.\mathbb{1}_\Gamma \vdash \chi_0(\mathbb{1}(s')) \circ \chi_0(\mathbb{1}(s)) \equiv \chi_0(\mathbb{1}(s' \circ s)) : \Theta.\mathbb{1}_\Theta}\ \textsf{compreh-}\mathbb{1}\textsf{-comp}$$

Rule $\mathbb{1}$-form corresponds to the action of $\mathbb{1}$ on the objects of $\mathcal{C}$. The action of $\mathbb{1}$ on morphisms gives $\mathbb{1}(s) : \mathcal{T}(\mathbb{1}_\Gamma, \mathbb{1}_\Delta)$ for each $s : \Gamma \to \Delta$ in $\mathcal{C}$. Rules compreh-$\mathbb{1}$-mor and compreh-$\mathbb{1}$-coh correspond to $\chi(\mathbb{1}(s)) : \mathcal{C}^{\to}(\pi_{\mathbb{1}_\Gamma}, \pi_{\mathbb{1}_\Delta})$. Rules compreh-$\mathbb{1}$-id and compreh-$\mathbb{1}$-comp correspond to $\mathbb{1}$, and consequently $\chi \circ \mathbb{1}$, preserving identity and composition.

From $\mathbb{1}$ preserves cartesian morphisms, we get the following rule using Lemma 6.12:

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma | \mathbb{1}_\Delta[s] \mathrel{\widetilde{\vdash}} i_{\mathbb{1},s} : \mathbb{1}_\Gamma}\ \textsf{sub-pres-}\mathbb{1}$$

From all projections of units having unique sections, we get:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \mathbb{1}\textsf{-intro}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash t : \mathbb{1}_\Gamma}{\Gamma \vdash t \equiv \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \mathbb{1}\textsf{-unique}$$

The rules concerning unit type are summarised in Fig. 6.4.

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathbb{1}_\Gamma \text{ type}}\ \mathbb{1}\textsf{-form} \qquad \frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \mathbb{1}\textsf{-intro} \qquad \frac{\Gamma \text{ ctx} \quad \Gamma \vdash t : \mathbb{1}_\Gamma}{\Gamma \vdash t \equiv \mathsf{tt}_\Gamma : \mathbb{1}_\Gamma}\ \mathbb{1}\textsf{-unique}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma | \mathbb{1}_\Delta[s] \mathrel{\widetilde{\vdash}} i_{\mathbb{1},s} : \mathbb{1}_\Gamma}\ \textsf{sub-pres-}\mathbb{1}$$

$$\frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma.\mathbb{1}_\Gamma \vdash \chi_0(\mathbb{1}(s)) : \Delta.\mathbb{1}_\Delta}\ \textsf{compreh-}\mathbb{1}\textsf{-mor} \qquad \frac{\Gamma, \Delta \text{ ctx} \quad \Gamma \vdash s : \Delta}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_{\mathbb{1}_\Delta} \circ \chi_0(\mathbb{1}(s)) \equiv s \circ \pi_{\mathbb{1}_\Gamma} : \Delta}\ \textsf{compreh-}\mathbb{1}\textsf{-coh}$$

$$\frac{\Gamma \text{ ctx}}{\Gamma.\mathbb{1}_\Gamma \vdash \chi_0(\mathbb{1}(1_\Gamma)) \equiv 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma}\ \textsf{compreh-}\mathbb{1}\textsf{-id}$$

$$\frac{\Gamma, \Delta, \Theta \text{ ctx} \quad \Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta}{\Gamma.\mathbb{1}_\Gamma \vdash \chi_0(\mathbb{1}(s')) \circ \chi_0(\mathbb{1}(s)) \equiv \chi_0(\mathbb{1}(s' \circ s)) : \Theta.\mathbb{1}_\Theta}\ \textsf{compreh-}\mathbb{1}\textsf{-comp}$$

FIGURE 6.4: Rules of the type theory regarding unit type.

Remark 6.16. Alternatively, we could have defined unit type with elimination and computation rules instead of Rule $\mathbb{1}$-unique similar to what is explained in Remark 6.16. We expect Rule $\mathbb{1}$-unique to be derivable from the elimination and computation rules if an appropriate notion of extensional identity type is added to the type theory.

6.4.2 Interpretation of Type Theory with Unit

In this section, we show the soundness of the derivation of the rules regarding unit type in Fig. 6.4 by giving an interpretation of the type theory expressed by the rules in Figs. 6.1 to 6.4 in any non-full comprehension category with unit as defined in Definition 6.8. The interpretation of the rules in Fig. 6.4 is as follows:

1. Rule $\mathbb{1}$-form is interpreted as the action of $\mathbb{1} : \mathcal{C} \to \mathcal{T}$ on the objects.
2. Rules $\mathbb{1}$-intro and Rule $\mathbb{1}$-unique are interpreted as the unique section $\mathsf{tt}_\Gamma : \|\Gamma\| \to \|\Gamma.\mathbb{1}_\Gamma\|$ given by Lemma 6.14.
3. Rule sub-pres-$\mathbb{1}$ is interpreted as the isomorphism $\|s\|^* \|\mathbb{1}_\Delta\| \cong \|\mathbb{1}_\Gamma\|$ given by Lemma 6.12.
4. Rules compreh-$\mathbb{1}$-mor and compreh-$\mathbb{1}$-coh are interpreted as $\chi(\mathbb{1}(\|s\|)) : \mathcal{C}^{\to}(\|\pi_{\mathbb{1}_\Gamma}\|, \|\pi_{\mathbb{1}_\Delta}\|)$, which corresponds to the following commuting square in $\mathcal{C}$:

$$\begin{tikzcd}
\|\Gamma.\mathbb{1}_\Gamma\| \arrow[r, "(\chi_0\circ\mathbb{1})\|s\|"] \arrow[d, "\|\pi_{\mathbb{1}_\Gamma}\|"'] & \|\Delta.\mathbb{1}_\Delta\| \arrow[d, "\|\pi_{\mathbb{1}_\Delta}\|"] \\
\|\Gamma\| \arrow[r, "\|s\|"'] & \|\Delta\|.
\end{tikzcd}$$

5. Rule compreh-$\mathbb{1}$-id is interpreted as $\chi(\mathbb{1}(1_{\|\Gamma\|})) : \mathcal{C}^{\to}(\|\pi_{\mathbb{1}_\Gamma}\|, \|\pi_{\mathbb{1}_\Gamma}\|)$, which corresponds to the following commuting square in $\mathcal{C}$:

$$\begin{tikzcd}
\|\Gamma.\mathbb{1}_\Gamma\| \arrow[r, "(\chi_0\circ\mathbb{1})1_{\|\Gamma\|}"] \arrow[d, "\|\pi_{\mathbb{1}_\Gamma}\|"'] & \|\Gamma.\mathbb{1}_\Gamma\| \arrow[d, "\|\pi_{\mathbb{1}_\Gamma}\|"] \\
\|\Gamma\| \arrow[r, "1_{\|\Gamma\|}"'] & \|\Gamma\|.
\end{tikzcd}$$

6. Rule compreh-$\mathbb{1}$-comp is interpreted as $\chi(\mathbb{1}(s' \circ s)) : \mathcal{C}^{\to}(\|\pi_{\mathbb{1}_\Gamma}\|, \|\pi_{\mathbb{1}_\Theta}\|)$, which corresponds to the following commuting square in $\mathcal{C}$:

$$\begin{tikzcd}
\|\Gamma.\mathbb{1}_\Gamma\| \arrow[r, "(\chi_0\circ\mathbb{1})\|s'\circ s\|"] \arrow[d, "\|\pi_{\mathbb{1}_\Gamma}\|"'] & \|\Theta.\mathbb{1}_\Theta\| \arrow[d, "\|\pi_{\mathbb{1}_\Theta}\|"] \\
\|\Gamma\| \arrow[r, "\|s'\circ s\|"'] & \|\Theta\|.
\end{tikzcd}$$

The interpretation of the rules in Figs. 6.1 to 6.3 is the same as discussed in Sections 5.1 and 6.3.

Remark 6.17. The proof of completeness is left for future work. Completeness implies that from the rules of Figs. 6.1 to 6.4, one can derive the following rule in the type theory, which corresponds to the result of Lemma 6.15:

$$\frac{\Gamma\ \mathsf{ctx}}{\Gamma \overset{\sim}{\vdash} \mathsf{tt}_\Gamma : \Gamma.\mathbb{1}_\Gamma}\ \textsf{unit-iso}$$

6.4.3 Bijection between Terms and Sections of Projections

Now that we have added unit type to the type theory, we discuss the requirements for having a bijection between terms of the form $\Gamma|\mathbb{1}_\Gamma \vdash t : A$ and the sections of the projection $\Gamma.A \vdash \pi_A : \Gamma$. Semantically, this corresponds to having:

$$\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A) \cong \mathcal{C}_\Gamma^{\rightarrow}(1_\Gamma, \pi_A)$$

for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$, in a comprehension category with unit (see Definition 6.8). In the type theory, this corresponds to having the following rules:

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma|\Gamma.t \circ \mathsf{tt}_\Gamma : A}\ \textsf{tm-to-ml}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma \vdash t : A}{\Gamma|\mathbb{1}_\Gamma \vdash t^{\mathsf{tm}} : A}\ \textsf{ml-to-tm}$$

$$\frac{\Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma|\mathbb{1}_\Gamma \vdash (\Gamma.t \circ \mathsf{tt}_\Gamma)^{\mathsf{tm}} \equiv t : A}\ \textsf{bij-tm-to-ml}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \Gamma.(t^{\mathsf{tm}}) \circ \mathsf{tt}_\Gamma \equiv t : A}\ \textsf{bij-ml-to-tm}$$

Rule tm-to-ml can already be derived from the rules in Figs. 6.1 to 6.4. This derivation is discussed in the following lemma.

Lemma 6.18. *In a type theory given by Figs. 6.1 to 6.4, we can derive the following rule:*

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma|\Gamma.t \circ \mathsf{tt}_\Gamma : A}\ \textit{tm-to-ml}$$

This means that for each term $\Gamma|\mathbb{1}_\Gamma \vdash t : A$, we have an MLTT style term given by $\Gamma.t \circ \mathsf{tt}_\Gamma$.

Proof. First, we show that given a context Γ, a type A in context Γ and $\Gamma|\mathbb{1}_\Gamma \vdash t : A$, $\Gamma.t \circ \mathsf{tt}_\Gamma$ is a context morphism from Γ to $\Gamma.A$.

$$\dfrac{\dfrac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \Gamma.\mathbb{1}_\Gamma}\ \mathbb{1}\textsf{-intro} \qquad \dfrac{\Gamma\ \mathsf{ctx} \quad \dfrac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma\ \mathsf{type}}\ \mathbb{1}\textsf{-form} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t : \Gamma.A}\ \textsf{ext-tm}}{\Gamma \vdash \Gamma.t \circ \mathsf{tt}_\Gamma : \Gamma.\mathbb{1}_\Gamma}\ \textsf{ctx-mor-comp}$$

Now we show that $\Gamma \vdash \Gamma.t \circ \mathsf{tt}_\Gamma \equiv 1_\Gamma : \Gamma$.

$$\dfrac{\dfrac{\dfrac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \Gamma.\mathbb{1}_\Gamma}\ \mathbb{1}\text{-intro} \quad \dfrac{\Gamma \vdash \mathbb{1}_\Gamma, A\ \mathsf{type} \quad \Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_A \circ \Gamma.t \equiv \pi_{\mathbb{1}_\Gamma} : \Gamma}\ \mathsf{ext\text{-}coh}}{\Gamma \vdash \pi_A \circ \Gamma.t \circ \mathsf{tt}_\Gamma \equiv \pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma : \Gamma}\ \mathsf{ctx\text{-}comp\text{-}cong\text{-}2} \quad \dfrac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma \equiv 1_\Gamma : \Gamma}\ \mathbb{1}\text{-intro}}{\Gamma \vdash \pi_A \circ \Gamma.t \circ \mathsf{tt}_\Gamma \equiv 1_\Gamma : \Gamma}\ \mathsf{ctx\text{-}eq\text{-}trans}$$

Hence we have the following rule.

$$\dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma|\Gamma.t \circ \mathsf{tt}_\Gamma : A}\ \mathsf{tm\text{-}to\text{-}ml}$$

Semantically this corresponds to showing $\chi(t) \circ \mathrm{tt}_\Gamma$ is a section of π_A, i.e. $\pi_A \circ \chi(t) \circ \mathrm{tt}_\Gamma = 1_\Gamma$. For each $\Gamma \in \mathcal{C}$, $A \in \mathcal{T}$ and $t \in \mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A)$, we have $\pi_A \circ \chi(t) \circ \mathrm{tt}_\Gamma = \pi_{\mathbb{1}_\Gamma} \circ \mathrm{tt}_\Gamma$, as $\chi(t)$ is a morphism in $\mathcal{C}^{\to}(\pi_{\mathbb{1}_\Gamma}, \pi_A)$. Using $\pi_{\mathbb{1}_\Gamma} \circ \mathrm{tt}_\Gamma = 1_\Gamma$ which we have from tt_Γ being a section of $\pi_{\mathbb{1}_\Gamma}$, we get $\pi_A \circ \chi(t) \circ \mathrm{tt}_\Gamma = 1_\Gamma$. □

$$\dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma \vdash t : A}{\Gamma|\mathbb{1}_\Gamma \vdash t^{\mathrm{tm}} : A}\ \mathsf{ml\text{-}to\text{-}tm}$$

$$\dfrac{\Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma|\mathbb{1}_\Gamma \vdash (\Gamma.t \circ \mathsf{tt}_\Gamma)^{\mathrm{tm}} \equiv t : A}\ \mathsf{bij\text{-}tm\text{-}to\text{-}ml} \qquad \dfrac{\Gamma \vdash t : A}{\Gamma \vdash \Gamma.(t^{\mathrm{tm}}) \circ \mathsf{tt}_\Gamma \equiv t : A}\ \mathsf{bij\text{-}ml\text{-}to\text{-}tm}$$

FIGURE 6.5: Rules of the type theory regarding the bijection between terms of the from $\Gamma|\mathbb{1}_\Gamma \vdash t : A$ and MLTT style terms of the form $\Gamma \vdash t : A$.

Rules ml-to-tm, bij-tm-to-ml and bij-ml-to-tm can be added to the rules in Figs. 6.1 to 6.4 to have the desired bijection between terms of the form $\Gamma|\mathbb{1}_\Gamma \vdash t : A$ and the MLTT style terms of the form $\Gamma \vdash t : A$. This rules needed to get this bijection are summarised in Fig. 6.5. In a type theory given by the rules in Figs. 6.1 to 6.5 certain terms coincide with the terms in MLTT. In particular, given context Γ and type A in context Γ, the terms of the form $\Gamma|\mathbb{1}_A \vdash t : A$ coincide with MLTT style terms of type A.

Remark 6.19. In a type theory that is expressed using rules from Figs. 6.1 to 6.5, context morphisms are built out of terms in the sense that given $\Gamma \vdash s : \Delta$ and $\Delta \vdash A$ type, there is a bijection between terms of the form $\Gamma|\mathbb{1}_\Gamma \vdash t : A[s]$ and context morphisms of the form $\Gamma \vdash s' : \Delta.A$ that satisfy $\Gamma \vdash \pi_A \circ s' \equiv s : \Delta$.

Proposition 6.20. *In a comprehension category with unit (Definition 6.8), if χ is fully faithful we have:*

$$\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A) \cong \mathcal{C}^{\to}_\Gamma(1_\Gamma, \pi_A).$$

Proof. We know from Lemma 6.15, that in a comprehension category with unit, for each $\Gamma \in \mathcal{C}$, $\Gamma \cong \Gamma.\mathbb{1}_\Gamma$. This implies:

$$\mathcal{C}^{\to}_\Gamma(1_\Gamma, \pi_A) \cong \mathcal{C}^{\to}_\Gamma(\pi_{\mathbb{1}_\Gamma}, \pi_A),$$

for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}_\Gamma$. From Proposition 3.20 we know that χ is fully faithful if and only if it is fibrewise fully faithful. This means that if χ is fully faithful, we get

$$\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A) \cong \mathcal{C}_\Gamma^{\rightarrow}(\pi_{\mathbb{1}_\Gamma}, \pi_A).$$

Hence, in a comprehension category with unit, if χ is fully faithful, then $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A) \cong \mathcal{C}_\Gamma^{\rightarrow}(1_\Gamma, \pi_A)$. $\square$

Remark 6.21. Assuming completeness of the rules in Figs. 6.1 to 6.4, Proposition 6.20 implies that if the rules corresponding to χ being fully faithful are added to the rules in Figs. 6.1 to 6.4, then the rules corresponding to $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A) \cong \mathcal{C}_\Gamma^{\rightarrow}(1_\Gamma, \pi_A)$ can be derived in the type theory. By the rules corresponding to χ being fully faithful we mean:

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma.A \vdash s : \Gamma.B \quad \Gamma.A \vdash \pi_B \circ s \equiv \pi_A : \Gamma}{\begin{array}{c}\Gamma | A \vdash \chi^{-1}(s) : B \\ \Gamma.A \vdash \Gamma.\chi^{-1}(s) \equiv s : \Gamma.B\end{array}}\ \chi\text{-full}$$

$$\frac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma | A \vdash t_1, t_2 : B \quad \Gamma.A \vdash \Gamma.t_1 \equiv \Gamma.t_2 : \Gamma.B}{\Gamma | A \vdash t_1 \equiv t_2 : B}\ \chi\text{-faithful}$$

The derivation of Rules tm-to-ml, ml-to-tm, bij-tm-to-ml and bij-ml-to-tm in a type theory that is given by the rules in Figs. 6.1 to 6.4 plus Rules χ-full and χ-faithful is presented in Appendix A.

Chapter 7

Discussion and Conclusion

In this section, we summarise the results of Chapter 6 and discuss them by comparing the type theory introduced in Section 6.1 to MLTT. We also compare the results to that of Ahrens et al. [ANW23] by comparing our type theory to BTT, the directed type theory introduced by them. We then mention some limitations and directions for future work.

In Chapter 6, we proposed a type theory as the internal language of comprehension categories by extracting the rules of this type theory from the semantics given by a comprehension category. Starting from a (weak) comprehension category without the requirement of χ preserving cartesian morphisms, we derived the rules in Figs. 6.1 and 6.5. We showed soundness by giving an interpretation of this type theory in every comprehension category.

We then showed that if we start from a comprehension category where χ preserves cartesian morphisms, we get the rules in Fig. 6.3 in addition to the previous rules. We also discussed that by adding the rules in Fig. 6.3, we require context morphisms to be built out of sections of the projection morphisms, which corresponds to context morphisms being built out of terms in MLTT.

Next, we discussed an appropriate notion for semantics of the unit type in a comprehension category that is not full, and extracted the rules regarding the unit type, Fig. 6.4, from the semantics. We again showed soundness by giving an interpretation of these rules in a comprehension category with unit. Finally, we discussed how adding the rules in Fig. 6.5 makes certain type morphisms in our type theory coincide with terms in MLTT, and how this is related to requiring the comprehension category to be full.

Comparison with MLTT In Section 6.1, we extracted the rules of a type theory from a (weak) comprehension category and in Section 6.2, we gave an interpretation of this type theory in every comprehension category.

The judgements of this type theory differ from MLTT in the following ways:

1. The first difference is that this type theory features explicit substitution in the syntax. This means that we have judgements of the form $\Gamma \vdash s : \Delta$, which is read as "s is context morphism from Γ to Δ".

2. The second difference is that the judgements regarding terms in this type theory are of the form $\Gamma | A \vdash t : B$, whereas in MLTT the judgements are of the form $\Gamma \vdash t : A$. The judgement $\Gamma \vdash t : A$ is interpreted as sections of the projection context morphisms in the category on contexts $\mathcal{C}$, where the judgement $\Gamma | A \vdash t : B$ is interpreted as a morphism in $\mathcal{T}_\Gamma(A, B)$. In a comprehension category with no requirement on the fullness of the comprehension category, there is no one-to-one correspondence between hom-sets of $\mathcal{T}_\Gamma$ and $\mathcal{C}/\Gamma$. This difference is expected since we extract the rules of the type theory from a comprehension category that is not necessarily full.

3. The third difference is that in this type theory, we do not have judgements regarding equality of contexts and types. The category of contexts $\mathcal{C}$ and types $\mathcal{T}$ are not necessarily strict. Hence, the type theory that is derived from such a comprehension category does not feature judgements regarding definitional equality of contexts and types. All the rules of the type theory should be considered up to renaming of the variables.

One of the main differences between the rules of this type theory and the rules of MLTT is that substitution in this type theory is not strictly functorial. We know from Proposition 3.39 that in MLTT, the composition of substitution is associative. We also know from Proposition 3.40 that in MLTT, $A[\mathsf{id}_A] = A$. In this type theory, however, these only hold up to isomorphism of types. This is expected since the rules of the type theory are derived from a non-split comprehension category and we know from Remark 3.17 that strictly functorial substitution can be interpreted in a split comprehension category.

Another difference between MLTT and this type theory is that in this type theory context morphisms are not built of terms. To have context morphisms be made of terms, one needs to derive the rules from a comprehension category where certain pullbacks exist in the category of contexts $\mathcal{C}$, particularly the ones discussed in Section 5.1.3. We showed in Section 6.3, that requiring these pullbacks to exist is equivalent to requiring χ to preserve cartesian morphisms. Hence, by adding the rules in Fig. 6.3 to the ones in Figs. 6.1 and 6.5, we get a type theory in which context morphisms are built out of MLTT style terms.

In our type theory, the unit type is given by the rules in Fig. 6.4. Unlike in MLTT, we have a rule that expresses how the unit type is preserved (up to isomorphism) under substitution. This is because we have explicit substitution in our type theory. In addition to this, we also have rules for coherence conditions regarding unit, context extension and comprehension. We also discussed how adding the rules in Fig. 6.5, gives a bijection between MLTT style terms and terms of this type theory (type morphisms).

Comparison with BTT We expect the type theory presented in Figs. 6.1 and 6.2 to be the one-dimensional restriction of the directed type theory introduced by Ahrens et al. [ANW23] (BTT). This is because the rules of BTT are extracted from comprehension bicategories which are the bicategorical generalisation of comprehension categories. Note that we do not include the rules in Figs. 6.3 to 6.5 in this comparison.

Both these type theories feature explicit substitution in the syntax and the terms of BTT are interpreted the same as the terms of our type theory. The syntax of BTT is extracted from weak comprehension bicategories, which are a weak version of comprehension bicategories where the comprehension does not necessarily preserve cartesian 1-cells and opcartesian 2-cells. In our case, this corresponds to the comprehension not preserving cartesian morphisms. Doing so, results in a type theory where context morphisms are not built of the sections of projection context morphisms.

It is discussed in [ANW23, section 11] how terms of BTT differ from the terms of other proposed directed type theories in the way they are interpreted, and propose a way to reconcile this mismatch. The terms of our type theory are interpreted similarly to the terms of BTT; hence, differ from terms of MLTT in their interpretations. We investigate the reconciliation of this mismatch in the one-dimensional sense by proposing how the unit type can be added to the syntax and discussing the requirements for having a bijection between the interpretations of our terms and MLTT style terms. We expect this result to be in line with how the mismatch between terms of BTT and other directed-type theories with terms interpreted similarly to MLTT could be reconciled.

Another difference between our type theory and BTT is that our type theory is extracted from a comprehension category where $\mathcal{C}$ has a terminal object. This results in having an empty context and rules regarding context morphisms into the empty context in the type theory.

In addition to these, the counterparts of Rules $\mathsf{lift\text{-}compreh}$, $\mathsf{lift\text{-}coh}$, $\mathsf{iid\text{-}coh}$ and $\mathsf{icomp\text{-}coh}$ are not present in BTT. We add these rules to reflect the comprehension of the cartesian lifts and the corresponding coherence conditions in the type theory.

Limitations and Future Work A limitation of this work is that we do not prove a completeness theorem for the type theory presented in Chapter 6. To partially compensate for this, we derive some results from the semantics in the type theory as well. This serves both as a sanity check for the extracted rules and as a way to compare the proofs in syntax and semantics. Proving the completeness is a possible direction for future work. Another possible direction for future work is to formalise the type theory and proofs of soundness and completeness in a proof assistant.

Lastly, we do not discuss type formers like Π-, Σ- and W- types, identity types and universes. Lindgren [Lin21] discusses that Π-types correspond to certain relative adjoints in a comprehension category which is not full. A possible way to extend this work, would be to extract the rules from a non-full comprehension category with products, defined by Lindgren [Lin21], and to extend this to Σ-types.

Conclusion In this thesis, we studied the interpretation of Martin-Löf type theory in full split comprehension categories and highlighted where the full and splitness assumptions are being used, by studying type theories that can be interpreted in general comprehension categories. For this, we started from the semantics given by a comprehension category and extracted a type theory such that all of the semantic structure is reflected in the type theory following the approach taken by Ahrens et al. [ANW23]. We then discussed adding the unit type to this type theory and the extra requirements that would make this type theory closer to MLTT.

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Appendix A

Extension to Remark 6.21

In this appendix, we show that if the rules corresponding to χ being fully faithful, Rules χ-full and χ-faithful, are added to the type theory described in Figs. 6.1 to 6.4, we can derive the rules regarding the bijection discussed in Section 6.4.3, Rules tm-to-ml, ml-to-tm, bij-tm-to-ml and bij-ml-to-tm.

The derivation of Rule tm-to-ml is already discussed in Lemma 6.18.

Notation A.1. For each type A in context Γ and $\Gamma|\mathbb{1}_\Gamma \vdash t : A$, we denote the corresponding MLTT style term $\Gamma.t \circ \mathrm{tt}_\Gamma$ as t^{ml}.

Lemma A.2. *If we have the following rules:*

$$\dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A, B\ \mathsf{type} \quad \Gamma.A \vdash s : \Gamma.B \quad \Gamma.A \vdash \pi_B \circ s \equiv \pi_A : \Gamma}{\begin{array}{c}\Gamma|A \vdash \chi^{-1}(s) : B \\ \Gamma.A \vdash \Gamma.\chi^{-1}(s) \equiv s : \Gamma.B\end{array}}\ \textit{χ-full}$$

which corresponds to χ being full, for each MLTT style term $\Gamma \vdash t : A$, we have $\Gamma|\mathbb{1}_\Gamma \vdash t \circ \pi_{\mathbb{1}_\Gamma} : A$. This corresponds to the rule:

$$\dfrac{\Gamma\ \mathsf{ctx} \quad \Gamma \vdash A\ \mathsf{type} \quad \Gamma \vdash t : A}{\Gamma|\mathbb{1}_\Gamma \vdash \chi^{-1}(t \circ \pi_{\mathbb{1}_\Gamma}) : A}\ \textit{ml-to-tm}$$

Proof. To get $\Gamma|\mathbb{1}_\Gamma \vdash \chi^{-1}(t \circ \pi_{\mathbb{1}_\Gamma}) : A$ we need to show $\Gamma.\mathbb{1}_\Gamma \vdash t \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A$ and $\Gamma.\mathbb{1}_\Gamma \vdash \pi_A \circ t \circ \pi_{\mathbb{1}_\Gamma} \equiv \pi_{\mathbb{1}_\Gamma} : \Gamma$. Then we use Rule χ-full to get the desired conclusion.

First we show $\Gamma.\mathbb{1}_\Gamma \vdash t \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A$:

$$\dfrac{\dfrac{\dfrac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma\ \mathsf{type}}\ \mathbb{1}\text{-from}}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_{\mathbb{1}_\Gamma} : \Gamma}\ \text{ext-proj} \qquad \Gamma \vdash t : \Gamma.A}{\Gamma.\mathbb{1}_\Gamma \vdash t \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A}\ \text{ctx-mor-comp}$$

Now we show $\Gamma.\mathbb{1}_\Gamma \vdash \pi_A \circ t \circ \pi_{\mathbb{1}_\Gamma} \equiv \pi_{\mathbb{1}_\Gamma} : \Gamma$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma\ \mathsf{ctx}}{\Gamma \vdash \mathbb{1}_\Gamma\ \mathsf{type}}\ \mathbb{1}\text{-from}}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_{\mathbb{1}_\Gamma} : \Gamma}\ \text{ext-proj} \qquad \Gamma \vdash \pi_A \circ t \equiv 1_\Gamma : \Gamma}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_A \circ t \circ \pi_{\mathbb{1}_\Gamma} \equiv 1_\Gamma \circ \pi_{\mathbb{1}_\Gamma} : \Gamma}\ \text{ctx-comp-cong-2} \qquad \Gamma.\mathbb{1}_\Gamma \vdash 1_\Gamma \circ \pi_{\mathbb{1}_\Gamma} \equiv \pi_{\mathbb{1}_\Gamma} : \Gamma}{\Gamma.\mathbb{1}_\Gamma \vdash \pi_A \circ t \circ \pi_{\mathbb{1}_\Gamma} \equiv \pi_{\mathbb{1}_\Gamma} : \Gamma}\ \text{ctx-eq-trans}$$

Finally, we have:

$$\frac{\Gamma.\mathbb{1}_\Gamma \vdash t \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A \quad \Gamma.\mathbb{1}_\Gamma \vdash \pi_A \circ t \circ \pi_{\mathbb{1}_\Gamma} \equiv \pi_{\mathbb{1}_\Gamma} : \Gamma}{\Gamma|\mathbb{1}_\Gamma \vdash \chi^{\to}(t \circ \pi_{\mathbb{1}_\Gamma}) : A} \text{ ml-to-tm}$$

Semantically this corresponds to showing that $t \circ \pi_{\mathbb{1}_\Gamma}$ is in $\mathcal{C}_\Gamma^{\to}(\pi_{\mathbb{1}_\Gamma}, \pi_A)$, and using χ being full to get a morphism in $\mathcal{T}_\Gamma(\mathbb{1}_\Gamma, A)$. □

Notation A.3. If we have Rule χ-full in the type theory, for $\Gamma \vdash t : A$, we denote the term $\Gamma|\mathbb{1}_\Gamma \vdash \chi^{-1}(t \circ \pi_{\mathbb{1}_\Gamma}) : A$ as t^{tm}.

Lemma A.4. *If we have Rules χ-full and χ-faithful*

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma.A \vdash s : \Gamma.B \quad \Gamma.A \vdash \pi_B \circ s \equiv \pi_A : \Gamma}{\begin{array}{c}\Gamma|A \vdash \chi^{-1}(s) : B \\ \Gamma.A \vdash \Gamma.\chi^{-1}(s) \equiv s : \Gamma.B\end{array}} \; \chi\text{-full}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash A, B \text{ type} \quad \Gamma|A \vdash t_1, t_2 : B \quad \Gamma.A \vdash \Gamma.t_1 \equiv \Gamma.t_2 : \Gamma.B}{\Gamma|A \vdash t_1 \equiv t_2 : B} \; \chi\text{-faithful}$$

which correspond to χ being full and faithful, in the type theory, then we have a bijection between terms of the form $\Gamma|\mathbb{1}_\Gamma \vdash t : A$ and the MLTT style terms of the form $\Gamma \vdash t^{\mathsf{ml}} : A$.

Proof. From Lemma 6.18, for each $\Gamma|\mathbb{1}_\Gamma \vdash t : A$ we have an MLTT style term $t^{\mathsf{ml}} := \Gamma.t \circ \mathsf{tt}_\Gamma$. Since we have the Rule χ-full, from Lemma A.2 we know that for each MLTT style term $\Gamma \vdash t : A$ we have a term $t^{\mathsf{tm}} := \chi^{-1}(t \circ \pi_{\mathbb{1}_\Gamma})$, where $\Gamma|\mathbb{1}_\Gamma \vdash t^{\mathsf{tm}} : A$. To show the bijection, we need to show the following two rules:

$$\frac{\Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma|\mathbb{1}_\Gamma \vdash (t^{\mathsf{ml}})^{\mathsf{tm}} \equiv t : A} \text{ tm-to-ml-bij} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash (t^{\mathsf{tm}})^{\mathsf{ml}} \equiv t : A} \text{ ml-to-tm-bij}$$

First, starting from an MLTT style term $\Gamma \vdash t : A$, from Rule χ-full we get $\Gamma|\mathbb{1}_\Gamma \vdash t^{\mathsf{tm}} : A$ and $\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t^{\mathsf{tm}} \equiv t \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A$. Using Rule tm-to-ml we get $\Gamma \vdash \Gamma.t^{\mathsf{tm}} \circ \mathsf{tt}_\Gamma : A$ and we need to show $\Gamma|\mathbb{1}_\Gamma \vdash (t^{\mathsf{ml}})^{\mathsf{tm}} \equiv t : A$. First we show:

$$\frac{\dfrac{\Gamma \text{ ctx}}{\Gamma \vdash \mathsf{tt}_\Gamma : \Gamma.\mathbb{1}_\Gamma} \,\mathbb{1}\text{-intro} \quad \Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t^{\mathsf{tm}} \equiv t \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t^{\mathsf{tm}} \circ \mathsf{tt}_\Gamma \equiv t \circ \pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma : \Gamma.A} \text{ ctx-comp-cong-2}$$

Next we show:

$$\frac{\dfrac{\Gamma \text{ ctx}}{\Gamma \vdash t : \Gamma.A} \,\mathbb{1}\text{-intro} \quad \Gamma \vdash \pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma \equiv 1_\Gamma : \Gamma}{\Gamma.\mathbb{1}_\Gamma \vdash t \circ \pi_{\mathbb{1}_\Gamma} \circ \mathsf{tt}_\Gamma \equiv t \circ 1_\Gamma : \Gamma.A} \text{ ctx-comp-cong-1}$$

Using the previous two results and Rule ctx-eq-trans we have $\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t^{\mathsf{tm}} \circ \mathsf{tt}_\Gamma \equiv t \circ 1_\Gamma : \Gamma.A$. Finally we have:

$$\frac{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t^{\mathsf{tm}} \circ \mathsf{tt}_\Gamma \equiv t \circ 1_\Gamma : \Gamma.A \quad \dfrac{\Gamma \vdash t : \Gamma.A}{\Gamma.\mathbb{1}_\Gamma \vdash t \circ 1_\Gamma \equiv t : \Gamma.A} \text{ ctx-id-unit-r}}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t^{\mathsf{tm}} \circ \mathsf{tt}_\Gamma \equiv t : \Gamma.A} \text{ ctx-eq-trans}$$

which means $\Gamma|\mathbb{1}_\Gamma \vdash (t^{\mathsf{ml}})^{\mathsf{tm}} \equiv t : A$.

Now we start from a term $\Gamma|\mathbb{1}_\Gamma \vdash t : A$. From Lemma 6.18 we get an MLTT style term $t^{\mathsf{ml}} := \Gamma.t \circ \mathrm{tt}_\Gamma$. From Rule χ-full we get $\Gamma|\mathbb{1}_\Gamma \vdash (t^{\mathsf{ml}})^{\mathsf{tm}} : A$ and $\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.(t^{\mathsf{ml}})^{\mathsf{tm}} \equiv t^{\mathsf{ml}} \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A$, which means that $\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.(t^{\mathsf{ml}})^{\mathsf{tm}} \equiv \Gamma.t \circ \mathrm{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A$. We also know from $\Gamma \overset{\sim}{\vdash} \mathrm{tt}_\Gamma : \Gamma.\mathbb{1}_\Gamma$ that $\Gamma.\mathbb{1}_\Gamma \vdash \mathrm{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} \equiv 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma$. Hence we get:

$$\dfrac{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.(t^{\mathsf{ml}})^{\mathsf{tm}} \equiv \Gamma.t \circ \mathrm{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} : \Gamma.A \quad \dfrac{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t : \Gamma.A \quad \Gamma.\mathbb{1}_\Gamma \vdash \mathrm{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} \equiv 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t \circ \mathrm{tt}_\Gamma \circ \pi_{\mathbb{1}_\Gamma} \equiv \Gamma.t \circ 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma}\,\textsf{ctx-comp-cong-1}}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.(t^{\mathsf{ml}})^{\mathsf{tm}} \equiv \Gamma.t \circ 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma}\,\textsf{ctx-eq-trans}$$

Finally, we have:

$$\dfrac{\dfrac{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.(t^{\mathsf{ml}})^{\mathsf{tm}} \equiv \Gamma.t \circ 1_{\Gamma.\mathbb{1}_\Gamma} : \Gamma.\mathbb{1}_\Gamma \quad \dfrac{\dfrac{\Gamma|\mathbb{1}_\Gamma \vdash t : A}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t : \Gamma.A}\,\textsf{ext-tm}}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.t \circ 1_{\Gamma.\mathbb{1}_\Gamma} \equiv \Gamma.t : \Gamma.A}\,\textsf{ctx-unit-id-r}}{\Gamma.\mathbb{1}_\Gamma \vdash \Gamma.(t^{\mathsf{ml}})^{\mathsf{tm}} \equiv \Gamma.t : \Gamma.A}\,\textsf{ctx-eq-trans}}{\Gamma|\mathbb{1}_\Gamma \vdash (t^{\mathsf{ml}})^{\mathsf{tm}} \equiv t : A}\,\chi\textsf{-faithful}$$

□