Modelling wave propagation over a current in a stratified fluid

Mathematical Physics

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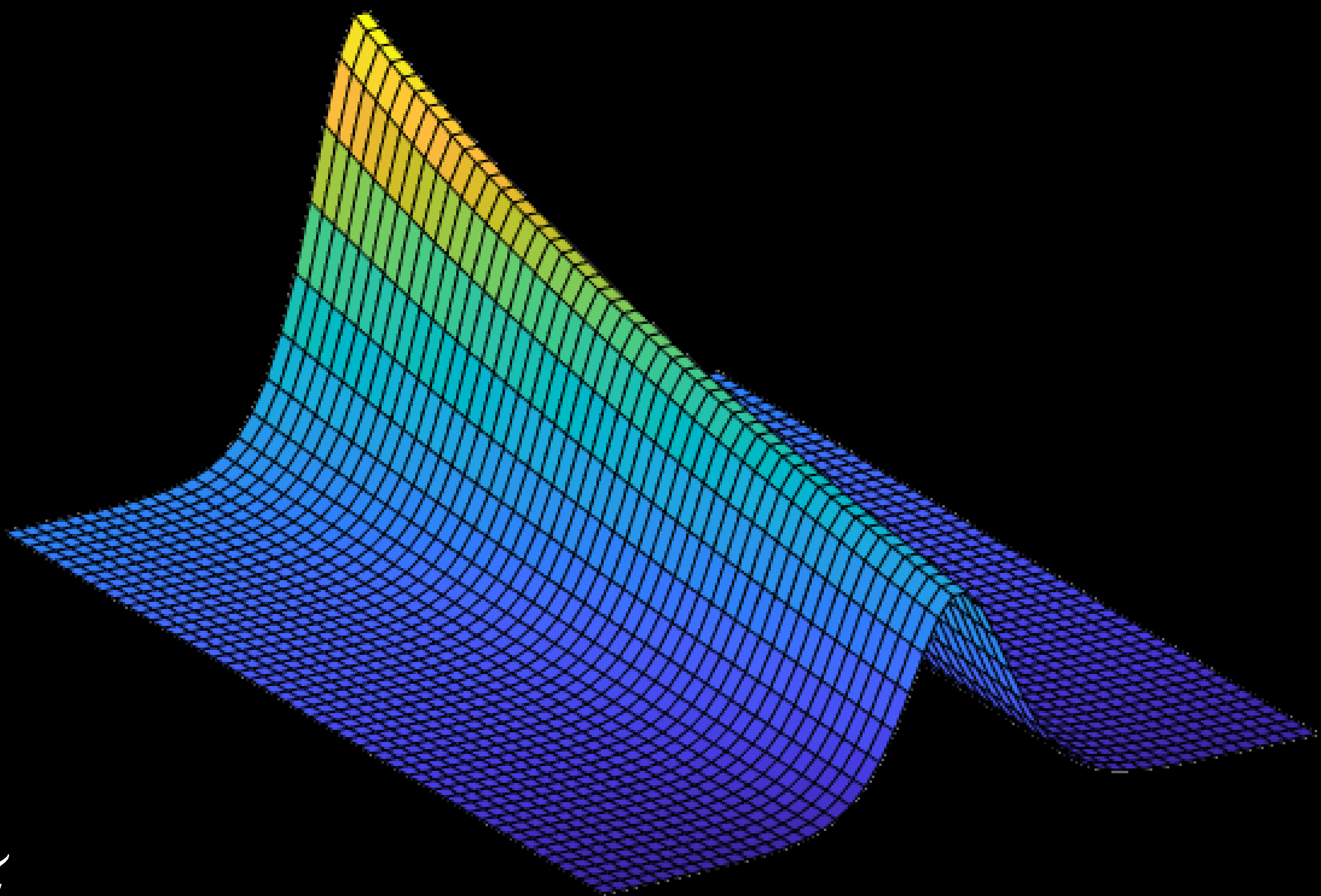

TU Delft

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by

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Summary

Layman Summary

The goal of this paper is to study how waves move over a fluid in with an underlying current Oceans displace debris and heat. Ocean models can be used to predict the movement of debris and even predict the effect of the movements of oceans on the climate. Modeling water can be improved by introducing more information from observations into the models. The goal of this paper is to combine the effect of varying density and an underlying current in a model for ocean water near the equator. Our model focuses on long, shallow waves. This allows us to simplify the model and solve it more easily. We combine methods from different sources to create a model for our problem. The model we create shows there are two types of solution waves: Periodic travelling waves and Solitons. Solitons are single waves that don't change shape through time. We find that density influences the amplitude of waves and the perturbation of the current. We find that the underlying current influences the wave profile's width as well as the amplitude.

Summary

Models of oceans can be used to predict the displacement of debris and even trace its path back to its origin. Oceans are a large influence on the weather and climate all over the world. Improving these models is therefore very useful.

There isn't a general equation that describes all water dynamics. Even if there would be, we would not have a computer good enough to make all the necessary calculations for the model. Though there will not be a perfect model, there is still a lot of room to improve the current models. The accuracy of water models, specifically for oceans, can be improved in different ways. We can increase the resolution to model smaller, more intricate behavior. The model can also be improved by coupling more different phenomena.

The goal of this paper is to combine existing methods for modeling the propagation of waves in a model that describes the propagation of waves over a current in a stratified fluid.

We assume the fluid to be inviscid and incompressible. We also assume there is no thermal conductivity. We choose to focus the model on long and shallow waves and changes in the current. These assumptions mean we can choose to base the governing equations on the Euler equations for inviscid fluids. Through non-dimensionalisation and scaling transformations, we transform our model to unitless equations.

We examine only the behavior of the leading order solutions by expanding the unknown variables as asymptotic series.

The final equations that describe the wave propagation belong to the family of Korteweg-deVries equations. One solution to these equations is a $\text{sech}^2(\theta)$ function. They represent soliton waves. These are solitary waves that hold their shape through the combination of dispersion and the non-linear character of the waves.

We find that the density influences the amplitude of waves and the perturbation of the current because it shows up as a multiplicative factor. We find that the underlying current influences the wave profile's width as well as the amplitude because it also shows up inside the θ term of the $\text{sech}^2(\theta)$ function.

Contents

1
Introduction

Many problems in the field of fluid mechanics remain partly or wholly unsolved. Many characteristics of water dynamics are still being studied; From wave propagation to turbulence, this is a very active field providing new insights regularly. Wave propagation in oceans is particularly interesting to model. Challenges are posed by the naturally occurring dynamics due to layers, currents, and temperature differences. The Equatorial undercurrent poses interesting interactions with the wave propagation and therefore ocean water around the equator is of specific interest.

The objective is to construct a model of wave propagation that can be applied to a body of ocean water governed by a prescribed density distribution over a prescribed current such as the Equatorial undercurrent. Large bodies of fluids such as oceans and the atmosphere are under the influence of the Coriolis Effect. Therefore we will take the Coriolis Effect into account as proposed by [2]. The model will exclude thermal changes to focus only on the influence of motion as proposed by [5]. The most important findings should be the behavior of waves and the influence of combining the prescribed density distribution and prescribed undercurrent.

To understand what kind of method we need to model the problem, we will start by analyzing existing methods for modeling wave propagation in chapter 2. Specifically, we will compare the techniques used by Johnson in his book on Mathematics for wave propagation[5] and the applications of these methods used by Geyer and Quirchmayer in their papers on modeling wave propagation over a current [3] and in a fluid with arbitrarily prescribed density distribution [2]. Next to studying the literature, we will also further specify the problem we want to solve. After comparing the steps in the literature and modifying them to our problem, it should be clear what steps we need to take to solve our problem. In chapter 3, we will use the steps defined in chapter 2 to solve our problem. Afterward, the behaviour of the solution in the combined method will be analysed.

2
Literature study

This chapter will start by giving a motivation for researching this particular modeling problem. We will also further specify our problem and any assumptions we need to make. This chapter works towards defining the steps we need to take to solve our problem. We do this by discussing the methods in closely linked literature. We will specify how to modify these methods based on our problem and assumptions. Lastly, we discuss what the literature shows what we can expect from the behaviour of the solutions. In chapter 3 we will then use the methods we learned and modified in this chapter to solve our modelling problem.

2.1. Motivation

In March 2014 a plane disappeared from the radar [1]. Using a model of the ocean near the presumed crash site, researchers were able to backtrack a portion of debris that was found at a possible crash site. A more detailed understanding of the coupling of different phenomena such as the influence of underlying currents on the surface waves could increase the accuracy of this model [8]. Next to transporting debris, oceans transport heat around the globe. Ocean currents transport heat around such as can be seen in the Atlantic Meridional Overturning Circulation(AMOC) [6]. As the atmosphere's temperature increases, this circulation weakens, transporting less heat north. This means that when we measure temperature changes over the coming years, in the northern hemisphere this heating will be slower. Meteorologists and climate scientists use coupled models of the ocean, atmosphere, land, and ice to predict changes in the climate [8]. When we improve the coupling of models we can improve our predictions of the effect of the ocean on our observations of climate change and then improve our climate predictions. Improving models of oceans depends on increasing the modeling resolution and computing power. However, where we still need to make improvements in our understanding of how fluids interact. A high-resolution model of a simplified model will rapidly increase error in the predictions over longer periods and any detail will be useless. Therefore, including more information about fluid dynamics in the model is a necessary step toward improvement.

2.2. Constructing Governing Equations and Boundary Conditions

We do not yet fully understand how fluids behave. For now, there are 2 main types of equations that describe fluid behavior: the Navier-Stokes equations for viscous fluids and the Euler's equations for inviscid fluids(fluids that are not viscous). For more explanation about these equations, we refer the reader to the book by Johnson [5], which was consulted extensively for this paper. These equations state how the forces applied to a fluid volume affect the velocity of the volume, based on Newton's second law. They are independent of any coordinate system or domain. The Navier-Stokes equations can also be reduced to Euler's equations under the assumption that the fluid is inviscid (it is not viscous) and assuming there is no heat conduction. We are going to choose one of these equations to use as a basis for our model.

The model should be detailed enough to be sufficient for our research goal, but should not include too much detail to work on in this single project. Therefore, we should then first select any assumptions that we make for our model. Based on these assumptions, we choose between the Euler and Navier-Stokes equations. Next, we can define the coordinate system and domain. Once we have those, we can define the equations we choose in this coordinate system and on this domain. We will finish by applying the assumptions and focus of the model.

2.2.1. Assumptions and Focus of the Model

Ocean water is salt water. Temperature and depth influence the salinity of the water which in turn influences the density. When the density varies with depth, the wave propagation also varies with depth. The temperature of the ocean changes drastically at around 200 to 1.000 meters deep(thermocline) [7]. The salinity changes rapidly around 150m deep(halocline). Because the density is a function of the temperature and salinity, we find the pycnocline where the density drastically changes. Ideally, we would like to be able to prescribe any density function. In contrast to Constantin and Johnson, who take the pycnocline to be infinitely thin in their paper [4] and prescribe a constant density for each layer of water, Geyer and Quirchmayer [2] choose to include an arbitrary prescribed density function that varies with depth in favor of a more general model, as can be seen in figure 2.2b.

As mentioned before, currents such as those part of the AMOC system are present all around the globe and we can expect them to be of great influence on the propagation of waves. Around the equator, there is a fairly one-directional stream, the EUC, which is highlighted by Constantin-Johnson [4]. Ideally, we would like to prescribe any underlying current as can be seen in [5]. Another paper by Geyer and Quirchmayer [3] will be consulted because of the improvements they have already made to this method by including layers of different densities next to an underlying current.

We will assume the water to be incompressible and inviscid, since we are interested in studying gravity waves. Under these assumptions and assuming there are no thermal changes, we could work with Euler's equations for inviscid fluids. Johnson [5] establishes a clear method for modeling water waves using Euler's equations, as can also be seen in the references from [2], [3] and [4]. We will now examine this method and contrast it to [2], [3] and [4]. With the obtained knowledge we will construct our guideline for modeling waves in a stratified fluid (a fluid that has a varying density with depth) over a current.

2.2.2. Defining the Coordinate System and Domain

First, we establish the coordinate system in which we define our Euler equations. Consistent with the other papers, we will also use the coordinate system from Johnson as illustrated in figure 2.1. Here we see illustrated a sphere that represents the earth and on its surface the origin of the coordinate system with $\bar{x}$ horizontally in Eastern direction, $\bar{y}$ vertically in Northern direction and $\bar{z}$ outwards pointing up.

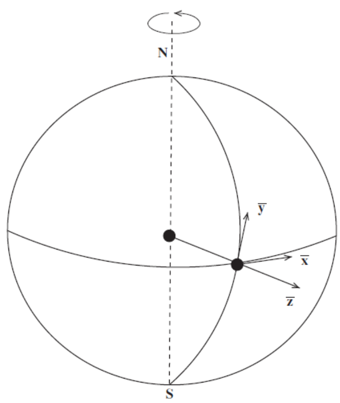
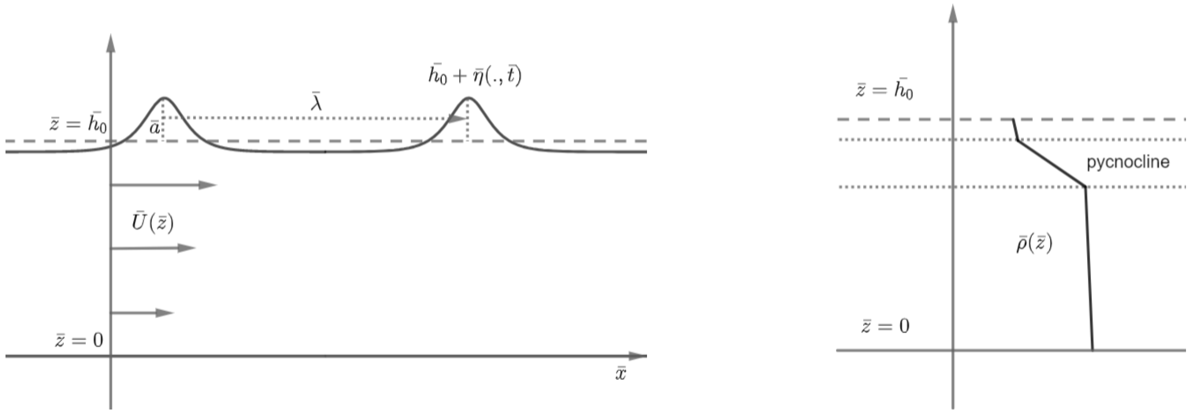


Figure 2.1: A sphere that represents the earth and on its surface the origin of the coordinate system with $\bar{x}$ horizontally in Eastern direction, $\bar{y}$ vertically in Northern direction and $\bar{z}$ outwards pointing up. Source:[4]

Adding a coordinate $\bar{t}$ for changes in time concludes our set of coordinates. Next we want to define our fluid domain, as illustrated in figure 2.2, on which our equations will be defined. For our model we will not couple the shape of the surface to the air flow above it, as suggested by [5] and [8]. Secondly, as the fluid body we are modelling is relatively far from any coast, the coasts are much farther apart than the ocean is deep, we will assume there are no boundaries present in both $\bar{x}$ directions. Thus we arrive at 2 boundaries. The fluid body is bounded from below by the ocean floor and above by the surface and there is a current present in the whole fluid domain as illustrated in figure 2.2a. The density function varies with depth as illustrated by an example in figure 2.2b.



(a) Fig. 2.2(a) illustrates the fluid domain in the $\bar{x}$, $\bar{z}$-plane, bounded by the ocean floor $\bar{z} = 0$ and the surface $\bar{z} = \bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$. $\bar{U}(\bar{z})$ represents an existing current on the entire fluid domain, varying with $\bar{z}$. The average water level is $\bar{h}_0$ and $\bar{\lambda}$ shows the distance between two consecutive wave peaks and $\bar{a}$ is the typical deviation of a peak from $\bar{h}_0$.

(b) Fig. 2.2(b) shows a density function $\bar{\rho}(\bar{z})$ over the fluid domain that increases with depth. The pycnocline is the region between the dotted lines near the surface where the density drastically changes.

Figure 2.2: The Fluid domain

2.2.3. Defining Euler Equations on the Domain

We have chosen the basis for our governing equations, namely the Euler equations, to fit our assumptions. Now that we have also defined our coordinate system and domain, we will state the Euler equations:

$$\begin{aligned}
\bar{\rho}(\bar{\mathfrak{u}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{u}}_{\bar{x}} + \bar{\mathfrak{v}}\bar{\mathfrak{u}}_{\bar{y}} + \bar{\mathfrak{w}}\bar{\mathfrak{u}}_{\bar{z}}) &= -\bar{\mathfrak{P}}_{\bar{x}} - 2\bar{\rho}\bar{\Omega}\bar{\mathfrak{w}} \\
\bar{\rho}(\bar{\mathfrak{v}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{v}}_{\bar{x}} + \bar{\mathfrak{v}}\bar{\mathfrak{v}}_{\bar{y}} + \bar{\mathfrak{w}}\bar{\mathfrak{v}}_{\bar{z}}) &= -\bar{\mathfrak{P}}_{\bar{y}} \\
\bar{\rho}(\bar{\mathfrak{w}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{w}}_{\bar{x}} + \bar{\mathfrak{v}}\bar{\mathfrak{z}}_{\bar{y}} + \bar{\mathfrak{w}}\bar{\mathfrak{w}}_{\bar{z}}) &= -\bar{\mathfrak{P}}_{\bar{z}} + 2\bar{\rho}\bar{\Omega}\bar{\mathfrak{u}} - \bar{g}
\end{aligned} \tag{2.1}$$

The left hand side of equations (2.1) represents the momentum of each fluid particle in $\bar{x}$, $\bar{y}$ and $\bar{z}$ direction. On the right hand side we see the body forces along the same directions. In these equations, $\bar{\mathfrak{u}}$, $\bar{\mathfrak{v}}$ and $\bar{\mathfrak{w}}$ represent the velocity in $\bar{x}$, $\bar{y}$ and $\bar{z}$ direction, respectively. The velocities dependent on all coordinates. ρ is the density of the fluid. $\bar{\Omega}$ represents the contribution of the Coriolis effect. Lastly, $\bar{\mathfrak{P}}$ represents the pressure that maintains the momentum.

We limit the applicability of this model to a small region near the equator because the fluid is uniform in y-direction. We can then approximate the Coriolis force as a constant [4]. Because the fluid is uniform in y-direction, we will continue to only look at the two other dimensions. Next, the model is also governed by the equation for mass conservation, leading to the following set of governing equations:

$$\begin{aligned}
\bar{\mathfrak{u}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{u}}_{\bar{x}} + \bar{\mathfrak{w}}\bar{\mathfrak{u}}_{\bar{z}} + 2\bar{\Omega}\bar{\mathfrak{w}} &= -\frac{1}{\bar{\rho}}\bar{\mathfrak{P}}_{\bar{x}} \\
\bar{\mathfrak{w}}_{\bar{t}} + \bar{\mathfrak{u}}\bar{\mathfrak{w}}_{\bar{x}} + \bar{\mathfrak{w}}\bar{\mathfrak{w}}_{\bar{z}} - 2\bar{\Omega}\bar{\mathfrak{u}} &= -\frac{1}{\bar{\rho}}\bar{\mathfrak{P}}_{\bar{z}} - \bar{g} \\
\bar{\rho}_{\bar{t}} + (\bar{\rho}\bar{\mathfrak{u}})_{\bar{x}} + (\bar{\rho}\bar{\mathfrak{w}})_{\bar{z}} &= 0
\end{aligned} \tag{2.2}$$

2.2.4. Boundary Conditions

Our two-dimensional model will only include boundaries at the surface and at the ocean floor. The surface is a free boundary, meaning its shape is not predetermined, and it is described by 2 boundary conditions. Firstly, the wave on the surface is maintained by a pressure difference that depends on the atmospheric pressure. The surface is formed by a fixed set of particles which are described by a second boundary condition. Lastly, the ocean floor is assumed to be impermeable and to not affect the horizontal velocity of passing water particles, thus not letting them penetrate the ocean bed. These boundary conditions are in turn described by the following equations:

$$\begin{aligned} \bar{\mathfrak{P}} &= \bar{P}_{\text{atm}} \text{ on } \bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t}) \\ \bar{\mathfrak{w}} &= \bar{\eta}_{\bar{t}} + \bar{u}\bar{\eta}_{\bar{x}} \text{ on } \bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t}) \\ \bar{\mathfrak{w}} &= 0 \text{ on } \bar{z} = 0 \end{aligned} \tag{2.3}$$

The first equation of boundary conditions (2.3) is the dynamic boundary equation. It is defined on the surface $\bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t})$. $\mathfrak{P}$ represents the pressure difference maintaining the wave and $\bar{P}_{\text{atm}}$ is the atmospheric pressure that is approximated to be constant over the entire surface. The second equation is the kinematic boundary condition, which is also defined on the surface. Here $\bar{\mathfrak{w}}$ represents the vertical velocity at the surface that depends on $\bar{\eta}$, which represents the wave's deviation from the surface. The last equation is defined at the ocean floor $\bar{z} = 0$, shows $\bar{\mathfrak{w}}$ is the vertical velocity is 0.

2.3. Including Current and Density Function

Equations 2.2 and 2.3 are the full set that represents our Boundary Value Problem(BVP). To include an arbitrary underlying current in our BVP, we separate the current $\bar{\mathfrak{u}}$ in a part that represents the background current $\bar{U}$ and a perturbation $\bar{u}$. Then we must also separate the pressure $\bar{\mathfrak{P}}$ into the pressure $\bar{P}$ that maintains the background current and the pressure $\bar{p}$ that maintains the perturbation. To include a non-constant density, that varies, as we described earlier, with $\bar{z}$ we will define $\bar{\rho} = \bar{\rho}(\bar{z})$. Thus we arrive at our final BVP:

$$\begin{aligned} \bar{u}_{\bar{t}} + (\bar{U} + \bar{u})\bar{u}_{\bar{x}} + \bar{w}(\bar{U} + \bar{u})_{\bar{z}} + 2\bar{\Omega}\bar{w} &= -\frac{1}{\bar{\rho}}\bar{p}_{\bar{x}} \\ \bar{w}_{\bar{t}} + (\bar{U} + \bar{u})\bar{w}_{\bar{x}} + \bar{w}\bar{w}_{\bar{z}} - 2\bar{\Omega}(\bar{U} + \bar{u}) &= -\frac{1}{\bar{\rho}}(\bar{P} + \bar{p})_{\bar{z}} - \bar{g} \\ \rho\bar{u}_{\bar{x}} + (\bar{\rho}\bar{w})_{\bar{z}} &= 0 \\ \\ \bar{P} + \bar{p} &= \bar{P}_{\text{atm}} \text{ on } \bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t}) \\ \bar{w} &= \bar{\eta}_{\bar{t}} + (\bar{U} + \bar{u})\bar{\eta}_{\bar{x}} \text{ on } \bar{z} = \bar{h}_0 + \bar{\eta}(\bar{x}, \bar{t}) \\ \bar{w} &= 0 \text{ on } \bar{z} = 0 \end{aligned} \tag{2.4}$$

This set of equations will be simplified to a problem that we can solve. Because together they form the BVP, and most transformations we perform to simplify and solve the problem will be performed on all these equations together, they will often be noted together in this group of 6 equations.

2.4. Simplifying and Solving the BVP

Now that we have constructed the BVP, we want to simplify the equations to a more general description similarly to [5]. Firstly, we transform the equations into a non-dimensional form. This ensures that the equations are independent of any subjective units. Secondly, we choose to scale the equations with respect to a parameter of interest. This means that in chapter 3, when we have a non-dimensional BVP, we choose a parameter (such as the average wave amplitude) and we look at how large the waves and currents behave relative to this parameter. By combining the non-dimensionalization and the scaling we obtain information about the waves and currents relative to the average wave amplitude instead of absolute distances of a specific unit. We will find later, as noted in [2], that a second set of scaling is necessary. It is important to note that this transformation will only be valid for modelling shallow water waves. This second scaling will be detailed in chapter 3.

We will thus arrive at a set of non-dimensional, scaled equations. From [5] we glean that these equations will not be easily solved on their own and therefore we will decide similarly to focus on leading-order equations by expressing the unknown variables as asymptotic series. We already chose the governing equations based on omitting relatively small interactions such as viscosity and slipping on the ocean floor, thus the higher-order solutions wouldn't give any useful information. Such an asymptotic series is not guaranteed to converge, however as we will truncate the series later it is still very useful. The leading order equations can easily be solved for the movement of the free surface, the horizontal speed, and the vertical speed.

These equations will belong to the family of equations called the Korteweg-deVries(KdV) equations. From the literature we expect KdV-type equations to have two types of travelling wave solutions: solitons and periodic travelling waves. A soliton is a single wave that does not change shape through time. The behavior of solutions of general KdV is discussed in more detail in [5]. At the end of chapter 3, when we will have derived our specific set of solutions, we will take a look at their behavior.

We have constructed a method to solve a problem that includes an arbitrary current and an arbitrary density distribution. We can use this method to solve the BVP (2.4) in chapter 3.

3

Methods and Results

This chapter applies the method we developed in chapter 2 to find the solution to the IBVP (2.4). First, the IBVP will be transformed to a new unit-less, scaled problem. Then we will look at the solutions of the leading order equations and their behavior.

3.1. Nondimensionalization

To generalize the IBVP (2.4) to a non-dimensional problem we need a transformation for each variable. The choice of transformation can include or exclude any information that you'd like to leave in or out. The following transformation, which is based on the transformation of [2], suits our needs.

$$\begin{aligned}
\bar{x} &= \bar{\lambda} x, & \bar{z} &= \bar{h}_0 z, & \bar{t} &= \frac{\bar{\lambda}}{\sqrt{\bar{g}\bar{h}_0}} t, \\
\bar{U} + \bar{u} &= \sqrt{\bar{g}\bar{h}_0}(U + u), & \bar{w} &= \sqrt{\bar{g}\bar{h}_0}\frac{\bar{h}_0}{\bar{\lambda}} w, & \bar{\eta} &= \bar{a}\eta, \\
\bar{P} + \bar{p} &= -\bar{g}\int_{\bar{h}_0}^{\bar{z}} \bar{\rho}(s)ds + \bar{g}\bar{h}_0\bar{\rho}(\bar{z})(P + p), & \bar{\Omega} &= \frac{\sqrt{\bar{g}\bar{h}_0}}{\bar{h}_0}\Omega, & & \\
\bar{\rho}(\bar{z}) &= \bar{\rho}(\bar{h}_0)\rho(z), & P_0 &= \frac{\bar{P}_{atm}}{\bar{g}\bar{h}_0\bar{\rho}(\bar{h}_0)}
\end{aligned} \tag{3.1}$$

These transformations were chosen in this particular way to retain exactly the information that we need. The transformations were found after trial and error, but one should always be careful that the units should disappear fully.

Next, we use the following dimensionless parameters: the amplitude and the shallowness parameter(left and right respectively).

$$\epsilon := \frac{\bar{a}}{\bar{h}_0} \qquad \delta := \frac{\bar{h}_0}{\bar{\lambda}} \tag{3.2}$$

Using these transformations, we arrive at the non-dimensional IBVP:

$$
\begin{aligned}
u_t + (U+u)u_x + w(U+u)_z + 2\Omega w &= -p_x \\
\delta^2(w_t + (U+u)w_x + ww_z) - 2\Omega(U+u) &= -\frac{(\rho(P+p))_z}{\rho} \\
u_x + \frac{(\rho w)_z}{\rho} &= 0 \\
P + p &= \frac{1}{\rho}\left(P_0 + \int_1^z \rho ds\right) \text{ on } z = 1 + \epsilon\eta(x,t) \\
w &= \epsilon(\eta_t + (U+u)\eta_x) \text{ on } z = 1 + \epsilon\eta(x,t) \\
w &= 0 \text{ on } z = 0
\end{aligned}
\tag{3.3}
$$

3.2. Transforming the Boundary Conditions

The boundary conditions of our dimensionless IBVP (3.3) are now still defined for a free boundary. It would be nice for further calculations to transform the boundary conditions to be defined on a fixed boundary. We use Taylor expansions of the dynamic and kinematic boundary conditions and obtain the following set of transformed boundary equations:

$$
\begin{aligned}
&P + p + \epsilon\eta(P+p)_z + \frac{\epsilon^2\eta^2}{2}(P+p)_{zz} = \\
&\quad P_0 + \epsilon\eta(1 - \rho'(1)P_0) - \frac{\epsilon^2\eta^2}{2}(-\rho'(1) + P_0[-\rho''(1) + 2(\rho('(1))^2) + \mathcal{O}(\epsilon^3) \text{ on } z = 1 \\
&\quad w = \epsilon(\eta_t + \eta_x(U+u) - \eta w_z) + \mathcal{O}(\epsilon^2) \text{ on } z = 1
\end{aligned}
\tag{3.4}
$$

To simplify the boundary conditions we use what we can learn from equation (3.3) in the situation where there is no perturbation (i.e. no waves). This means $u = 0, w = 0, p = 0, \eta = 0$.

$$
\begin{aligned}
-2\Omega U &= -\frac{(\rho P)_z}{\rho} \\
P(1) &= P_0 \\
P'(1) &= -\rho(1)P_0 - 2\Omega U(1) \\
P''(1) &= -\rho''(1)P_0 + 2(\rho'(1))^2 P_0 - 6\Omega U(1)\rho'(1) - 2\Omega U'(1)
\end{aligned}
\tag{3.5}
$$

We also use transformation (3.1) of ρ and z to find $\rho(z=1) = 1$. We substitute this information into the Taylor expansions of the dynamic and kinematic boundary conditions and into the Euler equations:

$$
\begin{aligned}
u_t + (U+u)u_x + w(U+u)_z + 2\Omega w &= -p_x \\
\delta^2(w_t + (U+u)w_x + ww_z) - 2\Omega u &= -\frac{(\rho p)_z}{\rho} \\
u_x + \frac{(\rho w)_z}{\rho} &= 0 \\
&p + \epsilon\eta p_z + \frac{\epsilon^2\eta^2}{2}p_{zz} = \\
&\epsilon\eta(1 + 2\Omega U(1)) + \frac{\epsilon^2\eta^2}{2}(-\rho'(1) + 2\Omega U'(1) + 6\Omega U(1)\rho'(1)) + \mathcal{O}(\epsilon^3) \text{ on } z = 1 \\
&w = \epsilon(\eta_t + \eta_x(U+u) - \eta w_z) + \mathcal{O}(\epsilon^2) \text{ on } z = 1 \\
&w = 0 \text{ on } z = 0
\end{aligned}
\tag{3.6}
$$

3.3. Scaling

From the new dynamic and kinematic boundary conditions of IBVP (3.6) we observe that p and w are proportional to ϵ. This informs scaling p and w with ϵ. We scale u with ϵ to satisfy the equation of mass conservation. Lastly, we'd like to also scale Ω. This scaling is validated in [2]. This leads to the following transformations:

$$\begin{aligned} u &\to \epsilon u, \\ w &\to \epsilon w, \\ p &\to \epsilon p, \\ \Omega &\to \epsilon \Omega_0 \end{aligned} \tag{3.7}$$

We apply this scaling to IBVP (3.6):

$$\begin{aligned} u_t + (U+u)u_x + w(U+\epsilon u)_z + 2\epsilon\Omega w &= -p_x \\ \delta^2(w_t + (U+\epsilon u)w_x + \epsilon w w_z) - 2\epsilon\Omega_0 u &= -\frac{(\rho p)_z}{\rho} \\ u_x + \frac{(\rho w)_z}{\rho} &= 0 \\ p + \epsilon\eta p_z + \frac{\epsilon^2\eta^2}{2}p_{zz} &= \\ \eta(1+2\epsilon\Omega_0 U(1)) + \frac{\epsilon\eta^2}{2}(-\rho'(1) + 2\epsilon\Omega_0 U'(1) + 6\epsilon\Omega_0 U(1)\rho'(1)) &+ \mathcal{O}(\epsilon^2) \text{ on } z=1 \\ w = \eta_t + \eta_x(U+\epsilon u) - \epsilon\eta w_z &+ \mathcal{O}(\epsilon^2) \text{ on } z=1 \\ w = 0 \text{ on } z = 0 & \end{aligned} \tag{3.8}$$

Geyer and Quirchmayer [2] suggest a second scaling to lose the shallowness parameter δ in favor of the amplitude parameter ϵ:

$$\begin{aligned} x &\to \frac{\delta}{\sqrt{\epsilon}}x, \\ z &\to z, \\ t &\to \frac{\delta}{\sqrt{\epsilon}}t, \\ P &\to P, \\ \eta &\to \eta, \\ u &\to u, \\ w &\to \frac{\sqrt{\epsilon}}{\delta}w \end{aligned} \tag{3.9}$$

As Geyer and Quirchmayer find out, this equation is only non-singular for $\delta << 1$, $\epsilon = \mathcal{O}(\delta^2)$. This condition informs us that the equations we derive can only model shallow water waves of small amplitude. We apply the second scaling (3.9) to IBVP (3.8):

$$
\begin{aligned}
u_t + (U+u)u_x + w(U+\epsilon u)_z + 2\epsilon\Omega w &= -p_x \\
\epsilon(w_t + (U+\epsilon u)w_x + \epsilon w w_z) - 2\epsilon\Omega_0 u &= -\frac{(\rho p)_z}{\rho} \\
u_x + \frac{(\rho w)_z}{\rho} &= 0 \\
p + \epsilon\eta p_z + \frac{\epsilon^2\eta^2}{2}p_{zz} &= \\
\eta(1+2\epsilon\Omega_0 U(1)) + \frac{\epsilon\eta^2}{2}(-\rho'(1) + 2\epsilon\Omega_0 U'(1) + 6\epsilon\Omega_0 U(1)\rho'(1)) &+ \mathcal{O}(\epsilon^2) \text{ on } z=1 \\
w &= \eta_t + \eta_x(U+\epsilon u) - \epsilon\eta w_z + \mathcal{O}(\epsilon^2) \text{ on } z=1 \\
w &= 0 \text{ on } z=0
\end{aligned}
\tag{3.10}
$$

3.4. Introducing New Coordinates

As previously discussed in chapter 2, we expect that the solutions of IBVP (2.4) are waves propagating at a constant speed. To follow these waves we introduce a coordinate frame that moves with the waves at constant speed c and focuses on right-traveling waves:

$$
\xi = x - ct, \tau = \epsilon t \tag{3.11}
$$

We transform the coordinates of (3.10) using (3.11):

$$
\begin{aligned}
\epsilon u_\tau + (U-c+\epsilon u)u_\xi + w(U+\epsilon u)_z + 2\epsilon\Omega_0 w &= -p_\xi \\
\epsilon(\epsilon w_\tau + (U-c+\epsilon u)w_\xi + \epsilon w w_z) - 2\epsilon\Omega_0 u &= -\frac{(\rho(z)p)_z}{\rho(z)} \\
u_\xi + \frac{(\rho(z)w)_z}{\rho(z)} &= 0 \\
p + \epsilon\eta p_z + \frac{\epsilon^2\eta^2}{2}p_{zz} &= \\
\eta(1+2\epsilon\Omega_0 U(1)) + \frac{\epsilon\eta^2}{2}(-\rho'(1)+2\epsilon\Omega_0 U'(1) + 6\epsilon\Omega_0 U(1)\rho'(1)) &+ \mathcal{O}(\epsilon^2) \text{ on } z=1 \\
w &= \epsilon\eta_\tau + (U-c+\epsilon u)\eta_\xi - \epsilon\eta w_z + \mathcal{O}(\epsilon^2) \text{ on } z=1 \\
w &= 0 \text{ on } z=0
\end{aligned}
\tag{3.12}
$$

3.5. Finding the Leading-Order solution and the Burns Condition

We'd like to solve IBVP (3.12) for η, u and w. However, as mentioned in the literature study, it is a valid choice to simplify the problem by performing the following asymptotic expansion in η, u, w, p and solve the leading order problem: $q \sim \sum_{n=0}^{\infty} q_n \epsilon^n$

Defining the Leading-Order Problem and the Wave Speed
The leading order problem then consists of the following equations:

$$\begin{aligned}(U-c)u_{0\xi} + U'w_0 &= -p_{0\xi}\\ 0 &= \frac{(\rho(z)p_o)_z}{\rho(z)}\\ u_{0\xi} + \frac{(\rho(z)w_0)_z}{\rho(z)} &= 0\end{aligned} \tag{3.13}$$

$$\begin{aligned}p_0 &= \eta_0 \text{ on } z=1\\ w_0 &= (U-c)\eta_{0\xi} \text{ on } z=1\\ w_0 &= 0 \text{ on } z=0\end{aligned}$$

Let us define:

$$I_n(z) = \int_0^z \frac{1}{(U-c)^n} ds \tag{3.14}$$

Combining the second equation and the dynamic boundary condition from (3.13) gives us a solution for p_0 and combining the first equation and the equation for mass conservation from (3.13), gives us a solution for w_0 and u_0:

$$\begin{aligned}p_0 &= \frac{\eta_0}{\rho(z)}\\ w_0 &= \eta_{0\xi}\frac{U-c}{\rho(z)}I_2(z)\\ u_0 &= -\frac{\eta_0}{\rho(z)}\left(\frac{1}{U-c} + U'I_2(z)\right)\end{aligned} \tag{3.15}$$

Until now the wave speed was arbitrary. When we combine the solution for w_0 with the kinetic boundary condition from (3.13) we find the following condition:

$$I_2(1) = \int_0^1 \frac{1}{(U-c)^2} ds = 1 \tag{3.16}$$

The wave speed of the solution therefore has to satisfy this condition (3.16), called the Burns condition.

p_0, w_0 and u_0 depend on η_0, which is still unknown. As it turns out, the first-order problem will give us all the information we need to find a solution for η_0. That means we do not need to look at any higher orders.

Solving the First-Order Problem
When we look at the first-order problem, we can also use our knowledge from the leading order solutions of u_0, w_0, and p_0. The first-order problem consists of the following equations:

$$\begin{aligned}(U-c)u_{1\xi} + w_1U' + u_{0\tau} + u_0u_{0\xi} + w_0u_{0z} + 2\Omega_0 w_0 &= -p_{1\xi}\\ (U-c)w_{0\xi} - 2\Omega_0 u_0 &= -\frac{(\rho(z)p_1)_z}{\rho(z)}\\ u_{1\xi} + \frac{(\rho(z)w_1)_z}{\rho(z)} &= 0\end{aligned} \tag{3.17}$$

$$\begin{aligned}p_1 &= \eta_1 + 2\eta_0\Omega_0 U(1) + \frac{\eta_0^2}{2}\rho'(1) \text{ on } z=1\\ w_1 &= \eta_{0\tau} + \eta_{1\xi}(U-c) + \eta_{0\xi}u_0 - \eta_0 w_{0z} \text{ on } z=1\\ w_1 &= 0 \text{ on } z=0\end{aligned}$$

From the first-order problem (3.17) we can reach a solution for η_0. Using the first equation and the equation for mass conservation in the first-order problem (3.17) we solve for w_1.

$$w_1(z) = \frac{U-c}{\rho(z)} \int_0^z \frac{\rho(s)}{(U-c)^2} [p_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} + 2\Omega_0 w_0] ds \tag{3.18}$$

This equation satisfies the last boundary condition in the first-order problem (3.17). Combine the second equation with the dynamic boundary condition we reach a solution for $p_{1\xi}$:

$$\begin{aligned} p_{1\xi} = \rho^{-1}(z)\Big[&- 2\Omega_0\eta_{0\xi} \int_1^z \frac{1}{U(s_1)-c} + U' I_2(s_1) ds_1 - \eta_{0\xi\xi\xi} \int_1^z (U(s_1)-c)^2 I_2(s_1) ds_1 \\ &+ \eta_{1\xi} + 2\eta_{0\xi}\Omega_0 U(1) + \eta_0\eta_{0\xi}\rho_z(1) \Big] \end{aligned} \tag{3.19}$$

We will evaluate (3.18) at $z=1$ and fill in the known variables from equations (3.15) and (3.19).

$$\begin{aligned} w_1(z=1) = (U(1)-c) \int_0^1 \frac{1}{(U(s_2)-c)^2} \Big[&- 2\Omega_0\eta_{0\xi} \int_1^z \frac{1}{U(s_1)-c} + U' I_2(s_1) ds_1 \\ &- \eta_{0\xi\xi\xi} \int_1^z (U(s_1)-c)^2 I_2(s_1) ds_1 \\ &+ \eta_{1\xi} + 2\eta_{0\xi}\Omega_0 U(1) + \eta_0\eta_{0\xi}\rho_z(1) \\ &- \eta_{0\tau}\Big(\frac{1}{U(s_2)-c} + U'(s_2) I_2(s_2)\Big) \\ &+ \eta_0\eta_{0\xi} \frac{1}{\rho(s_2)} \Big(\frac{1}{U(s_2)-c} + U'(s_2) I_2(x_2)\Big)^2 \\ &- \eta_0\eta_{0\xi} (U(s_2)-c) I_2(s_2) \cdot \frac{d}{ds_2} \left[\frac{(\frac{1}{U(s_2)-c} + U'(s_2) I_2(x_2))}{\rho(s_2)} \right] \\ &+ \eta_{0\xi} 2\Omega_0 (U(s_2)-c) I_2(s_2) \Big] ds_2 \end{aligned} \tag{3.20}$$

We can later simplify this result further. First, we will look at the kinematic boundary condition from the first-order problem (3.17) to gain a different expression for $w_1(z=1)$. We fill in the unknowns to get:

$$\begin{aligned} w_1(z=1) = &\eta_{1\xi}(U(1)-c) \\ &+ \eta_{0\tau} \\ &- \eta_0\eta_{0\xi} \left(\frac{1}{U(1)-c} + U'(1) + \frac{d}{dz}[(U(z)-c)I_2(z)]|_{z=1} - \rho'(1)(U(1)-c) \right) \end{aligned} \tag{3.21}$$

We now have 2 equations for $w_1(z=1)$, (3.20) and (3.21), which we can equate. This results in an equation with 2 unknowns η_1 and η_0:

$$\begin{aligned}
\int_0^1 \frac{1}{(U(s_2)-c)^2}\Bigg[&- \eta_{0\xi} 2\Omega_0 \int_1^{s_2} \frac{1}{U(s_1)-c} + U' I_2(s_1) ds_1 \\
&- \eta_{0\xi\xi\xi} \int_1^{s_2} (U(s_1)-c)^2 I_2(s_1) ds_1 \\
&+ \eta_{1\xi} + 2\eta_{0\xi}\Omega_0 U(1) + \eta_0 \eta_{0\xi} \rho_z(1) \\
&- \eta_{0\tau}(\frac{1}{U(s_2)-c} + U'(s_2) I_2(s_2)) \\
&+ \eta_0 \eta_{0\xi} \frac{1}{\rho(s_2)} (\frac{1}{U(s_2)-c} + U'(s_2) I_2(x_2))^2 \\
&- \eta_0 \eta_{0\xi} (U(s_2)-c) I_2(s_2) * \frac{d}{ds_2}\left[\frac{\frac{1}{U(s_2)-c} + U'(s_2) I_2(x_2)}{\rho(s_2)}\right] \\
&+ \eta_{0\xi} 2\Omega_0 (U(s_2)-c) I_2(s_2) \Bigg] ds_2 \\
&- \eta_{1\xi} \\
&- \eta_{0\tau} \frac{1}{U(1)-c} \\
&+ \eta_0 \eta_{0\xi} \left(\frac{1}{(U(1)-c)^2} + \frac{U'(1)}{U(1)-c} + \frac{1}{U(1)-c} \frac{d}{dz}\Big[(U(z)-c) I_2(z)\Big]|_{z=1} - \rho'(1) \right) \\
&= 0
\end{aligned} \tag{3.22}$$

Working this out we get an equation of the shape:

$$0 = A \cdot \eta_{1\xi} + B \cdot \eta_{0\tau} + C \cdot \eta_0 \eta_{0\xi} + D \cdot \eta_{0\xi\xi\xi} + \Omega_0 \cdot E \cdot \eta_{0\xi} \tag{3.23}$$

We will look at the simplified coëfficients A, B, C, D and E. As is our luck, the η_1 term will cancel out, resulting in an PDE for η_0 with coefficients:

$$\begin{aligned}
A &= 0 \\
B &= -2I_3(1) \\
C &= 3\int_0^1 \frac{1}{\rho(U(s_2)-c)^4} ds_2 - 2\int_0^1 \frac{\rho'(s_2) I_2(s_2)}{\rho^2 (U(s_2)-c)^2} ds_2 - \int_0^1 \frac{\rho'(s_2) U'(s_2) I_2(s_2)}{\rho^2 (U(s_2)-c)} ds_2 + \frac{U'(1)}{U(1)-c} \\
D &= J_1 = \int_0^1 \frac{1}{(U(s_2)-c)^2} \int_1^{s_2} (U(s_1)-c)^2 I_2(s_1) ds_1 ds_2 \\
E &= 2U(1) - c
\end{aligned} \tag{3.24}$$

From equations (3.15) we can find the solutions for u_0 and w_0. These three equations are the leading order approximations of the free surface, the horizontal and the vertical velocity where $q \sim q_0 + \mathcal{O}(\epsilon)$.

3.6. Validation and Checking the Results

In an attempt to validate our results we will compare them to the results from [5] and [2].

To compare to [5] we set $\rho = 1$, $\Omega = 0$. We get the following coefficients:

$$\begin{aligned}
A &= 0 \\
B &= -2I_3(1) \\
C &= 3I_4(1) + \frac{U'(1)}{U(1)-c} \\
D &= J_1 \\
E &\text{ vanishes as } \Omega_0 = 0
\end{aligned} \tag{3.25}$$

The coefficient C is inconsistent with [5]. Therefore we can conclude that it is incorrect. Because E vanished we cannot check its validity.

To compare to [2] we set $U(z) = 0$, $c = 1$, $P = 0$. We get the following coefficients:

$$\begin{aligned} A &= 0 \\ B &= 2 \\ C &= 3\int_0^1 \frac{1}{\rho(s_2)} ds_2 + \frac{1}{\rho(0)} + 1 \\ D &= \frac{1}{3} \\ E &= -2 \end{aligned} \tag{3.26}$$

As we expected C to be incorrect, that explains why it is inconsistent with [2]. E is consistent. However, another example that would give the same result would be $E = 2(U(1) - c)$. As we cannot cross check it with Johnson, we cannot conclude whether it is correct.

3.7. Solution Behaviour

Because we concluded that the coefficient C is incorrect, we will try to adjust it enough to be able to analyse its behaviour by assuming $C = 3\int_0^1 \frac{1}{\rho(U(s_2)-c)^4} ds_2 + \nu(\rho)$ where $\nu(\rho) = 2 - \rho_z(1) + \int_0^1 (\rho_z + \frac{(z\rho)_z}{\rho^2}) dz$ as defined in [2]. This is a reasonable, though not verifiable, assumption as we expect the C to be dependent on ρ and $U(z)$.

Equation (3.23) has two types of solutions as we discussed in chapter 2, solitons and periodic traveling waves. This report will focus on the soliton solutions, but the reader can take a look at [2] on how to find the periodic traveling wave solutions.

We continue by solving the PDE for $\eta(\xi, \tau)$ by first transforming $\eta(\xi, \tau)$ to $\phi(\xi - c\tau)$ from which we obtain an ODE that we can solve. The solution to the ODE is:

$$\begin{aligned} \phi(\xi - c\tau) &= \frac{K}{L}\text{sech}^2\left(\frac{\sqrt{K}}{2}(\xi - c\tau)\right) \\ K &= -\frac{1}{J1}(2cI_3(1) + 2\Omega_0(2U(1) - c)) \\ L &= \frac{1}{J1}\left(\int_0^1 \frac{1}{\rho(U-c)^4} ds_2 + \nu(\rho)\right) \end{aligned} \tag{3.27}$$

Because coefficient C from solution (3.24) was estimated, we can say the result is correct up to a scaling factor. We have found an expression for $\phi(\xi - c\tau)$. This expression does not have any unknown coefficients (the integration steps used were based on [2]). Therefore, the initial condition is $\phi(\xi - c \cdot 0) = \frac{K}{L}\text{sech}^2\left(\frac{\sqrt{K}}{2}(\xi)\right)$. Thus we can conclude we found the solution to our IBVP.

3.7.1. Comparing the Solution with Other Sources

We will compare solution (3.27) to the result from [2] which includes a variable density, but no underlying current and the result from [5] that does include an underlying current but no variable density.

Variable density, no underlying current:

$$\begin{aligned} K &= 6(c + \Omega_0) \\ L &= \nu(\rho) \end{aligned} \tag{3.28}$$

Underlying current, no variable density:

$$K = -\frac{2cI_3(1)}{J1}$$
$$L = \frac{I_4(1)}{J1} \tag{3.29}$$

Consistent with (3.28), the density has influence on the amplitude of (3.27) because it is only part of the scaling coefficient. Next, what is also consistent with (3.29) is the influence of U as a multiplicative factor, thus influencing the amplitude, but also its influence on the width of the wave profile by its inclusion inside the sech$^2(\theta)$ term.

Next the solution for $u_0(\xi,\tau,z) = u(\xi - c\tau, z)$ and $w_0(\xi,\tau,z) = w(\xi - c\tau, z)$ can be obtained through the relation (3.15):

$$u(\xi - c\tau, z) = -\frac{\phi}{\rho(z)}\left(\frac{1}{U-c} + U' I_2(z)\right)$$
$$w(\xi - c\tau, z) = \phi' \frac{U-c}{\rho(z)} I_2(z) \tag{3.30}$$

3.7.2. Plotting for Linear density and Linear current

In this section we will plot a wave profile and observe changes in current velocity. Before we do this we will define some parameters. Firstly, we choose Ω_0 based on observations from [4], where they choose $\Omega_0 = 2.5 \cdot 10^{-4}$. Next we can prescribe the density and the current. In [2], Geyer and Quirchmayer already take a look at a realistic density function and in [4] Constantin and Johnson take a look at a realistic current that replicates the effect of the EUC. Because we are mostly interested in observing the combined effect of the current and varying density, we will choose a linear density and linear current function. A constant current would also only displace the body of fluid, thus not affecting the wave profile or velocity in an interesting way.

Wave Profile

We plot the wave profile for the following values

$$\Omega_0 = 2.5 \cdot 10^{-4}$$
$$\rho(z) = (1 - z) + 1$$
$$U(z) = 0.5z + 0.1 \tag{3.31}$$

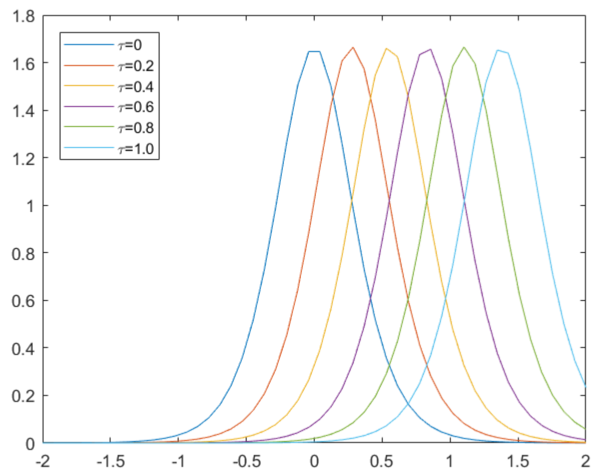
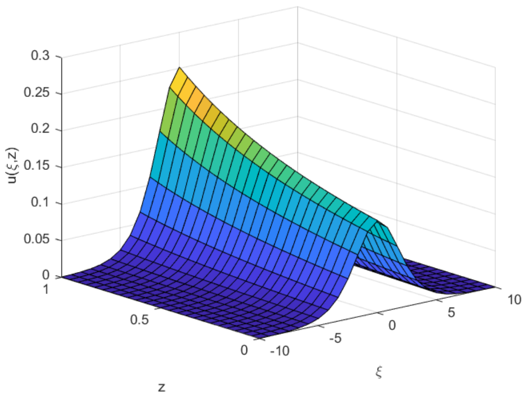


Figure 3.1: The wave profile of ϕ,(3.27), plotted for different values of τ. Source code: Appendix A

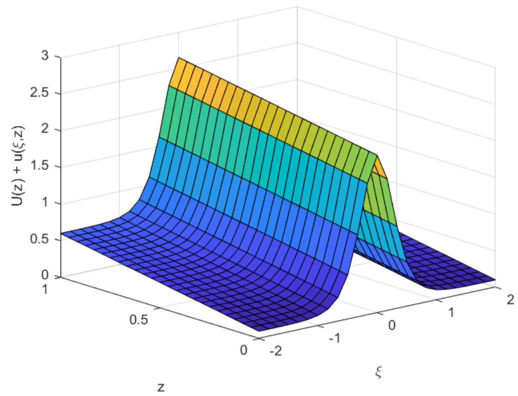
We observe in figure 3.1 that for these values of τ, the profile remains constant as we expected from soliton solutions of KdV-type equations.

Current velocity

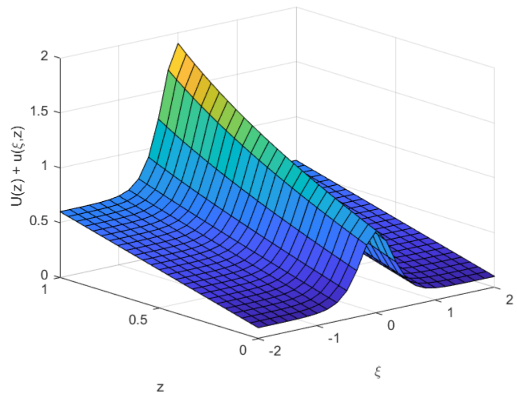
Next, we will observe how combining the varying density with the effect of the underlying current has changed the velocity profile of $u(\xi, z)$. For this plot we choose the same values for Ω_0, $\rho(z)$ and $U(z)$. We will plot the velocity profile at time $\tau = 0$.



(a) The velocity $U(z) + u(\xi, z)$ in the fluid body influenced by a density function that varies with depth $\rho(z)$ as presented in [2]



(b) The velocity $U(z) + u(\xi, z)$ in the fluid body influenced by a current $U(z)$ as presented in [5]



(c) The velocity $U(z) + u(\xi, z)$ in the fluid body with current $U(z)$ influenced by a density function that varies with depth $\rho(z)$ as presented in equation (3.27)

Figure 3.2: This figure shows the different velocity $U(z) + u(\xi, z)$ for linear $\rho(z)$ and linear $U(z)$ at time $t = 0$. Source code: Appendix A.

In figure 3.2 we observe the effect of $\rho(z)$ and $U(z)$ on the velocity in the fluid body. The values on the $U(z) + u(\xi, z)$ axis in figure 3.2c are inaccurate because of the incorrect coefficient in equation (3.24). Because this coefficient is only a scaling factor in front of the solution (3.27) the overall figure still gives a good representation of the combined effect of $\rho(z)$ and $U(z)$.

In figure 3.2a we observe the effect of the linear density function ρ as it is presented in [2]. We see how the current is stronger at point $\xi = 0$ where the wave originates and weakens further from the origin. We observe that the velocity $u(\xi, z)$ decreases with z as ρ increases. The same effect can be seen in figure 3.2c.

Next, in figure 3.2b we observe the effect of a linear underlying current. We see again that the current is stronger near the origin of the wave at $\xi = 0$. Faster currents mean a taller and slimmer peak in the wave profile and in the current profile. In contrast to figure 3.2a, the velocity does not slope down as drastically.

In figure 3.2c we observe that the effects we previously noted about figure 3.2a and 3.2b are both present. The profile is slimmer and it also slopes down more drastically as ρ increases.

4

Conclusion and Discussion

4.1. Conclusion

In this report, we obtained a model for wave propagation over current in a continuously stratified fluid. The main goal was to combine methods for a variable density and methods for an underlying current. This was achieved by studying existing literature such as ([2], [3]) from Geyer and Quirchmayer and ([4], [5]) from Constantin and Johnson. Starting from the Euler equations, we generalised the model and focused on the leading order equations whih can be written in a closed form as a KdV type equation (3.23):

$$0 = B \cdot \eta_{0\tau} + C \cdot \eta_0 \eta_{0\xi} + D \cdot \eta_{0\xi\xi\xi} + \Omega_0 \cdot E \cdot \eta_{0\xi} \tag{4.1}$$

with coefficients (3.24). Though the coefficients were shown to be incorrect, the solutions could still be analysed. The solution equation (3.27):

$$\begin{aligned} \phi(\xi - c\tau) &= \frac{K}{L}\text{sech}^2\left(\frac{\sqrt{K}}{2}(\xi - c\tau)\right) \\ K &= -\frac{1}{J1}(2cI_3(1) + 2\Omega_0(2U(1) - c)) \\ L &= \frac{1}{J1}\left(\int_0^1 \frac{1}{\rho(U-c)^4} ds_2 + \nu(\rho)\right) \end{aligned} \tag{4.2}$$

is a solitary wave that keeps its shape as we observed in 3.1. The variable density and the underlying current influenced the wave profile in our model similar to how they influence the velocity separately. This can be observed from the equation 3.27 and also from figure 3.1.

This new model can be used to model waves over a current in a stratified fluid. This improvement is a small step in the right direction to improve the correctness and effectiveness of models for oceans which in turn improves the predictions we want to make using these models.

4.2. Discussion

Firstly, we should stress that the coefficients calculated for (4.1) were shown to be incorrect. However, because the incorrect coefficient would only scale the solution in (3.27), section 3.7.2 still gives useful information. It still shows the general influence of the variable density and the underlying current. The next step would thus be to iron out any mistakes in calculating the precise coefficients.

As mentioned in chapter 3, we have only looked at the soliton solutions of the problem. A next step would be to also specify the periodic travelling wave solutions. In section 3.7.2 we look at the wave propagation and velocity when influenced by a linear density function $\rho(z)$ and a linear current function $U(z)$. We stated that [2] and [4] already take a more specific look at realistic density and current functions, therefore this was not included in this report. A different interesting step is modelling waves over more intricate density and current functions. When modelling these waves, we calculate the wave speed through the Burns condition. For functions of higher order, solving the Burns condition 3.16 takes a much longer time and therefore this step was not included in this paper. Creating a faster program to model for higher order current functions would be a good expansion of the model created for this report. When analysing the solutions, the stability of the solutions is an important characteristic. This paper does not answer the question of whether the solutions we find are stable, but the method to find out whether they are stable can be based on the comments on stability in [2].

After finding a solution, the usual next step in the modelling process is to verify the model. So far we have compared the solution to solutions of similar models from different sources. However, for real verification the theoretical solution should be compared using different approaches. The problem we have defined at the beginning of this paper makes many assumptions, as stated in 2. Thus comparing the solution with measured data is not viable for this problem.

This study was performed with the focus to combine specific existing methods. The focus of the literature study was thus firstly to understand these methods. Any follow-up research in comparing with different types of methods would therefore be recommended. Next, the literature study showed that the model can be improved in more ways then the one in this report. Further research could therefore focus on for example coupling the dynamics of the air above the surface with the model of the body of water. We've plotted the wave profile of ϕ and the velocity u, but we have not yet taken a look at the effect of ρ on the horizontal velocity. Though we have mentioned how to calculate w_0 from (3.15), actually plotting this solution would be a next step to take.

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A

Source Code

Wave Profile and Figure 3.2c

The first piece of code was used to plot figures 3.1 with variable ”resolution”=50 and 3.2c with ”resolution”=25. You can prescribe the current and density functions at the beginning. c is calculated separately(code later in the appendix).

```
%% Input parameters
syms z s
resolution=25;
Omega=(2.5).*10.^-4;

L = 2;

K = 4;

x = linspace(-L,L,resolution);
T = linspace(0,K,6)
Z= linspace(1,0,resolution);

%Define U here
o=0;
p=0.5;
q=0.1;
U = @(z) o.*z.^2+p.*z+q
dU=matlabFunction(diff(U(z),z))
c=7/20 + sqrt(17)/4; %computed using code defined separately, satisfies Burns condition

%Define rho here
w=1
y=1
rho=@(z) w.*(1-z)+y
dr=matlabFunction( diff(rho(z)) )
dzr=matlabFunction(diff(z*rho(z)))

%Defining functions

%%J1
fun1=@(z2,z1,z) ((U(z1)-c).^2)./(((U(z)-c).^2).*((U(z2)-c).^2));
J1=integral3(@(z2,z1,z) fun1(z2,z1,z) , 0 , 1, @(z2) z2 , 1 , 0 , @(z1,z2) z1)

%%I31
fun2=@(z) 1./(U(z)-c).^3;
I31=integral(@(z) fun2(z),0,1)

%%I2
fun4=@(z) 1./(U(z)-c).^2;
I2=@(z) integral(@(z) fun4(z),0,z)
I2(1)

%%I41
fun3=@(z) 1./(rho(z).*(U(z)-c).^4);
```

```
Ir41=integral(@(z) fun3(z),0,1);

%% etaeta_0 coefficient
nu=@(z) dr()+dzr(z)./rho(z);
e=3.*Ir41 + 2-dr()+ integral(@(z) nu(z),0,1); %eta*eta0 coefficient

% Analytic solution
A=(-2.*c.*I31+2.*Omega.*(2.*U(1)-c))./J1
B=(e)./(3.*J1);

phi = @(b) A/B*sech( sqrt(A)./2*(b)).^2;
u=@(b,z) -(phi(b)./rho(z)).*((1./(U(z)-c))+dU().*I2(z));
u(1,1);

%%plots

figure(1);
plot(x-c*0,phi(x-c*0),x,phi(x-c*0.2),x,phi(x-c*0.4),x,phi(x-c*0.6),x,phi(x-c*0.8),x,phi(x-c
    *1.0))
legend("\tau=0", "\tau=0.2", "\tau=0.4", "\tau=0.6", "\tau=0.8", "\tau=1.0",'Location','
    northwest')

figure(2);
t=0;
[X1,Z1] = meshgrid(x-c*t,Z);
F=zeros(resolution); %This is slow, but it works. An alternative is better
for i=1:1:resolution
    for j=1:1:resolution
        F(j,i)=u(x(i)-c*t,Z(j));
    end
end
surf(X1,Z1,U(Z1)+F)
%set(gca,'visible','off')
%grid off;
az_angle=-40;
el_angle=20;

FS = '\fontname{Palatino} ';
xlabel([FS '\xi']);

ylabel([FS 'z']);

zlabel([FS 'U(z) + u(\xi,z)']);
view(az_angle,el_angle);
```

Figure 3.2a

This code was used to plot figure 3.2a, so it does not take into account a prescribed current $U(z)$. It is similar to the previous code. The code to create the plot is the same.

```
%% Input parameters
syms z
resolution=25;
o=0;
p=0.5;
q=0.1;
U = @(z) o.*z.^2+p.*z+q;
dU=matlabFunction(diff(U(z),z));

Omega=(2.5).*10.^-4;

L = 10;

T = 20;

x = linspace(-L,L,resolution);

t = linspace(0,T,301);

Z= linspace(1,0,resolution);

```

```
c = 0.1;


w=1;
y=1;

rho=@(z) w.*(1-z)+y
%dr= diff(rho,z);
dr=matlabFunction( diff(rho(z)) )

dzr=matlabFunction(diff(z*rho(z)))
fun6=@(z) dr()+ dzr(z)./(rho(z)).^2;
A=6.*(c+Omega)
B=2.*log(1+w)./w+1

% Analytic solution

step = 30;
t=0;
[X1,Z1] = meshgrid(x-c*t,Z);

phi = @(b) A/B*sech( sqrt(A)/2*(b)).^2;
u=@(b,z) phi(b)./rho(z);

% Plots
figure(1);
plot(x-c*t,phi(x-c*t))
figure(2);
surf(X1,Z1,u(X1-c*t,Z1));
az_angle=-40;
el_angle=20;
FS = '\fontname{Palatino} ';
xlabel([FS '\xi']);

ylabel([FS 'z']);

zlabel([FS 'u(\xi,z)']);
view(az_angle,el_angle);
```

Figure 3.2b

This code was used to plot figure 3.2b, so it does not take into account a prescribed density function(we set $\rho = 1$). It is similar to the previous code. The code to create the plot is the same.

```
%% Input parameters
syms z s
resolution=25;
Omega=(2.5).*10.^-4;

L = 2;

T = 4;

x = linspace(-L,L,resolution);

t = linspace(0,T,301);
Z= linspace(0,1,resolution);



%U
o=0;
p=0.5;
q=0.1;
U = @(z) o.*z.^2+p.*z+q;
dU=matlabFunction(diff(U(z),z))
c=7/20 + sqrt(17)/4;

%rho
w=0;
y=1;
```

```
rho=@(z) w.*(1-z)+y
dr=matlabFunction( diff(rho(z)) )
dzr=matlabFunction(diff(z*rho(z)))

%Defining functions

%%J1
fun1=@(z2,z1,z) ((U(z1)-c).^2)./(((U(z)-c).^2).*((U(z2)-c).^2));
J1=integral3(@(z2,z1,z) fun1(z2,z1,z) , 0 , 1, @(z2) z2 , 1 , 0 , @(z1,z2) z1)

%%I31
fun2=@(z) 1./(U(z)-c).^3;
I31=integral(@(z) fun2(z),0,1)

%%I2
fun4=@(z) 1./(U(z)-c).^2;
I2=@(z) integral(@(z) fun4(z),0,z)
I2(1)

%% etaeta_0 coefficient
fun3=@(z) 1./(U(z)-c).^4;
I41=integral(@(z) fun3(z),0,1)
e=3.*I41


% Analytic solution
A=-2.*c.*I31./J1
B=I41./J1

step = 30;
t=0;
[X1,Z1] = meshgrid(x-c*t,Z);

phi = @(b) A/B*sech( sqrt(A)./2*(b)).^2;
u=@(b,z) -(phi(b)./rho(z)).*((1./(U(z)-c))+dU().*I2(z));
u(1,1)
%%plots
figure(1);
plot(x-c*t,phi(x-c*t))
figure(2);

F=zeros(resolution); %This is slow, but it works. An alternative is better
for i=1:1:resolution
    for j=1:1:resolution
        F(j,i)=u(x(i)-c*t,Z(j));
    end
end
surf(X1,Z1,U(Z1)+F)
az_angle=-40;
el_angle=20;

FS = '\fontname{Palatino} ';
xlabel([FS '\xi']);

ylabel([FS 'z']);

zlabel([FS 'U(z) + u(\xi,z)']);
view(az_angle,el_angle);
```

Finding the wave speed

The following code was used to compute the wave speed c. It finds the wave speed by solving the Burns condition as defined in equations (3.16).

```
%% Input parameters
syms z s

%Define U here
o=0;
p=0.5;
q=0.1;
```

```
U=@(z) o.*z.^2+p.*z+q
dU=diff(U(z),z)

cU = o.*z.^2+p.*z+q
Burnsintegrand=(cU-s)^(-2)
Burnscond=int(Burnsintegrand,z,0,1);
c1=solve(Burnscond==1,s);%this returns 2 solutions
c=c1(2) %we choose the positive solution here, but there are 2.
```