Advances in Stochastic Duality for Interacting Particle Systems: from many to few

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COLOPHON

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ADVANCES IN STOCHASTIC DUALITY FOR INTERACTING PARTICLE SYSTEMS: FROM MANY TO FEW

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus Prof.dr.ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op
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door

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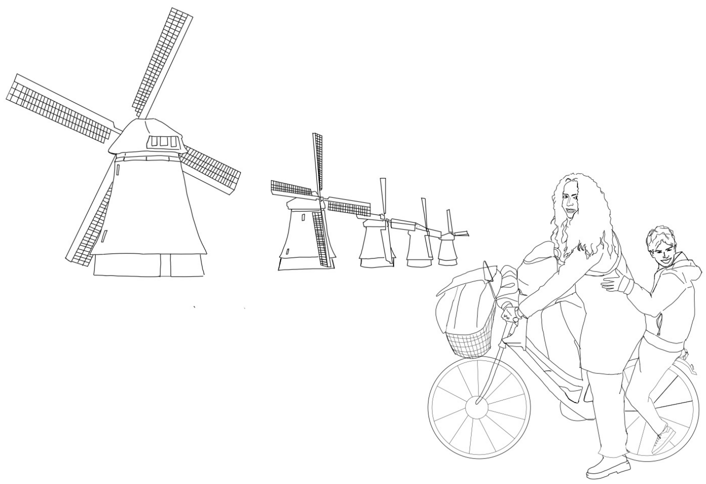
Alla mia mami, che mi accompagna dal primo giorno



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Part I

Introduction

Chapter 1

An informal introduction and motivations

When introducing the field of interacting particle systems (IPS), it is necessary to take a step back and start with a quick overview of statistical mechanics. Indeed, IPS, as studied nowadays, arose as an independent research area within probability theory from some of the fundamental questions that physicists attempted to answer since the end of the 19^{th} century. This short chapter has the aim of framing this thesis within a more general context, though, not having the ambition of providing a complete and precise picture of the physical motivations and perspectives behind the field of IPS.

Statistical mechanics is the area of physics where the emergence of macroscopic laws starting from a microscopic kinetic description of particles is studied. Any physical system can be studied at different scales, and here we distinguish two of them:

i) the macroscopic scale, where the system is studied as a "whole". The system is then described in terms of continuous variables, such as density, pressure and temperature, which evolve according to some deterministic partial differential equations (PDEs);

ii) the microscopic scale, namely the level at which we can discern the enormous amount of molecules of the system, which rapidly move and collide with each other, following (quantum) Hamiltonian evolution.

Establishing a connection between the microscopic and the macroscopic world was the goal of the founding fathers of statistical mechanics, Boltzmann, Gibbs and Maxwell. In the foundational work of statistical mechanics the emphasis was on the study of equilibrium and the connection between the microscopic world and the macroscopic laws of equilibrium thermodynamics. Later on, Boltzmann, via the introduction of the Boltzmann equation, tried to make further progress on these ideas also in the realm of non–equilibrium. Understanding the connection between micro and macro laws in the context of non-equilibrium systems is nowadays still largely an open problem and a subject of intense investigation. Only close to equilibrium, there is a general formalism known as *linear response theory*. Far from equilibrium, the study of this problem is mostly model driven, where one distinguishes stochastic models (the subject of this thesis) and deterministic models (like dynamical systems).

Equilibrium systems are characterized by reversibility: the evolution of all the particles together is reversible in time and the detailed-balance relation is satisfied. A typical equilibrium situation can be obtained by putting a system in contact with thermal baths at the same temperature: in such systems there are no macroscopic currents, such as heat current. In this setting there is a well established formalism to describe the distribution of the micro-states compatible with thermodynamic macro-parameters such as density, temperature and pressure. These parameters determine the equilibrium states which are characterized by time-reversal invariance and described by the Boltzmann-Gibbs distribution

$$\mathbb{P}(X = x) = \frac{1}{Z_\beta} e^{-\beta E(x)}. \tag{1.0.1}$$

Here $e^{-\beta E(x)}$ is the Boltzmann weight, β is the inverse temperature, $E(x)$ is the energy of the micro-state x and Z_β is the partition function. The scaling procedure to pass from the micro to the macro world via such probabilities is referred to as thermodynamic limit. Further, from these distributions, the law of equilibrium thermodynamics can be obtained, as well as phenomena like phase transitions and symmetry breaking.

In contrast, in non-equilibrium statistical mechanics, there is not such a universal method to determine the relevant probability measures which allow to pass from the micro to the macro world. There are two forms of non–

equilibrium: relaxation to equilibrium and non–equilibrium caused by an external driving. In the latter, namely in driven systems, one obtains in the long limit a steady state, called *stationary non–equilibrium state*. This differs from equilibrium steady states by the absence of detailed balance and by the presence of currents or, equivalently, by the breaking of time reversal symmetry. At present, deriving universal properties of such non–equilibrium steady states in various settings is one of the main research subject of non–equilibrium statistical mechanics. Examples of *driving mechanisms* pushing a system out of equilibrium are:

i) *boundary–driving*: open systems in contact with thermal bath/reservoirs working at different temperatures/densities.

ii) *bulk–driving*: systems undergoing the action of an external field which pushes the particles in a preferential direction.

iii) *activity of particles*: systems where particles have an internal state that may change over time.

In such systems, there is a net current of quantities such as heat or particle-density in a preferential direction. Preferential direction is a manifestation of time reversal breaking. Another signature of non-equilibrium is the presence of long-range correlations in the non–equilibrium stationary state, which are expected to be universal and not depending on a specific choice of the parameters of the system (in contrast to equilibrium, where, for system with short range interactions, long-range correlations only appear at critical points).

Relaxation to equilibrium refers to how a system converges to equilibrium when starting from a non-equilibrium state. At the macro–scale, there are various equations available describing relaxation to equilibrium and most of them can be easily derived with heuristic arguments. A prominent example is the heat (or diffusion) equation, which can be derived from conservation of mass together with Fick's law relating the current to the gradient of the conserved quantity (such as particle density). It is however a completely different problem to derive such equations from the microscopic dynamics. In particular, for realistic molecular motion (Hamiltonian dynamics), this is in general still a completely open problem, and only for very special systems, such as hard spheres, a rigorous derivation of macroscopic equations can be obtained. However, the variety of interesting phenomena emerging from systems of particles out of equilibrium pushes the study of non-equilibrium steady states. Typical examples of such phenomena are transport phenomena of heat or mass, with possible uphill diffusions and dissipation, and hydrodynamics motion with turbulence or formation of shocks.

Both in the context of equilibrium and out–of–equilibrium statistical mechanics, interacting particle systems are simplified microscopic models which provide a fruitful framework in which the transition from micro to macro can be made mathematically rigorous. IPS were introduced in the 70s by Spitzer as microscopic models of particles based on the two following simplifications:

i) Particles move and interact in a random way such that the whole configuration is a Markov process, i.e. the future evolution of the system depends only on the present state and not on the past. The Hamiltonian description is thus abandoned in favor of stochasticity in the motion mimicking the complicated microscopic dynamics.

ii) The physical space is discretized, i.e. the particles evolve on graphs such as the Euclidean lattices $\mathbb{Z}^d$, $d \geq 1$.

The only stochasticity in real microscopic deterministic dynamics comes from the initial condition. This, combined with the chaotic motion of particles, leads to the same type of ergodicity that is present in IPS. However, such ergodicity in IPS has been introduced artificially by the randomness in the motion of the particles. Even if these simplifications emphasize the toy-nature of IPS, still such caricatures of the real microscopic dynamics grasp the essential properties of the true physical interactions. Most importantly, the randomness assigned to the motion of the particles makes the rigorous derivation of macroscopic equations possible. In this probabilistic framework, the derivation of macro equations can be thought of as an infinite dimensional law of large numbers (at the trajectory level) and the procedure is called *hydrodynamic limit*. In the spirit of this analogy, one can then view the limit theorems for fluctuations around the hydrodynamic limit as an infinite dimensional analogues of the central limit theorems and the probabilities to deviate from the macro equations as an infinite dimensional large deviation result. The scaling results that we can obtain from IPS are expected to be universal and to be shared by large classes of systems, thus reinforcing the motivation to study IPS.

When turning to out–of–equilibrium scenarios, IPS in contact with reservoirs working at different densities, IPS driven by external fields and systems of interacting active particles, turn out to be analyzable models as well. In some cases, some out–of–equilibrium IPS are even exactly solvable models. The Bethe-Ansatz method and the matrix formulation method in the context of boundary driven IPS allowed to obtain explicit expressions for the non-equilibrium steady states of very specific models. However, these models are very special and the aforementioned methods are not robust enough to include, for instance, spatial inhomogeneities, which is one of the main themes in this thesis.

A class of IPS slightly more general than exactly solvable models are systems satisfying stochastic duality. Stochastic duality is a useful tool in probability theory which allows to study a Markov process (the one that interests you) via another Markov process, called dual process, which is hopefully easier to be studied. The connection between the two processes is established via a function, the so-called duality function which takes configurations of both processes as input. In the context of IPS, one of the typical simplifications provided by stochastic duality is that a system with an infinite number of particles can be studied via a finite number of particles. Notice that in order to perform the transition from micro to macro, it is necessary, as already understood at the beginning of the evolution of statistical mechanics, to consider systems with a large number of particles. Thus, being able to reduce the scaling limit problem to a finite system is a big advantage. When the dual system with a finite number of particles is a copy of the original process, duality is referred to as self-duality.

Even though IPS satisfying stochastic (self-)duality are still special, they are less special than exactly solvable models. Indeed duality is robust enough to still hold when modelling spatial inhomogeneities. We can then take advantage of duality in space inhomogeneous settings and obtain a closed form of the correlations of the systems: the time evolution of time dependent n-th order correlations will depend only on the initial correlations up to order n, and not on higher order correlations. Even if these closed forms may not be exactly computable, they still provide interesting information, such as universal properties of the system, and they can help in performing the transition from micro to macro. For instance, the study of the expectation of the rescaled empirical density field of a self-dual IPS in a spatial inhomogeneous setting, simplifies, by self-duality, to the study of the scaling behavior of one single space-inhomogeneous particle, i.e. to an invariance principle.

The simplification *from many to few particles* is nowadays a standard practice for classical IPS and more generally in studying scaling limits of Markov processes. However, it is less standard in the context of space inhomogeneous settings for which the literature is quite poor. When speaking of space inhomogeneities, the underlying physical idea is that we would like to incorporate in the model the presence of impurities and defects in the underlying environment where the particles evolve or modelling media composed by multiple materials with different characteristics. There are several ways of modelling the space inhomogeneities depending on the situations that one is trying to capture. Two of them are reported below.

i) *Random Environments*: often, the presence of inhomogeneities in a medium are modeled with an extra source of randomness, the so-called random environment. When performing the rescaling from micro to macro, one might obtain homogenization results: under certain assumptions the extra-randomness miming the inhomogeneities will then homogenize into a deterministic macroscopic quantity.

ii) *Multi-layer systems*: in some cases, the spatial inhomogeneities are caused by the presence of several layers in the media where the particle evolves in each of them with different characteristics. Multi-layer systems appeared also in the context of both active particles and population dynamics with seed-banks, where individuals are in either active or dormant state.

Pushing stochastic duality in this direction, i.e., studying interacting particle systems in space inhomogeneous settings, is the first aim of this thesis. More specifically, we want to extend stochastic duality to space inhomogeneous settings and use the simplification *from many to few*, both to have detailed information on the microscopic properties of such microscopic IPS in and out–of–equilibrium, and to perform rigorously the transition from micro to macro.

The second main goal of this thesis is to extend the notion stochastic self-duality beyond discrete underlying particles space. Namely, we want to get rid of the simplifications that particles move on $\mathbb{Z}^d$ and to be able to formulate duality in the form *from many to few* for particles evolving in the continuum, e.g. on $\mathbb{R}^d$. In discrete settings, self-duality functions are products over lattice sites of polynomials in the number of particles at each site, depending on the number of dual particles (the number of dual particles corresponds to the degree of the polynomial). The self-duality functions are usually categorized in "classical" self-duality functions, corresponding to (modified) factorial moments, and "orthogonal" self-duality functions, which are products of orthogonal polynomials, where orthogonality is with respect to an underlying reversible product measure. The language and formulation of duality in terms of number of particles at discrete lattice sites clearly breaks down in many natural settings of particles moving in the continuum. It is therefore important to develop a more general approach to self-duality that can lead to results also in the continuum, on general state spaces.

Chapter 2

Probabilistic introduction to interacting particle systems: a duality perspective

In this chapter we introduce the field of interacting particle systems from a probabilistic point of view. The focus will be on models satisfying stochastic duality, a technique that will be explained in detail later.

2.1 Notation and general terminology

An *interacting particle system* (IPS) is a collection of elements, the particles, that move randomly and are subjected to some interaction rules. Thus, it is a collection of coupled random processes. In many cases, we are interested in modelling spaces that resemble regular crystalline structures, and the Euclidean lattice $\mathbb{Z}^d$ is suitable for that purpose. The points of $\mathbb{Z}^d$ are thus the points in the physical space. The choice of $\mathbb{Z}^d$ as physical space can be also viewed as a model simplification.

The simplest particle system model is a system of independent random walks on $\mathbb{Z}^d$: each particle evolves as an independent continuous–time jumping Markov process on the Euclidean lattice and no interaction takes place. The particle system is described via the so-called configuration process (which is a Markov process), namely as the collection

$$\{\eta_t,\ t \geq 0\} = \{(\eta_t(x))_{x\in\mathbb{Z}^d}, t \geq 0\},$$

where, for any $x \in \mathbb{Z}^d$ and $t \geq 0$, the variable $\eta_t(x)$ denotes the number of particles at time t at the location x. In more interesting cases, on top of the independent Markovian dynamics of the particles, one super-imposes an interaction rule: in this way the evolution of an individual particle is no longer Markovian but what is still Markovian in many relevant interacting particle systems is the evolution of the composite state of the process, namely the configuration process $(\eta_t)_{t\geq 0}$. Denoting by $\mathcal{X}$ the state space of $(\eta_t)_{t\geq 0}$ (which in the case of independent random walks is $\mathbb{N}^{\mathbb{Z}^d}$ endowed with the product topology) and letting $f : \mathcal{X} \to \mathbb{R}$ be a continuous and bounded function, we can take advantage of two mathematical objects encapsulating the Markovian nature of the configuration process:

i) the semigroup $\{S_t, t \geq 0\}$ of $\{\eta_t,\ t \geq 0\}$ given by

$$S_t f(\eta) := \mathbb{E}_\eta[f(\eta_t)],\ \eta \in \mathcal{X},$$

where $\mathbb{E}_\eta[\cdot]$ denotes the expectation with respect to the law of the configuration process;

ii) the generator $\mathcal{L}$ of $\{\eta_t,\ t \geq 0\}$ given by

$$\mathcal{L}f(\eta) := \lim_{t\to 0} \frac{S_t f(\eta) - f(\eta)}{t}$$

supposing that for the continuous and bounded function f under consideration the above limit exists. The limit above has to be interpreted in the sense of the norm of an appropriate function space that depends on the specific model.

The semigroup provides the expected evolution of an observable $f : \mathcal{X} \to \mathbb{R}$ of the particle system, while the generator provides the expected infinitesimal change of an observable f.

2.2 Stationary measures and hydrodynamics

As already mentioned in the previous chapter, some of the goals of statistical mechanics are to study equilibrium behaviors of physical systems and how equilibria states are attained, the emergence of macroscopic dynamics starting from complicated microscopic ones, and the non-equilibrium properties in transport phenomena. At this point, we can formulate these problems in a mathematical way.

2.2.1 Steady states

Studying the equilibria states of the particle systems means to find the invariant probability measures of the configurations process: i.e., find a probability measure μ on $\mathcal{X}$ for which

$$\mu = \mu S_t \quad \forall t \geq 0,$$

where the evolved measure μS_t is defined via the relation $\int f \mathrm{d}\mu S_t = \int S_t f \mathrm{d}\mu$ for any $f : \mathcal{X} \to \mathbb{R}$ continuous and bounded. In terms of the generator $\mathcal{L}$ of the process, a probability measure μ is invariant if and only if

$$\int \mathcal{L} f \mathrm{d}\mu = 0$$

for any f in the domain of $\mathcal{L}$. Moreover, we say that a stationary measure μ is also reversible if for any continuous and bounded functions $f, g : \mathcal{X} \to \mathbb{R}$

$$\int (S_t f) g \mathrm{d}\mu = \int (S_t g) f \mathrm{d}\mu,$$

which, in terms of the generator $\mathcal{L}$, is equivalent to require $\mathcal{L}$ to be self-adjoint in $L^2(\mu)$.

Other basic questions related to the equilibrium states are:

i) under which conditions is there a unique invariant measure for the system?

ii) can the domain of attraction of an invariant measure μ be identified? In other words, which are the probability measures ν on $\mathcal{X}$ such that $\nu S_t \to \mu$ as $t \to \infty$?

These classical questions were among the first to be addressed in the literature of interacting particle systems. We refer the reader to [126] for an extensive treatment of several IPS.

2.2.2 Non-equilibrium steady states

In this thesis, for many models on $\mathbb{Z}^d$ under consideration the stationary (in some cases reversible) measures will be known or easily obtained from standard knowledge in the literature of IPS. The situation is different when we will look at *out–of–equilibrium* systems. A typical way to model a particle system out–of–equilibrium is the following: you let the interacting particles evolve on a finite chain, say $V_N := \{1, \ldots, N\}$, and one couples the left and the right end points of the chain, respectively the sites 1 and N, with a left and a right reservoir. *Reservoirs* are mechanisms that inject and absorb particles from the systems at some prescribed rates. If the two reservoirs work at different densities, then the system is out–of–equilibrium. Systems coupled with reservoirs are called *boundary driven systems* and studying and identifying the stationary state, which is called non-equilibrium steady state, is typically a much harder problem than studying the invariant measures for systems of particles in $\mathbb{Z}^d$ (see, e.g., [107]). To fix some nomenclature, the boundary driven systems will be also referred as *open systems*, where the word "open"refers to the fact that particles exit and enter the *bulk* (the chain V_N) of the system, thus exchanging mass or energy with the outside. On the other hand, systems of particles evolving on $\mathbb{Z}^d$ will be referred to as *closed systems*.

Supposing that for a certain boundary driven system one is able to prove that there exists a unique non-equilibrium steady state μ, typical objects of study are the non-equilibrium *stationary profile* $\left(\mathbb{E}_\mu[\eta(x)]\right)_{x \in V_N}$, the *stationary truncated n-point correlations* $\mathbb{E}_\mu\left[\prod_{i=1}^n (\eta(x_i) - \mathbb{E}_\mu[\eta(x_i)])\right]$ and whether the stationary current of particles satisfies the *Fick's law of transport*, namely if the flux of particles at stationarity goes from the reservoir working at higher density towards the reservoirs working at lower density. All these quantities and problems are non-trivial due to the fact that in many models the non-equilibrium steady state μ is not known explicitly.

2.2.3 Hydrodynamics

As already mentioned at the beginning of this section, another challenge is to rigorously derive a macroscopic law starting from the microscopic dynamics modeled via an interacting particle system. The general idea here is that IPS are models at a microscopic scale, in the sense that when modelling the motion and the interaction of the particles we are implicitly assuming that we are zooming in a certain material or physical phenomenon to be able to see these small entities and slowing down the time to be able to follow their trajectories. These two actions, the zooming in the space and the slowing–down of time, are performed from our human point of view, the macroscopic scale. To understand better this separation of scale, let us look at a concrete example. Let us start by reasoning at the macroscopic scale (basically describing what we see) and let us study the motion of a drop of ink in a big glass full of water. First, the size of the glass has to be large when compared with the amount of ink injected, in the sense that we want to reproduce a situation where the boundary conditions imposed by the presence of the glass are negligible for the motion of the ink. To find a model for the evolution of the density ρ_t of ink, two aspects must be observed and mathematically formulated. First the conservation in time of the total mass of ink, i.e.,

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \rho_t(x)\mathrm{d}x &= -\int_{\partial V} \nabla J_t \cdot \hat{n} \\ &= -\int_V \nabla \cdot J_t(x)\mathrm{d}x\end{aligned}$$

where V is an arbitrary control volume inside the glass of water with smooth boundary ∂V and outward normal vector $\hat{n}$, J_t is the flux of ink and the second equality follows from the divergence theorem. In other words, the above formula is telling that the variation of mass of ink in a certain control volume is equal to the amount of flux of ink that enters and leaves the system. Secondly, empirical observations tell us that the flux of ink goes against the gradient of concentration of ink, the so-called Fick's law mentioned above. Thus we have

$$J_t(x) = -D\nabla\rho_t(x),$$

where D is the so-called diffusion coefficient. Putting together the two formulas above and using the arbitrariness of the control volume V we get the following partial differential equation (PDE),

$$\frac{\partial}{\partial t}\rho_t = D\Delta\rho_t,$$

which is known under the name of *diffusion equation* or *heat equation* (indeed, besides the diffusion of mass, it also models diffusion of heat, in which case one speaks of Fourier's law instead of Fick's law).

At this point one is led to the fundamental question whether the above PDE, that is nowadays standard and stands at the basis of many engineering simulations, can be rigorously derived starting from a microscopic dynamics of particles. Thus, considering the simplest particle systems consisting of independent simple symmetric random walks on $\mathbb{Z}^d$ and keeping in mind the separation of scales discussed before, we need to take care of two things:

i) find a way to rescale space, time and mass to go from the microscopic perspective, when looking at the particles, to the macroscopic point of view of humans;

ii) find a way to study the collective behavior of the particles as a whole.

These two goals are achieved introducing a scaling parameter N and the rescaled empirical density field X_t^N, i.e. the random measure on $\mathbb{R}^d$ given by

$$\mathsf{X}_t^N := \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\delta_{\frac{x}{N}}\,\eta_{tN^2}(x)\,. \tag{2.2.1}$$

As we can see from the above definition, this is a measure on $\mathbb{R}^d$ given by a sum of Dirac deltas at the locations of the particles, and each of these deltas is weighted by the number of particles at that location. Being the number of particles at various times the result of the stochastic evolution of the particle system, X_t^N is a random measure. Moreover, the scaling parameter N is used for three purposes, as mentioned earlier:

i) to rescale the mass of each particle by a factor $\frac{1}{N^d}$;

ii) to squeeze the space by a factor $\frac{1}{N}$;

iii) to speed up the time by a factor N^2.

These three operations are responsible for the zooming-out procedure that allows to pass from the microscopic to the macroscopic scale.

Thus, we are after the following mathematical question: can we prove rigorously that when sending N to infinity, the rescaled empirical density field properly converges to a macroscopic deterministic measure X_t, which is absolutely continuous with respect to the Lebesgue measure

$$\mathsf{X}_t(\mathrm{d}u) = \rho_t(u)\mathrm{d}u,$$

and whose density ρ_t is the solution of some physically relevant PDE (like the diffusion equation previously introduced)? This limiting procedure is called *hydrodynamic limit* and the PDE solved by the macroscopic density is called *hydrodynamic* equation. These names come from the fact that typical limiting equations are indeed the ones used in hydrodynamic theory, like the diffusion equation. Notice that the rescaled space-time used when defining the rescaled empirical density field is the so–called parabolic rescaling: the diffusion equation is indeed invariant under this rescaling and reflects the fact that a single random walk in time N^2t travels $O(Nt)$ distance. Thus the rescaling to be adopted will depend on the particle system under consideration and on the macroscopic quantity and phenomenon that we are intended to study. Even for the simplest system of independent random walks the first rigorous results about hydrodynamic limits are rather recent. Establishing hydrodynamic limits for a comprehensive class of IPS and macroscopic laws is nowadays extremely active and vibrant research area. We refer the reader to the books [47] and [106] for several techniques and case studies.

2.3 A classical model: the exclusion process

After having introduced some of the problems that will be addressed in this thesis, it is now time to introduce one of the most classical and studied IPS: the *simple symmetric exclusion process* (SSEP). The SSEP was introduced by Spitzer in [156] as a model for a lattice gas at infinite temperature. More precisely, the particles move as independent random walks on $\mathbb{Z}^d$, but they are prevented by the (physical) constraint that they cannot share the same space location. Thus, on the dynamics of the independent random walks, the so-called *exclusion rule* is super-imposed: maximum one particle is allowed in each site, thus, jumps towards already occupied sites are suppressed.

The state space of SSEP is given by $\mathcal{X} = \{0,1\}^{\mathbb{Z}^d}$ and the generator by

$$\mathcal{L}f(\eta) = \sum_{\substack{\{x,y\}\subseteq\mathbb{Z}^d,\\ |x-y|=1}} \left\{ \begin{array}{c} \eta(x)(1-\eta(y))\,(f(\eta^{x,y})-f(\eta)) \\ +\,\eta(y)(1-\eta(x))\,(f(\eta^{y,x})-f(\eta)) \end{array} \right\}, \tag{2.3.1}$$

where $f : \mathcal{X} \to \mathbb{R}$ is a local (depending on a finite number of sites only) and bounded function and $\eta^{x,y}$ denotes the configuration where a particle has been moved from x to y. The above generator is describing the following dynamics of particles:

i) a particle at $x \in \mathbb{Z}^d$ waits an exponential time with parameter $2d$;

ii) after this waiting time, the particle chooses a neighboring site y (i.e., such that $|y-x| = 1$) with uniform probability given by $\frac{1}{2d}$;

iii) if the chosen site y is empty, the particle jumps there, otherwise it stays at x.

When turning to the open SSEP, i.e. the boundary driven version of the particle system, the state space is given by $\mathcal{X}^{\text{open}} = \{0,1\}^{V_N}$ with $V_N = \{1,\dots,N\}$ and the generator is given by

$$\mathcal{L}^{\text{open}}f(\eta) = \mathcal{L}^{\text{bulk}}f(\eta) + \mathcal{L}^{L,R}f(\eta)\,. \tag{2.3.2}$$

The generator $\mathcal{L}^{\text{bulk}}$ describes the bulk part of the dynamics and is given by

$$\mathcal{L}^{\text{bulk}}f(\eta) = \sum_{\substack{\{x,y\}\subseteq V_N,\\ |x-y|=1}} \left\{ \begin{array}{c} \eta(x)(1-\eta(y))\,(f(\eta^{x,y})-f(\eta)) \\ +\,\eta(y)(1-\eta(x))\,(f(\eta^{y,x})-f(\eta)) \end{array} \right\}$$

and the boundary part of the dynamics is described by the generator $\mathcal{L}^{L,R}$ as follows:

$$\mathcal{L}^{L,R} f(\eta) = \mathcal{L}_L f(\eta) + \mathcal{L}_R f(\eta) \,, \tag{2.3.3}$$

with

$$\begin{aligned}\mathcal{L}_L f(\eta) &= \eta(1)\,(1-\theta_L)\,(f(\eta^{1,-}) - f(\eta)) \\ &\quad + \theta_L\,(1-\eta(1))\,(f(\eta^{1,+}) - f(\eta))\end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned}\mathcal{L}_R f(\eta) &= \eta(N)\,(1-\theta_R)\left(f(\eta^{N,-}) - f(\eta)\right) \\ &\quad + \theta_R(1-\eta(N))\left(f(\eta^{N,+}) - f(\eta)\right) \,,\end{aligned} \tag{2.3.5}$$

where $\eta^{x,-} \in \mathcal{X}$, resp. $\eta^{x,+} \in \mathcal{X}$, denotes the configuration obtained from η by removing, resp. adding, a particle from, resp. to, site $x \in V$. In the above dynamics, creation and annihilation of particles occur at sites $x = 1$ and $x = N$ due to the interaction with a reservoir. The parameters $\theta_L,\ \theta_R \in [0,1]$ are the so-called *reservoirs densities*, and if $\theta_L \neq \theta_R$ the system is driven out of equilibrium.

Even if the interaction in the SSEP is quite simple, solving the problems described during this section for the SSEP, or its open version, is harder than in the systems of independent random walks. However, since the moment when the process was introduced, a special property for the SSEP was found out: the SSEP satisfies the *self-duality* property and, thanks to that, many quantities of interest can be computed quite explicitly.

Self-duality is a special instance of the so-called *stochastic duality* property and it can be viewed as a certain degree of exact solvability of the model. *Stochastic duality* is at the core of this thesis, and in the next section, after introducing it mathematically and showing some of the simplifications that provides, we will show how it can be used to solve some problems for the SSEP.

2.4 Stochastic duality

Stochastic duality is a probabilistic property that connects two Markov processes, allowing to study one process via the other one. The connection is established via a function, the *duality function*, defined on the product of the state spaces of the two connected Markov processes and which can be viewed as an observable of them. This tool becomes interesting when one of the two process is "much simpler" than the other one and the duality function is a useful observable of the original Markov process. Let us be more precise providing the mathematical definition of stochastic duality.

Definition 2.4.1. *Let $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ be two Markov processes with state space $\mathcal{X}$ and $\hat{\mathcal{X}}$, and let $D : \hat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ be a measurable function. $(\eta_t)_{t\geq 0}$ satisfies stochastic duality with dual process $(\xi_t)_{t\geq 0}$ and duality function D if*

$$\mathbb{E}_\eta \left[D(\xi, \eta_t)\right] = \hat{\mathbb{E}}_\xi \left[D(\xi_t, \eta)\right] , \qquad \forall t > 0, \eta \in \mathcal{X},\ \xi \in \hat{\mathcal{X}} \tag{2.4.1}$$

where $\mathbb{E}_\eta$ denotes expectation w.r.t. $(\eta_t)_{t\geq 0}$ starting at $\eta \in \mathcal{X}$ and $\hat{\mathbb{E}}_\xi$ denotes the expectation w.r.t. $(\xi_t)_{t\geq 0}$ starting at $\xi \in \hat{\mathcal{X}}$.

2.4.1 Examples

Some of the typical simplifications provided by stochastic duality are listed below via several interesting examples.

i) *From reflecting to absorbing*: this if the first historical example of stochastic duality and it is due to Lévy (see [130]). Let $(X_t)_{t\geq 0}$ be a Brownian motion on $[0,\infty)$, reflected at the origin and denote by $\mathbb{E}_x$ the expectation with respect to $(X_t)_{t\geq 0}$ starting from $x > 0$. Let $(Y_t)_{t\geq 0}$ be a Brownian motion on $[0,\infty)$, absorbed at the origin, and denote by $\hat{\mathbb{E}}_y$ the expectation with respect to $(Y_t)_{t\geq 0}$ starting from $y > 0$. These two processes are in duality relation with respect to the function $D(y, x) = \mathbf{1}_{\{x\leq y\}}$, i.e.

$$\mathbb{E}_x[D(y, X_t)] = \hat{\mathbb{E}}_y[D(Y_t, x)] \qquad \forall t \geq 0, x, y > 0.$$

ii) *From a complicated initial condition to a simpler one*: when the two processes connected by the stochastic duality property are two copies of the same process, we speak of *self-duality*. The simplification that *self-duality* provides lies in the fact that a Markov process starting from a complicated initial condition (think of infinitely many particles at time 0) can be studied via a copy of the same process but with a simpler initial condition (with one or a few particles). The self-duality property is satisfied by the classical example introduced above: the simple symmetric exclusion process. Indeed, denote by $\mathbb{E}_\eta$ the path-space expectation of the process $(\eta_t)_{t\geq 0}$ defined via the generator given in (2.3.1) and starting from $\eta \in \{0,1\}^{\mathbb{Z}^d}$. Let $D(\eta,\xi) = \prod_{x\in\mathbb{Z}^d} \mathbf{1}_{\{\eta(x)\geq\xi(x)\}}$, then the self-duality relation (see, e.g., [156]) reads as

$$\mathbb{E}_\eta[D(\xi,\eta_t)] = \mathbb{E}_\xi[D(\xi_t,\eta)] \quad \forall t \geq 0,\ \eta,\xi \in \{0,1\}^{\mathbb{Z}^d} \tag{2.4.2}$$

where $(\xi_t)_{t\geq 0}$ is a copy of $(\eta_t)_{t\geq 0}$ and is still referred to as the dual process. The above relation becomes interesting when dual process $(\xi_t)_{t\geq 0}$ is such that $\xi(x) \leq \eta(x)$ for any $x \in \mathbb{Z}^d$, i.e. ξ is a sub-configuration of η, and $\sum_{x\in\mathbb{Z}^d} \xi(x) < \infty$, i.e. the total number of particles in the dual process is finite. In particular, when $\sum_{x\in\mathbb{Z}^d} \xi(x) = 1$, i.e. there is only one particle in the dual system, we are left with a single random walk which is not subjected to any interaction rule due to the absence of other particles. Denote by $\mathbb{E}_x^{\mathrm{RW}}$ the path-space expectation with respect to the random walk starting from $x \in \mathbb{Z}^d$ evolving according to the generator

$$\mathcal{L}^{\mathrm{RW}} g(x) = \sum_{y:|x-y|=1} (g(y) - g(x)). \tag{2.4.3}$$

It then follows that relation (2.4.2) with $\xi = \delta_x$ can be rewritten as

$$\mathbb{E}_\eta[\eta_t(x)] = \mathbb{E}_x^{\mathrm{RW}}[\eta(X_t)]. \tag{2.4.4}$$

The above relation tells that the expectation in the SSEP process of the number of particles at time t at the location x is equal to an expectation with respect to a simple symmetric random walk starting at x of the number of particles at time 0 in the location of this random walk at time t. Later, we will see how this relation will be helpful when studying hydrodynamic limits.

iii) *From continuous to discrete variables*: in some cases, stochastic duality allows to study a process evolving in the continuum via a process evolving on a discrete, and thus simpler, space. This is the case for the so called Brownian momentum process (BMP), a system of diffusion processes subjected to a time-dependent magnetic field, which emerges as a *high-temperature* limit of an Hamiltonian dynamics. More precisely, let $G = (V, E)$ be a graph with vertexes in V and edges in E, then the BMP process $x_t = (x_t(1), \ldots, x_t(|V|))$ on $\mathbb{R}^{|V|}$ is defined via the generator

$$\mathcal{L}^{\mathrm{BMP}} = \sum_{(i,j)\in E} \left(x(i)\frac{\partial}{\partial x(j)} - x(j)\frac{\partial}{\partial x(i)} \right)^2 .$$

This is a model of heat conduction satisfying Fourier's law. The dual process is the so-called *symmetric inclusion process* (SIP) on the space $\mathbb{N}^V$, whose generator is given by

$$\mathcal{L}^{\mathrm{SIP}} f(\xi) = \sum_{(i,j)\in E} \left\{ \begin{array}{c} \xi(i)\left(\frac{1}{2} + \xi(j)\right)\left(f(\xi^{i,j}) - f(\xi)\right) \\ +\xi(j)\left(\frac{1}{2} + \xi(i)\right)\left(f(\xi^{j,i}) - f(\xi)\right) \end{array} \right\}$$

where $\xi \in \mathbb{N}^V$, $\xi^{i,j}$ represents the configuration where a particle has jumped from i to j and $f : \mathbb{N}^V \to \mathbb{R}$ is a bounded function. The duality function is given by (see, e.g., [32])

$$D(x,\xi) = \prod_{i\in V} \frac{x(i)^{2\xi(i)}}{(2\xi(i)-1)!!}.$$

iv) *From evolutions forward in time to evolutions backward in time*: this is a typical simplification provided by duality which has been often used in models of population genetics and genetics evolution (see, e.g., [61]). The simplest example of process satisfying this instance of duality is the following. Let $(Y_t)_{t\geq 0}$ be the diffusion process on $[0,1]$ evolving according to the following stochastic differential equation:

$$\mathrm{d}Y_t = \sqrt{Y_t(1-Y_t)}\,\mathrm{d}B_t$$

where $(B_t)_{t\geq 0}$ denotes a standard Brownian motion. The process $(Y_t)_{t\geq 0}$ models the evolution of the fraction of individuals of type A in a two-type population on large space-time scales. Denote by $\mathbb{E}_y$ the expectation with respect to $(Y_t)_{t\geq 0}$ starting from $y \in (0,1)$. Let $(N_t)_{t\geq 0}$ be the death process on $\mathbb{N} = \{1,2,\ldots\}$ where transitions from n to $n-1$ occur at rate $\binom{n}{2}$. Denote by $\hat{\mathbb{E}}_n$ the expectation with respect to $(N_t)_{t\geq 0}$ starting from $n \in \mathbb{N}$. Then, these two processes are in duality relation with respect to the function $D(n, y) = y^n$, namely

$$\mathbb{E}_y[(Y_t)^n] = \hat{\mathbb{E}}_n[y^{N_t}] \quad \forall t \geq 0, y \in (0,1), n \in \mathbb{N}.$$

The above expectation on the left hand side can be interpreted as the probability that n individuals from an infinite population are of type A at time t, thus, as a quantity resulting from an evolution forward in time. On the other hand, the above expectation on the right hand side can be read as the probability that the ancestors at time 0 of the n individuals are of type A, thus as a quantity resulting from an evolution backward in time.

As a direct and simple application of this duality relation, we compute the fixation probability in the process $(Y_t)_{t\geq 0}$. Indeed, denoting by $\mathbb{P}_y$ the path-space probability of the process $(Y_t)_{t\geq 0}$ starting from $y \in (0,1)$, we have

$$\mathbb{P}_y(Y_\infty = 1) = \mathbb{E}_y[Y_\infty] = \mathbb{E}_y[D(1, Y_\infty)] = \hat{\mathbb{E}}_1[D(N_\infty, y)] = y,$$

where in the third step we used duality and the last step follows immediately from the definition $(N_t)_{t\geq 0}$. In the above computation we can appreciate how stochastic duality can simplify the derivation of meaningful results for Markov processes.

v) *From boundary driven systems to a system with absorbing boundaries*: as explained before, a typical way to model out–of–equilibrium systems consists in adding to the model reservoirs working at different densities. It turns out that, for several systems, the model with the reservoirs is dual to the model where the reservoirs are substituted by absorbing sites, which are much simpler mechanisms. The first example of this instance of duality goes back to the so–called Kipnis-Marchioro-Presutti (KMP) model (see [107]), a model of energy redistribution. Turning to particle systems (see, e.g., [32]), we have that the boundary driven process $(\eta_t)_{t\geq 0}$ with generator given in (2.3.2) is dual to the process $(\xi_t)_{t\geq 0}$ on the extended chain $V_N \cup \{0, N+1\}$ which evolves as $(\eta_t)_{t\geq 0}$ on the bulk V_N and where the reservoirs are replaced by the absorbing sites $\{0, N+1\}$. More precisely, the state space of $(\xi_t)_{t\geq 0}$ is $\widehat{\mathcal{X}} = \{0,1\}^{V_N \cup \{0,N+1\}}$ and its generator is given by

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}^{\text{bulk}} + \widehat{\mathcal{L}}^{L,R}\ , \tag{2.4.5}$$

where, for all bounded functions $f : \widehat{\mathcal{X}} \to \mathbb{R}$,

$$\widehat{\mathcal{L}}^{\text{bulk}} f(\xi) = \sum_{\substack{\{x,y\}\subseteq V_N,\\ |x-y|=1}} \left\{ \begin{array}{c} \xi(x)\,(1-\xi(y))\,(f(\xi^{x,y}) - f(\xi)) \\ +\,\xi(y)\,(1-\xi(x))\,(f(\xi^{y,x}) - f(\xi)) \end{array} \right\},$$

and

$$\begin{aligned} \widehat{\mathcal{L}}^{L,R} f(\xi) &= \widehat{\mathcal{L}}_L f(\xi) + \widehat{\mathcal{L}}_R f(\xi) \\ &= \xi(1)\left(f(\xi^{1,L}) - f(\xi)\right) + \xi(N)\left(f(\xi^{N,R}) - f(\xi)\right)\ , \end{aligned}$$

with $\xi^{x,y} = \xi - \delta_x + \delta_y \in \widehat{\mathcal{X}}$.

The duality function is given by

$$D(\eta, \xi) = \theta_L^{\xi(0)} \left(\prod_{x \in V_N} \mathbf{1}_{\{\eta(x) \geq \xi(x)\}} \right) \theta_R^{\xi(N+1)},$$

where we recall that θ_L and θ_R are the reservoirs densities appearing in the generator in (2.3.2).

vi) *From a deterministic to a stochastic evolution*: in some cases, stochastic duality connects a system evolving in a deterministic way to a stochastic process. The simplest example of this instance of duality is the following. Let $(\eta_t)_{t\geq 0}$ with $\eta_t = (\eta_t(x))_{x\in\mathbb{Z}}$ be the configuration process of a system of simple symmetric independent random walks on $\mathbb{Z}$ with generator given by

$$\mathcal{L} f(\eta) = \sum_{\substack{\{x,y\}\subseteq \mathbb{Z},\\ |x-y|=1}} \left\{ \begin{array}{c} \eta(x)\,(f(\eta^{x,y}) - f(\eta)) \\ +\,\eta(y)\,(f(\eta^{y,x}) - f(\eta)) \end{array} \right\}.$$

Consider the deterministic process $(\zeta_t)_{t\geq 0}$ on $[0,\infty)^{\mathbb{Z}}$ which is the solution of the following system of linear ODE's:

$$\frac{\mathrm{d}\zeta_t(x)}{\mathrm{d}t} = \sum_{y:|y-x|=1} (\zeta_t(y) - \zeta_t(x)).$$

Then, $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ are in duality relation (see, e.g., [35]) with respect to the duality function

$$D(\eta,\zeta) = \prod_{x\in\mathbb{Z}} \zeta(x)^{\eta(x)}.$$

vii) *Stochastic representation of PDEs solutions*: this instance of duality is highly connected to the previous item and refers to the duality relation between deterministic evolution described via PDEs or SPDEs and associated evolutions of stochastic processes. The simplest example in this case is an instance of the so-called Feynman–Kac formula (see, e.g., [63]): i.e., let ρ_t be the solution of the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\rho_t = \frac{1}{2}\Delta\rho_t \\ \rho_0 = \bar{\rho} \end{cases}$$

and let $(B_t)_{t\geq 0}$ be a standard Brownian motion, we then have the following relation

$$\rho_t(x) = \mathbb{E}_x^{\mathrm{BM}}[\bar{\rho}(B_t)] \tag{2.4.6}$$

where $\mathbb{E}_x^{\mathrm{BM}}$ denotes the expectation of the process $(B_t)_{t\geq 0}$ starting from $x\in\mathbb{R}$. Here the processes in duality relation are $(\rho_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ and the duality function is given by

$$D(\rho, B) = \rho(B).$$

2.4.2 General principles to obtain dualities

At this point, we hope the reader is convinced of how stochastic duality can be versatile and useful in the study of Markov processes. If this is not yet the case, the powerfulness of this technique will become evident while further reading the manuscript.

However, less satisfactory will be the answers to questions of the type "how can I find a dual process?"or "which are necessary and sufficient conditions for a Markov process to satisfy stochastic duality?" These are very difficult and general questions, that will not be addressed in this work, at least not in such a generality. Using the words of A. Etheridge: "finding dual processes is something of a black art" (see [61, p. 519]). Big progresses in developing a general theory for stochastic duality have been recently obtained via the so-called algebraic approach to duality (see, e.g., [83]), which relies on a deep connection between stochastic duality and representations of Lie algebras.

Some general principles that will be used in this thesis regard the relation between duality functions and symmetries of the Markov process under consideration, i.e. operators S that commute its generator $\mathcal{L}$

$$S\mathcal{L} = \mathcal{L}S,$$

and intertwinings of two Markov generators $\mathcal{L}_1$, $\mathcal{L}_2$, i.e. operators Λ such that

$$\mathcal{L}_1\Lambda = \Lambda\mathcal{L}_2.$$

Before informally explaining these general principles to obtain dualities relation, we notice how, for many Markov processes satisfying stochastic duality, the duality relation can be expressed in terms of generators instead of expectations as done in Definition 2.4.1. More precisely, let $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ be two Markov processes appearing in Definition 2.4.1 and denote by $\mathcal{L}$ and $\mathcal{L}^{\mathrm{dual}}$ their respective generator. Then, assuming that $D(\xi,\cdot)$ is in the domain of $\mathcal{L}$ for all $\xi\in\hat{\mathcal{X}}$ and $D(\cdot,\eta)$ in the domain of $\mathcal{L}^{\mathrm{dual}}$ for all for all $\eta\in\mathcal{X}$, the duality relation (2.4.1) can be restated as

$$\mathcal{L}D(\xi,\cdot)(\eta) = \mathcal{L}^{\mathrm{dual}}D(\cdot,\eta)(\xi), \qquad \forall \eta\in\mathcal{X},\ \xi\in\hat{\mathcal{X}}. \tag{2.4.7}$$

On the left–hand side the operator $\mathcal{L}$ is acting on the η–variable of the duality function, while on the right hand side $\mathcal{L}^{\mathrm{dual}}$ is acting on the ξ–variable.

Below, we list informally some general principles to obtain duality or self-duality functions.

i) Suppose that (2.4.7) holds and let S be a symmetry of $\mathcal{L}$, then $\hat{D} := SD$ with S acting on the right entry of D is also a duality function (see, e.g., [146]). Indeed, we have

$$\mathcal{L}\hat{D}(\eta, \cdot)(\eta) = S\mathcal{L}D(\eta, \cdot)(\eta) = S\mathcal{L}^{\text{dual}}D(\cdot, \eta)(\xi) = \mathcal{L}^{\text{dual}}\hat{D}(\cdot, \eta)(\xi),$$

where in the first equality we used the definition of $\hat{D}$ and the fact that S is a symmetry of $\mathcal{L}$, in the second equality the duality relation with respect to D and in the last equality the fact that $\mathcal{L}^{\text{dual}}$ and S act on two different entries of D.

ii) Let $(\eta_t)_{t\geq 0}$ be a reversible Markov process with reversible measure μ, then

$$D(\eta, \eta') := \frac{1}{\mu(\eta)}\delta_{\eta,\eta'}$$

is a self-duality function, called *cheap self-duality* (see, e.g., [37]). Thus, if S is a symmetry of the generator of $(\eta_t)_{t\geq 0}$, $\hat{D} := SD$ is a self-duality function as well.

iii) Let $\mathcal{L}_1, \mathcal{L}_2$ be two Markov generators and let Λ be an intertwiner. If Λ can be written as a kernel operator in an $L^2(\mu)$ space where μ is a reversible measure of the process with generator $\mathcal{L}$, i.e.

$$\Lambda f(\eta) = \int f(\eta')D(\eta, \eta')\mathrm{d}\mu(\eta),$$

then the corresponding kernel is a duality function for the processes with generators $\mathcal{L}_1, \mathcal{L}_2$ (see e.g. Lemma 2.1 in Groenevelt [90]).

After this brief introduction to stochastic duality, in the next section we specify on which instances of duality we focus in this thesis and we provide some first computations where duality plays a key role.

2.5 From many to few: self-dual systems and their boundary driven counterparts

In this thesis the focus will be on the simplifications provided by items ii) and v) above. These two instances of stochastic duality allow to provide meaningful information of a system with *many* particles via a system with a *few* particles. The simplification *from many to few* is at the core of all the results and of the investigations in this thesis. Moreover, the focus will be on particle systems that are conservative and consistent or such that they are in duality relation with a *conservative* and *consistent* system. By *conservative* we mean that the total number of particles is conserved by the dynamics. *Consistency* is a more delicate property that, in words, means that the action of removing uniformly at random a particle commutes with the dynamics of the particle system.

In this section we present some classical interacting particle systems satisfying self-duality; additionally, we discuss their duality functions and stochastic duality of their boundary driven counterparts.

2.5.1 Three self-dual systems: classical and orthogonal dualities

For $\sigma \in \{-1, 0, 1\}$ consider the Markov process with state space

$$\eta := \{\eta(x)\}_{x\in\mathbb{Z}} \in \mathcal{X} = \begin{cases} \{0,1\}^{\mathbb{Z}^d}, & \text{if } \sigma = -1, \\ \mathbb{N}_0^{\mathbb{Z}^d}, & \text{if } \sigma = 0, 1, \end{cases}$$

and with generator given by

$$\mathcal{L}f(\eta) = \sum_{\substack{\{x,y\}\subseteq\mathbb{Z}^d, \\ |x-y|=1}} \left\{ \begin{array}{c} \eta(x)(1+\sigma\eta(y))\,(f(\eta^{x,y}) - f(\eta)) \\ +\,\eta(y)(1+\sigma\eta(x))\,(f(\eta^{y,x}) - f(\eta)) \end{array} \right\}. \tag{2.5.1}$$

For $\sigma = -1$ we obtain the previously introduced SSEP, for $\sigma = 0$ the system of simple symmetric independent random walks (SSIRW) and for $\sigma = 1$ the so-called simple symmetric *inclusion process* (SSIP). While for $\sigma = 0$ the particles do not interact, for $\sigma = 1$ the interaction is opposite with respect to the one described before for the SSEP: there is no restriction on the total number of particles per site and particles have a higher chance to jump to sites with more particles. If for $\sigma = -1$ the particles are subject to a repulsive interaction, for $\sigma = 1$ there is an attractive type of interaction.

In order to determine the reversible measures of such systems it is enough to impose the so–called *detailed balance condition*, namely look for measures μ on $\mathcal{X}$ such that

$$\mu(\eta)c(\eta,\eta') = \mu(\eta')c(\eta',\eta)$$

where $c(\eta,\eta')$ denotes the rate of the transition $\eta \to \eta'$. A direct computation shows that, fixed $\sigma \in \{-1,0,1\}$, there exists a one-parameter family of reversible product measures

$$\{\mu_\theta = \bigotimes_{x\in\mathbb{Z}^d} \nu_{x,\theta} : \ \theta \in \Theta\}$$

with $\Theta = [0,1]$ if $\sigma = -1$ and $\Theta = [0,\infty)$ if $\sigma \in \{0,1\}$, and with marginals given by

$$\nu_{x,\theta} = \nu_\theta = \begin{cases} \text{Bernoulli}\,(\theta), & \sigma = -1, \\ \text{Poisson}\,(\theta), & \sigma = 0, \\ \text{Negative–Binomial}\,(1, \frac{\theta}{1+\theta}), & \sigma = 1. \end{cases} \tag{2.5.2}$$

Moreover, the so-called *classical self-duality relation* holds, i.e., for all configurations η, $\xi \in \mathcal{X}$ and for all times $t \geq 0$,

$$\mathbb{E}_\eta[D(\xi,\eta_t)] = \mathbb{E}_\xi[D(\xi_t,\eta)], \tag{2.5.3}$$

with $\{\xi(t) : \ t \geq 0\}$ and $\{\eta(t) : \ t \geq 0\}$ two copies of the process with generator given in (2.5.1) and self-duality function $D\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ given by

$$D(\xi,\eta) := \prod_{x\in\mathbb{Z}^d} d(\xi(x),\eta(x)), \tag{2.5.4}$$

with

$$d(k,n) := \frac{n!}{(n-k)!}\frac{1}{w(k)}\mathbf{1}_{\{k\leq n\}} \tag{2.5.5}$$

and

$$w(k) := \begin{cases} \frac{\Gamma(1+k)}{\Gamma(1)}, & \sigma = 1, \\ 1, & \sigma = -1, 0. \end{cases} \tag{2.5.6}$$

The word *classical* refers to the fact that the duality functions are products of falling factorial polynomials and for $\sigma = -1$ they reduce to the duality functions originally found by Spitzer (see [156]).

There is a second type of self-duality functions for this class of system: the so–called *orthogonal dualities*. These are products of polynomials parametrized by the dual configurations and that satisfy the following orthogonality relation with respect to the reversible measure μ_θ of the systems under consideration: i.e.

$$\int D_\theta^{\mathrm{or}}(\xi,\eta)D_\theta^{\mathrm{or}}(\xi',\eta)\mathrm{d}\mu_\theta = 0 \qquad \text{if } \xi \neq \xi'.$$

More precisely,

$$D_\theta^{or}(\xi,\eta) = \prod_{x\in\mathbb{Z}^d} d_\theta^{or}(\xi(x),\eta(x))$$

where, for all $k,n \in \mathbb{N}_0$,

$$d_\theta^{or}(k,n) \;=\; (-\theta)^k \times \begin{cases} {}_2F_1\left[\begin{matrix} -k & -n \\ & -1 \end{matrix}; \frac{1}{\theta}\right] & \sigma = -1 \\ {}_2F_0\left[\begin{matrix} -k & -n \\ & - \end{matrix}; -\frac{1}{\theta}\right] & \sigma = 0 \\ {}_2F_1\left[\begin{matrix} -k & -n \\ & 1 \end{matrix}; -\frac{1}{\theta}\right] & \sigma = 1\,. \end{cases} \tag{2.5.7}$$

In other words, these jointly factorized orthogonal dualities consist of products of hypergeometric functions of the following two types: either

$$ {}_2F_0\left[\begin{matrix}-k & -n\\ & -\end{matrix}; -u\right] := \sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{n!}{(n-\ell)!}\mathbf{1}_{\{\ell\le n\}}\right)u^{\ell} \tag{2.5.8}$$

or

$$ {}_2F_1\left[\begin{matrix}-k & -n\\ & v\end{matrix}; u\right] := \sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{\Gamma(v)}{\Gamma(v+\ell)}\frac{n!}{(n-\ell)!}\mathbf{1}_{\{\ell\le n\}}\right)u^{\ell}\,, \tag{2.5.9}$$

with $k, n \in \mathbb{N}_0$ and $u, v \in \mathbb{R}$. The orthogonality relation for the single–site duality function then reads as follows for all $k, \ell \in \mathbb{N}_0$,

$$\sum_{n=0}^{\infty} d_\theta^{or}(k,n)\, d_\theta^{or}(\ell,n)\, \nu_\theta(n) = \mathbf{1}_{\{k=\ell\}}\|d_\theta(k,\cdot)\|^2_{L^2(\nu_\theta)}\,.$$

More specifically, these orthogonal single-site self-duality functions are Kravchuk polynomials for SEP ($\sigma = -1$), Charlier polynomials for IRW ($\sigma = 0$) and Meixner polynomials for SIP ($\sigma = 1$) (see e.g. [78]). Because in this setting there exists a one-parameter family of stationary product measures for each of the three particle systems, this corresponds to the existence of a one-parameter family of orthogonal duality functions.

2.5.2 The boundary driven counterparts

When turning to the boundary driven counterparts of the SSEP, SIRW and SSIP, i.e. the IPS on the finite chain $V_N := \{1, \dots, N\}$ coupled with a left and a right reservoir, the state space is

$$\eta := \{\eta(x)\}_{x\in\mathbb{Z}} \in \mathcal{X} = \begin{cases}\{0,1\}^{V_N}, & \text{if } \sigma = -1,\\ \mathbb{N}_0^{V_N}, & \text{if } \sigma = 0, 1,\end{cases}$$

and the generator is

$$\mathcal{L}^{\text{open}} f(\eta) = \mathcal{L}^{\text{bulk}} f(\eta) + \mathcal{L}^{L,R} f(\eta)\,. \tag{2.5.10}$$

The generator $\mathcal{L}^{\text{bulk}}$ describes the bulk part of the dynamics and is given by

$$\mathcal{L}^{\text{bulk}} f(\eta) = \sum_{\substack{\{x,y\}\subseteq V_N,\\ |x-y|=1}} \left\{\begin{matrix} \eta(x)(1+\sigma\eta(y))\,(f(\eta^{x,y}) - f(\eta))\\ +\,\eta(y)(1+\sigma\eta(x))\,(f(\eta^{y,x}) - f(\eta))\end{matrix}\right\}\,.$$

and the boundary part of the dynamics is described by the generator $\mathcal{L}^{L,R}$ as follows:

$$\mathcal{L}^{L,R} f(\eta) = \mathcal{L}_L f(\eta) + \mathcal{L}_R f(\eta)\,, \tag{2.5.11}$$

with

$$\begin{aligned}\mathcal{L}_L f(\eta) &= \eta(1)\,(1+\sigma\theta_L)\,(f(\eta^{1,-}) - f(\eta))\\ &\quad + \theta_L\,(1+\sigma\eta(1))\,(f(\eta^{1,+}) - f(\eta))\end{aligned} \tag{2.5.12}$$

and

$$\begin{aligned}\mathcal{L}_R f(\eta) &= \eta(N)\,(1+\sigma\theta_R)\left(f(\eta^{N,-}) - f(\eta)\right)\\ &\quad + \theta_R(1+\sigma\eta(N))\left(f(\eta^{N,+}) - f(\eta)\right)\,,\end{aligned} \tag{2.5.13}$$

where $\eta^{x,-} \in \mathcal{X}$, resp. $\eta^{x,+} \in \mathcal{X}$, denotes the configuration obtained from η by removing, resp. adding, a particle from, resp. to, site $x \in V$. In the above dynamics, creation and annihilation of particles occurs at sites $x = 1$ and $x = N$ due to the interaction with a reservoir.

For each choice of $\sigma \in \{-1,0,1\}$, it has been proved in [32] that a particle system with purely absorbing reservoirs is dual to the corresponding system in contact with reservoirs. In the dual systems, particles hop on the extended chain $V_N \cup \{L,R\}$ following the same bulk dynamics as the particle systems with generators in (2.5.10) but having $\{L,R\}$ as absorbing sites.

More in detail, $\{\xi_t : t \geq 0\}$ denotes such particle systems having

$$\widehat{\mathcal{X}} = \mathcal{X} \times \mathbb{N}_0^{\{L,R\}} \tag{2.5.14}$$

as configuration space and infinitesimal generator $\widehat{\mathcal{L}}$ given by

$$\widehat{\mathcal{L}} f(\xi) = \widehat{\mathcal{L}}^{\text{bulk}} f(\xi) + \widehat{\mathcal{L}}^{L,R} f(\xi)\ , \tag{2.5.15}$$

where, for all bounded functions $f : \widehat{\mathcal{X}} \to \mathbb{R}$,

$$\widehat{\mathcal{L}}^{\text{bulk}} f(\xi) = \sum_{\substack{\{x,y\}\subseteq V_N,\\ |x-y|=1}} \left\{ \begin{array}{c} \xi(x)\,(1+\sigma\xi(y))\,(f(\xi^{x,y}) - f(\xi)) \\ +\,\xi(y)\,(1+\sigma\xi(x))\,(f(\xi^{y,x}) - f(\xi)) \end{array} \right\} ,$$

and

$$\begin{aligned} \widehat{\mathcal{L}}^{L,R} f(\xi) &= \widehat{\mathcal{L}}_L f(\xi) + \widehat{\mathcal{L}}_R f(\xi) \\ &= \xi(1)\left(f(\xi^{1,L}) - f(\xi)\right) + \ \xi(N)\left(f(\xi^{N,R}) - f(\xi)\right)\ , \end{aligned}$$

with, for all $x, y \in V \cup \{L,R\}$, $\xi^{x,y} = \xi - \delta_x + \delta_y \in \widehat{\mathcal{X}}$.

The stochastic duality relation between the processes with generators given in (2.5.10) and (2.5.15) hold with the duality function $D^{c\ell} : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ defined as follows: for all configurations $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$,

$$D^{c\ell}(\xi,\eta) = d_L^{c\ell}(\xi(L)) \times \left(\prod_{x\in V} d^{c\ell}(\xi(x),\eta(x)) \right) \times d_R^{c\ell}(\xi(R))\ ,$$

where, for all $k,n \in \mathbb{N}_0$, $d^{c\ell}(k,n)$ is given in (2.5.5) and

$$d_L^{c\ell}(k) = (\theta_L)^k \qquad \text{and} \qquad d_R^{c\ell}(k) = (\theta_R)^k\ . \tag{2.5.16}$$

After having recalled these known duality results that constitute the starting point for the research contained in this thesis, we provide in the next two subsections some applications of these results. More precisely, we first use self-duality for the closed systems with generator given in (2.5.1) to study the expectation of the empirical density field. Second, we use the duality result for the boundary driven system with generator given in (2.5.10) to compute the stationary profile.

2.5.3 Self-duality for closed systems: hydrodynamics

In this section we use stochastic duality to study macroscopic fields via the scaling limit of dual particles. More precisely we focus on the rescaled empirical density field X_t^N defined in (2.2.1) and associated to the particle systems with generator given in (2.5.1). Stochastic duality with one single dual particle, a random walk, implies that the hydrodynamic limit can be determined by the scaling limit of such random walk.

For any $\sigma \in \{-1,0,1\}$, as already mentioned before for the SSEP, if we consider an initial configuration with one particle only, i.e. $\eta \in \mathcal{X}$ such that $\sum_{x\in\mathbb{Z}^d} \eta(x) = 1$ we are left with a random walk with generator $\mathcal{L}^{\text{RW}}$ given in (2.4.3). It is well known the so-called *invariance principle*:

> Let $(X_t)_{t\geq 0}$ be the random walk with generator given in (2.4.3) and starting from the origin. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Then
>
> $$\left(\frac{1}{N} X_{N^2 t}\right)_{t\geq 0} \to (B_t)_{t\geq 0} \qquad \text{as n} \to \infty$$
>
> where, here, $\to$ denotes the weak-convergence in path space.

Notice that the invariance principle implies that for any bounded and continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \mathbb{E}^{RW}\left[f\left(\frac{1}{N}X_{N^2t}\right)\right] = \mathbb{E}^{\mathrm{BM}}[f(B_t)]\ , \quad \forall t \geq 0.$$

Moreover, recall that for a continuous and bounded function $\bar{\rho} : \mathbb{R}^d \to \mathbb{R}_+$ we have the stochastic representation of the solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\rho_t = \frac{1}{2}\Delta\rho_t \\ \rho_0 = \bar{\rho} \end{cases} \tag{2.5.17}$$

given in (2.4.6).

Before stating the result, we need to assume that the initial distribution of the particle system converges to the macroscopic profile $\bar{\rho}$.

Definition 2.5.1 (Consistency of the initial conditions)**.** *We say that a sequence of probabilities $\{\nu_N\}_{N\in\mathbb{N}}$ on $\mathcal{X}$ is* consistent to a continuous macroscopic profile $\bar{\rho} : \mathbb{R}^d \to \mathbb{R}$ *if the following convergence*

$$\nu_N\left(\left\{\eta \in \mathcal{X} : \left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\eta(x) - \int_{\mathbb{R}^d} G(u)\,\bar{\rho}(u)\,\mathrm{d}u\right| > \delta\right\}\right) \underset{N\to\infty}{\longrightarrow} 0 \tag{2.5.18}$$

holds for all $G \in \mathscr{S}(\mathbb{R}^d)$ and $\delta > 0$.

$\mathscr{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions, i.e., C^∞ functions of which all derivatives converge to zero at infinity faster than any polynomial. $\mathscr{S}(\mathbb{R}^d)$, endowed with a suitable topology, is chosen to be the space of test function for the sequence empirical density fields $(\mathsf{X}^N)_N$ which indeed are viewed as elements in $\mathscr{S}'(\mathbb{R}^d)$, the set of Schwartz distributions. For a probability measure ν on $\mathcal{X}$, we denote by $\mathbb{E}_\nu$ the expectation for the process $(\eta_t)_{t\geq 0}$ with generator given in (2.5.1), initially distributed according to ν.

Proposition 2.5.2. *Let $\bar{\rho} : \mathbb{R}^d \to \mathbb{R}_+$ a bounded and continuous function, such that* $\sup_{x\in\mathbb{R}^d}\rho(x) \leq 1$ *for $\sigma = -1$. Let $(\nu_N)_{N\in\mathbb{N}}$ be a sequence of probability measures on $\mathcal{X}$ such that $\int_X \eta(x)\mathrm{d}\nu_N(\eta) = \bar{\rho}(\frac{x}{N})$ for any $x \in \mathbb{Z}^d$ and $N \in \mathbb{N}$. Let $(\mathsf{X}^N)_{N\in\mathbb{N}}$ be the sequence of rescaled density fields associated to the IPS with generator given in* (2.5.1) *and $\sigma \in \{-1, 0, 1\}$. Then*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_N}[\mathsf{X}_t^N(G)] = \int_{\mathbb{R}^d} \rho_t(x)G(x)\mathrm{d}x$$

for any $G \in \mathscr{S}(\mathbb{R}^d)$, $t \geq 0$, where $(\rho_t)_{t\geq 0}$ is the unique continuous and bounded solution of (2.5.17).

Proof. For all $\sigma \in \{-1, 0, 1\}$, the self-duality relation in (2.5.3) when the dual initial state ξ has one-particle only, i.e. $\sum_{x\in\mathbb{Z}^d}\xi(x) = 1$, rewrites as in (2.4.4), i.e.

$$\mathbb{E}_\eta[\eta_t(x)] = \mathbb{E}_x^{\mathrm{RW}}[\eta(X_t)].$$

Notice that sequence ν_N is a consistent initial condition, and using stochastic duality we obtain

$$\mathbb{E}_{\nu_N}[\eta_{N^2t}(x)] = \int_X \mathbb{E}_\eta[\eta_{tN^2}(x)]\mathrm{d}\nu_N(\eta) = \int_X \mathbb{E}_x^{\mathrm{RW}}[\eta(X_{N^2t})]\mathrm{d}\nu_N(\eta) = \mathbb{E}_x^{\mathrm{RW}}\left[\bar{\rho}\left(\frac{1}{N}X_{N^2t}\right)\right]$$

and from the translation invariance of the random walk we have

$$\mathbb{E}_{\nu_N}[\eta_{N^2t}(x)] = \mathbb{E}_0^{\mathrm{RW}}\left[\bar{\rho}\left(\frac{1}{N}X_{N^2t} + \frac{x}{N}\right)\right].$$

The invariance principle then implies

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}_{\nu_N}[\mathsf{X}_t^N(G)] &= \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G\left(\frac{x}{N}\right)\mathbb{E}_0^{\mathrm{RW}}\left[\bar{\rho}\left(\frac{1}{N}X_{N^2t} + \frac{x}{N}\right)\right] \\
&= \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G\left(\frac{x}{N}\right)\mathbb{E}_0^{\mathrm{BM}}\left[\bar{\rho}\left(B_t + \frac{x}{N}\right)\right] \\
&= \lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G\left(\frac{x}{N}\right)\rho_t\left(\frac{x}{N}\right) = \int_{\mathbb{R}^d} G(x)\rho_t(x)\mathrm{d}x
\end{aligned}$$

concluding the proof, where we used (2.4.6) in the third equality. □

Notice that in the proof we used the translation invariance of the law of the walk. When the transition rates of the random walk are inhomogeneous, and more precisely space dependent, the translation invariance is lost and the invariance principle from the origin will not be enough to prove the convergence of the expectation of the empirical density field. This issue is discussed in detail in the next chapter of the thesis.

2.5.4 Duality for boundary driven systems: non-equilibrium steady state and stationary profile

In this section we provide a simple application of duality to the systems with generator given in (2.5.10).

For these systems it can be proved that, for all $\theta_L, \theta_R > 0$, i.e. also out–of–equilibrium, there exists a unique stationary state μ_{stat}. However, unless $\sigma = 0$, or the equilibrium case $\theta_L = \theta_R$ for $\sigma \in \{-1, 1\}$, μ_{stat} does not take an easy and product form and, for $\sigma = 1$, μ_{stat} is not explicitly known in the literature as in the case $\sigma = 1$ for which a matrix formulation for μ_{stat} is available (see, e.g., [55]). However, stochastic duality with one dual absorbed particle provides in a simple way the stationary profile, namely the quantities

$$\theta_N(x) := \mathbb{E}_{\mu_{\text{stat}}}[\eta(x)] \tag{2.5.19}$$

for all $x \in \{1, \dots, N\}$. Here, the sub-index N in $\theta_N(x)$ denotes the size of the chain.

Recall indeed the generator of the dual process given in (2.5.15). When the initial dual configuration ξ contains only one particle, i.e. $\sum_{x\in\{0,\dots,N+1\}} \xi(x) = 1$ the process $(\xi_t)_{t\geq 0}$ reduces to a purely absorbed random walk $(X_t^{\text{abs}})_{t\geq 0}$ evolving as a simple symmetric random walk on V_N and absorbed at $\{0, N+1\}$. The duality relation between the particle systems described by the generators in (2.5.10) for $\sigma \in \{-1, 0, 1\}$ and $(X_t^{\text{abs}})_{t\geq 0}$ then reads

$$\mathbb{E}_\eta[\eta_t(x)] = \mathbb{E}_x^{\text{RWabs}}\left[\theta_L^{\mathbf{1}_{\{0\}}(X_t^{\text{abs}})}\theta_R^{\mathbf{1}_{\{N+1\}}(X_t^{\text{abs}})}\mathbf{1}_{\{0,N+1\}}(X_t^{\text{abs}}) + \eta(X_t^{\text{abs}})\mathbf{1}_{V_N}(X_t^{\text{abs}})\right], \tag{2.5.20}$$

where $\mathbb{E}_x^{\text{RWabs}}[\cdot]$ denotes the expectation of the process $(X_t^{\text{abs}})_{t\geq 0}$ starting from x.

Proposition 2.5.3. *The stationary profile in the non–equilibrium steady state (see* (2.5.19)*) is given by*

$$\theta_N(x) = \theta_R + \left(1 - \frac{x}{N+1}\right)(\theta_L - \theta_R).$$

Proof. Recalling the definition of $\theta_N(x)$, by the duality relation (2.5.20) and the uniqueness of the steady state μ_{stat}, we have

$$\begin{aligned}
\theta_N(x) = \mathbb{E}_{\mu_{\text{stat}}}[\eta(x)] &= \int \lim_{t\to\infty} \mathbb{E}_\eta[\eta_t(x)] \mathrm{d}\mu_{\text{stat}}(\eta) \\
&= \int \lim_{t\to\infty} \mathbb{E}_x^{\text{RWabs}}\left[\theta_L^{\mathbf{1}_{\{0\}}(X_t^{\text{abs}})}\theta_R^{\mathbf{1}_{\{N+1\}}(X_t^{\text{abs}})}\mathbf{1}_{\{0,N+1\}}(X_t^{\text{abs}}) + \eta(X_t^{\text{abs}})\mathbf{1}_{V_N}(X_t^{\text{abs}})\right] \mathrm{d}\mu_{\text{stat}}(\eta) \\
&= \int \mathbb{E}_x^{\text{RWabs}}\left[\theta_L^{\mathbf{1}_{\{0\}}(X_\infty^{\text{abs}})}\theta_R^{\mathbf{1}_{\{N+1\}}(X_\infty^{\text{abs}})}\right] \mathrm{d}\mu_{\text{stat}}(\eta)
\end{aligned}$$

and since the expression inside the integral above does not depend on the integration variable η, we obtain

$$\begin{aligned}
\theta_N(x) &= \mathbb{E}_x^{\text{RWabs}}\left[\theta_L^{\mathbf{1}_{\{0\}}(X_\infty^{\text{abs}})}\theta_R^{\mathbf{1}_{\{N+1\}}(X_\infty^{\text{abs}})}\right] \\
&= \theta_L \hat{p}_\infty(x, 0) + \theta_R(1 - \hat{p}_\infty(x, 0))
\end{aligned}$$

where $\hat{p}_\infty(x, 0)$ is the probability that the random walk $(X_t^{\text{abs}})_{t\geq 0}$ starting from x is absorbed at 0. The absorption probabilities $(\hat{p}_\infty(x, 0))_{x\in V_N}$ satisfy

$$\mathcal{L}^{\text{RWabs}} \hat{p}_\infty(\cdot, 0)(x) = 0,$$

where $\mathcal{L}^{\text{RWabs}}$ denotes the generator of $(X_t^{\text{abs}})_{t\geq 0}$, i.e. $x \to \hat{p}_\infty(x, 0)$ is an harmonic function for $\mathcal{L}^{\text{RWabs}}$. This provides the following linear system of $N + 2$ equations

$$\begin{cases}
\hat{p}_\infty(0, 0) = 1 \\
\hat{p}_\infty(x + 1, 0) - 2\hat{p}_\infty(x, 0) - \hat{p}_\infty(x - 1, 0) = 0, \qquad x \in V_N \\
\hat{p}_\infty(1, 0) = 0
\end{cases} \tag{2.5.21}$$

which is solved by

$$\hat{p}_\infty(x, 0) = \left(1 - \frac{x}{N+1}\right)$$

concluding the proof. □

As a direct consequence of the computation we just performed, we can investigate the behavior of the stationary current in the boundary driven particle systems.

For $x \in V_N$, let us denote by $c(\eta, \eta^{x,x+1})$ the rate of the transition from the configuration η to the configuration $\eta^{x,x+1}$ where a particle, if any, has jumped from x to $x+1$. The instantaneous current on the edge $\{x, x+1\}$ at time t is then defined as

$$J_{x,x+1}(t) := c(\eta_t, \eta_t^{x,x+1}) - c(\eta_t, \eta_t^{x+1,x}).$$

For the particle systems under consideration, we have

$$\begin{aligned} J^N_{x,x+1}(t) &= \eta_t(x)(1+\sigma\eta_t(x+1)) - \eta_t(x+1)(1+\sigma\eta_t(x)) \\ &= \eta_t(x) - \eta_t(x+1), \end{aligned}$$

i.e., the instantaneous current on the edge $\{x, x+1\}$ is equal to the difference of the occupation variables at these sites, exhibiting the so-called *gradient* behavior. If then we define the stationary current as $J^{N,\text{stat}}_{x,x+1} := \mathbb{E}_{\mu_{\text{stat}}}[J^N_{x,x+1}(t)]$, we obtain, in view of Proposition 2.5.3,

$$\begin{aligned} J^{N,\text{stat}}_{x,x+1} &= \theta_N(x) - \theta_N(x+1) \\ &= -\frac{1}{N}(\theta_R - \theta_L), \end{aligned}$$

i.e., the stationary current is constant in space and proportional to the difference of the reservoirs' densities. The Fick's law is thus satisfied at the microscopic scale. Moreover, this is the case also at the macroscopic scale. Indeed if we define the macroscopic stationary current as

$$J(u) := \lim_{N\to\infty} N J_N(\lfloor uN \rfloor, \lfloor uN \rfloor + 1), \quad u \in (0,1)$$

and the macroscopic stationary profile as

$$\rho(u) := \lim_{N\to\infty} \rho_N(\lfloor uN \rfloor), \quad u \in (0,1)$$

we obtain $J(u) = \theta_L - \theta_R$ and $\rho(u) = \theta_R + (1-u)\theta_L$, leading to the standard Fick's relation

$$J(u) = -\nabla\rho(u).$$

Chapter 3

Research problems and outline of the thesis

In this thesis, two general problems will be addressed:

i) Extending the self-duality and duality results for spatial inhomogeneous versions of the processes described by the generator in (2.5.1) and consequently applying these duality relations to rigorously prove physically meaningful properties of inhomogeneous IPS.

ii) Generalize the notion of self-duality to systems of particles evolving in the continuum and more specifically on $\mathbb{R}^d$.

In Section 3.1 we introduce the topic of inhomogeneous IPS, while in Section 3.2 we introduce a new formulation of self-duality that makes sense in the continuum.

3.1 Inhomogeneous closed and open IPS

In this thesis we consider two cases of space-inhomogeneous evolutions: processes in random environment and processes in a multi-layer system.

3.1.1 Random environments and stochastic homogenizations

In many cases, the medium where the particles evolve is microscopically irregular due to the presence of impurities and defects. A random environment is an external source of randomness added to the evolution of particles, aimed at capturing the presence of impurities and defects in the material where the evolution takes place. A motivation to introduce this extra stochasticity could be the following. In many situations the presence of impurities is unavoidable and an essential feature of nature. However, in experiments, it is rarely the case that such impurities can be measured and mapped in the system. Such measurements would be typically invasive and most probably would affect the set-up. Moreover, whenever an experiment has to be repeated in several samples, mapping every time the impurities would be costly and unfeasible. Each sample would have its own impurities located in a particular way. It is then natural to model such impurities at random (as a random environment), thus treating the observed (stochastic) dynamics as a statistical realization of an ensemble where the local properties of the dynamics are sampled according to an external source of randomness modeled by a probability measure $\mathcal{P}$. One then investigates under which conditions on $\mathcal{P}$ the presence of random impurities homogenize when passing from the micro to the macro picture of the evolution. In other words, even if the microscopic picture may reveal many irregularities, in many cases those are invisible at the macroscopic scale: space inhomogeneous microscopic motion of particles may rescale to a homogeneous macroscopic diffusion whose diffusion characteristic, the diffusion matrix, depends only on the law $\mathcal{P}$ and not on the specific realization of the environment.

The exclusion process with random conductances One of the most studied random environments is obtained by assigning random conductances $\boldsymbol{\omega} = \{\omega_{\{x,y\}}, |x-y| = 1\}$ to the edges of $\mathbb{Z}^d$. We then say that the jump rate of a particle from x in $\mathbb{Z}^d$ to a nearest neighboring site y is given by $\omega_{\{x,y\}}$. In the context of the most studied IPS, the symmetric exclusion process, we thus obtain the following generator

$$\mathcal{L}^\omega f(\eta) = \sum_{\substack{\{x,y\}\subseteq\mathbb{Z}^d,\\ |x-y|=1}} \omega_{\{x,y\}} \left\{ \begin{array}{c} \eta(x)(1-\eta(y))\,(f(\eta^{x,y})-f(\eta)) \\ +\,\eta(y)(1-\eta(x))\,(f(\eta^{y,x})-f(\eta)) \end{array} \right\}, \tag{3.1.1}$$

where we remark that the weights are random $\omega_{\{x,y\}}$ and sampled according to some probability $\mathcal{P}$. We denote the exclusion process with random conductance with generator given in (3.1.1) by SEP(ω). In order to have homogenization when studying the hydrodynamic limit of the SEP(ω) in random environment, some sort of averaging has to occur for the random conductances: namely, the ergodic theorem has to play a role when rescaling the system. The conditions on $\mathcal{P}$ are then:

i) $\mathcal{P}$ is invariant and ergodic under translation in $\mathbb{Z}^d$.

Moreover, in order to guarantee non-degeneracy of the limiting diffusion, one assumes also uniform ellipticity for the random conductances: i.e.

ii) there exists $C, c > 0$ such that $\mathcal{P}(c \le \omega_{\{x,y\}} \le C) = 1$ for any $|x-y| = 1$.

The quenched invariance principle for the random conductance model In order to understand how the above assumptions ensure stochastic homogenization, let us consider a single continuous-time random walk in the same random environment. For simplicity let us stick to $d = 1$, where all the quantities of interest can be computed explicitly. This paragraph serves as an illustration of the ideas behind the stochastic homogenization result we are going to study and is written for the sake of explanation. However, when $d > 1$, the situation is more complicated and analogous explicit expressions are not available.

Let use denote by $X^\omega = \{X_t^\omega,\ t \ge 0\}$, abbreviated by RW($\omega$), the Markov process starting with law P_z^ω, $z \in \mathbb{Z}$ denoting the starting point, on $D([0,\infty),\mathbb{R})$ (and corresponding expectation E_z^ω) and evolving on $\mathbb{Z}$ according to the generator given by

$$A^\omega f(x) := \sum_{\substack{y\in\mathbb{Z}\\ |y-x|=1}} \omega_{xy}\,(f(y)-f(x))\ , \qquad x \in \mathbb{Z}\,, \tag{3.1.2}$$

where $f : \mathbb{Z} \to \mathbb{R}$ is a bounded function. Such a model is called *random conductance model* and is very well studied in the literature (see, e.g., [23]). Under the above conditions we have the following result (see, e.g., [23]).

Proposition 3.1.1 (Quenched invariance principle)**.** *For $\mathcal{P}$–a.e. ω, under P_0^ω, $\{\frac{1}{N}X_{tN^2}^\omega,\ t \ge 0\}$ convergences in law to a Brownian motion with diffusion constant given by*

$$2D(\mathcal{P}) = 2\mathbb{E}_{\mathcal{P}}\left[1/\omega_{0,1}\right]^{-1}.$$

Proof. Let us define

$$\psi(\omega,x) := \begin{cases} D(\mathcal{P})\sum_{i=0}^{x-1}\frac{1}{\omega_{\{i,i+1\}}}, & x>0\\ 0, & x=0\\ D(\mathcal{P})\sum_{i=x}^{-1}\frac{1}{\omega_{\{i,i+1\}}}, & x<0. \end{cases}$$

It is easy to check that $x \to \psi(\omega,x)$ is a harmonic function for A^ω. We thus have the following decomposition

$$X_t^\omega = \psi(\omega, X_t^\omega) + \mathfrak{X}(\omega, X_t^\omega),$$

where $\psi(\omega, X_t^\omega)$ is a martingale w.r.t. the natural filtration of X^ω. By the martingale central limit theorem, (see, e.g. [92]), and ergodicity of $\mathcal{P}$, it follows that $\mathcal{P}$-a.s., under P_0^ω, $\frac{1}{N}\psi(\omega, X_{tN^2}^\omega)$ converges in law to a Brownian motion with limiting diffusion constant given by

$$\mathcal{D} = \lim_{t\to\infty}\frac{1}{t}\int E_0^\omega[\psi^2(\omega,X_t^\omega)]\mathcal{P}(\mathrm{d}\omega) = \lim_{n\to\infty}\frac{1}{n\delta}\int E_0^\omega[\psi^2(\omega,X_{n\delta}^\omega)]\mathcal{P}(\mathrm{d}\omega).$$

The above limit can be computed in the following way. First notice that

$$\int E_0^\omega[\psi^2(\omega,X_{n\delta}^\omega)]\mathcal{P}(\mathrm{d}\omega) = \int E_0^\omega\left[\left(\sum_{k=0}^{n-1}\psi(\omega,X_{(k+1)\delta}^\omega)-\psi(\omega,X_{(k)\delta}^\omega)\right)^2\right]\mathcal{P}(\mathrm{d}\omega)$$

$$= n \int E_0^\omega[\psi^2(\omega, X_\delta^\omega)]\mathcal{P}(\mathrm{d}\omega)$$

where the last step follows by stationarity and orthogonality of the increments. Thus, for any $\delta > 0$,

$$\mathcal{D} = \frac{1}{\delta} \int E_0^\omega[\psi^2(\omega, X_\delta^\omega)]\mathcal{P}(\mathrm{d}\omega)$$

and in particular

$$\begin{aligned}\mathcal{D} &= \lim_{\delta\to 0} \frac{1}{\delta} \int E_0^\omega[\psi^2(\omega, X_\delta^\omega)]\mathcal{P}(\mathrm{d}\omega) = \int A^\omega \psi^2(\omega, \cdot)(x)\mathcal{P}(\mathrm{d}\omega) \\ &= \int \Big(\omega_{\{0,1\}}\psi^2(\omega, 1) + \omega_{\{0,-1\}}\psi^2(\omega, -1)\Big)\mathcal{P}(\mathrm{d}\omega) = 2\mathbb{E}_{\mathcal{P}}\left[1/\omega_{0,1}\right]^{-1} = 2D(\mathcal{P}).\end{aligned}$$

To conclude the proof, it remains to show that for $\mathcal{P}$-a.e. ω, under P_0^ω,

$$\lim_{N\to\infty} \frac{1}{N}\mathfrak{X}(\omega, X_{tN^2}^\omega) = 0$$

which, in turns, follows from

$$\mathfrak{X}(\omega, x) = o(|x|), \tag{3.1.3}$$

i.e. the sub-linearity of the corrector $\mathfrak{X}$. Indeed, by the central limit theorem for martingales we have that $\psi(\omega, X_{tN^2}^\omega) = O(N)$ and if (3.1.3) holds, then $X_{tN^2}^\omega = O(N)$ which in turn implies that $\mathfrak{X}(\omega, X_{tN^2}^\omega) = o(N)$. But (3.1.3) follows directly from the ergodic theorem, indeed, for $\mathcal{P}$-a.e. ω

$$\lim_{x\to+\infty} \frac{x - \psi(\omega, x)}{x} = \lim_{x\to+\infty} D(\mathcal{P})\left(\frac{1}{D(\mathcal{P})} - \frac{1}{x}\sum_{i=0}^{x-1} \frac{1}{\omega_{\{x,x+1\}}}\right) = 0$$

and similarly for $x \to -\infty$. □

The quenched hydrodynamic limit Let us now turn to the stochastic homogenization problem for the SEP(ω). The key observation is that self-duality still holds for this model with the same duality functions given in (5.3.3).

Proposition 3.1.2 (Self-duality with one dual particle)**.** RW(ω) *and SEP(ω) are in duality relation w.r.t. the function*

$$D(\eta, x) = \eta(x).$$

The computation is simple and instructive.

Proof. It is enough to check that for any $\eta \in \{0, 1\}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$

$$\mathcal{L}^\omega D(\cdot, x)(\eta) = A^\omega D(\eta, \cdot)(x).$$

We have

$$\begin{aligned}\mathcal{L}^\omega D(\cdot, x)(\eta) &= \sum_{\substack{\{z,y\}\subseteq\mathbb{Z}^d,\\ |z-y|=1}} \omega_{\{z,y\}} \left\{ \begin{array}{c} \eta(z)(1 - \eta(y))\,(D(\eta^{z,y}, x) - D(\eta, x)) \\ +\,\eta(y)(1 - \eta(z))\,(D(\eta^{y,z}, x) - D(\eta, x)) \end{array} \right\}, \\ &= \sum_{\substack{y\in\mathbb{Z}^d,\\ |x-y|=1}} \omega_{\{x,y\}} \left\{ \begin{array}{c} \eta(x)(1 - \eta(y))\,(\eta(x) - 1 - \eta(x)) \\ +\,\eta(y)(1 - \eta(x))\,(\eta(x) + 1 - \eta(x)) \end{array} \right\}, \\ &= \sum_{\substack{y\in\mathbb{Z}^d,\\ |x-y|=1}} \omega_{\{x,y\}}(\eta(y) - \eta(x)) = \sum_{\substack{y\in\mathbb{Z}^d,\\ |x-y|=1}} \omega_{\{x,y\}}(D(\eta, y) - D(\eta, x)) = A^\omega D(\eta, \cdot)(x)\end{aligned}$$

where in the first line we used the definition of $\mathcal{L}^\omega$, in the second and in the fourth equality the definition of the duality function and in the last equality the definition of A^ω. Notice that all the terms in the sum with $z \neq x$ give a contribute equal to zero being $D(\eta^{z,y}, x) = D(\eta, x)$. The proof is concluded. □

The following homogenization result for the one-dimensional SEP (ω) was proved in [139].

Proposition 3.1.3 (See [139], Theorem 3)**.** *Let $\bar{\rho} : \mathbb{R} \to \mathbb{R}_+$ a bounded and continuous function, such that $\sup_{x\in\mathbb{R}} \rho(x) \leq 1$. Let $(\nu_N)_{N\in\mathbb{N}}$ be a sequence of probability measures on $\mathcal{X}$ such that $\int_X \eta(x) \mathrm{d}\nu_N(\eta) = \bar{\rho}(\frac{x}{N})$ for any $x \in \mathbb{Z}$ and $N \in \mathbb{N}$. Let $(\mathsf{X}^N)_{N\in\mathbb{N}}$ be the sequence of rescaled density fields associated to the SEP(ω). Then, for $\mathcal{P}$–a.e. environment ω*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_N}[\mathsf{X}_t^N(G)] = \int_{\mathbb{R}} \rho_t(x) G(x) \mathrm{d}x$$

for any $G \in C_c^\infty(\mathbb{R})$, $t \geq 0$, where $(\rho_t)_{t\geq 0}$ is the unique continuous and bounded solution of

$$\begin{cases} \frac{\partial}{\partial t}\rho_t = 2D(\mathcal{P})\Delta\rho_t \\ \rho_0 = \bar{\rho} \end{cases} \tag{3.1.4}$$

with $D(\mathcal{P}) = \frac{1}{\mathbb{E}_{\mathcal{P}}[1/\omega_{\{0,1\}}]}$.

The idea of the proof is to exploit the duality relation given in Proposition (3.1.2) to transfer the homogenization problem of the SEP(ω) to an homogenization problem for RW(ω). By following the same computations as in the proof of Proposition (2.5.2), we obtain that

$$\begin{aligned} \mathbb{E}_{\nu_N}[\mathsf{X}_t^N(G)] &= \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G\left(\frac{x}{N}\right) E_x^\omega \left[\bar{\rho}\left(\frac{1}{N} X_{tN^2}^\omega\right)\right] \\ &= \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} E_x^\omega \left[G\left(\frac{1}{N} X_{tN^2}^\omega\right)\right] \bar{\rho}\left(\frac{x}{N}\right) \end{aligned}$$

where in the last step we used reversibility of RW(ω) w.r.t. the counting measure. Denoting by $\mathbb{E}_x^{\mathrm{BM(D(\mathcal{P}))}}$ the expectation w.r.t. the Brownian motion starting from x and with diffusion constant $2D(\mathcal{P})$, one is thus left with proving that for $\mathcal{P}$-a.e. ω the following L^1-convergence

$$\frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left| E_x^\omega \left[G\left(\frac{X_{tN^2}^\omega}{N}\right)\right] - \mathbb{E}_{\frac{x}{N}}^{\mathrm{BM(D(\mathcal{P}))}}[G(B_t)] \right| \underset{N\to\infty}{\longrightarrow} 0\,, \qquad t \geq 0\,, \tag{3.1.5}$$

holds for all $G : \mathbb{R} \to \mathbb{R}$ continuous with compact support. Notice, however, that the stochastic homogenization result given in Proposition 3.1.1 is not in general enough to prove the convergence in (3.1.5), since a simultaneous control over all the starting points of the random walks is needed. In [139], (3.1.5) is achieved by proving the convergence of the associated generators of RW(ω) and of the Brownian motion with diffusivity given by $2D(\mathcal{P})$. However, this strategy works only in the one dimensional setting.

One of the first aims of this thesis is to extend these ideas to a different type of physically relevant random environment for the exclusion process and to make the connection between the homogenization result for one single particle to the homogenization result for the IPS rigorous (and in any dimension $d \geq 1$), strengthening the quenched invariance principle. Moreover, we are interested in proving a path-space hydrodynamic limit, i.e. the convergence of the density field random trajectories, a stronger convergence than the convergence of the expectation of the density field at fixed times.

Boundary driven systems in a quenched environment When moving to the study of inhomogeneous systems out–of–equilibrium, it is natural to consider boundary driven IPS in random environment. More specifically, the open SEP with random conductances is the Markov process with state space given by $\mathcal{X}^{\mathrm{open}} = \{0,1\}^{V_N}$ with $V_N = \{1,\ldots,N\}$ and the generator given by

$$\mathcal{L}_\omega^{\mathrm{open}} f(\eta) = \mathcal{L}_\omega^{\mathrm{bulk}} f(\eta) + \mathcal{L}_\omega^{L,R} f(\eta)\,. \tag{3.1.6}$$

The bulk generator $\mathcal{L}_\omega^{\mathrm{bulk}}$ is defined as

$$\mathcal{L}_\omega^{\mathrm{bulk}} f(\eta) = \sum_{\substack{\{x,y\}\subseteq V_N, \\ |x-y|=1}} \omega_{\{x,y\}} \left\{ \begin{array}{c} \eta(x)(1-\eta(y))\,(f(\eta^{x,y}) - f(\eta)) \\ +\,\eta(y)(1-\eta(x))\,(f(\eta^{y,x}) - f(\eta)) \end{array} \right\}.$$

The boundary part of the dynamics is described by the generator $\mathcal{L}_\omega^{L,R}$ as follows:

$$\mathcal{L}_\omega^{L,R} f(\eta) = \omega_{\{L,1\}} \mathcal{L}_L f(\eta) + \omega_{\{N,R\}} \mathcal{L}_R f(\eta) \,, \tag{3.1.7}$$

where $\mathcal{L}_L$ and $\mathcal{L}_R$ are given, respectively, in (2.5.12) and in (2.5.13). We are then interested in understanding and studying properties of the non-equilibrium steady state and homogenizations effects for relevant quantities such as the current and microscopic profile. Even if the matrix formulation (see, e.g., [57]) giving the explicit steady correlations for the homogeneous boundary driven SEP may still work in presence of the conductances, it certainly breaks down when allowing more than one particle per site. We are thus interested in investigating universal properties on the out–of–equilibrium steady states correlations in the most general quenched random environment setting which still preserve the validity of duality relations of the type given in Section (2.5.2).

3.1.2 Multi-diffusivity

The second space-inhomogeneous evolution that we consider refers to scenarios in which the inhomogeneities still have some degree of geometric regularity. More specifically, we focus on motion of particles in layered materials, where each layer has its own conductivity properties. The goal is then to understand which type of dynamics is followed by the total density of particles in such materials, which is expected to violate the Fick's law.

Typical physical examples of evolution in multi-porous media are the following:

i) diffusion of thermal energy or chemical diffusion of dissolved solutes in layered biological materials (such as animals skin, see, e.g. [135]);

ii) diffusion in polycrystal materials (see, e.g., [159]);

iii) dislocation-pipe diffusion (see, e.g., [59]).

The typical macroscopic model for diffusion is given by the following PDE

$$\frac{\partial \rho}{\partial t} = D \Delta \rho,$$

where ρ is the concentration and $D > 0$ is the diffusion coefficient. As already mentioned, this model is based on the Fick's law of transport and it is applicable to homogeneous, isotropic and isothermal media, with a single family of diffusion paths. During the second half of the past century, physicists established experimentally the limitations of Fick's law for several situations such as the ones described in the list above. A single macroscopic diffusivity is not realistic in these examples and averaging the distinct microscopic properties of the media in a unique macroscopic diffusivity D may lead to very rough estimates not useful in applications. To overcome the limitations of Fick's law, during the 70s some new models have been proposed. A very simple one, that turned out to be very successful in applications and well matching empirical data, is the so-called *double diffusivity model* (D-D model).

The D-D model (see, e.g., [3]), is a model for materials or environments with two distinct diffusivity properties, thus with two families of diffusion paths. Let ρ_0 and ρ_1 be the concentrations of each family of paths and denote by J_0 and J_1 the corresponding fluxes. For each single family, we still impose the Fick's law, namely

$$J_0 = -D_0 \nabla \rho_0$$

and

$$J_1 = -D_1 \nabla \rho_1,$$

where $D_0, D_1 > 0$ are the two distinct diffusion coefficients of the material. The conservation of mass of each single diffusion path gives

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot J_0 = q$$

and

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot J_1 = -q,$$

where the function $q = q(\rho_0, \rho_1)$ models the mass transfer between the two diffusivity paths. The first expression that has been used and studied for q is given by

$$q = -k_0 \rho_0 - k_1 \rho_1$$

where $k_0 > 0$ and $k_1 < 0$.

Putting everything together, we obtain the following system of coupled PDEs, called indeed the D-D model:

$$\begin{cases} \frac{\partial \rho_0}{\partial t} = D_0 \Delta \rho_0 - k_0 \rho_0 - k_1 \rho_1 \\ \frac{\partial \rho_1}{\partial t} = D_1 \Delta \rho_0 + k_0 \rho_0 + k_1 \rho_1. \end{cases}$$

As one can immediately see, the total concentration

$$\rho = \rho_0 + \rho_1$$

does not satisfy a standard diffusion equation with a diffusion constant $D = f(D_1, D_2)$ for some function f. By simple computations, one finds that ρ satisfies

$$\frac{\partial^2 \rho}{\partial t} + (k_0 - k_1)\frac{\partial \rho}{\partial t} = (k_0 D_1 - k_1 D_0)\Delta \rho + (D_0 + D_1)\frac{\partial \Delta \rho}{\partial t} - D_0 D_1 \Delta(\Delta \rho).$$

Even though the literature for this macroscopic model is wide and touches different fields, such as material science and financial mathematics, the D-D has not been derived from a microscopic model of interacting particles. Studying multi–layer IPS which are the natural microscopic counterpart of the D-D model and exploring the violation of the Fick's law of such IPS out–of–equilibrium are some of the goals of this thesis. Additional motivations to study multi–layer IPS comes from population genetics, where individuals can be either active or dormant (see, e.g., [124]), and from models of interacting active random walks with an internal state that changes randomly (e.g. activity, internal energy) and that determines their diffusion rate and or drift (see, e.g. [114]).

3.2 Duality results for systems of particles evolving in the continuum

As already mentioned, the second main goal of this thesis is to generalize the notion of stochastic (self-)duality beyond $\mathbb{Z}^d$. However, the known classical and orthogonal self-duality functions previously introduced are given by products over lattice sites of polynomials in the number of particles at each site. This formulation of duality clearly does not make sense in very natural settings such as systems of interacting Brownian motions or more general interacting Markov processes in the continuum. Even for one of the simplest examples such as independent Brownian motions, it is not immediately clear how to formulate and obtain self-duality. The naive approach of using the scaling limit of self-dualities of independent random walkers does not lead to useful results. However, it is very natural to expect that all the classical discrete systems with self-duality properties have counterparts in the continuum.

Thus, we need to develop a more general and abstract approach to self-duality that can work on very general state spaces. At first, one has to find a language in which the basic duality properties of discrete systems, including the orthogonal dualities, can be restated in such a way that they make sense in the continuum. Secondly, one has to understand under which assumptions these generalized relations are valid, hoping to include many more systems in the class of self-dual Markov processes. These two steps are part of the contributions of this thesis.

In Section 3.2.1 we revisit self-duality for independent random walkers and link it to factorial moment measures of point processes. This allows us to rewrite the self-duality relation in such a way that it makes sense for independent Markov processes on general state spaces, provided a symmetry condition is fulfilled.

3.2.1 Self-Duality for Independent Random Walkers on a Finite Set: Revisited

The section "interpolates" between the usual notation in the IPS literature and that for point processes. We start by considering a system of independent random walks on a finite set, for which duality and self-duality properties are well-known (see, e.g., [84, 47]). First, we revisit these duality results in the language of labelled particles. This will provide us with a notational framework in which these known duality relations are cast in a language that makes sense in a much more general setting, namely independent Markov processes on a general state space. In this way, the reader is prepared (via a convenient and easy case) to the general framework that we build in Chapter 8.

Let E be a finite set and $(\eta_t)_{t\geq 0}$, $\eta_t = (\eta_t(x))_{x\in E}$, be the Markov process on $\mathbb{N}_0^E$ generated by

$$Lf(\eta) = \sum_{x,y\in E} \eta(x)c(x,y)(f(\eta^{x,y}) - f(\eta))$$

for $f : E \to \mathbb{R}$ and $c : E \times E \to \mathbb{R}_+$ a symmetric function ($c(x,x) = 0$ for any $x \in E$ without loss of generality). We denote by $p_t(x,y)$ the transition probability of a single random walk, which is a symmetric function due to the symmetry of the rates $c : E \times E \to \mathbb{R}_+$.

Recalling the definition of the *classical self-duality functions* given in (2.5.4), the self-duality relation for the system of independent walkers then reads as follows

$$\mathbb{E}_\eta \left(D(\xi, \eta_t)\right) = \mathbb{E}_\xi \left(D(\xi_t, \eta)\right) \tag{3.2.1}$$

for all $\eta, \xi \in \mathbb{N}_0^E$ and $t \geq 0$, where $\mathbb{E}_\eta$ denotes the expectation in the configuration process started from η.

In the next paragraph, by a change of notation, we reformulate the relation (3.2.1) with one dual particle (i.e., $\xi = \delta_x$) in such a way that it is meaningful in contexts more general than random walks on a finite set, namely also in the continuum. Thus we get rid of the configuration process notation.

A new framework for self-duality Let $\mathcal{X} := (\mathcal{X}_0(1), \dots, \mathcal{X}_0(N))$ be an arbitrary labelling of the initial positions of the particles which are in total $N < \infty$. We then denote $\mathcal{X}_t$ the positions of these particles at time $t \geq 0$, with $\mathcal{X}_t(i)$ the position of the i-th particle at time $t \geq 0$. The correspondence between the labeled system $(\mathcal{X}_t)_{t\geq 0}$ and the previously introduced configuration process $(\eta_t)_{t\geq 0}$ is given by $\eta_t(x) = \sum_{i=1}^N \mathbf{1}_{\{\mathcal{X}_t(i)=x\}}$.

We describe the system also via the point configuration $\sum_{i=1}^N \delta_{\mathcal{X}_t(i)}$. Notice that in this discrete setting, this is simply a change of notation for the configuration: indeed, for $x \in E$, we have $\left(\sum_{i=1}^N \delta_{\mathcal{X}_t(i)}\right)(\{x\}) = \eta_t(x)$. For the generalization of self-duality in Chapter 8, it is convenient to identify η_t with the point configuration

$$\eta_t = \sum_{i=1}^N \delta_{\mathcal{X}_t(i)}.$$

This is the same as identifying a measure η on the finite E with the vector $\eta(\{x\}), x \in E$. The advantage of this change of notation is that it generalizes to arbitrary measurable state spaces E, and it also allows to produce a simple but insightful proof of the self-duality (3.2.1).

Let us start with self-duality with a single dual particle, i.e. (2.5.4) with dual configuration $\xi = \delta_x$, which reads as

$$\mathbb{E}_\eta(\eta_t(\{x\})) = \mathbb{E}_x^{\mathrm{IRW}}\left(\eta_0(\{Y_t\})\right) = \sum_{y\in E} p_t(x,y)\eta(\{y\}),$$

where $\mathbb{E}_x^{\mathrm{IRW}}$ denotes the expectation with respect to random walk with transition rates $c(x,y)$ starting at $x \in E$.

Let us denote by $\mathbb{E}_\mathcal{X}(\eta_t)$ the measure defined as $\mathbb{E}_\mathcal{X}\left(\eta_t\right)(A) := \mathbb{E}_\mathcal{X}\left(\eta_t(A)\right)$ for $A \subset E$, where $\mathbb{E}_\mathcal{X}$ denotes the expectation when starting $\mathcal{X}_t(1), \dots, \mathcal{X}_t(N)$ at $\mathcal{X}$. We then have

$$\mathbb{E}_\mathcal{X}\left(\eta_t\right) = \mathbb{E}_\mathcal{X}\left(\sum_{i=1}^N \delta_{\mathcal{X}_t(i)}\right) = \sum_{i=1}^N \mathbb{E}_\mathcal{X}(\delta_{\mathcal{X}_t(i)}) = \sum_{i=1}^N \mathbb{E}_{\mathcal{X}_0(i)}^{\mathrm{IRW}}(\delta_{\mathcal{X}_t(i)}) \tag{3.2.2}$$

where in the third equality in (3.2.2) we used that the particles are independent, i.e., the distribution of the position of the i-th particle is only depending on its initial position $\mathcal{X}_0(i)$ and not on the other particles. Using that $\mathbb{E}_{\mathcal{X}_0(i)}^{\mathrm{IRW}}(\delta_{\mathcal{X}_t(i)}) = \sum_{y\in E} p_t(\mathcal{X}_0(i), y)\delta_y$, we can rewrite (3.2.2) as

$$\mathbb{E}_\mathcal{X}\left(\eta_t\right) = \sum_{i=1}^N \sum_{y\in E} p_t(\mathcal{X}_0(i), y)\delta_y = \sum_{y\in E} \delta_y \sum_i p_t(y, \mathcal{X}_0(i)) = \sum_{y\in E}\left(\int p_t(y,z)\eta_0(\mathrm{d}z)\right)\delta_y$$

where in the fourth equality we used the symmetry of the transition probabilities $p_t(x,y)$. If we denote by $\lambda(\mathrm{d}y)$ the counting measure on E we obtain

$$\left(\mathbb{E}_\mathcal{X}\left(\eta_t\right)\right)(\mathrm{d}y) = \left(\int p_t(y,z)\eta_0(\mathrm{d}z)\right)\lambda(\mathrm{d}y). \tag{3.2.3}$$

The above reformulation of the self-duality relation with one dual particle now makes sense on general measurable state spaces E.

Reformulation of Self-Duality with n Dual Particles As a next step we want to generalize (3.2.3) to the case of n dual particles. In order to do so, given a point configuration $\eta = \sum_{i=1}^N \delta_{x_i}$, we introduce the n-th factorial measure of η (see, e.g., [120, Eq. (4.5)]), which is given by

$$\eta^{(n)} = \sum_{1\le i_1,\dots,i_n\le N}^{\neq} \delta_{(x_{i_1},\dots,x_{i_n})}. \tag{3.2.4}$$

Using the notation adopted in [120], the superscript $\neq$ indicates summation over n-tuples with pairwise different entries and where an empty sum is defined as zero. The reason why the measure in (3.2.4) is called falling factorial is clearly explained by the elementary combinatorial lemma below, where the relation with the classical dualities defined in (2.5.4) (consisting of products of falling factorial polynomials) is given. We leave the simple proof to the reader.

Lemma 3.2.1. *Let $\eta = \sum_{i=1}^N \delta_{x_i}$. Then, for all $(y_1, \dots, y_n) \in E^n$, we have*

$$\eta^{(n)}(\{(y_1,\dots,y_n)\}) = D\left(\sum_{k=1}^n \delta_{y_k}, \eta\right), \tag{3.2.5}$$

where $D(\cdot\,,\cdot)$ is the self-duality polynomial function given in (2.5.4). *As a consequence, the n-th factorial measure can be rewritten as follows*

$$\eta^{(n)} = \sum_{y_1,\dots,y_n\in E} \delta_{(y_1,\dots,y_n)} D\left(\sum_{k=1}^n \delta_{y_k}, \eta\right). \tag{3.2.6}$$

We can then generalize (3.2.3) to the expectation of the n-th factorial measure measure $\eta_t^{(n)}$ of the point configuration valued process $\eta_t = \sum_i \delta_{X_t(i)}$ introduced above.

Proposition 3.2.2. *Let λ be the counting measure on E. Then, for all $t > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_{\mathcal{X}}(\eta_t^{(n)})(\mathrm{d}(y_1 \dots y_n)) = \left(\int_{E^n} \prod_{i=1}^n p_t(y_i, z_i)\eta_0^{(n)}(\mathrm{d}(z_1, \dots z_n))\right)\lambda^{\otimes n}(\mathrm{d}(y_1,\dots,y_n)) \tag{3.2.7}$$

Proof. Let $f : E^n \to \mathbb{R}$. We then have

$$\mathbb{E}_{\mathcal{X}}\left(\int f(y_1,\dots,y_n)\eta_t^{(n)}(\mathrm{d}(y_1\dots y_n))\right) = \sum_{1\le i_1,\dots,i_n\le N}^{\neq} \mathbb{E}_{\mathcal{X}} f(\mathcal{X}_t(i_1),\dots,\mathcal{X}_t(i_n)) \tag{3.2.8}$$

$$= \sum_{1\le i_1,\dots,i_n\le N}^{\neq} \int f(y_1,\dots,y_n)\prod_{k=1}^n p_t(\mathcal{X}_0(i_k), y_k)\prod_{k=1}^n \lambda(\mathrm{d}y_k)$$

$$= \sum_{1\le i_1,\dots,i_n\le N}^{\neq} \int f(y_1,\dots,y_n)\prod_{k=1}^n p_t(y_k, \mathcal{X}_0(i_k))\prod_{k=1}^n \lambda(\mathrm{d}y_k)$$

$$= \int f(y_1,\dots,y_n)\left(\int \prod_{k=1}^n p_t(y_k, z_k)\eta_0^{(n)}(\mathrm{d}(z_1\dots z_n))\right)\prod_{k=1}^n \lambda(\mathrm{d}y_k).$$

where we used (3.2.4) in the first and the last equality, the independence of the particles in the second equality and the of the transition probabilities in the third equality. Because f is arbitrary, this proves (3.2.7) □

Remark 3.2.3. *1. Equation* (3.2.7) *holds for each system of independent reversible random walks where the reversible measure λ_{rev} for the single random walk is used in place of the counting measure λ.*

2. Without assuming the symmetry of the rates $c : E \times E \to \mathbb{R}$, from (3.2.8) *and the independence of the particles, we still have the relation*

$$\mathbb{E}_{\mathcal{X}}\left(\int_{E^n} f\mathrm{d}\eta_t^{(n)}\right) = \int_{E^n} \mathbb{E}^{\mathrm{IRW}}_{y_1,\dots,y_n}(f(Y_t(1),\dots,Y_t(n)))\,\eta_0^{(n)}(\mathrm{d}(y_1\dots,y_n)), \tag{3.2.9}$$

where $f : E^n \to \mathbb{R}$ is a permutation invariant function and $\mathbb{E}^{IRW}_{y_1,\dots,y_n}$ denotes expectation with respect to n independent random walkers initially starting from $(y_1,\dots,y_n)$. Equation (3.2.9) *has to be read as a self-intertwining relation and it will be generalized in Section 8.2.2.*

iii) For any $(y_1,\dots,y_n)\in E^n$, (3.2.7) *implies*

$$\mathbb{E}_{\mathcal{X}}(\eta_t^{(n)}(\{(y_1,\dots,y_n)\}) = \mathbb{E}^{IRW}_{y_1,\dots,y_n}\left(\eta_0^{(n)}(\{(Y_t(1),\dots,Y_t(n))\})\right)$$

which, in view of (3.2.5)*, reads as*

$$\mathbb{E}_{\mathcal{X}}\left(D\left(\sum_{k=1}^n \delta_{y_k},\eta_t\right)\right) = \mathbb{E}^{IRW}_{y_1,\dots,y_n}\left(D\left(\sum_{k=1}^n \delta_{Y_t(i)},\eta\right)\right),$$

which is precisely the classical self-duality relation given in (3.2.1).

Orthogonal Self-Duality In this paragraph we turn to orthogonal self-dualities for random walks in a finite set. In [146], [78] and [76] it has been shown (using, respectively, generating functions method, three–term recurrence relations and algebraic methods) that, for all $\theta>0$, the following self-duality relation holds

$$\mathbb{E}_\eta(D_\theta(\xi,\eta_t)) = \mathbb{E}_\xi(D_\theta(\xi_t,\eta)) \tag{3.2.10}$$

with respect to the self-duality functions

$$D_\theta(\xi,\eta) = \prod_{x\in S} d^{\mathrm{or}}_{\xi(\{x\})}(\eta(\{x\});\theta). \tag{3.2.11}$$

$\{d_n^{\mathrm{or}}(\,\cdot\,;\theta)\}_{n\in\mathbb{N}}$ are the Charlier polynomials, i.e. the polynomials satisfying the following orthogonality relation

$$\int d_n^{\mathrm{or}}(\eta(\{x\});\theta)d_m^{\mathrm{or}}(\eta(\{x\});\theta)\rho_\theta(\mathrm{d}\eta) = \mathbf{1}_{\{n=m\}}\frac{n!}{\theta^n}$$

with $\rho_\theta = \otimes_{x\in E}\rho_{x,\theta}$ and $\rho_{x,\theta} = \mathrm{Poisson}(\theta)$ for each $x\in S$. We refer to the functions in (3.2.11) as *orthogonal self-dualities*. Let $[n] := \{1,\dots,n\}$ and $\xi = \sum_{i=1}^n \delta_{y_i}$. In this setting, the relation between orthogonal and classical dualities is simple and given by (see [76, Remark 4.2])

$$D_\theta(\xi,\eta) = \sum_{\xi'\le\xi}(-\theta)^{|\xi|-|\xi'|}\binom{\xi}{\xi'}D(\xi',\eta) = \sum_{I\subset[n]}(-\theta)^{n-|I|}D\left(\sum_{i\in I}\delta_{y_i},\eta\right) \tag{3.2.12}$$

from which it follows that (3.2.10) is a direct consequence of (3.2.1) and the independence of the particles. We can now reformulate the self-duality relation (3.2.10) in terms of a point configuration notation. First we introduce the orthogonalized version of the falling factorial measure associated to a point configuration $\eta = \sum_{i=1}^N \delta_{x_i}$, namely

$$\eta^{(n),\theta}(\mathrm{d}(x_1,\dots,x_n)) := \sum_{r=0}^n (-\theta)^{n-r}\sum_{I\subset[n]:|I|=r}\eta^{(r)}(\mathrm{d}(x_1,\dots,x_n)_I)\otimes\lambda^{\otimes(n-r)}(\mathrm{d}(x_1,\dots,x_n)_{[n]\setminus I}), \tag{3.2.13}$$

where λ denotes the counting measure, $\int f_0\,\mathrm{d}\eta^{(0)} := f_0$ for all $f_0\in\mathbb{R}$ and $(x_1,\dots,x_n)_I$ denotes the subvector of $(x_1,\dots,x_n)$ with components in $I\subset[n]$. The relation between $\eta^{(n),\theta}$ and the orthogonal self-dualities is expressed in the following result.

Lemma 3.2.4. *Let* $\eta=\sum_{i=1}^N\delta_{x_i}$*. Then, for all* $(y_1,\dots,y_n)\in E^n$*, we have*

$$\eta^{(n),\theta}(\{(y_1,\dots,y_n)\}) = D_\theta\left(\sum_{i=1}^n\delta_{y_i},\eta\right) \tag{3.2.14}$$

where $D_\theta(\,\cdot\,,\,\cdot\,)$ *is the orthogonal self-duality given in* (3.2.12)*. As a consequence*

$$\eta_t^{(n),\theta} = \sum_{y_1,\dots,y_n\in E} D_\theta\left(\sum_{i=1}^n\delta_{y_i},\eta\right)\delta_{(y_1,\dots,y_n)}.$$

Proof. For $I\subset[n]$ with $|I|=r$, we have, using (3.2.5),

$$\begin{aligned}D\left(\sum_{i\in I}y_i,\eta\right) &= \eta^{(r)}((y_1,\dots,y_n)_I) = \int \mathbf{1}_{(y_1,\dots,y_n)_I}(x_1,\dots,x_r)\eta^{(r)}(\mathrm{d}(x_1,\dots,x_r))\\ &= \int \mathbf{1}_{(y_1,\dots,y_n)}(x_1,\dots,x_n)\eta^{(r)}(\mathrm{d}(x_1,\dots,x_n)_I)\otimes\lambda^{\otimes(n-r)}(\mathrm{d}(x_1,\dots,x_n)_{[n]\setminus I}).\end{aligned}$$

Therefore, (3.2.14) follows from (3.2.12). □

We then state the analogue of Proposition 3.2.2 for $\eta^{(n),\theta}$ in a notation which makes sense in the context of general measurable state space E. The result follows from (3.2.7) combined with the definition of $\eta^{(n),\theta}$ and the reversibility of λ for the single random walk: we omit here the simple proof and we refer to Section 8.2.3 for the proof of the self-intertwining formulation of this result in a much more general setting.

Proposition 3.2.5. *For all $t > 0$ and $n \in \mathbb{N}$*

$$\mathbb{E}_{\mathcal{X}}(\eta_t^{(n),\theta})(\mathrm{d}(y_1, \ldots, y_n)) = \left(\int_{E^n} \prod_{i=1}^{n} p_t(y_i, x_i) \eta_0^{(n),\theta}(\mathrm{d}(x_1, \ldots, x_n)) \right) \lambda^{\otimes n}(\mathrm{d}(y_1, \ldots, y_n)). \tag{3.2.15}$$

It was observed in [78] (just above equation (8) in [78]), that the orthogonal self-dualities given in (3.2.11) coincide with the polynomials obtained by the Gram-Schmidt orthogonalization procedure initialized with the classical duality functions given in (2.5.4). In the present context, the Gram-Schmidt orthogonalization applied to (2.5.4) is (3.2.12). However, so far, no proof was provided of the fact that the orthogonalization procedure applied to classical self-duality functions leads again to self-duality functions. In Chapter 8, we prove, in a much more general context, that if we properly orthogonalize a self-intertwiner which is a generalized falling factorial polynomial, we get a generalized orthogonal polynomial which is again a self-intertwiner. The proof boils down to show the commutation of the semigroup of the point configuration process with the linear map of the orthogonalization procedure, i.e. that the orthogonalization procedure is a symmetry. From the self-intertwining relations just mentioned follows both classical and orthogonal self-duality relations. The self-intertwiner related to the generalized falling factorial polynomials is introduced in Section 8.2.2 below and the connection between self-intertwining and classical self-dualities is explained in Section 8.3.1.

3.2.2 Boundary driven systems in the continuum

The final goal of this thesis is to initiate the analysis of duality for *boundary-driven systems in the continuum*, starting from the case of independent particles. To achieve our goal, a proper definition of the action of reservoirs in the continuum has to be considered. In the interval $[0, 1]$, the naive idea would be to study a system of independent Brownian motions that are absorbed at the boundaries 0 and 1, with additional creation of particles at 0 and 1. However, as it was noticed in [20], this approach does not work, because in the continuum particles put at the boundary would immediately leave via that same boundary. In [20] the *boundary-driven Brownian gas* on $[0, 1]$ has been defined as the sum of two independent processes: one process modeling the evolution of the particles initially present in the system and moving as independent Brownian motions absorbed at 0 and at 1; and another Poisson point process adding particles on $(0, 1)$ with well-chosen intensity. The creation of particles no longer takes place at the boundaries only, instead particles are created everywhere in $(0, 1)$ with an intensity that guarantees the prescribed densities of the reservoirs. One of the goals of this thesis is to establish in the setting of the boundary driven Brownian gas, the kind of duality results proved in [32, 76] for discrete boundary driven systems (see Section section: duality boundry driven). To do this, we use the set-up introduced in Chapter 8 for closed systems in the continuum and extend it to the boundary driven Brownian gas.

3.3 Organization of the thesis

The rest of the thesis is subdivided in three parts.

Part II is dedicated to hydrodynamic limits in space–inhomogeneous settings and contains Chapter 4 and Chapter 5.

In Chapter 4 we introduce a new random environment for the exclusion process in $\mathbb{Z}^d$ obtained by assigning a maximal occupancy to each site. This maximal occupancy is allowed to randomly vary among sites, and partial exclusion occurs. Under the assumption of ergodicity under translation and uniform ellipticity of the environment, we derive a quenched hydrodynamic limit in path space by using the mild solution approach. To this purpose, we prove, employing the technology developed for the random conductance model, a homogenization result in the form of an arbitrary starting point quenched invariance principle for a single particle in the same environment, which is a result of independent interest. The self-duality property of the partial exclusion process allows us to transfer this homogenization result to the particle system.

Chapter 4 is based on [74], a joint work with Frank Redig (TU Delft) and Federico Sau (IST Austria).

In Chapter 5 we consider three classes of interacting particle systems on $\mathbb{Z}$: independent random walks, the symmetric exclusion process, and the symmetric inclusion process. Particles are allowed to switch their jump rate (the rate identifies the *type* of particle) between 1 (*fast particles*) and $\epsilon \in [0, 1]$ (*slow particles*). The switch between the two jump rates happens at rate $\gamma \in (0, \infty)$. In the exclusion process, the interaction is such that each site can be occupied by at most one particle of each type. In the inclusion process, the interaction takes places between particles of the same type at different sites and between particles of different type at the same site. We derive the macroscopic limit equations for the three systems, obtained after scaling space by N^{-1}, time by N^2, the switching rate by N^{-2}, and letting $N \to \infty$. The limit equations for the macroscopic densities associated to the fast and slow particles is the previously introduced double diffusivity model. We provide a discussion on the solution of the D-D model, thereby connecting mathematical literature applied to material science and to financial mathematics.

Chapter 5 is based on the first part of [75], a joint work with Cristian Giardinà (Modena and Reggio Emilia Uni.), Frank den Hollander (Leiden Uni.), Shubhamoy Nandan(Leiden Uni.) and Frank Redig (TU Delft).

Part III is dedicated to the study of out–of–equilibrium properties of boundary driven systems and contains Chapter 6 and Chapter 7.

In Chapter 6, we consider symmetric partial exclusion and inclusion processes in a general graph in contact with reservoirs, where we allow both for edge disorder and well-chosen site disorder. We extend the classical dualities to this context and then we derive new orthogonal polynomial dualities. From the classical dualities, we derive the uniqueness of the non-equilibrium steady state and obtain correlation inequalities. Starting from the orthogonal polynomial dualities, we show universal properties of n-point correlation functions in the non-equilibrium steady state for systems with at most two different reservoir parameters, such as a chain with reservoirs at left and right ends.

Chapter 6 is based on [76], a joint work with Frank Redig (TU Delft) and Federico Sau (IST Austria).

In Chapter 7, in order to investigate the microscopic out–of–equilibrium properties of the model introduced in Chapter 5, we analyse the system on $[N] = \{1, \ldots, N\}$, adding boundary reservoirs at sites 1 and N of fast and slow particles, respectively. Inside $[N]$ particles move as in the models of Chapter 5, but now particles are injected and absorbed at sites 1 and N with prescribed rates that depend on the particle type. We compute the steady-state density profile and the steady-state current. It turns out that uphill diffusion is possible, i.e., the total flow can be in the direction of increasing total density. This phenomenon, which cannot occur in a single-type particle system, is a violation of Fick's law made possible by the switching between types. We rescale the microscopic steady-state density profile and steady-state current and obtain the steady-state solution of a boundary-value problem for the double diffusivity model.

Chapter 7 is based on the second part of [75], a joint work with Cristian Giardinà (Modena and Reggio Emilia Uni.), Frank den Hollander (Leiden Uni.), Shubhamoy Nandan(Leiden Uni.) and Frank Redig (TU Delft).

Part IV is dedicated to the theoretical study of stochastic duality for closed and open systems of particles evolving in the continuum and contains Chapter 8 and Chapter 9.

In Chapter 8, we derive intertwining relations for a broad class of conservative particle systems both in discrete and continuous setting. Using the language of point process theory, we are able to derive a new framework in which duality and intertwining can be formulated for particle systems evolving in general spaces. These new intertwining relations are formulated with respect to factorial and orthogonal polynomials. Our novel approach unites all the previously found self-dualities in the context of conservative discrete particle systems and provides new duality results for several interacting systems in the continuum, such as interacting Brownian motions. We also introduce a process, consisting of interacting random walks in the continuum, for which our method applies and yields generalized Meixner polynomials as orthogonal self-intertwiners.

Chapter 8 is based on [73], a joint work with Sabine Jansen (LMU Munich), Frank Redig (TU Delft) and Stefan Wagner (LMU Munich).

Finally, in Chapter 9, inspired by the recent work of Bertini and Posta [20], who introduced the boundary driven Brownian gas on $[0, 1]$, we study boundary driven systems of independent particles in a general setting, including particles jumping on finite graphs and diffusion processes on bounded domains in $\mathbb{R}^d$. We prove duality with a dual process that is absorbed at the boundaries, thereby creating a general framework that unifies dualities for boundary driven systems in the discrete and continuum setting. We use duality first to show that from any initial condition the systems evolve to the unique invariant measure, which is a Poisson point process with intensity the solution of a Dirichlet problem. Second, we show how the boundary driven Brownian gas arises as the diffusive scaling limit of a system of independent random walks coupled to reservoirs with properly rescaled intensity.

Chapter 9 is based on [36], a joint work with Gioia Carinci (Modena and Reggio Emilia Uni.), Cristian Giardinà (Modena and Reggio Emilia Uni.) and Frank Redig (TU Delft).

Part II

Hydrodynamics for inhomogeneous interacting particle systems: a duality approach

Chapter 4

Hydrodynamics for the partial exclusion process in random environment

4.1 Introduction

In recent years there has been extensive study of the scaling limit of random walks in both static and dynamic random environment. In this realm, the *random conductance model* (RCM) takes a prominent place. Various analytic tools have been developed to prove scaling properties such as quenched invariance principles, local central limit theorems as well as detailed estimates on the random walks such as heat kernel bounds (see, e.g., [23] for an overview on the subject).

A natural next step is to consider interacting particle systems in random environment, where particles model transport of mass or energy, while the random environments model, as explained in Section 3.1.1, impurities or defects in the conducting material. The macroscopic effects of the environment may be studied through scaling limits such as hydrodynamic limits, fluctuations and large deviations around the hydrodynamic limit, as well as via the study of non-equilibrium behavior of systems coupled to reservoirs which, in random environment, is still a challenge.

Due to the presence of the random environment, these systems are typically *non-gradient* and standard gradients methods to study the hydrodynamic behavior do not carry on. Nevertheless, interacting particle systems with (self-)duality are especially suitable to make the step from single-particle scaling limits towards the derivation of the macroscopic equation for the many-particle system. Indeed, in such systems (see Section 3.1.1 for the one dimensional case), the macroscopic equation can be guessed from the behavior of the expectation of the local particle density which, in turn, amounts to understand the scaling behavior of a single "dual" particle. However, this intuitive "transference principle" from the scaling limit of one random walker to the macroscopic equation has to be made rigorous.

4.1.1 Model

In this chapter, we introduce a random environment for the exclusion process in $\mathbb{Z}^d$ obtained by assigning a maximal occupancy $\alpha_x \in \mathbb{N}$ to each site $x \in \mathbb{Z}^d$ and we study its hydrodynamic limit.

In what follows, we refer to *random environment* as the collection $\boldsymbol{\alpha} = \{\alpha_x,\ x \in \mathbb{Z}^d\}$, for which we assume the following.

Assumption 4.1.1 (Ergodicity and uniform ellipticity of $\boldsymbol{\alpha}$)**.** *We fix a constant* $\mathfrak{c} \in \mathbb{N}$ *for which the random environment* $\boldsymbol{\alpha} = \{\alpha_x,\ x \in \mathbb{Z}^d\}$ *is chosen according to a distribution* $\mathcal{P}$ *on* $\{1, ..., \mathfrak{c}\}^{\mathbb{Z}^d}$*, which is stationary and ergodic under translations* $\{\tau_x,\ x \in \mathbb{Z}^d\}$ *in* $\mathbb{Z}^d$.

In particular, all realizations $\boldsymbol{\alpha}$ of the random environment satisfy the following uniform upper and lower bounds:

$$1 \le \alpha_x \le \mathfrak{c}\,, \qquad x \in \mathbb{Z}^d\,. \tag{4.1.1}$$

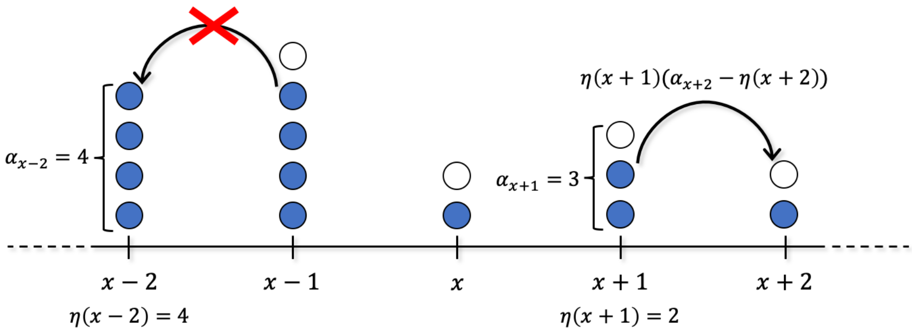


Figure 4.1: Schematic description of the one-dimensional partial exclusion process in the environment $\boldsymbol{\alpha} = \{\alpha_x,\ x \in \mathbb{Z}\}$, where $\alpha_x \in \mathbb{N}$ denotes the maximal occupancy of site $x \in \mathbb{Z}$.

Let us introduce the exclusion process in the environment $\boldsymbol{\alpha}$ (see Figure 4.1) and indicate the configuration of particles by $\eta = \{\eta(x),\ x \in \mathbb{Z}^d\}$, consisting of a collection of occupation variables indexed by the sites of $\mathbb{Z}^d$. These variables indicate the number of particles at each site, i.e.,

$$\eta(x) := \text{number of particles at } x\,.$$

We define the configuration space $\mathcal{X}^\alpha$ (endowed with the product topology) as

$$\mathcal{X}^\alpha := \Pi_{x\in\mathbb{Z}^d}\{0, ..., \alpha_x\}\ ; \tag{4.1.2}$$

here the superscript emphasizes the dependence of the configuration space on the realization of the environment. Hence, given a realization $\boldsymbol{\alpha}$ of the random environment, the *partial (simple) exclusion process in the environment* $\boldsymbol{\alpha}$, abbreviated by SEP($\boldsymbol{\alpha}$), is the Markov process on $\mathcal{X}^\alpha$ whose generator acts on bounded cylindrical functions φ : $\mathcal{X}^\alpha \to \mathbb{R}$, i.e., functions which depend only on a finite number of occupation variables, as follows (all throughout the chapter, $|\cdot|$ will always denote the Euclidean norm):

$$L^\alpha\varphi(\eta) = \sum_{\substack{\{x,y\}\subseteq\mathbb{Z}^d,\\ |x-y|=1}} \left\{ \begin{array}{c} \eta(x)(\alpha_y - \eta(y))\,(\varphi(\eta^{x,y}) - \varphi(\eta)) \\ +\,\eta(y)(\alpha_x - \eta(x))\,(\varphi(\eta^{y,x}) - \varphi(\eta)) \end{array} \right\}. \tag{4.1.3}$$

In the above formula, $\eta^{x,y}$ denotes the configuration obtained from η by removing a particle (if any) from the site x and adding a particle to the site y, i.e.,

$$\eta^{x,y} = \begin{cases} \eta - \delta_x + \delta_y & \text{if } \eta(x) \geq 1 \text{ and } \eta(y) < \alpha_y \\ \eta & \text{otherwise}\,. \end{cases} \tag{4.1.4}$$

Condition (4.1.1) ensures the existence of the process (see, e.g., [126, Chapter 1]), which we call $\{\eta_t,\ t \geq 0\}$, defined via the generator (4.1.3). We highlight that SEP($\boldsymbol{\alpha}$) is a inhomogeneous variant of the partial exclusion process considered in [152] (see also [84]), where $\alpha_x = m$ for any $x \in \mathbb{Z}^d$ and m is a natural number, while, for the choice $\alpha_x = 1$ for any $x \in \mathbb{Z}^d$, we recover the simple symmetric exclusion process in $\mathbb{Z}^d$ (see, e.g., [126]). Moreover, if there is only one particle in the system, no interaction takes place and we are left with a single random walk in the environment $\boldsymbol{\alpha}$, that we call *random walk in the random environment* $\boldsymbol{\alpha}$, abbreviated by RW($\boldsymbol{\alpha}$). More precisely, RW($\boldsymbol{\alpha}$) is the Markov process $\{X_t^\alpha,\ t \geq 0\}$ on $\mathbb{Z}^d$ with law P^α induced by the infinitesimal generator given by

$$A^\alpha f(x) := \sum_{\substack{y\in\mathbb{Z}^d\\ |y-x|=1}} \alpha_y\,(f(y) - f(x))\ , \tag{4.1.5}$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is a bounded function. For all $x \in \mathbb{Z}^d$, let $X^{\alpha,x} = \{X_t^{\alpha,x},\ t \geq 0\}$ denote the random walk RW($\boldsymbol{\alpha}$) started in $x \in \mathbb{Z}^d$.

4.1.2 Quenched hydrodynamics and discussion of related literature

The main result of this chapter, Theorem 4.2.2, states that, under Assumption 4.1.1, for almost every realization of the environment α, the path-space hydrodynamic limit of SEP(α) is a deterministic diffusion equation with a non-degenerate diffusion matrix not depending on the realization of the environment. To this purpose, we run through the following steps. First, we show that SEP(α) is dual to RW(α) and we express the occupation variables of SEP(α) at time t as mild solutions of a lattice stochastic partial differential equation, linear in the drift. Then we show that the microscopic disorder α undergoes a homogenization effect, in the form of a quenched invariance principle for the random walks RW(α). In conclusion, we transfer this homogenization effect from the random walk to the interacting particle system via the aforementioned duality. To the essence, this transference principle boils down to the following two requirements:

(i) *Consistency of the initial conditions* (see Definition 4.2.1 below) stating, roughly speaking, that a law of large numbers holds for the initial particle densities;

(ii) The validity of a quenched homogenization result for the random walks RW(α) in the form of an *arbitrary starting point quenched invariance principle* (see (4.3.1) below).

The mild solution approach to hydrodynamic limits in random environment has been initiated in [139] in $\mathbb{Z}^d$ with $d = 1$ and further developed to any dimension and with less restrictive conditions in [64]. Hence, the idea of deriving the hydrodynamic limit in random environment from a homogenization result for the dual random walk is not new. These works, though, lack of a proof of path space tightness for the empirical density fields of the particle system, as more classical tightness criteria such as Aldous-Rebolledo and Censov (see, respectively, e.g., [106] and [47]) do not apply when employing a mild solution representation for the density fields.

On the other hand, along with the derivation of the limiting hydrodynamic equation, the proof of tightness for particle systems in random environment has been obtained in several works by introducing the so-called *corrected empirical density field*, an auxiliary process for which the evolution equation "closes" and the aforementioned tightness criteria apply. Thus, one has to face the extra step consisting in proving that the empirical density field and the corrected one are close in a suitable sense. The idea of the corrected empirical density field has been introduced in [99] for the exclusion process with random conductances on $\mathbb{Z}^d$ with $d = 1$ and later extended to the d-dimensional torus in [87], with $d \geq 1$, and more general geometries in [98]. The construction of the corrected empirical density field as in [87] is general enough to apply, by employing the convergence of either the random walk generators or the associated Dirichlet forms, also to different contexts, like in [68] for a one-dimensional subdiffusive exclusion process, [66] for a zero range process with random conductances and our context of site-varying maximal occupancy exclusion process. However, we believe that a general strategy to establish tightness and the hydrodynamic limit for sequences of tempered distribution-valued mild solutions may be of help when stochastic convolutions, although not being martingales, ensure a stronger space-time regularity of the stochastic processes as in the context, e.g., of Gaussian SPDEs. In [147], in which the hydrodynamic limit of the simple exclusion process in presence of dynamic random conductances is studied, a criterion for relative compactness, based on the notion of uniform stochastic continuity, has been presented. We apply this criterion to our context of partial exclusion, which has the advantages to directly apply to the sequence of mild solutions and avoid the introduction of the auxiliary sequence of the corrected empirical density fields.

Next to the problem of ensuring relative compactness for the empirical density, another main challenge in the study of scaling limits of particle systems in random environment is to prove a homogenization result for the underlying environment. To get the desired homogenization result we employ, via a suitable random time change, several concepts and results developed in the context of the random conductance model (RCM) (see, e.g., [23]). So far, the technology developed in the last two decades for RCM has not been employed in the context of particle systems in random environment, other types of convergence being preferred. In particular, either Γ-convergence (see, e.g., [98]) or two-scale convergence (see, e.g., [65, 67]) were employed to recover quenched hydrodynamic limits for the simple exclusion process in more general settings than RCM with uniformly elliptic conductances.

For the RW(α) under Assumption 4.1.1, one does not need such a level of generality and it is natural to try to use the existing quenched invariance principles for the random conductance model. However all quenched invariance principles for RCM (see, e.g., [7, 10, 8, 15, 155]) are derived for the walk starting at the origin, which is, in general, too weak as a convergence to ensure the quenched hydrodynamic limit for the particle system. To fill the gap between quenched invariance principle and quenched hydrodynamic limit, a homogenization result involving the random walks RW(α) starting from *all* spatial locations suffices. To this purpose, we choose to extend the quenched invariance principle valid for the random walk starting from the origin to walks starting from arbitrary sequences of starting points; we believe the latter to be a result of interest in its own right. Note that this strengthening is not trivial due to the lack of translation invariance of the law of the random walk in quenched random environment.

The problem of deriving quenched arbitrary starting point invariance principles has been posed in [148] and only recently solved in [42] for the static random conductance model on the supercritical percolation cluster. In our context of random environment $\boldsymbol{\alpha}$, in order to prove the quenched invariance principle with arbitrary starting positions for the dual random walk, we use the formalism and ideas from [42].

The connection between the quenched invariance principle in RCM and hydrodynamics in random environment seems to be promising, at least in the case of particle systems with self-duality, and this gives hope, for future works, to obtain path-space hydrodynamic limit also in degenerate environments. In conclusion, we remark that other strategies than self-duality to prove hydrodynamic limits for interacting particle systems in random environment are available and rely on the non-gradient methods (see, e.g., [132]) and methods based on Riemann-characteristics for hyperbolic concentration laws (see, e.g., [12]).

The remaining of the chapter is organized as follows. In Section 4.2 we state the main theorem – the quenched hydrodynamic limit in path space – and explain the strategy of the proof in more detail. Section 4.3 is devoted to the arbitrary starting point quenched invariance principle and Section 4.4 to the proof of the hydrodynamic limit. The proofs of some auxiliary results stated in the body of the chapter are collected in separate sections at the end of the chapter.

4.2 Main result and strategy of the proof

As observable of the macroscopic behavior of the interacting particle system, we consider the empirical density fields, indicated, for all $N \in \mathbb{N}$, by $\mathsf{X}^N = \{\mathsf{X}^N_t,\ t \geq 0\}$. Given, for a fixed realization of the environment $\boldsymbol{\alpha}$, a sequence of probability measures $\nu^{\boldsymbol{\alpha}} = \{\nu^{\boldsymbol{\alpha}}_N\}_{N\in\mathbb{N}}$ on the configuration space $\mathcal{X}^{\boldsymbol{\alpha}}$, for all $N \in \mathbb{N}$, the empirical density field X^N is a measure-valued process obtained as a function of the system $\eta = \{\eta_t,\ t \geq 0\}$ as follows:

$$\mathsf{X}^N_t := \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \delta_{\frac{x}{N}}\, \eta_{tN^2}(x)\ , \tag{4.2.1}$$

where η is the process SEP($\boldsymbol{\alpha}$) introduced in Section 4.1 initially distributed as $\nu^{\boldsymbol{\alpha}}_N$. We refer to $\mathbb{P}^{\boldsymbol{\alpha}}_{\nu^{\boldsymbol{\alpha}}_N}$ as the probability measure on the Skorokhod space $\mathcal{D}([0,\infty), \mathcal{X}^{\boldsymbol{\alpha}})$ of such process and let $\mathbb{E}^{\boldsymbol{\alpha}}_{\nu^{\boldsymbol{\alpha}}_N}$ denote the corresponding expectation, while $\mathbb{P}^{\boldsymbol{\alpha}}_\eta$ and $\mathbb{E}^{\boldsymbol{\alpha}}_\eta$ indicate the law and the corresponding expectation, respectively, of the process starting from the configuration η. We note that the definition (4.2.1) encodes a space-time diffusive rescaling of the microscopic system. Moreover, due to the uniform upper bound in (4.1.1) on the maximal occupancies, we view (as done, e.g., in the textbook [47, Chapter 2]) the empirical density fields as processes in $\mathcal{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$; here, $\mathscr{S}'(\mathbb{R}^d)$ denotes the topological dual of the Schwartz class of smooth and rapidly decreasing functions $\mathscr{S}(\mathbb{R}^d)$ and $\mathcal{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ the Skorokhod space of $\mathscr{S}'(\mathbb{R}^d)$-valued càdlàg trajectories. For further details on the construction and topologies of these spaces, we refer to, e.g., [47, Chapter 2, Section 6], [138], as well as [101, Chapter 2, Section 4]. Hence, for all $t \geq 0$, the action of X^N_t on the test function $G \in \mathscr{S}(\mathbb{R}^d)$ is given by

$$\mathsf{X}^N_t(G) := \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\, \eta_{tN^2}(x)\ . \tag{4.2.2}$$

Let us remark that this choice of the functional spaces $\mathcal{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$, while being standard when studying fluctuation fields, is less canonical in the context of hydrodynamic limits (cf., e.g., [106]). The motivation behind this choice is twofold. On the one side, the nuclear structure of the pair $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ allows, in Section 4.4.1, to employ Mitoma's tightness criterion for processes in $\mathcal{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$, see [138]. On the other side, in Section 4.4.2, we need that $\mathscr{S}(\mathbb{R}^d)$ is dense and invariant under the action of the semigroup on $C_0(\mathbb{R}^d)$ – the Banach space of continuous and vanishing at infinity functions endowed with the supremum norm – of the d-dimensional Brownian motion $\{B^\Sigma_t,\ t \geq 0\}$ with diffusion matrix $\Sigma \in \mathbb{R}^{d\times d}$, i.e., the strongly continuous and contraction semigroup $\{\mathcal{S}^\Sigma_t,\ t \geq 0\}$ on $C_0(\mathbb{R}^d)$ associated to the following second-order differential operator

$$\mathcal{A}^\Sigma = \tfrac{1}{2} \nabla \cdot (\Sigma\, \nabla)\ .$$

As our goal is to study the limit of the N-th empirical density field X^N as N goes to infinity, we need to require that the initial particle configurations suitably rescale to a macroscopic profile. We make this requirement precise in the following definition, in which $\mathscr{P}(\mathcal{X}^\alpha)$ denotes the space of probability measures on $\mathcal{X}^\alpha$.

Definition 4.2.1 (Consistency of the initial conditions)**.** *We say that, for a given environment $\boldsymbol{\alpha}$, a sequence of probabilities $\nu^\alpha := \{\nu_N^\alpha\}_{N\in\mathbb{N}}$ in $\mathscr{P}(\mathcal{X}^\alpha)$ is* consistent to a continuous macroscopic profile $\bar\rho : \mathbb{R}^d \to [0,1]$ *if the following convergence*

$$\nu_N^\alpha\left(\left\{\eta \in \mathcal{X}^\alpha \,:\, \left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\eta(x) - \int_{\mathbb{R}^d} G(u)\,\mathbb{E}_{\mathcal{P}}[\alpha_0]\,\bar\rho(u)\,\mathrm{d}u\right| > \delta\right\}\right) \underset{N\to\infty}{\longrightarrow} 0 \tag{4.2.3}$$

holds for all $G \in \mathscr{S}(\mathbb{R}^d)$ and $\delta > 0$.

We are ready to state our main theorem, whose proof is deferred to Section 4.4 below.

Theorem 4.2.2 (Hydrodynamic limit in quenched random environment)**.** *Let $\bar\rho : \mathbb{R}^d \to [0,1]$ be a continuous macroscopic profile and, for all realizations of the environment $\boldsymbol{\alpha}$, let $\nu^\alpha = \{\nu_N^\alpha\}_{N\in\mathbb{N}}$ be a sequence of probabilities on $\mathscr{P}(\mathcal{X}^\alpha)$. Recall Definition 4.2.1, define*

$$\mathfrak{C} := \left\{\boldsymbol{\alpha} \in \{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d} \,:\, \nu^\alpha \text{ is consistent with } \bar\rho\right\}, \tag{4.2.4}$$

and assume that $\mathcal{P}(\mathfrak{C}) = 1$.

Then, there exists two measurable subsets $\mathfrak{A}$ and $\mathfrak{B} \subseteq \{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d}$ with $\mathcal{P}(\mathfrak{A}) = \mathcal{P}(\mathfrak{B}) = 1$ (given, respectively, in (4.2.16) *and* (4.3.11) *below) such that, for all $\boldsymbol{\alpha} \in \mathfrak{A} \cap \mathfrak{B} \cap \mathfrak{C}$ and for all $T > 0$, we have the following weak convergence in $\mathcal{D}([0,T], \mathscr{S}'(\mathbb{R}^d))$:*

$$\left\{\mathsf{X}_t^N,\ t \in [0,T]\right\} \underset{N\to\infty}{\Longrightarrow} \left\{\pi_t^\Sigma,\ t \in [0,T]\right\}, \tag{4.2.5}$$

where the empirical density fields $\{\mathsf{X}_t^N,\ t \in [0,T]\}_{N\in\mathbb{N}}$ are given as in (4.2.1) *and*

$$\pi_t^\Sigma(\mathrm{d}u) := \mathbb{E}_{\mathcal{P}}[\alpha_0]\,\rho_t^\Sigma(u)\,\mathrm{d}u\,, \tag{4.2.6}$$

with $\{\rho_t^\Sigma,\ t \ge 0\}$ being the unique strong solution in $\mathbb{R}^d$ to

$$\begin{cases} \partial_t\rho &= \tfrac{1}{2}\nabla\cdot(\Sigma\,\nabla\rho) \\ \rho_0 &= \bar\rho\,. \end{cases} \tag{4.2.7}$$

In particular, the diffusion matrix $\Sigma \in \mathbb{R}^{d\times d}$ in (4.2.7) *and given in Proposition 4.3.4 below is non-degenerate, symmetric, positive-definite and does not depend on the particular realization of the environment.*

Remark 4.2.3 (Existence and uniqueness of the limit)**.** *Let $C_b(\mathbb{R}^d)$ denote the Banach space of continuous and bounded functions from $\mathbb{R}^d$ to $\mathbb{R}$ endowed with the supremum norm. It is well-known (see, e.g., [63, Chapter 2, Section 3.1, Theorem 1]) that, $\bar\rho$ being bounded and continuous, the strong solution $\{\rho_t^\Sigma,\ t \ge 0\}$ to* (4.2.7) *exists, is unique and admits the following stochastic representation in terms of the contraction and strongly continuous semigroup of Brownian motion $\{B_t^\Sigma,\ t \ge 0\}$ on $C_b(\mathbb{R}^d)$, still referred to – with a slight abuse of notation – as $\{\mathcal{S}_t^\Sigma,\ t \ge 0\}$:*

$$\rho_t^\Sigma = \mathcal{S}_t^\Sigma\bar\rho\,, \qquad t \ge 0\,. \tag{4.2.8}$$

Moreover, by [96, Theorem 1.4], there exists a unique element $\{\pi_t,\ t \ge 0\}$ in the space of $\mathscr{S}'(\mathbb{R}^d)$-valued continuous trajectories $C([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ (see, e.g., [96], [101, Chapter 2, Section 4]) such that either one of the following two identities hold for all $t \ge 0$ and $G \in \mathscr{S}(\mathbb{R}^d)$:

$$\pi_t(G) = \pi^{\bar\rho}(G) + \int_0^t \pi_s(\mathcal{A}^\Sigma G)\,\mathrm{d}s \qquad \text{or} \qquad \pi_t(G) = \pi^{\bar\rho}(\mathcal{S}_t^\Sigma G)\,, \tag{4.2.9}$$

where

$$\pi^{\bar\rho}(\mathrm{d}u) := \mathbb{E}_{\mathcal{P}}[\alpha_0]\,\bar\rho(u)\,\mathrm{d}u\,. \tag{4.2.10}$$

As a consequence of (4.2.8) *and*

$$\int_{\mathbb{R}^d} \mathcal{S}_t^\Sigma G(u)\,H(u)\,\mathrm{d}u = \int_{\mathbb{R}^d} G(u)\,\mathcal{S}_t^\Sigma H(u)\,\mathrm{d}u\,, \qquad G \in \mathscr{S}(\mathbb{R}^d)\,,\ H \in C_b(\mathbb{R}^d)\,,\ t \ge 0\,, \tag{4.2.11}$$

such a unique element must coincide with $\{\pi_t^\Sigma,\ t \ge 0\}$ in (4.2.6).

Before discussing the strategy of proof of Theorem 4.2.2, we present an ergodic theorem (Lemma 4.2.5 below) of importance at various stages of the chapter; in particular, this allows us to exhibit in Proposition 4.2.6 below a class of initial distributions for SEP(α) which verify the assumption of Theorem 4.2.2. Preliminarily, we need the following definition.

Definition 4.2.4. *A subset $\mathcal{F}$ of $C_0(\mathbb{R}^d)$ is said to be* equicontinuous *if*

$$\lim_{\delta\downarrow 0} \sup_{\substack{u,v\in\mathbb{R}^d\\|u-v|<\delta}} \sup_{F\in\mathcal{F}} |F(u)-F(v)| = 0 \tag{4.2.12}$$

holds, bounded *if*

$$\sup_{F\in\mathcal{F}} \sup_{u\in\mathbb{R}^d} |F(u)| < \infty \tag{4.2.13}$$

holds, and uniformly integrable *if*

$$\sup_{F\in\mathcal{F}} |F(u)| \le f(u)\ , \qquad u\in\mathbb{R}^d\ , \tag{4.2.14}$$

holds for some function $f\in L^1(\mathbb{R}^d)\cap C_0(\mathbb{R}^d)$.

Lemma 4.2.5. *Under Assumption 4.1.1 on the environment, for $\mathcal{P}$-a.e. realization of the environment $\boldsymbol{\alpha}$, the following holds:*

> *For all equicontinuous, bounded and uniformly integrable subsets $\mathcal{F}$ of $C_0(\mathbb{R}^d)$ (see Definition 4.2.4), we have*
>
> $$\sup_{F\in\mathcal{F}} \left| \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} F(\tfrac{x}{N})\,\alpha_x - \mathbb{E}_{\mathcal{P}}[\alpha_0]\int_{\mathbb{R}^d} F(u)\,\mathrm{d}u \right| \underset{N\to\infty}{\longrightarrow} 0\ . \tag{4.2.15}$$

The proof of Lemma 4.2.5 can be found in Section 4.6 below. Moreover, we find convenient to define

$$\mathfrak{A} := \left\{\boldsymbol{\alpha}\in\{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d} : \text{the claim in Lemma 4.2.5 holds for } \boldsymbol{\alpha}\right\}\ . \tag{4.2.16}$$

By a detailed balance computation, it is simple to check that the following product measures

$$\nu_p^{\boldsymbol{\alpha}} = \bigotimes_{x\in\mathbb{Z}^d} \text{Binomial}(\alpha_x, p)\ , \tag{4.2.17}$$

are reversible measures for SEP($\boldsymbol{\alpha}$), for all parameters $p\in[0,1]$. In general, if the parameter p depends on the site $x\in\mathbb{Z}^d$, the corresponding Bernoulli product measures are not invariant for the exclusion dynamics. Nevertheless, as shown in Proposition 4.2.6 below, such probability measures with slowly varying parameter satisfy the assumptions of Theorem 4.2.2.

Proposition 4.2.6. *For all $\boldsymbol{\alpha}\in\mathfrak{A}$ (see (4.2.16)) and for all continuous profiles $\bar\rho:\mathbb{R}^d\to[0,1]$, the sequence of probabilities $\{\nu_N^{\boldsymbol{\alpha},\bar\rho}\}_{N\in\mathbb{N}}$ in $\mathscr{P}(\mathcal{X}^{\boldsymbol{\alpha}})$ given, for all $N\in\mathbb{N}$, by*

$$\nu_N^{\boldsymbol{\alpha},\bar\rho} := \bigotimes_{x\in\mathbb{Z}^d} \text{Binomial}(\alpha_x, \bar\rho(\tfrac{x}{N})) \tag{4.2.18}$$

is consistent with the continuous profile $\bar\rho$ (Definition 4.2.1), thus, satisfying the assumption of Theorem 4.2.2.

Proof. Note that, for all realizations of the environment $\boldsymbol{\alpha}$, $N\in\mathbb{N}$ and $x\in\mathbb{Z}^d$, one has

$$\mathbb{E}^{\boldsymbol{\alpha}}_{\nu_N^{\boldsymbol{\alpha},\bar\rho}}[\eta(x)] = \alpha_x\bar\rho(\tfrac{x}{N}) \qquad\text{and}\qquad \mathbb{E}^{\boldsymbol{\alpha}}_{\nu_N^{\boldsymbol{\alpha},\bar\rho}}\left[\left(\eta(x)-\alpha_x\bar\rho(\tfrac{x}{N})\right)^2\right] = \alpha_x\bar\rho(\tfrac{x}{N})\left(1-\bar\rho(\tfrac{x}{N})\right)\ . \tag{4.2.19}$$

Hence, by Chebyshev's inequality, for all $\delta>0$ and $G\in\mathscr{S}(\mathbb{R}^d)$,

$$\nu_N^{\boldsymbol{\alpha},\bar\rho}\left(\left\{\eta\in\mathcal{X}^{\boldsymbol{\alpha}} : \left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\left(\eta(x)-\alpha_x\bar\rho(\tfrac{x}{N})\right)\right| > \delta\right\}\right) \underset{N\to\infty}{\longrightarrow} 0 \tag{4.2.20}$$

holds true for all $\boldsymbol{\alpha}$. With the observation that, for all functions $G\in\mathscr{S}(\mathbb{R}^d)$ and continuous profiles $\bar\rho:\mathbb{R}^d\to[0,1]$, the product of G and $\bar\rho$ is continuous, bounded and integrable, Lemma 4.2.5 yields the desired result for $\boldsymbol{\alpha}\in\mathfrak{A}$. □

4.2.1 Duality

For all given environments $\boldsymbol{\alpha}$, SEP($\boldsymbol{\alpha}$) and RW($\boldsymbol{\alpha}$), besides being the latter a particular instance of the former when the system consists of only one particle, are connected through the notion of *stochastic duality*, or, shortly, *duality*. This notion occurs in various contexts (see, e.g., [126]) and, in the particular case of interacting particle systems, turns useful when quantities of a many-particle system may be studied in terms of quantities of a simpler, typically a-few-particle, system. Moreover, when this duality relation is established between two copies of the same Markov process, one speaks about *self-duality*.

SEP($\boldsymbol{\alpha}$) is a self-dual Markov process, meaning that there exists a function $D^\alpha : \mathcal{X}^\alpha_f \times \mathcal{X}^\alpha \to \mathbb{R}$ (with $\mathcal{X}^\alpha_f$ being the subset of configurations in $\mathcal{X}^\alpha$ with finitely-many particles), called *self-duality function*, given by

$$D^\alpha(\xi,\eta) := \prod_{x\in\mathbb{Z}^d} \frac{\eta(x)!}{(\eta(x)-\xi(x))!}\,\frac{(\alpha_x-\xi(x))!}{\alpha_x!}\mathbf{1}_{\{\xi(x)\le\eta(x)\}}\ ,$$

for which the following self-duality relation holds: for all $\xi \in \mathcal{X}^\alpha_f$ and $\eta \in \mathcal{X}^\alpha$,

$$L^\alpha D^\alpha(\cdot,\eta)(\xi) = L^\alpha D^\alpha(\xi,\cdot)(\eta)\ . \tag{4.2.21}$$

In particular, the l.h.s. corresponds to apply the generator L^α to the function $D(\cdot,\eta)$ and evaluate the resulting function at ξ; similarly for the r.h.s.. This property was proven for the first time in [152] for the homogeneous partial exclusion, i.e., for $\alpha_x = m \in \mathbb{N}$ for all $x \in \mathbb{Z}^d$, (see also [84]) and extends to the random environment context.

We are interested in a particular instance of this self-duality property, namely when the dual configuration consists in a single particle configuration, i.e., $\xi = \delta_x$ for some $x \in \mathbb{Z}^d$. In this case the function $D^\alpha(\delta_x,\eta) =: D^\alpha(x,\eta)$ reads

$$D^\alpha(x,\eta) = \tfrac{\eta(x)}{\alpha_x} \tag{4.2.22}$$

and the self-duality relation reduces to

$$A^\alpha D^\alpha(\cdot,\eta)(x) = L^\alpha D^\alpha(x,\cdot)(\eta)\ , \tag{4.2.23}$$

which may be checked by a straightforward computation. Relation (4.2.23) has to be interpreted as a duality relation between SEP($\boldsymbol{\alpha}$) and RW($\boldsymbol{\alpha}$) with duality function D^α given in (4.2.22).

Notice that the generator A^α is, in view of Assumption 4.1.1, a bounded operator on both Banach spaces $\ell^\infty(\mathbb{Z}^d,\boldsymbol{\alpha})$ and $\ell^1(\mathbb{Z}^d,\boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ plays the role of reference measure on $\mathbb{Z}^d$ assigning to each site $x \in \mathbb{Z}^d$ the positive value α_x. Likewise, A^α is a bounded operator on the weighted Hilbert space $\ell^2(\mathbb{Z}^d,\boldsymbol{\alpha})$ whose inner product is defined as

$$\langle f,g\rangle := \sum_{x\in\mathbb{Z}^d} f(x)\,g(x)\,\alpha_x\ . \tag{4.2.24}$$

With a slight abuse of notation, we continue to use $\langle\cdot,\cdot\rangle$ also for the bilinear map on $\ell^1(\mathbb{Z}^d,\boldsymbol{\alpha})\times\ell^\infty(\mathbb{Z}^d,\boldsymbol{\alpha})$ defined by the r.h.s. of (4.2.24); moreover, we let A_α and $\{S^\alpha_t,\ t\ge 0\}$ denote the generator and corresponding semigroup associated to RW($\boldsymbol{\alpha}$), indistinguishably of the Banach space they act on.

As it follows from a detailed balance relation, RW($\boldsymbol{\alpha}$) is reversible with respect to the weighted counting measure $\boldsymbol{\alpha}$. More precisely, A^α is self-adjoint in $\ell^2(\mathbb{Z}^d,\boldsymbol{\alpha})$ and, moreover, for all $f \in \ell^1(\mathbb{Z}^d,\boldsymbol{\alpha})$ (resp. $\ell^2(\mathbb{Z}^d,\boldsymbol{\alpha})$) and $g \in \ell^\infty(\mathbb{Z}^d,\boldsymbol{\alpha})$ (resp. $\ell^2(\mathbb{Z}^d,\boldsymbol{\alpha})$) and for all $t \ge 0$, we have

$$\langle S^\alpha_t f, g\rangle = \langle f, S^\alpha_t g\rangle\ , \tag{4.2.25}$$

or, equivalently,

$$\alpha_x\,p^\alpha_t(x,y) = \alpha_y\,p^\alpha_t(y,x)\ , \qquad x,y\in\mathbb{Z}^d\ ,\ t\ge 0\ , \tag{4.2.26}$$

for the corresponding transition probabilities.

4.2.2 Strategy of the proof

The self-duality relation (4.2.23) suggests that the limiting collective behavior of the particle density is connected to the limiting behavior of the diffusively rescaled RW($\boldsymbol{\alpha}$). Let us describe the strategy of the proof of our main result and the role of this connection.

Mild solution representation

As a first observation, by following closely [139] and [64], for all realizations of the environment $\boldsymbol{\alpha}$, we apply Dynkin's formula to the bounded cylindrical functions $\{D^\alpha(x,\cdot) : \mathcal{X}^\alpha \to \mathbb{R}\}_{x\in\mathbb{Z}^d}$ given in (4.2.22): for all initial configurations $\eta \in \mathcal{X}^\alpha$, we have

$$D^\alpha(x,\eta_t) = D^\alpha(x,\eta) + \int_0^t L^\alpha D^\alpha(x,\cdot)(\eta_s)\,\mathrm{d}s + M_t^\alpha(x)\,, \qquad x\in\mathbb{Z}^d\,,\ t\ge 0\,, \tag{4.2.27}$$

where $\{M_t^\alpha(x), t \ge 0\}_{x\in\mathbb{Z}^d}$ is a family of martingales w.r.t. the natural filtration of the process whose joint law is characterized in terms of their predictable quadratic covariations (see (4.4.2)–(4.4.3) below; for an explicit construction of these martingales, see Section 4.5 below). We remark that in (4.2.27) above L^α acts on the function $D^\alpha(\cdot,\cdot)$ w.r.t. the η-variables. We recall from (4.2.23) that the function $D^\alpha : \mathbb{Z}^d \times \mathcal{X}^\alpha \to \mathbb{R}$ of the joint system is a duality function between SEP($\boldsymbol{\alpha}$) and RW($\boldsymbol{\alpha}$). Hence, by using (4.2.23), we rewrite (4.2.27) as

$$D^\alpha(x,\eta_t) = D^\alpha(x,\eta) + \int_0^t A^\alpha D^\alpha(\cdot,\eta_s)(x)\,\mathrm{d}s + M_t^\alpha(x)\,, \qquad x\in\mathbb{Z}^d\,,\ t\ge 0\,, \tag{4.2.28}$$

yielding a system (indexed by $x\in\mathbb{Z}^d$) of linear – in the drift – stochastic integral equations. As a consequence, the solution of this system may be represented as a mild solution by considering the semigroup $\{S_t^\alpha, t\ge 0\}$ associated to the generator A^α of RW($\boldsymbol{\alpha}$), i.e., we have

$$D^\alpha(x,\eta_t) = S_t^\alpha D^\alpha(\cdot,\eta)(x) + \int_0^t S_{t-s}^\alpha\,\mathrm{d}M_s^\alpha(x)\,, \qquad x\in\mathbb{Z}^d\,,\ t\ge 0\,, \tag{4.2.29}$$

where

$$\int_0^t S_{t-s}^\alpha\,\mathrm{d}M_s^\alpha(x) := \int_0^t \sum_{y\in\mathbb{Z}^d} P^\alpha(X_{t-s}^{\alpha,x} = y)\,\mathrm{d}M_s^\alpha(y) \tag{4.2.30}$$

(for a definition of $X^{\alpha,x}$ and its law, see the end of Section 4.1.1; for a proof of the absolute convergence of the latter infinite sum, we refer the reader to Lemma 4.5.2 below).

Combining the definitions (4.2.1)–(4.2.2) and (4.2.22) with the mild solution representation in (4.2.29), we rewrite the empirical density fields, for all test functions $G\in\mathscr{S}(\mathbb{R}^d)$, as follows:

$$\begin{aligned}
\mathsf{X}_t^N(G) &= \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\,D^\alpha(x,\eta_{tN^2})\,\alpha_x \\
&= \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\,S_{tN^2}^\alpha D^\alpha(\cdot,\eta_0)(x)\,\alpha_x + \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} G(\tfrac{x}{N})\left(\int_0^{tN^2} S_{tN^2-s}^\alpha\,\mathrm{d}M_s^\alpha(x)\right)\alpha_x\,.
\end{aligned}$$

Furthermore, because both A^α and the corresponding semigroup are self-adjoint in $\ell^2(\mathbb{Z}^d,\boldsymbol{\alpha})$ (see (4.2.25)), we obtain:

$$\begin{aligned}
\mathsf{X}_t^N(G) &= \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} S_{tN^2}^{N,\alpha}G(\tfrac{x}{N})\,D^\alpha(x,\eta_0)\,\alpha_x + \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(\int_0^{tN^2} S_{tN^2-s}^{N,\alpha}G(\tfrac{x}{N})\,\mathrm{d}M_s^\alpha(x)\right)\alpha_x \\
&= \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d} S_{tN^2}^{N,\alpha}G(\tfrac{x}{N})\,\eta_0(x) + \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(\int_0^{tN^2} S_{tN^2-s}^{N,\alpha}G(\tfrac{x}{N})\,\mathrm{d}M_s^\alpha(x)\right)\alpha_x \\
&= \mathsf{X}_0^N(S_{tN^2}^{N,\alpha}G) + \int_0^t \mathrm{d}\mathsf{M}_s^N(S_{tN^2-s}^{N,\alpha}G)\,,
\end{aligned} \tag{4.2.31}$$

where we adopted the shorthand, for all $G\in\mathscr{S}(\mathbb{R}^d)$,

$$\int_0^t \mathrm{d}\mathsf{M}_s^N(S_{tN^2-s}^{N,\alpha}G) := \frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(\int_0^{tN^2} S_{tN^2-s}^{N,\alpha}G(\tfrac{x}{N})\,\mathrm{d}M_s^\alpha(x)\right)\alpha_x\,, \tag{4.2.32}$$

with

$$S_t^{N,\alpha}G(\tfrac{x}{N}) := S_t^\alpha G(\tfrac{\cdot}{N})(x)\,, \qquad x\in\mathbb{Z}^d\,,\ t\ge 0\,. \tag{4.2.33}$$

Hence, we obtain in (4.2.31) the same decomposition as in, e.g., [139, 64, 68, 147], in which the empirical density field is written as a sum of its expectation (the first term on the r.h.s. of (4.2.31)), and "noise"(the second term), which is not a martingale.

From the arbitrary starting point invariance principle towards the path space hydrodynamic limit

As in those works, our first aim is to prove that, for $\mathcal{P}$-a.e. $\boldsymbol{\alpha}$, the finite-dimensional distributions of the empirical density fields converge in probability to those of the solution of the hydrodynamic equation (4.2.7). Moreover, since convergence in probability of finite-dimensional distributions is implied by the convergence in probability of single marginals, it suffices to prove convergence of one-dimensional distributions. In particular, we will show in Section 4.4 below that, for all $G \in \mathscr{S}(\mathbb{R}^d)$, $t \geq 0$ and $\delta > 0$,

$$\mathbb{P}^{\boldsymbol{\alpha}}_{\nu^{\boldsymbol{\alpha}}_N}\left(\left|\int_0^{tN^2} \mathrm{d}\mathsf{M}^N_s(S^{N,\boldsymbol{\alpha}}_{tN^2-s}G)\right| > \delta\right) \underset{N\to\infty}{\longrightarrow} 0 \tag{4.2.34}$$

holds for all environments $\boldsymbol{\alpha}$, and that (recall (4.2.10))

$$\mathbb{P}^{\boldsymbol{\alpha}}_{\nu^{\boldsymbol{\alpha}}_N}\left(\left|\mathsf{X}^N_0(S^{N,\boldsymbol{\alpha}}_{tN^2}G) - \pi^{\bar{\rho}}(\mathcal{S}^{\Sigma}_t G)\right| > \delta\right) \underset{N\to\infty}{\longrightarrow} 0 \tag{4.2.35}$$

holds for $\mathcal{P}$-a.e. environment $\boldsymbol{\alpha}$. Hence, provided that $\{\mathsf{X}^N_t,\ t \geq 0\}$ is relatively compact in $\mathcal{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ and that all limit points belong to $C([0,\infty), \mathscr{S}'(\mathbb{R}^d))$, (4.2.34)–(4.2.35) and the uniqueness result in Remark 4.2.3 would then yield a quenched (w.r.t. the environment law $\mathcal{P}$) convergence in probability of finite-dimensional distributions for the empirical density fields.

More specifically, the convergence in (4.2.34) (whose proof is close in spirit to that in all other related works) relies on Chebyshev's inequality and the uniform upper bound (4.1.1) on the environment $\boldsymbol{\alpha}$. This result is established in Section 4.4.2 below. For what concerns (4.2.35), as done in the aforementioned references, the idea is to go through a homogenization result which ensures convergence – in a sense to be made precise – of semigroups for $\mathcal{P}$-a.e. $\boldsymbol{\alpha}$. In particular, provided $\boldsymbol{\alpha}$ is an environment for which the following L^1-convergence

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^{N,\boldsymbol{\alpha}}_{tN^2}G(\tfrac{x}{N}) - \mathcal{S}^{\Sigma}_t G(\tfrac{x}{N})\right|\alpha_x \underset{N\to\infty}{\longrightarrow} 0\,, \qquad t \geq 0\,, \tag{4.2.36}$$

holds for all $G \in \mathscr{S}(\mathbb{R}^d)$, Markov's inequality, the uniform boundedness of the occupation variables $\{\eta(x),\ x \in \mathbb{Z}^d\}$ and (4.2.36) yield

$$\mathbb{P}^{\boldsymbol{\alpha}}_{\nu^{\boldsymbol{\alpha}}_N}\left(\left|\mathsf{X}^N_0(S^{N,\boldsymbol{\alpha}}_{tN^2}G) - \mathsf{X}^N_0(\mathcal{S}^{\Sigma}_t G)\right| > \delta\right) \underset{N\to\infty}{\longrightarrow} 0\,, \qquad t \geq 0\,, \tag{4.2.37}$$

for that same environment $\boldsymbol{\alpha}$ and all test functions $G \in \mathscr{S}(\mathbb{R}^d)$. By combining (4.2.37) – which will hold for $\mathcal{P}$-a.e. $\boldsymbol{\alpha}$ – with the assumption of $\mathcal{P}$-a.s. consistency of initial conditions (see the statement of Theorem 4.2.2 and Definition 4.2.1), we obtain (4.2.35) for $\mathcal{P}$-a.e. $\boldsymbol{\alpha}$. All the details of the proof of (4.2.35) may be found in Proposition 4.4.4 below.

In view of these considerations, the proof of convergence of the finite dimensional distributions of the empirical density fields boils down to show (4.2.34) and (4.2.36). Several methods have been developed in, e.g., [139, 64, 68, 67] to obtain (4.2.36). The road we follow here is to derive (4.2.36) from quenched invariance principle results for random conductance models (RCM) (see, e.g., [23]) in the following two steps:

(i) By viewing our random walks $\mathrm{RW}(\boldsymbol{\alpha})$ as random time changes of suitable RCM, we derive from well-known analogous results in the context of RCM, a *quenched invariance principle* for the random walk $\mathrm{RW}(\boldsymbol{\alpha})$ *started from the origin*.

(ii) By means of the space-time Hölder equicontinuity of the semigroups $\{S^{N,\boldsymbol{\alpha}}_t,\ t \geq 0\}_{N\in\mathbb{N}}$ (see (4.2.33) for its definition), heat kernel upper bounds and building on the ideas in [42, Appendix A.2], we obtain: for $\mathcal{P}$-a.e. realization of the environment $\boldsymbol{\alpha}$,

For all $T > 0$ and $G \in C_0(\mathbb{R}^d)$,

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{Z}^d}\left|S^{N,\boldsymbol{\alpha}}_{tN^2}G(\tfrac{x}{N}) - \mathcal{S}^{\Sigma}_t G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0\,,$$

holds true.

(4.2.38)

Relating the above convergence of Markov semigroups to the weak convergence in path-space of the corresponding Markov processes, it is straightforward to check that (4.2.38) implies the weak convergence of the finite dimensional distributions of RW(α) with *arbitrary starting positions*, i.e., for $\mathcal{P}$-a.e. $\boldsymbol{\alpha}$,

For all $u \in \mathbb{R}^d$ and for any sequence of points $\{x_N\}_{N\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that $\frac{x_N}{N} \to u$ as $N \to \infty$,

$$E^\alpha\left[G_1\left(\frac{X^{\alpha,x_N}_{t_1N^2}}{N}\right)\cdots G_n\left(\frac{X^{\alpha,x_N}_{t_nN^2}}{N}\right)\right] \underset{N\to\infty}{\longrightarrow} \mathtt{E}\left[G_1\left(B^{\Sigma,u}_{t_1}\right)\cdots G_n\left(B^{\Sigma,u}_{t_n}\right)\right]$$

holds true for all $n \in \mathbb{N}$, $0 \le t_1 < \ldots < t_n$ and $G_1, \ldots, G_n \in C_0(\mathbb{R}^d)$, where $\{B^{\Sigma,u}_t := B^\Sigma_t + u,\ t \ge 0\}$ is the Brownian motion introduced in Section 4.2 started from $u \in \mathbb{R}^d$.

Moreover, as a direct consequence of the heat kernel upper bound in Proposition 4.3.7 below, the tightness of the random walks $\{\frac{1}{N}X^{\alpha,x_N}_{tN^2},\ t \ge 0\}_{N\in\mathbb{N}}$ in $\mathcal{D}([0,\infty),\mathbb{R}^d)$ can also be derived (see, e.g., [147, Lemma C.3]). In view of this implication, we will refer to (4.2.38) as the *arbitrary starting point invariance principle*. See also Theorem 4.3.1 below for a slightly more precise statement regarding the convergence in (4.2.38) and Remark 4.3.8 below for a discussion on the equivalence between (4.2.38) and the weak convergence in path-space of the corresponding Markov processes; for a general result on the fact that the convergence in (4.2.38) implies convergence of the corresponding Markov processes we refer the interested reader to [116, Theorem 4.29].

As shown in Corollary 4.3.2 below, the convergence in (4.2.38) implies, in particular, for $\mathcal{P}$-a.e. $\boldsymbol{\alpha}$ and for all $T > 0$ and $G \in \mathscr{S}(\mathbb{R}^d)$,

$$\sup_{t\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left|S^{N,\alpha}_{tN^2} G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_t G(\tfrac{x}{N})\right| \alpha_x \underset{N\to\infty}{\longrightarrow} 0\,. \tag{4.2.39}$$

Note that the above convergence differs from (4.2.36) by the uniformity of the convergence over bounded intervals of times.

The results (4.2.38) and (4.2.39) are stronger than what is strictly needed for the proof of convergence of finite dimensional distributions of the empirical density fields, but they turn out to be very useful in the proof of relative compactness of the probability distributions of

$$\left\{\mathsf{X}^N_t,\ t \in [0,T]\right\}_{N\in\mathbb{N}} \tag{4.2.40}$$

in $\mathcal{D}([0,T],\mathscr{S}'(\mathbb{R}^d))$. Indeed, because the random walk RW(α) semigroups enter in the decomposition of the empirical density fields, it has to be expected that some sort of equicontinuity in time of such semigroups is needed for the sequence (4.2.40) to be tight. This intuition can be made rigorous by means of a combination of the tightness criteria developed in [138, Theorem 4.1] and [147, Appendix B], which apply directly to the empirical density fields decomposed as mild solutions. We refer the reader to Section 4.4.1 below for all the details on the proof of tightness.

4.3 Arbitrary starting point quenched invariance principle

This section is devoted to the proof of a quenched homogenization result for the dual random walk in random environment $\boldsymbol{\alpha}$, RW(α) with generator A_α given in (4.1.5) and corresponding semigroup $\{S^\alpha_t,\ t \ge 0\}$. More precisely, we will prove the following theorem:

Theorem 4.3.1 (Arbitrary starting point quenched invariance principle)**.** *There exists a measurable subset* $\mathfrak{B} \subseteq \{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d}$ *(defined in* (4.3.11) *below) with* $\mathcal{P}(\mathfrak{B}) = 1$ *and such that, for all environments* $\boldsymbol{\alpha} \in \mathfrak{B}$*, for all* $T > 0$ *and* $G \in C_0(\mathbb{R}^d)$*,* (4.2.38) *holds, i.e.,*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{Z}^d} \left|S^{N,\alpha}_{tN^2} G(\tfrac{x}{N}) - \mathcal{S}^\Sigma_t G(\tfrac{x}{N})\right| \underset{N\to\infty}{\longrightarrow} 0\,. \tag{4.3.1}$$

The proof of the above theorem is deferred to Section 4.3.3 below, and goes through the proof of three intermediate results: the quenched invariance principle for the random walk started from the origin (see Proposition 4.3.4 in Section 4.3.1 below), the space-time equicontinuity of the random walk semigroups (see Proposition 4.3.6 in Section 4.3.2 below) and heat kernel upper bounds (see Proposition 4.3.7 in Section 4.3.2 below).

As a consequence of Theorem 4.3.1 above and Lemma 4.2.5, and recalling from there the characterizations of the subsets $\mathfrak{B}$ and $\mathfrak{A} \subseteq \{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d}$, respectively, we obtain:

Corollary 4.3.2. *For all environments $\boldsymbol{\alpha} \in \mathfrak{A} \cap \mathfrak{B}$, for all $T > 0$ and $G \in \mathscr{S}(\mathbb{R}^d)$, (4.2.39) holds, i.e.,*

$$\sup_{t\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left| S^{N,\alpha}_{tN^2} G(\tfrac{x}{N}) - \mathcal{S}^{\Sigma}_t G(\tfrac{x}{N}) \right| \alpha_x \underset{N\to\infty}{\longrightarrow} 0 \,. \tag{4.3.2}$$

The proof of the above corollary – whose main ideas are adapted from [147, Proposition 5.3] – is postponed to Section 4.6 below.

4.3.1 Quenched invariance principle for RW(α) starting from the origin

For all realizations $\boldsymbol{\alpha}$ of the environment, the random walk RW(α), $X^{\alpha,0} = \{X^{\alpha,0}_t,\, t \geq 0\}$ – with generator given in (4.1.5) and with the origin of $\mathbb{Z}^d$ as starting position – can be viewed as a random time change of a specific RCM, i.e., the continuous-time random walk $X^{\omega,0} = \{X^{\omega,0}_t,\, t \geq 0\}$, abbreviated by RW($\boldsymbol{\omega}$) and with law $P^{\boldsymbol{\omega}}$ (and corresponding expectation $E^{\boldsymbol{\omega}}$), starting from the origin of $\mathbb{Z}^d$ and evolving on $\mathbb{Z}^d$ according to the generator given by

$$A^{\omega} f(x) := \sum_{\substack{y\in\mathbb{Z}^d\\ |y-x|=1}} \omega_{xy}\,(f(y) - f(x)) \;, \qquad x \in \mathbb{Z}^d \;, \tag{4.3.3}$$

where $f : \mathbb{Z}^d \to \mathbb{R}$ is a bounded function and

$$\omega_{xy} := \alpha_x \alpha_y \;, \qquad \forall\, x, y \in \mathbb{Z}^d \text{ such that } |x-y| = 1 \,. \tag{4.3.4}$$

Indeed, when in position $x \in \mathbb{Z}^d$, the walk $X^{\alpha,0}$ spends there an exponential holding time with parameter λ^{α}_x given by

$$\lambda^{\alpha}_x = \sum_{\substack{y\in\mathbb{Z}^d\\ |y-x|=1}} \alpha_y \;, \tag{4.3.5}$$

and then jumps to a neighbor of x, say z, with probability $r_{\alpha}(x,z)$ given by

$$r^{\alpha}(x,z) = \frac{\alpha_z}{\lambda^{\alpha}_x} \,. \tag{4.3.6}$$

The corresponding quantities (4.3.5) and (4.3.6) for the walk $X^{\omega,0}$ are given, respectively, by

$$\lambda^{\omega}_x = \sum_{\substack{y\in\mathbb{Z}^d\\ |y-x|=1}} \alpha_x\alpha_y = \alpha_x \lambda^{\alpha}_x \;, \tag{4.3.7}$$

and

$$r^{\omega}(x,z) = \frac{\alpha_x\alpha_z}{\alpha_x\lambda^{\alpha}_x} = r^{\alpha}(x,z) \,. \tag{4.3.8}$$

Hence, if we define the random time change $\{R(t), t \geq 0\}$ by

$$R(t) := \int_0^t \alpha_{X^{\omega,0}_s}\, \mathrm{d}s \;, \tag{4.3.9}$$

then, in law,

$$\{X^{\omega,0}_{R^{-1}(t)},\, t \geq 0\} = \{X^{\alpha,0}_t,\, t \geq 0\} \;,$$

where R^{-1} is the inverse of the continuous piecewise linear and increasing bijection $R : [0,\infty) \to [0,\infty)$.

In what follows, we let Ω denote the space of all conductances $\boldsymbol{\omega}$ with $\omega_{xy} \in \{1, ..., \mathfrak{c}^2\}$ endowed with the Borel σ-algebra induced by the discrete topology. Recall the definition of $\mathcal{P}$ in Assumption 4.1.1. We then let $\mathcal{Q}$ be the probability measure on Ω for which, for all measurable $\mathcal{U} \subseteq \Omega$,

$$\mathcal{Q}(\mathcal{U}) = \mathcal{P}\left(\boldsymbol{\alpha} \in \{1, \ldots, \mathfrak{c}\}^{\mathbb{Z}^d} : \begin{array}{c} \exists\, \boldsymbol{\omega} \in \mathcal{U} \text{ s.t. } \omega_{xy} = \alpha_x\alpha_y \\ \forall\, x, y \in \mathbb{Z}^d \text{ with } |x-y| = 1 \end{array} \right) . \tag{4.3.10}$$

We remark that the measure $\mathcal{Q}$ inherits the invariance and ergodicity under space translations from $\mathcal{P}$ (see Assumption 4.1.1). We then have the following result, taken from [155, Theorem 1.1 and Remark 1.3].

Theorem 4.3.3 (Quenched invariance principle for RW(ω) started from the origin [155])**.** *The quenched invariance principle holds for the random walk* RW(ω) *started from the origin with a limiting non-degenerate covariance matrix Λ, i.e., for Q-a.e. environment ω and for all $T > 0$, the following convergence in law in the Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^d)$ holds*

$$\left\{\frac{X^{\omega,0}_{tN^2}}{N},\, t \in [0,T]\right\} \underset{N\to\infty}{\Longrightarrow} \left\{B^{\Lambda}_t,\, t\in[0,T]\right\} ,$$

where the r.h.s. is a Brownian motion on $\mathbb{R}^d$ starting at the origin with a non-degenerate covariance matrix $\Lambda \in \mathbb{R}^{d\times d}$ independent of the realization of the environment ω.

We remark that [155] and [133] were the first two works in which the quenched invariance principle for RCM with ergodic and uniformly elliptic conductances was proven for any dimension $d \geq 1$. We refer to, e.g., [18, 134, 24, 8, 15] as a partial list for further results in which the uniform ellipticity assumption on the conductances has been replaced by more general conditions on the conductance moments.

In order to get the quenched invariance principle for the random walk RW(α), we only need to check that the random time change defined in (4.3.9) properly rescales. In the proof of the following result, we follow closely Section 6.2 in [?].

Proposition 4.3.4 (Quenched invariance principle for RW(α) started from the origin)**.** *The quenched invariance principle holds for the random walk* RW(α) *started from the origin with a limiting non-degenerate covariance matrix $\Sigma := \frac{1}{\mathbb{E}_{\mathcal{P}}[\alpha_0]}\Lambda$. Here Λ is the covariance matrix appearing in Theorem 4.3.3. In particular, the covariance matrix Σ does not depend on the specific realization of the environment α, but only on the law $\mathcal{P}$.*

Remark 4.3.5. *For later purposes, we define*

$$\mathfrak{B} := \left\{\alpha \in \{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d} : \text{the invariance principle for } \mathrm{RW}(\alpha) \text{ in Proposition 4.3.4 holds}\right\} . \tag{4.3.11}$$

Proof. Consider the random walk $X^{\omega,0}$ starting from the origin and the corresponding process of the environment α as seen from the random walk $X^{\omega,0}$, i.e.,

$$\left\{\tau_{X^{\omega,0}_t}\,\alpha,\; t \geq 0\right\} \subseteq \{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d} . \tag{4.3.12}$$

By our Assumption 4.1.1 and [50, Lemma 4.3], $\mathcal{P}$ is an invariant (actually reversible) and ergodic law for the process in (4.3.12). Hence, recalling the random time change $\{R(t),\, t \geq 0\}$ defined in (4.3.9), Birkhoff's ergodic theorem for the process in (4.3.12) yields, for $\mathcal{P}$-a.e. environment α,

$$\lim_{t\to\infty} \frac{R(t)}{t} = \mathbb{E}_{\mathcal{P}}\left[\alpha_0\right] . \tag{4.3.13}$$

Because $R : [0,\infty) \to [0,\infty)$ is a strictly increasing bijection, (4.3.13) is, in turn, equivalent to

$$\lim_{t\to\infty} \frac{R^{-1}(t)}{t} = \frac{1}{\mathbb{E}_{\mathcal{P}}\left[\alpha_0\right]} . \tag{4.3.14}$$

The conclusion of the theorem follows from the argument in Section 6.2 in [?] as soon as we prove that, for all $t > 0$ and $\epsilon > 0$, we have, for $\mathcal{P}$-a.e. α (recall from (4.3.4) that $\omega = \omega(\alpha)$ with $\omega_{xy} = \alpha_x\alpha_y$ for all $x, y \in \mathbb{Z}^d$ with $|x-y| = 1$),

$$\limsup_{N\to\infty} P^{\omega}\left(\left|\frac{X^{\omega,0}_{R^{-1}(tN^2)} - X^{\omega,0}_{\frac{1}{\mathbb{E}_{\mathcal{P}}[\alpha_0]}tN^2}}{N}\right| > \epsilon\right) = 0 , \tag{4.3.15}$$

where P^{ω} denotes the law of X^{ω}.

We are, thus, left with the proof of (4.3.15). Fix $t > 0$ and $\epsilon > 0$. Then, for all $\delta > 0$, we have

$$P^{\omega}\left(\left|\frac{X^{\omega,0}_{R^{-1}(tN^2)} - X^{\omega,0}_{\frac{1}{\mathbb{E}_{\mathcal{P}}[\alpha_0]}tN^2}}{N}\right| > \epsilon\right)$$

$$\leq P^\omega\left(\left|\frac{X^{\omega,0}_{R^{-1}(tN^2)} - X^{\omega,0}_{\frac{1}{\mathbb{E}_{\mathcal{P}}[\alpha_0]}tN^2}}{N}\right| > \epsilon, \left|\frac{R^{-1}(tN^2)}{N^2} - \frac{t}{\mathbb{E}_{\mathcal{P}}[\alpha_0]}\right| \leq \delta\right)$$
$$+ P^\omega\left(\left|\frac{R^{-1}(tN^2)}{N^2} - \frac{t}{\mathbb{E}_{\mathcal{P}}[\alpha_0]}\right| > \delta\right). \tag{4.3.16}$$

The second term on the r.h.s. of (4.3.16) goes to zero as $N \to \infty$ by (4.3.14), while the first term is bounded above by

$$P^\omega\left(\sup_{\substack{|s-r|\leq\delta\\ r,s\leq T}}\left|\frac{X^{\omega,0}_{sN^2} - X^{\omega,0}_{rN^2}}{N}\right| > \epsilon\right). \tag{4.3.17}$$

for some positive $T = T(t, \mathfrak{c})$ independent of $N \in \mathbb{N}$. Due to Theorem 4.3.3 and the continuity of the trajectories of the limit process, the expression in (4.3.17) vanishes as $N \to \infty$ and $\delta \to 0$, i.e.,

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty} P^\omega\left(\sup_{\substack{|s-r|\leq\delta\\ r,s\leq T}}\left|\frac{X^{\omega,0}_{sN^2} - X^{\omega,0}_{rN^2}}{N}\right| > \epsilon\right) = 0\,. \tag{4.3.18}$$

Indeed, let $\tilde{X}^{\omega,0} = \{\tilde{X}^{\omega,0}_t,\ t \in [0,T]\}$ denote the piecewise linear interpolation of the jump process $X^{\omega,0}$. Then, due to the continuity of the trajectories of the limiting Brownian motion in Theorem 4.3.3, the same theorem holds with $C([0,T],\mathbb{R}^d)$ (the Banach space of continuous functions from $[0,T]$ to $\mathbb{R}^d$ endowed with the supremum norm; see, e.g., [22, Chapter 8]) and $\tilde{X}^{\omega,0}$ in place of $\mathcal{D}([0,T],\mathbb{R}^d)$ and $X^{\omega,0}$, respectively. By Prohorov's theorem (see, e.g., [22, Theorem 6.2]) and the characterization of tightness of probability measures on $C([0,T],\mathbb{R}^d)$ (see, e.g., [22, Theorem 8.2]), we have, for all $\epsilon > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty} P^\omega\left(\sup_{\substack{|s-r|\leq\delta\\ r,s\leq T}}\left|\frac{\tilde{X}^{\omega,0}_{sN^2} - \tilde{X}^{\omega,0}_{rN^2}}{N}\right| > \epsilon\right) = 0\,. \tag{4.3.19}$$

For all $\delta > 0$ and $\epsilon > 0$, $X^{\omega,0}$ being a nearest-neighbor random walk implies that

$$\limsup_{N\to\infty} P^\omega\left(\sup_{\substack{|s-r|\leq\delta\\ r,s\leq T}}\left|\frac{\tilde{X}^{\omega,0}_{sN^2} - \tilde{X}^{\omega,0}_{rN^2}}{N}\right| > \epsilon\right) = \limsup_{N\to\infty} P^\omega\left(\sup_{\substack{|s-r|\leq\delta\\ r,s\leq T}}\left|\frac{X^{\omega,0}_{sN^2} - X^{\omega,0}_{rN^2}}{N}\right| > \epsilon\right)$$

holds true. This and (4.3.19) yield (4.3.18), thus, concluding the proof of the proposition. □

4.3.2 Hölder equicontinuity of the semigroup and heat kernel upper bounds for RW(α)

In this section, $\boldsymbol{\alpha}$ is an arbitrary realization of the environment. We start by proving that the family of semigroups corresponding to the diffusively rescaled random walks RW($\boldsymbol{\alpha}$) are Hölder equicontinuous in both space and time variables. It is well-known (see, e.g., [157, 52] as references in the context of graphs) that Hölder equicontinuity of solutions to parabolic partial differential equations may be derived from parabolic Harnack inequalities (see, e.g., [52, Definition 1.6]). In our context, for all bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$, the parabolic partial difference equation that $S^{\alpha}_{\cdot} f(\cdot) = \left\{S^{\alpha}_t f(x),\ t \geq 0,\ x \in \mathbb{Z}^d\right\}$ solves reads as follows:

$$\alpha_x \frac{\partial}{\partial t}\psi(t,x) = \sum_y \alpha_x\alpha_y(\psi(t,y) - \psi(t,x))\,, \qquad t \geq 0\,, \quad x \in \mathbb{Z}^d\,, \tag{4.3.20}$$

with initial condition $\psi(0,\cdot) = f$. By applying the Moser iteration scheme as in [52, Section 2], we recover the parabolic Harnack inequality ([52, Theorem 2.1]) for positive solutions of (4.3.20). We note that $\boldsymbol{\alpha}$, viewed as a σ-finite measure on $\mathbb{Z}^d$ and due to the assumption of uniform ellipticity, plays the role of speed measure (cf. m in [52, Section 1.1]; see also [9, Remark 1.5] for an analogous discussion).

In conclusion, by applying the aforementioned parabolic Harnack inequality as, e.g., in [52, Proposition 4.1] and [157, Theorem 1.31], we obtain the following result:

Proposition 4.3.6 (Hölder equicontinuity of semigroups)**.** *There exists $C > 0$ and $\gamma > 0$ such that, for all realizations α of the environment, for all $N \in \mathbb{N}$ and for all $G \in C_0(\mathbb{R}^d)$, we have*

$$\left|S^{N,\alpha}_{tN^2}G(\tfrac{x}{N}) - S^{N,\alpha}_{sN^2}G(\tfrac{y}{N})\right| \leq C \sup_{u\in\mathbb{R}^d}|G(u)| \left(\frac{\sqrt{|t-s|} \vee |\frac{x}{N} - \frac{y}{N}|}{\sqrt{t \wedge s}}\right)^{\gamma} \tag{4.3.21}$$

for all $s, t > 0$ and $x, y \in \mathbb{Z}^d$.

The second result is an upper bound for the heat kernel of the random walk RW(α), i.e.,

$$q^{\alpha}_t(x,y) := \frac{1}{\alpha_y} P^{\alpha}(X^{\alpha,x}_t = y) \equiv \frac{p^{\alpha}_t(x,y)}{\alpha_y} \ . \tag{4.3.22}$$

More precisely, we need to ensure that the tails of the heat kernels satisfy a uniform integrability condition. To this aim, many results of heat kernel upper bounds which have been established in the literature, such as Gaussian upper bounds (see, e.g., [13, Theorem 2.3]), would suffice. Here, we follow Nash-Davies' method as in Section 3 in [38] applied to our context. Indeed, by [38, Theorem 3.25], if Nash inequality in [38, Eq. (3.18)] holds true, then there exists a constant $c' > 0$ depending only on the dimension $d \geq 1$ and $\mathfrak{c}$ such that

$$q^{\alpha}_t(x,y) \leq \frac{c'}{1 \vee \sqrt{t^d}} e^{-D(2t;x,y)} \ , \tag{4.3.23}$$

where

$$D(r;x,y) := \sup_{\psi\in\ell^{\infty}(\mathbb{Z}^d,\alpha)} \left(|\psi(x) - \psi(y)| - r\Gamma(\psi)^2\right) \tag{4.3.24}$$

and

$$\Gamma(\psi)^2 := \sup_{x\in\mathbb{Z}^d} \left\{ \sum_{y:|y-x|=1} \frac{\alpha_y}{2} \left(e^{\psi(y)-\psi(x)} - 1\right)^2 \right\} \ , \tag{4.3.25}$$

with the above quantity corresponding to the equation one line above [38, Theorem 3.9]. For what concerns Nash inequality, since $\alpha(x)\alpha(y) \geq 1$ for all $x, y \in \mathbb{Z}^d$, we have, for all $f \in \ell^1(\mathbb{Z}^d, \alpha)$,

$$\begin{aligned}
\mathcal{E}_{\alpha}(f,f) &:= \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{y:|y-x|=1} \alpha(x)\alpha(y) \left(f(y) - f(x)\right)^2 \\
&\geq \frac{1}{2} \sum_{x\in\mathbb{Z}^d} \sum_{y:|y-x|=1} \left(f(y) - f(x)\right)^2 \\
&\geq C \, \|f\|^{2+\frac{4}{d}}_{\ell^2(\mathbb{Z}^d,\nu)} \, \|f\|^{-\frac{4}{d}}_{\ell^1(\mathbb{Z}^d,\lambda)} \ ,
\end{aligned} \tag{4.3.26}$$

where λ is the counting measure on $\mathbb{Z}^d$. Note that for the last inequality above we used Nash inequality for the continuous-time simple random walk (see, e.g., [157, Eq. (1.8)]), with the constant $C > 0$ depending only on the dimension $d \geq 1$. Due to the assumed uniform ellipticity of α (see Assumption 4.1.1), the equivalence of the norms $\|\cdot\|_{\ell^p(\mathbb{Z}^d,\lambda)}$ and $\|\cdot\|_{\ell^p(\mathbb{Z}^d,\alpha)}$ together with (4.3.26) yield

$$\mathcal{E}_{\alpha}(f,f) \geq C \, \mathfrak{c}^{-(1+\frac{2}{d})} \, \|f\|^{2+\frac{4}{d}}_{\ell^2(\mathbb{Z}^d,\alpha)} \, \|f\|^{-\frac{4}{d}}_{\ell^1(\mathbb{Z}^d,\alpha)} \ , \tag{4.3.27}$$

which corresponds to [38, Eq. (3.18)] with $A = \left(C^{-1}\mathfrak{c}^{1+\frac{2}{d}}\right)$, $\nu = d$ and $\delta = 0$. Therefore, we get (4.3.23) by [38, Theorem 3.25] for $\rho = 1$.

Finally, by arguing as in the proof of Lemma 1.9 in [157] and by the uniform ellipticity of α, we obtain the following proposition:

Proposition 4.3.7 (Heat kernel upper bound)**.** *There exists a constant $c > 0$ depending only on $d \geq 1$ and $\mathfrak{c}$ such that, for all environments α, $t > 0$ and $x, y \in \mathbb{Z}^d$, the following upper bound holds:*

$$P^{\alpha}\left(X^{\alpha,x}_t = y\right) \leq \frac{c}{1 \vee \sqrt{t^d}} e^{-\frac{|x-y|}{1\vee\sqrt{t}}} . \tag{4.3.28}$$

4.3.3 Proof of Theorem 4.3.1

Let us conclude the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. First we prove that, for all $\boldsymbol{\alpha} \in \mathfrak{B}$ (see (4.3.11)), for all $t \geq 0$ and $G \in C_0(\mathbb{R}^d)$, we have

$$\sup_{x\in\mathbb{Z}^d} \left| S^{N,\boldsymbol{\alpha}}_{tN^2} G(\tfrac{x}{N}) - \mathcal{S}^{\Sigma}_t G(\tfrac{x}{N}) \right| \underset{N\to\infty}{\longrightarrow} 0 . \tag{4.3.29}$$

We follow the ideas in [42, Appendix A.2]. For all $u \in \mathbb{R}^d$ and $\varepsilon > 0$, let $\mathcal{B}_\varepsilon(u)$ (resp. $\overline{\mathcal{B}_\varepsilon(u)}$) denote the open (resp. closed) Euclidean ball of radius $\varepsilon > 0$ centered in $u \in \mathbb{R}^d$. Moreover, for all $\boldsymbol{\alpha}$, we define

$$\sigma^N_\varepsilon(u) := \inf\left\{ t \geq 0 : \frac{X^{\boldsymbol{\alpha},0}_{tN^2}}{N} \in \mathcal{B}_\varepsilon(u) \right\} \quad \text{and} \quad \sigma_\varepsilon(u) := \inf\left\{ t \geq 0 : B^{\Sigma}_t \in \mathcal{B}_\varepsilon(u) \right\}$$

to be the first hitting times of $\mathcal{B}_\varepsilon(u)$ of the random walks and Brownian motion, respectively. Then, as a consequence of Proposition 4.3.4 (see also Remark 4.3.5) and the strong Markov property of both processes, we have, for all $\boldsymbol{\alpha} \in \mathfrak{B}$, for all $t \geq 0$, $T > 0$ and $G \in C_0(\mathbb{R}^d)$,

$$\sum_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)} \cap \frac{\mathbb{Z}^d}{N}} E^{\boldsymbol{\alpha}}\left[G\left(\frac{X^{\boldsymbol{\alpha},y}_{tN^2}}{N} \right) \right] P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\frac{y}{N} \right) \underset{N\to\infty}{\longrightarrow} \int_{\overline{\mathcal{B}_\varepsilon(u)}} \mathrm{E}\left[G(B^{\Sigma}_t + v) \right] \mathrm{P}_{\varepsilon,u,T}(\mathrm{d}v) , \tag{4.3.30}$$

where

$$P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\tfrac{y}{N} \right) := P^{\boldsymbol{\alpha}}\left(\frac{X^{\boldsymbol{\alpha},0}_{\sigma^N_\varepsilon(u)}}{N} = \frac{y}{N} \middle| \sigma^N_\varepsilon(u) < T \right) \quad \text{and} \quad \mathrm{P}_{\varepsilon,u,T}(\mathrm{d}v) := \mathrm{P}(B^{\Sigma}_{\sigma_\varepsilon(u)} = \mathrm{d}v \,|\, \sigma_\varepsilon(u) < T) .$$

Let $\{x_N\}_{N\in\mathbb{N}} \subseteq \mathbb{Z}^d$ be such that $\frac{x_N}{N} \to u$ as $N \to \infty$. Then, by the triangle inequality, we have, for all $\varepsilon > 0$,

$$\begin{aligned}
&\left| S^{N,\boldsymbol{\alpha}}_{tN^2} G(\tfrac{x_N}{N}) - \mathcal{S}^{\Sigma}_t (\tfrac{x_N}{N}) \right| \\
&\leq \left| S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{x_N}{N}\right) - \sum_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)} \cap \frac{\mathbb{Z}^d}{N}} E^{\boldsymbol{\alpha}}\left[G\left(\frac{X^{\boldsymbol{\alpha},y}_{tN^2}}{N} \right) \right] P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\frac{y}{N} \right) \right| \\
&+ \left| \sum_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)} \cap \frac{\mathbb{Z}^d}{N}} E^{\boldsymbol{\alpha}}\left[G\left(\frac{X^{\boldsymbol{\alpha},y}_{tN^2}}{N} \right) \right] P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\frac{y}{N} \right) - \int_{\overline{\mathcal{B}_\varepsilon(u)}} \mathrm{E}\left[G(B^{\Sigma}_t + v) \right] \mathrm{P}_{\varepsilon,u,T}(\mathrm{d}v) \right| \\
&+ \left| \int_{\overline{\mathcal{B}_\varepsilon(u)}} \mathrm{E}\left[G(B^{\Sigma}_t + v) \right] \mathrm{P}_{\varepsilon,u,T}(\mathrm{d}v) - \mathcal{S}^{\Sigma}_t G(\tfrac{x_N}{N}) \right| .
\end{aligned} \tag{4.3.31}$$

As for the first term on the r.h.s. above, for all environments $\boldsymbol{\alpha} \in \mathfrak{B}$, we have, by Hölder's inequality,

$$\begin{aligned}
&\left| S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{x_N}{N}\right) - \sum_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)} \cap \frac{\mathbb{Z}^d}{N}} E^{\boldsymbol{\alpha}}\left[G\left(\frac{X^{\boldsymbol{\alpha},y}_{tN^2}}{N} \right) \right] P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\frac{y}{N} \right) \right| \\
&\leq \sum_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)} \cap \frac{\mathbb{Z}^d}{N}} \left| S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{x_N}{N}\right) - S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{y}{N}\right) \right| P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\frac{y}{N} \right) \leq \sup_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)}} \left| S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{x_N}{N}\right) - S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{y}{N}\right) \right| .
\end{aligned}$$

The above upper bound, since $\frac{x_N}{N} \to u$ as $N \to \infty$, yields, by Proposition 4.3.6 and the uniform boundedness of the function $G \in C_0(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \left| S^{N,\boldsymbol{\alpha}}_{tN^2} G\left(\tfrac{x_N}{N}\right) - \sum_{\frac{y}{N} \in \overline{\mathcal{B}_\varepsilon(u)} \cap \frac{\mathbb{Z}^d}{N}} E^{\boldsymbol{\alpha}}\left[G\left(\frac{X^{\boldsymbol{\alpha},y}_{tN^2}}{N} \right) \right] P^{\boldsymbol{\alpha}}_{\varepsilon,u,T}\left(\frac{y}{N} \right) \right| = 0 \tag{4.3.32}$$

for all environments $\boldsymbol{\alpha} \in \mathfrak{B}$ and $t \geq 0$. A similar argument employing the uniform continuity of $G \in C_0(\mathbb{R}^d)$ and the translation invariance of the law of Brownian motion ensures

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \left| \int_{\overline{\mathcal{B}_\varepsilon(u)}} \mathtt{E}\left[G(B_t^\Sigma + v)\right] \mathtt{P}_{\varepsilon,u,T}(\mathrm{d}v) - \mathcal{S}_t^\Sigma G(\tfrac{x_N}{N}) \right| = 0 \tag{4.3.33}$$

for all $t \geq 0$. By combining (4.3.30)–(4.3.33), we obtain, for all $\boldsymbol{\alpha} \in \mathfrak{B}$, for all $G \in C_0(\mathbb{R}^d)$, $t \geq 0$, $u \in \mathbb{R}^d$ and approximating points $\frac{x_N}{N} \to u$,

$$\left| S_{tN^2}^{N,\alpha} G(\tfrac{x_N}{N}) - \mathcal{S}_t^\Sigma G(\tfrac{x_N}{N}) \right| \underset{N \to \infty}{\longrightarrow} 0 \,. \tag{4.3.34}$$

In order to go from pointwise (i.e., (4.3.34)) to uniform convergence over points in $\mathbb{Z}^d$ (i.e., (4.3.29)), we crucially use the heat kernel upper bound in Proposition 4.3.7 and the Hölder equicontinuity in Proposition 4.3.6. First, note that proving (4.3.29) for continuous and compactly supported functions $G \in C_c(\mathbb{R}^d)$ suffices, due to the density of $C_c(\mathbb{R}^d)$ in $C_0(\mathbb{R}^d)$ and the contractivity of the semigroups $\{S_{tN^2}^{N,\alpha}, t \geq 0\}$ and $\{\mathcal{S}_t^\Sigma, t \geq 0\}$ w.r.t. the supremum norms on $\frac{\mathbb{Z}^d}{N}$ and $\mathbb{R}^d$, respectively. Hence, for all $G \in C_c(\mathbb{R}^d)$ and for all compact sets $\mathcal{K} \subseteq \mathbb{R}^d$, (4.3.34), Proposition 4.3.6 and the uniform continuity of $\mathcal{S}_t^\Sigma G \in C_0(\mathbb{R}^d)$ imply, for all $\boldsymbol{\alpha} \in \mathfrak{B}$,

$$\sup_{\frac{x}{N} \in \mathcal{K} \cap \frac{\mathbb{Z}^d}{N}} \left| S_{tN^2}^{N,\alpha} G(\tfrac{x}{N}) - \mathcal{S}_t^\Sigma G(\tfrac{x}{N}) \right| \underset{N \to \infty}{\longrightarrow} 0 \,. \tag{4.3.35}$$

Letting $\mathrm{supp}(G) \subseteq \mathbb{R}^d$ denote the compact support of $G \in C_c(\mathbb{R}^d)$, we have

$$\begin{aligned} &\sup_{\frac{x}{N} \in \mathcal{K}^c \cap \frac{\mathbb{Z}^d}{N}} \left| S_{tN^2}^{N,\alpha} G(\tfrac{x}{N}) - \mathcal{S}_t^\Sigma G(\tfrac{x}{N}) \right| \\ &\qquad \leq \sup_{u \in \mathbb{R}^d} |G(u)| \sup_{\frac{x}{N} \in \mathcal{K}^c \cap \frac{\mathbb{Z}^d}{N}} \left\{ P^\alpha \left(\frac{X_{tN^2}^{\alpha,x}}{N} \in \mathrm{supp}(G) \right) + \mathtt{P}(B_t^\Sigma + x \in \mathrm{supp}(G)) \right\} . \end{aligned} \tag{4.3.36}$$

Thus, by the heat kernel upper bounds for RW($\boldsymbol{\alpha}$) (Proposition 4.3.7) and analogous bounds for the non-degenerate Brownian motion $\{B_t^\Sigma, t \geq 0\}$, we can choose $\mathcal{K} \subseteq \mathbb{R}^d$ such that the r.h.s. in (4.3.36) is arbitrarily small. This yields (4.3.29) for all $\boldsymbol{\alpha} \in \mathfrak{B}$.

To go from (4.3.29) to (4.3.1) in which the convergence is uniform over bounded intervals of time, we apply [62, Chapter 1, Theorem 6.1]. Indeed, for all realizations of the environment $\boldsymbol{\alpha}$, the semigroups $\{S_t^{N,\alpha},\ t \geq 0\}_{N \in \mathbb{N}}$ and $\{\mathcal{S}_t^\Sigma,\ t \geq 0\}$ are strongly continuous contraction semigroups in the Banach spaces $\{C_0(\frac{\mathbb{Z}^d}{N})\}_{N \in \mathbb{N}}$ and $C_0(\mathbb{R}^d)$ (endowed with the corresponding supremum norms), respectively; moreover, the projections $\pi_N : C_0(\mathbb{R}^d) \to C_0(\frac{\mathbb{Z}^d}{N})$ given by $\pi_N G(\frac{x}{N}) := G(\frac{x}{N})$ are linear and such that $\sup_{N \in \mathbb{N}} \|\pi_N\|_N = 1 < \infty$, with $\|\pi_N\|_N$ denoting the operator norm of π_N. □

Remark 4.3.8 (Equivalent formulations of the arbitrary starting point quenched invariance principle)**.** *If one assumes, for a given realization of the environment $\boldsymbol{\alpha}$, the invariance principle for the random walk* RW($\boldsymbol{\alpha}$) *with arbitrary starting positions, i.e.,*

> *For all $T > 0$, for any macroscopic point $u \in \mathbb{R}^d$ and for any sequence of points $\{x_N\}_{N \in \mathbb{N}} \subseteq \mathbb{Z}^d$ such that $\frac{x_N}{N} \to u$ as $N \to \infty$, the laws of $\{\frac{1}{N} X_{tN^2}^{\alpha,x_N},\ t \in [0,T]\}_{N \in \mathbb{N}}$, the diffusively rescaled* RW($\boldsymbol{\alpha}$) *started from $\frac{x_N}{N}$, converge weakly to the law of $\{B_t^{\Sigma,u} := B_t^\Sigma + u,\ t \in [0,T]\}$, the Brownian motion started from $u \in \mathbb{R}^d$ and with a non-degenerate covariance matrix Σ independent of the realization of the environment $\boldsymbol{\alpha}$* (4.3.37)

then (4.3.34) *follows immediately by the uniform continuity of $\mathcal{S}_t^\Sigma G \in C_0(\mathbb{R}^d)$. By the same argument used in the final part of the proof of Theorem 4.3.1 above (i.e., the part of the proof immediately after* (4.3.34) *involving the heat kernel upper bound in Proposition 4.3.7 and the Hölder equicontinuity in Proposition 4.3.6) one gets the convergence in* (4.2.38)*. Therefore, in view of the discussion just after* (4.2.38)*, we obtain that, under Assumption 4.1.1,* (4.3.37) *and* (4.2.38) *are equivalent.*

Remark 4.3.9 (Quenched local CLT)**.** *As already mentioned,* (4.3.1)*, namely the arbitrary starting point quenched invariance principle for the diffusively rescaled random walks* RW($\boldsymbol{\alpha}$)*, is stronger than the quenched invariance principle for* RW($\boldsymbol{\alpha}$) *starting from the origin. Another well-known strengthening of the quenched invariance principle is the* quenched local central limit theorem *(see, e.g., [**?** , Theorem 1.11 and Remark 1.12], which applies to* *our context) for* RW($\boldsymbol{\alpha}$)*: if we denote by k_t^Σ the heat kernel of the Brownian motion started at the origin, it holds that, for $\mathcal{P}$-a.e. environment $\boldsymbol{\alpha}$ and for any ℓ, $T > 0$ and $\delta > 0$,*

$$\lim_{N\to\infty} \sup_{\left|\frac{y}{N}\right|<\ell} \sup_{t\in[\delta,T]} \left| N^d P^{\boldsymbol{\alpha}}\left(\frac{X^{\boldsymbol{\alpha},0}_{tN^2}}{N} = \frac{y}{N}\right) - k_t^\Sigma(\tfrac{y}{N})\right| = 0 \,. \tag{4.3.38}$$

The proof of (4.3.38) *resembles that of Theorem 4.3.1 and, thus, one may wonder whether* (4.3.38) *directly yields* (4.3.1)*. However,* (4.3.38) *does not seem to be of help when proving* (4.3.29)*, being the supremum over space in the arrival point and not in the starting point – fixed to be the origin – and being the supremum over time only on bounded intervals away from $t = 0$.*

4.4 Proof of the hydrodynamic limit

In this section we present the proof of Theorem 4.2.2, which consists of two steps: ensuring tightness of the empirical density fields and establishing convergence of their finite dimensional distributions to the unique solution of (4.2.7). In both steps, we use the following representation for the renormalized occupation variables: for all realizations of the environment $\boldsymbol{\alpha}$, there exists a probability space such that a.s., for all initial configurations $\eta \in \mathcal{X}^{\boldsymbol{\alpha}}$, for all $x \in \mathbb{Z}^d$ and $t \geq 0$,

$$\tfrac{\eta_t(x)}{\alpha_x} = S_t^{\boldsymbol{\alpha}}(\tfrac{\eta(\cdot)}{\alpha_\cdot})(x) + \int_0^t S^{\boldsymbol{\alpha}}_{t-s} \mathrm{d}M_s^{\boldsymbol{\alpha}}(x) \,, \tag{4.4.1}$$

where $\{M_\cdot^{\boldsymbol{\alpha}}(x),\ x \in \mathbb{Z}^d\}$ is a family of square integrable martingales w.r.t. the natural filtration of SEP($\boldsymbol{\alpha}$) (see also (4.2.29)–(4.2.30)), whose predictable quadratic covariations are given by

$$\langle M^{\boldsymbol{\alpha}}(x), M^{\boldsymbol{\alpha}}(y)\rangle_t = -\mathbf{1}_{\{|x-y|=1\}} \int_0^t \alpha_x\alpha_y \left(\tfrac{\eta_s(x)}{\alpha_x} + \tfrac{\eta_s(y)}{\alpha_y} - 2\tfrac{\eta_s(x)}{\alpha_x}\tfrac{\eta_s(y)}{\alpha_y}\right) \mathrm{d}s \tag{4.4.2}$$

for $x, y \in \mathbb{Z}^d$ with $x \neq y$, and

$$\langle M^{\boldsymbol{\alpha}}(x), M^{\boldsymbol{\alpha}}(x)\rangle_t = -\sum_{\substack{y\in\mathbb{Z}^d\\ |y-x|=1}} \langle M^{\boldsymbol{\alpha}}(x), M^{\boldsymbol{\alpha}}(y)\rangle_t \tag{4.4.3}$$

for $x \in \mathbb{Z}^d$. The identity in (4.4.1) expresses the solution of the following infinite system of stochastic differential equations (cf. (4.2.22)–(4.2.23))

$$\begin{cases} \mathrm{d}(\tfrac{\eta_t(\cdot)}{\alpha_\cdot})(x) &= A^{\boldsymbol{\alpha}}(\tfrac{\eta_{t^-}(\cdot)}{\alpha_\cdot})(x)\,\mathrm{d}t + \mathrm{d}M_t^{\boldsymbol{\alpha}}(x)\,, \quad x \in \mathbb{Z}^d\,,\ t \geq 0 \\ \qquad \tfrac{\eta_0(x)}{\alpha_x} &= \tfrac{\eta(x)}{\alpha_x}\,, \qquad\qquad\qquad\qquad\qquad x \in \mathbb{Z}^d\,, \end{cases}$$

as a mild solution (see, e.g., [145, Chapter 6, Section 1]). The rigorous proof of the identity in (4.4.1) – in which the r.h.s. contains infinite summations – is postponed to Section 4.5 below. The idea of the proof is to first provide a so-called "ladder representation" for SEP($\boldsymbol{\alpha}$) in terms of a symmetric exclusion process which allows at most one particle per site; then obtain a mild solution representation analogous to the one in (4.4.1) for such "ladder" exclusion process as done in, e.g., [139, 64, 147]. The same strategy can be applied to rigorously verify the identities in (4.4.2)–(4.4.3). We refer to Section 4.5 for further details.

4.4.1 Tightness

In the proof of tightness for the empirical density fields we employ the uniform convergence over time of the semigroups established in Theorem 4.3.1 and Corollary 4.3.2. Tightness in quenched random environment, which by Mitoma's tightness criterion [138] follows from tightness of the following real-valued processes

$$\left\{\mathsf{X}_t^N(G),\ t \in [0,T]\right\}_{N\in\mathbb{N}}\,, \qquad \forall\, G \in \mathscr{S}(\mathbb{R}^d)\,, \tag{4.4.4}$$

has been established via the strategy of employing corrected empirical density fields ([99, 87, 68, 66] and [98]). In what follows, we opt for a different strategy by applying the tightness criterion developed in [147, Appendix B], which, for convenience of the reader, we report below.

Theorem 4.4.1 (Tightness criterion [147, Theorem B.4]). *For a fixed $T > 0$, let $\left\{\mathsf{Z}_t^N,\ t \in [0,T]\right\}_{N\in\mathbb{N}}$ be a family of real-valued stochastic processes with laws $\{\mathscr{P}^N\}_{N\in\mathbb{N}}$. Then, this family is tight in the Skorokhod space $\mathcal{D}([0,T],\mathbb{R})$ if the following conditions hold:*

(T1) *For all t in a dense subset of $[0,T]$ which includes T,*

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\mathscr{P}^N\left(\left|\mathsf{Z}_t^N\right| > \ell\right) = 0\ .$$

(T2) *For all $\varepsilon > 0$, there exists $h_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that, for all $N \geq N_\varepsilon$, there exist deterministic functions $\psi_\varepsilon^N, \psi_\varepsilon : [0,h_\varepsilon] \to [0,\infty)$ and non-negative values ϕ_ε^N satisfying the following properties:*

(i) *The functions ψ_ε^N are non-decreasing.*

(ii) *For all $h \in [0,h_\varepsilon]$ and $t \in [0,T]$, we have*

$$\mathscr{P}^N\left(\left|\mathsf{Z}_{t+h}^N - \mathsf{Z}_t^N\right| > \varepsilon\middle|\mathscr{F}_t^N\right) \leq \psi_\varepsilon^N(h)\ , \quad \text{a.s.}\ ,$$

where $\left\{\mathscr{F}_t^N, t \geq 0\right\}$ denotes the natural filtration of $\left\{\mathsf{Z}_t^N, t \geq 0\right\}$.

(iii) *For all $h \in [0,h_\varepsilon]$, we have $\psi_\varepsilon^N(h) \leq \psi_\varepsilon(h) + \phi_\varepsilon^N$.*

(iv) *$\phi_\varepsilon^N \to 0$ as $N \to \infty$.*

(v) *$\psi_\varepsilon(h) \to 0$ as $h \to 0$.*

As we show in the proof of Proposition 4.4.2 below, this criterion, the semigroup convergence in Theorem 4.3.1 and the following mild solution representation of the empirical density fields (see also (4.2.31))

$$\mathsf{X}_{t+h}^N(G) = \mathsf{X}_t^N(S_{hN^2}^{N,\alpha}G) + \int_{tN^2}^{(t+h)N^2} \mathrm{d}\mathsf{M}_s^N(S_{(t+h)N^2-s}^{N,\alpha}G)\ , \quad t,h > 0\ ,\ G \in \mathscr{S}(\mathbb{R}^d)\ , \tag{4.4.5}$$

yield tightness directly for the processes in (4.4.4).

Proposition 4.4.2 (Tightness). *For all environments $\boldsymbol{\alpha} \in \mathfrak{A} \cap \mathfrak{B}$ (see (4.2.16) and (4.3.11)) and for all $T > 0$, the sequence*

$$\left\{\mathsf{X}_t^N,\ t \in [0,T]\right\}_{N\in\mathbb{N}}$$

is tight in $\mathcal{D}([0,T],\mathscr{S}'(\mathbb{R}^d))$. As a consequence, $\{\mathsf{X}_t^N,\ t \geq 0\}_{N\in\mathbb{N}}$ is tight in $\mathcal{D}([0,\infty),\mathscr{S}'(\mathbb{R}^d))$.

Proof. All throughout the proof, we fix $\boldsymbol{\alpha} \in \mathfrak{A} \cap \mathfrak{B}$. As mentioned above, it suffices to show that conditions (T1) and (T2) in Theorem 4.4.1 hold for

$$\left\{\mathsf{Z}_t^N,\ t \in [0,T]\right\}_{N\in\mathbb{N}} = \left\{\mathsf{X}_t^N(G),\ t \in [0,T]\right\}_{N\in\mathbb{N}}\ , \tag{4.4.6}$$

for all $G \in \mathscr{S}(\mathbb{R}^d)$. Because (T1) is a consequence of Proposition 4.4.4 below, it suffices to show (T2). To this purpose, we fix $G \in \mathscr{S}(\mathbb{R}^d)$ and set, for all $\varepsilon > 0$, $h \geq 0$ and $N \in \mathbb{N}$,

$$\psi_\varepsilon^N(h) := \frac{C}{\varepsilon^2}\sup_{h'\in[0,h]}\sup_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N}) - S_{h'N^2}^{N,\alpha}G(\tfrac{x}{N})\right| \tag{4.4.7}$$

$$\psi_\varepsilon(h) := \frac{C}{\varepsilon^2}\sup_{h'\in[0,h]}\sup_{u\in\mathbb{R}^d}\left|G(u) - \mathcal{S}_{h'}^{\Sigma}G(u)\right| \tag{4.4.8}$$

and

$$\phi_\varepsilon^N := \frac{C}{\varepsilon^2}\sup_{t\in[0,T]}\sup_{x\in\mathbb{Z}^d}\left|S_{tN^2}^{N,\alpha}G(\tfrac{x}{N}) - \mathcal{S}_t^{\Sigma}G(\tfrac{x}{N})\right|\ , \tag{4.4.9}$$

where

$$C := \sup_{N\in\mathbb{N}}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N})\right|\alpha_x \tag{4.4.10}$$

is a constant independent of $N \in \mathbb{N}$ and, since $\boldsymbol{\alpha} \in \mathfrak{A}$ (see (4.2.16)), finite. As a consequence of the triangle inequality, Theorem 4.3.1 and the continuity of $h \in [0,\infty) \mapsto \mathcal{S}_h^{\Sigma}G \in C_0(\mathbb{R}^d)$, the functions in (4.4.7)–(4.4.9) satisfy the conditions in items (i), (ii), (iv) and (v) of the tightness criterion in Theorem 4.4.1. In the remainder of the proof, we verify also the remaining condition (iii) in that theorem.

By (4.4.5) and the triangle inequality, we have, for all $t, h \geq 0$ and $N \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_{t+h}(G) - \mathsf{X}^N_t(G)\right| > \varepsilon \middle| \mathcal{F}^N_t\right) &\leq \mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_t(S^{N,\alpha}_{hN^2}G - G)\right| > \frac{\varepsilon}{2}\middle|\mathcal{F}^N_t\right) \\ &\quad + \mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\int_{tN^2}^{(t+h)N^2} \mathrm{d}\mathsf{M}^N_s(S^{N,\alpha}_{(t+h)N^2-s}G)\right| > \frac{\varepsilon}{2}\middle|\mathcal{F}^N_t\right) ,\end{aligned} \tag{4.4.11}$$

with $\mathcal{F}^N_t := \sigma\{\mathsf{X}^N_s,\ s \leq t\}$. The boundedness of the occupation variables of SEP(α), the convergence in (4.3.2) and the continuity of $h \in [0,\infty) \mapsto \mathcal{S}^{\Sigma}_h G \in C_0(\mathbb{R}^d)$ allows us to choose $h_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that the first term on the r.h.s. in (4.4.11) equals zero for all $h \in [0, h_\varepsilon]$, $N \geq N_\varepsilon$ and $t \geq 0$, i.e.,

$$\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_t(S^{N,\alpha}_{hN^2}G - G)\right| > \frac{\varepsilon}{2}\middle|\mathcal{F}^N_t\right) = 0 , \qquad h \in [0, h_\varepsilon] ,\ N \geq N_\varepsilon . \tag{4.4.12}$$

As for the second term on the r.h.s. in (4.4.11), by Chebyshev's inequality and the first inequality in (4.4.14) below, we obtain, for all $h \in [0, h_\varepsilon]$, $N \geq N_\varepsilon$ and $t \geq 0$,

$$\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\int_{tN^2}^{(t+h)N^2} \mathrm{d}\mathsf{M}^N_s(S^{N,\alpha}_{(t+h)N^2-s}G)\right| > \frac{\varepsilon}{2}\middle|\mathcal{F}^N_t\right) \leq \psi^N_\varepsilon(h) , \qquad \text{a.s.} . \tag{4.4.13}$$

By combining (4.4.11)–(4.4.13), condition (iii) in Theorem 4.4.1 holds true for the process (4.4.6), thus yielding the desired result. □

4.4.2 Convergence of finite dimensional distributions

In the following proposition – which is an adaptation of, e.g., [139, Lemma 12], [64, Lemma 3.1], [147, Lemma 5.1] – we prove (4.2.34). To this purpose, recall the definitions of $\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}$, $\mathbb{P}^{\alpha}_{\eta}$, $\mathbb{E}^{\alpha}_{\nu^{\alpha}_N}$ and $\mathbb{E}^{\alpha}_{\eta}$ at the beginning of Section 4.2 (below (4.2.1)).

Lemma 4.4.3. *For any given realization of the environment $\boldsymbol{\alpha}$, for all $N \in \mathbb{N}$, $G \in \mathscr{S}(\mathbb{R}^d)$, $\eta \in \mathcal{X}^{\alpha}$ and $t \geq 0$, we have*

$$\begin{aligned}&\mathbb{E}^{\alpha}_{\eta}\left[\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\alpha_x\int_0^{tN^2} S^{N,\alpha}_{tN^2-s}G(\tfrac{x}{N})\,\mathrm{d}M^{\alpha}_s(x)\right)^2\right] \\ &\qquad\leq \frac{1}{2N^d}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(\left|G(\tfrac{x}{N})\right|^2 - \left|S^{N,\alpha}_{tN^2}G(\tfrac{x}{N})\right|^2\right)\alpha_x \leq \frac{1}{2N^d}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N})\right|^2\alpha_x .\end{aligned} \tag{4.4.14}$$

As a consequence of (4.4.14) *and the uniformity of the upper bound w.r.t. $\eta \in \mathcal{X}^{\alpha}$, we further get*

$$\mathbb{E}^{\alpha}_{\nu^{\alpha}_N}\left[\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\alpha_x\int_0^{tN^2} S^{N,\alpha}_{tN^2-s}G(\tfrac{x}{N})\,\mathrm{d}M^{\alpha}_s(x)\right)^2\right] \underset{N\to\infty}{\longrightarrow} 0 , \tag{4.4.15}$$

where $\{\nu^{\alpha}_N\}_{N\in\mathbb{N}}$ is the sequence of probability measures on $\mathcal{X}^{\alpha}$ given in Theorem 4.2.2.

Proof. A simple computation employing the explicit form of the predictable quadratic covariations of the martingales (4.4.2)–(4.4.3) yields

$$\begin{aligned}&\mathbb{E}^{\alpha}_{\eta}\left[\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\alpha_x\int_0^{tN^2} S^{N,\alpha}_{tN^2-s}G(\tfrac{x}{N})\,\mathrm{d}M^{\alpha}_s(x)\right)^2\right] \\ &= \int_0^{tN^2}\frac{1}{N^{2d}}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}}\left(S^{N,\alpha}_{tN^2-s}G(\tfrac{x}{N}) - S^{N,\alpha}_{tN^2-s}G(\tfrac{y}{N})\right)^2\alpha_x\alpha_y\,\mathbb{E}^{\alpha}_{\eta}\left[\left(\tfrac{\eta_s(x)}{\alpha_x} + \tfrac{\eta_s(y)}{\alpha_y} - 2\tfrac{\eta_s(x)}{\alpha_x}\tfrac{\eta_s(y)}{\alpha_y}\right)\right]\mathrm{d}s .\end{aligned}$$

Because a.s. $0 \le \left(\frac{\eta_s(x)}{\alpha_x} + \frac{\eta_s(y)}{\alpha_y} - 2\frac{\eta_s(x)}{\alpha_x}\frac{\eta_s(y)}{\alpha_y}\right) \le 2$, we further get

$$\begin{aligned}
&\mathbb{E}^{\alpha}_{\eta}\left[\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\alpha_x \int_0^{tN^2} S^{N,\alpha}_{tN^2-s} G(\tfrac{x}{N})\,\mathrm{d}M^{\alpha}_s(x)\right)^2\right] \\
&\le \frac{1}{N^d}\int_0^{tN^2} \frac{1}{N^d}\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}} \alpha_x\alpha_y \left(S^{N,\alpha}_{tN^2-s}G(\tfrac{x}{N}) - S^{N,\alpha}_{tN^2-s}G(\tfrac{y}{N})\right)^2 \mathrm{d}s \\
&= \frac{1}{N^d}\int_0^{tN^2} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^{N,\alpha}_{tN^2-s}G(\tfrac{x}{N})\right|^2\alpha_x\right)\mathrm{d}s \\
&= \frac{1}{N^d}\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left(\left|G(\tfrac{x}{N})\right|^2 - \left|S^{N,\alpha}_{tN^2}G(\tfrac{x}{N})\right|^2\right)\alpha_x\right) \le \frac{1}{N^d}\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N})\right|^2\alpha_x\right).
\end{aligned}$$

In view of Lemma 4.2.5, $\limsup_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|G(\tfrac{x}{N})\right|^2\alpha_x < \infty$, thus, concluding the proof. □

Since, with probability one, only one particle jumps at the time, for all environments α, and for all $T > 0$ and $G \in \mathscr{S}(\mathbb{R}^d)$, we have

$$\mathbb{E}^{\alpha}_{\nu^{\alpha}_N}\left[\sup_{t\in[0,T]}\left|\mathsf{X}^N_{t^+}(G) - \mathsf{X}^N_{t^-}(G)\right|\right] \le \frac{2\sup_{u\in\mathbb{R}^d}|G(u)|}{N^d} \underset{N\to\infty}{\longrightarrow} 0\,. \tag{4.4.16}$$

By combining this with the relative compactness of $\{\mathsf{X}^N_t,\ t\ge 0\}$ in $\mathcal{D}([0,T],\mathscr{S}'(\mathbb{R}^d))$ (see Proposition 4.4.2), we obtain that all limit points of $\{\mathsf{X}^N_t,\ t\ge 0\}$ belong to $C([0,T],\ \mathscr{S}'(\mathbb{R}^d))$. Hence, Remark 4.2.3 and the following proposition conclude the proof of Theorem 4.2.2.

Proposition 4.4.4. *Recall the definitions* (4.2.16)*,* (4.3.11) *and* (4.2.4)*, and fix* $\alpha \in \mathfrak{A}\cap\mathfrak{B}\cap\mathfrak{C}$*. Then, for all* $\delta > 0$*,* $t \ge 0$ *and* $G \in \mathscr{S}(\mathbb{R}^d)$*, we have*

$$\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_t(G) - \pi^{\Sigma}_t(G)\right| > \delta\right) \underset{N\to\infty}{\longrightarrow} 0\,, \tag{4.4.17}$$

where $\{\pi^{\Sigma}_t,\ t\ge 0\}$ *is given in* (4.2.6).

Proof. Due to the uniform boundedness of the environments α (Assumption 4.1.1) and the decomposition (4.2.31) of the empirical density fields, we obtain, for all $\delta > 0$,

$$\begin{aligned}
&\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_t(G) - \pi^{\Sigma}_t(G)\right| > \delta\right) \\
&\qquad \le \mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_0(S^{N,\alpha}_{tN^2}G) - \pi^{\Sigma}_t(G)\right| > \frac{\delta}{2}\right) + \mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\int_0^{tN^2}\mathrm{d}\mathsf{M}^N_s(S^{N,\alpha}_{tN^2-s}G)\right| > \frac{\delta}{2}\right).
\end{aligned} \tag{4.4.18}$$

Hence, by Chebychev's inequality and Lemma 4.4.3, the second term on the r.h.s. in (4.4.18) vanishes as $N\to\infty$. Concerning the first term on the r.h.s. in (4.4.18), in view of $\pi^{\Sigma}_t(G) = \pi^{\bar\rho}(\mathcal{S}^{\Sigma}_t G)$ (see Remark 4.2.3), we proceed as follows:

$$\begin{aligned}
\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_0(S^{N,\alpha}_{tN^2}G) - \pi^{\Sigma}_t(G)\right| > \frac{\delta}{2}\right) &\le \mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_0(S^{N,\alpha}_{tN^2}G) - \mathsf{X}^N_0(\mathcal{S}^{\Sigma}_t G)\right| > \frac{\delta}{4}\right) \\
&\quad + \mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_0(\mathcal{S}^{\Sigma}_t G) - \pi^{\bar\rho}(\mathcal{S}^{\Sigma}_t G)\right| > \frac{\delta}{4}\right).
\end{aligned} \tag{4.4.19}$$

For the first term on the r.h.s. in (4.4.19), by Markov's inequality and the uniform boundedness of the occupation variables $\{\eta(x),\ x\in\mathbb{Z}^d\}$, we obtain

$$\mathbb{P}^{\alpha}_{\nu^{\alpha}_N}\left(\left|\mathsf{X}^N_0(S^{N,\alpha}_{tN^2}G) - \mathsf{X}^N_0(\mathcal{S}^{\Sigma}_t G)\right| > \frac{\delta}{4}\right) \le \frac{4}{\delta}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^{N,\alpha}_{tN^2}G(\tfrac{x}{N}) - \mathcal{S}^{\Sigma}_t G(\tfrac{x}{N})\right|\alpha_x\,.$$

In turn, this latter upper bound vanishes for all environments $\alpha\in\mathfrak{A}\cap\mathfrak{B}$, for all $G\in\mathscr{S}(\mathbb{R}^d)$ and $t\ge 0$, in view of Corollary 4.3.2. The second term on the r.h.s. in (4.4.19) vanishes because $\mathscr{S}(\mathbb{R}^d)$ is invariant under the action of the Brownian motion semigroup and because of the assumed consistency of the initial conditions (see Definition 4.2.1) for $\alpha\in\mathfrak{C}$ (see (4.2.4)). This concludes the proof. □

4.5 Mild solution and ladder construction

In this section we derive the mild solution representation for SEP($\boldsymbol{\alpha}$). More in detail, we start from a so-called $\boldsymbol{\alpha}$-ladder symmetric exclusion process (see, e.g., [84]), we obtain the a.s. mild solution representation as in e.g. [64, Section 3] and [147, Proposition 4.1] for this ladder counterpart and, then, by means of a projection which preserves the Markov property, we derive an a.s. mild solution representation for SEP($\boldsymbol{\alpha}$).

Let us fix a realization of the environment $\boldsymbol{\alpha}$ satisfying Assumption 4.1.1. Then, we define

$$\{\bar{\mathcal{N}}_\cdot(\{(x,i),(y,j)\}) : x,y \in \mathbb{Z}^d \text{ with } |x-y|=1,\ i\in\{1,\dots,\alpha_x\},\ j\in\{1,\dots,\alpha_y\}\}. \tag{4.5.1}$$

to be a family of independent and identically distributed compensated Poisson processes with intensity one.

We denote by $(\bar{\mathsf{N}}, \mathsf{F}, \{\mathsf{F}_t : t\geq 0\}, \mathsf{P})$ the probability space on which this compensated Poisson processes are defined. This randomness will be responsible (see Lemma 4.5.1 below) for the stirring construction (see, e.g., [126, p. 399]) of the so-called *ladder symmetric exclusion process with parameter* $\boldsymbol{\alpha} \in \{1,\dots,\mathfrak{c}\}^{\mathbb{Z}^d}$, the particle system with configuration space

$$\tilde{\mathcal{X}}^{\boldsymbol{\alpha}} = \{\tilde{\eta} : \tilde{\eta}(x,i)\in\{0,1\} \text{ for all } x\in\mathbb{Z}^d \text{ and } i\in\{1,\dots,\alpha_x\}\} \tag{4.5.2}$$

and with infinitesimal generator $\tilde{L}^{\boldsymbol{\alpha}}$ acting on bounded cylindrical functions $\tilde{\varphi} : \tilde{\mathcal{X}}^{\boldsymbol{\alpha}} \to \mathbb{R}$ as follows:

$$\tilde{L}^{\boldsymbol{\alpha}}\tilde{\varphi}(\tilde{\eta}) \ = \sum_{\substack{\{x,y\}\in\mathbb{Z}^d\\|x-y|=1}} \tilde{L}^{\boldsymbol{\alpha}}_{xy}\tilde{\varphi}(\tilde{\eta})\ , \tag{4.5.3}$$

where

$$\begin{aligned}\tilde{L}^{\boldsymbol{\alpha}}_{xy}\tilde{\varphi}(\tilde{\eta}) = \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y}\Big\{&\tilde{\eta}(x,i)\,(1-\tilde{\eta}(y,j))\,(\tilde{\varphi}(\tilde{\eta}^{(x,i),(y,j)})-\tilde{\varphi}(\tilde{\eta}))\\ &+\tilde{\eta}(y,j)\,(1-\tilde{\eta}(x,i))\,(\tilde{\varphi}(\tilde{\eta}^{(y,j),(x,i)})-\tilde{\varphi}(\tilde{\eta}))\Big\}\ .\end{aligned}$$

Here $\tilde{\eta}^{(x,i),(y,j)}$ denotes, also in this context, the configuration obtained from $\tilde{\eta}\in\tilde{\mathcal{X}}^{\boldsymbol{\alpha}}$ by removing a particle at position (x,i) and placing it on (y,j).

This process may be considered as a special case of a symmetric exclusion process on the set $\tilde{\mathbb{Z}}^d = \{(x,i),\ x\in\mathbb{Z}^d,\ i\in\{1,\dots,\alpha_x\}\}$. For this reason and from the uniform boundedness assumption of the environment, we obtain the following representation of $\{\tilde{\eta}_t,\ t\geq 0\}$, whose proof is completely analogous to the one of, e.g., [64, Section 3] and [147, Proposition 4.3]. We restate this result below for convenience of the reader.

Lemma 4.5.1 (Mild solution for the ladder exclusion)**.** *Fix an environment* $\boldsymbol{\alpha}\in\{1,\dots,\mathfrak{c}\}^{\mathbb{Z}^d}$*. For* P*-a.e. realization of the compensated Poisson processes* $\{\bar{\mathcal{N}}_\cdot(\{\cdot,\cdot\})\}$ *and for all initial configurations* $\tilde{\eta}\in\tilde{\mathcal{X}}_{\alpha}$*, we have, for all* $(x,i)\in\tilde{\mathbb{Z}}^d$ *and* $t\geq 0$*,*

$$\tilde{\eta}_t(x,i) = \tilde{S}^{\boldsymbol{\alpha}}_t\tilde{\eta}_0(x,i) + \int_0^t \tilde{S}^{\boldsymbol{\alpha}}_{t-s}\,\mathrm{d}\tilde{M}^{\boldsymbol{\alpha}}_s(x,i)\ . \tag{4.5.4}$$

In the above formula, $\{\tilde{S}^{\boldsymbol{\alpha}}_t,\ t\geq 0\}$*, resp.* $\{\tilde{p}^{\boldsymbol{\alpha}}_t(\cdot,\cdot),\ t\geq 0\}$*, corresponds to the transition semigroup, resp. probabilities, associated to the continuous-time random walk on* $\tilde{\mathbb{Z}}^d$ *whose infinitesimal generator* $\tilde{A}^{\boldsymbol{\alpha}}$ *is given below:*

$$\tilde{A}^{\boldsymbol{\alpha}}f(x,i) \ = \sum_{\substack{y\in\mathbb{Z}^d\\|y-x|=1}}\sum_{j=1}^{\alpha_y}(f(y,j)-f(x,i))\ , \qquad (x,i)\in\tilde{\mathbb{Z}}^d\ ,$$

where $f:\tilde{\mathbb{Z}}^d \to \mathbb{R}$ *is a bounded function. Moreover, for all* $(x,i)\in\tilde{\mathbb{Z}}^d$ *and* $t,s\geq 0$*,*

$$\mathrm{d}\tilde{M}^{\boldsymbol{\alpha}}_s(x,i)\equiv \mathrm{d}\tilde{M}^{\boldsymbol{\alpha}}_s((x,i),\tilde{\eta}_{s^-}) := \sum_{\substack{y\in\mathbb{Z}^d\\|y-x|=1}}\sum_{j=1}^{\alpha_y}(\tilde{\eta}_{s^-}(y,j)-\tilde{\eta}_{s^-}(x,i))\,\mathrm{d}\bar{\mathcal{N}}_s(\{(x,i),(y,j)\})\ , \tag{4.5.5}$$

and

$$\int_0^t \tilde{S}^{\boldsymbol{\alpha}}_{t-s}\,\mathrm{d}\tilde{M}^{\boldsymbol{\alpha}}_s(x,i) := \sum_{y\in\mathbb{Z}^d}\sum_{j=1}^{\alpha_y}\int_0^t \tilde{p}^{\boldsymbol{\alpha}}_{t-s}((x,i),(y,j))\,\mathrm{d}\tilde{M}^{\boldsymbol{\alpha}}_s(y,j)\ ,$$

where the above time-integrals are Lebesgue-Stieltjes integrals w.r.t. the realizations of the compensated Poisson processes. Furthermore, the infinite summations in (4.5.4) *are* P*-a.s. – for all times and initial configurations – absolutely convergent.*

We leave to the reader to check that, P-a.s., for all times $t \geq 0$ and initial configurations $\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha}$, the predictable quadratic covariations of the martingales $\{\tilde{M}_t^{\alpha}(\cdot),\, t \geq 0\}$ in (4.5.4) read as

$$\langle \tilde{M}^{\alpha}(x,i), \tilde{M}^{\alpha}(y,j)\rangle_t = -\mathbf{1}_{\{|x-y|=1\}} \int_0^t (\tilde{\eta}_s(x,i) - \tilde{\eta}_s(y,j))^2 \, \mathrm{d}s \tag{4.5.6}$$

for $(x,i), (y,j) \in \tilde{\mathbb{Z}}^d$ with $x \neq y$, and

$$\langle \tilde{M}^{\alpha}(x,i), \tilde{M}^{\alpha}(x,i)\rangle_t = -\sum_{\substack{y \in \mathbb{Z}^d \\ |x-y|=1}} \sum_{j=1}^{\alpha_y} \langle \tilde{M}^{\alpha}(x,i), \tilde{M}^{\alpha}(y,j)\rangle_t \tag{4.5.7}$$

for $(x,i) \in \tilde{\mathbb{Z}}^d$.

In the following lemma, we show how to obtain SEP(α), with generator given in (4.1.3), from the ladder symmetric exclusion process with parameter α (see, e.g., [84] for further details on this construction). By combining this result with Lemma 4.5.1, we obtain a mild solution representation of SEP(α) which employs the same randomness used to define the ladder process.

Lemma 4.5.2 (Mild solution for SEP(α))**.** *Fix an environment $\alpha \in \{1, \ldots, \mathfrak{c}\}^{\mathbb{Z}^d}$.*

Let $\eta \in \mathcal{X}^{\alpha}$ and $\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha}$ be configurations satisfying the following relation:

$$\eta(x) = \sum_{i=1}^{\alpha_x} \tilde{\eta}(x,i)\,, \qquad x \in \mathbb{Z}^d\,. \tag{4.5.8}$$

Let $\{\tilde{\eta}_t,\, t \geq 0\}$ be the ladder symmetric exclusion process with parameter α, started from $\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha}$ presented above and represented as in Lemma 4.5.1. Then, the stochastic process $\{\eta_t,\, t \geq 0\}$ taking values in $\mathcal{X}^{\alpha}$ defined in terms of $\{\tilde{\eta}_t,\, t \geq 0\}$ as follows

$$\eta_t(x) := \sum_{i=1}^{\alpha_x} \tilde{\eta}_t(x,i)\,, \qquad t \geq 0\,,\ x \in \mathbb{Z}^d\,, \tag{4.5.9}$$

is a Markov process with infinitesimal generator L^{α} as given in (4.1.3) *and started from $\eta \in \mathcal{X}^{\alpha}$.*

Moreover, for P*-a.e. realization of the compensated Poisson processes in* (4.5.1) *and for all initial configurations $\eta \in \mathcal{X}^{\alpha}$, we have (cf. the definition of the semigroup $\{S_t^{\alpha},\, t \geq 0\}$ in Section 4.2.2, as well as* (4.2.29)–(4.2.30)*)*

$$\left(\tfrac{\eta_t}{\alpha}\right)(x) = S_t^{\alpha}\left(\tfrac{\eta}{\alpha}\right)(x) + \int_0^t S_{t-s}^{\alpha}\, \mathrm{d}M_s^{\alpha}(x)\,, \qquad t \geq 0\,,\ x \in \mathbb{Z}^d\,, \tag{4.5.10}$$

where

$$\mathrm{d}M_s^{\alpha}(x) := \frac{1}{\alpha_x} \sum_{i=1}^{\alpha_x} \mathrm{d}\tilde{M}_s^{\alpha}((x,i))\,, \qquad x \in \mathbb{Z}^d\,, \tag{4.5.11}$$

with $\{\tilde{M}_t^{\alpha}(\cdot),\, t \geq 0\}$ being the martingales given in (4.5.5) *and defined in terms of the ladder exclusion process $\{\tilde{\eta}_t,\, t \geq 0\}$ started from any configuration $\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha}$ related to $\eta \in \mathcal{X}^{\alpha}$ as in* (4.5.8)*; furthermore, the predictable quadratic covariations of the martingales in* (4.5.11) *are those given in* (4.4.2)–(4.4.3)*.*

Proof. Arguing as in [84, Theorem 4.2(a)], the process $\{\eta_t,\, t \geq 0\}$ defined in (4.5.9) is Markov; furthermore, it is simple to check that, by uniqueness in law of the solution to the martingale problem associated to (L^{α}, η) (see, e.g., [126, Chapter 1]), its infinitesimal generator is L^{α} (we refer to [84, Section 4.1] for further details).

As for the second part of the claim, by definition of the process $\{\eta_t,\, t \geq 0\}$ in terms of the process $\{\tilde{\eta}_t,\, t \geq 0\}$ and formula (4.5.4), we obtain, P-a.s., for all $x \in \mathbb{Z}^d$ and $t \geq 0$, the following expression for $\eta_t(x)$:

$$\begin{aligned} \eta_t(x) &:= \sum_{i=1}^{\alpha_x} \tilde{\eta}_t(x,i) \\ &= \sum_{i=1}^{\alpha_x} \sum_{y \in \mathbb{Z}^d} \sum_{j=1}^{\alpha_y} \left(\tilde{p}_t^{\alpha}((x,i),(y,j))\, \tilde{\eta}_0(y,i) + \int_0^t \tilde{p}_{t-s}^{\alpha}((x,i),(y,j))\, \mathrm{d}\tilde{M}_s^{\alpha}(y,j) \right). \end{aligned} \tag{4.5.12}$$

Since the infinite summations above are absolutely convergent, we may re-order them so to obtain:

$$\eta_t(x) \;=\; \sum_{y\in\mathbb{Z}^d} \mathcal{Y}_t(y)\;,$$

where

$$\mathcal{Y}_t(y) := \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y} \tilde{p}_t^{\alpha}((x,i),(y,j))\,\tilde{\eta}_0(y,j) + \int_0^t \sum_{i=1}^{\alpha_x}\sum_{j=1}^{\alpha_y} \tilde{p}_{t-s}^{\alpha}((x,i),(y,j))\,\mathrm{d}\tilde{M}_s^{\alpha}(y,j)\;. \tag{4.5.13}$$

We observe that, for all sites $x, y \in \mathbb{Z}^d$ and labels $i, i' \in \{1, \ldots, \alpha_x\}$, $j, j' \in \{1, \ldots, \alpha_y\}$, $\tilde{p}_t^{\alpha}((x,i),(y,j)) = \tilde{p}_t^{\alpha}((x,i'),(y,j'))$; in other words, the transition probabilities $\tilde{p}_t^{\alpha}(\cdot,\cdot)$ do not depend on the labels, but only on the sites. Therefore, we define $\tilde{p}_t^{\alpha}(x,y) := \tilde{p}_t^{\alpha}((x,i),(y,j))$. If we combine this with the definition of $\eta_0(y) := \sum_{j=1}^{\alpha_y} \tilde{\eta}_0(y,j)$, the expression in (4.5.13) rewrites as follows:

$$\mathcal{Y}_t(y) = \alpha_x\,\tilde{p}_t^{\alpha}(x,y)\,\eta_0(y) \;+\; \int_0^t \alpha_x\,\tilde{p}_{t-s}^{\alpha}(x,y)\,\sum_{j=1}^{\alpha_y} \mathrm{d}\tilde{M}_s^{\alpha}((y,j),\tilde{\eta}_{s^-})\;.$$

Recalling from Section 4.2.2 the definition of transition probabilities $\{p_t^{\alpha}(\cdot,\cdot),\, t \geq 0\}$ associated to RW($\boldsymbol{\alpha}$) and after observing that

$$p_t^{\alpha}(x,y) = \sum_{j=1}^{\alpha_y} \tilde{p}_t^{\alpha}((x,i),(y,j)) = \alpha_y\,\tilde{p}_t^{\alpha}(x,y)\;, \tag{4.5.14}$$

the proof of the identity (4.5.10) is concluded.

In order to recover the predictable quadratic covariations (4.4.2)–(4.4.3) for the martingales $\{M_t^{\alpha}(\cdot),\, t \geq 0\}$, it suffices to combine (4.5.11) with (4.5.6)–(4.5.7) and (4.5.9); we leave the details to the reader. □

We take the construction and (4.5.9) in Lemma 4.5.2 as a definition of our partial exclusion process SEP($\boldsymbol{\alpha}$). In particular, we consider the process $\{\eta_t,\, t \geq 0\}$ as a Markov functional of the ladder process $\{\tilde{\eta}_t,\, t \geq 0\}$, whose evolution, in turn, is prescribed in Lemma 4.5.1 in terms of the compensated Poisson processes $\{\tilde{\mathcal{N}}(\cdot,\cdot)\}$ in (4.5.1) and its initial configuration $\tilde{\eta}_0 \in \tilde{\mathcal{X}}^{\alpha}$.

However, to any given SEP($\boldsymbol{\alpha}$)-configuration $\eta \in \mathcal{X}^{\alpha}$ there may correspond, in general, many "compatible ladder configurations", namely configurations $\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha}$ of the following type:

$$\left\{\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha} : \sum_{i=1}^{\alpha_x} \tilde{\eta}(x,i) = \eta(x) \text{ for all } x \in \mathbb{Z}^d\right\}\;.$$

Therefore, when we say that the particle system $\{\eta_t,\, t \geq 0\}$ starts from the configuration $\eta \in \mathcal{X}^{\alpha}$, we first need to specify how to initialize the underlying ladder process and, then, unequivocally follow the Poissonian source of randomness yielding (4.5.10) and (4.5.11). We will always assume that, given an initial configuration $\eta \in \mathcal{X}^{\alpha}$, the compatible ladder configurations $\tilde{\eta} \in \tilde{\mathcal{X}}^{\alpha}$ are chosen according to some probability distribution *independent* of the compensated Poisson processes in (4.5.1). We can, for instance, make the deterministic choice of filling up the ladders at each site starting from bottom to top.

4.6 Proofs of auxiliary results

In order to fix notation, for all compact subsets $\mathcal{K} \subseteq \mathbb{R}^d$, $C_b(\mathcal{K})$ (resp. $C_c(\mathcal{K})$) denotes the space of continuous and bounded (resp. compactly supported) functions from $\mathcal{K}$ to $\mathbb{R}$ endowed with the supremum norm, while $\mathcal{M}_+(\mathcal{K})$ denotes the space of non-negative finite Borel measures on $\mathcal{K}$ endowed with the weak* topology w.r.t. $C_b(\mathcal{K})$. Moreover, for all $\mu \in \mathcal{M}_+(\mathcal{K})$ and $F \in C_b(\mathcal{K})$, we define

$$\mu(F) := \int_{\mathcal{K}} F(u)\,\mu(\mathrm{d}u)\;. \tag{4.6.1}$$

4.6.1 Proof of Lemma 4.2.5

Proof of Lemma 4.2.5. The methodology of the proof is inspired by [28, Theorem 8.2.18].

By applying [67, Proposition 3.2] to the integrable function $g : \boldsymbol{\alpha} \in \{1, \ldots, \mathfrak{c}\}^{\mathbb{Z}^d} \mapsto \alpha_0 \in \mathbb{R}$, there exists a (translation invariant) measurable subset $\mathfrak{A} \subseteq \{1, \ldots, \mathfrak{c}\}^{\mathbb{Z}^d}$ such that $\mathcal{P}(\mathfrak{A}) = 1$ holds, as well as

$$\left| \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} G(\tfrac{x}{N})\, \alpha_x - \mathbb{E}_{\mathcal{P}}[\alpha_0] \int_{\mathbb{R}^d} G(u)\, \mathrm{d}u \right| \underset{N\to\infty}{\longrightarrow} 0 \tag{4.6.2}$$

hold for all $\boldsymbol{\alpha} \in \mathfrak{A}$ and $G \in C_c(\mathbb{R}^d)$, the subspace of $C_0(\mathbb{R}^d)$ of compactly supported functions.

In the remainder of this proof, $\boldsymbol{\alpha} \in \mathfrak{A}$; moreover, we define

$$\mathsf{Y}^{N,\boldsymbol{\alpha}} := \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \delta_{\frac{x}{N}}\, \alpha_x \qquad \text{and} \qquad \mathsf{Y} := \mathbb{E}_{\mathcal{P}}[\alpha_0]\, \mathrm{d}u \tag{4.6.3}$$

as elements in $\mathscr{S}'(\mathbb{R}^d)$.

Recall from the proof of Theorem 4.3.1 the definitions of the open and closed Euclidean balls $\mathcal{B}_\ell(u)$ and $\overline{\mathcal{B}_\ell(u)}$. Then, for all $\ell > 0$, since the restriction map $|_{\overline{\mathcal{B}_\ell(0)}} : C_c(\mathbb{R}^d) \to C_c(\overline{\mathcal{B}_\ell(0)})$ is onto and since $C_c(\overline{\mathcal{B}_\ell(0)}) \equiv C_b(\overline{\mathcal{B}_\ell(0)})$, (4.6.2) implies that, for all $\boldsymbol{\alpha} \in \mathfrak{A}$, $\mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell$ weakly converge as non-negative finite Borel measures as $N \to \infty$ to Y_ℓ, where

$$\mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell(\mathrm{d}u) := \frac{1}{N^d} \sum_{\frac{x}{N} \in \overline{\mathcal{B}_\ell(0)}} \delta_{\frac{x}{N}}(\mathrm{d}u)\, \alpha_x \qquad \text{and} \qquad \mathsf{Y}_\ell(\mathrm{d}u) := \mathbb{E}_{\mathcal{P}}[\alpha_0]\, \mathbf{1}_{\{u \in \overline{\mathcal{B}_\ell(0)}\}}\, \mathrm{d}u\,. \tag{4.6.4}$$

By the compactness of $\overline{\mathcal{B}_\ell(0)} \subseteq \mathbb{R}^d$, for all $\delta > 0$, there exists a finite sub-cover $\mathcal{U}_\ell(\delta) := \{\mathcal{B}_\delta(u_i)\}_{i=1}^n$ of open balls of radius $\delta > 0$ (with $n = n(\delta) \in \mathbb{N}$). Moreover, by defining recursively $V_1 := \mathcal{B}_\delta(u_1) \cap \overline{\mathcal{B}_\ell(0)}$ and $V_i := \{\mathcal{B}_\delta(u_i) \cap \overline{\mathcal{B}_\ell(0)}\} \setminus V_{i-1}$, it is simple to check that the pairwise disjoint sets $\mathcal{V}_\ell(\delta) := \{V_i\}_{i=1}^n$ cover $\overline{\mathcal{B}_\ell(0)}$ and $\mathsf{Y}_\ell(\partial V_i) = 0$ for all $i = 1, \ldots, n$, where ∂V_i denotes the boundary of V_i in the subspace topology on $\overline{\mathcal{B}_\ell(0)}$. Hence,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} \left| \mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell(F) - \mathsf{Y}_\ell(F) \right| \le{}& \sup_{F\in\mathcal{F}} \sum_{i=1}^n \frac{1}{N^d} \sum_{\frac{x}{N}\in V_i} \left| F(\tfrac{x}{N}) - F(u_i) \right| \alpha_x \\
&+ \sup_{F\in\mathcal{F}} \sup_{u\in\mathbb{R}^d} |F(u)| \sum_{i=1}^n \left| \frac{1}{N^d} \sum_{\frac{x}{N}\in V_i} \alpha_x - \int_{V_i} \mathbb{E}_{\mathcal{P}}[\alpha_0]\, \mathrm{d}u \right| \\
&+ \sup_{F\in\mathcal{F}} \sum_{i=1}^n \mathbb{E}_{\mathcal{P}}[\alpha_0] \int_{V_i} |F(u) - F(u_i)|\, \mathrm{d}u\,.
\end{aligned} \tag{4.6.5}$$

The boundedness of $\mathcal{F}$ (see (4.2.13)), and the convergence (recall that $\mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell$ weakly converges to Y_ℓ as $N \to \infty$ as well as $\mathsf{Y}_\ell(\partial V_i) = 0$ for all $i = 1, \ldots, n$)

$$\mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell(V_i) \underset{N\to\infty}{\longrightarrow} \mathsf{Y}_\ell(V_i)\,, \qquad i = 1, \ldots, n\,, \tag{4.6.6}$$

ensure that, for all $\delta > 0$, the second term on the r.h.s. in (4.6.5) vanishes as $N \to \infty$:

$$\sup_{F\in\mathcal{F}} \sup_{u\in\mathbb{R}^d} |F(u)| \sum_{i=1}^n \left| \frac{1}{N^d} \sum_{\frac{x}{N}\in V_i} \alpha_x - \int_{V_i} \mathbb{E}_{\mathcal{P}}[\alpha_0]\, \mathrm{d}u \right| \underset{N\to\infty}{\longrightarrow} 0\,. \tag{4.6.7}$$

The first and third terms on the r.h.s. in (4.6.5) are both bounded above by

$$\sup_{\substack{u,v\in\mathbb{R}^d\\ |u-v|<\delta}} \sup_{F\in\mathcal{F}} |F(u) - F(v)| \left\{ \mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell(\overline{\mathcal{B}_\ell(0)}) + \mathsf{Y}_\ell(\overline{\mathcal{B}_\ell(0)}) \right\}\,; \tag{4.6.8}$$

hence, by the definition (4.2.12) of equicontinuity of the subset $\mathcal{F} \subseteq C_0(\mathbb{R}^d)$ and

$$\limsup_{N\to\infty} \mathsf{Y}^{N,\boldsymbol{\alpha}}_\ell(\overline{\mathcal{B}_\ell(0)}) + \mathsf{Y}_\ell(\overline{\mathcal{B}_\ell(0)}) = 2\mathsf{Y}_\ell(\overline{\mathcal{B}_\ell(0)}) < \infty\,, \tag{4.6.9}$$

we obtain

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\sup_{F\in\mathcal{F}}\left\{\sum_{i=1}^n \frac{1}{N^d}\sum_{\frac{x}{N}\in V_i}\left|F(\tfrac{x}{N})-F(u_i)\right|\alpha_x+\sum_{i=1}^n \mathbb{E}_{\mathcal{P}}[\alpha_0]\int_{V_i}|F(u)-F(u_i)|\,\mathrm{d}u\right\}=0\,. \tag{4.6.10}$$

Hence, (4.6.7) and (4.6.10) combined with (4.6.5) yield, for all $\ell>0$,

$$\sup_{F\in\mathcal{F}}\left|\mathsf{Y}^{N,\alpha}_\ell(F)-\mathsf{Y}_\ell(F)\right|\underset{N\to\infty}{\longrightarrow}0\,. \tag{4.6.11}$$

The uniform integrability assumption (see (4.2.14)) and the upper bound $\alpha_x\le\mathfrak{c}<\infty$ (see Assumption 4.1.1) ensure

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\sup_{F\in\mathcal{F}}\left\{\frac{1}{N^d}\sum_{|\frac{x}{N}|>\ell}\left|F(\tfrac{x}{N})\right|\alpha_x+\mathbb{E}_{\mathcal{P}}[\alpha_0]\int_{\{|u|>\ell\}}|F(u)|\,\mathrm{d}u\right\}=0\,. \tag{4.6.12}$$

The triangle inequality

$$\begin{aligned}\sup_{F\in\mathcal{F}}\left|\mathsf{Y}^{N,\alpha}(F)-\mathsf{Y}(F)\right|\le\ &\sup_{F\in\mathcal{F}}\left|\mathsf{Y}^{N,\alpha}_\ell(F)-\mathsf{Y}_\ell(F)\right|\\ &+\sup_{F\in\mathcal{F}}\left\{\frac{1}{N^d}\sum_{|\frac{x}{N}|>\ell}\left|F(\tfrac{x}{N})\right|\alpha_x+\mathbb{E}_{\mathcal{P}}[\alpha_0]\int_{\{|u|>\ell\}}|F(u)|\,\mathrm{d}u\right\}\,,\end{aligned} \tag{4.6.13}$$

which holds for all $\ell>0$ and $N\in\mathbb{N}$, combined with (4.6.11) and (4.6.12), yields the desired result. □

4.6.2 Proof of Corollary 4.3.2

Proof of Corollary 4.3.2. In what follows, let $\boldsymbol{\alpha}$ be an environment in the subset $\mathfrak{A}\cap\mathfrak{B}\subseteq\{1,\ldots,\mathfrak{c}\}^{\mathbb{Z}^d}$ (see (4.2.16) and (4.3.11)). Fix $T>0$ and $G\in\mathscr{S}(\mathbb{R}^d)\subseteq C_0(\mathbb{R}^d)$. Let G^+ and G^- be the positive and negative parts of G ($G=G^+-G^-$); then, $G^\pm\in L^1(\mathbb{R}^d)\cap C_0(\mathbb{R}^d)$ (hence they satisfy (4.3.1)) and there exist functions $H^\pm\in\mathscr{S}(\mathbb{R}^d)$ (see, e.g., [147, Proposition 5.3] for an explicit construction) such that

$$0\le G^\pm(u)\le H^\pm(u)\,,\qquad u\in\mathbb{R}^d\,. \tag{4.6.14}$$

As a consequence, there exist constants $C^\pm>0$ such that

$$\sup_{0\le t\le T}|\mathcal{S}^\Sigma_t G^\pm(u)|\le\frac{C^\pm}{1+|u|^{2d}}\,,\qquad u\in\mathbb{R}^d\,. \tag{4.6.15}$$

This follows from the bounds (4.6.14), the fact that $\mathcal{S}^\Sigma_t$ acts as convolution with a non-degenerate Gaussian kernel and the use of Fourier transformation in $\mathscr{S}(\mathbb{R}^d)$. Moreover, because of the uniform continuity of $G^\pm$ and the contractivity of the semigroup in $C_0(\mathbb{R}^d)$, we have

$$\sup_{t\in[0,T]}\sup_{|u-v|<\delta}\left|\mathcal{S}^\Sigma_t G^\pm(u)-\mathcal{S}^\Sigma_t G^\pm(v)\right|\ \le\ \sup_{t\in[0,T]}\sup_{|u-v|<\delta}\left|G^\pm(u)-G^\pm(v)\right|\underset{\delta\to0}{\longrightarrow}0\,.$$

As a consequence, for all $T>0$, both subsets of $C_0(\mathbb{R}^d)$ given by

$$\mathcal{F}_{[0,T]}(G^\pm):=\left\{\mathcal{S}^\Sigma_t G^\pm\in C_0(\mathbb{R}^d):t\in[0,T]\right\} \tag{4.6.16}$$

satisfy the assumptions in Lemma 4.2.5. Therefore, since $\boldsymbol{\alpha}\in\mathfrak{A}$, Lemma 4.2.5 ensures that, for all $G\in\mathscr{S}(\mathbb{R}^d)$ and $T>0$, we have

$$\sup_{t\in[0,T]}\left|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N})\,\alpha_x-\int_{\mathbb{R}^d}\mathcal{S}^\Sigma_t G^\pm(u)\,\mathbb{E}_{\mathcal{P}}[\alpha_0]\,\mathrm{d}u\right|\underset{N\to\infty}{\longrightarrow}0\,. \tag{4.6.17}$$

Let us now prove

$$\sup_{t\in[0,T]}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|S^{N,\alpha}_{tN^2}G^\pm(\tfrac{x}{N})-\mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N})\right|\alpha_x\underset{N\to\infty}{\longrightarrow}0\,, \tag{4.6.18}$$

from which (4.3.2) follows.

Since $|c| = c + 2\max\{-c, 0\}$ for all $c \in \mathbb{R}$, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left| S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}) - \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) \right| \alpha_x \\
&\le \sup_{t\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left(S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}) - \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) \right) \alpha_x \\
&+ \sup_{t\in[0,T]} \frac{2}{N^d} \sum_{x\in\mathbb{Z}^d} \max\left\{ \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) - S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}), 0 \right\} \alpha_x \,. \qquad (4.6.19)
\end{aligned}$$

As for the first term in the r.h.s. above, by detailed balance (see (4.2.26)), $\sum_{x\in\mathbb{Z}^d} p^\alpha_{tN^2}(y,x) = 1$, as well as $\int_{\mathbb{R}^d} \mathcal{S}^\Sigma_t G^\pm(u)\,\mathrm{d}u = \int_{\mathbb{R}^d} G^\pm(u)\,\mathrm{d}u$, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \left| \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left(S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}) - \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) \right) \alpha_x \right| \\
&= \sup_{t\in[0,T]} \left| \frac{1}{N^d} \sum_{y\in\mathbb{Z}^d} G^\pm(\tfrac{y}{N})\, \alpha_y \sum_{x\in\mathbb{Z}^d} p^\alpha_{tN^2}(y,x) - \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \mathcal{S}^\Sigma_{t-s} G^\pm(\tfrac{x}{N})\, \alpha_x \right| \\
&\le \left| \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} G^\pm(\tfrac{x}{N})\, \alpha_x - \int_{\mathbb{R}^d} G^\pm(u)\, \mathbb{E}_{\mathcal{P}}[\alpha_0]\, \mathrm{d}u \right| \\
&+ \sup_{t\in[0,T]} \left| \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N})\, \alpha_x - \int_{\mathbb{R}^d} \mathcal{S}^\Sigma_t G^\pm(u)\, \mathbb{E}_{\mathcal{P}}[\alpha_0]\, \mathrm{d}u \right| \;;
\end{aligned}$$

thus, the first expression on the r.h.s. of (4.6.19) vanishes as $N \to \infty$ by (4.6.17).

Moreover, we have, for all $N \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\sup_{t\in[0,T]} \max\left\{ \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) - S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}),\, 0 \right\} \alpha_x \le \sup_{t\in[0,T]} \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N})\, \alpha_x \,. \qquad (4.6.20)$$

Therefore, for all $\ell > 0$ and combining (4.6.20) and (4.6.15), we obtain

$$\begin{aligned}
&\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{2}{N^d} \sum_{x\in\mathbb{Z}^d} \max\left\{ \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) - S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}),\, 0 \right\} \alpha_x \\
&\le \limsup_{N\to\infty} \sup_{t\in[0,T]} \sup_{|\frac{x}{N}|\le\ell} \left| \mathcal{S}^\Sigma_t G^\pm(\tfrac{x}{N}) - S^{N,\alpha}_{tN^2} G^\pm(\tfrac{x}{N}) \right| \frac{1}{N^d} \sum_{|\frac{x}{N}|\le\ell} \alpha_x \qquad (4.6.21) \\
&+ \limsup_{N\to\infty} \frac{2}{N^d} \sum_{|\frac{x}{N}|>\ell} \frac{C^\pm\, \alpha_x}{1 + |\frac{x}{N}|^{2d}} \,. \qquad (4.6.22)
\end{aligned}$$

By Theorem 4.3.1 applied to the functions $G^\pm$ and $\sup_{N\in\mathbb{N}} \frac{1}{N^d} \sum_{|\frac{x}{N}|\le\ell} \alpha_x < \infty$, (4.6.21) equals zero for all $\ell > 0$, while (4.6.22) vanishes as $\ell \to \infty$. This concludes the proof. □

Chapter 5

Switching interacting particle systems: hydrodynamics

5.1 Introduction

Section 5.1.1 provides the background and the motivation for the chapter. Section 5.2 defines the model. Section 5.3 identifies the dual and the stationary measures. Section 5.3.1 gives a brief outline of the remainder of the chapter.

5.1.1 Background and motivation

As explained in the introduction of this thesis, interacting particle systems are used to model and analyse properties of *non-equilibrium systems*, such as macroscopic profiles, long-range correlations and macroscopic large deviations. Some models have additional structure, such as duality or integrability properties, which allow for a study of the fine details of non-equilibrium steady states, such as microscopic profiles and correlations. Examples include zero-range processes, exclusion processes, and models that fit into the algebraic approach to duality, such as inclusion processes and related diffusion processes, or models of heat conduction, such as the Kipnis-Marchioro-Presutti model [32, 56, 55, 84, 107]. Most of these models have indistinguishable particles of which the total number is conserved, and so the relevant macroscopic quantity is the *density* of particles.

Turning to more complex models of non-equilibrium, various exclusion processes with *multi-type particles* have been studied [70, 71, 115], as well as reaction-diffusion processes [27, 29, 47, 48, 49], where non-linear reaction-diffusion equations are obtained in the hydrodynamic limit, and large deviations around such equations have been analysed. In this chapter, we focus on a reaction-diffusion model that on the one hand is simple enough so that via duality a complete microscopic analysis of the non-equilibrium profiles can be carried out, but on the other hand exhibits interesting phenomena, such as *violation of the Fick's law*. In our model we have two types of particles, *fast* and *slow*, that jump at rate 1 and $\epsilon \in [0, 1]$, respectively. Particles of identical type are allowed to interact via exclusion or inclusion. There is no interaction between particles of different type that are at different sites. Each particle can change type at a rate that is adapted to the particle interaction (exclusion or inclusion), and is therefore interacting with particles of different type at the same site. An alternative and equivalent view is to consider two layers of particles, where the layer determines the jump rate (rate 1 for bottom layer, rate ϵ for top layer) and where on each layer the particles move according to exclusion or inclusion, and to let particles change layer at a rate that is appropriately chosen in accordance with the interaction. In the limit as $\epsilon \downarrow 0$, particles are immobile on the top layer.

We show that the *hydrodynamic limit* of all three dynamics is a linear reaction-diffusion system known under the name of *double diffusivity model*, namely,

$$\begin{cases} \partial_t \rho_0 = \Delta \rho_0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t \rho_1 = \epsilon \Delta \rho_1 + \Upsilon(\rho_0 - \rho_1), \end{cases} \tag{5.1.1}$$

where ρ_i, $i \in \{0, 1\}$, are the macroscopic densities of the two types of particles, and $\Upsilon \in (0, \infty)$ is the scaled switching rate. The above system was introduced in [3] to model polycrystal diffusion (more generally, diffusion in inhomogeneous porous media) and dislocation pipe diffusion, with the goal to overcome the restrictions imposed by Fick's law. Non-Fick behaviour is immediate from the fact that the total density $\rho = \rho_0 + \rho_1$ does not satisfy the classical diffusion equation.

The double diffusivity model was studied extensively in the PDE literature [4, 95, 94], while its discrete counterpart was analysed in terms of a single random walk switching between two layers [93]. The same macroscopic model was studied independently in the mathematical finance literature in the context of switching diffusion processes [161]. Thus, we have a family of interacting particle systems whose macroscopic limit is relevant in several contexts. Another context our three dynamics fit into are models of interacting active random walks with an internal state that changes randomly (e.g. activity, internal energy) and that determines their diffusion rate and or drift [53, 77, 91, 114, 131, 143, 6, 111].

An additional motivation to study two-layer models comes from population genetics. Individuals live in colonies, carry different genetics types, and can be either active or dormant. While active, individuals resample by adopting the type of a randomly sampled individual in the same colony, and migrate between colonies by hopping around. Active individuals can become dormant, after which they suspend resampling and migration, until they become active again. Dormant individuals reside in what is called a *seed bank*. The overall effect of dormancy is that extinction of types is slowed down, and so genetic diversity is enhanced by the presence of the seed bank. A wealth of phenomena can occur, depending on the parameters that control the rates of resampling, migration, falling asleep and waking up [25, 89]. Dormancy not only affects the long-term behaviour of the population quantitatively. It may also lead to qualitatively different equilibria and time scales of convergence. For a panoramic view on the role of dormancy in the life sciences, we refer the reader to [124].

5.2 Three models

For $\sigma \in \{-1, 0, 1\}$ we introduce an interacting particle system on $\mathbb{Z}$ where the particles randomly switch their jump rate between two possible values, 1 and ϵ, with $\epsilon \in [0, 1]$. For $\sigma = -1$ the particles are subject to the exclusion interaction, for $\sigma = 0$ the particles are independent, while for $\sigma = 1$ the particles are subject to the inclusion interaction. Let

$$\eta_0(x) := \text{number of particles at site } x \text{ jumping at rate } 1,$$
$$\eta_1(x) := \text{number of particles at site } x \text{ jumping at rate } \epsilon.$$

The configuration of the system is

$$\eta := \{\eta(x)\}_{x\in\mathbb{Z}} \in \mathcal{X} = \begin{cases} \{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}, & \text{if } \sigma = -1, \\ \mathbb{N}_0^{\mathbb{Z}} \times \mathbb{N}_0^{\mathbb{Z}}, & \text{if } \sigma = 0, 1, \end{cases}$$

where

$$\eta(x) := (\eta_0(x), \eta_1(x)), \qquad x \in \mathbb{Z}.$$

We call $\eta_0 = \{\eta_0(x)\}_{x\in\mathbb{Z}}$ and $\eta_1 = \{\eta_1(x)\}_{x\in\mathbb{Z}}$ the configurations of *fast particles*, respectively, *slow particles*. When $\epsilon = 0$ we speak of *dormant particles* (see Fig. 5.2).

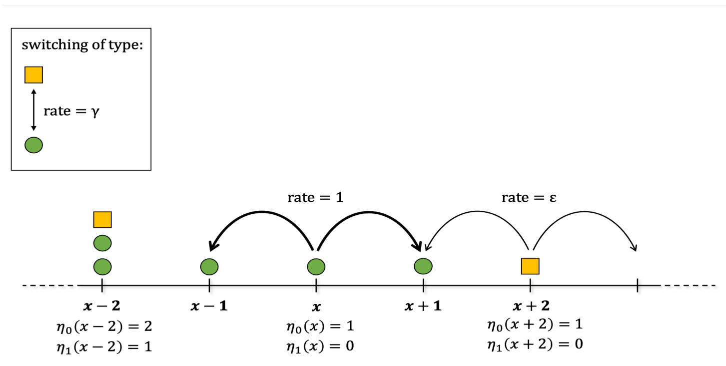


Figure 5.1: Representation via *slow* and *fast* particles moving on the one-layer graph $\mathbb{Z}$ ($\sigma = 0$).

Definition 5.2.1. [Switching interacting particle systems] For $\epsilon \in [0, 1]$ and $\gamma \in (0, \infty)$, let $L_{\epsilon,\gamma}$ be the generator

$$L_{\epsilon,\gamma} := L_0 + \epsilon L_1 + \gamma L_{0\updownarrow 1}, \tag{5.2.1}$$

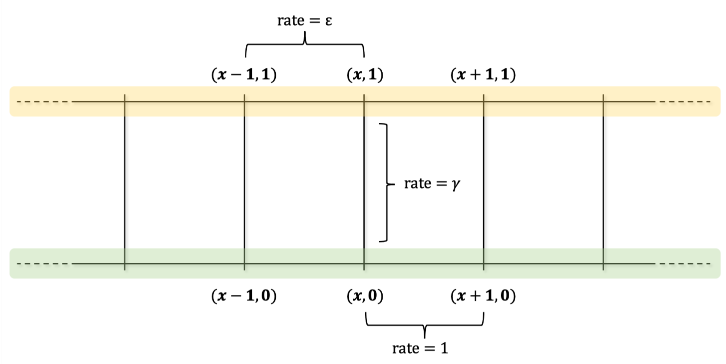


Figure 5.2: Representation via particles moving on the two-layer graph $\mathbb{Z} \times I$ ($\sigma = 0$).

acting on bounded cylindrical functions $f : \mathcal{X} \to \mathbb{R}$ as

$$\begin{aligned}(L_0 f)(\eta) = \sum_{|x-y|=1} \Big\{ &\eta_0(x)(1+\sigma\eta_0(y))\,[f((\eta_0-\delta_x+\delta_y,\eta_1))-f(\eta)] \\ &+ \eta_0(y)(1+\sigma\eta_0(x))\,[f((\eta_0+\delta_x-\delta_y,\eta_1))-f(\eta)]\Big\},\\ (L_1 f)(\eta) = \sum_{|x-y|=1} \Big\{ &\eta_1(x)(1+\sigma\eta_1(y))\,[f((\eta_0,\eta_1-\delta_x+\delta_y))-f(\eta)] \\ &+ \eta_1(y)(1+\sigma\eta_1(x))\,[f((\eta_0,\eta_1+\delta_x-\delta_y))-f(\eta)]\Big\},\\ (L_{0\updownarrow 1} f)(\eta) = \gamma \sum_{x\in\mathbb{Z}^d} \Big\{ &\eta_0(x)(1+\sigma\eta_1(x))\,[f((\eta_0-\delta_x,\eta_1+\delta_x))-f(\eta)] \\ &+ \eta_1(x)(1+\sigma\eta_0(x))\,[f((\eta_0+\delta_x,\eta_1-\delta_x))-f(\eta)]\Big\}.\end{aligned}$$

The Markov process $\{\eta(t) \colon t \geq 0\}$ on state space $\mathcal{X}$ with

$$\eta(t) := \{\eta(x,t)\}_{x\in\mathbb{Z}} = \{(\eta_0(x,t),\eta_1(x,t))\}_{x\in\mathbb{Z}},$$

hopping rates $1, \epsilon$ and switching rate γ is called *switching exclusion process* for $\sigma = -1$, *switching random walks* for $\sigma = 0$ (see Fig. 5.2), and *switching inclusion process* for $\sigma = 1$. ♠

5.3 Duality and stationary measures

The systems defined in (5.2.1) can be equivalently formulated as jump processes on the graph (see Fig. 5.2) with vertex set $\{(x,i) \in \mathbb{Z}^d \times I\}$, with $I = \{0,1\}$ labelling the two layers, and edge set given by the nearest-neighbour relation

$$(x,i) \sim (y,j) \quad \text{when} \quad \begin{cases} |x-y| = 1 \text{ and } i = j, \\ x = y \text{ and } |i-j| = 1. \end{cases}$$

In this formulation the particle configuration is

$$\eta = (\eta_i(x))_{(x,i)\in\mathbb{Z}\times I}$$

and the generator L is given by

$$\begin{aligned}(Lf)(\eta) = \sum_{i\in I}\sum_{|x-y|=1} &\epsilon^i \eta_i(x)(1+\sigma\eta_i(y))\,[f(\eta-\delta_{(x,i)}+\delta_{(y,i)})-f(\eta)] \\ &+ \epsilon^i \eta_i(y)(1+\sigma\eta_i(x))\,[f(\eta-\delta_{(y,i)}+\delta_{(x,i)})-f(\eta)] \\ + \sum_{i\in I}\gamma\sum_{x\in\mathbb{Z}} &\eta_i(x)(1+\sigma\eta_{1-i})\,[f(\eta-\delta_{(x,i)}+\delta_{(x,1-i)})-f(\eta)].\end{aligned} \tag{5.3.1}$$

Thus, a single particle (when no other particles are present) is subject to two movements:

(i) *Horizontal movement*: In layer $i = 0$ and $i = 1$ the particle performs a nearest-neighbour random walk on $\mathbb{Z}$ at rate 1, respectively, ϵ.

(ii) *Vertical movement*: The particle switches layer at the same site at rate γ.

It is well known (see e.g. [146]) that for these systems there exists a one-parameter family of reversible product measures

$$\{\mu_\theta = \otimes_{(x,i)\in\mathbb{Z}\times I}\nu_{(x,i),\theta} \colon \theta\in\Theta\}$$

with $\Theta = [0,1]$ if $\sigma = -1$ and $\Theta = [0,\infty)$ if $\sigma \in \{0,1\}$, and with marginals given by

$$\nu_{(x,i),\theta} = \begin{cases} \text{Bernoulli}\,(\theta), & \sigma = -1,\\ \text{Poisson}\,(\theta), & \sigma = 0,\\ \text{Negative–Binomial}\,(1, \frac{\theta}{1+\theta}), & \sigma = 1. \end{cases} \tag{5.3.2}$$

Moreover, the *classical self-duality relation* holds, i.e., for all configurations $\eta,\ \xi \in \mathcal{X}$ and for all times $t\geq 0$,

$$\mathbb{E}_\eta[D(\xi,\eta_t)] = \mathbb{E}_\xi[D(\xi_t,\eta)],$$

with $\{\xi(t) : t\geq 0\}$ and $\{\eta(t) : t\geq 0\}$ two copies of the process with generator given in (5.2.1) and self-duality function $D\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ given by

$$D(\xi,\eta) := \prod_{(x,i)\in\mathbb{Z}^d\times I} d(\xi_i(x),\eta_i(x)), \tag{5.3.3}$$

with

$$d(k,n) := \frac{n!}{(n-k)!}\frac{1}{w(k)}\mathbf{1}_{\{k\leq n\}} \tag{5.3.4}$$

and

$$w(k) := \begin{cases} \frac{\Gamma(1+k)}{\Gamma(1)}, & \sigma = 1,\\ 1, & \sigma = -1,0. \end{cases} \tag{5.3.5}$$

Remark 5.3.1. [Possible extensions] Note that we could allow for more than two layers, for inhomogeneous rates and for non-nearest neighbour jumps as well, and the same duality relation would still hold (see e.g. [74] for an inhomogeneous version of the exclusion process). More precisely, let $\{\omega_i(\{x,y\})\}_{x,y\in\mathbb{Z}}$ and $\{\alpha_i(x)\}_{x\in\mathbb{Z}}$ be collections of bounded weights for $i\in I_M = \{0,1,\ldots,M\}$ with $M<\infty$. Then the interacting particle systems with generator

$$\begin{aligned}(L_{D,\gamma}f)(\eta) = {}& \sum_{i=0}^{M} D_i \sum_{|x-y|=1} \omega_i(\{x,y\})\Big\{\eta_i(x)\,(\alpha_i(y)+\sigma\eta_i(y))\,[f(\eta-\delta_{(x,i)}+\delta_{(y,i)})-f(\eta)]\\ &\qquad + \eta_i(y)\,(\alpha_i(x)+\sigma\eta_i(x))\,[f(\eta-\delta_{(y,i)}+\delta_{(x,i)})-f(\eta)]\Big\}\\ &+\sum_{i=0}^{M-1}\gamma_{\{i,i+1\}}\sum_{x\in\mathbb{Z}}\Big\{\eta_i(x)\,[f(\eta-\delta_{(x,i)}+\delta_{(x,i+1)})-f(\eta)]\\ &\qquad + \eta_{i+1}(x)\,[f(\eta-\delta_{(x,i+1)}+\delta_{(x,i)})-f(\eta)]\Big\},\end{aligned} \tag{5.3.6}$$

with $\eta = (\eta_i(x))_{(x,i)\in\mathbb{Z}\times I_M}$, $\{D_i\}_{i\in I_M}$ a bounded decreasing collection of weights in $[0,1]$ and $\gamma_{\{i,i+1\}}\in(0,\infty)$, are still self-dual with duality function as in (5.3.3), but with I replaced by I_M and single-site duality functions given by $d_{(x,i)}(k,n) = \frac{n!}{(n-k)!}\frac{1}{w_{(x,i)}(k)}\mathbf{1}_{\{k\leq n\}}$ with

$$w_{(x,i)}(k) := \begin{cases} \dfrac{\alpha_i(x)!}{(\alpha_i(x)-k)!}\mathbf{1}_{\{k\leq\alpha_i(x)\}}, & \sigma=-1,\\ \alpha_i(x)^k, & \sigma = 0,\\ \dfrac{\Gamma(\alpha_i(x)+k)}{\Gamma(\alpha_i(x))}, & \sigma = 1. \end{cases}$$

We prefer to stick to the two-layer homogeneous setting in order not to introduce extra notations. However, it is straightforward to extend many of our results to the inhomogeneous multi-layer model. ♠

Duality is a key tool in the study of detailed properties of interacting particle systems, since it allows for explicit computations. It has been used widely in the literature (see, e.g., [126, 47]). In the next section, *self-duality* (which implies microscopic closure of the evolution equation for the empirical density field) will be used to derive the hydrodynamic limit of the switching interacting particle systems described above. More precisely, we will use self-duality with one and two dual particles to compute the expectation of the evolution of the occupation variables and of the two-point correlations. These are needed, respectively, to control the expectation and the variance of the density field.

5.3.1 Outline

Section 5.4 identifies and analyses the *hydrodynamic limit* of the system in Definition 5.2.1 after scaling space, time and switching rate diffusively. In doing so, we exhibit a class of interacting particle systems whose microscopic dynamics scales to a macroscopic dynamics called the double diffusivity model. In Section 5.5, we provide a discussion on the solutions of this model, thereby connecting mathematical literature applied to material science and to financial mathematics.

5.4 The hydrodynamic limit

In this section we scale space, time and switching diffusively, so as to obtain a *hydrodynamic limit*. In Section 5.4.1 we scale space by $1/N$, time by N^2, the switching rate by $1/N^2$, introduce scaled microscopic empirical distributions, and let $N \to \infty$ to obtain a system of macroscopic equations. In Section 5.5 we recall some known results for this system, namely, there exists a unique solution that can be represented in terms of an underlying diffusion equation or, alternatively, via a Feynman-Kac formula involving the switching diffusion process.

5.4.1 From microscopic to macroscopic

Let $N \in \mathbb{N}$, and consider the scaled generator L_{ϵ,γ_N} (recall (5.2.1)) with $\gamma_N = \Upsilon/N^2$ for some $\Upsilon \in (0, \infty)$, i.e., the reaction term is slowed down by a factor N^2 in anticipation of the diffusive scaling we are going to consider.

In order to study the collective behaviour of the particles after scaling of space and time, we introduce the following empirical density fields, which are Radon measure-valued càdlàg (i.e., right-continuous with left limits) processes:

$$\mathsf{X}_0^N(t) := \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_0(x, tN^2)\,\delta_{x/N}, \qquad \mathsf{X}_1^N(t) := \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_1(x, tN^2)\,\delta_{x/N},$$

where δ_y stands for the Dirac measure at $y \in \mathbb{R}$. In order to derive the hydrodynamic limit for the switching interacting particle systems, we need the following set of assumptions. In the following we denote by $C_c^\infty(\mathbb{R})$ the space of infinitely differentiable functions with values in $\mathbb{R}$ and compact support, by $C_b(\mathbb{R};\sigma)$ the space of bounded and continuous functions with values in $\mathbb{R}_+$ for $\sigma \in \{0, 1\}$ and with values in $[0, 1]$ for $\sigma = -1$, by $C_0(\mathbb{R})$ the space of continuous functions vanishing at infinity, by $C_0^2(\mathbb{R})$ the space of twice differentiable functions vanishing at infinity and by M the space of Radon measure on $\mathbb{R}$.

Assumption 5.4.1. [Compatible initial conditions] Let $\bar\rho_i \in C_b(\mathbb{R};\sigma)$ for $i \in \{0, 1\}$ be two given functions, called initial macroscopic profiles. We say that a sequence $(\mu_N)_{N\in\mathbb{N}}$ of measures on $\mathcal{X}$ is a sequence of compatible initial conditions when:

(i) For any $i \in \{0, 1\}$, $g \in C_c^\infty(\mathbb{R})$ and $\delta > 0$,

$$\lim_{N\to\infty} \mu_N\left(\left|\langle \mathsf{X}_i^N(0), g\rangle - \int_{\mathbb{R}} \mathrm{d}x\, \bar\rho_i(x)g(x)\right| > \delta\right) = 0.$$

(ii) There exists a constant $C < \infty$ such that

$$\sup_{(x,i)\in\mathbb{Z}\times I} \mathbb{E}_{\mu_N}[\eta_i(x)^2] \le C. \tag{5.4.1}$$

♠

Note that Assumption 5.4.1(ii) is the same as employed in [40, Theorem 1, Assumption (b)] and is trivial for the exclusion process.

Theorem 5.4.1. [Hydrodynamic scaling] *Let $\bar\rho_0, \bar\rho_1 \in C_b(\mathbb{R};\sigma)$ be two initial macroscopic profiles, and let $(\mu_N)_{N\in\mathbb{N}}$ be a sequence of compatible initial conditions. Let $\mathbb{P}_{\mu_N}$ be the law of the measure-valued process*

$$\{X^N(t)\colon\ t \ge 0\}, \qquad X^N(t) := (X_0^N(t), X_1^N(t)),$$

induced by the initial measure μ_N. Then, for any $T, \delta > 0$ and $g \in C_c^\infty(\mathbb{R})$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\langle \mathsf{X}_i^N(t), g\rangle - \int_{\mathbb{R}} \mathrm{d}x\, \rho_i(x,t)g(x)\right| > \delta\right) = 0, \qquad i \in I,$$

where ρ_0 and ρ_1 are the unique continuous and bounded strong solutions of the system

$$\begin{cases}\partial_t\rho_0 = \Delta\rho_0 + \Upsilon(\rho_1 - \rho_0),\\ \partial_t\rho_1 = \epsilon\Delta\rho_1 + \Upsilon(\rho_0 - \rho_1),\end{cases} \tag{5.4.2}$$

with initial conditions

$$\begin{cases}\rho_0(x, 0) = \bar{\rho}_0(x),\\ \rho_1(x, 0) = \bar{\rho}_1(x).\end{cases} \tag{5.4.3}$$

Proof. The proof follows the standard route presented in [153, Section 8] (see also [47, 40]). We still explain the main steps because the two-layer setup is not standard. First of all, note that the macroscopic equation (5.4.2) can be straightforwardly identified by computing the action of the rescaled generator $L^N = L_{\epsilon,\Upsilon/N^2}$ on the cylindrical functions $f_i(\eta) := \eta_i(x),\ i \in \{0, 1\}$, namely ,

$$(L^N f_i)(\eta) = \epsilon^i\,[\eta_i(x+1) - 2\eta_i(x) + \eta_i(x-1)] + \frac{\Upsilon}{N^2}\,[\eta_{1-i}(x) - \eta_i(x)]$$

and hence, for any $g \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\int_0^{tN^2} \mathrm{d}s\, L^N\left(\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_i(x,s)\right) &= \int_0^{tN^2} \mathrm{d}s\,\frac{\epsilon^i}{N}\sum_{x\in\mathbb{Z}} \eta_i(x,s)\,\tfrac{1}{2}\,[g((x+1)/N) - 2g(x/N) + g((x-1)/N)]\\ &+ \int_0^{tN^2} \mathrm{d}s\,\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\frac{\Upsilon}{N^2}\,[\eta_{1-i}(x,s) - \eta_i(x,s)],\end{aligned}$$

where we moved the generator of the simple random walk to the test function by using reversibility w.r.t. the counting measure. By the regularity of g, we thus have

$$\int_0^{tN^2} \mathrm{d}s\, L^N\left(\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_i(x,s)\right) = \int_0^t \mathrm{d}s\,\langle \mathsf{X}_i^N(s), \epsilon^i\Delta g\rangle + \int_0^t \mathrm{d}s\,\Upsilon\,[\langle \mathsf{X}_{1-i}^N(s), g\rangle - \langle \mathsf{X}_i^N(s), g\rangle] + o(\tfrac{1}{N^2}),$$

which is the discrete counterpart of the weak formulation of the right-hand side of (5.4.2), i.e., $\int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}x\,\rho_i(x,s)\Delta g(x) + \Upsilon\int_0^t \mathrm{d}s\int_{\mathbb{R}} \mathrm{d}x\,[\rho_{1-i}(x,s) - \rho_i(x,s)]\,g(x)$. Thus, as a first step, we show that

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\langle \mathsf{X}_i^N(t), g\rangle - \langle \mathsf{X}_i^N(0), g\rangle - \int_0^t \mathrm{d}s\,\langle \mathsf{X}_i^N(s), \epsilon^i\Delta g\rangle - \int_0^t \mathrm{d}s\,\Upsilon\,[\langle \mathsf{X}_{1-i}^N(s) - \mathsf{X}_i^N(s), g\rangle]\right| > \delta\right) \quad = \quad 0.$$

In order to prove the above convergence we employ the Dynkin's martingale formula for Markov processes (see, e.g., [153, Theorem 4.8]), which gives that the process defined as

$$M_i^N(g,t) := \langle \mathsf{X}_i^N(t), g\rangle - \langle \mathsf{X}_i^N(0), g\rangle - \int_0^{tN^2} \mathrm{d}s\, L^N\left(\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_i(x,s)\right)$$

is a martingale w.r.t. the natural filtration generated by the process $\{\eta_t\}_{t\geq 0}$ and with predictable quadratic variation expressed in terms of the carré du champ, i.e.,

$$\langle M_i^N(g,t), M_i^N(g,t)\rangle = \int_0^{tN^2} \mathrm{d}s\,\mathbb{E}_{\mu_N}\left[\Gamma_i^N(g,s)\right]$$

with

$$\Gamma_i^N(g,s) = L^N\left(\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_i(x,s)\right)^2 - 2\left(\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_i(x,s)\right) L^N\left(\frac{1}{N}\sum_{x\in\mathbb{Z}} g(x/N)\,\eta_i(x,s)\right).$$

We then have, by Chebyshev's inequality and Doob's martingale inequality (see, e.g., [102, Section1.3]),

$$\begin{aligned}&\mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\langle \mathsf{X}_i^N(t), g\rangle - \langle \mathsf{X}_i^N(0), g\rangle - \int_0^t \mathrm{d}s\,\langle \mathsf{X}_i^N(s), \epsilon^i\Delta g\rangle - \int_0^t \mathrm{d}s\,\Upsilon\,[\langle \mathsf{X}_{1-i}^N(s), g\rangle - \langle \mathsf{X}_i^N(s), g\rangle]\right| > \delta\right)\\ &\leq \frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left[\sup_{t\in[0,T]}\left|M_i^N(g,s)\right|^2\right] \leq \frac{4}{\delta^2}\mathbb{E}_{\mu_N}\left[\left|M_i^N(g,T)\right|^2\right] = \frac{4}{\delta^2}\mathbb{E}_{\mu_N}\left[\langle M_i^N(g,T), M_i^N(g,T)\rangle^2\right]\end{aligned}$$

$$= \frac{4}{\delta^2 N^2}\mathbb{E}_{\mu_N}\left[\int_0^{N^2T} \mathrm{d}s \sum_{x\in\mathbb{Z}^d} \eta_i(x,s)(1+\sigma\eta_i(x\pm 1,s))\left(g\left(\frac{x\pm 1}{N}\right)-g\left(\frac{x}{N}\right)\right)^2\right] \tag{5.4.4}$$
$$+ \frac{4\Upsilon}{\delta^2 N^4}\mathbb{E}_{\mu_N}\left[\int_0^{N^2T} \mathrm{d}s \sum_{x\in\mathbb{Z}^d}(\eta_i(x,s)+\eta_{1-i}(x,s)+2\sigma\eta_i(x,s)\eta_{1-i}(x,s))g^2\left(\frac{x}{N}\right)\right],$$

where in the last equality we explicitly computed the carré du champ. Let $k\in\mathbb{N}$ be such that the support of g is in $[-k,k]$. Then, by the regularity of g, (5.4.4) is bounded by

$$\frac{4}{\delta^2 N^2}(N^2T)(2k+1)N\frac{\|\nabla g\|_\infty^2}{N^2}\sup_{x\in\mathbb{Z},\ s\in[0,N^2T]}\mathbb{E}_{\mu_N}\left[\eta_i(x,s)(1+\sigma\eta_i(x+1,s))\right]$$
$$+\frac{4\Upsilon}{\delta^2N^4}(N^2T)(2k+1)N\|g\|_\infty^2\sup_{x\in\mathbb{Z},\ s\in[0,N^2T]}\mathbb{E}_{\mu_N}\left[\eta_i(x,s)+\eta_{1-i}(x,s)+2\sigma\eta_i(x,s)\eta_{1-i}(x,s)\right]. \tag{5.4.5}$$

We now show that, as a consequence of (5.4.1), for any $(x,i),\ (y,j)\in\mathbb{Z}\times I$ and $s\geq 0$,

$$\mathbb{E}_{\mu_N}\left[\eta_i(x,s)\right]\leq C,\quad \mathbb{E}_{\mu_N}\left[\eta_i(x,s)\eta_j(y,s)\right]\leq C, \tag{5.4.6}$$

from which we obtain

$$\mathbb{P}_{\mu_N}\left(\sup_{t\in[0,T]}\left|\langle \mathsf{X}_i^N(t),g\rangle-\langle\mathsf{X}_i^N(0),g\rangle-\int_0^t \mathrm{d}s\,\langle\mathsf{X}_i^N(s),\epsilon^i\Delta g\rangle-\int_0^t\mathrm{d}s\,\Upsilon\left[\langle\mathsf{X}_{1-i}^N(s),g\rangle-\langle\mathsf{X}_i^N(s),g\rangle\right]\right|>\delta\right)$$
$$\leq\frac{8T}{\delta^2N}(2k+1)\|\nabla g\|_\infty^2C+\Upsilon\frac{16T}{\delta^2N}(2k+1)\|g\|_\infty^2C,$$

and the desired convergence follows. In order to prove (5.4.6), first of all note that, by the Cauchy-Schwartz inequality, it follows from (5.4.1) that, for any $(x,i),\ (y,j)\in\mathbb{Z}\times I$,

$$\mathbb{E}_{\mu_N}\left[\eta_i(x)\eta_j(y)\right]\leq C. \tag{5.4.7}$$

Moreover, recalling the duality functions given in (5.3.3) and defining the configuration $\xi=\delta_{(x,i)}+\delta_{(y,j)}$ for $(x,i)\neq(y,j)$, we have that $D(\xi,\eta_t)=\eta_i(x,t)\eta_j(y,t)$ and thus, using the classical self-duality relation,

$$\mathbb{E}_{\mu_N}\left[\eta_i(x,t)\eta_j(y,t)\right]=\mathbb{E}_{\mu_N}[D(\xi,\eta_t)]=\int_X\mathbb{E}_\eta[D(\xi,\eta_t)]\,\mathrm{d}\mu_N(\eta)$$
$$=\int_X\mathbb{E}_\xi[D(\xi_t,\eta)]\,\mathrm{d}\mu_N(\eta)=\mathbb{E}_\xi\left[\mathbb{E}_{\mu_N}[D(\xi_t,\eta)]\right].$$

Labeling the particles in the dual configuration as (X_t,i_t) and (Y_t,j_t) with initial conditions $(X_0,i_0)=(x,i)$ and $(Y_0,j_0)=(y,j)$, we obtain

$$\mathbb{E}_{\mu_N}\left[\eta_i(x,t)\eta_j(y,t)\right]=\mathbb{E}_\xi\left[\mathbb{E}_{\mu_N}[\eta_{i_t}(X_t)\eta_{j_t}(Y_t)\mathbf{1}_{(X_t,i_t)\neq(Y_t,j_t)}]+\mathbb{E}_{\mu_N}[\eta_{i_t}(X_t)(\eta_{i_t}(X_t)-1)\mathbf{1}_{(X_t,i_t)=(Y_t,j_t)}]\right]$$
$$\leq\mathbb{E}_\xi\left[\mathbb{E}_{\mu_N}[\eta_{i_t}(X_t)\eta_{j_t}(Y_t)]\right]\leq\mathbb{E}_\xi\left[\sup_{(x,i),(y,j)\in\mathbb{Z}\times\{0,1\}}\mathbb{E}_{\mu_N}[\eta_i(x)\eta_j(y)]\right]\leq C, \tag{5.4.8}$$

where we used (5.4.7) in the last inequality. Similarly, for $\xi=\delta_{(x,i)}$ and (X_t,i_t) the dual particle with initial condition $(X_0,i_0)=(x,i)$, we have that $\mathbb{E}_{\mu_N}\left[\eta_i(x,t)\right]\leq\mathbb{E}_{\mu_N}\left[D(\xi,\eta_t)\right]=\mathbb{E}_\xi[\mathbb{E}_{\mu_N}[\eta_{i_t}(X_t)]]$. Using that $\eta_i(x)\leq\eta_i(x)^2$ for any $(x,i)\in\mathbb{Z}\times I$ and using (5.4.1), we obtain (5.4.6). The proof is concluded after showing the following:

(i) Tightness holds for the sequence of distributions of the processes $\{\mathsf{X}_i^N\}_{N\in\mathbb{N}}$, denoted by $\{Q_N\}_{N\in\mathbb{N}}$.

(ii) All limit points coincide and are supported by the unique path $\mathsf{X}_i(t,\mathrm{d}x)=\rho_i(x,t)\,\mathrm{d}x$, with ρ_i the unique weak (and in particular strong) bounded and continuous solution of (5.4.2).

While for (i) we provide an explanation, we skip the proof of (ii) because it is standard and is based on PDE arguments, namely, the existence and the uniqueness of the solutions in the class of continuous-time functions with values in $C_b(\mathbb{R},\sigma)$ (we refer to [153, Lemma 8.6 and 8.7] for further details), and the fact that Assumption 5.4.1(i) ensures that the initial condition of (5.4.2) is also matched.

Tightness of the sequence $\{Q_N\}_{N\in\mathbb{N}}$ follows from the compact containment condition on the one hand, i.e., for any $\delta>0$ and $t>0$ there exists a compact set $K\subset M$, with M the space of Radon measures, such that

$$\mathbb{P}_{\mu_N}(\mathsf{X}_i^N(t)\in K)>1-\delta\quad\forall N\in\mathbb{N},$$

and the equi-continuity condition on the other hand, i.e.,

$$\limsup_{N\to\infty} \mathbb{P}_{\mu_N}(\omega(\mathsf{X}_i^N,\delta,T)) \geq \mathfrak{e}) \leq \mathfrak{e}$$

for $\omega(\alpha,\delta,T) := \sup\{d_M(\alpha(s),\alpha(t)) : s,t\in[0,T], |s-t|\leq\delta\}$ with d_M the metric on Radon measures defined as

$$d_M(\nu_1,\nu_2) := \sum_{j\in\mathbb{N}} 2^{-j}\left(1\wedge\left|\int_{\mathbb{R}}\phi_j \mathrm{d}\nu_1 - \int_{\mathbb{R}}\phi_j \mathrm{d}\nu_2\right|\right)$$

for an appropriately chosen sequence of functions $(\phi_j)_{j\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$. We refer to [153, Section A.10] for details on the above metric and to the proof of [153, Lemma 8.5] for the equi-continuity condition. We conclude by proving the compact containment condition. Define

$$K := \left\{\nu\in M \ s.t. \ \exists\, k\in\mathbb{N} \ s.t. \ \nu[-\ell,\ell]\leq A(2\ell+1)\ell^2 \ \forall\ \ell\in[k,\infty]\cap\mathbb{N}\right\}$$

with $A>0$ such that $\frac{C\pi}{6A}<\delta$. By [153, Proposition A.25], we have that K is a pre-compact subset of M. Moreover, by the Markov inequality and Assumption 5.4.1(ii), it follows that

$$\begin{aligned}\mathbb{P}_{\mu_N}(\mathsf{X}_i^N(t)\in\bar{K}^c) &\leq \sum_{\ell\in\mathbb{N}}\mathbb{P}_{\mu_N}\left(\mathsf{X}_i^N(t)([-\ell,\ell])\geq A(2\ell+1)\ell^2\right)\leq\sum_{\ell\in\mathbb{N}}\frac{1}{A(2\ell+1)\ell^2}\mathbb{E}_{\mu_N}\left[\mathsf{X}_i^N(t)([-\ell,\ell])\right]\\ &= \sum_{\ell\in\mathbb{N}}\frac{1}{A(2\ell+1)\ell^2}\sum_{x\in[-\ell,\ell]\cap\frac{\mathbb{Z}}{N}}\mathbb{E}_{\mu_N}\left[\eta_i(x,tN^2)\right]\leq\sum_{\ell\in\mathbb{N}}\frac{1}{A(2\ell+1)\ell^2}\frac{2\ell N+1}{N}C\leq\frac{C}{A}\sum_{\ell\in\mathbb{N}}\frac{1}{\ell^2}<\delta,\end{aligned}$$

and, thus, $\mathbb{P}_{\mu_N}(\mathsf{X}_i^N(t)\in K)>1-\delta$ for any N, concluding the proof. □

Remark 5.4.2. [Total density] (i) If ρ_0,ρ_1 are smooth enough and satisfy (5.4.2), then by taking extra derivatives we see that the total density $\rho:=\rho_0+\rho_1$ satisfies the *thermal telegrapher equation*

$$\partial_t\left(\partial_t\rho+2\Upsilon\rho\right) = -\epsilon\Delta(\Delta\rho)+(1+\epsilon)\Delta\left(\partial_t\rho+\Upsilon\rho\right), \tag{5.4.9}$$

which is second order in ∂_t and fourth order in ∂_x (see [4, 95] for a derivation). Note that (5.4.9) shows that the total density does not satisfy the usual diffusion equation. This fact will be investigated in detail in the next section, where we will analyse the non-Fick property of ρ.
(ii) If $\epsilon=1$, then the total density ρ satisfies the *heat equation* $\partial_t\rho=\Delta\rho$.
(iii) If $\epsilon=0$, then (5.4.9) reads

$$\partial_t\left(\partial_t\rho+2\Upsilon\rho\right)=\Delta\left(\partial_t\rho+\Upsilon\rho\right),$$

which is known as the *strongly damped wave equation*. The term $\partial_t(2\lambda\rho)$ is referred to as frictional damping, the term $\Delta(\partial_t\rho)$ as Kelvin-Voigt damping (see [39]). ♠

Remark 5.4.3. [Literature] We mention in passing that in [111] hydrodynamic scaling of interacting particles with internal states has been considered in a different setting and with a different methodology. ♠

5.5 Existence, uniqueness and representation of the solution of the double diffusivity model

The existence and uniqueness of a continuous-time solution $(\rho_0(t),\,\rho_1(t))$ with values in $C_b(\mathbb{R},\sigma)$ of the system in (5.4.2) can be proved by standard Fourier analysis. Below we recall some known results that have a more probabilistic interpretation.

Stochastic representation of the solution. The system in (5.4.2) fits in the realm of switching diffusions (see e.g. [161]), which are widely studied in the mathematical finance literature. Indeed, let $\{i_t\colon\ t\geq 0\}$ be the pure jump process on state space $I=\{0,1\}$ that switches at rate Υ, whose generator acting on bounded functions $g\colon\ I\to\mathbb{R}$ is

$$(Ag)(i) := \Upsilon(g(1-i)-g(i)), \qquad i\in I.$$

Let $\{X_t\colon\ t\geq 0\}$ be the stochastic process on $\mathbb{R}$ solving the stochastic differential equation

$$\mathrm{d}X_t = \psi(i_t)\,\mathrm{d}W_t,$$

where $W_t = B_{2t}$ with $\{B_t\colon\ t \geq 0\}$ standard Brownian motion, and $\psi\colon\ I \to \{D_0, D_1\}$ is given by

$$\psi := D_0\,\mathbf{1}_{\{0\}} + D_1\,\mathbf{1}_{\{1\}},$$

with $D_0 = 1$ and $D_1 = \epsilon$ in our setting. Let $\mathcal{L} = \mathcal{L}_{\epsilon,\Upsilon}$ be the generator defined by

$$(\mathcal{L}f)(x,i) := \lim_{t\downarrow 0} \frac{1}{t}\, \mathbb{E}_{x,i}[f(X_t, i_t) - f(x,i)]$$

for $f\colon\ \mathbb{R}\times I \to \mathbb{R}$ such that $f(\cdot, i) \in C_0^2(\mathbb{R})$. Then, via a standard computation (see e.g. [81, Eq.(4.4)]), it follows that

$$\begin{aligned}(\mathcal{L}f)(x,i) &= \psi(i)(\Delta f)(x,i) + \Upsilon[f(x,1-i) - f(x,i)]\\ &= \begin{cases}\Delta f(x,0) + \Upsilon\,[f(x,1) - f(x,0)], & i = 0,\\ \epsilon\Delta f(x,1) + \Upsilon\,[f(x,0) - f(x,1)], & i = 1.\end{cases}\end{aligned}$$

We therefore have the following result that corresponds to [81, Chapter 5, Section 4, Theorem 4.1](see also [161, Theorem 5.2]).

Theorem 5.5.1. [Stochastic representation of the solution] *Suppose that $\bar{\rho}_i\colon\ \mathbb{R} \to \mathbb{R}$ for $i \in I$ are continuous and bounded. Then* (5.4.2) *has a unique solution given by*

$$\rho_i(x,t) = \mathbb{E}_{(x,i)}[\bar{\rho}_{i_t}(X_t)], \qquad i \in I.$$

Note that if there is only one particle in the system (5.2.1), then we are left with a single random walk, say $\{Y_t\colon\ t \geq 0\}$, whose generator, denoted by A, acts on bounded functions $f\colon\ \mathbb{Z}\times I \to \mathbb{R}$ as

$$(\mathsf{A}f)(y,i) = \psi(i)\left[\sum_{z\sim y}[f(z,i) - f(y,i)]\right] + \Upsilon\,[f(y,1-i) - f(y,i)].$$

After we apply the generator to the function $f(y,i) = y$, we get

$$(Af)(y,i) = 0,$$

i.e., the position of the random walk is a martingale. Computing the quadratic variation via the carré du champ, we find

$$\mathsf{A}(Y_t^2) = \psi(i_t)[(Y_t+1)^2 - Y_t^2] + \psi(i_t)[(Y_t-1)^2 - Y_t^2] = 2\psi(i_t).$$

Hence the predictable quadratic variation is given by

$$\int_0^t \mathrm{d}s\, 2\psi(i_s).$$

Note that for $\epsilon = 0$ the latter equals the total amount of time the random walk is not *dormant* up to time t.

When we diffusively scale the system (scaling the reaction term was done at the beginning of Section 5.4), the quadratic variation becomes

$$\int_0^{tN^2} \mathrm{d}s\,\psi(i_{N,s}) = \int_0^t \mathrm{d}r\,\psi(i_r).$$

As a consequence, we have the following invariance principle:

Given the path of the process $\{i_t\colon\ t \geq 0\}$,

$$\lim_{N\to\infty} \frac{Y_{N^2 t}}{N} = W_{\int_0^t \mathrm{d}r\,\sqrt{\psi(i_r)}},$$

where $W_t = B_{2t}$ with $\{B_t\colon\ t \geq 0\}$ is standard Brownian motion.

Thus, if we knew the path of the process $\{i_r\colon\ r \geq 0\}$, then we could express the solution of the system in (5.4.2) in terms of a time-changed Brownian motion. However, even though $\{i_r\colon\ r \geq 0\}$ is a simple flipping process, we cannot say much explicitly about the random time $\int_0^t \mathrm{d}r\,\sqrt{\psi(i_r)}$. We therefore look for a simpler formula, where the relation to a Brownian motion with different velocities is more explicit. We achieve this by looking at the resolvent of the generator $\mathcal{L}$. In the following, we denote by $\{S_t,\, t \geq 0\}$ the semigroup on $C_b(\mathbb{R})$ of $\{W_t\,:\ t \geq 0\}$.

Proposition 5.5.2. [Resolvent] *Let $f\colon\ \mathbb{R}\times I\to\mathbb{R}$ be a bounded and smooth function. Then, for $\lambda>0$, $\epsilon\in(0,1]$ and $i\in I$,*

$$\begin{aligned}
&(\lambda \boldsymbol{I}-\mathcal{L})^{-1}f(x,i)\\
&=\int_0^\infty \mathrm{d}t\,\frac{1}{\epsilon^i}\,\mathrm{e}^{-\frac{1+\epsilon}{\epsilon}\ell(\Upsilon,\lambda)t}\left(\cosh(tc_\epsilon(\Upsilon,\lambda))+\frac{1-\epsilon}{\epsilon}\ell_\epsilon(\Upsilon,\lambda)\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\lambda)}\right)(S_tf(\cdot,i))(x)\\
&+\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\frac{1+\epsilon}{\epsilon}\ell(\Upsilon,\lambda)t}\left(\frac{\Upsilon}{\epsilon}\sinh(tc_\epsilon(\Upsilon,\lambda))\right)(S_tf(\cdot,1-i))(x),
\end{aligned}\tag{5.5.1}$$

where $c_\epsilon(\Upsilon,\lambda)=\sqrt{\left(\frac{1-\epsilon}{\epsilon}\right)^2\ell(\Upsilon,\lambda)^2+\frac{\Upsilon^2}{\epsilon}}$ and $\ell(\Upsilon,\lambda)=\frac{\Upsilon+\lambda}{2}$, while for $\epsilon=0$,

$$(\lambda \boldsymbol{I}-\mathcal{L})^{-1}f(x,i)=\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\lambda\frac{2\Upsilon+\lambda}{\Upsilon+\lambda}t}\left(\left(\tfrac{\Upsilon}{\lambda+\Upsilon}\right)^i(S_tf(\cdot,0))(x)+\left(\tfrac{\Upsilon}{\Upsilon+\lambda}\right)^{i+1}(S_tf(\cdot,1))(x)\right).\tag{5.5.2}$$

Proof. The proof is split into two parts.

Case $\epsilon>0$. We can split the generator $\mathcal{L}$ as

$$\mathcal{L}=\psi(i)\tilde{\mathcal{L}}=\psi(i)\left(\Delta+\frac{1}{\psi(i)}A\right)=\psi(i)(\Delta+\tilde{A}),$$

i.e., we decouple X_t and i_t in the action of the generator. We can now use the Feynman-Kac formula to express the resolvent of the operator $\mathcal{L}$ in terms of the operator $\tilde{\mathcal{L}}$. Denoting by $\tilde{\mathbb{E}}$ the expectation of the process with generator $\tilde{\mathcal{L}}$, we have, for $\lambda\in\mathbb{R}$,

$$(\lambda \boldsymbol{I}-\mathcal{L})^{-1}f(x,i)=\left(\frac{\lambda \boldsymbol{I}}{\psi}-\tilde{\mathcal{L}}\right)^{-1}\left(\frac{f(x,i)}{\psi(i)}\right)=\int_0^\infty \mathrm{d}t\,\tilde{\mathbb{E}}_{(x,i)}\left[\mathrm{e}^{-\int_0^t \mathrm{d}s\,\frac{\lambda}{\psi(i_s)}}\,\frac{f(X_t,i_t)}{\psi(i_t)}\right],$$

and by the decoupling of X_t and i_t under $\tilde{\mathcal{L}}$, we get

$$\begin{aligned}
&(\lambda \boldsymbol{I}-\mathcal{L})^{-1}f(x,i) &(5.5.3)\\
&=\int_0^\infty \mathrm{d}t\,\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda\int_0^t \mathrm{d}s\,\frac{1}{\psi(i_s)}}\,\frac{\mathbf{1}_{\{0\}}(i_t)}{\psi(i_t)}\right](S_tf(\cdot,0))(x)+\int_0^\infty \mathrm{d}t\,\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda\int_0^t\frac{1}{\psi(i_s)}}\,\frac{\mathbf{1}_{\{1\}}(i_t)}{\psi(i_t)}\right](S_tf(\cdot,1))(x) &(5.5.4)\\
&=\int_0^\infty \mathrm{d}t\,\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda\int_0^t \mathrm{d}s\,\frac{1}{\psi(i_s)}}\,\mathbf{1}_{\{0\}}(i_t)\right](S_tf(\cdot,0))(x)+\frac{1}{\epsilon}\int_0^\infty \mathrm{d}t\,\tilde{\mathbb{E}}_i\left[\mathrm{e}^{-\lambda\int_0^t \mathrm{d}s\,\frac{1}{\psi(i_s)}}\,\mathbf{1}_{\{1\}}(i_t)\right](S_tf(\cdot,1))(x). &(5.5.5)
\end{aligned}$$

Defining

$$A:=\begin{bmatrix}-\Upsilon & \Upsilon\\ \Upsilon & -\Upsilon\end{bmatrix},\qquad \psi_\epsilon:=\begin{bmatrix}1 & 0\\ 0 & \epsilon\end{bmatrix},$$

and using again the Feynman-Kac formula, we have

$$(\lambda \boldsymbol{I}-\mathcal{L})^{-1}\begin{bmatrix}f(x,0)\\ f(x,1)\end{bmatrix}=\int_0^\infty \mathrm{d}t\,K_\epsilon(t,\lambda)\begin{bmatrix}(S_tf(\cdot,0))(x)\\ (S_tf(\cdot,1))(x)\end{bmatrix}$$

with $K_\epsilon(t,\lambda)=\mathrm{e}^{t\psi_\epsilon^{-1}(-\lambda \boldsymbol{I}+A)}\psi_\epsilon^{-1}$.

Using the explicit formula for the exponential of a 2×2 matrix (see e.g. [19, Corollary 2.4]), we obtain

$$\mathrm{e}^{t\psi_\epsilon^{-1}(-\lambda \boldsymbol{I}+A)}=\mathrm{e}^{-\frac{1+\epsilon}{\epsilon}\ell(\Upsilon,\lambda)t}\begin{bmatrix}\cosh(tc_\epsilon(\Upsilon,\lambda))+\frac{1-\epsilon}{\epsilon}\ell(\Upsilon,\lambda)\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)} & \Upsilon\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)}\\ \frac{\Upsilon}{\epsilon}\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)} & \cosh(tc_\epsilon(\Upsilon,\lambda))-\frac{1-\epsilon}{\epsilon}\ell(\Upsilon,\lambda)\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)}\end{bmatrix}\tag{5.5.6}$$

with $c_\epsilon(\Upsilon,\lambda)=\sqrt{\left(\frac{1-\epsilon}{\epsilon}\right)^2\ell(\Upsilon,\lambda)^2+\frac{\Upsilon^2}{\epsilon}}$ and $\ell(\Upsilon,\lambda)=\frac{\Upsilon+\lambda}{2}$, from which we obtain (5.5.1).

Case $\epsilon=0$. We derive $K_0(t,\lambda)$ by taking the limit $\epsilon\downarrow 0$ in the previous expression, i.e., $K_0(t,\lambda)=\lim_{\epsilon\downarrow 0}K_\epsilon(t,\lambda)$. We thus have that $K_0(t,\lambda)$ is equal to

$$\lim_{\epsilon\downarrow 0}\mathrm{e}^{-\frac{1+\epsilon}{\epsilon}\ell(\Upsilon,\lambda)t}\begin{bmatrix}\cosh(tc_\epsilon(\Upsilon,\lambda))+\frac{1-\epsilon}{\epsilon}\ell(\Upsilon,\lambda)\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)} & \frac{\Upsilon}{\epsilon}\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)}\\ \frac{\Upsilon}{\epsilon}\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)} & \frac{1}{\epsilon}\cosh(tc_\epsilon(\Upsilon,\lambda))-\frac{1-\epsilon}{\epsilon^2}\ell(\Upsilon,\lambda)\frac{\sinh(tc_\epsilon(\Upsilon,\lambda))}{c_\epsilon(\Upsilon,\lambda)}\end{bmatrix}$$

$$= \mathrm{e}^{-\lambda \frac{2\Upsilon+\lambda}{\Upsilon+\lambda} t} \begin{bmatrix} 1 & \frac{\Upsilon}{\Upsilon+\lambda} \\ \frac{\Upsilon}{\Upsilon+\lambda} & \left(\frac{\Upsilon}{\Upsilon+\lambda}\right)^2 \end{bmatrix},$$

from which (5.5.2) follows. □

Remark 5.5.3. [Symmetric layers] Note that for $\epsilon = 1$ we have

$$(\lambda \boldsymbol{I} - \mathcal{L})^{-1} f(x,i) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\lambda t} \left(\tfrac{1+\mathrm{e}^{-2\Upsilon t}}{2} (S_t f(\cdot, i))(x) + \tfrac{1-\mathrm{e}^{-2\Upsilon t}}{2} (S_t f(\cdot, 1-i))(x) \right).$$

♠

We conclude this section by noting that the system in (5.4.2) was studied in detail in [4, 95]. By taking Fourier and Laplace transforms and inverting them, it is possible to deduce explicitly the solution, which is expressed in terms of solutions to the classical heat equation. More precisely, using formula [95, Eq.2.2], we have that

$$\rho_0(x,t) \quad = \quad \mathrm{e}^{-\Upsilon t} (S_t \bar{\rho}_0)(x) \; + \; \frac{\Upsilon}{1-\epsilon} \mathrm{e}^{-\Upsilon t} \int_{\epsilon t}^{t} \mathrm{d}s \left(\left(\frac{s-\epsilon t}{t-s} \right)^{1/2} I_1(\upsilon(s)) \, (S_s \bar{\rho}_0)(x) + I_0(\upsilon(s)) \, (S_s \bar{\rho}_1)(x) \right) \tag{5.5.7}$$

and

$$\rho_1(x,t) \quad = \quad \mathrm{e}^{-\Upsilon t} (S_{\epsilon t} \bar{\rho}_1)(x) \; + \; \frac{\Upsilon}{1-\epsilon} \mathrm{e}^{-\sigma t} \int_{\epsilon t}^{t} \mathrm{d}s \left(\left(\frac{s-\epsilon t}{t-s} \right)^{-1/2} I_1(\upsilon(s)) \, (S_s \bar{\rho}_1)(x) + I_0(\upsilon(s)) \, (S_s \bar{\rho}_0)(x) \right), \tag{5.5.8}$$

where $\upsilon(s) = \frac{2\Upsilon}{1-\epsilon}((t-s)(s-\epsilon t))^{1/2}$, and $I_0(\cdot)$ and $I_1(\cdot)$ are the modified Bessel functions.

Part III

Boundary driven inhomogeneous interacting particle systems: a duality approach

Chapter 6

Orthogonal polynomial duality of boundary driven particle systems and non-equilibrium correlations

6.1 Introduction

Exactly solvable models have played an important role in the understanding of fundamental properties of non-equilibrium steady states such as the presence of long-range correlations and the non-locality of large deviation free energies [55, 57, 21, 86]. As explained in Chapter 1, an important class of particle systems which is slightly broader than exactly solvable models are the models which satisfy self-duality or, more generally, duality properties. Recall that (see Section 2.5.2) such systems when coupled to appropriate reservoirs are dual to systems where the reservoirs are replaced by absorbing boundaries, and the computation of n-point correlation functions in the original system reduces to the computation of absorption probabilities in a dual system with n particles. Even when these absorption probabilities cannot be obtained in closed form, e.g. when Bethe ansatz is not available, still the connection between the non-equilibrium system coupled to reservoirs and the absorbing dual turns out to be very useful to obtain macroscopic properties such as the hydrodynamic limit, fluctuations, mixing and propagation of chaos and local equilibrium (see e.g. [88, 117, 141, 79, 82]).

In recent works (self-)duality with orthogonal polynomials (see Section 2.5.1) has been studied in several particle systems including generalized symmetric exclusion processes (SEP), symmetric inclusion process (SIP) and associated diffusion processes such as the Brownian momentum process. Orthogonal polynomials in the occupation number variables are a natural extension of the higher order correlation functions studied in SEP in [57]. Orthogonal polynomial duality is very useful in the study of fluctuation fields [11, 40], identifies a set of functions with positive time dependent correlations and is useful in the study of speed of relaxation to equilibrium [35]. So far, orthogonal polynomial duality has not been obtained in the context of boundary driven systems.

In this chapter we start extending the classical dualities from [32] for a generalized class of boundary driven systems, where we allow both for edge disorder and well-chosen site disorder. We then use a symmetry of the dual absorbing system in order to derive duality with orthogonal polynomials for these systems.

More precisely, we consider three classes of interacting particle systems: partial symmetric exclusion [74] where we allow edge-dependent conductances and a site-varying maximal occupancy, symmetric inclusion where we allow edge-dependent conductances and a site-varying "attraction parameter", and independent walkers. We couple these systems to two reservoirs, with reservoir parameters θ_L and θ_R. The precise meaning of the reservoir parameters θ_L and θ_R will be explained in detail later; for the moment one can think of them – roughly – as being proportional to the densities of left and right reservoirs, respectively. Moreover, the bulk system can be defined on any graph. Hence, our setting includes the standard one of a chain coupled to reservoirs at left and right ends, but it is in no way restricted to that setting. The only important geometrical requirement is the presence of precisely two reservoirs. When $\theta_L = \theta_R = \theta$ the system is in equilibrium, with a unique reversible product measure μ_θ. When $\theta_L \neq \theta_R$ the system evolves towards a unique non-equilibrium stationary measure μ_{θ_L,θ_R}. At stationarity, by means of classical dualities with a dual system that has two absorbing sites, corresponding to the reservoirs in the original system, we obtain correlation inequalities, thereby extending and strengthening those from [85]. In particular, the dual particle system dynamics does not depend on the reservoir parameters θ_L and θ_R.

Next, for the same pair of boundary driven and purely absorbing systems, we introduce orthogonal polynomial dualities. The orthogonal duality functions are in product form and the factors associated to the bulk sites are the same orthogonal polynomials as those appearing for the same particle systems not coupled to reservoirs (see e.g. [78, 146]), while the remaining factors corresponding to the absorbing sites have a form depending on the reservoir parameters. The orthogonal polynomials carry themselves a parameter θ which corresponds to the equilibrium reversible product measure μ_θ w.r.t. which they are orthogonal.

We then give various applications of these orthogonal polynomial dualities to properties of correlation functions in the non-equilibrium stationary measure μ_{θ_L,θ_R}. First we prove that the correlations of order n of the occupation variables at different locations $x_1, \ldots, x_n$, as well as the cumulants of order n, are of the form $(\theta_L - \theta_R)^n$ multiplied by a universal function ψ which depends only on $x_1, \ldots, x_n$ and the dual particle system dynamics, thus, not depending on θ_L and θ_R. We prove, in fact, a stronger result, namely that whenever the system is started from a local equilibrium product measure, then, at any later time $t > 0$, the n-point correlations are of the form $(\theta_L - \theta_R)^n$ multiplied by a universal function ψ_t which, again, does not depend on the reservoir parameters θ_L and θ_R, but only on the dual system dynamics.

Finally, we relate the joint moment generating function of the occupation variables to an expectation in the absorbing dual. Despite the fact that this quantity can in general not be obtained in analytic form, the relation is useful, both from the point of view of simulations, as well as from the point of view of computing macroscopic limits such as density fluctuation fields and large deviations of the density profile.

6.1.1 Summary of main results, related works and organization of the chapter

As a conclusion of this introduction, we summarize more schematically here, for the convenience of the reader, our main contributions in relation to previous works and how we organize the rest of the chapter.

We introduce a class of boundary driven particle systems in a general inhomogeneous framework – generalizing, in particular, those considered in, e.g., [32, 57] – showing that classical dualities may be extended beyond homogeneous systems. As a first main result, employing these classical dualities, we show that *correlations* of interacting systems are *monotone in time* when starting from suitable local equilibrium product measures. As a consequence, we deduce a family of correlation inequalities, improving on those established for homogeneous symmetric exclusion and inclusion processes in, e.g., [83, 85, 126].

As a second main result, in our context of boundary driven systems, we derive the *orthogonal polynomial dualities*, previously studied in [79, 146, 34, 90] for closed systems. To this purpose, we develop a new method, which is of independent interest and is based on the relation between orthogonal and classical duality functions. For further details, we refer to the discussion following Theorem 6.4.1.

As a third main result, by suitably tilting these orthogonal dualities, we show that *n-point non-equilibrium stationary correlations* and *cumulants* exhibit a *universal factorized structure*, one factor consisting in a simple expression in the reservoir parameters and the other factor depending only on the underlying geometry of the system. This result holds for both boundary driven exclusion and inclusion processes in presence of edge and site disorder. In particular, for these more general systems, this result recovers the same structure previously obtained for the boundary driven one-dimensional SEP in [57] by means of the explicit knowledge via matrix formulation of the non-equilibrium steady state.

The rest of the chapter is organized as follows. In Section 6.2 we define the boundary driven particle systems and their dual absorbing processes as well as introducing the classical duality functions. In Section 6.3 we study properties and correlation inequalities for the equilibrium and non-equilibrium stationary measures. In Section 6.4 we derive orthogonal duality functions between the boundary driven and the absorbing systems. In Section 6.5 we obtain the aforementioned universal expression for the higher order correlations in the non-equilibrium steady state. In the same section, the same structure is recovered for more general correlations at finite times when started from a local equilibrium product measure. Section 6.6 is devoted to a relation between weighted exponential generating functions of the occupation variables at stationarity and the correlation functions obtained in the previous section. In conclusion, Section 6.7 contains part of the proof of Theorem 7.3.1 in Section 6.3.

6.2 Setting

In this section, we start by introducing the common geometry and the disorder on which the particle dynamics takes place. Then, we couple this "bulk" system to two reservoirs at possibly different densities.

6.2.1 Boundary driven particle systems

We consider three particle systems with either an exclusion, inclusion or no interaction. All these systems will evolve on a set of sites $V = \{1, \ldots, N\}$ ($N \in \mathbb{N}$) and the rate of particle exchanges between two sites x and $y \in V$ will be proportional to some given (symmetric) conductance $\omega_{\{x,y\}} \in [0, \infty)$. Sites x and $y \in V$ for which $\omega_{\{x,y\}} \neq 0$ will be considered as connected, indicated by $x \sim y$. In what follows, we will assume that $\omega_{\{x,x\}} = 0$ for all $x \in V$ and that the induced graph $(V, \sim)$ is connected. We will further attach to each site $x \in V$ a value $\alpha_x \in \mathbb{N}$. While the conductances $\boldsymbol{\omega} = \{\omega_{\{x,y\}} : x, y \in V\}$ represent the bond disorder, the collection $\boldsymbol{\alpha} = \{\alpha_x : x \in V\}$ stands for the site disorder. This disorder may be thought, e.g., as a realization of a random environment (see, e.g., [141, 74]); however, our work in this chapter is not focusing on homogenization properties arising from the randomness of the disorder. Instead, we consider the disorder as deterministic and parameterizing the model all throughout the chapter.

The set V endowed with the disorder $(\boldsymbol{\omega}, \boldsymbol{\alpha})$ is referred to as *bulk* of the system. This bulk is in contact with a left and a right reservoir through respectively site 1 and site $N \in V$. Particle exchanges between the bulk sites and the reservoirs is tuned by a set of non-negative parameters ω_L, ω_R, θ_L, θ_R, α_L and α_R as explained in the paragraph below.

Particle dynamics

In this setting, for each choice of the parameter $\sigma \in \{-1, 0, 1\}$, we introduce a boundary driven particle system $\{\eta_t : t \geq 0\}$ as a Markov process with $\mathcal{X}$, given by

$$\mathcal{X} = \begin{cases} \prod_{x \in V} \{0, \ldots, \alpha_x\} & \text{if } \sigma = -1 \\ \prod_{x \in V} \{0, 1, \ldots\} = \mathbb{N}_0^V & \text{otherwise} , \end{cases}$$

denoting the configuration space, with $\eta \in \mathcal{X}$ standing for a particle configuration and with $\eta(x)$ indicating the number of particles at site $x \in V$ for the configuration $\eta \in \mathcal{X}$. The particle dynamics is described by the infinitesimal generator $\mathcal{L}$, whose action on bounded functions $f : \mathcal{X} \to \mathbb{R}$ reads as follows:

$$\mathcal{L} f(\eta) = \mathcal{L}^{\text{bulk}} f(\eta) + \mathcal{L}^{L,R} f(\eta) . \tag{6.2.1}$$

In the above expression, the generator $\mathcal{L}^{\text{bulk}}$ describes the bulk part of the dynamics and is given by

$$\mathcal{L}^{\text{bulk}} f(\eta) = \sum_{x \sim y} \omega_{\{x,y\}} \, \mathcal{L}_{\{x,y\}} f(\eta) \tag{6.2.2}$$

where the summation above runs over the unordered pairs of sites and with the single-bond generator $\mathcal{L}_{\{x,y\}}$ given by

$$\begin{aligned} \mathcal{L}_{\{x,y\}} f(\eta) = {} & \eta(x) \, (\alpha_y + \sigma \eta(y)) \, (f(\eta^{x,y}) - f(\eta)) \\ & + \eta(y) \, (\alpha_x + \sigma \eta(x)) \, (f(\eta^{y,x}) - f(\eta)) , \end{aligned}$$

where $\eta^{x,y} = \eta - \delta_x + \delta_y \in \mathcal{X}$, i.e. the configuration in which a particle (if any) has been removed from $x \in V$ and placed at $y \in V$. The boundary part of the dynamics is described by the generator $\mathcal{L}^{L,R}$ in (6.2.1) as follows:

$$\mathcal{L}^{L,R} f(\eta) = \omega_L \, \mathcal{L}_L f(\eta) + \omega_R \, \mathcal{L}_R f(\eta) , \tag{6.2.3}$$

with

$$\begin{aligned} \mathcal{L}_L f(\eta) = {} & \eta(1) \, (\alpha_L + \sigma \alpha_L \theta_L) \, (f(\eta^{1,-}) - f(\eta)) \\ & + \alpha_L \theta_L \, (\alpha_1 + \sigma \eta(1)) \, (f(\eta^{1,+}) - f(\eta)) \end{aligned} \tag{6.2.4}$$

and

$$\mathcal{L}_R f(\eta) = \eta(N)\,(\alpha_R + \sigma\alpha_R\theta_R)\left(f(\eta^{N,-}) - f(\eta)\right) + \alpha_R\theta_R\left(\alpha_N + \sigma\eta(N)\right)\left(f(\eta^{N,+}) - f(\eta)\right)\,, \tag{6.2.5}$$

where $\eta^{x,-} \in \mathcal{X}$, resp. $\eta^{x,+} \in \mathcal{X}$, denotes the configuration obtained from η by removing, resp. adding, a particle from, resp. to, site $x \in V$. In the above dynamics, creation and annihilation of particles occurs at sites $x = 1$ and $x = N$ due to the interaction with a reservoir.

We note that, depending on the choice of the value $\sigma \in \{-1, 0, 1\}$ in the definition of the generator $\mathcal{L}$ in (6.2.1), we recover either the *symmetric partial exclusion process* (SEP) for $\sigma = -1$, a system of *independent random walkers* (IRW) for $\sigma = 0$ or the *symmetric inclusion process* (SIP) for $\sigma = 1$ in contact with left and right reservoirs and in presence of disorder.

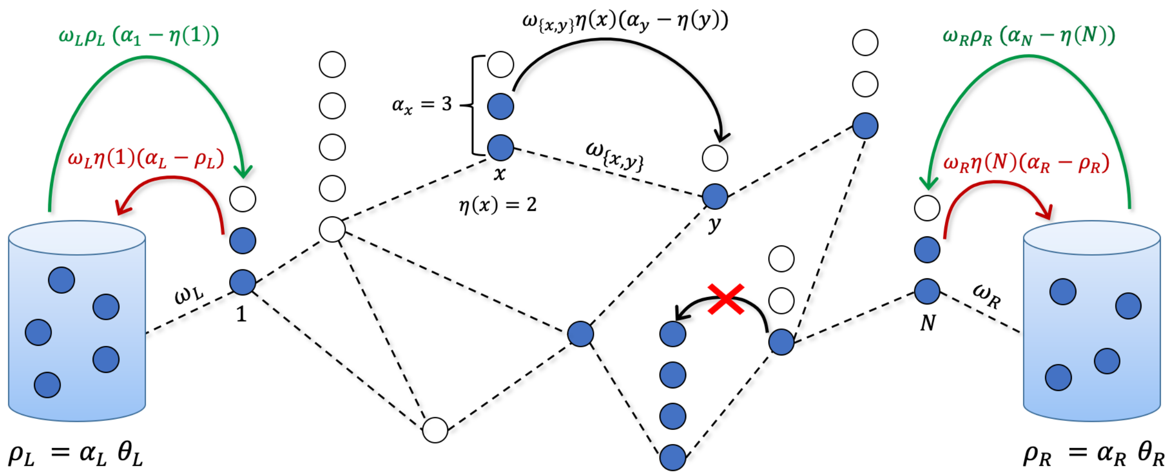


Figure 6.1: Schematic description of the partial exclusion process (SEP) dynamics in contact with left and right reservoirs.

The parameters $\boldsymbol{\alpha} = \{\alpha_x : x \in V\} \subset \mathbb{N}$ have the interpretation of *maximal occupancies* for SEP ($\sigma = -1$) of the sites of V (see Fig. 6.1). For IRW ($\sigma = 0$) and SIP ($\sigma = 1$), $\alpha_x \in \mathbb{N}$ stands for the *site attraction parameter* of the site $x \in V$. We observe that the choice $\boldsymbol{\alpha} \subset \mathbb{N}$ rather than $(0, \infty)$ is needed only in the context of the exclusion process; however, for the sake of uniformity of notation, we adopt $\mathbb{N}$-valued site parameters $\boldsymbol{\alpha}$ for all three choices of $\sigma \in \{-1, 0, 1\}$.

Moreover, while ω_L and ω_R shall be interpreted as conductances between the boundaries and the associated bulk sites, the parameters $\alpha_L > 0$ and $\alpha_R > 0$ are the boundary analogues of the bulk site parameters $\boldsymbol{\alpha}$. The parameters θ_L and θ_R are responsible for the scaling of the reservoirs' densities ρ_L and ρ_R, i.e.

$$\rho_L = \alpha_L\theta_L \quad \text{and} \quad \rho_R = \alpha_R\theta_R\,, \tag{6.2.6}$$

and, for this reason, we refer to them as *scale parameters*. In particular, while in general we only require that

$$\theta_L\,,\,\theta_R \in [0, \infty)\,,$$

for the case of the exclusion process ($\sigma = -1$), we need to further impose

$$\theta_L\,,\,\theta_R \in [0, 1]$$

to prevent the rates in (6.2.4) and (6.2.5) to become negative.

Remark 6.2.1 (NOTATIONAL COMPARISON WITH [32]). *If we choose $\omega_{\{x,y\}} = \mathbf{1}_{\{|x-y|=1\}}$ and*

$$\alpha_x = \begin{cases} 2j & \text{if } \sigma = -1 \text{ for some } 2j \in \mathbb{N} \\ 1 & \text{if } \sigma = 0 \\ 2k & \text{if } \sigma = 1 \text{ for some } k > 0\,, \end{cases} \tag{6.2.7}$$

for all $x, y \in V$, we recover exactly the same bulk dynamics of the models studied in [32]. For what concerns the reservoir dynamics, the authors of [32] employ the following notation (see e.g. [32, Figs. 1–2])

$$\alpha := \alpha_L \theta_L \qquad \beta := \alpha_R + \sigma\alpha_R\theta_R$$
$$\gamma := \alpha_R \theta_R \qquad \delta := \alpha_L + \sigma\alpha_L\theta_L \;.$$

However, we believe that the parametrization of the bulk-boundary interaction through α_L, α_R, θ_L and θ_R yields more transparent formulas as, for instance, for the duality functions in presence of disorder.

Remark 6.2.2 (MORE GENERAL RESERVOIRS GEOMETRIES). *We emphasize that our results may be stated for boundary driven particle systems with the same bulk dynamics – as described by the generator $\mathcal{L}$ – and a more general boundary part of the dynamics, in which creation and annihilation of particles due to the reservoir interaction occur at more than two bulk sites. More precisely, the results stated in this section and Sections 6.3 and 6.4 below – namely, the duality relations and the correlation inequalities – naturally extend if $\mathcal{L}^{L,R}$ in* (6.2.1),(6.2.3) *is replaced by*

$$\mathcal{L}^{\mathrm{res}} f(\eta) = \sum_{x \in V} \omega_x^{\mathrm{res}} \mathcal{L}_x^{\mathrm{res}} f(\eta) \;,$$

with

$$\begin{aligned} \mathcal{L}_x^{\mathrm{res}} f(\eta) := &\, \eta(x)\left(\alpha_x^{\mathrm{res}} + \sigma\alpha_x^{\mathrm{res}}\theta_x^{\mathrm{res}}\right)\left(f(\eta^{x,-}) - f(\eta)\right) \\ &+ \alpha_x^{\mathrm{res}}\theta_x^{\mathrm{res}}\left(\alpha_x + \sigma\eta(x)\right)\left(f(\eta^{x,+}) - f(\eta)\right) \;, \end{aligned}$$

for some set of non-negative parameters $\boldsymbol{\alpha}^{\mathrm{res}} = \{\alpha_x^{\mathrm{res}} : x \in V\}$, $\boldsymbol{\theta}^{\mathrm{res}} = \{\theta_x^{\mathrm{res}} : x \in V\}$ and $\boldsymbol{\omega}^{\mathrm{res}} = \{\omega_x^{\mathrm{res}} : x \in V\}$. Also the results in Sections 6.5 and 6.6 below extend to this more general boundary dynamics as long as the scale parameters $\boldsymbol{\theta}^{\mathrm{res}} = \{\theta_x^{\mathrm{res}} : x \in V\}$ take at most two values, say θ_L and θ_R.

6.2.2 Duality

In this section, for each one of the particle systems presented in the section above, we derive two types of duality relations with a particle system in contact with purely absorbing boundaries. Recall that by duality relation for the particle system $\{\eta_t : t \geq 0\}$ on $\mathcal{X}$, we mean that there exist a *dual* particle system $\{\xi_t : t \geq 0\}$ on $\widehat{\mathcal{X}}$ and a measurable function $D : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ – referred to as *duality function* – for which the following relation holds: for all configurations $\eta \in \mathcal{X}$, $\xi \in \widehat{\mathcal{X}}$ and times $t \geq 0$, we have

$$\widehat{\mathbb{E}}_\xi\left[D(\xi_t, \eta)\right] = \mathbb{E}_\eta\left[D(\xi, \eta_t)\right] \;, \tag{6.2.8}$$

where $\widehat{\mathbb{E}}_\xi$, resp. $\mathbb{E}_\eta$, denotes expectation w.r.t. the law $\widehat{\mathbb{P}}_\xi$ of $\{\xi_t : t \geq 0\}$ with initial condition $\xi_0 = \xi$, resp. the law $\mathbb{P}_\eta$ of $\{\eta_t : t \geq 0\}$ with initial condition $\eta_0 = \eta$. More in general, for a given probability measure μ on $\mathcal{X}$, $\mathbb{E}_\mu$ denotes the expectation w.r.t. the law $\mathbb{P}_\mu$ of $\{\eta_t : t \geq 0\}$ initially distributed according to μ. Notice that, with a slight abuse of notation, when we write $\mathbb{E}_\mu\left[D(\xi, \eta)\right]$ we mean $\sum_{\eta \in \mathcal{X}} D(\xi, \eta)\, \mu(\eta)$.

Moreover, recall that if $\widehat{\mathcal{L}}$ and $\mathcal{L}$ denote the infinitesimal generators associated to the processes $\{\xi_t : t \geq 0\}$ and $\{\eta_t : t \geq 0\}$ respectively, the duality relation (6.2.8) is equivalent to the following relation: for all configurations $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$, we have

$$\widehat{\mathcal{L}}_{\mathrm{left}} D(\xi, \eta) = \mathcal{L}_{\mathrm{right}} D(\xi, \eta) \;, \tag{6.2.9}$$

where the subscript “left”, resp. “right”, indicates that the generator acts as an operator on the function $D(\cdot, \cdot)$, viewed as a function of the left, resp. right, variables. More precisely,

$$\widehat{\mathcal{L}}_{\mathrm{left}} D(\xi, \eta) = \widehat{\mathcal{L}} D(\cdot, \eta)(\xi) \quad \text{and} \quad \mathcal{L}_{\mathrm{right}} D(\xi, \eta) = \mathcal{L} D(\xi, \cdot)(\eta) \;.$$

In what follows, first we present the dual particle systems and, then, the duality relations. More specifically, we study in Sections 6.2.2 and 6.4 below, duality relations with two types of duality functions, which we call, respectively, “classical” and “orthogonal” for reasons that will be explained below.

Dual particle system with purely absorbing boundaries

For each choice of $\sigma \in \{-1, 0, 1\}$, we define a particle system with purely absorbing reservoirs, which we prove to be dual (see Propositions 6.2.3 and 6.4.1 below) to the corresponding system in contact with reservoirs of Section 6.2.1. For such systems, particles hop on $V \cup \{L, R\}$ following the same bulk dynamics as the particle systems of Section 6.2.1 but having $\{L, R\}$ as absorbing sites. More in detail, $\{\xi_t : t \geq 0\}$ denotes such particle systems having

$$\widehat{\mathcal{X}} = \mathcal{X} \times \mathbb{N}_0^{\{L,R\}} \tag{6.2.10}$$

as configuration space and infinitesimal generator $\widehat{\mathcal{L}}$ given by

$$\widehat{\mathcal{L}} f(\xi) = \widehat{\mathcal{L}}^{\text{bulk}} f(\xi) + \widehat{\mathcal{L}}^{L,R} f(\xi) \,, \tag{6.2.11}$$

where, for all bounded functions $f : \widehat{\mathcal{X}} \to \mathbb{R}$,

$$\begin{aligned}\widehat{\mathcal{L}}^{\text{bulk}} f(\xi) &= \sum_{x \sim y} \omega_{\{x,y\}}\, \widehat{\mathcal{L}}_{\{x,y\}} f(\xi) \\ &= \sum_{x \sim y} \omega_{\{x,y\}} \left\{ \begin{array}{c} \xi(x)\, (\alpha_y + \sigma \xi(y))\, (f(\xi^{x,y}) - f(\xi)) \\ + \xi(y)\, (\alpha_x + \sigma \xi(x))\, (f(\xi^{y,x}) - f(\xi)) \end{array} \right\} ,\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathcal{L}}^{L,R} f(\xi) &= \omega_L\, \widehat{\mathcal{L}}_L f(\xi) + \omega_R\, \widehat{\mathcal{L}}_R f(\xi) \\ &= \omega_L\, \alpha_L\, \xi(1) \left(f(\xi^{1,L}) - f(\xi) \right) + \omega_R\, \alpha_R\, \xi(N) \left(f(\xi^{N,R}) - f(\xi) \right) ,\end{aligned}$$

with, for all $x, y \in V \cup \{L, R\}$, $\xi^{x,y} = \xi - \delta_x + \delta_y \in \widehat{\mathcal{X}}$.

For all configurations $\xi \in \widehat{\mathcal{X}}$, let $|\xi|$ denote the total number of particles of the configuration ξ, i.e.

$$|\xi| := \xi(L) + \xi(R) + \sum_{x \in V} \xi(x) \,. \tag{6.2.12}$$

Once the total number of particles is fixed, due to the conservation of particles under the dynamics, the assumption of connectedness of the graph $(V, \sim)$ (see Section 6.2.1) and the positivity of ω_L and ω_R, the particle system $\{\xi_t : t \geq 0\}$ is irreducible on

$$\widehat{\mathcal{X}}_n := \left\{ \xi \in \widehat{\mathcal{X}} : |\xi| = n \right\}$$

whenever $n = |\xi_0|$ and admits a unique stationary measure fully supported on configurations

$$\left\{ \xi \in \widehat{\mathcal{X}}_n : \xi(x) = 0 \text{ for all } x \in V \right\} ,$$

i.e. all particles will get eventually absorbed in the sites $\{L, R\}$. Furthermore, the evolution of the particle systems $\{\xi_t : t \geq 0\}$ does *not* depend on θ_L and θ_R, but only on the following set of parameters:

$$\boldsymbol{\omega} = \{\omega_{\{x,y\}} : x, y \in V\} \,, \tag{6.2.13}$$

$$\boldsymbol{\alpha} = \{\alpha_x : x \in V\} \qquad \text{and} \qquad \{\omega_L, \omega_R, \alpha_L, \alpha_R\} \,. \tag{6.2.14}$$

For this reason, in the sequel we will refer to $V \cup \{L, R\}$ endowed with the above parameters as the *underlying geometry* of our particle systems.

Classical dualities

In this section, we generalize to the disordered setting the duality relations already appearing in e.g. [32]. In particular, these duality functions are in factorized – jointly in the original and dual configuration variables – form over all sites, i.e., for all $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$,

$$D(\xi, \eta) = d_L(\xi(L)) \times \left(\prod_{x \in V} d_x(\xi(x), \eta(x)) \right) \times d_R(\xi(R)) \,, \tag{6.2.15}$$

with the factors $\{d_x(\cdot,\cdot) : x \in V\} \cup \{d_L(\cdot), d_R(\cdot)\}$ named *single-site duality functions*. As already mentioned in the introduction, we refer to them as "classical" because the duality functions consist in weighted factorial moments of the occupation variables of the configuration η generalizing to IRW and SIP the renown duality relations for the symmetric simple exclusion process, see e.g. [126, Theorem 1.1, p. 363].

The precise form of these classical duality functions is contained in the following proposition. The proof of this duality relation boils down to directly check identity (6.2.9) and we omit it being it a straightforward rewriting of the proof of [32, Theorem 4.1]. We remark that in (6.2.17) below and in the rest of the chapter, we adopt the conventions $0^0 := 1$, $\frac{\Gamma(v+\ell)}{\Gamma(v)} := v(v+1)\cdots(v+\ell-1)$ for $v \geq 0$ and $\ell \in \mathbb{N}_0$, while

$$\frac{\Gamma(v+\ell)}{\Gamma(v)} := \begin{cases} 1 & \text{if } \ell = 0 \\ v(v+1)\cdots(v+\ell-1) & \text{if } \ell \in \{1,...,|v|\} \\ 0 & \text{otherwise ,} \end{cases}$$

for $v \in \mathbb{Z} \cap (-\infty, 0)$ and $\ell \in \mathbb{N}_0$.

Proposition 6.2.3 (CLASSICAL DUALITY FUNCTIONS)**.** *For each choice of $\sigma \in \{-1,0,1\}$, let $\mathcal{L}$ and $\widehat{\mathcal{L}}$ be the infinitesimal generators given in* (6.2.1) *and* (6.2.11)*, respectively, associated to the particle systems $\{\eta_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$. Then the duality relations in* (6.2.8) *and* (6.2.9) *hold with the duality function $D^{c\ell} : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ defined as follows: for all configurations $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$,*

$$D^{c\ell}(\xi,\eta) = d_L^{c\ell}(\xi(L)) \times \left(\prod_{x \in V} d_x^{c\ell}(\xi(x),\eta(x)) \right) \times d_R^{c\ell}(\xi(R)) \,,$$

where, for all $x \in V$ and $k, n \in \mathbb{N}_0$,

$$d_x^{c\ell}(k,n) = \frac{n!}{(n-k)!} \frac{1}{w_x(k)} \mathbf{1}_{\{k \leq n\}} \tag{6.2.16}$$

and

$$d_L^{c\ell}(k) = (\theta_L)^k \qquad \text{and} \qquad d_R^{c\ell}(k) = (\theta_R)^k \,, \tag{6.2.17}$$

where

$$w_x(k) \;=\; \begin{cases} \dfrac{\alpha_x!}{(\alpha_x - k)!} \mathbf{1}_{\{k \leq \alpha_x\}} & \text{if } \sigma = -1 \\ \alpha_x^k & \text{if } \sigma = 0 \\ \dfrac{\Gamma(\alpha_x + k)}{\Gamma(\alpha_x)} & \text{if } \sigma = 1 \,. \end{cases} \tag{6.2.18}$$

6.3 Equilibrium and non-equilibrium stationary measures

The long run behavior of the boundary driven particle systems of Section 6.2.1, encoded in their stationary measures, is explicitly known when the particle systems are *not* in contact with the reservoirs. Indeed, if $\omega_L = \omega_R = 0$, the particle systems $\{\eta_t : t \geq 0\}$ admit a one-parameter family of stationary – actually reversible – product measures

$$\{\mu_\theta = \otimes_{x \in V} \nu_{x,\theta} : \theta \in \Theta\} \tag{6.3.1}$$

with $\Theta = [0,1]$ if $\sigma = -1$ (SEP) and $\Theta = [0,\infty)$ if $\sigma = 0$ (IRW) and $\sigma = 1$ (SIP) and marginals given, for all $x \in V$, by

$$\nu_{x,\theta} \sim \begin{cases} \text{Binomial}(\alpha_x, \theta) & \text{if } \sigma = -1 \\ \text{Poisson}(\alpha_x \theta) & \text{if } \sigma = 0 \\ \text{Negative-Binomial}(\alpha_x, \frac{\theta}{1+\theta}) & \text{if } \sigma = 1 \,. \end{cases} \tag{6.3.2}$$

More concretely, for all $n \in \mathbb{N}_0$,

$$\nu_{x,\theta}(n) \;=\; \frac{w_x(n)}{z_{x,\theta}} \frac{\left(\frac{\theta}{1+\sigma\theta}\right)^n}{n!} \,, \tag{6.3.3}$$

with the functions $\{w_x : x \in V\}$ as given in (6.2.18) and

$$z_{x,\theta} \ = \ \begin{cases} (1-\theta)^{-\alpha_x} & \text{if } \sigma = -1 \\ e^{\alpha_x \theta} & \text{if } \sigma = 0 \\ (1+\theta)^{\alpha_x} & \text{if } \sigma = 1 \ , \end{cases} \tag{6.3.4}$$

where, for $\sigma = -1$, we set $\nu_{x,1}(n) := \mathbf{1}_{\{n=\alpha_x\}}$. Reversibility of these product measures for the dynamics induced by $\mathcal{L}$ in (6.2.1) follows by a standard detailed balance computation (see e.g. [32] for an analogous statement with site-independent parameters $\boldsymbol{\alpha}$). We note that, in analogy with (6.2.6), the parameterization of these product measures and corresponding marginals is chosen in such a way that the density of particles

$$\rho_x := \mathbb{E}_{\mu_\theta}[\eta(x)] \tag{6.3.5}$$

at site $x \in V$ w.r.t. μ_θ is given by the product of α_x and θ, i.e.

$$\rho_x = \alpha_x \theta \ , \quad x \in V \ . \tag{6.3.6}$$

6.3.1 Equilibrium stationary measure

In presence of interaction with only one of the two reservoirs, e.g. $\omega_L > 0$ and $\omega_R = 0$ and with scale parameters given by θ_L and θ_R, respectively, the same detailed balance computation shows that the systems have μ_θ (see (6.3.1)) with $\theta = \theta_L$ as reversible product measures. The stationary measures remain the same as long as the systems are in contact with both reservoirs, i.e. $\omega_L, \omega_R > 0$, and the two reservoirs are given equal scale parameters $\theta_L = \theta_R \in \Theta$. We refer to such stationary measures as *equilibrium stationary measures*.

6.3.2 Non-equilibrium stationary measures

As for *non-equilibrium stationary measures*, i.e. the stationary measures of the particle systems when $\omega_L, \omega_R > 0$ and $\theta_L \neq \theta_R$, none of the measures $\{\mu_\theta = \otimes_{x \in V} \nu_{x,\theta} : \theta \in \Theta\}$ above is stationary. However, for each of the particle systems, the non-equilibrium stationary measure exists, is unique and we denote it by μ_{θ_L,θ_R}. Moreover, while for the case of independent random walkers μ_{θ_L,θ_R} is in product form, for the case of exclusion and inclusion particle systems in non-equilibrium μ_{θ_L,θ_R} is non-product and has non-zero two-point correlations. This is the content of Theorem 7.3.1 below. In particular, the result on two-point correlations (item (b)) will be complemented with the study of the signs of such correlations in Theorem 6.3.4 and Lemma 6.3.5 below. We recall that, for the special case of the exclusion process with $\boldsymbol{\alpha} = \{\alpha_x : x \in V\}$ satisfying $\alpha_x = 1$ for all $x \in V$ and with nearest-neighbor unitary conductances, i.e.

$$\omega_{\{x,y\}} = \mathbf{1}_{\{|x-y|=1\}} \ , \quad x, y \in V \ ,$$

the unique non-product non-equilibrium stationary measure μ_{θ_L,θ_R} has been characterized in terms of a matrix formulation (see e.g. [55] and [125, Part III. Section 3]). Goal of Section 6.5 below is to provide a partial characterization of the non-equilibrium stationary measure of these systems by expressing suitably centered factorial moments – related to the orthogonal duality functions of Section 6.4 below – in terms of the product of a suitable power of $(\theta_L - \theta_R)$ and a coefficient which does not depend on neither θ_L nor θ_R.

In what follows, for all $x \in V$, we introduce the non-equilibrium stationary profile of the classical duality functions:

$$\bar{\theta}_x := \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\frac{\eta(x)}{\alpha_x}\right] = \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{c\ell}(\delta_x, \eta)\right] \ . \tag{6.3.7}$$

We recall that $\widehat{\mathbb{P}}_\xi$ denotes the law of the dual particle system started from the deterministic configuration $\xi \in \widehat{\mathcal{X}}$. Then, by stationarity and duality (Proposition 6.2.3), we obtain, for all $x \in V$,

$$\bar{\theta}_x = \lim_{t\to\infty} \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\frac{\eta_t(x)}{\alpha_x}\right] = \widehat{p}_\infty(\delta_x, \delta_L)\,\theta_L + \widehat{p}_\infty(\delta_x, \delta_R)\,\theta_R = \theta_R + \widehat{p}_\infty(\delta_x, \delta_L)\,(\theta_L - \theta_R) \ , \tag{6.3.8}$$

where, for all $\xi, \xi' \in \widehat{\mathcal{X}}$, $\widehat{p}_\infty(\xi, \xi') := \lim_{t\to\infty} \widehat{p}_t(\xi, \xi')$, with

$$\widehat{p}_t(\xi, \xi') := \widehat{\mathbb{P}}_\xi(\xi_t = \xi') \ .$$

Equivalently, stationarity and duality imply that $\{\bar{\theta}_x\,:\,x\in V\}$ solves the following difference equations: for all $x \in V$,

$$\sum_{y\in V} \omega_{\{x,y\}}\,\alpha_y\,(\bar{\theta}_y-\bar{\theta}_x)+\mathbf{1}_{\{x=1\}}\,\omega_L\,\alpha_L\,(\theta_L-\bar{\theta}_1)+\mathbf{1}_{\{x=N\}}\,\omega_R\,\alpha_R\,(\theta_R-\bar{\theta}_N)=0\,. \tag{6.3.9}$$

Consequently, because of the connectedness of $(V,\sim)$, if $\theta_L=\theta_R$, then $\bar{\theta}_x=\theta_L=\theta_R$ for all $x\in V$, while $\theta_L\neq\theta_R$ implies that there exist $x,y\in V$ such that $\bar{\theta}_x\neq\bar{\theta}_y$ and, moreover, that $\bar{\theta}_x>0$ for all $x\in V$.

Remark 6.3.1 (NON-EQUILIBRIUM STATIONARY PROFILE FOR A CHAIN)**.** *In the particular instance of a chain, i.e.*

$$\omega_{\{x,y\}}>0 \quad \textit{if and only if} \quad |x-y|=1\,,$$

the solution to the system (6.3.9) *is given by:*

$$\begin{aligned}\bar{\theta}_x &= \theta_R+\widehat{p}_\infty(\delta_x,\delta_L)\,(\theta_L-\theta_R)\\ &= \theta_R+\left(\frac{\frac{1}{\omega_R\alpha_R\alpha_N}+\sum_{y=x}^{N-1}\frac{1}{\omega_{\{y,y+1\}}\alpha_y\alpha_{y+1}}}{\frac{1}{\omega_L\alpha_L\alpha_1}+\left(\sum_{y=1}^{N-1}\frac{1}{\omega_{\{y,y+1\}}\alpha_y\alpha_{y+1}}\right)+\frac{1}{\omega_R\alpha_R\alpha_N}}\right)(\theta_L-\theta_R)\ .\end{aligned}$$

If, additionally, the conductances and site parameters $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are constant, $\alpha_L=\alpha_R=\alpha_x$ and $\omega_L=\omega_R=\omega_{\{x,x+1\}}$, the profile $x\mapsto\bar{\theta}_x$ is linear (cf. [32, Eq. (4.24)]):

$$\bar{\theta}_x=\theta_R+\left(1-\frac{x}{N+1}\right)(\theta_L-\theta_R)\ . \tag{6.3.10}$$

Before stating the main result of this section, we introduce the following definition.

Definition 6.3.2 (LOCAL EQUILIBRIUM PRODUCT MEASURE)**.** *Given $\bar{\boldsymbol{\theta}}:=\{\bar{\theta}_x\,:\,x\in V\}$ the stationary profile introduced in* (6.3.8)*, we define the following product measure*

$$\mu_{\bar{\boldsymbol{\theta}}}:=\otimes_{x\in V}\,\nu_{x,\bar{\theta}_x}\ , \tag{6.3.11}$$

and refer to it as the local equilibrium product measure.

Theorem 6.3.3. *For each choice of $\sigma\in\{-1,0,1\}$ and provided that $\omega_L\vee\omega_R>0$, for all $\theta_L,\theta_R\in\Theta$ there exists a unique stationary measure μ_{θ_L,θ_R} for the particle system $\{\eta_t\,:\,t\geq 0\}$. Moreover,*

(a) *If $\sigma=0$* (IRW)*, the stationary measure μ_{θ_L,θ_R} is in product form and is given by*

$$\mu_{\theta_L,\theta_R}=\mu_{\bar{\boldsymbol{\theta}}}\ . \tag{6.3.12}$$

(b) *If either $\sigma=-1$* (SEP) *or $\sigma=1$* (SIP) *and, additionally, $\omega_L,\omega_R>0$ and $\theta_L\neq\theta_R$, there exists $x,y\in V$ with $x\neq y$ for which*

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\left(\frac{\eta(x)}{\alpha_x}-\bar{\theta}_x\right)\left(\frac{\eta(y)}{\alpha_y}-\bar{\theta}_y\right)\right]\neq 0\ .$$

As a consequence, the unique non-equilibrium stationary measure μ_{θ_L,θ_R} is not in product form.

Proof. The proof of existence and uniqueness of the stationary measure μ_{θ_L,θ_R} is trivial for the exclusion process, which is a finite state irreducible Markov chain. We postpone the proof for the case of independent random walkers and inclusion process to Section 6.7. Although this result is standard, it does not appear, to the best of our knowledge, in the literature.

For what concerns item (a) in which $\sigma=0$, let us compute, for all $\xi\in\widehat{\mathcal{X}}$,

$$\sum_{\eta\in\mathcal{X}}\mathcal{L}_{\mathrm{right}}D^{c\ell}(\xi,\eta)\,\mu_{\bar{\boldsymbol{\theta}}}(\eta)\ .$$

By duality, the following relation (cf. e.g. [146])

$$\sum_{n\in\mathbb{N}_0}d_x^{c\ell}(k,n)\,\nu_{x,\bar{\theta}_x}(n)=(\bar{\theta}_x)^k\ , \tag{6.3.13}$$

which holds for all $x \in V$ and $k \in \mathbb{N}_0$ if $\sigma \in \{0,1\}$ while $k \in \{0,\ldots,\alpha_x\}$ if $\sigma = -1$, we obtain, for all $\xi \in \widehat{\mathcal{X}}$,

$$\sum_{\eta \in \mathcal{X}} \mathcal{L}_{\text{right}} D^{c\ell}(\xi,\eta)\, \mu_{\bar{\theta}}(\eta) = \sum_{\eta \in \mathcal{X}} \widehat{\mathcal{L}}_{\text{left}} D^{c\ell}(\xi,\eta)\, \mu_{\bar{\theta}}(\eta)$$
$$= \sum_{x \in V} \left(\sum_{\eta \in \mathcal{X}} D^{c\ell}(\xi - \delta_x, \eta)\, \mu_{\bar{\theta}}(\eta) \right) \xi(x) \left\{ \begin{array}{l} \sum_{y \in V} \omega_{\{x,y\}}\, \alpha_y\, (\bar{\theta}_y - \bar{\theta}_x) \\ +\mathbf{1}_{\{x=1\}}\, \omega_L\, \alpha_L\, (\theta_L - \bar{\theta}_1) \\ +\mathbf{1}_{\{x=N\}}\, \omega_R\, \alpha_R\, (\theta_R - \bar{\theta}_N) \end{array} \right\} = 0\,,$$

where the last identity follows from (6.3.9). Because the products of Poisson distributions are completely characterized by their factorial moments $\{D^{c\ell}(\xi,\cdot) : \xi \in \widehat{\mathcal{X}}\}$, we get (7.3.1).

For item (b) in which $\sigma \neq 0$, let us suppose by contradiction that all two-point correlations are zero, i.e. for all $x, y \in V$ with $x \neq y$,

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\frac{\eta(x)}{\alpha_x} \frac{\eta(y)}{\alpha_y} \right] = \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{c\ell}(\delta_x + \delta_y, \eta) \right] = \bar{\theta}_x \bar{\theta}_y\,. \tag{6.3.14}$$

If we use the following shortcut

$$\bar{\theta}''_x := \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{c\ell}(2\delta_x, \eta) \right]\,,$$

by stationarity, duality and (6.3.14), we obtain, for all $x \in V$,

$$\sum_{\eta \in \mathcal{X}} \mathcal{L}_{\text{right}} D^{c\ell}(2\delta_x,\eta)\, \mu_{\theta_L,\theta_R}(\eta) = \sum_{\eta \in \mathcal{X}} \widehat{\mathcal{L}}_{\text{left}} D^{c\ell}(2\delta_x,\eta)\, \mu_{\theta_L,\theta_R}(\eta)$$
$$= 2 \sum_{y \in V} \omega_{\{x,y\}} \alpha_y (\bar{\theta}_x\, \bar{\theta}_y - \bar{\theta}''_x) + 2 \left\{ \mathbf{1}_{\{x=1\}} \omega_L \alpha_L (\theta_L \bar{\theta}_1 - \bar{\theta}''_1) + \mathbf{1}_{\{x=N\}} \omega_R \alpha_R (\theta_R \bar{\theta}_N - \bar{\theta}''_N) \right\} = 0\,.$$

By adding and subtracting

$$2 \left\{ \sum_{y \in V} \omega_{\{x,y\}} \alpha_y (\bar{\theta}_x)^2 + \mathbf{1}_{\{x=1\}} \omega_L \alpha_L (\bar{\theta}_1)^2 + \mathbf{1}_{\{x=N\}} \omega_R \alpha_R (\bar{\theta}_N)^2 \right\}$$

to the identity above and by relation (6.3.9), we get

$$\left((\bar{\theta}_x)^2 - \bar{\theta}''_x \right) 2 \left\{ \sum_{y \in V} \omega_{\{x,y\}} \alpha_y + \mathbf{1}_{\{x=1\}} \omega_L \alpha_L + \mathbf{1}_{\{x=N\}} \omega_R \alpha_R \right\} = 0\,.$$

Because the above identity holds for all $x \in V$ and by the positivity of the expression in curly brackets due to the connectedness of $(V, \sim)$, we get

$$\bar{\theta}''_x = (\bar{\theta}_x)^2\,, \quad \text{for all} \quad x \in V\,. \tag{6.3.15}$$

In view of (6.3.14), (6.3.15), stationarity of μ_{θ_L,θ_R} and duality, we have

$$\sum_{\eta \in \mathcal{X}} \mathcal{L}_{\text{right}} D^{c\ell}(\delta_x + \delta_y,\eta)\, \mu_{\theta_L,\theta_R}(\eta) = \sum_{\eta \in \mathcal{X}} \widehat{\mathcal{L}}_{\text{left}} D^{c\ell}(\delta_x + \delta_y,\eta)\, \mu_{\theta_L,\theta_R}(\eta)$$
$$= \bar{\theta}_y \left\{ \begin{array}{l} \sum_{z \in V} \omega_{\{x,z\}}\, \alpha_z\, (\bar{\theta}_z - \bar{\theta}_x) \\ +\mathbf{1}_{\{x=1\}}\, \omega_L\, \alpha_L\, (\theta_L - \bar{\theta}_1) \\ +\mathbf{1}_{\{x=N\}}\, \omega_R\, \alpha_R\, (\theta_R - \bar{\theta}_N) \end{array} \right\} + \bar{\theta}_x \left\{ \begin{array}{l} \sum_{z \in V} \omega_{\{y,z\}}\, \alpha_z\, (\bar{\theta}_z - \bar{\theta}_y) \\ +\mathbf{1}_{\{y=1\}}\, \omega_L\, \alpha_L\, (\theta_L - \bar{\theta}_1) \\ +\mathbf{1}_{\{y=N\}}\, \omega_R\, \alpha_R\, (\theta_R - \bar{\theta}_N) \end{array} \right\} + \sigma \omega_{\{x,y\}} (\bar{\theta}_x - \bar{\theta}_y)^2$$
$$= \sigma \omega_{\{x,y\}} (\bar{\theta}_x - \bar{\theta}_y)^2\,. \tag{6.3.16}$$

Therefore, because $\sigma \in \{-1, 1\}$, as a consequence of the connectedness of $(V, \sim)$, we have

$$\sum_{x \sim y} \left(\sum_{\eta \in \mathcal{X}} \mathcal{L}_{\text{right}} D^{c\ell}(\delta_x + \delta_y,\eta)\, \mu_{\theta_L,\theta_R}(\eta) \right) = \sigma \sum_{x \sim y} \omega_{\{x,y\}}\, (\bar{\theta}_x - \bar{\theta}_y)^2 = 0 \tag{6.3.17}$$

if and only if

$$\bar{\theta}_x = \bar{\theta}_y \,, \quad \text{for all} \quad x, y \in V \,. \tag{6.3.18}$$

However, because $\theta_L \neq \theta_R$, the latter condition (6.3.18) contradicts the claim below (6.3.9) .

□

6.3.3 Two-point correlations in the non-equilibrium steady state

In the following theorem we prove that as soon as the system has interaction, i.e. $\sigma \in \{-1, 1\}$, the local equilibrium product measure expectations of classical duality functions decrease (resp. increase) for exclusion (resp. inclusion) in the course of time. This implies, in particular, negative (resp. positive) two-point correlations for exclusion (resp. inclusion) particle systems. This strengthens previous results on correlation inequalities in [85], indeed here we obtain strict inequalities. The proof of this theorem is based on Lemma 6.3.5 below, which is of interest in itself because it provides an explicit expression of the l.h.s. in (6.3.19).

Theorem 6.3.4 (sign of two-point correlations)**.** *If $\omega_L, \omega_R > 0$ and $\xi \in \widehat{\mathcal{X}}$ is such that $\sum_{x \in V} \xi(x) \geq 2$, then, for all $\theta_L, \theta_R \in \Theta$ with $\theta_L \neq \theta_R$ and $t > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_{\mu_{\bar{\theta}}} \left[D^{c\ell}(\xi, \eta_t) \right] \begin{cases} < 0 & \text{if } \sigma = -1 \\ > 0 & \text{if } \sigma = 1 \,. \end{cases} \tag{6.3.19}$$

As a consequence, for all $x, y \in V$ with $x \neq y$,

$$\mathbb{E}_{\mu_{\theta_L, \theta_R}} \left[\left(\frac{\eta(x)}{\alpha_x} - \bar{\theta}_x \right) \left(\frac{\eta(y)}{\alpha_y} - \bar{\theta}_y \right) \right] \begin{cases} < 0 & \text{if } \sigma = -1 \\ > 0 & \text{if } \sigma = 1 \,. \end{cases}$$

Proof. The local equilibrium product measures $\mu_{\bar{\theta}}$ satisfy the hypothesis of Lemma 6.3.5 below (cf. (6.3.13)). Then, by the claim after (6.3.9) and the assumption $\theta_L \neq \theta_R$, (6.3.19) is recovered as a consequence of the first equality of (6.3.21) from the same lemma. □

Lemma 6.3.5. *For all $n \in \mathbb{N}$, let μ be a probability measure on $\mathcal{X}$ such that*

$$\mathbb{E}_\mu \left[D^{c\ell}(\xi, \eta) \right] = H(\xi, \bar{\boldsymbol{\theta}}) \tag{6.3.20}$$

holds for all $\xi \in \widehat{\mathcal{X}}$ with $|\xi| \leq n$, where $\bar{\boldsymbol{\theta}} = \{\bar{\theta}_x : x \in V\}$ and, for all $\boldsymbol{\theta} = \{\theta_x : x \in V\} \subset \Theta$,

$$H(\xi, \boldsymbol{\theta}) := (\theta_L)^{\xi(L)} \left(\prod_{x \in V} (\theta_x)^{\xi(x)} \right) (\theta_R)^{\xi(R)} \,.$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_\mu \left[D^{c\ell}(\xi, \eta_t) \right] &= \sigma \sum_{x \sim y} \omega_{\{x,y\}} \left(\bar{\theta}_y - \bar{\theta}_x \right)^2 \widehat{\mathbb{E}}_\xi \left[\frac{\xi_t(x)}{\bar{\theta}_x} \frac{\xi_t(y)}{\bar{\theta}_y} \mathbb{E}_\mu \left[D^{c\ell}(\xi_t, \eta) \right] \right] \\ &= \sigma \sum_{x \sim y} \omega_{\{x,y\}} \widehat{\mathbb{E}}_\xi \left[\left(\bar{\theta}_y - \bar{\theta}_x \right)^2 \partial^2_{\theta_x \theta_y} H(\xi_t, \bar{\boldsymbol{\theta}}) \right] \end{aligned} \tag{6.3.21}$$

holds for all $\xi \in \mathcal{X}$ with $|\xi| \leq n$ and $t \geq 0$.

Proof. By duality, we obtain, for all $\xi \in \widehat{\mathcal{X}}$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_\mu \left[D^{c\ell}(\xi, \eta_t) \right] = \sum_{\eta \in \mathcal{X}} \mathcal{L}_{\text{right}} \mathbb{E}_\eta \left[D^{c\ell}(\xi, \eta_t) \right] \mu(\eta) \\ &= \sum_{\eta \in \mathcal{X}} \mathcal{L}_{\text{right}} \widehat{\mathbb{E}}_\xi \left[D^{c\ell}(\xi_t, \eta) \right] \mu(\eta) = \sum_{\eta \in \mathcal{X}} \widehat{\mathbb{E}}_\xi \left[\widehat{\mathcal{L}}_{\text{left}} D^{c\ell}(\xi_t, \eta) \right] \mu(\eta) \\ &= \sum_{x \in V} \widehat{\mathbb{E}}_\xi \left[\xi_t(x) \left\{ \begin{array}{l} \sum_{y \in V} \omega_{\{x,y\}} \, \alpha_y \left(\mathbb{E}_\mu \left[D^{c\ell}(\xi_t^{x,y}, \eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi_t, \eta) \right] \right) \\ + \mathbf{1}_{\{x=1\}} \omega_L \, \alpha_L \left(\mathbb{E}_\mu \left[D^{c\ell}(\xi_t^{1,L}, \eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi_t, \eta) \right] \right) \\ + \mathbf{1}_{\{x=N\}} \omega_R \, \alpha_R \left(\mathbb{E}_\mu \left[D^{c\ell}(\xi_t^{N,R}, \eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi_t, \eta) \right] \right) \end{array} \right\} \right] \end{aligned}$$

$$+\,\sigma \sum_{x\in V} \widehat{\mathbb{E}}_\xi \left[\sum_{y\in V} \omega_{\{x,y\}} \xi_t(x)\,\xi_t(y) \left(\mathbb{E}_\mu \left[D^{c\ell}(\xi_t^{x,y},\eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi_t,\eta) \right] \right) \right] .$$

By (6.3.20), for all $x, y \in V$ and $\xi \in \widehat{\mathcal{X}}$ with $|\xi| \le n$, we have

$$\mathbb{E}_\mu \left[D^{c\ell}(\xi^{x,y},\eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta) \right] = \frac{\mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta) \right]}{\bar\theta_x} \left(\bar\theta_y - \bar\theta_x \right) ,$$

and, similarly,

$$\mathbb{E}_\mu \left[D^{c\ell}(\xi^{1,L},\eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta) \right] = \frac{\mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta) \right]}{\bar\theta_1} \left(\theta_L - \bar\theta_1 \right)$$

$$\mathbb{E}_\mu \left[D^{c\ell}(\xi^{N,R},\eta) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta) \right] = \frac{\mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta) \right]}{\bar\theta_N} \left(\theta_R - \bar\theta_N \right) .$$

As a consequence, we further obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta_t) \right] &= \sum_{\eta\in\mathcal{X}} \mathcal{L}_{\mathrm{right}} \mathbb{E}_\eta \left[D^{c\ell}(\xi,\eta_t) \right] \mu(\eta) \\ &= \sum_{x\in V} \widehat{\mathbb{E}}_\xi \left[\frac{\xi_t(x)}{\bar\theta_x} \, \mathbb{E}_\mu \left[D^{c\ell}(\xi_t,\eta) \right] \left\{ \begin{array}{l} \sum_{y\in V} \omega_{\{x,y\}}\, \alpha_y \left(\bar\theta_y - \bar\theta_x \right) \\ +\mathbf{1}_{\{x=1\}} \omega_L \alpha_L \left(\theta_L - \bar\theta_1 \right) \\ +\mathbf{1}_{\{x=N\}} \omega_R \alpha_R \left(\theta_R - \bar\theta_N \right) \end{array} \right\} \right] \\ &\quad + \sigma \sum_{x\sim y} \omega_{\{x,y\}} \left(\bar\theta_y - \bar\theta_x \right)^2 \widehat{\mathbb{E}}_\xi \left[\frac{\xi_t(x)}{\bar\theta_x} \frac{\xi_t(y)}{\bar\theta_y} \mathbb{E}_\mu \left[D^{c\ell}(\xi_t,\eta) \right] \right] . \end{aligned}$$

The observation that each of the expressions between curly brackets above equals zero because of the choice of the scale parameters $\{\bar\theta_x : x \in V\}$ (cf. (6.3.7) and (6.3.9)) concludes the proof. □

Remark 6.3.6. (a) *For all $\xi \in \widehat{\mathcal{X}}$ with $\sum_{z\in V} \xi(z) \ge 2$, for all times $t > 0$ and for all sites $x, y \in V$, the geometric assumption on the connectedness of $(V, \sim)$ implies that*

$$\widehat{\mathbb{P}}_\xi \left(\xi_t(x)\xi_t(y) > 0 \right) > 0 .$$

As a consequence, the sign of the time derivative in (6.3.21) *for $\xi \in \widehat{\mathcal{X}}$ with $\sum_{z\in V} \xi(z) \ge 2$ and for $t > 0$ is determined by $\sigma \in \{-1, 0, 1\}$. In particular, if the probability measure μ and the configuration $\xi \in \widehat{\mathcal{X}}$ are given as in Theorem 6.3.4, the convergence*

$$\mathbb{E}_\mu \left[D^{c\ell}(\xi,\eta_t) \right] \underset{t\to\infty}{\longrightarrow} \mathbb{E}_{\mu_{\theta_L,\theta_R}} \left[D^{c\ell}(\xi,\eta) \right]$$

is strictly monotone *in time: decreasing for $\sigma = -1$ and increasing for $\sigma = 1$.*

(b) *In the particular situation in which $\xi = \delta_x + \delta_y$ for some $x, y \in V$ and the probability measure μ satisfies the hypothesis of Theorem 6.3.4 for $n \ge 2$, the expression in* (6.3.21) *further simplifies yielding, for all $t > 0$,*

$$\begin{aligned} &\mathbb{E}_\mu \left[D^{c\ell}(\delta_x + \delta_y,\eta_t) \right] - \bar\theta_x \bar\theta_y \\ &= \mathbb{E}_\mu \left[D^{c\ell}(\delta_x + \delta_y,\eta_t) \right] - \mathbb{E}_\mu \left[D^{c\ell}(\delta_x + \delta_y,\eta_0) \right] \\ &= \sigma \int_0^t \sum_{z\sim w} \omega_{\{z,w\}} \left(\bar\theta_w - \bar\theta_z \right)^2 \widehat{\mathbb{E}}_{\xi=\delta_x+\delta_y} \left[\xi_s(z)\xi_s(w) \right] \mathrm{d}s \\ &= \sigma \int_0^t \sum_{z\sim w} \omega_{\{z,w\}} \left(\bar\theta_w - \bar\theta_z \right)^2 \widehat{\mathbb{P}}_{\xi=\delta_x+\delta_y} \left(\xi_s(z) = 1 \text{ and } \xi_s(w) = 1 \right) \mathrm{d}s . \end{aligned} \tag{6.3.22}$$

If, additionally, we impose

$$\alpha_x = \alpha_L = \alpha_R \qquad \text{and} \qquad \omega_L = \omega_R = 1 \qquad \text{and} \qquad \omega_{\{x,y\}} = \mathbf{1}_{\{|x-y|=1\}} ,$$

for all $x, y \in V$, we further get (cf. (6.3.10)*)*

$$\mathbb{E}_\mu \left[D^{c\ell}(\delta_x + \delta_y,\eta_t) \right] - \bar\theta_x \bar\theta_y = \sigma \frac{(\theta_L - \theta_R)^2}{(N+1)^2} \int_0^t \widehat{\mathbb{P}}_{\xi=\delta_x+\delta_y} \left(\sum_{z=1}^{N-1} \xi_s(z)\xi_s(z+1) = 1 \right) \mathrm{d}s . \tag{6.3.23}$$

6.4 Orthogonal dualities

By orthogonal dualities we refer to a specific subclass of duality functions $D(\xi,\eta)$ in the form (6.2.15). This subclass consists of jointly factorized functions whose each "bulk" single-site duality function

$$(k,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mapsto d_x(k,n) \in \mathbb{R}$$

is a family of polynomials in the n-variables and orthogonal w.r.t. a suitable probability measure ν_x on $\mathbb{N}_0$, i.e. for all $k,\ell \in \mathbb{N}_0$,

$$\sum_{n=0}^{\infty} d_x(k,n)\, d_x(\ell,n)\, \nu_x(n) = \mathbf{1}_{\{k=\ell\}} \|d_x(k,\cdot)\|^2_{L^2(\nu_x)} \, .$$

Orthogonal duality functions for exclusion, inclusion and independent particle systems with no interaction with reservoirs have been first introduced in [78] by direct computations and then characterized in [146] through generating function techniques. There, the dual particle system has the same law of the original particle system; therefore, orthogonal dualities are actually self-dualities. Moreover, for each $\sigma \in \{-1,0,1\}$, these jointly factorized orthogonal dualities consist of products of hypergeometric functions of the following two types: either

$$ {}_2F_0\left[\begin{matrix} -k & -n \\ & - \end{matrix}; -u\right] := \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{n!}{(n-\ell)!} \mathbf{1}_{\{\ell \le n\}} \right) u^\ell \tag{6.4.1}$$

or

$$ {}_2F_1\left[\begin{matrix} -k & -n \\ & v \end{matrix}; u\right] := \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{\Gamma(v)}{\Gamma(v+\ell)} \frac{n!}{(n-\ell)!} \mathbf{1}_{\{\ell \le n\}} \right) u^\ell \, , \tag{6.4.2}$$

with $k,n \in \mathbb{N}_0$ and $u,v \in \mathbb{R}$. More specifically, these orthogonal single-site self-duality functions are Kravchuk polynomials for SEP ($\sigma=-1$), Charlier polynomials for IRW ($\sigma=0$) and Meixner polynomials for SIP ($\sigma=1$) (see e.g. [108]). It turns out that such single-site self-duality functions are orthogonal families w.r.t. the single-site marginals of the stationary (actually reversible) product measures of the corresponding particle system; in particular, Kravchuk polynomials are orthogonal w.r.t. Binomial distributions, Charlier polynomials w.r.t. Poisson distributions and Meixner polynomials w.r.t. Negative Binomial distributions. More precisely, because in this setting there exists a one-parameter family of stationary product measures for each of the three particle systems (see also Section 6.3 above), this corresponds to the existence of a one-parameter family of orthogonal duality functions.

This correspondence between orthogonal duality functions and stationary measures may suggest that, knowing a stationary measure of a particle system, an orthogonal family of observables of this system would correspond, in general, to duality functions. This program, however, besides not being generally verifiable, does not apply to the case of particle systems in contact with reservoirs, for which the non-equilibrium stationary measures are, generally speaking, not in product form and not explicitly known (see also Section 6.3.2).

Nevertheless, from an algebraic point of view (see e.g. [83]), new duality relations may be generated from the knowledge of a duality relation and a *symmetry* of one of the two generators involved in the duality relation. In brief, given the following duality relation

$$\widehat{\mathcal{L}}_{\text{left}} D(\xi,\eta) = \mathcal{L}_{\text{right}} D(\xi,\eta)$$

for all $\xi \in \widehat{\mathcal{X}}$, $\eta \in \mathcal{X}$, and a symmetry $\widehat{\mathcal{K}}$ for the generator $\widehat{\mathcal{L}}$, i.e., for all $f : \widehat{\mathcal{X}} \to \mathbb{R}$ and $\xi \in \widehat{\mathcal{X}}$,

$$\widehat{\mathcal{K}}\, \widehat{\mathcal{L}} f(\xi) = \widehat{\mathcal{L}} \widehat{\mathcal{K}} f(\xi) \, , \tag{6.4.3}$$

then, if $F(\widehat{\mathcal{K}})$ with $F : \mathbb{R} \to \mathbb{R}$ is a well-defined operator, the function $(F(\widehat{\mathcal{K}}))_{\text{left}} D(\xi,\eta)$ is a duality function between $\mathcal{L}$ and $\widehat{\mathcal{L}}$. Indeed, for all $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$, we have

$$\begin{aligned} \widehat{\mathcal{L}}_{\text{left}} (F(\widehat{\mathcal{K}}))_{\text{left}} D(\xi,\eta) &= (F(\widehat{\mathcal{K}}))_{\text{left}} \widehat{\mathcal{L}}_{\text{left}} D(\xi,\eta) \\ &= (F(\widehat{\mathcal{K}}))_{\text{left}} \mathcal{L}_{\text{right}} D(\xi,\eta) \\ &= \mathcal{L}_{\text{right}} (F(\widehat{\mathcal{K}}))_{\text{left}} D(\xi,\eta) \, . \end{aligned}$$

This latter approach is the one we follow here (Theorem 6.4.1 below) to recover a one-parameter family of orthogonal duality functions for boundary driven particle systems. Its proof combines two ingredients: first, as already proved in [37], we observe that the so-called *annihilation operator on* $V \cup \{L, R\}$ given, for all $f : \widehat{\mathcal{X}} \to \mathbb{R}$, by

$$\widehat{\mathcal{K}} f(\xi) = \widehat{\mathcal{K}}^{\text{bulk}} f(\xi) + \widehat{\mathcal{K}}^{L,R} f(\xi) , \tag{6.4.4}$$

where

$$\widehat{\mathcal{K}}^{\text{bulk}} f(\xi) = \sum_{x\in V} \widehat{\mathcal{K}}_x f(\xi) = \sum_{x\in V} \xi(x)\, f(\xi - \delta_x)$$

and

$$\widehat{\mathcal{K}}^{L,R} f(\xi) = \widehat{\mathcal{K}}_L f(\xi) + \widehat{\mathcal{K}}_R f(\xi) = \xi(L)\, f(\xi - \delta_L) + \xi(R)\, f(\xi - \delta_R) ,$$

is a symmetry for the generator $\widehat{\mathcal{L}}$ associated to the particle systems with purely absorbing reservoirs and defined in (6.2.11). Then, we obtain the candidate orthogonal dualities by acting with suitable exponential functions of this symmetry $\widehat{\mathcal{K}}$ on the classical duality functions appearing in Proposition 6.2.3. We recall that in (6.4.6) below, the convention $0^0 := 1$ holds.

Theorem 6.4.1 (ORTHOGONAL DUALITY FUNCTIONS). *For each choice of* $\sigma \in \{-1, 0, 1\}$*, let* $\mathcal{L}$ *and* $\widehat{\mathcal{L}}$ *be the infinitesimal generators given in* (6.2.1) *and* (6.2.11)*, respectively, associated to the particle systems* $\{\eta_t : t \geq 0\}$ *and* $\{\xi_t : t \geq 0\}$*. Then the duality relations in* (6.2.8) *and* (6.2.9) *hold with the duality functions* $D^{or}_\theta : \widehat{\mathcal{X}} \times \mathcal{X} \to \mathbb{R}$ *defined, for all* $\theta \in \Theta$*, as follows: for all configurations* $\eta \in \mathcal{X}$ *and* $\xi \in \widehat{\mathcal{X}}$*,*

$$D^{or}_\theta(\xi, \eta) = d^{or}_{L,\theta}(\xi(L)) \times \left(\prod_{x\in V} d^{or}_{x,\theta}(\xi(x), \eta(x)) \right) \times d^{or}_{R,\theta}(\xi(R))$$

where, for all $x \in V$ *and* $k, n \in \mathbb{N}_0$*,*

$$d^{or}_{x,\theta}(k, n) \;=\; (-\theta)^k \times \begin{cases} {}_2F_1\left[\begin{matrix} -k \;\; -n \\ -\alpha_x \end{matrix} ; \frac{1}{\theta}\right] & \sigma = -1 \\ {}_2F_0\left[\begin{matrix} -k \;\; -n \\ - \end{matrix} ; -\frac{1}{\theta\alpha_x}\right] & \sigma = 0 \\ {}_2F_1\left[\begin{matrix} -k \;\; -n \\ \alpha_x \end{matrix} ; -\frac{1}{\theta}\right] & \sigma = 1 , \end{cases} \tag{6.4.5}$$

and

$$d^{or}_{L,\theta}(k) = (\theta_L - \theta)^k \qquad \text{and} \qquad d^{or}_{R,\theta}(k) = (\theta_R - \theta)^k \;. \tag{6.4.6}$$

Proof. We start with the observation that, for each $\sigma \in \{-1, 0, 1\}$, the commutation relation (6.4.3) between the annihilation operator $\widehat{\mathcal{K}}$ in (6.4.4) and the generator $\widehat{\mathcal{L}}$ (6.2.11) holds (for a detailed proof, we refer to e.g. [37, Section 5]).

As a consequence, for all $\theta \in \Theta$, the following function

$$(e^{-\theta\widehat{\mathcal{K}}})_{\text{left}} D^{c\ell}(\xi, \eta) \tag{6.4.7}$$

is a duality function between $\mathcal{L}$ and $\widehat{\mathcal{L}}$. In particular, recalling the definitions of single-site classical duality functions in (6.2.16)–(6.2.17) and hypergeometric functions in (6.4.1)–(6.4.2), due to the factorized form of both symmetry $e^{-\theta\widehat{\mathcal{K}}}$ and classical duality function, the combination of

$$(e^{-\theta\widehat{\mathcal{K}}_L}) d^{c\ell}_L(k) = \sum_{\ell=0}^{k} \binom{k}{\ell} d^{c\ell}_L(\ell)\, (-\theta)^{k-\ell} = (\theta_L - \theta)^k$$

$$(e^{-\theta\widehat{\mathcal{K}}_R}) d^{c\ell}_R(k) = \sum_{\ell=0}^{k} \binom{k}{\ell} d^{c\ell}_R(\ell)\, (-\theta)^{k-\ell} = (\theta_R - \theta)^k$$

and

$$(e^{-\theta\widehat{\mathcal{K}}_x})_{\text{left}} d_x^{c\ell}(k,n) = \sum_{\ell=0}^{k} \binom{k}{\ell} d_x^{c\ell}(\ell,n)\,(-\theta)^{k-\ell} = (-\theta)^k \begin{cases} {}_2F_1\left[\begin{matrix} -k & -n \\ & -\alpha_x \end{matrix} ; \frac{1}{\theta}\right] & \sigma=-1 \\ {}_2F_0\left[\begin{matrix} -k & -n \\ & - \end{matrix} ; -\frac{1}{\theta\alpha_x}\right] & \sigma=0 \\ {}_2F_1\left[\begin{matrix} -k & -n \\ & \alpha_x \end{matrix} ; -\frac{1}{\theta}\right] & \sigma=1\,, \end{cases} \tag{6.4.8}$$

for all $x \in V$, concludes the proof. □

The above method to derive the orthogonal duality functions may be summarized as consisting in the application on the classical duality functions of a suitable symmetry on the "left" dual variables ξ. This approach differs from all those previously employed in the context of closed systems: e.g., [78] is based on solving suitable recurrence relations, [146] on computing generating functions, while [34] on acting with suitable unitary symmetries on the "right" variables η. The main advantage of our method is that it works in both contexts of closed and open systems with no substantial alteration, since the annihilation operator is a commutator of the dual generator in both situations.

Remark 6.4.2. *To provide the reader with a further interpretation of orthogonal dualities, we note that the following formula connecting orthogonal and classical dualities is reminiscent of the Newton binomial formula:*

$$d_{x,\theta}^{or}(k,n) = \sum_{\ell=0}^{k} \binom{k}{\ell} d_x^{c\ell}(\ell,n)\,(-\theta)^{k-\ell}\,. \tag{6.4.9}$$

In particular, setting $\theta = 0$ and recalling the convention $0^0 := 1$,

$$d_{x,\theta=0}^{or}(k,n) = \sum_{\ell=0}^{k} \binom{k}{\ell} d_x^{c\ell}(\ell,n)\,(-0)^{k-\ell} = d_x^{c\ell}(k,n)\,, \tag{6.4.10}$$

i.e., the classical duality functions, $D^{c\ell}(\xi,\eta)$, may be seen as a particular instance of the orthogonal duality functions if the scale parameter $\theta \in \Theta$ is set equal to zero, $D_{\theta=0}^{or}(\xi,\eta)$ (cf. [146, §4.1.1 & §4.1.2]).

Remark 6.4.3 (ORTHOGONALITY RELATIONS)**.** *In general, the orthogonal duality functions of Theorem 6.4.1 are* not *orthogonal w.r.t. the stationary measure of the particle dynamics in non-equilibrium. In fact, for each choice of $\sigma \in \{-1,0,1\}$ and $\theta \in \Theta$, the orthogonal duality function $D_\theta^{or}(\xi,\eta)$ gives rise to an orthogonal basis $\{e_\xi : \xi \in \widehat{\mathcal{Y}}\}$ of $L^2(\mathcal{X},\mu_\theta)$, where μ_θ is given in* (6.3.1)*,*

$$e_\xi := D_\theta^{or}(\xi,\cdot) \qquad \text{and} \qquad \widehat{\mathcal{Y}} := \{\xi \in \widehat{\mathcal{X}} : \xi(L) = \xi(R) = 0\}\,. \tag{6.4.11}$$

In equilibrium, i.e. $\theta_L = \theta_R = \theta \in \Theta$, we have seen (see Section 6.3) that the measure μ_θ is stationary for the particle system $\{\eta_t : t \geq 0\}$. In non-equilibrium, i.e. $\theta_L \neq \theta_R$, μ_θ fails to be stationary. Nevertheless, the aforementioned orthogonality relations still hold in both contexts, regardless of the stationarity of μ_θ.

As an immediate consequence of Theorem 6.4.1, we can compute the following expectations of the orthogonal duality functions.

Proposition 6.4.4. *Let $b \in \mathbb{R}$ such that*

$$\theta := \theta_R + b(\theta_L - \theta_R) \in \Theta\,. \tag{6.4.12}$$

Then, for all $t \geq 0$ and for all configurations $\xi \in \widehat{\mathcal{X}}$, we have

$$\mathbb{E}_{\mu_\theta}\left[D_\theta^{or}(\xi,\eta_t)\right] = (\theta_L - \theta_R)^{|\xi|}\phi_{t,b}(\xi)\,, \tag{6.4.13}$$

where μ_θ is the product measure (cf. (6.3.1)*) with scale parameter $\theta = \theta_R + b\,(\theta_L - \theta_R)$ and*

$$\phi_{t,b}(\xi) := (-b)^{|\xi|}\,\widehat{\mathbb{E}}_\xi\left[\left(\frac{b-1}{b}\right)^{\xi_t(L)} \mathbf{1}_{\{\xi_t(L)+\xi_t(R)=|\xi|\}}\right]\,.$$

Moreover, for all configurations $\xi \in \widehat{\mathcal{X}}$, we have

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{or}_\theta(\xi,\eta)\right] = (\theta_L-\theta_R)^{|\xi|}\,\phi_b(\xi)\ , \tag{6.4.14}$$

where

$$\phi_b(\xi) := (-b)^{|\xi|}\,\widehat{\mathbb{E}}_\xi\left[\left(\frac{b-1}{b}\right)^{\xi_\infty(L)}\right].$$

In particular, $\phi_{b,t}$ and ϕ_b do not depend on neither θ_L nor θ_R, but only on b, $\sigma \in \{-1,0,1\}$ and the underlying geometry of the system.

Proof. As a consequence of duality (Theorem 6.4.1), we obtain

$$\begin{aligned}
\mathbb{E}_{\mu_\theta}\left[D^{or}_\theta(\xi,\eta_t)\right] &= \widehat{\mathbb{E}}_\xi\left[\mathbb{E}_{\mu_\theta}\left[D^{or}_\theta(\xi_t,\eta)\right]\right]\\
&= \widehat{\mathbb{E}}_\xi\left[(\theta_L-\theta)^{\xi_t(L)}\,(\theta_R-\theta)^{\xi_t(R)}\,\mathbf{1}_{\{\xi_t(L)+\xi_t(R)=|\xi|\}}\right]\\
&= \widehat{\mathbb{E}}_\xi\left[(\theta_L-\theta)^{\xi_t(L)}\,(\theta_R-\theta)^{|\xi|-\xi_t(L)}\,\mathbf{1}_{\{\xi_t(L)+\xi_t(R)=|\xi|\}}\right]\\
&= (\theta_R-\theta)^{|\xi|}\,\widehat{\mathbb{E}}_\xi\left[(\theta_L-\theta)^{\xi_t(L)}\,(\theta_R-\theta)^{-\xi_t(L)}\,\mathbf{1}_{\{\xi_t(L)+\xi_t(R)=|\xi|\}}\right]\ ,
\end{aligned} \tag{6.4.15}$$

where in the second identity we have used orthogonality of the single-site duality functions $d^{or}_{x,\theta}(k,\cdot)$ w.r.t. the marginal $\nu_{x,\theta}$ (see also Remark 6.4.3) and the observation that

$$d^{or}_{x,\theta}(0,\cdot) \equiv 1\ ,\quad x \in V\ .$$

Inserting $\theta = \theta_R + b(\theta_L-\theta_R)$ (cf. (6.4.12)) in the last line of (6.4.15), we get (6.4.13). By sending $t \to \infty$, the uniqueness of the stationary measure yields (6.4.14). □

Remark 6.4.5. *For the choice $b = \frac{1}{2}$ and, thus, $\theta = \frac{\theta_L+\theta_R}{2}$,* (6.4.13) *and* (6.4.14) *further simplify as*

$$\mathbb{E}_{\mu_\theta}\left[D^{or}_\theta(\xi,\eta_t)\right] = \left(\frac{\theta_L-\theta_R}{2}\right)^{|\xi|}\widehat{\mathbb{E}}_\xi\left[(-1)^{|\xi|-\xi_t(L)}\,\mathbf{1}_{\{\xi_t(L)+\xi_t(R)=|\xi|\}}\right] \tag{6.4.16}$$

and

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{or}_\theta(\xi,\eta)\right] = \left(\frac{\theta_L-\theta_R}{2}\right)^{|\xi|}\widehat{\mathbb{E}}_\xi\left[(-1)^{|\xi|-\xi_\infty(L)}\right]\ . \tag{6.4.17}$$

6.5 Higher order correlations in non-equilibrium

In this section, we study higher order space correlations for the non-equilibrium stationary measures presented in Section 6.3. In particular, we show in Theorem 6.5.1 below, by using the orthogonal duality functions of Section 6.4, that the n-point correlation functions in non-equilibrium may be factorized into a first term, namely $(\theta_L-\theta_R)^n$, and a second term, which we call ψ and which is independent of the values θ_L and θ_R. This result may be seen as a higher order generalization of the decomposition obtained for the simple symmetric exclusion process in [57, Eqs. (2.3)–(2.8)]. There the authors exploit the matrix formulation of the non-equilibrium stationary measure to recover the explicit expression for the first, second and third order correlation functions.

While the coefficients ψ in (6.5.3) for the case of independent random walkers are identically zero (see item (b) after Theorem 6.5.1 below), for the interacting case ($\sigma \in \{-1,1\}$) they are expressed in terms of absorption probabilities of both interacting and independent dual particles. These absorption probabilities – apart from some special instances, see e.g. [57] and [32, Section 6.1] – are not explicitly known. Nonetheless, Theorem 6.5.1 – and the related Theorem 6.5.6 – highlight the common structure of the higher order correlations for all three particle systems considered in this chapter. In particular, this common structure arises for all values of the parameters θ_L and $\theta_R \in \Theta$ and with all disorders $(\boldsymbol{\omega},\boldsymbol{\alpha})$ and parameters $\{\omega_L,\omega_R,\alpha_L,\alpha_R\}$ as in (6.2.13)–(6.2.14). Moreover, along the same lines, we show that all higher order space correlations at any finite time $t > 0$ for the particle system started from suitable product measures exhibit the same structure. This is the content of Theorem 6.5.6 in Section 6.5.2 below. In fact, we derive Theorem 6.5.1 on the structure of stationary correlations from the more general result stated in Theorem 6.5.6, whose proof is deferred to Section 6.5.3.

6.5.1 Stationary non-equilibrium correlations and cumulants

For each choice of $\sigma \in \{-1,0,1\}$, we recall that μ_{θ_L,θ_R} denotes the non-equilibrium stationary measure of the particle system $\{\eta_t : t \geq 0\}$ with generator $\mathcal{L}$ given in (6.2.1). Moreover, let us recall the definition of $\{\bar{\theta}_x : x \in V\}$ in (6.3.7) and introduce the following ordering of dual configurations: for all $\xi \in \widehat{\mathcal{X}}$,

$$\zeta \leq \xi \quad \text{if and only if} \quad \zeta \in \widehat{\mathcal{X}} \quad \text{and} \quad \begin{array}{ll} \zeta(L) \leq \xi(L)\,, & \zeta(R) \leq \xi(R) \\ \zeta(x) \leq \xi(x)\,, & \text{for all } x \in V\,. \end{array} \tag{6.5.1}$$

Analogously, we say that $\zeta < \xi$ if $\zeta \leq \xi$ and at least one of the inequalities in (6.5.1) is strict. Finally, given $\xi, \zeta \in \widehat{\mathcal{X}}$, let $\xi \pm \zeta$ denote the configuration with $\xi(x) \pm \zeta(x)$ particles at site x, for all $x \in V \cup \{L, \mathbb{R}\}$, as long as $\xi \pm \zeta \in \widehat{\mathcal{X}}$.

In what follows, for all choices of $\sigma \in \{-1,0,1\}$, $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{E}}$ denote the law and expectation, respectively, of the dual process with either exclusion ($\sigma = -1$), inclusion ($\sigma = 1$) or no interaction ($\sigma = 0$), while we adopt $\widehat{\mathbb{P}}^{\mathrm{IRW}}$ and $\widehat{\mathbb{E}}^{\mathrm{IRW}}$ to refer to the law and corresponding expectation, respectively, of the dual process consisting of non-interacting random walks ($\sigma = 0$).

Theorem 6.5.1 (stationary correlation functions). *For all $n \in \mathbb{N}$ with $n \leq |V|$ and for all $x_1, \ldots, x_n \in V$ with $x_i \neq x_j$ if $i \neq j$, by setting*

$$\xi = \delta_{x_1} + \cdots + \delta_{x_n}\,,$$

we have

$$\begin{aligned} \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\prod_{i=1}^{n}\left(\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_{x_i}\right)\right] &= (\theta_L - \theta_R)^n\, \psi(\xi) \\ &= (\theta_L - \theta_R)^n\, \psi(\delta_{x_1} + \cdots + \delta_{x_n})\,, \end{aligned} \tag{6.5.2}$$

where

$$\psi(\xi) = \sum_{\zeta \leq \xi} (-1)^{|\xi| - |\zeta|}\, \widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi - \zeta)_\infty(L) = |\xi - \zeta|)\, \widehat{\mathbb{P}}_\zeta(\zeta_\infty(L) = |\zeta|)\,. \tag{6.5.3}$$

In particular, $\psi(\xi) \in \mathbb{R}$ and it does not depend on neither θ_L nor θ_R, but only on $\sigma \in \{-1,0,1\}$ and the underlying geometry (see Eqs. (6.2.13)–(6.2.14)*) of the system.*

As an immediate consequence we have the following corollary on the stationary non-equilibrium joint cumulants.

Corollary 6.5.2 (joint cumulants). *For all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in V$ with $x_i \neq x_j$ if $i \neq j$, let $\kappa(\delta_{x_1} + \ldots + \delta_{x_n})$ denote the joint cumulant of the random variables*

$$\left\{\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_{x_i} : x_1, \ldots, x_n \in V\right\}\,.$$

Then, we have

$$\kappa(\delta_{x_1} + \cdots + \delta_{x_n}) = (\theta_L - \theta_R)^n\, \varphi(\delta_{x_1} + \cdots + \delta_{x_n})\,,$$

where $\varphi(\delta_{x_1} + \cdots + \delta_{x_n}) \in \mathbb{R}$ does not depend on neither θ_L nor θ_R, but only on $\sigma \in \{-1,0,1\}$ and the underlying geometry of the system.

Proof. After recalling that

$$\kappa(\delta_{x_1} + \cdots + \delta_{x_n}) = \sum_{\gamma \in \mathcal{T}} (|\gamma| - 1)!\,(-1)^{|\gamma|-1} \prod_{U \in \gamma} \mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\prod_{y \in U}\left(\frac{\eta(y)}{\alpha_y} - \bar{\theta}_y\right)\right]\,,$$

where $\mathcal{T} = \mathcal{T}(\{x_1, \ldots, x_n\})$ denotes the set of partitions of $\{x_1, \ldots, x_n\} \subset V$, the result follows by (6.5.2) with $\varphi(\{x_1, \ldots, x_n\})$ given by

$$\varphi(\delta_{x_1} + \cdots + \delta_{x_n}) = \sum_{\gamma \in \mathcal{T}} (|\gamma| - 1)!\,(-1)^{|\gamma|-1} \prod_{U \in \gamma} \psi(U)\,,$$

where $\psi(U) := \psi(\sum_{x \in U} \delta_x)$. □

Properties of the function ψ

We collect below some further properties of the coefficients ψ in (6.5.2):

(a) For all $\sigma \in \{-1,0,1\}$, if $|\xi| = 0$, i.e. the dual configuration is empty, then $\psi(\xi) = 1$.

(b) For $\sigma = 0$, $\psi(\xi) = 0$ for all $\xi \in \widehat{\mathcal{X}}$ such that $|\xi| \geq 1$.

(c) For all $\sigma \in \{-1,0,1\}$ and for all $x \in V$, $\psi(\delta_x) = 0$.

(d) If $\sigma \in \{-1,1\}$ and $\theta_L \neq \theta_R$, as a consequence of Theorem 6.3.4 and $(\theta_L - \theta_R)^2 > 0$, $\psi(\delta_x + \delta_y)$ is negative for $\sigma = -1$ and positive for $\sigma = 1$ for all $x, y \in V$.

(e) Because $\psi(\delta_{x_1} + \cdots + \delta_{x_n})$ depends only on the underlying geometry of the system and not on θ_L, θ_R, exchanging the role of θ_L and θ_R does not affect the value of the stationary n-point correlation functions if $n \in \mathbb{N}$ is even, while it involves only a change of sign if $n \in \mathbb{N}$ is odd. More precisely, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in V$,

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\prod_{i=1}^n \left(\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_{x_i}\right)\right] = (-1)^n \, \mathbb{E}_{\mu_{\theta_R,\theta_L}}\left[\prod_{i=1}^n \left(\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_{x_i}\right)\right] .$$

(f) As we will see in the course of the next section 6.5.2, $\psi(\xi)$ in (6.5.2)–(6.5.3) can be defined for any $\xi \in \widehat{\mathcal{X}}$ and equivalently expressed in terms of a parameter $b \in \mathbb{R}$. More precisely, given $\xi \in \widehat{\mathcal{X}}$ and $b \in \mathbb{R}$, we have

$$\psi(\xi) = \sum_{\zeta \leq \xi} (-1)^{|\xi|-|\zeta|} \left(\prod_{x \in V} \binom{\xi(x)}{\zeta(x)} (\widehat{p}_\infty(\delta_x, \delta_L) - b)^{\xi(x)-\zeta(x)} \right) \widehat{\mathbb{E}}_\zeta \left[(1-b)^{\zeta_\infty(L)} (-b)^{\zeta_\infty(R)} \right] . \tag{6.5.4}$$

Notice that, by setting $\xi = \delta_{x_1} + \cdots + \delta_{x_n}$ with $x_i \neq x_j$ if $i \neq j$, all the binomial coefficients in (6.5.4) are equal to one. The choice $b = 0$ corresponds then to the expression on the l.h.s. of (6.5.2), while choosing $b = 1$ leads to

$$\psi(\xi) = \sum_{\zeta \leq \xi} (-1)^{|\zeta|} \, \widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi-\zeta)_\infty(R) = |\xi - \zeta|) \, \widehat{\mathbb{P}}_\zeta(\zeta_\infty(R) = |\zeta|) . \tag{6.5.5}$$

In particular, since $\psi(\xi)$ does not depend on b, we have that

$$\frac{\mathrm{d}\psi(\xi)}{\mathrm{d}b} = 0 , \tag{6.5.6}$$

which is an equation giving information on the absorption probabilities. If we consider, for instance, the case $\xi = \delta_x + \delta_y$ with $x \neq y$, (6.5.4) and (6.5.6) yield

$$2\,\widehat{\mathbb{P}}_{\xi=\delta_x+\delta_y}(\xi_\infty(L) = 2) + \widehat{\mathbb{P}}_{\xi=\delta_x+\delta_y}(\xi_\infty(L) = 1) = \widehat{p}_\infty(\delta_x, \delta_L) + \widehat{p}_\infty(\delta_y, \delta_L) , \tag{6.5.7}$$

which corresponds to the recursive relation found in [37, Proposition 5.1]. More generally, by matching the two expressions of $\psi(\xi)$ for $\xi = \delta_{x_1} + \cdots + \delta_{x_n}$ with $x_i \neq x_j$ if $i \neq j$, in (6.5.3) and (6.5.5), the relation that we find is

$$\begin{aligned} &\widehat{\mathbb{P}}_\xi\,(\xi_\infty(L) = |\xi|) - (-1)^{|\xi|}\,\widehat{\mathbb{P}}_\xi\,(\xi_\infty(R) = |\xi|) \\ &= \sum_{\zeta < \xi} \left\{ \begin{array}{l} \widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi-\zeta)_\infty(R) = |\xi-\zeta|)\,\widehat{\mathbb{P}}_\zeta(\zeta_\infty(R) = |\zeta|) \\ -(-1)^{|\xi|}\,\widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi-\zeta)_\infty(L) = |\xi-\zeta|)\,\widehat{\mathbb{P}}_\zeta(\zeta_\infty(L) = |\zeta|) \end{array} \right\} . \end{aligned}$$

In other words, the above equation relates the probabilities of having all $|\xi|$ dual particles absorbed at the same end with a linear combination of analogous probabilities for systems with a strictly smaller number of particles.

6.5.2 Correlations at finite times and proof of Theorem 6.5.1

Theorem 6.5.1 follows from a more general result. This is the content of Theorem 6.5.6 below. There, we show that a decomposition reminiscent of that in (6.5.2) holds also for expectations at some fixed positive time of generalizations of the n-point correlation functions of Theorem 6.5.1 when the particle system starts from a suitable product measure. The aforementioned generalizations of the correlation functions are constructed by suitably recombining the orthogonal duality functions of Section 6.4 so to obtain a family of functions orthogonal w.r.t. what we call "interpolating product measures" given in the following definition.

Definition 6.5.3 (interpolating product measures)**.** *We call* interpolating product measure *with interpolating parameters*

$$\boldsymbol{\beta} = \{\beta_x : x \in V\} \tag{6.5.8}$$

the measure given by

$$\mu_{\theta_L,\theta_R,\boldsymbol{\beta}} := \otimes_{x\in V}\, \nu_{x,\theta_x}\ , \tag{6.5.9}$$

with

$$\theta_x := \theta_R + \beta_x(\theta_L - \theta_R)\ , \tag{6.5.10}$$

where the marginals $\{\nu_{x,\theta} : x \in V\}$ *appearing in* (6.5.9) *are those given in* (6.3.2) *and* $\boldsymbol{\beta}$ *in* (6.5.8)–(6.5.10) *is chosen such that, for each choice of* $\sigma \in \{-1, 0, 1\}$*, the product measure* $\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}$ *is a probability measure, i.e., for all* $x \in V$*, the following conditions hold:*

$$\beta_x \in \mathbb{R} \qquad and \qquad \theta_x = \theta_R + \beta_x(\theta_L - \theta_R) \in \Theta\ . \tag{6.5.11}$$

In particular, if we choose

$$\beta_x = \widehat{p}_\infty(\delta_x, \delta_L) =: \bar{\beta}_x\ , \quad x \in V\ ,$$

as corresponding interpolating product measure we recover the local equilibrium product measure $\mu_{\bar{\boldsymbol{\theta}}}$ (Definition 6.3.2):

$$\mu_{\theta_L,\theta_R,\bar{\boldsymbol{\beta}}} = \mu_{\bar{\boldsymbol{\theta}}}\ . \tag{6.5.12}$$

Let us now introduce what we call the "interpolating orthogonal functions".

Definition 6.5.4 (interpolating orthogonal functions)**.** *Recalling the definition of orthogonal polynomial dualities in* (6.4.5)–(6.4.6) *and the definition of interpolating parameters* $\boldsymbol{\beta}$ *in* (6.5.8)*, we define the* interpolating orthogonal function *with interpolating parameters* $\boldsymbol{\beta}$ *as follows:*

$$D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}}(\xi,\eta) := d^{or}_{L,\theta_L}(\xi(L)) \times \left(\prod_{x\in V} d^{or}_{x,\theta_x}(\xi(x),\eta(x))\right) \times d^{or}_{R,\theta_R}(\xi(R))\ , \tag{6.5.13}$$

where the parameters $\{\theta_x : x \in V\}$ *are defined in terms of* θ_L*,* θ_R *and* $\boldsymbol{\beta}$ *as in* (6.5.10)*.*

In analogy with (6.5.12), we define

$$D^{or}_{\bar{\boldsymbol{\theta}}}(\xi,\eta) := D^{or}_{\theta_L,\theta_R,\bar{\boldsymbol{\beta}}}(\xi,\eta) = d^{or}_{L,\theta_L}(\xi(L)) \times \left(\prod_{x\in V} d^{or}_{x,\bar{\theta}_x}(\xi(x),\eta(x))\right) \times d^{or}_{R,\theta_R}(\xi(R))\ . \tag{6.5.14}$$

Remark 6.5.5. *We note that, despite the analogy in notation, in general these functions are* not *duality functions for the particle system* $\{\eta_t : t \geq 0\}$*, unless we assume the system to be at equilibrium, i.e.* $\theta_L = \theta_R = \theta \in \Theta$*. Only in the latter case,* $D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}}(\xi,\eta) = D^{or}_{\theta}(\xi,\eta)$ *for all choices of* $\boldsymbol{\beta}$*.*

With the definition (6.5.13), we have (cf. Remark 6.4.3) that

$$D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}}(\xi,\cdot) = 0\ , \qquad \text{if} \quad \xi \in \widehat{\mathcal{X}} \setminus \widehat{\mathcal{Y}}\ , \tag{6.5.15}$$

and that the family of functions

$$\left\{D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}}(\xi,\cdot) : \xi \in \widehat{\mathcal{Y}}\right\}$$

is an orthogonal basis in $L^2(\mathcal{X}, \mu_{\theta_L,\theta_R,\boldsymbol{\beta}})$. Now we are ready to state the main result of this section, whose Theorem 6.5.1 is a particular instance.

Theorem 6.5.6. *Let us consider two set of interpolating parameters*

$$\boldsymbol{\beta} = \{\beta_x : x \in V\} \quad \textit{and} \quad \boldsymbol{\beta}' = \{\beta'_x : x \in V\}$$

both satisfying (6.5.10). *Then, for all $\xi \in \widehat{\mathcal{Y}} \subset \widehat{\mathcal{X}}$ and $t \geq 0$, we have*

$$\mathbb{E}_{\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}}\left[D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'}(\xi,\eta_t)\right] = (\theta_L - \theta_R)^{|\xi|}\,\psi_{t,\boldsymbol{\beta},\boldsymbol{\beta}'}(\xi)\,, \tag{6.5.16}$$

where

$$\psi_{t,\boldsymbol{\beta},\boldsymbol{\beta}'}(\xi) := \sum_{\zeta\leq\xi}(-1)^{|\xi|-|\zeta|}\left(\prod_{x\in V}\binom{\xi(x)}{\zeta(x)}(\beta'_x)^{\xi(x)-\zeta(x)}\,\widehat{\mathbb{E}}_\zeta\left[\mathbf{1}_{\{\zeta_t(R)=0\}}\left(\prod_{x\in V}(\beta_x)^{\zeta_t(x)}\right)\right]\right)\,, \tag{6.5.17}$$

and $\psi_{t,\boldsymbol{\beta},\boldsymbol{\beta}'}(\xi)$ does not depend on neither θ_L nor θ_R, but only on $\boldsymbol{\beta}$, $\boldsymbol{\beta}'$, $\sigma \in \{-1,0,1\}$ and the underlying geometry of the system. Moreover, by sending t to infinity in (6.5.16) *we obtain, for all $\xi \in \widehat{\mathcal{Y}} \subset \widehat{\mathcal{X}}$,*

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'}(\xi,\eta)\right] = (\theta_L - \theta_R)^{|\xi|}\,\psi_{\boldsymbol{\beta}'}(\xi)\,, \tag{6.5.18}$$

where

$$\psi_{\boldsymbol{\beta}'}(\xi) := \sum_{\zeta\leq\xi}(-1)^{|\xi|-|\zeta|}\left(\prod_{x\in V}\binom{\xi(x)}{\zeta(x)}(\beta'_x)^{\xi(x)-\zeta(x)}\right)\widehat{\mathbb{P}}_\zeta\left[\zeta_\infty(L) = |\zeta|\right]\,. \tag{6.5.19}$$

Again, $\psi_{\boldsymbol{\beta}'}(\xi)$ is independent of θ_L and θ_R.

Remark 6.5.7. *From the proof of Theorem 6.5.6, the results of the theorem extend to configurations $\xi \in \widehat{\mathcal{X}} \setminus \widehat{\mathcal{Y}}$ and, by* (6.5.15)*,*

$$\psi_{t,\boldsymbol{\beta},\boldsymbol{\beta}'}(\xi) = 0\,, \qquad \textit{if} \quad \xi \in \widehat{\mathcal{X}} \setminus \widehat{\mathcal{Y}}\,. \tag{6.5.20}$$

Before moving to the next section, Section 6.5.3, in which we provide the proof of Theorem 6.5.6, we show how this latter result implies Theorem 6.5.1.

Proof of Theorem 6.5.1. Recall that, by the definitions of hypergeometric functions (6.4.1)–(6.4.2) and of single-site orthogonal duality functions in (6.4.5), we have, for all $n \in \mathbb{N}$ and $\eta \in \mathcal{X}$,

$$D^{or}_{\boldsymbol{\theta}}(\delta_{x_1} + \cdots + \delta_{x_n},\eta) = \prod_{i=1}^n\left(\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_i\right) \tag{6.5.21}$$

anytime $x_1,\ldots,x_n \in V$ with $x_i \neq x_j$ if $i \neq j$. By choosing for any $x \in V$, $\beta'_x = \widehat{p}_\infty(\delta_x,\delta_L)$, the result follows immediately from Theorem 6.5.6. □

Probabilistic interpretation of the function ψ

Theorem 6.5.1 may be seen as a particular instance of Theorem 6.5.6 with the choice $t = \infty$, $\xi \in \widehat{\mathcal{X}}$ consisting of finitely many particles all sitting at different sites in the bulk and $\beta'_x = \widehat{p}_\infty(\delta_x,\delta_L)$ for every $x \in V$. In fact, Theorem 6.5.6 extends the relation (6.5.2) to all $\xi \in \widehat{\mathcal{X}}$, i.e.

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[D^{or}_{\boldsymbol{\theta}}(\xi,\eta)\right] = (\theta_L - \theta_R)^{|\xi|}\,\psi(\xi)\,, \tag{6.5.22}$$

with,

$$\psi(\xi) := \sum_{\zeta\leq\xi}\binom{\xi}{\zeta}(-1)^{|\xi|-|\zeta|}\,\widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi-\zeta)_\infty(L) = |\xi-\zeta|)\,\widehat{\mathbb{P}}_\zeta\left(\zeta_\infty(L) = |\zeta|\right)\,, \tag{6.5.23}$$

where $\binom{\xi}{\zeta} := \prod_{x\in V}\binom{\xi(x)}{\zeta(x)}$ and $\widehat{\mathbb{P}}^{\mathrm{IRW}}$ refers to the law of the dual process for $\sigma = 0$, consisting of non-interacting random walks.

In order to obtain a more probabilistic interpretation of (6.5.23), we define

(a) the probability measure γ_ξ on $\widehat{\mathcal{X}}$ given by

$$\gamma_\xi(\zeta) = \frac{\binom{\xi}{\zeta}}{2^{|\xi|}} \mathbf{1}_{\{\zeta \leq \xi\}}, \tag{6.5.24}$$

i.e. the distribution of uniformly chosen sub-configuration of ξ (i.e. $\zeta \leq \xi$);

(b) the function $\Psi_\xi : \widehat{\mathcal{X}} \to \mathbb{R}$ given by

$$\Psi_\xi(\zeta) := \mathbf{1}_{\{\zeta \leq \xi\}} \, \widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi-\zeta)_\infty(L) = |\xi-\zeta|)\, \widehat{\mathbb{P}}_\zeta\,(\zeta_\infty(L) = |\zeta|) \ ,$$

i.e., the function that assigns to any $\zeta \leq \xi$ the probability that, in a system composed by the superposition of the configuration ζ of *interacting dual particles* and the configuration $\xi - \zeta$ of *independent dual random walks*, independent between each other, all the particles are eventually absorbed at L.

The function $\psi(\xi)$ in (6.5.23) can, then, be rewritten as follows:

$$\psi(\xi) = 2^{|\xi|} \sum_{\zeta \in \widehat{\mathcal{X}}} (-1)^{|\xi-\zeta|}\, \Psi_\xi(\zeta)\, \gamma_\xi(\zeta) \,.$$

Similarly, for all $t \geq 0$, $\xi \in \widehat{\mathcal{X}}$ and for the special choice

$$\boldsymbol{\beta} = \boldsymbol{\beta}' \quad \text{and} \quad \beta_x = \beta'_x = \widehat{p}_\infty(\delta_x, \delta_L) \,,$$

the identity in (6.5.16) yields, as a particular case,

$$\mathbb{E}_{\mu_\theta}\left[D^{or}_{\boldsymbol{\theta}}(\xi, \eta_t)\right] = (\theta_L - \theta_R)^{|\xi|}\, \psi_t(\xi) \,, \tag{6.5.25}$$

where

$$\begin{aligned} \psi_t(\xi) &:= \sum_{\zeta \leq \xi} (-1)^{\xi-\zeta}\, \widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((|\xi| - |\zeta|)_\infty(L) = |\xi-\zeta|)\, \widehat{\mathbb{E}}_\zeta\left[\widehat{\mathbb{P}}^{\mathrm{IRW}}_{\zeta_t}(\zeta_\infty(L) = |\zeta|)\right] \\ &= 2^{|\xi|} \sum_{\zeta \in \widehat{\mathcal{X}}} (-1)^{|\xi|-|\zeta|} \Psi_{t,\xi}(\zeta)\, \gamma_\xi(\zeta) \,, \end{aligned} \tag{6.5.26}$$

where the integral in the last identity is w.r.t. the probability measure γ_ξ defined in (6.5.24) and

$$\Psi_{t,\xi}(\zeta) := \mathbf{1}_{\{\zeta \leq \xi\}} \widehat{\mathbb{P}}^{\mathrm{IRW}}_{\xi-\zeta}((\xi-\zeta)_\infty(L) = |\xi| - |\zeta|)\, \widehat{\mathbb{E}}_\zeta\left[\widehat{\mathbb{P}}^{\mathrm{IRW}}_{\zeta_t}(\zeta_\infty(L) = |\zeta|)\right].$$

6.5.3 Proof or Theorem 6.5.6

We prove Theorem 6.5.6 in two steps.

First we obtain a formula to relate the functions $D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'}(\xi, \eta)$ in (6.5.13) appearing in the statement of Proposition 6.5.6 to the orthogonal duality functions $D^{or}_\theta(\xi, \eta)$ in Section 6.4, for some $\theta \in \Theta$.

Lemma 6.5.8. *For each choice of $\sigma \in \{-1, 0, 1\}$ and $b \in \mathbb{R}$, we define*

$$\theta := \theta_R + b(\theta_L - \theta_R) \,. \tag{6.5.27}$$

Then, for all configurations $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$,

$$D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'}(\xi, \eta) = \sum_{\zeta \leq \xi} (\theta_L - \theta_R)^{|\xi|-|\zeta|}\, (-1)^{|\xi|-|\zeta|}\, E_{\boldsymbol{\beta}',b}(\zeta, \xi)\, D^{or}_\theta(\zeta, \eta) \,,$$

where $E_{\boldsymbol{\beta}',b}(\zeta, \xi)$ is defined as

$$E_{\boldsymbol{\beta}',b}(\zeta, \xi) := E_{L,b}(\zeta(L), \xi(L)) \times \left(\prod_{x \in V} E_{x,\beta'_x,b}(\zeta(x), \xi(x))\right) \times E_{R,b}(\zeta(R), \xi(R)) \,, \tag{6.5.28}$$

where, for all $x \in V$,

$$E_{x,\beta'_x,b}(\ell,k) := \binom{k}{\ell}(\beta'_x - b)^{k-\ell}\mathbf{1}_{\{\ell\le k\}}\ ,$$

and

$$E_{L,b}(\ell,k) := \binom{k}{\ell}(1-b)^{k-\ell}\mathbf{1}_{\{\ell\le k\}}$$
$$E_{R,b}(\ell,k) := \binom{k}{\ell}(-b)^{k-\ell}\mathbf{1}_{\{\ell\le k\}}\ .$$

Proof. By definition of the orthogonal duality functions in Theorem 6.4.1 (see also (6.4.7)) and of the functions $D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'}$ in (6.5.13), we have

$$D^{or}_\theta = \left(e^{-\theta\widehat{\mathcal{K}}}\right)_{\text{left}} D^{c\ell}$$

and

$$D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'} = \left(e^{-\theta_L\widehat{\mathcal{K}}_L-\left(\sum_{x\in V}\theta'_x\widehat{\mathcal{K}}_x\right)-\theta_R\widehat{\mathcal{K}}_R}\right)_{\text{left}} D^{c\ell}\quad ,$$

where

$$\theta'_x := \theta_R + \beta'_x\,(\theta_L-\theta_R)\ ,\quad x\in V\ .$$

Next, we get

$$\begin{aligned}D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'} &= \left(e^{-\theta_L\widehat{\mathcal{K}}_L-\left(\sum_{x\in V}\theta'_x\widehat{\mathcal{K}}_x\right)-\theta_R\widehat{\mathcal{K}}_R+\theta\widehat{\mathcal{K}}}\right)_{\text{left}}\left(e^{-\theta\widehat{\mathcal{K}}}\right)_{\text{left}} D^{c\ell}\\ &= \left(e^{-(\theta_L-\theta)\widehat{\mathcal{K}}_L-\left(\sum_{x\in V}(\theta'_x-\theta)\widehat{\mathcal{K}}_x\right)-(\theta_R-\theta)\widehat{\mathcal{K}}_R}\right)_{\text{left}} D^{or}_\theta\ ,\end{aligned}$$

where the latter identity is a consequence of the fact that all the operators $\{\widehat{\mathcal{K}}_x : x\in V\}\cup\{\widehat{\mathcal{K}}_L,\widehat{\mathcal{K}}_R\}$ commute. The expressions in terms of $(\theta_L-\theta_R)$ of the parameters $\{\theta'_x : x\in V\}$ in (6.5.11) and θ in (6.5.27) yield the final result. □

Then, we derive an analogue of Theorem 6.5.6 for the orthogonal duality functions.

Lemma 6.5.9. *For each choice of $\sigma\in\{-1,0,1\}$ and $b\in\mathbb{R}$ and $\theta\in\mathbb{R}$ as in* (6.5.27) *and such that $\theta\in\Theta$, we have, for all configurations $\zeta\in\widehat{\mathcal{X}}$,*

$$\mathbb{E}_{\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}}\left[D^{or}_\theta(\zeta,\eta_t)\right] = (\theta_L-\theta_R)^{|\zeta|}\,\phi_{t,\boldsymbol{\beta},b}(\zeta)\ , \tag{6.5.29}$$

where $\phi_{t,\boldsymbol{\beta},b}(\zeta)\in\mathbb{R}$ is defined as

$$\phi_{t,\boldsymbol{\beta},b}(\zeta) := \widehat{\mathbb{E}}_\zeta\left[(1-b)^{\zeta_t(L)}\times\left(\prod_{x\in V}(\beta_x-b)^{\zeta_t(x)}\right)\times(-b)^{\zeta_t(R)}\right] \tag{6.5.30}$$

and, in particular, it does not depend on neither θ_L nor θ_R, but only on $\boldsymbol{\beta}$, b, $\sigma\in\{-1,0,1\}$ and the underlying geometry of the system.

Proof. Recall the definition of $\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}$ in (6.5.9) and of the scale parameters $\{\theta_x : x\in V\}$ in (6.5.10). By duality (Theorem 6.4.1), we have

$$\begin{aligned}\mathbb{E}_{\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}}\left[D^{or}_\theta(\zeta,\eta_t)\right] &= \sum_{\zeta'\in\widehat{\mathcal{X}}}\widehat{p}_t(\zeta,\zeta')\,\mathbb{E}_{\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}}\left[D^{or}_\theta(\zeta',\eta)\right]\\ &= \sum_{\zeta'\in\widehat{\mathcal{X}}}\widehat{p}_t(\zeta,\zeta')\left\{(\theta_L-\theta)^{\zeta'(L)}\times\left(\prod_{x\in V}(\theta_x-\theta)^{\zeta'(x)}\right)\times(\theta_R-\theta)^{\zeta'(R)}\right\}\ ,\end{aligned}$$

where this last identity is a consequence of

$$\sum_{n\in\mathbb{N}_0} d^{or}_{x,\theta}(k,n)\, \nu_{x,\theta_x}(n) = (\theta_x-\theta)^k$$

for all $x \in V$ and $k \in \{0,\dots,\alpha_x\}$ if $\sigma = -1$ and $k \in \mathbb{N}_0$ if $\sigma \in \{0,1\}$ (see e.g. [146]). We obtain (6.5.29) with the function $\phi_{t,\boldsymbol{\beta},b}$ as in (6.5.30) by rewriting in terms of the parameters $\boldsymbol{\beta}$ and b the expression above between curly brackets. □

A combination of Lemma 6.5.8 and Lemma 6.5.9 concludes the proof of Theorem 6.5.6. Indeed,

$$\begin{aligned}\mathbb{E}_{\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}}\left[D^{or}_{\theta_L,\theta_R,\boldsymbol{\beta}'}(\xi,\eta_t)\right] &= \sum_{\zeta\le\xi}(\theta_L-\theta_R)^{|\xi|-|\zeta|}\,(-1)^{|\xi|-|\zeta|}\,E_{\boldsymbol{\beta}',b}(\zeta,\xi)\,\mathbb{E}_{\mu_{\theta_L,\theta_R,\boldsymbol{\beta}}}\left[D^{or}_{\theta}(\zeta,\eta_t)\right]\\ &= (\theta_L-\theta_R)^{|\xi|}\sum_{\zeta\le\xi}(-1)^{|\xi|-|\zeta|}E_{\boldsymbol{\beta}',b}(\zeta,\xi)\,\phi_{t,\boldsymbol{\beta},b}(\zeta)\,,\end{aligned}$$

which yields (6.5.16) with $\psi_{t,\boldsymbol{\beta},\boldsymbol{\beta}'}(\xi)$ given by

$$\psi_{t,\boldsymbol{\beta},\boldsymbol{\beta}'}(\xi) = \sum_{\zeta\in\widehat{\mathcal{X}}}(-1)^{|\xi|-|\zeta|}E_{\boldsymbol{\beta}',b}(\zeta,\xi)\,\phi_{t,\boldsymbol{\beta},b}(\zeta)\,. \tag{6.5.31}$$

We note that, because the l.h.s. in (6.5.16) and $(\theta_L-\theta_R)^{|\xi|}$ do not depend on the parameter $b \in \mathbb{R}$, the whole expression in (6.5.31) is independent of b, and in particular, we obtain (6.5.16) for the choice $b = 0$. By passing to the limit as t goes to infinity on both sides in (6.5.16), by uniqueness of the stationary measure μ_{θ_L,θ_R}, we obtain (6.5.18)–(6.5.19).

6.6 Exponential moments and generating functions

In this section we use the fact that the orthogonal dualities have explicit and simple generating functions in order to produce a formula for the joint moment generating function of the occupation variables in the non-equilibrium stationary state, in terms of the absorbing dual started from a random configuration ξ of which the distribution is related to the reservoir parameters. We recall that $\Theta = [0,1]$ if $\sigma = -1$ and $\Theta = [0,\infty)$ if $\sigma \in \{0,1\}$.

Theorem 6.6.1. *Let $\boldsymbol{\lambda} = \{\lambda_x : x \in V\} \in \mathbb{R}^N$ be such that, for all $x \in V$,*

$$\Lambda_x := 1 + \frac{\lambda_x}{1+\sigma\lambda_x(1+\bar\theta_x)} \ge 0\,, \tag{6.6.1}$$

and

$$\kappa_x := \frac{\lambda_x(\theta_L-\theta_R)}{1+\sigma\lambda_x\,(1-(\theta_L-\theta_R))} \in \Theta\,. \tag{6.6.2}$$

Then, we have

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\prod_{x\in V}(\Lambda_x)^{\eta(x)}\right] = \left(\prod_{x\in V}J_{\theta_L,\theta_R,\lambda_x}\right)\mathbb{E}_{\mu_{\boldsymbol{\kappa}}}[\psi], \tag{6.6.3}$$

and, for all $t \ge 0$,

$$\mathbb{E}_{\mu_{\bar\theta}}\left[\prod_{x\in V}(\Lambda_x)^{\eta_t(x)}\right] = \left(\prod_{x\in V}J_{\theta_L,\theta_R,\lambda_x}\right)\mathbb{E}_{\mu_{\boldsymbol{\kappa}}}[\psi_t], \tag{6.6.4}$$

where ψ and ψ_t are given in (6.5.2) *and* (6.5.26)*, respectively, $\mu_{\boldsymbol{\kappa}} = \otimes_{x\in V}\nu_{x,\kappa_x}$ is the probability measure defined in* (6.3.1) *with parameters $\boldsymbol{\kappa} = \{\kappa_x : x \in V\}$, viewed as a probability measure on $\widehat{\mathcal{X}}$ concentrated on $\widehat{\mathcal{Y}}$, and*

$$J_{\theta_L,\theta_R,\lambda_x} := \begin{cases} e^{\alpha_x\lambda_x(\bar\theta_x+(\theta_L-\theta_R))} & \text{if } \sigma = 0\\ \left(\dfrac{1+\sigma\lambda_x(1+\bar\theta_x)}{1+\sigma\lambda_x(1-(\theta_L-\theta_R))}\right)^{\sigma\alpha_x} & \text{if } \sigma \in \{-1,1\}\,.\end{cases} \tag{6.6.5}$$

Remark 6.6.2 (CONDITIONS (6.6.1) & (6.6.2)). *Condition* (6.6.1) *is obtained for*

$$\lambda_x \subset \begin{cases} \left(-\infty, \dfrac{1}{1+\theta_x}\right] \cup \left[\dfrac{1}{\theta_x}, \infty\right) & \text{if } \sigma = -1 \\ [-1, \infty) & \text{if } \sigma = 0 \\ \left(-\infty, -\dfrac{1}{1+\theta_x}\right] \cup \left[-\dfrac{1}{2+\theta_x}, \infty\right) & \text{if } \sigma = 1\ , \end{cases} \tag{6.6.6}$$

while condition (6.6.2) *for*

(i) *Case* $\theta_L - \theta_R \geq 0$ *:*

$$\frac{1}{\lambda_x} \subset \begin{cases} [1, \infty) & \text{if } \sigma = -1 \\ [0, \infty) & \text{if } \sigma = 0 \\ [\theta_L - \theta_R - 1, \infty) & \text{if } \sigma = 1\ , \end{cases} \tag{6.6.7}$$

(ii) *Case* $\theta_L - \theta_R \leq 0$ *:*

$$\frac{1}{\lambda_x} \subset \begin{cases} (-\infty, 1 - \theta_L + \theta_R] & \text{if } \sigma = -1 \\ (-\infty, -1) & \text{if } \sigma = 0 \\ (-\infty, \theta_L - \theta_R - 1] & \text{if } \sigma = 1. \end{cases} \tag{6.6.8}$$

We devote the remaining of this section to the proof of Theorem 6.6.1. To this purpose, let us recall the definition of $\{w_x : x \in V\}$ and $\{z_{x,\cdot} : x \in V\}$ in (6.2.18) and (6.3.4), respectively.

Definition 6.6.3 (SINGLE-SITE GENERATING FUNCTIONS). *For each choice of* $\sigma \in \{-1, 0, 1\}$*, for all* $x \in V$ *and for all functions* $f : \mathbb{N}_0 \to \mathbb{R}$*, we define*

$$(\Upsilon_x f)(\lambda) := \sum_{k=0}^{\infty} \frac{w_x(k)}{k!} \frac{\left(\frac{\lambda}{1+\sigma\lambda}\right)^k}{z_{x,\lambda}} f(k), \tag{6.6.9}$$

$$(\Upsilon_L f)(\lambda) := \sum_{k=0}^{\infty} \frac{(\alpha_L \lambda)^k}{k!} f(k)\, e^{-\alpha_L \lambda}$$

and

$$(\Upsilon_R f)(\lambda) := \sum_{k=0}^{\infty} \frac{(\alpha_R \lambda)^k}{k!} f(k)\, e^{-\alpha_R \lambda}$$

for all $\lambda \in \mathbb{R}$ *such that the above series absolutely converge. Moreover, we define*

$$\Upsilon := \Upsilon_L \otimes (\otimes_{x \in V} \Upsilon_x) \otimes \Upsilon_R\ , \tag{6.6.10}$$

acting on functions $f : \mathbb{N}_0^{N+2} \to \mathbb{R}$.

Remark 6.6.4. *If* $\lambda \in \Theta$ *then, for all* $x \in V$ *and* $f : \mathbb{N}_0 \to \mathbb{R}$*,*

$$(\Upsilon_x f)(\lambda) = \mathbb{E}_{\nu_{x,\lambda}}[f]\ ,$$

where ν_{x,λ_x} *is given in* (6.3.3).

As a first step, we investigate the action of the operators $\{\Upsilon_x : x \in V\}$ on the duality functions.

Lemma 6.6.5 (DUALITY AND GENERATING FUNCTIONS). *For each choice of* $\sigma \in \{-1, 0, 1\}$*, for all* $\theta \in \Theta$ *and for all* $x \in V$*,*

$$(\Upsilon_x)_{left}\, d_x^{c\ell}(\cdot, n)(\lambda) = \frac{\left(1 + \frac{\lambda}{1+\sigma\lambda}\right)^n}{z_{x,\lambda}} \tag{6.6.11}$$

and

$$(\Upsilon_x)_{left}\, d^{or}_{x,\theta}(\cdot,n)(\lambda) = \frac{\left(1+\frac{\lambda}{1+\sigma\lambda(1+\theta)}\right)^n}{z_{x,\lambda(1+\theta)}} \ . \tag{6.6.12}$$

Moreover

$$\Upsilon_{left} D^{or}_{\theta}(\cdot,\eta)(\boldsymbol{\lambda}) = e^{-\alpha_L\lambda_L(1+\theta-\theta_L)} \left(\prod_{x\in V} \frac{\left(1+\frac{\lambda_x}{1+\sigma\lambda_x(1+\theta)}\right)^{\eta(x)}}{z_{x,\lambda_x(1+\theta)}}\right) e^{-\alpha_R\lambda_R(1+\theta-\theta_R)} \ ,$$

and, analogously,

$$\Upsilon_{left} D^{or}_{\bar{\boldsymbol{\theta}}}(\cdot,\eta)(\boldsymbol{\lambda}) = e^{-\alpha_L\lambda_L} \left(\prod_{x\in V} \frac{\left(1+\frac{\lambda_x}{1+\sigma\lambda_x(1+\bar{\theta}_x)}\right)^{\eta(x)}}{z_{x,\lambda_x(1+\bar{\theta}_x)}}\right) e^{-\alpha_R\lambda_R} \ . \tag{6.6.13}$$

Remark 6.6.6. *In order to guarantee the absolute convergence of the series in the definition of the operators Υ in Definition 6.6.3, for the case $\sigma = 1$ we have to choose λ and θ such that*

$$\left|\frac{\theta\lambda}{1+\lambda}\right| < 1 \ .$$

Proof. By (6.4.10), we prove (6.6.12) from which, by setting $\theta = 0$, (6.6.11) follows. By definition of Υ_x in (6.6.9), relation (6.4.9) and the form of the functions $\{w_x : x \in V\}$ (see (6.2.18)), we obtain

$$\begin{aligned}
(\Upsilon_x)_{\text{left}}\, d^{or}_{x,\theta}(\cdot,n)(\lambda) &= \sum_{k=0}^{\infty} \frac{w_x(k)}{k!} \frac{\left(\frac{\lambda}{1+\sigma\lambda}\right)^k}{z_{x,\lambda}} d^{or}_{x,\theta}(k,n) \\
&= \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{\left(\frac{\lambda}{1+\sigma\lambda}\right)^\ell}{z_{x,\lambda}} \sum_{k=\ell}^{\infty} \frac{w_x(k)}{w_x(\ell)(k-\ell)!} \left(\frac{-\theta\lambda}{1+\sigma\lambda}\right)^{k-\ell} \\
&= \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{\left(\frac{\lambda}{1+\sigma\lambda}\right)^\ell}{z_{x,\lambda}} F_x(\theta,\lambda,\ell) \ ,
\end{aligned}$$

where, as long as $\left|\frac{\theta\lambda}{1+\lambda}\right| < 1$ if $\sigma = 1$ and for all $\lambda \in \mathbb{R}$ otherwise,

$$F_x(\theta,\lambda,\ell) = \begin{cases} \left(\dfrac{1+\sigma\lambda(1+\theta)}{1+\sigma\lambda}\right)^{-(\sigma\alpha_x+\ell)} & \text{if } \sigma \in \{-1,1\} \\ e^{-\alpha_x\theta\lambda} & \text{if } \sigma = 0 \ . \end{cases}$$

□

Proof of Theorem 6.6.1. We start by proving (6.6.4). First, by (6.6.13), the l.h.s. in (6.6.4) equals

$$\begin{aligned}
\text{l.h.s. in (6.6.4)} &= \mathbb{E}_{\mu_{\bar{\theta}}}\left[\Upsilon_{\text{left}} D^{or}_{\bar{\boldsymbol{\theta}}}(\cdot,\eta_t)(\boldsymbol{\lambda})\right] e^{\alpha_L\lambda_L+\alpha_R\lambda_R} \left(\prod_{x\in V} z_{x,\lambda_x(1+\bar{\theta}_x)}\right) \\
&= \Upsilon\left((\theta_L-\theta_R)^{|\cdot|}\, \psi_t(\cdot)\right)(\boldsymbol{\lambda})\, e^{\alpha_L\lambda_L+\alpha_R\lambda_R} \left(\prod_{x\in V} z_{x,\lambda_x(1+\bar{\theta}_x)}\right) \ ,
\end{aligned}$$

where in the second identity we have exchanged Υ_{left} and the expectation w.r.t. η – two operators acting on different variables – together with (6.5.25). By the definition of Υ (cf. Definition 6.6.3) and (6.5.20) (cf. (6.4.11)), we further get

$$\text{l.h.s. in (6.6.4)} = \sum_{\xi\in\widehat{\mathcal{Y}}} \left(\prod_{x\in V} \frac{w_x(\xi(x))}{(\xi(x))!} \frac{\left(\frac{\lambda_x(\theta_L-\theta_R)}{1+\sigma\lambda_x}\right)^{\xi(x)}}{z_{x,\lambda_x}}\right) \psi_t(\xi) \left(\prod_{x\in V} z_{x,\lambda_x(1+\bar{\theta}_x)}\right) \ ,$$

which, by the definition of $\mu_{\boldsymbol{\kappa}}$ (cf. the statement of the theorem), equals

$$\text{l.h.s. in (6.6.4)} = \left(\prod_{x \in V} \frac{z_{x,\lambda_x(1+\bar{\theta}_x)} z_{x,\kappa_x}}{z_{x,\lambda_x}} \right) \sum_{\xi \in \widehat{\mathcal{X}}} \mu_{\boldsymbol{\kappa}}(\xi)\, \psi_t(\xi) \; .$$

The explicit form of $\{z_{x,\cdot} : x \in V\}$ given in (6.3.4) yields (6.6.4). Sending $t \to \infty$ in (6.6.4), by the uniqueness of the stationary measure, we obtain (6.6.3). □

6.7 Existence and uniqueness of the stationary measure

In this section, we treat with full details the issue of existence and uniqueness of the stationary measure for IRW and SIP in equilibrium and non-equilibrium. In what follows we take either $\sigma = 0$ or $\sigma = 1$.

We recall that a probability measure μ on the countable space $\mathcal{X}$ (endowed with the discrete topology) is the unique stationary measure for the particle system $\{\eta_t : t \geq 0\}$ if, for all bounded functions $f : \mathcal{X} \to \mathbb{R}$ and for all probability measures μ' on $\mathcal{X}$, the following holds:

$$\lim_{t \to \infty} \mathbb{E}_{\mu'}\left[f(\eta_t)\right] = \mathbb{E}_\mu\left[f(\eta)\right] \; . \tag{6.7.1}$$

Out of all probability measures μ on $\mathcal{X}$, we say that μ is *tempered* if it is characterized by the integrals

$$\mathbb{E}_\mu\left[D^{c\ell}(\xi,\eta)\right] \; , \quad \text{for all} \quad \xi \in \widehat{\mathcal{X}} \; .$$

To the purpose of determining whether a probability measure μ is tempered or not, we adopt the following strategy. First, we recall that the functions $\{D^{c\ell}(\xi,\cdot) : \xi \in \widehat{\mathcal{X}}\}$ are weighted products of factorial moments of the variables $\{\eta(x) : x \in V\}$ (see Proposition 6.2.3). Then, we express these weighted factorial moments in terms of moments. We conclude by means of a multidimensional Carleman's condition.

By following the aforementioned ideas, we provide in the following lemma a sufficient condition for a measure to be tempered.

Lemma 6.7.1. *Let μ be a probability measure on $\mathcal{X}$. If there exists $\theta \in \Theta = [0,\infty)$ such that*

$$\mathbb{E}_\mu\left[D^{c\ell}(\xi,\eta)\right] \leq \theta^{|\xi|} \tag{6.7.2}$$

for all $\xi \in \widehat{\mathcal{X}}$, then μ is tempered.

Proof. Let us start by expressing the moments of $\eta(x)$ in terms of single-site classical duality functions in (6.2.16): for all $x \in V$ and for all $k, n \in \mathbb{N}_0$,

$$n^k = \sum_{\ell=0}^{k} \left\{ {k \atop \ell} \right\} d_x^{c\ell}(\ell, n)\, w_x(\ell) \; ,$$

where $\left\{ {k \atop \ell} \right\}$ denotes the Stirling number of the second kind given by

$$\left\{ {k \atop \ell} \right\} = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} j^k \; . \tag{6.7.3}$$

In view of (6.7.2), we obtain

$$\begin{aligned} \mathbb{E}_\mu\left[(\eta(x))^k\right] &= \sum_{\ell=0}^{k} \left\{ {k \atop \ell} \right\} \mathbb{E}_\mu\left[D^{c\ell}(\ell\delta_x,\eta)\right] w_x(\ell) \\ &\leq \sum_{\ell=0}^{k} \frac{w_x(\ell)}{\ell!} \mathbb{E}_\mu\left[D^{c\ell}(\ell\delta_x,\eta)\right] \sum_{j=0}^{\ell} \binom{\ell}{j} j^k \\ &\leq k^k \sum_{\ell=0}^{k} \frac{(2\theta)^\ell}{\ell!} w_x(\ell) \; . \end{aligned}$$

By recalling the definition of $w_x(\ell)$ in (6.2.18), in both cases with $\sigma = 0$ and $\sigma = 1$, we get

$$\mathbb{E}_\mu\left[(\eta(x))^k\right] \leq (a_x k)^k , \tag{6.7.4}$$

for all $k \in \mathbb{N}$, with $a_x = (1 + 2\theta\alpha_x)$ for $\sigma = 0$ and $a_x = \lfloor\alpha_x\rfloor!(1 + 2\theta)^{\lfloor\alpha_x\rfloor+1}$ for $\sigma = 1$. Therefore, if $m_x(k) := \mathbb{E}_\mu\left[(\eta(x))^k\right]$, (6.7.4) yields

$$\sum_{k=1}^\infty m_x(2k)^{-\frac{1}{2k}} \geq \frac{1}{a_x} \sum_{k=1}^\infty \frac{1}{2k} = \infty .$$

Because the above condition holds for all $x \in V$, the multidimensional Carleman condition (see e.g. [150, Theorem 14.19]) applies. Hence, μ is completely characterized by the moments $\{m_x(k) : x \in V, k \in \mathbb{N}\}$ and, in turn, is tempered. □

Now, by means of duality, we observe that, for all $\eta \in \mathcal{X}$ and $\xi \in \widehat{\mathcal{X}}$ with $|\xi| = k$,

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}_\eta\left[D^{c\ell}(\xi, \eta_t)\right] &= \lim_{t\to\infty} \widehat{\mathbb{E}}_\xi\left[D^{c\ell}(\xi_t, \eta)\right] \\ &= \sum_{\ell=0}^k \theta_L^\ell\, \theta_R^{k-\ell}\, \widehat{\mathbb{P}}_\xi\left(\xi_\infty = \ell\delta_L + (k-\ell)\delta_R\right) .\end{aligned} \tag{6.7.5}$$

We note that the expression above does not depend on $\eta \in \mathcal{X}$ and, moreover,

$$\lim_{t\to\infty} \mathbb{E}_\eta\left[D^{c\ell}(\xi, \eta_t)\right] \leq (\theta_L \vee \theta_R)^{|\xi|}$$

for all $\xi \in \widehat{\mathcal{X}}$. Therefore, by Lemma 6.7.1, there exists a unique probability measure $\mu_\star$ on $\mathcal{X}$ such that

$$\mathbb{E}_{\mu_\star}\left[D^{c\ell}(\xi, \eta)\right] = \sum_{\ell=0}^{|\xi|} \theta_L^\ell\, \theta_R^{|\xi|-\ell}\, \widehat{\mathbb{P}}_\xi\left(\xi_\infty = \ell\delta_L + (|\xi|-\ell)\delta_R\right) .$$

Furthermore, because the convergence in (6.7.5) for all $\xi \in \widehat{\mathcal{X}}$ implies convergence of all marginal moments and because the limiting measure is uniquely characterized by these limiting moments, then, for all $f : \mathcal{X} \to \mathbb{R}$ bounded and for all $\eta \in \mathcal{X}$, we have

$$\lim_{t\to\infty} \mathbb{E}_\eta\left[f(\eta_t)\right] = \mathbb{E}_{\mu_\star}\left[f(\eta)\right] . \tag{6.7.6}$$

By dominated convergence, (6.7.6) yields, for all probability measures μ on $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}_\mu\left[f(\eta_t)\right] = \mathbb{E}_\mu\left[\lim_{t\to\infty} \mathbb{E}_\eta\left[f(\eta_t)\right]\right] = \mathbb{E}_{\mu_\star}\left[f(\eta)\right] ,$$

i.e. $\mu_\star$ is the unique stationary measure of the process $\{\eta_t : t \geq 0\}$. □

Chapter 7

Boundary driven switching interacting particle systems: scaling limits, uphill diffusion and boundary layer

7.1 Introduction

In this chapter we consider a finite version of the switching interacting particle systems introduced in Definition 5.2.1 to which boundary reservoirs are added. From the point of view of non-equilibrium systems driven by boundary reservoirs, switching interacting particle systems have not been studied. On the one hand, such systems have both reaction and diffusion and therefore exhibit a richer non-equilibrium behaviour. On the other hand, the macroscopic equations are linear and exactly solvable in one dimension, and so these systems are simple enough to make a detailed microscopic analysis possible. As explained in Chapter 5, the system can be viewed as an interacting particle system on two layers. Therefore duality properties are available, which allows for a detailed analysis of the system coupled to reservoirs, dual to an absorbing system. In one dimension the analysis of the microscopic density profile reduces to a computation of the absorption probabilities of a simple random walk on a two-layer system absorbed at the left and right boundaries. From the analytic solution, we can identify both the density profile and the current in the system. This leads to two interesting phenomena. The first phenomenon is *uphill diffusion* (see e.g. [43, 44, 45, 51, 113]), i.e., in a well-defined parameter regime the current can go against the particle density gradient: when the total density of particles at the left end is higher than at the right end, the current can still go from right to left. The second phenomenon is *boundary-layer behaviour*: in the limit as $\epsilon \downarrow 0$, in the macroscopic stationary profile the densities in the top and bottom layer are equal, which for unequal boundary conditions in the top and bottom layer results in a *discontinuity* in the stationary profile. Corresponding to this jump in the macroscopic system, we identify a boundary layer of size $\sqrt{\epsilon}\,\log(1/\epsilon)$ in the microscopic system where the densities are unequal. The quantification of the *size* of this boundary layer is an interesting corollary of the exact macroscopic stationary profile that we obtain from the microscopic system via duality.

7.2 The system with boundary reservoirs

Section 7.2.1 defines the model. Section 7.2.2 identifies the dual and the stationary measures. Section 7.3 derives the non-equilibrium density profile, both for the microscopic system and the macroscopic system, and offers various simulations. In Section 7.4 we compute the stationary horizontal current of slow and fast particles both for the microscopic system and the macroscopic system. Section 7.5 shows that in the macroscopic system, for certain choices of the rates, there can be a flow of particles uphill, i.e., against the gradient imposed by the reservoirs. Thus, as a consequence of the competing driving mechanisms of slow and fast particles, we can have a flow of particles from the side with lower density to the side with higher density.

7.2.1 Model

We consider the same system as in Definition 5.2.1, but restricted to $V := \{1, \dots, N\} \subset \mathbb{Z}$. In addition, we set $\hat{V} := V \cup \{L, R\}$ and attach a *left-reservoir* to L and a *right-reservoir* to R, both for fast and slow particles. To be more precise, there are four reservoirs (see Fig. 7.2):

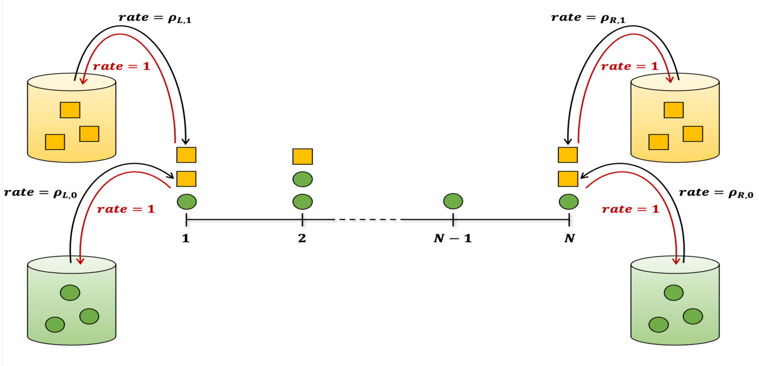


Figure 7.1: Representation via *slow* and *fast* particles moving on V.Case $\sigma = 0$, $\epsilon > 0$.

(i) For the fast particles, a left-reservoir at L injects fast particles at $x = 1$ at rate $\rho_{L,0}(1 + \sigma\eta_0(1, t))$ and a right-reservoir at R injects fast particles at $x = N$ at rate $\rho_{R,0}(1 + \sigma\eta_0(N, t))$. The left-reservoir absorbs fast particles at rate $1 + \sigma\rho_{L,0}$, while the right-reservoir does so at rate $1 + \sigma\rho_{R,0}$.

(ii) For the slow particles, a left-reservoir at L injects slow particles at $x = 1$ at rate $\rho_{L,1}(1 + \sigma\eta_1(1, t))$ and a right-reservoir at R injects slow particles at $x = N$ at rate $\rho_{R,1}(1 + \sigma\eta_1(N, t))$. The left-reservoir absorbs fast particles at rate $1 + \sigma\rho_{L,1}$, while the right-reservoir does so at rate $1 + \sigma\rho_{R,1}$.

Inside V, the particles move as before.

For $i \in I$, $x \in V$ and $t \geq 0$, let $\eta_i(x, t)$ denote the number of particles in layer i at site x at time t. For $\sigma \in \{-1, 0, 1\}$, the Markov process $\{\eta(t) \colon\ t \geq 0\}$ with

$$\eta(t) = \{\eta_0(x, t), \eta_1(x, t)\}_{x\in V}$$

has state space

$$\mathcal{X} = \begin{cases} \{0, 1\}^V \times \{0, 1\}^V, & \sigma = -1, \\ \mathbb{N}_0^V \times \mathbb{N}_0^V, & \sigma = 0, 1, \end{cases}$$

and generator

$$L := L_{\epsilon,\gamma,N} = L^{\text{bulk}} + L^{\text{res}} \tag{7.2.1}$$

with

$$L^{\text{bulk}} := L_0^{\text{bulk}} + \epsilon L_1^{\text{bulk}} + \gamma L_{0\updownarrow 1}^{\text{bulk}} \tag{7.2.2}$$

acting on bounded cylindrical functions $f\colon\ \mathcal{X} \to \mathbb{R}$ as

$$(L_0^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N-1} \Big\{\eta_0(x)(1 + \sigma\eta_0(x + 1)) \left[f(\eta_0 - \delta_x + \delta_{x+1}, \eta_1) - f(\eta_0, \eta_1)\right]$$

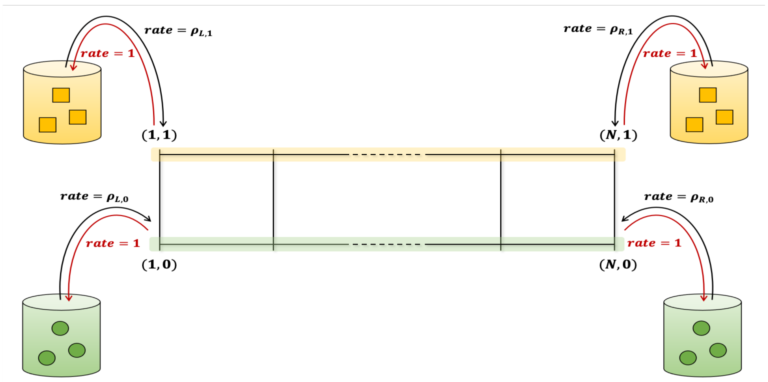


Figure 7.2: Representation via particles moving on $V \times I$. Case $\sigma = 0$, $\epsilon > 0$.

$$+ \eta_0(x+1)(1+\sigma\eta_0(x))\left[f(\eta_0 - \delta_{x+1} + \delta_x, \eta) - f(\eta_0, \eta_1)\right]\Big\},$$

$$(L_1^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N-1} \Big\{\eta_1(x)(1+\sigma\eta_1(x+1))\left[f(\eta_0, \eta_1 - \delta_x + \delta_{x+1}) - f(\eta_0, \eta_1)\right]$$
$$+ \eta_1(x+1)(1+\sigma\eta_1(x))\left[f(\eta_0, \eta_1 - \delta_{x+1} + \delta_x) - f(\eta_0, \eta_1))\right]\Big\},$$

$$(L_{0\updownarrow 1}^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N} \Big\{\eta_0(x)(1+\sigma\eta_1(x))\left[f(\eta_0 - \delta_x, \eta_1 + \delta_x) - f(\eta_0, \eta_1)\right]$$
$$+ \eta_1(x)(1+\sigma\eta_0(x))\left[f(\eta_0 + \delta_x, \eta_1 - \delta_x) - f(\eta_0, \eta_1))\right]\Big\},$$

and

$$L^{\text{res}} := L_0^{\text{res}} + L_1^{\text{res}} \tag{7.2.3}$$

acting as

$$(L_0^{\text{res}} f)(\eta) = \eta_0(1)(1+\sigma\rho_{L,0})\left[f(\eta_0 - \delta_1, \eta_1) - f(\eta_0, \eta_1)\right]$$
$$+ \rho_{L,0}(1+\sigma\eta_0(1))\left[f(\eta_0 + \delta_1, \eta_1) - f(\eta_0, \eta_1)\right]$$
$$+ \eta_0(N)(1+\sigma\rho_{R,0})\left[f(\eta_0 - \delta_N, \eta_1) - f(\eta_0, \eta_1)\right] + \rho_{R,0}(1+\sigma\eta_0(N))\left[f(\eta_0 + \delta_N, \eta) - f(\eta_0, \eta_1)\right],$$

$$(L_1^{\text{res}} f)(\eta) = \eta_1(1)(1+\sigma\rho_{L,1})\left[f(\eta_0, \eta_1 - \delta_1) - f(\eta_0, \eta_1)\right]$$
$$+ \rho_{L,1}(1+\sigma\eta_1(1))\left[f(\eta_0, \eta_1 + \delta_1) - f(\eta_0, \eta_1)\right]$$
$$+ \eta_1(N)(1+\sigma\rho_{R,1})\left[f(\eta_0, \eta_1 - \delta_N) - f(\eta_0, \eta_1)\right] + \rho_{R,1}(1+\sigma\rho_{R,N})\left[f(\eta_0, \eta_1 + \delta_N) - f(\eta_0, \eta_1)\right].$$

7.2.2 Duality

In [32] it was shown that the partial exclusion process, a system of independent random walks and the symmetric inclusion processes on a finite set V, coupled with proper left and right reservoirs, are dual to the same particle system but with the reservoirs replaced by absorbing sites. As remarked in [76], the same result holds for more general geometries, consisting of inhomogeneous rates (site and edge dependent), and for many proper reservoirs. Our model is a particular instance of the case treated in [76, Remark 2.2]), because we can think of the rate as conductances attached to the edges.

More precisely, we consider the system where particles jump on two copies of

$$\hat{V} := V \cup \{L, R\}$$

and follow the same dynamics as before in V, but with the reservoirs at L and R absorbing. We denote by ξ the configuration

$$\xi = (\xi_0, \xi_1) := (\{\xi_0(x)\}_{x\in\hat{V}}, \{\xi_1(x)\}_{x\in\hat{V}}),$$

where $\xi_i(x)$ denotes the number of particles at site x in layer i. The state space is $\hat{\mathcal{X}} = \mathbb{N}_0^{\hat{V}} \times \mathbb{N}_0^{\hat{V}}$, and the generator is

$$\hat{L} := \hat{L}_{\epsilon,\gamma,N} = \hat{L}^{\text{bulk}} + \hat{L}^{L,R} \tag{7.2.4}$$

with

$$\hat{L}^{\text{bulk}} := \hat{L}_0^{\text{bulk}} + \epsilon\hat{L}_1^{\text{bulk}} + \gamma\hat{L}_{0\updownarrow 1}^{\text{bulk}}$$

acting on cylindrical functions $f\colon\ \mathcal{X} \to \mathbb{R}$ as

$$(\hat{L}_0^{\text{bulk}} f)(\xi) = \sum_{x=1}^{N-1} \Big\{\xi_0(x)(1+\sigma\xi_0(x+1))\left[f(\xi_0 - \delta_x + \delta_{x+1}, \xi_1) - f(\xi_0, \xi_1)\right]$$
$$+ \xi_0(x+1)(1+\sigma\xi_0(x))\left[f(\xi_0 - \delta_{x+1} + \delta_x, \xi_1) - f(\xi_0, \xi_1)\right]\Big\},$$

$$(\hat{L}_1^{\text{bulk}} f)(\xi) = \sum_{x=1}^{N-1} \Big\{ \xi_1(x)(1+\sigma\xi_1(x+1)) \left[f(\xi_0, \xi_1 - \delta_x + \delta_{x+1}) - f(\xi_0,\xi_1) \right]$$
$$+ \xi_1(x+1)(1+\sigma\xi_1(x)) \left[f(\xi_0, \xi_1 - \delta_{x+1} + \delta_x) - f(\xi_0,\xi_1) \right] \Big\},$$
$$(\hat{L}_{0\updownarrow 1}^{\text{bulk}} f)(\eta) = \sum_{x=1}^{N} \Big\{ \xi_0(x)(1+\sigma\xi_1(x)) \left[f(\xi_0 - \delta_x, \xi_1 + \delta_x) - f(\xi_0,\xi_1) \right]$$
$$+ \xi_1(x)(1+\sigma\xi_0(x)) \left[f(\xi_0 + \delta_x, \xi_1 - \delta_x) - f(\xi_0,\xi_1) \right] \Big\},$$

and

$$\hat{L}^{L,R} = \hat{L}_0^{L,R} + \hat{L}_1^{L,R}$$

acting as

$$(\hat{L}_0^{L,R} f)(\xi) = \xi_0(1) \left[f(\xi_0 - \delta_1, \xi_1) - f(\xi_0,\xi_1) \right] + \xi_0(N) \left[f(\xi_0 - \delta_N, \xi_1) - f(\xi_0,\xi_1) \right],$$
$$(\hat{L}_1^{L,R} f)(\xi) = \xi_1(1) \left[f(\xi_0, \xi_1 - \delta_1) - f(\xi_0,\xi_1) \right] + \xi_1(N) \left[f(\xi_0, \xi_1 - \delta_N) - f(\xi_0,\xi_1) \right].$$

Proposition 7.2.1. [Duality] [32, Theorem 4.1] and [76, Proposition 2.3] *The Markov processes*

$$\{\eta(t)\colon\ t \ge 0\}, \qquad \eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V},$$
$$\{\xi(t)\colon\ t \ge 0\}, \qquad \xi(t) = \{\xi_0(x,t), \xi_1(x,t)\}_{x\in \hat{V}},$$

with generators L in (7.2.1) *and $\hat{L}$ in* (7.2.4) *are dual. Namely, for all configurations $\eta \in \mathcal{X}$, $\xi \in \hat{\mathcal{X}}$ and times $t \ge 0$,*

$$\mathbb{E}_\eta[D(\xi, \eta_t)] = \mathbb{E}_\xi[D(\xi_t, \eta)],$$

where the duality function is given by

$$D(\xi,\eta) := \left(\prod_{i\in I} d_{(L,i)}(\xi_i(L)) \right) \times \left(\prod_{x\in V} d(\xi_i(x), \eta_i(x)) \right) \times \left(\prod_{i\in I} d_{(R,i)}(\xi_i(R)) \right),$$

where, for $k, n \in \mathbb{N}$ and $i \in I$, $d(\cdot,\cdot)$ is given in (5.3.4) *and*

$$d_{(L,i)}(k) = (\rho_{L,i})^k\,, \qquad d_{(R,i)}(k) = (\rho_{R,i})^k\,.$$

The proof boils down to checking that the relation

$$\hat{L}D(\cdot,\eta)(\xi) = LD(\xi,\cdot)(\eta)$$

holds for any $\xi \in \mathcal{X}$ and $\xi \in \hat{\mathcal{X}}$, as follows from a rewriting of the proof of [32, Theorem 4.1].

Remark 7.2.2. [Choice of reservoir rates] (i) Note that we have chosen the reservoir rates to be 1 both for fast and slow particles. We did this because we view the reservoirs as an external mechanism that injects and absorbs neutral particles, while the particles assume their type as soon as they are in the bulk of the system. In other words, in the present context we view the change of the rate in the two layers as a change of the viscosity properties of the medium is which the particles evolve, instead of a property of the particles themselves.
(ii) If we would tune the reservoir rate of the slow particles to be ϵ, then the duality relation mentioned above would still holds, with the difference that the dual system would have ϵ as the rate of absorption for the slow particles. This change of the reservoir rates does not affect our results on the non-Fick properties of the model (see Section 7.5 below) and on the size of the boundary layer (see Section 7.6 below). Indeed, the limiting macroscopic properties we get by changing the rate of the reservoir of the slow particles are the same as the ones we derive later (i.e., the macroscopic boundary-value problem is the same for any choice of reservoir rate). Note that we do not rescale the reservoir rate when we rescale the system to pass from microscopic to macroscopic, which implies that our macroscopic equation has a Dirichlet boundary condition (see (7.3.40) below). ♠

Also in the context of boundary-driven systems, duality is an essential tool to perform explicit computations. We refer to [107] and [32], where duality for boundary-driven systems was used to compute the stationary profile, by looking at the absorption probabilities of the dual. This is the approach we will follow in the next section. We remark that, for the inclusion process and for generalizations of the exclusion process, duality is the only available tool to characterize properties of the non-equilibrium steady state (such as the stationary profile), whereas other more direct methods (such as the matrix formulation in e.g. [55]) are not applicable in this setting.

7.3 Non-equilibrium stationary profile

Also the existence and uniqueness of the non-equilibrium steady state has been established in [76, Theorem 3.3] for general geometries, and the argument in that paper can be easily adapted to our setting.

Theorem 7.3.1. [Stationary measure] [76, Theorem 3.3(a)] *For $\sigma \in \{-1,0,1\}$ there exists a unique stationary measure μ_{stat} for $\{\eta(t)\colon\ t \geq 0\}$. Moreover, for $\sigma = 0$ and for any values of $\{\rho_{L,0},\ \rho_{L,1},\ \rho_{R,0},\ \rho_{R,1}\}$,*

$$\mu_{stat} = \prod_{(x,i)\in V\times I} \nu_{(x,i)}, \qquad \nu_{(x,i)} = \text{Poisson}(\theta_{(x,i)}), \tag{7.3.1}$$

while, for $\sigma \in \{-1,1\}$, μ_{stat} is in general not in product form, unless $\rho_{L,0} = \rho_{L,1} = \rho_{R,0} = \rho_{R,1}$, for which

$$\mu_{stat} = \prod_{(x,i)\in V\times I} \nu_{(x,i),\theta}, \tag{7.3.2}$$

where $\nu_{(x,i),\theta}$ is given in (6.3.2).

Proof. For $\sigma = -1$, the existence and uniqueness of the stationary measure is trivial by the irreducibility and the finiteness of the state space of the process. For $\sigma \in \{0,1\}$, recall from [76, Appendix A] that a probability measure μ on $\mathcal{X}$ is said to be tempered if it is characterized by the integrals $\{\mathbb{E}_\mu[D(\xi,\eta)]\ :\ \xi \in \hat{\mathcal{X}}\}$ and that if there exists a $\theta \in [0,\infty)$ such that $\mathbb{E}_\mu[D(\xi,\eta)] \leq \theta^{|\xi|}$ for any $\xi \in \hat{\mathcal{X}}$. By means of duality we have that, for any $\eta \in \mathcal{X}$ and $\xi \in \hat{\mathcal{X}}$,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_\eta[D(\xi,\eta_t)] = \lim_{t\to\infty} \hat{\mathbb{E}}_\xi[D(\xi_t,\eta)] \\
&= \sum_{i_0=0}^{|\xi|} \sum_{i_{0,L}=0}^{i_0} \sum_{j_{1,L}=0}^{|\xi|-i_0} \rho_{L,0}^{i_{0,L}} \rho_{R,0}^{i_0-i_{0,L}} \rho_{L,1}^{i_{1,L}} \rho_{R,1}^{|\xi|-i_0-i_{1,L}} &(7.3.3)\\
&\qquad \times \hat{\mathbb{P}}_\xi\left(\xi_\infty = i_{0,L}\delta_{(L,0)} + (i_0 - i_{0,L})\delta_{(R,0)} + i_{1,L}\delta_{(L,1)} + (|\xi| - i_0 - i_{1,L})\delta_{(R,1)}\right), &(7.3.4)
\end{aligned}$$

from which we conclude that $\lim_{t\to\infty} \mathbb{E}_\eta[D(\xi,\eta_t)] \leq \max\{\rho_{L,0},\rho_{R,0},\rho_{L,1},\rho_{R,1}\}^{|\xi|}$. Let μ_s be the unique tempered probability measure such that for any $\xi \in \hat{\mathcal{X}}$, $\mathbb{E}_{\mu_{stat}}[D(\xi,\eta)]$ coincides with (7.3.3). From the convergence of the marginal moments in (7.3.3) we conclude that, for any $f : \mathcal{X} \to \mathbb{R}$ bounded and for any $\eta \in \mathcal{X}$,

$$\lim_{t\to\infty} \mathbb{E}_\eta[f(\eta_t)] = \mathbb{E}_{\mu_{stat}}[f(\eta)].$$

Thus, a dominated convergence argument yields that for any probability measure μ on $\mathcal{X}$,

$$\lim_{t\to\infty} \mathbb{E}_\mu[f(\eta_t)] = \mathbb{E}_{\mu_{stat}}[f(\eta)],$$

giving that μ_{stat} is the unique stationary measure. The explicit expression in (7.3.1) and (7.3.2) follows from similar computations as in [32], while, arguing by contradiction as in the proof of [76, Theorem 3.3], we can show that the two-point truncated correlations are non-zero for $\sigma \in \{-1,1\}$ whenever at least two reservoir parameters are different. □

Stationary microscopic profile and absorption probability

In this section we provide an explicit expression for the stationary microscopic density of each type of particle. To this end, let μ_{stat} be the unique non-equilibrium stationary measure of the process

$$\{\eta(t)\colon\ t \geq 0\}, \qquad \eta(t) := \{\eta_0(x,t),\eta_1(x,t)\}_{x\in V},$$

and let $\{\theta_0(x),\theta_1(x)\}_{x\in V}$ be the stationary microscopic profile, i.e., for $x \in V$ and $i \in I$,

$$\theta_i(x) = \mathbb{E}_{\mu_{stat}}[\eta_i(x,t)]. \tag{7.3.5}$$

Write $\mathbb{P}_\xi$ (and $\mathbb{E}_\xi$) to denote the law (and the expectation) of the dual Markov process

$$\{\xi(t)\colon\ t \geq 0\}, \qquad \xi(t) := \{\xi_0(x,t),\xi_1(x,t)\}_{x\in\hat{V}},$$

starting from $\xi = \{\xi_0(x),\xi_1(x)\}_{x\in\hat{V}}$. For $x \in V$, set

$$\begin{aligned}
\vec{p}_x &:= [\ \hat{p}(\delta_{(x,0)},\delta_{(L,0)}) \quad \hat{p}(\delta_{(x,0)},\delta_{(L,1)}) \quad \hat{p}(\delta_{(x,0)},\delta_{(R,0)}) \quad \hat{p}(\delta_{(x,0)},\delta_{(R,1)})\ ]^T,\\
\vec{q}_x &:= [\ \hat{p}(\delta_{(x,1)},\delta_{(L,0)}) \quad \hat{p}(\delta_{(x,1)},\delta_{(L,1)}) \quad \hat{p}(\delta_{(x,1)},\delta_{(R,0)}) \quad \hat{p}(\delta_{(x,1)},\delta_{(R,1)})\ ]^T,
\end{aligned} \tag{7.3.6}$$

where

$$\hat{p}(\xi,\tilde{\xi}) = \lim_{t\to\infty}\mathbb{P}_\xi(\xi(t)=\tilde{\xi}), \quad \xi=\delta_{(x,i)} \text{ for some } (x,i)\in V\times I,\ \tilde{\xi}\in\{\delta_{(L,0)},\delta_{(L,1)},\delta_{(R,0)},\delta_{(R,1)}\}, \tag{7.3.7}$$

and let

$$\vec{\rho} := [\ \rho_{(L,0)} \quad \rho_{(L,1)} \quad \rho_{(R,0)} \quad \rho_{(R,1)}\ ]^T. \tag{7.3.8}$$

Note that $\hat{p}(\delta_{(x,i)},\cdot)$ is the probability of the dual process, starting from a single particle at site x at layer $i\in I$, of being absorbed at one of the four reservoirs. Using Proposition 7.2.1 and Theorem 7.3.1, we obtain the following.

Corollary 7.3.2. [Dual representation of stationary profile] *For $x\in V$, the microscopic stationary profile is given by*

$$\begin{aligned}\theta_0(x) &= \vec{p}_x\cdot\vec{\rho},\\ \theta_1(x) &= \vec{q}_x\cdot\vec{\rho},\end{aligned} \qquad x\in\{1,\dots,N\}, \tag{7.3.9}$$

where $\vec{p}_x,\vec{q}_x$ and $\vec{\rho}$ are as in (7.3.6)–(7.3.8).

We next compute the absorption probabilities associated to the dual process in order to obtain a more explicit expression for the stationary microscopic profile $\{\theta_0(x),\theta_1(x)\}_{x\in V}$. The absorption probabilities $\hat{p}(\cdot,\cdot)$ of the dual process satisfy

$$(\hat{L}\hat{p})(\cdot,\tilde{\xi})(\xi)=0 \qquad \forall\,\xi\in\hat{\mathcal{X}},$$

where $\hat{L}$ is the dual generator defined in (7.2.4), i.e., they are harmonic functions for the generator $\hat{L}$.

In matrix form, the above translates into the following systems of equations:

$$\begin{aligned}
\vec{p}_1 &= \frac{1}{2+\gamma}(\vec{p}_0+\vec{p}_2)+\frac{\gamma}{2+\gamma}\vec{q}_1,\\
\vec{q}_1 &= \frac{\epsilon}{(1+\epsilon)+\gamma}\vec{q}_2+\frac{1}{(1+\epsilon)+\gamma}\vec{q}_0+\frac{\gamma}{(1+\epsilon)+\gamma}\vec{p}_1,\\
\vec{p}_x &= \frac{1}{2+\gamma}(\vec{p}_{x-1}+\vec{p}_{x+1})+\frac{\gamma}{2+\gamma}\vec{q}_x, && x\in\{2,\dots,N-1\},\\
\vec{q}_x &= \frac{\epsilon}{2\epsilon+\gamma}(\vec{q}_{x-1}+\vec{q}_{x+1})+\frac{\gamma}{2\epsilon+\gamma}\vec{p}_x, && x\in\{2,\dots,N-1\},\\
\vec{p}_N &= \frac{1}{2+\gamma}(\vec{p}_{N-1}+\vec{p}_{N+1})+\frac{\gamma}{2+\gamma}\vec{q}_N,\\
\vec{q}_N &= \frac{\epsilon}{(1+\epsilon)+\gamma}\vec{q}_{N-1}+\frac{1}{(1+\epsilon)+\gamma}\vec{q}_{N+1}+\frac{\gamma}{(1+\epsilon)+\gamma}\vec{p}_N,
\end{aligned} \tag{7.3.10}$$

where

$$\begin{aligned}\vec{p}_0 &:= [\ 1\quad 0\quad 0\quad 0\ ]^T, & \vec{q}_0 &:= [\ 0\quad 1\quad 0\quad 0\ ]^T,\\ \vec{p}_{N+1} &:= [\ 0\quad 0\quad 1\quad 0\ ]^T, & \vec{q}_{N+1} &:= [\ 0\quad 0\quad 0\quad 1\ ]^T.\end{aligned}$$

We divide the analysis of the absorption probabilities into two cases: $\epsilon=0$ and $\epsilon>0$.

Case $\epsilon=0$.

Proposition 7.3.3. [Absorption probability for $\epsilon=0$] *Consider the dual process*

$$\{\xi(t)\colon\ t\geq 0\}, \qquad \xi(t)=\{\xi_0(x,t),\xi_1(x,t)\}_{x\in V},$$

with generator $\hat{L}_{\epsilon,\gamma,N}$ (see (7.2.4)*) with $\epsilon=0$. Then for the dual process, starting from a single particle, the absorption probabilities $\hat{p}(\cdot,\cdot)$ (see* (7.3.7)*) are given by*

$$\begin{aligned}
\hat{p}(\delta_{(x,0)},\delta_{(L,0)}) &= \frac{1+\gamma}{1+2\gamma}\left(\frac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma}-\frac{1+2\gamma}{1+N+2N\gamma}x\right),\\
\hat{p}(\delta_{(x,0)},\delta_{(L,1)}) &= \frac{\gamma}{1+2\gamma}\left(\frac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma}-\frac{1+2\gamma}{1+N+2N\gamma}x\right),\\
\hat{p}(\delta_{(x,0)},\delta_{(R,0)}) &= \frac{1+\gamma}{1+2\gamma}\left(\frac{-\gamma}{1+N+2N\gamma}+\frac{1+2\gamma}{1+N+2N\gamma}x\right),\\
\hat{p}(\delta_{(x,0)},\delta_{(R,1)}) &= \frac{\gamma}{1+2\gamma}\left(\frac{-\gamma}{1+N+2N\gamma}+\frac{1+2\gamma}{1+N+2N\gamma}x\right),
\end{aligned} \tag{7.3.11}$$

$$
\begin{aligned}
\hat{p}(\delta_{(1,1)},\delta_{(L,0)}) &= \frac{\gamma(N-\gamma+2N\gamma)}{(1+2\gamma)(1+N+2N\gamma)}, \quad \hat{p}(\delta_{(1,1)},\delta_{(L,1)}) = \frac{1+N+(1+3N)\gamma-(1-2N)\gamma^2}{(1+2\gamma)(1+N+2N\gamma)},\\
\hat{p}(\delta_{(1,1)},\delta_{(R,0)}) &= \frac{\gamma(1+\gamma)}{(1+2\gamma)(1+N+2N\gamma)}, \quad \hat{p}(\delta_{(1,1)},\delta_{(R,1)}) = \frac{\gamma^2}{(1+2\gamma)(1+N+2N\gamma)},
\end{aligned}
\tag{7.3.12}
$$

and

$$
\hat{p}(\delta_{(x,1)},\delta_{(\beta,i)}) = \hat{p}(\delta_{(x,0)},\delta_{(\beta,i)}), \qquad x\in\{2,\ldots,N-1\},\ (\beta,i)\in\{L,R\}\times I, \tag{7.3.13}
$$

and

$$
\begin{aligned}
\hat{p}(\delta_{(N,1)},\delta_{(L,0)}) &= \hat{p}(\delta_{(1,1)},\delta_{(R,0)}), \quad \hat{p}(\delta_{(N,1)},\delta_{(L,1)}) = \hat{p}(\delta_{(1,1)},\delta_{(R,1)}),\\
\hat{p}(\delta_{(N,1)},\delta_{(R,0)}) &= \hat{p}(\delta_{(1,1)},\delta_{(L,0)}), \quad \hat{p}(\delta_{(N,1)},\delta_{(R,1)}) = \hat{p}(\delta_{(1,1)},\delta_{(L,1)}).
\end{aligned}
\tag{7.3.14}
$$

Proof. Note that, for $\epsilon = 0$, from the linear system in (7.3.10) we get

$$
\begin{aligned}
&\vec{p}_{x+1}-\vec{p}_x = \vec{p}_x - \vec{p}_{x-1},\\
&\vec{q}_x = \vec{p}_x,
\end{aligned}
\qquad x\in\{2,\ldots,N-1\}. \tag{7.3.15}
$$

Thus, if we set $\vec{c} = \vec{p}_2 - \vec{p}_1$, then it suffices to solve the following 4 linear equations with 4 unknowns $\vec{p}_1, \vec{c}, \vec{q}_1, \vec{q}_N$:

$$
\begin{aligned}
\vec{p}_1 &= \frac{1}{2+\gamma}(\vec{p}_0+\vec{p}_1+\vec{c}) + \frac{\gamma}{2+\gamma}\vec{q}_1,\\
\vec{q}_1 &= \frac{1}{1+\gamma}\vec{q}_0 + \frac{\gamma}{1+\gamma}\vec{p}_1,\\
\vec{p}_1+(N-1)\vec{c} &= \frac{1}{2+\gamma}(\vec{p}_1+(N-2)\vec{c}+\vec{p}_{N+1}) + \frac{\gamma}{2+\gamma}\vec{q}_N,\\
\vec{q}_N &= \frac{1}{1+\gamma}\vec{q}_{N+1} + \frac{\gamma}{1+\gamma}(\vec{p}_1+(N-1)\vec{c}).
\end{aligned}
\tag{7.3.16}
$$

Solving the above equations we get the desired result. □

As a consequence, we obtain the stationary microscopic profile for the original process $\{\eta(t)\colon\ t\geq 0\}$, $\eta(t) = \{\eta_0(x,t),\eta_1(x,t)\}_{x\in V}$ when $\epsilon = 0$.

Theorem 7.3.4. [Stationary microscopic profile for $\epsilon = 0$]
The stationary microscopic profile $\{\theta_0(x),\theta_1(x)\}_{x\in V}$ (see (7.3.5)) for the process $\{\eta(t)\colon\ t\geq 0\}$ with $\eta(t) = \{\eta_0(x,t),\eta_1(x,t)\}_{x\in V}$ with generator $L_{\epsilon,\gamma,N}$ (see (7.2.1)) and $\epsilon = 0$ is given by

$$
\begin{aligned}
\theta_0(x) = \frac{1+\gamma}{1+2\gamma}&\Big[\Big(\tfrac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma} - \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{L,0} + \Big(\tfrac{-\gamma}{1+N+2N\gamma} + \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{R,0}\Big]\\
&+ \frac{\gamma}{1+2\gamma}\Big[\Big(\tfrac{(1+N)+(1+2N)\gamma}{1+N+2N\gamma} - \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{(L,1)} + \Big(\tfrac{-\gamma}{1+N+2N\gamma} + \tfrac{1+2\gamma}{1+N+2N\gamma}\,x\Big)\rho_{(R,1)}\Big]
\end{aligned}
\tag{7.3.17}
$$

and

$$
\begin{aligned}
\theta_1(1) &= \frac{\gamma}{1+\gamma}\theta_0(1) + \frac{1}{1+\gamma}\rho_{(L,1)},\\
\theta_1(x) &= \theta_0(x), \qquad\qquad x\in\{2,\ldots,N-1\},\\
\theta_1(N) &= \frac{\gamma}{1+\gamma}\theta_0(N) + \frac{1}{1+\gamma}\rho_{(R,1)}.
\end{aligned}
\tag{7.3.18}
$$

Proof. The proof directly follows from Corollary 7.3.2 and Proposition 7.3.3. □

Case $\epsilon > 0$. We next compute the absorption probabilities for the dual process and the stationary microscopic profile for the original process when $\epsilon > 0$.

Proposition 7.3.5. [Absorption probability for $\epsilon > 0$] *Consider the dual process*

$$
\{\xi(t)\colon\ t\geq 0\}, \qquad \xi(t) = \{\xi_0(x,t),\xi_1(x,t)\}_{x\in V},
$$

with generator $\hat{L}_{\epsilon,\gamma}$ (see (7.2.4)) with $\epsilon > 0$. Let $\hat{p}(\cdot,\cdot)$ (see (7.3.7)) be the absorption probabilities of the dual process starting from a single particle, and let $(\vec{p}_x, \vec{q}_x)_{x\in V}$ be as defined in (7.3.6). Then

$$\begin{aligned} \vec{p}_x &= \vec{c}_1\, x + \vec{c}_2 + \epsilon(\vec{c}_3\, \alpha_1^x + \vec{c}_4\, \alpha_2^x), \\ \vec{q}_x &= \vec{c}_1\, x + \vec{c}_2 - (\vec{c}_3\, \alpha_1^x + \vec{c}_4\, \alpha_2^x), \end{aligned} \qquad x \in V, \tag{7.3.19}$$

where α_1, α_2 are the two roots of the equation

$$\epsilon\alpha^2 - (\gamma(1+\epsilon) + 2\epsilon)\,\alpha + \epsilon = 0, \tag{7.3.20}$$

and $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ are vectors that depend on the parameters $N, \epsilon, \alpha_1, \alpha_2$ (see (7.6.18) for explicit expressions).

Proof. Applying the transformation

$$\vec{r}_x := \vec{p}_x + \epsilon\vec{q}_x, \qquad \vec{s}_x := \vec{p}_x - \vec{q}_x, \tag{7.3.21}$$

we see that the system in (7.3.10) decouples in the bulk (i.e., the interior of V), and

$$\vec{r}_x = \frac{1}{2}(\vec{r}_{x+1} + \vec{r}_{x-1}), \qquad \vec{s}_x = \frac{\epsilon}{\gamma(1+\epsilon) + 2\epsilon}(\vec{s}_{x+1} + \vec{s}_{x-1}), \qquad x \in \{2,\ldots,N-1\}. \tag{7.3.22}$$

The solution of the above system of recursion equations takes the form

$$\vec{r}_x = \vec{A}_1 x + \vec{A}_2, \quad \vec{s}_x = \vec{A}_3\alpha_1^x + \vec{A}_4\alpha_2^x, \tag{7.3.23}$$

where α_1, α_2 are the two roots of the equation

$$\epsilon\alpha^2 - (\gamma(1+\epsilon) + 2\epsilon)\,\alpha + \epsilon = 0. \tag{7.3.24}$$

Rewriting the four boundary conditions in (7.3.10) in terms of the new transformations, we get

$$[\ \vec{A}_1 \quad \vec{A}_2 \quad \vec{A}_3 \quad \vec{A}_4\ ] = (1+\epsilon)(M_\epsilon^{-1})^T, \tag{7.3.25}$$

where M_ϵ is given by

$$M_\epsilon := \begin{bmatrix} 0 & 1 & \epsilon & \epsilon \\ 1-\epsilon & 1 & (\epsilon-1)\alpha_1 - \epsilon & (\epsilon-1)\alpha_2 - \epsilon \\ N+1 & 1 & \epsilon\alpha_1^{N+1} & \epsilon\alpha_2^{N+1} \\ N+\epsilon & 1 & -\alpha_1^N(\epsilon\alpha_1 + (1-\epsilon)) & -\alpha_2^N(\epsilon\alpha_2 + (1-\epsilon)) \end{bmatrix}. \tag{7.3.26}$$

Since $\vec{p}_x = \frac{1}{1+\epsilon}(\vec{r}_x + \epsilon\vec{s}_x)$ and $\vec{q}_x = \frac{1}{1+\epsilon}(\vec{r}_x - \vec{s}_x)$, by setting

$$\vec{c}_i = \frac{1}{1+\epsilon}\vec{A}_i, \quad i \in \{1,2,3,4\},$$

we get the desired identities. □

Without loss of generality, from here onwards, we fix the choices of the roots α_1 and α_2 of the quadratic equation in (7.3.20) as

$$\alpha_1 = 1 + \frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right) - \sqrt{\left[1+\frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right)\right]^2 - 1}, \quad \alpha_2 = 1 + \frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right) + \sqrt{\left[1+\frac{\gamma}{2}\left(1+\frac{1}{\epsilon}\right)\right]^2 - 1}. \tag{7.3.27}$$

Note that, for any $\epsilon, \gamma > 0$, we have

$$\alpha_1\alpha_2 = 1. \tag{7.3.28}$$

As a corollary, we get the expression for the stationary microscopic profile of the original process.

Theorem 7.3.6. [Stationary microscopic profile for $\epsilon > 0$]
The stationary microscopic profile $\{\theta_0(x), \theta_1(x)\}_{x\in V}$ (see (7.3.5)) for the process $\{\eta(t)\colon\ t \geq 0\}$ and $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$ with generator $L_{\epsilon,\gamma,N}$ (see (7.2.1)) with $\epsilon > 0$ is given by

$$\begin{aligned} \theta_0(x) &= (\vec{c}_1 \,.\, \vec{\rho})x + (\vec{c}_2 \,.\, \vec{\rho}) + \epsilon(\vec{c}_3 \,.\, \vec{\rho})\alpha_1^x + \epsilon(\vec{c}_4 \,.\, \vec{\rho})\alpha_2^x, \\ \theta_1(x) &= (\vec{c}_1 \,.\, \vec{\rho})x + (\vec{c}_2 \,.\, \vec{\rho}) - (\vec{c}_3 \,.\, \vec{\rho})\alpha_1^x - (\vec{c}_4 \,.\, \vec{\rho})\alpha_2^x, \end{aligned} \qquad x \in V, \tag{7.3.29}$$

where $(\vec{c}_i)_{1\leq i\leq 4}$ are as in (7.6.18), and

$$\vec{\rho} := [\ \rho_{(L,0)} \quad \rho_{(L,1)} \quad \rho_{(R,0)} \quad \rho_{(R,1)}\ ]^T.$$

Proof. The proof follows directly from Corollary 7.3.2 and Proposition 7.3.5. □

Remark 7.3.7. [Symmetric layers] For $\epsilon = 1$, the inverse of the matrix M_ϵ in the proof of Proposition 7.3.5 takes a simpler form. This is because for $\epsilon = 1$ the system is fully symmetric. In this case, the explicit expression of the stationary microscopic profile is given by

$$\begin{aligned}\theta_0(x) =& \frac{1}{2}\left(\frac{N+1-x}{N+1} + \frac{\alpha_2^{N+1-x}-\alpha_1^{N+1-x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{L,0} + \frac{1}{2}\left(\frac{x}{N+1} + \frac{\alpha_2^{x}-\alpha_1^{x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{R,0} \\ &+ \frac{1}{2}\left(\frac{N+1-x}{N+1} - \frac{\alpha_2^{N+1-x}-\alpha_1^{N+1-x}}{\alpha_2^{N+1}\alpha_1^{N+1}}\right)\rho_{(L,1)} + \frac{1}{2}\left(\frac{x}{N+1} - \frac{\alpha_2^{x}-\alpha_1^{x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{(R,1)}\end{aligned} \tag{7.3.30}$$

and

$$\begin{aligned}\theta_1(x) =& \frac{1}{2}\left(\frac{N+1-x}{N+1} - \frac{\alpha_2^{N+1-x}-\alpha_1^{N+1-x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{L,0} + \frac{1}{2}\left(\frac{x}{N+1} - \frac{\alpha_2^{x}-\alpha_1^{x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{R,0} \\ &+ \frac{1}{2}\left(\frac{N+1-x}{N+1} + \frac{\alpha_2^{N+1-x}-\alpha_1^{N+1-x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{(L,1)} + \frac{1}{2}\left(\frac{x}{N+1} + \frac{\alpha_2^{x}-\alpha_1^{x}}{\alpha_2^{N+1}-\alpha_1^{N+1}}\right)\rho_{(R,1)}.\end{aligned} \tag{7.3.31}$$

However, note that

$$\theta_0(x) + \theta_1(x) = 2[(\vec{c}_1.\vec{\rho})x + (\vec{c}_2.\vec{\rho})] - (1-\epsilon)[(\vec{c}_3\,.\,\vec{\rho})\alpha_1^x - (\vec{c}_4\,.\,\vec{\rho})\alpha_2^x],$$

which is linear in x only when $\epsilon = 1$, and

$$\theta_0(x) - \theta_1(x) = (1+\epsilon)[(\vec{c}_3\,.\,\vec{\rho})\alpha_1^x + (\vec{c}_4\,.\,\vec{\rho})\alpha_2^x],$$

which is purely exponential in x. ♠

Stationary macroscopic profile and boundary-value problem

In this section we rescale the finite-volume system with boundary reservoirs, in the same way as was done for the infinite-volume system in Section 5.4 when we derived the hydrodynamic limit (i.e., space is scaled by $1/N$ and the switching rate γ_N is scaled such that $\gamma_N N^2 \to \Upsilon > 0$), and study the validity of Fick's law at stationarity on macroscopic scale. Before we do that, we justify below that the current scaling of the parameters is indeed the proper choice, in the sense that we obtain non-trivial pointwise limits (macroscopic stationary profiles) of the microscopic stationary profiles found in previous sections, and that the resulting limits (when $\epsilon > 0$) satisfy the stationary boundary-value problem given in (5.4.2) with boundary conditions $\rho_0^{\text{stat}}(0) = \rho_{L,0}$, $\rho_0^{\text{stat}}(1) = \rho_{R,0}$, $\rho_1^{\text{stat}}(0) = \rho_{L,1}$ and $\rho_1^{\text{stat}}(1) = \rho_{R,1}$.

We say that *the macroscopic stationary profiles* are given by functions $\rho_i^{\text{stat}} : (0,1) \to \mathbb{R}$ for $i \in I$ if, for any $y \in (0,1)$,

$$\lim_{N\to\infty} \theta_0^{(N)}(\lceil yN \rceil) = \rho_0^{\text{stat}}(y), \quad \lim_{N\to\infty} \theta_1^{(N)}(\lceil yN \rceil) = \rho_1^{\text{stat}}(y). \tag{7.3.32}$$

Theorem 7.3.8. [Stationary macroscopic profile] *Let $(\theta_0^{(N)}(x), \theta_1^{(N)}(x))_{x\in V}$ be the stationary microscopic profile (see* (7.3.5)*) for the process $\{\eta(t)\colon\ t \geq 0\}$, $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$ with generator $L_{\epsilon,\gamma_N,N}$ (see* (7.2.1)*), where γ_N is such that $\gamma_N N^2 \to \Upsilon$ as $N \to \infty$ for some $\Upsilon > 0$. Then, for each $y \in (0,1)$, the pointwise limits (see Fig. 7.3)*

$$\rho_0^{\text{stat}}(y) := \lim_{N\to\infty} \theta_0^{(N)}(\lceil yN \rceil), \quad \rho_1^{\text{stat}}(y) := \lim_{N\to\infty} \theta_1^{(N)}(\lceil yN \rceil), \tag{7.3.33}$$

exist and are given by

$$\begin{aligned}\rho_0^{\text{stat}}(y) &= \rho_{L,0} + (\rho_{R,0} - \rho_{L,0})y, \quad && y \in (0,1), \\ \rho_1^{\text{stat}}(y) &= \rho_0^{\text{stat}}(y), && y \in (0,1),\end{aligned} \tag{7.3.34}$$

when $\epsilon = 0$, while

$$\begin{aligned}\rho_0^{\text{stat}}(y) =& \frac{\epsilon}{1+\epsilon}\left[\frac{\sinh[B_{\epsilon,\Upsilon}(1-y)]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(L,0)} - \rho_{(L,1)}) + \frac{\sinh[B_{\epsilon,\Upsilon}\,y]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(R,0)} - \rho_{(R,1)})\right] \\ &+ \frac{1}{1+\epsilon}\left[\rho_{(R,0)}\,y + \rho_{(L,0)}(1-y)\right] + \frac{\epsilon}{1+\epsilon}\left[\rho_{(R,1)}\,y + \rho_{(L,1)}(1-y)\right],\end{aligned} \tag{7.3.35}$$

$$\begin{aligned}\rho_1^{\text{stat}}(y) = {} & \frac{1}{1+\epsilon}\left[\frac{\sinh\left[B_{\epsilon,\Upsilon}(1-y)\right]}{\sinh\left[B_{\epsilon,\Upsilon}\right]}\left(\rho_{(L,1)}-\rho_{(L,0)}\right)+\frac{\sinh\left[B_{\epsilon,\Upsilon}\,y\right]}{\sinh\left[B_{\epsilon,\Upsilon}\right]}\left(\rho_{(R,1)}-\rho_{(R,0)}\right)\right] \\ & + \frac{1}{1+\epsilon}\left[\rho_{(R,0)}\,y+\rho_{(L,0)}\,(1-y)\right]+\frac{\epsilon}{1+\epsilon}\left[\rho_{(R,1)}\,y+\rho_{(L,1)}\,(1-y)\right],\end{aligned} \tag{7.3.36}$$

when $\epsilon > 0$, where $B_{\epsilon,\Upsilon} := \sqrt{\Upsilon(1+\frac{1}{\epsilon})}$. Moreover, when $\epsilon > 0$, the two limits in (7.3.33) *are uniform in* $(0,1)$.

Proof. For $\epsilon = 0$, it easily follows from (7.3.17) plus the fact that $\gamma_N N^2 \to \Upsilon > 0$ and $\frac{\lceil yN\rceil}{N} \to y$ uniformly in $(0,1)$ as $N\to\infty$, that

$$\lim_{N\to\infty}\sup_{y\in(0,1)}\left|\theta_0^{(N)}(\lceil yN\rceil)-\left[\rho_{(L,0)}+\left(\rho_{(R,0)}-\rho_{(L,0)}\right)y\right]\right|=0,$$

and since $\theta_1(x) = \theta_0(x)$ for all $x \in \{2,\dots,N-1\}$, for fixed $y\in(0,1)$, we have

$$\lim_{N\to\infty}\theta_1^{(N)}(\lceil yN\rceil)=\rho_0^{\text{stat}}(y).$$

When $\epsilon > 0$, since $\gamma_N N^2 \to \Upsilon > 0$ as $N\to\infty$, we note the following:

$$\begin{aligned}&\gamma_N \overset{N\to\infty}{\longrightarrow} 0,\\ &\lim_{N\to\infty}\alpha_1=\lim_{N\to\infty}\alpha_2=1,\\ &\lim_{N\to\infty}\alpha_1^N=\mathrm{e}^{-B_{\epsilon,\Upsilon}},\quad \lim_{N\to\infty}\alpha_2^N=\mathrm{e}^{B_{\epsilon,\Upsilon}}.\end{aligned}\tag{7.3.37}$$

Consequently, from the expressions of $(\vec{c}_i)_{1\le i\le 4}$ defined in (7.6.18), we also have

$$\begin{aligned}&\lim_{N\to\infty}N\vec{c}_1=\frac{1}{1+\epsilon}\begin{bmatrix}-1 & -\epsilon & 1 & \epsilon\end{bmatrix}^T,\ \lim_{N\to\infty}\vec{c}_2=\frac{1}{1+\epsilon}\begin{bmatrix}1 & \epsilon & 0 & 0\end{bmatrix}^T,\\ &\lim_{N\to\infty}\vec{c}_3=\frac{1}{1+\epsilon}\begin{bmatrix}\frac{\mathrm{e}^{B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & -\frac{\mathrm{e}^{B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & -\frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & \frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}}\end{bmatrix}^T,\\ &\lim_{N\to\infty}\vec{c}_4=\frac{1}{1+\epsilon}\begin{bmatrix}-\frac{\mathrm{e}^{-B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & \frac{\mathrm{e}^{-B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & \frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & -\frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}}\end{bmatrix}^T.\end{aligned}\tag{7.3.38}$$

Combining the above equations with (7.3.29), and the fact that $\frac{\lceil yN\rceil}{N}\to y$ uniformly in $(0,1)$ as $N\to\infty$, we get the desired result. □

Remark 7.3.9. [Non-uniform convergence] Note that for $\epsilon > 0$ both stationary macroscopic profiles, when extended continuously to the closed interval $[0,1]$, match the prescribed boundary conditions. This is different from what happens for $\epsilon = 0$, where the continuous extension of ρ_1^{stat} to the closed interval $[0,1]$ equals $\rho_0^{\text{stat}}(y) = \rho_{L,0} + (\rho_{R,0}-\rho_{L,0})y$, which does not necessarily match the prescribed boundary conditions unless $\rho_{(L,1)} = \rho_{(L,0)}$ and $\rho_{(R,1)} = \rho_{(R,0)}$. Moreover, as can be seen from the proof above, for $\epsilon > 0$, the convergence of θ_i to ρ_i is uniform in $[0,1]$, i.e.,

$$\lim_{N\to\infty}\sup_{y\in[0,1]}\left|\rho_0^{\text{stat}}(y)-\theta_0^{(N)}(\lceil yN\rceil)\right|=0,\quad \lim_{N\to\infty}\sup_{y\in[0,1]}\left|\rho_1^{\text{stat}}(y)-\theta_1^{(N)}(\lceil yN\rceil)\right|=0,$$

while for $\epsilon = 0$, the convergence of θ_1 to ρ_1 is not uniform in $[0,1]$ when either $\rho_{(L,0)} \neq \rho_{(L,1)}$ or $\rho_{(R,0)} \neq \rho_{(R,1)}$.

Also, if $\rho_i^{\text{stat},\epsilon}(\cdot)$ denotes the macroscopic profile defined in (7.3.35)–(7.3.36), then for $\epsilon > 0$ and $i\in\{0,1\}$, we have

$$\lim_{\epsilon\to 0}\rho_i^{\text{stat},\epsilon}(y)\to\rho_i^{\text{stat},0}(y)\tag{7.3.39}$$

for fixed $y\in(0,1)$ and $i\in\{0,1\}$, where $\rho_i^{\text{stat},0}(\cdot)$ is the corresponding macroscopic profile in (7.3.34) for $\epsilon = 0$. However, this convergence is also not uniform for $i = 1$ when $\rho_{(L,0)} \neq \rho_{(L,1)}$ or $\rho_{(R,0)} \neq \rho_{(R,1)}$. ♠

In view of the considerations in Remark 7.3.9, we next concentrate on the case $\epsilon > 0$. The following result tells us that for $\epsilon > 0$ the stationary macroscopic profiles satisfy a stationary PDE with fixed boundary conditions and also admit a stochastic representation in terms of an absorbing switching diffusion process.

Theorem 7.3.10. [Stationary boundary value problem] *Consider the boundary value problem*

$$\begin{cases}0=\Delta u_0+\Upsilon(u_1-u_0),\\ 0=\epsilon\Delta u_1+\Upsilon(u_0-u_1),\end{cases}\tag{7.3.40}$$

with boundary conditions

$$\begin{cases} u_0(0) = \rho_{L,0},\ u_0(1) = \rho_{R,0}, \\ u_1(0) = \rho_{L,1},\ u_1(1) = \rho_{R,1}, \end{cases} \tag{7.3.41}$$

where $\epsilon, \Upsilon > 0$*, and the four boundary parameters* $\rho_{(L,0)}, \rho_{(L,1)}, \rho_{(R,0)}, \rho_{(R,1)}$ *are also positive. Then the PDE admits a unique strong solution given by*

$$u_i(y) = \rho_i^{\text{stat}}(y), \qquad y \in [0,1], \tag{7.3.42}$$

where $(\rho_0^{\text{stat}}(\cdot), \rho_1^{\text{stat}}(\cdot))$ *are as defined in (7.3.33). Furthermore,* $(\rho_0^{\text{stat}}(\cdot), \rho_1^{\text{stat}}(\cdot))$ *has the stochastic representation*

$$\rho_i^{\text{stat}}(y) = \mathbb{E}_{(y,i)}[\Phi_{i_\tau}(X_\tau)], \tag{7.3.43}$$

where $\{i_t\colon\ t \geq 0\}$ *is the pure jump process on state space* $I = \{0,1\}$ *that switches at rate* Υ*, the functions* $\Phi_0, \Phi_1\colon\ I \to \mathbb{R}_+$ *are defined as*

$$\Phi_0 = \rho_{(L,0)}\,\mathbf{1}_{\{0\}} + \rho_{(R,0)}\,\mathbf{1}_{\{1\}}, \qquad \Phi_1 = \rho_{(L,1)}\,\mathbf{1}_{\{0\}} + \rho_{(R,1)}\,\mathbf{1}_{\{1\}},$$

$\{X_t\colon\ t \geq 0\}$ *is the stochastic process* $[0,1]$ *that satisfies the SDE*

$$\mathrm{d}X_t = \psi(i_t)\,\mathrm{d}W_t$$

with $W_t = B_{2t}$ *and* $\{B_t\colon\ t \geq 0\}$ *standard Brownian motion, the switching diffusion process* $\{(X_t, i_t)\colon\ t \geq 0\}$ *is killed at the stopping time*

$$\tau := \inf\{t \geq 0 :\ X_t \in I\},$$

and $\psi\colon\ I \to \{1, \epsilon\}$ *is given by* $\psi := \mathbf{1}_{\{0\}} + \epsilon\,\mathbf{1}_{\{1\}}$.

Proof. It is straightforward to verify that for $\epsilon > 0$ the macroscopic profiles ρ_0, ρ_1 defined in (7.3.35)–(7.3.36) are indeed uniformly continuous in $(0,1)$ and thus can be uniquely extended continuously to $[0,1]$, namely, by defining $\rho_i^{\text{stat}}(0) = \rho_{(L,i)}$, $\rho_i^{\text{stat}}(1) = \rho_{(R,i)}$ for $i \in I$. Also $\rho_i^{\text{stat}} \in C^\infty([0,1])$ for $i \in I$ and satisfy the stationary PDE (7.3.40), with the boundary conditions specified in (7.3.41).

The stochastic representation of a solution of the system in (7.3.40) follows from [81, p385, Eq.(4.7)]. For the sake of completeness, we give the proof of uniqueness of the solution of (7.3.40). Let $u = (u_0, u_1)$ and $v = (v_0, v_1)$ be two solutions of the stationary reaction diffusion equation with the specified boundary conditions in (7.3.41). Then $(w_0, w_1) := (u_0 - v_0, u_1 - v_1)$ satisfies

$$\begin{cases} 0 = \Delta w_0 + \Upsilon(w_1 - w_0), \\ 0 = \epsilon\Delta w_1 + \Upsilon(w_0 - w_1), \end{cases} \tag{7.3.44}$$

with boundary conditions

$$w_0(0) = w_0(1) = w_1(0) = w_1(1) = 0. \tag{7.3.45}$$

Multiplying the two equations in (7.3.44) with w_0 and w_1, respectively, and using the identity

$$w_i \Delta w_i = \nabla \cdot (w_i \nabla w_i) - |\nabla w_i|^2, \quad i \in I,$$

we get

$$\begin{cases} 0 = \nabla \cdot (w_0 \nabla w_0) - |\nabla w_0|^2 + \Upsilon(w_1 - w_0)w_0, \\ 0 = \epsilon\nabla \cdot (w_1 \nabla w_1) - \epsilon|\nabla w_1|^2 + \Upsilon(w_0 - w_1)w_1. \end{cases} \tag{7.3.46}$$

Integrating both equations by parts over $[0,1]$, we get

$$\begin{aligned} 0 &= -[w_0(1)\nabla w_0(1) - w_0(0)\nabla w_0(0)] - \int_0^1 dy\, |\nabla w_0(y)|^2 + \Upsilon \int_0^1 dy\, (w_1(y) - w_0(y))w_0(y), \\ 0 &= -\epsilon[w_1(1)\nabla w_1(1) - w_1(0)\nabla w_1(0)] - \epsilon \int_0^1 dy\, |\nabla w_1(y)|^2 + \Upsilon \int_0^1 dy\, (w_0(y) - w_1(y))w_1(y). \end{aligned} \tag{7.3.47}$$

Adding the above two equations and using the zero boundary conditions in (7.3.45), we have

$$\int_0^1 dy\, |\nabla w_0(y)|^2 + \epsilon \int_0^1 dy\, |\nabla w_1(y)|^2 + \Upsilon \int_0^1 dy\, [w_1(y) - w_0(y)]^2 = 0. \tag{7.3.48}$$

Since both w_0 and w_1 are continuous and $\epsilon > 0, \Upsilon > 0$, it follows that

$$w_0 = w_1, \quad \nabla w_0 = \nabla w_1 = 0, \tag{7.3.49}$$

and so $w_0 = w_1 \equiv 0$. □

Note that, as a result of Theorem 7.3.10, the four absorption probabilities of the switching diffusion process $\{(X_t, i_t) \,:\, t \geq 0\}$ starting from $(y, i) \in [0,1] \times I$ are indeed the respective coefficients of $\rho_{(L,0)}, \rho_{(L,1)}, \rho_{(R,0)}, \rho_{(R,1)}$ appearing in the expression of $\rho_i^{\text{stat}}(y)$. Furthermore note that, as a consequence of Theorem 7.3.10 and the results in [95, Section 3], the time-dependent boundary-value problem

$$\begin{cases} \partial_t \rho_0 = \Delta \rho_0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t \rho_1 = \epsilon \Delta \rho_1 + \Upsilon(\rho_0 - \rho_1), \end{cases} \tag{7.3.50}$$

with initial conditions

$$\begin{cases} \rho_0(x,0) = \bar{\rho}_0(x), \\ \rho_1(x,0) = \bar{\rho}_1(x), \end{cases} \tag{7.3.51}$$

and boundary conditions

$$\begin{cases} \rho_0(0,t) = \rho_{L,0},\ \rho_0(1,t) = \rho_{R,0}, \\ \rho_1(0,t) = \rho_{L,1},\ \rho_1(1,t) = \rho_{R,1}, \end{cases} \tag{7.3.52}$$

admits a unique solution given by

$$\begin{cases} \rho_0(x,t) = \rho_0^{\text{hom}}(x,t) + \rho_0^{\text{stat}}(x), \\ \rho_1(x,t) = \rho_1^{\text{hom}}(x,t) + \rho_1^{\text{stat}}(x), \end{cases} \tag{7.3.53}$$

where

$$\rho_0^{\text{hom}}(x,t) = \mathrm{e}^{-\Upsilon t} h_0(x,t) + \frac{\Upsilon}{1-\epsilon} \mathrm{e}^{-\Upsilon t} \int_{\epsilon t}^{t} \mathrm{d}s \left(\left(\frac{s - \epsilon t}{t - s} \right)^{1/2} I_1(\upsilon(s)) h_0(x,s) + I_0(\upsilon(s)) h_1(x,s) \right), \tag{7.3.54}$$

$$\rho_1^{\text{hom}}(x,t) = \mathrm{e}^{-\Upsilon t} h_1(x,\epsilon t) + \frac{\Upsilon}{1-\epsilon} \mathrm{e}^{-\Upsilon t} \int_{\epsilon t}^{t} \mathrm{d}s \left(\left(\frac{s - \epsilon t}{t - s} \right)^{-1/2} I_1(\upsilon(s)) h_1(x,s) + I_0(\upsilon(s)) h_0(x,s) \right), \tag{7.3.55}$$

$\upsilon(s) = \frac{2\Upsilon}{1-\epsilon}((t-s)(s-\epsilon t))^{1/2}$, $I_0(\cdot)$ and $I_1(\cdot)$ are the modified Bessel functions, $h_0(x,t)$, $h_1(x,t)$ are the solutions of

$$\begin{cases} \partial_t h_0 = \Delta h_0, \\ \partial_t h_1 = \Delta h_1, \\ h_0(x,0) = \bar{\rho}_0(x) - \rho_0^{\text{stat}}(x), \\ h_1(x,1) = \bar{\rho}_1(x) - \rho_1^{\text{stat}}(x), \\ h_0(0,t) = h_0(1,t) = h_1(0,t) = h_1(1,t) = 0, \end{cases} \tag{7.3.56}$$

and $\rho_0^{\text{stat}}(x)$, $\rho_1^{\text{stat}}(x)$ are given in (7.3.36).

We conclude this section by proving that the solution of the time-dependent boundary-value problem in (7.3.50) converges to the stationary profile in (7.3.36).

Proposition 7.3.11. [Convergence to stationary profile] *Let $\rho_0^{\text{hom}}(x,t)$ and $\rho_1^{\text{hom}}(x,t)$ be as in* (7.3.54) *and* (7.3.55)*, respectively, i.e., the solutions of the boundary-value problem* (7.3.50) *with zero boundary conditions and initial conditions given by $\rho_0^{\text{hom}}(x,0) = \bar{\rho}_0(x) - \rho_0^{\text{stat}}(x)$ and $\rho_1^{\text{hom}}(x,0) = \bar{\rho}_1(x) - \rho_1^{\text{stat}}(x)$. Then, for any $k \in \mathbb{N}$,*

$$\lim_{t\to\infty} \left[\|\rho_0^{\text{hom}}(x,t)\|_{C^k(0,1)} + \|\rho_1^{\text{hom}}(x,t)\|_{C^k(0,1)} \right] = 0.$$

Proof. We start by showing that

$$\lim_{t\to\infty} \left[\|\rho_0^{\text{hom}}(x,t)\|_{L^2(0,1)} + \|\rho_1^{\text{hom}}(x,t)\|_{L^2(0,1)} \right] = 0. \tag{7.3.57}$$

Multiply the first equation of (7.3.50) by ρ_0 and the second equation by ρ_1. Integration by parts yields

$$\begin{cases} \partial_t \left(\int_0^1 \mathrm{d}x\, \rho_0^2 \right) = -\int_0^1 \mathrm{d}x\, |\partial_x \rho_0|^2 + \Upsilon \int_0^1 \mathrm{d}x\, (\rho_1 \rho_0 - \rho_0^2), \\ \partial_t \left(\int_0^1 \mathrm{d}x\, \rho_1^2(x,t) \right) = -\epsilon \int_0^1 \mathrm{d}x\, |\partial_x \rho_1|^2 + \Upsilon \int_0^1 \mathrm{d}x\, (\rho_0 \rho_1 - \rho_1^2). \end{cases} \tag{7.3.58}$$

Summing the two equations and defining $E(t) := \int_0^1 \mathrm{d}x\, (\rho_0(x,t)^2 + \rho_1(x,t)^2)$, we obtain

$$\partial_t E(t) = -\left(\int_0^1 \mathrm{d}x\, |\partial_x \rho_0|^2 + \epsilon \int_0^1 \mathrm{d}x\, |\partial_x \rho_1|^2 \right) - \Upsilon \int_0^1 \mathrm{d}x\, (\rho_0 - \rho_1)^2. \tag{7.3.59}$$

By the Poincaré inequality (i.e., $\int_0^1 \mathrm{d}x\, |\partial_x \rho_i(x,t)|^2 \geq C_p \int_0^1 \mathrm{d}x\, |\rho_i(x,t)|^2$, with $C_p > 0$) we have $\partial_t E(t) \leq -\epsilon C_p E(t)$, from which we obtain

$$E(t) \leq \mathrm{e}^{-C_p t} E(0),$$

and hence (7.3.57).

From [142, Theorem 2.1] it follows that

$$A := \begin{bmatrix} \Delta - \Upsilon & \Upsilon \\ \Upsilon & \epsilon\Delta - \Upsilon \end{bmatrix},$$

with domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$, generates a semigroup $\{\mathcal{S}_t\colon\ t \geq 0\}$. If we set $\vec{\rho}(t) = \mathcal{S}_t(\vec{\rho}^{\,\mathrm{hom}})$, with $\vec{\rho}^{\,\mathrm{hom}} = \vec{\rho} - \vec{\rho}^{\,\mathrm{stat}}$, then by the semigroup property we have

$$\vec{\rho}(t) = \mathcal{S}_{t-1}(\mathcal{S}_{1/k})^k(\vec{\rho}^{\,\mathrm{hom}}), \qquad t \geq 1,$$

and hence $A^k\vec{\rho}(t) = \mathcal{S}_{t-1}(A\mathcal{S}_{1/k})^k(\vec{\rho}^{\,\mathrm{hom}})$. If we set $\vec{p} := (A\mathcal{S}_{1/k})^k(\vec{\rho}^{\,\mathrm{hom}})$, then we obtain, by [142, Theorem 5.2(d)],

$$\|A^k\vec{\rho}(t)\|_{L^2(0,1)} \leq \|\mathcal{S}_{t-1}\vec{p}\|_{L^2(0,1)},$$

where $\lim_{t\to\infty} \|\mathcal{S}_{t-1}\vec{p}\|_{L^2(0,1)} = 0$ by the first part of the proof. The compact embedding

$$D(A^k) \hookrightarrow H^{2k}(0,1) \hookrightarrow C^k(0,1), \qquad k \in \mathbb{N},$$

concludes the proof. □

7.4 The stationary current

In this section we compute the expected current in the non-equilibrium steady state that is induced by different densities at the boundaries. We consider the microscopic and macroscopic systems, respectively.

Microscopic system. We start by defining the notion of current. The microscopic currents are associated with the edges of the underlying two-layer graph. In our setting, we denote by $\mathcal{J}^0_{x,x+1}(t)$ and $\mathcal{J}^1_{x,x+1}(t)$ the instantaneous current through the horizontal edge $(x, x+1)$, $x \in V$, of the bottom layer, respectively, top layer at time t. Obviously,

$$\mathcal{J}^0_{x,x+1}(t) = \eta_0(x,t) - \eta_0(x+1,t), \qquad \mathcal{J}^1_{x,x+1}(t) = \epsilon[\eta_1(x,t) - \eta_1(x+1,t)].$$

We are interested in the stationary currents $J^0_{x,x+1}$, respectively, $J^1_{x,x+1}$, which are obtained as

$$J^0_{x,x+1} = \mathbb{E}_{stat}[\eta_0(x) - \eta_0(x+1)], \qquad J^1_{x,x+1} = \epsilon\mathbb{E}_{stat}[\eta_1(x) - \eta_1(x+1)], \tag{7.4.1}$$

where $\mathbb{E}_{stat}$ denotes expectation w.r.t. the unique invariant probability measure of the microscopic system $\{\eta(t)\colon\ t \geq 0\}$ with $\eta(t) = \{\eta_0(x,t), \eta_1(x,t)\}_{x\in V}$. In other words, $J^0_{x,x+1}$ and $J^1_{x,x+1}$ give the average flux of particles of type 0 and type 1 across the bond $(x, x+1)$ due to diffusion.

Of course, the average number of particle at each site varies in time also as a consequence of the reaction term:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[\eta_0(x,t)] &= \mathbb{E}[\mathcal{J}^0_{x-1,x}(t) - \mathcal{J}^0_{x,x+1}(t)] + \gamma(\mathbb{E}[\eta_1(x,t)] - \mathbb{E}[\eta_0(x,t)]), \\ \frac{d}{dt}\mathbb{E}[\eta_1(x,t)] &= \mathbb{E}[\mathcal{J}^1_{x-1,x}(t) - \mathcal{J}^1_{x,x+1}(t)] + \gamma(\mathbb{E}[\eta_0(x,t)] - \mathbb{E}[\eta_1(x,t)]). \end{aligned}$$

Summing these equations, we see that there is no contribution of the reaction part to the variation of the average number of particles at site x:

$$\frac{d}{dt}\mathbb{E}[\eta_0(x,t) + \eta_1(x,t)] = \mathbb{E}[\mathcal{J}_{x-1,x}(t) - \mathcal{J}_{x,x+1}(t)].$$

The sum

$$J_{x,x+1} = J^0_{x,x+1} + J^1_{x,x+1}, \tag{7.4.2}$$

with $J^0_{x,x+1}$ and $J^1_{x,x+1}$ defined in (7.4.1), will be called the *stationary current* between sites at $x, x+1$, $x \in V$, which is responsible for the variation of the total average number of particles at each site, regardless of their type.

Proposition 7.4.1. [Stationary microscopic current] *For $x \in \{2, \dots, N-1\}$ the stationary currents defined in* (7.4.1) *are given by*

$$J^0_{x,x+1} = -\tfrac{1+\gamma}{1+N+2N\gamma}[\rho_{(R,0)} - \rho_{(L,0)}] - \tfrac{\gamma}{1+N+2N\gamma}[\rho_{(R,1)} - \rho_{(L,1)}], \quad J^1_{x,x+1} = 0, \tag{7.4.3}$$

when $\epsilon = 0$ and by

$$\begin{aligned} J^0_{x,x+1} &= -\vec{c}_1 \cdot \vec{\rho} - \epsilon[(\vec{c}_3 \cdot \vec{\rho})\alpha_1^x(\alpha_1 - 1) + (\vec{c}_4 \cdot \vec{\rho})\alpha_2^x(\alpha_2 - 1)], \\ J^1_{x,x+1} &= -\epsilon\vec{c}_1 \cdot \vec{\rho} + \epsilon[(\vec{c}_3 \cdot \vec{\rho})\alpha_1^x(\alpha_1 - 1) + (\vec{c}_4 \cdot \vec{\rho})\alpha_2^x(\alpha_2 - 1)], \end{aligned} \tag{7.4.4}$$

when $\epsilon > 0$, where $\vec{c}_1, \vec{c}_3, \vec{c}_4$ are the vectors defined in (7.6.18) *of Section 7.6.1, and α_1, α_2 are defined in* (7.3.27). *As a consequence, the current $J_{x,x+1} = J^0_{x,x+1} + J^1_{x,x+1}$ is independent of x and is given by*

$$J_{x,x+1} = -\tfrac{1+\gamma}{1+N+2N\gamma}[\rho_{(R,0)} - \rho_{(L,0)}] - \tfrac{\gamma}{1+N+2N\gamma}[\rho_{(R,1)} - \rho_{(L,1)}] \tag{7.4.5}$$

when $\epsilon = 0$ and

$$J_{x,x+1} = -(1+\epsilon)\left[C_1\,(\rho_{R,0} - \rho_{L,0}) + \epsilon\, C_2\,(\rho_{R,1} - \rho_{L,1})\right] \tag{7.4.6}$$

when $\epsilon > 0$, where

$$\begin{aligned} C_1 &= \frac{[\alpha_1(1-\epsilon)(\alpha_1^{N-1} - 1) + \epsilon\,(\alpha_1^{N+1} - 1)]}{\alpha_1(1-\epsilon)(\alpha_1^{N-1} - 1)(N+1) + 2\epsilon\,(\alpha_1^{N+1} - 1)(N+\epsilon)}, \\ C_2 &= \frac{(\alpha_1^{N+1} - 1)}{\alpha_1(1-\epsilon)(\alpha_1^{N-1} - 1)(N+1) + 2\epsilon\,(\alpha_1^{N+1} - 1)(N+\epsilon)}. \end{aligned} \tag{7.4.7}$$

Proof. From (7.4.1) we have

$$J^0_{x,x+1} = \theta_0(x) - \theta_0(x+1), \qquad J^1_{x,x+1} = \epsilon[\theta_1(x) - \theta_1(x+1)], \tag{7.4.8}$$

where $\theta_0(\cdot), \theta_1(\cdot)$ are the average microscopic profiles. Thus, when $\epsilon = 0$, the expressions of $J^0_{x,x+1}, J^1_{x,x+1}$ and consequently $J_{x,x+1}$ follow directly from (7.3.17).

For $\epsilon > 0$, using the expressions of $\theta_0(\cdot), \theta_1(\cdot)$ in (7.3.29), we see that

$$\begin{aligned} J^0_{x,x+1} &= \theta_0(x) - \theta_0(x+1) = -\vec{c}_1 \cdot \vec{\rho} - \epsilon[(\vec{c}_3 \cdot \vec{\rho})\alpha_1^x(\alpha_1 - 1) + (\vec{c}_4 \cdot \vec{\rho})\alpha_2^x(\alpha_2 - 1)], \\ J^1_{x,x+1} &= \epsilon[\theta_1(x) - \theta_1(x+1)] = -\epsilon\vec{c}_1 \cdot \vec{\rho} + \epsilon[(\vec{c}_3 \cdot \vec{\rho})\alpha_1^x(\alpha_1 - 1) + (\vec{c}_4 \cdot \vec{\rho})\alpha_2^x(\alpha_2 - 1)], \end{aligned} \tag{7.4.9}$$

where $\vec{c}_1, \vec{c}_3, \vec{c}_4$ are the vectors defined in (7.6.18) of Section 7.6.1, and α_1, α_2 are defined in (7.3.27). Adding the two equations, we also have

$$J_{x,x+1} = J^0_{x,x+1} + J^1_{x,x+1} = -(1+\epsilon)\,\vec{c}_1 \cdot \vec{\rho} = (1+\epsilon)\left[C_1\,(\rho_{R,0} - \rho_{L,0}) + \epsilon\, C_2\,(\rho_{R,1} - \rho_{L,1})\right], \tag{7.4.10}$$

where C_1, C_2 are as in (7.4.7). □

Macroscopic system. The microscopic current scales like $1/N$. Indeed, the currents associated to the two layers in the macroscopic system can be obtained from the microscopic currents, respectively, by defining

$$J^0(y) = \lim_{N\to\infty} N J^0_{\lfloor yN\rfloor, \lfloor yN\rfloor+1}, \qquad J^1(y) = \lim_{N\to\infty} N J^1_{\lfloor yN\rfloor, \lfloor yN\rfloor+1}. \tag{7.4.11}$$

Below we justify the existence of the two limits and thereby provide explicit expressions for the macroscopic currents.

Proposition 7.4.2. [Stationary macroscopic current] *For $y \in (0,1)$ the stationary currents defined in* (7.4.11) *are given by*

$$J^0(y) = -\left[(\rho_{R,0} - \rho_{L,0})\right], \quad J^1(y) = 0, \tag{7.4.12}$$

when $\epsilon = 0$ and by

$$\begin{aligned} J^0(y) = {} & \frac{\epsilon B_{\epsilon,\Upsilon}}{1+\epsilon}\left[\frac{\cosh[B_{\epsilon,\Upsilon}(1-y)]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(L,0)} - \rho_{(L,1)}) - \frac{\cosh[B_{\epsilon,\Upsilon}\, y]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(R,0)} - \rho_{(R,1)})\right] \\ & - \frac{1}{1+\epsilon}\left[(\rho_{(R,0)} - \rho_{(L,0)}) + \epsilon(\rho_{(R,1)} - \rho_{(L,1)})\right] \end{aligned} \tag{7.4.13}$$

and

$$J^1(y) = -\frac{\epsilon B_{\epsilon,\Upsilon}}{1+\epsilon}\left[\frac{\cosh[B_{\epsilon,\Upsilon}(1-y)]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(L,0)}-\rho_{(L,1)}) - \frac{\cosh[B_{\epsilon,\Upsilon}\,y]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(R,0)}-\rho_{(R,1)})\right] - \frac{\epsilon}{1+\epsilon}\left[(\rho_{(R,0)}-\rho_{(L,0)}) + \epsilon(\rho_{(R,1)}-\rho_{(L,1)})\right] \tag{7.4.14}$$

when $\epsilon > 0$*. As a consequence, the total current* $J(y) = J^0(y) + J^1(y)$ *is constant and is given by*

$$J(y) = -\left[(\rho_{R,0}-\rho_{L,0}) + \epsilon(\rho_{R,1}-\rho_{L,1})\right]. \tag{7.4.15}$$

Proof. For $\epsilon = 0$ the claim easily follows from the expressions of $J^0_{x,x+1}, J^1_{x,x+1}$ given in (7.4.3) and the fact that $\gamma_N \to 0$ as $N \to \infty$.

When $\epsilon > 0$, we first note the following:

$$\begin{aligned} &\gamma_N N^2 \overset{N\to\infty}{\longrightarrow} \Upsilon > 0, \\ &\lim_{N\to\infty}\alpha_1 = \lim_{N\to\infty}\alpha_2 = 1, \\ &\lim_{N\to\infty} N(\alpha_1 - 1) = -B_{\epsilon,\Upsilon},\ \lim_{N\to\infty} N(\alpha_2-1) = B_{\epsilon,\Upsilon}, \\ &\lim_{N\to\infty}\alpha_1^N = \mathrm{e}^{-B_{\epsilon,\Upsilon}}, \quad \lim_{N\to\infty}\alpha_2^N = \mathrm{e}^{B_{\epsilon,\Upsilon}}. \end{aligned} \tag{7.4.16}$$

Consequently, from the expressions for $(\vec{c}_i)_{1\le i\le 4}$ defined in (7.6.18), we also have

$$\begin{aligned} &\lim_{N\to\infty} N\vec{c}_1 = \frac{1}{1+\epsilon}\begin{bmatrix} -1 & -\epsilon & 1 & \epsilon \end{bmatrix}^T, \\ &\lim_{N\to\infty}\vec{c}_3 = \frac{1}{1+\epsilon}\begin{bmatrix} \frac{\mathrm{e}^{B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & -\frac{\mathrm{e}^{B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & -\frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & \frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} \end{bmatrix}^T, \\ &\lim_{N\to\infty}\vec{c}_4 = \frac{1}{1+\epsilon}\begin{bmatrix} -\frac{\mathrm{e}^{-B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & \frac{\mathrm{e}^{-B_{\epsilon,\Upsilon}}}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & \frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} & -\frac{1}{\mathrm{e}^{B_{\epsilon,\Upsilon}}-\mathrm{e}^{-B_{\epsilon,\Upsilon}}} \end{bmatrix}^T. \end{aligned} \tag{7.4.17}$$

Combining the above equations with (7.4.4), we have

$$\begin{aligned} J^0(y) &= \lim_{N\to\infty} N J^0_{\lfloor yN\rfloor,\lfloor yN\rfloor+1} \\ &= -\epsilon B_{\epsilon,\Upsilon}\Big[\Big(\lim_{N\to\infty}\vec{c}_4\cdot\vec{\rho}\Big)\mathrm{e}^{B_{\epsilon,\Upsilon}y} - \Big(\lim_{N\to\infty}\vec{c}_3\cdot\vec{\rho}\Big)\mathrm{e}^{-B_{\epsilon,\Upsilon}y}\Big] - \Big(\lim_{N\to\infty}N\vec{c}_1\cdot\vec{\rho}\Big) \\ &= \frac{\epsilon B_{\epsilon,\Upsilon}}{1+\epsilon}\left[\frac{\cosh[B_{\epsilon,\Upsilon}(1-y)]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(L,0)}-\rho_{(L,1)}) - \frac{\cosh[B_{\epsilon,\Upsilon}\,y]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(R,0)}-\rho_{(R,1)})\right] \\ &\quad - \frac{1}{1+\epsilon}\left[(\rho_{(R,0)}-\rho_{(L,0)}) + \epsilon(\rho_{(R,1)}-\rho_{(L,1)})\right] \end{aligned} \tag{7.4.18}$$

and, similarly,

$$\begin{aligned} J^1(y) &= \lim_{N\to\infty} N J^1_{\lfloor yN\rfloor,\lfloor yN\rfloor+1} \\ &= \epsilon B_{\epsilon,\Upsilon}\Big[\Big(\lim_{N\to\infty}\vec{c}_4\cdot\vec{\rho}\Big)\mathrm{e}^{B_{\epsilon,\Upsilon}y} - \Big(\lim_{N\to\infty}\vec{c}_3\cdot\vec{\rho}\Big)\mathrm{e}^{-B_{\epsilon,\Upsilon}y}\Big] - \epsilon\Big(\lim_{N\to\infty}N\vec{c}_1\cdot\vec{\rho}\Big) \\ &= -\frac{\epsilon B_{\epsilon,\Upsilon}}{1+\epsilon}\left[\frac{\cosh[B_{\epsilon,\Upsilon}(1-y)]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(L,0)}-\rho_{(L,1)}) - \frac{\cosh[B_{\epsilon,\Upsilon}\,y]}{\sinh[B_{\epsilon,\Upsilon}]}(\rho_{(R,0)}-\rho_{(R,1)})\right] \\ &\quad - \frac{\epsilon}{1+\epsilon}\left[(\rho_{(R,0)}-\rho_{(L,0)}) + \epsilon(\rho_{(R,1)}-\rho_{(L,1)})\right]. \end{aligned} \tag{7.4.19}$$

Adding $J^0(y)$ and $J^1(y)$, we obtain the total current

$$J(y) = J^0(y) + J^1(y) = -\left[(\rho_{R,0}-\rho_{L,0}) + \epsilon(\rho_{R,1}-\rho_{L,1})\right], \tag{7.4.20}$$

which is indeed independent of y. □

Remark 7.4.3. [Currents] Combining the expressions for the density profiles and the current, we see that

$$J^0(y) = -\frac{d\rho_0}{dy}(y), \qquad J^1(y) = -\epsilon\frac{d\rho_1}{dy}(y).$$

♠

7.5 Discussion: Fick's law and uphill diffusion

In this section we discuss the behaviour of the boundary-driven system as the parameter ϵ is varied. For simplicity we restrict our discussion to the macroscopic setting, although similar comments hold for the microscopic system as well.

In view of the previous results, we can rewrite the equations for the densities $\rho_0(y,t), \rho_1(y,t)$ as

$$\begin{cases} \partial_t \rho_0 = -\nabla J^0 + \Upsilon(\rho_1 - \rho_0), \\ \partial_t \rho_1 = -\nabla J^1 + \Upsilon(\rho_0 - \rho_1), \\ J_0 = -\nabla \rho_0, \\ J_1 = -\epsilon \nabla \rho_1, \end{cases}$$

which are complemented with the boundary values (for $\epsilon > 0$)

$$\begin{cases} \rho_0(0,t) = \rho_{L,0}, \ \rho_0(1,t) = \rho_{R,0}, \\ \rho_1(0,t) = \rho_{L,1}, \ \rho_1(1,t) = \rho_{R,1}. \end{cases}$$

We will be concerned with the total density $\rho = \rho_0 + \rho_1$, whose evolution equation does not contain the reaction part, and is given by

$$\begin{cases} \partial_t \rho = -\nabla J, \\ J = -\nabla(\rho_0 + \epsilon \rho_1), \end{cases} \tag{7.5.1}$$

with boundary values

$$\begin{cases} \rho(0,t) = \rho_L = \rho_{L,0} + \rho_{R,0}, \\ \rho(1,t) = \rho_R = \rho_{R,0} + \rho_{R,1}. \end{cases} \tag{7.5.2}$$

Non-validity of Fick's law. From (7.5.1) we immediately see that Fick's law of mass transport is satisfied if and only if $\epsilon = 1$. When we allow diffusion and reaction of slow and fast particles, i.e., $0 \le \epsilon < 1$, Fick's law breaks down, since the current associated to the total mass is not proportional to the gradient of the total mass. Rather, the current J is the sum of a contribution J^0 due to the diffusion of fast particles of type 0 (at rate 1) and a contribution J^1 due to the diffusion of slow particles of type 1 (at rate ϵ). Interestingly, the violation of Fick's law opens up the possibility of several interesting phenomena that we discuss in what follows.

Equal boundary densities with non-zero current. In a system with diffusion and reaction of slow and fast particles we may observe a *non-zero current when the total density has the same value at the two boundaries*. This is different from what is observed in standard diffusive systems driven by boundary reservoirs, where in order to have a stationary current it is necessary that the reservoirs have different chemical potentials, and therefore different densities, at the boundaries.

Let us, for instance, consider the specific case when $\rho_{L,0} = \rho_{R,1} = 2$ and $\rho_{L,1} = \rho_{R,0} = 4$, which indeed implies equal densities at the boundaries given by $\rho_L = \rho_R = 6$. The density profiles and currents are displayed in Fig. 7.3 for two values of ϵ, which shows the comparison between the Fick-regime $\epsilon = 1$ (left panels) and the non-Fick-regime with very slow particles $\epsilon = 0.001$ (right panels).

On the one hand, in the Fick-regime the profile of both types of particles interpolates between the boundary values, with a slightly non-linear shape that has been quantified precisely in (7.3.35)–(7.3.36). Furthermore, in the same regime $\epsilon = 1$, the total density profile is flat and the total current J vanishes because $J^0(y) = -J^1(y)$ for all $y \in [0,1]$.

On the other hand, in the non-Fick-regime with $\epsilon = 0.001$, the stationary macroscopic profile for the fast particles interpolates between the boundary values almost linearly (see (7.3.39)), whereas the profile for the slow particles is non-monotone: it has two bumps at the boundaries and in the bulk closely follows the other profile. This non-monotonicity in the profile of the slow particles is due to the non-uniform convergence in the limit $\epsilon \downarrow 0$, as pointed out in the last part of Remark 7.3.9. As a consequence, the total density profile is not flat and has two bumps at the boundaries. Most strikingly, the total current is $J = -2$, since now the current of the bottom layer J^0 is dominating, while the current of the bottom layer J^1 is small (order ϵ).

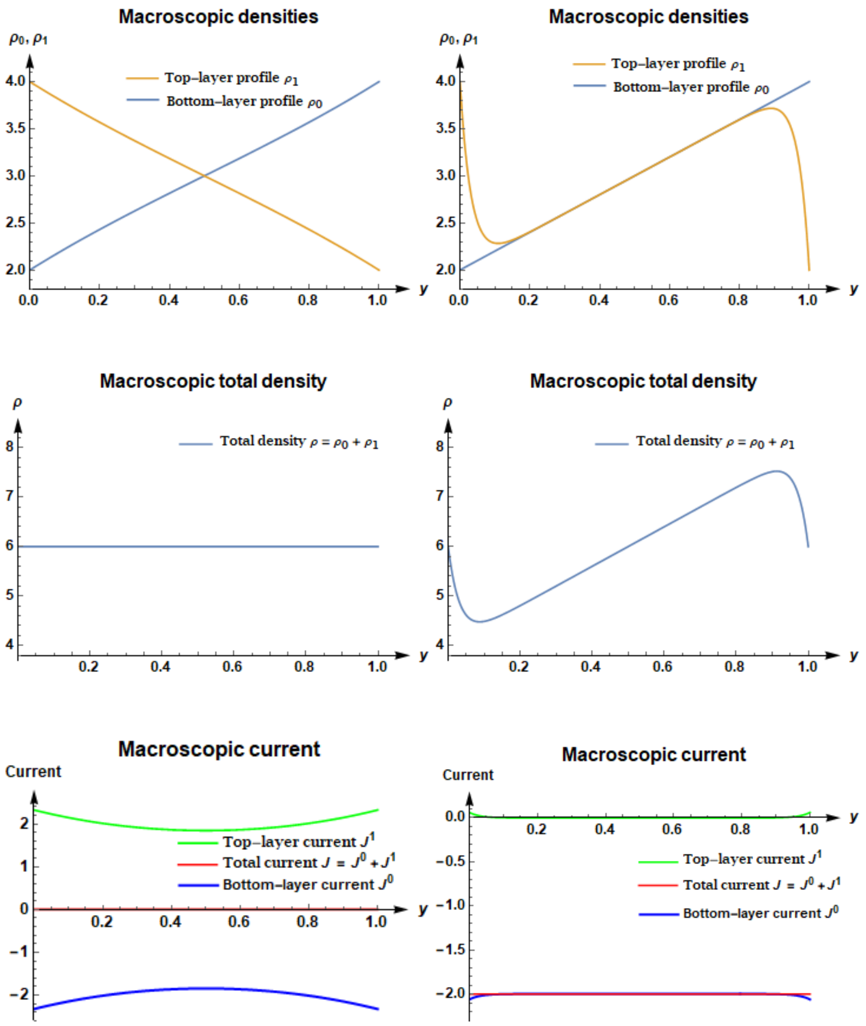


Figure 7.3: Macroscopic profiles of the densities for slow and fast particles (top panels), macroscopic profile of the total density (central panels), and the currents (bottom panels). Here, $\rho_{(L,0)} = 2$, $\rho_{(L,1)} = 4$, $\rho_{(R,0)} = 4$ and $\rho_{(R,1)} = 2$, $\Upsilon = 1$. For the panels in the left column, $\epsilon = 1$ and for the panels in the right column, $\epsilon = 0.001$.

Unequal boundary densities with uphill diffusion. As argued earlier, since the system does not always obey Fick's law, by tuning the parameters $\rho_{(L,0)}$, $\rho_{(L,1)}$, $\rho_{(R,0)}$, $\rho_{(R,1)}$ and ϵ, we can push the system into a regime where the total current is such that $J < 0$ and the total densities are such that $\rho_R < \rho_L$, where $\rho_R = \rho_{(R,0)} + \rho_{(R,1)}$ and $\rho_L = \rho_{(L,0)} + \rho_{(L,1)}$. In this regime, *the current goes uphill*, since the total density of particles at the right is lower than at the left, yet the average current is negative.

For an illustration, consider the case when $\rho_{L,1} = 6, \rho_{R,0} = 4$ and $\rho_{L,0} = \rho_{R,1} = 2$, which implies $\rho_L = 8$ and $\rho_R = 6$ and thus $\rho_R < \rho_L$. The density profiles and currents are shown in Fig. 7.4 for two values of ϵ, in particular, a comparison between the Fick-regime $\epsilon = 1$ (left panels) and the non-Fick-regime with very slow particles $\epsilon = 0.001$ (right panels). As can be seen in the figure, when $\epsilon = 1$, the system obeys Fick's law: the total density linearly interpolates between the two total boundary densities 8 and 6, respectively. The average total stationary current is positive as predicted by Fick's law. However, in the *uphill* regime, the total density is non-linear and the gradient of the total density is not proportional to the total current, violating Fick's law. The total current is negative and is effectively dominated by the current of the fast particles. It will be shown later that the transition into the uphill regime happens at the critical value $\epsilon = \frac{|\rho_{(R,0)} - \rho_{(L,0)}|}{|\rho_{(R,1)} - \rho_{(L,1)}|} = \frac{1}{2}$. In the limit $\epsilon \downarrow 0$ the total density profile and the current always get dominated in the bulk by the profile and the current of the fast particles, respectively. When $\epsilon < \frac{1}{2}$, even though the density of the slow particles makes the total density near the boundaries such that $\rho_R < \rho_L$, it is not strong enough to help the system overcome the domination of the fast particles in the bulk, and so the effective total current goes in the same direction as the current of the fast particles, producing an uphill current.

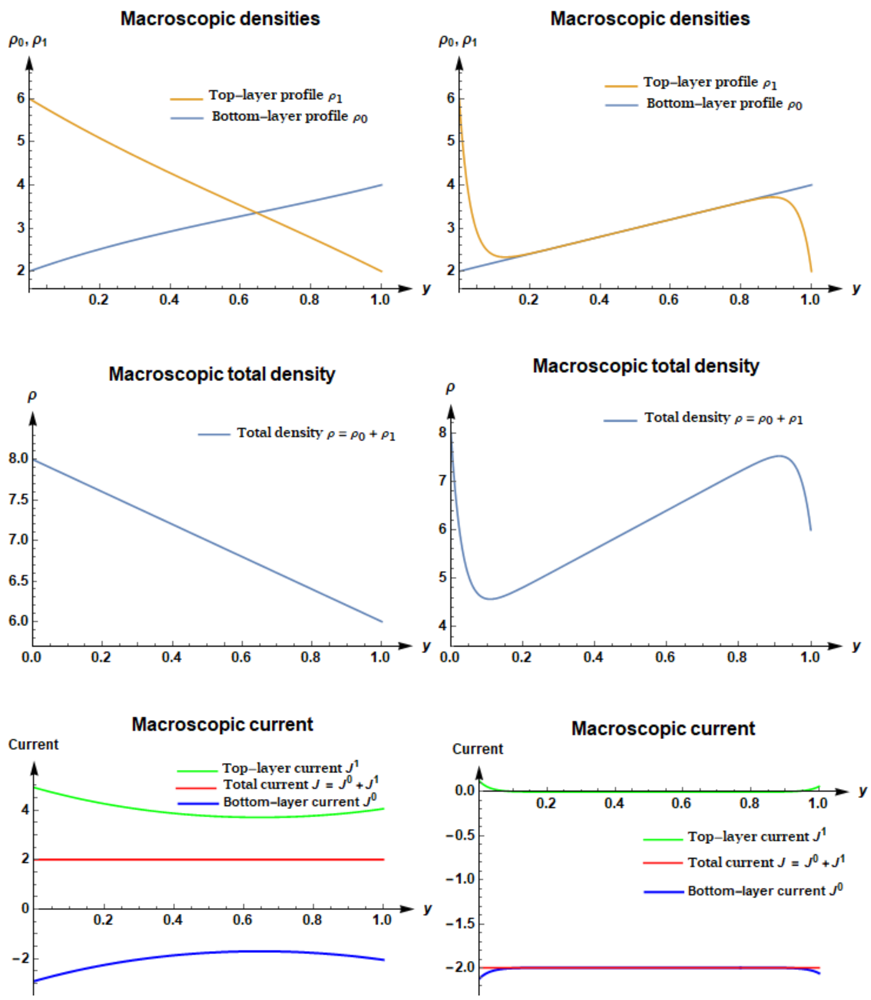
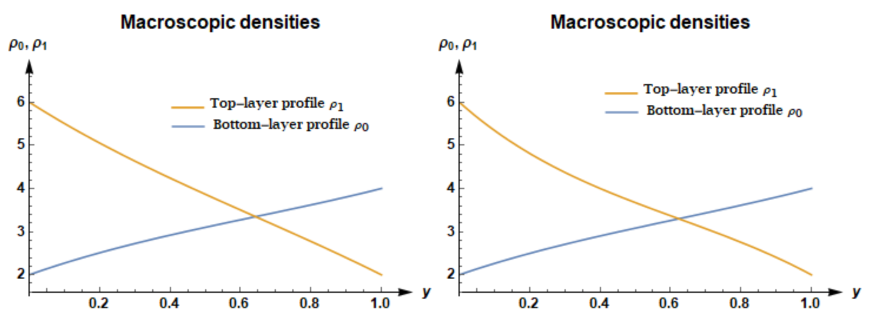


Figure 7.4: Macroscopic profiles of the densities for slow and fast particles (top panels), macroscopic profile of the total density (central panels), and the currents (bottom panels). Here, $\rho_{(L,0)} = 2$, $\rho_{(L,1)} = 6$, $\rho_{(R,0)} = 4$ and $\rho_{(R,1)} = 2$, $\Upsilon = 1$. For the panels in the left column, $\epsilon = 1$ and for the panels in the right column, $\epsilon = 0.001$.

The transition between downhill and uphill. We observe that for the choice of reservoir parameters $\rho_{L,1} = 6, \rho_{R,0} = 4$ and $\rho_{L,0} = \rho_{R,1} = 2$, the change from downhill to uphill diffusion occurs at $\epsilon = \frac{|\rho_{(R,0)} - \rho_{(L,0)}|}{|\rho_{(R,1)} - \rho_{(L,1)}|} = \frac{1}{2}$. The density profiles and currents are shown in Fig. 7.5 for two additional values of ϵ, one in the "mild" downhill regime $J > 0$ for $\epsilon = 0.75$ (left panels), the other in the "mild" uphill regime $J < 0$ for $\epsilon = 0.25$ (right panels). In the uphill regime (right panel), i.e., when $\epsilon = 0.75$, the "mild" non-linearity of the total density profile is already visible, indicating the violation of Fick's law.



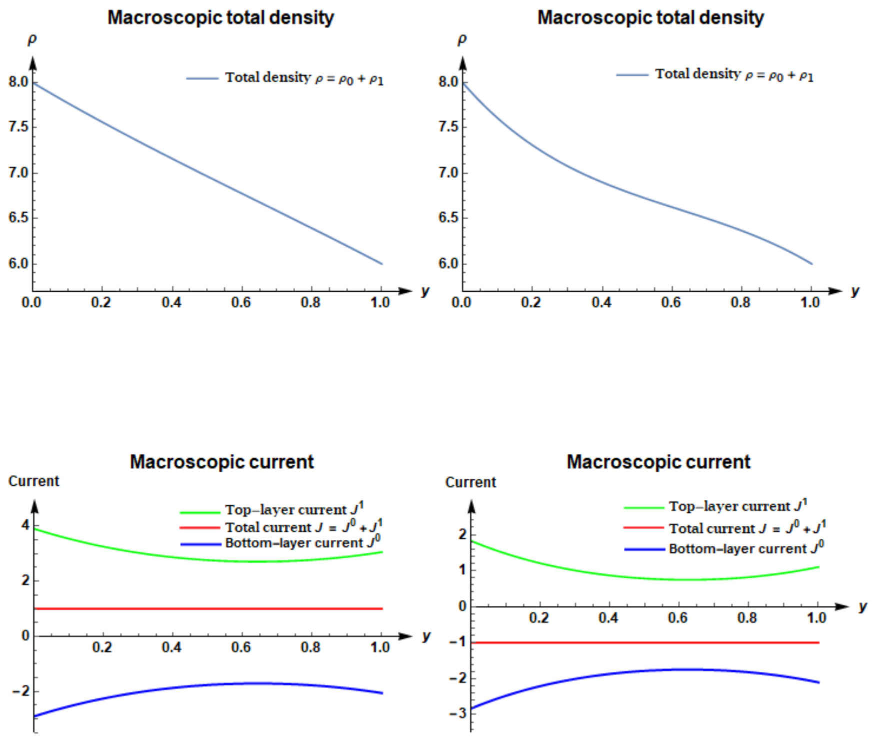


Figure 7.5: Macroscopic profiles of the densities for slow and fast particles (top panels), macroscopic profile of the total density (central panels), and the currents (bottom panels) in the "mild" downhill and the "mild" uphill regime. Here, $\rho_{(L,0)} = 2$, $\rho_{(L,1)} = 6$, $\rho_{(R,0)} = 4$ and $\rho_{(R,1)} = 2$, $\Upsilon = 1$. For the panels in the left column, $\epsilon = 0.75$ and for the panels in the right column, $\epsilon = 0.25$.

Identification of the uphill regime. We define the notion of uphill current below and identify the parameter ranges for which uphill diffusion occurs.

Definition 7.5.1. [Uphill diffusion] For parameters $\rho_{(L,0)}, \rho_{(L,1)}, \rho_{(R,0)}, \rho_{(R,1)}$ and $\epsilon > 0$, we say the system has an uphill current in stationarity if the total current J and the difference between the total density of particles in the right and the left side of the system given by $\rho_R - \rho_L$ have the same sign, where it is understood that $\rho_R = \rho_{(R,0)} + \rho_{(R,1)}$ and $\rho_L = \rho_{(L,0)} + \rho_{(L,1)}$. ♠

Proposition 7.5.2. [Uphill regime] *Let* $a_0 := \rho_{(R,0)} - \rho_{(L,0)}$ *and* $a_1 := \rho_{(R,1)} - \rho_{(L,1)}$. *Then the macroscopic system admits an uphill current in stationarity if and only if*

$$a_0^2 + (1+\epsilon)\, a_0 a_1 + \epsilon a_1^2 < 0. \tag{7.5.3}$$

If, furthermore, $\epsilon \in [0,1]$, *then*

(i) either

$$a_0 + a_1 > 0 \text{ with } a_0 < 0,\ a_1 > 0$$

or

$$a_0 + a_1 < 0 \text{ with } a_0 > 0,\ a_1 < 0,$$

(ii) $\epsilon \in [0, -\frac{a_0}{a_1}]$.

Proof. Note that, by (7.4.15), there is an uphill current if and only if $a_0 + a_1$ and $a_0 + \epsilon a_1$ have opposite signs. In other words, this happens if and only if

$$(a_0 + a_1)(a_0 + \epsilon\, a_1) = a_0^2 + (1+\epsilon)\, a_0 a_1 + \epsilon a_1^2 < 0.$$

The above constraint forces $a_0 a_1 < 0$. Further simplification reduces the parameter regime to the following four cases:

- $a_0 + a_1 > 0$ with $a_0 < 0$, $a_1 > 0$ and $\epsilon < -\frac{a_0}{a_1}$,
- $a_0 + a_1 < 0$ with $a_0 > 0$, $a_1 < 0$ and $\epsilon < -\frac{a_0}{a_1}$,
- $a_0 + a_1 > 0$ with $a_0 > 0$, $a_1 < 0$ and $\epsilon > -\frac{a_0}{a_1}$,
- $a_0 + a_1 < 0$ with $a_0 < 0$, $a_1 > 0$ and $\epsilon > -\frac{a_0}{a_1}$.

Under the assumption $\epsilon \in [0, 1]$, only the first two of the above four cases survive. □

7.6 The width of the boundary layer

We have seen that for $\epsilon = 0$ the microscopic density profile of the fast particles $\theta_0(x)$ linearly interpolates between $\rho_{L,0}$ and $\rho_{R,0}$, whereas the density profile of the slow particles satisfies $\theta_1(x) = \theta_0(x)$ for all $x \in \{2, \dots, N-1\}$. In the macroscopic setting this produces a continuous macroscopic profile $\rho_0^{\text{stat}}(y) = \rho_{L,0} + (\rho_{R,0} - \rho_{L,0})y$ for the bottom-layer, while the top-layer profile develops two discontinuities at the boundaries when either $\rho_{(L,0)} \neq \rho_{(L,1)}$ or $\rho_{(R,0)} \neq \rho_{(R,1)}$. In particular,

$$\rho_1^{\text{stat}}(y) \to [\rho_{L,0} + (\rho_{R,0} - \rho_{L,0})y]\, \mathbf{1}_{(0,1)}(y) + \rho_{L,1} \mathbf{1}_{\{1\}}(y) + \rho_{R,1} \mathbf{1}_{\{0\}}(y), \quad \epsilon \downarrow 0,$$

for $y \in [0, 1]$. For small but positive ϵ, the curve is smooth and the discontinuity is turned into a boundary layer. In this section we investigate the width of the left and the right boundary layers as $\epsilon \downarrow 0$. To this end, let us define

$$W_L := |\rho_{(L,0)} - \rho_{(L,1)}|, \quad W_R := |\rho_{(R,0)} - \rho_{(R,1)}|. \tag{7.6.1}$$

Note that, the profile ρ_1 develops a left boundary layer if and only if $W_L > 0$ and, similarly, a right boundary layer if and only if $W_R > 0$.

Definition 7.6.1. We say that the *left boundary layer* is of size $f_L(\epsilon)$ if there exists $C > 0$ such that, for any $c > 0$,

$$\lim_{\epsilon \downarrow 0} \frac{R_L(\epsilon, c)}{f_L(\epsilon)} = C,$$

where $R_L(\epsilon, c) = \sup\left\{y \in (0, \tfrac{1}{2}) : \left|\frac{d^2}{dy^2}\rho_1^{\text{stat}}(y)\right| \geq c\right\}$. Analogously, we say that the *right boundary layer* is of size $f_R(\epsilon)$ if there exists $C > 0$ such that, for any $c > 0$,

$$\lim_{\epsilon \downarrow 0} \frac{1 - R_R(\epsilon, c)}{f_R(\epsilon)} = C,$$

where $R_R(\epsilon, c) = \inf\left\{y \in (\tfrac{1}{2}, 1) : \left|\frac{d^2}{dy^2}\rho_1^{\text{stat}}(y)\right| \geq c\right\}$.

The widths of the two boundary layers essentially measure the deviation of the top-layer density profile (and therefore also the total density profile) from the bulk linear profile corresponding to the case $\epsilon = 0$. In the following proposition we estimate the sizes of the two boundary layers.

Proposition 7.6.2. [Width of boundary layers] *The widths of the two boundary layers are given by*

$$f_L(\epsilon) = f_R(\epsilon) = \sqrt{\epsilon}\, \log(1/\epsilon), \tag{7.6.2}$$

where $f_L(\epsilon), f_R(\epsilon)$ are defined as in Definition 7.6.1.

Proof. Note that, to compute $f_L(\epsilon)$, it suffices to keep $W_L > 0$ fixed and put $W_R = 0$, where W_L, W_R are as in (7.6.1). Let $\bar{y}(\epsilon, c) \in (0, \frac{1}{2})$ be such that, for some constant $c > 0$,

$$\left|\frac{d^2}{dy^2}\rho_1^{\text{stat}}(y)\right| \geq c, \tag{7.6.3}$$

or equivalently, since $\epsilon\Delta\rho_1 = \Upsilon(\rho_1 - \rho_0)$,

$$|\rho_1^{\text{stat}}(y) - \rho_0^{\text{stat}}(y)| \geq \frac{c\epsilon}{\Upsilon}. \tag{7.6.4}$$

Recalling the expressions of $\rho_0^{\text{stat}}(\cdot)$ and $\rho_1^{\text{stat}}(\cdot)$ for positive ϵ given in (7.3.35)–(7.3.36), we get

$$\left|\frac{\sinh\left[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}(1-y)\right]}{\sinh\left[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}\right]}\left(\rho_{(L,0)}-\rho_{(L,1)}\right)+\frac{\sinh\left[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}\,y\right]}{\sinh\left[\sqrt{\Upsilon(1+\frac{1}{\epsilon})}\right]}\left(\rho_{(R,0)}-\rho_{(R,1)}\right)\right|\geq\frac{c\epsilon}{\Upsilon}. \tag{7.6.5}$$

Using (7.6.1) plus the fact that $W_R = 0$, and setting $B_{\epsilon,\Upsilon} := \sqrt{\Upsilon\left(1+\frac{1}{\epsilon}\right)}$, we see that

$$\sinh\left[B_{\epsilon,\Upsilon}(1-y)\right]\geq\frac{c\epsilon}{\Upsilon W_L}\sinh\left[B_{\epsilon,\Upsilon}\right]. \tag{7.6.6}$$

Because $\sinh(\cdot)$ is strictly increasing, (7.6.6) holds if and only if

$$\bar{y}(\epsilon,c)\leq 1-\frac{1}{B_{\epsilon,\Upsilon}}\sinh^{-1}\left[\frac{c\epsilon}{\Upsilon W_L}\sinh\left(\tfrac{B_{\epsilon,\Upsilon}}{2}\right)\right]. \tag{7.6.7}$$

Thus, for small $\epsilon > 0$ we have

$$R_L(\epsilon,c)=1-\frac{1}{B_{\epsilon,\Upsilon}}\sinh^{-1}\left[\frac{c\epsilon}{\Upsilon W_L}\sinh\left(\tfrac{B_{\epsilon,\Upsilon}}{2}\right)\right], \tag{7.6.8}$$

where $R_L(\epsilon, c)$ is defined as in Definition 7.6.1. Since $\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$ for $x \in \mathbb{R}$, we obtain

$$\begin{aligned}R_L(\epsilon,c)&=\frac{\sqrt{\epsilon}}{\sqrt{\Upsilon(1+\epsilon)}}\log\left[\frac{N_{\epsilon,\Upsilon}+\sqrt{N_{\epsilon,\Upsilon}^2+1}}{\epsilon C N_{\epsilon,\Upsilon}+\sqrt{(\epsilon C N_{\epsilon,\Upsilon})^2+1}}\right]\\&=\frac{\sqrt{\epsilon}}{\sqrt{\Upsilon(1+\epsilon)}}\log(1/\epsilon)+\frac{\sqrt{\epsilon}}{\sqrt{\Upsilon(1+\epsilon)}}\log\left[\frac{1+\sqrt{1+(1/N_{\epsilon,\Upsilon})^2}}{C+\sqrt{C^2+(1/(\epsilon N_{\epsilon,\Upsilon}))^2}}\right]\\&=\frac{\sqrt{\epsilon}}{\sqrt{\Upsilon(1+\epsilon)}}\log(1/\epsilon)+R_{\epsilon,\Upsilon,W_L},\end{aligned} \tag{7.6.9}$$

where $N_{\epsilon,\Upsilon} := \sinh\left(\frac{B_{\epsilon,\Upsilon}}{2}\right)$, $C := \frac{c}{\Upsilon W_L}$, and the error term is

$$R_{\epsilon,\Upsilon,W_L}:=\frac{\sqrt{\epsilon}}{\sqrt{\Upsilon(1+\epsilon)}}\log\left[\frac{1+\sqrt{1+(1/N_{\epsilon,\Upsilon})^2}}{C+\sqrt{C^2+(1/(\epsilon N_{\epsilon,\Upsilon}))^2}}\right].$$

Note that, since $\epsilon N_{\epsilon,\Upsilon} \to \infty$ as $\epsilon \downarrow 0$, we have

$$\lim_{\epsilon\downarrow 0}\frac{R_{\epsilon,\Upsilon,W_L}}{\sqrt{\epsilon}}=\frac{1}{\sqrt{\Upsilon}}\log(1/C)<\infty. \tag{7.6.10}$$

Hence, combining (7.6.9)–(7.6.10), we get

$$\lim_{\epsilon\downarrow 0}\frac{R_L(\epsilon,c)}{\sqrt{\epsilon}\log(1/\epsilon)}=\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{\Upsilon(1+\epsilon)}}+\lim_{\epsilon\downarrow 0}\frac{R_{\epsilon,\Upsilon,W_L}}{\sqrt{\epsilon}\log(1/\epsilon)}=\frac{1}{\sqrt{\Upsilon}} \tag{7.6.11}$$

and so, by Definition 7.6.1, $f_L(\epsilon) = \sqrt{\epsilon}\log(1/\epsilon)$.

Similarly, to compute $f_R(\epsilon)$, we first fix $W_L = 0$, $W_R > 0$ and note that, for some $c > 0$, we have, by using (7.6.5),

$$|\partial^2\rho_1^{\text{stat}}(y)|\geq c\quad\text{ if and only if }\quad\sinh\left[B_{\epsilon,\Upsilon}\,y\right]\geq\tfrac{c\epsilon}{\Upsilon W_R}\sinh\left[B_{\epsilon,\Upsilon}\right]. \tag{7.6.12}$$

Hence, by appealing to the strict monotonicity of $\sinh(\cdot)$, we obtain

$$R_R(\epsilon,c)=\inf\left\{y\in(\tfrac{1}{2},1):\left|\frac{d^2}{dy^2}\rho_1^{\text{stat}}(y)\right|\geq c\right\}=\frac{1}{B_{\epsilon,\Upsilon}}\sinh^{-1}\left[\frac{c\epsilon}{\Upsilon W_R}\sinh\left(\tfrac{B_{\epsilon,\Upsilon}}{2}\right)\right]. \tag{7.6.13}$$

Finally, by similar computations as in (7.6.9)–(7.6.11), we see that

$$\lim_{\epsilon\downarrow 0}\frac{1-R_R(\epsilon,c)}{\sqrt{\epsilon\log(1/\epsilon)}}=\frac{1}{\sqrt{\Upsilon}} \tag{7.6.14}$$

and hence $f_R(\epsilon) = \sqrt{\epsilon}\log(1/\epsilon)$. □

7.6.1 Inverse of the boundary-layer matrix

The inverse of the matrix M_ϵ defined in (7.3.26) is given by (α_1 and α_2 are as in (7.3.27))

$$M_\epsilon^{-1} := \frac{1}{Z}\begin{bmatrix} -m_{13} & -m_{14} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31}(\alpha_2) & m_{32}(\alpha_2) & m_{33}(\alpha_2) & m_{34}(\alpha_2) \\ -m_{31}(\alpha_1) & -m_{32}(\alpha_1) & -m_{33}(\alpha_1) & -m_{34}(\alpha_1) \end{bmatrix}, \tag{7.6.15}$$

where

$$\begin{aligned} Z &:= \alpha_1^{N+1}[\alpha_2(1-\epsilon)(\alpha_2^{N-1}+1) + 2\epsilon(\alpha_2^{N+1}+1)]\,[\alpha_2(1+N)(1-\epsilon)(\alpha_2^{N-1}-1) + 2\epsilon(N+\epsilon)(\alpha_2^{1+N}-1)], \\ m_{13} &:= \alpha_1^{N+1}[\alpha_2(1-\epsilon)(\alpha_2^{N-1}+1) + 2\epsilon(\alpha_2^{N+1}+1)]\,[\alpha_2(1-\epsilon)(\alpha_2^{N-1}-1) + \epsilon(\alpha_2^{N+1}-1)], \\ m_{14} &:= \epsilon\,\alpha_1^{N+1}[\alpha_2(1-\epsilon)(\alpha_2^{N-1}+1) + 2\epsilon(\alpha_2^{N+1}+1)]\,(\alpha_2^{N+1}-1), \\ m_{21} &:= (1+N)(1-\epsilon)^2(\alpha_2^{N-1}-\alpha_1^{N-1}) - \epsilon(1-\epsilon)^2(\alpha_2-\alpha_1) \\ &\qquad + \epsilon^2(1+2N+\epsilon)(\alpha_2^{N+1}-\alpha_1^{N+1}) + \epsilon(1-\epsilon)(2+3N+\epsilon)(\alpha_2^N-\alpha_1^N), \\ m_{22} &:= \epsilon\,[(1-\epsilon)(1+N)(\alpha_2^N-\alpha_1^N) + \epsilon(1+2N+\epsilon)(\alpha_2^{N+1}-\alpha_1^{N+1})], \\ m_{23} &:= \epsilon\,(1-\epsilon)[(N+\epsilon)(\alpha_2-\alpha_1) - (1-\epsilon)(\alpha_2^N-\alpha_1^N) - \epsilon(\alpha_2^{N+1}-\alpha_1^{N+1})], \\ m_{24} &:= -\epsilon(1-\epsilon)[(1+N)(\alpha_2-\alpha_1) + \epsilon\,(\alpha_2^{N+1}-\alpha_1^{N+1})], \end{aligned} \tag{7.6.16}$$

and the polynomials $m_{31}(z), m_{32}(z), m_{33}(z), m_{34}(z)$ are defined as

$$\begin{aligned} m_{31}(z) &:= -(1-\epsilon)^2\,z - \epsilon\,(1-\epsilon) + (1-\epsilon)(N+\epsilon)\,z^N - \epsilon(1-2N-3\epsilon)\,z^{N+1}, \\ m_{32}(z) &:= -(1-\epsilon)(1+N)z^N - \epsilon\,(1-\epsilon) - \epsilon(1+2N+\epsilon)\,z^{N+1}, \\ m_{33}(z) &:= (1-\epsilon)^2\,z^N + \epsilon\,(1-\epsilon)\,z^{N+1} - (1-\epsilon)(N+\epsilon)\,z + \epsilon(1-2N-3\epsilon), \\ m_{34}(z) &:= (1+N)(1-\epsilon)\,z + \epsilon\,(1-\epsilon)\,z^{N+1} + \epsilon(1+2N+\epsilon). \end{aligned} \tag{7.6.17}$$

We remark that most of the terms appearing in the inverse simplify because of (7.3.28). We define the four vectors $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ as the respective rows of M_ϵ^{-1}, i.e.,

$$\begin{aligned} \vec{c}_1 &:= (M_\epsilon^{-1})^T\vec{e}_1, \quad \vec{c}_2 := (M_\epsilon^{-1})^T\vec{e}_2, \\ \vec{c}_3 &:= (M_\epsilon^{-1})^T\vec{e}_3, \quad \vec{c}_4 := (M_\epsilon^{-1})^T\vec{e}_4, \end{aligned} \tag{7.6.18}$$

where

$$\begin{aligned} \vec{e}_1 &:= [\,1 \quad 0 \quad 0 \quad 0\,]^T, \quad \vec{e}_2 := [\,0 \quad 1 \quad 0 \quad 0\,]^T, \\ \vec{e}_3 &:= [\,0 \quad 0 \quad 1 \quad 0\,]^T, \quad \vec{e}_4 := [\,0 \quad 0 \quad 0 \quad 1\,]^T. \end{aligned}$$

Part IV

From many to few in the continuum: closed and open systems

Chapter 8

Intertwining and Duality for Consistent Markov Processes

8.1 Introduction

As mentioned in the introduction of this manuscript, the language and formulation of duality in terms of occupation variables at discrete lattice sites clearly breaks down in many natural settings of e.g. particles moving in the continuum, such as interacting Brownian motions or more general interacting Markov processes. In this chapter we develop a more general approach to self-duality that can lead to results also in the continuum, on very general state spaces. First we find a language in which the basic duality properties of discrete systems, including the orthogonal dualities, can be restated in such a way that they make sense in the continuum. Second we understand under which assumptions these generalized relations are valid, including many more systems in the class of self-dual Markov processes.

8.1.1 The role of consistency

In [37], the notion of consistency (see also [107]) was connected to self-duality in the context of discrete interacting particle systems. In particular, for the three basic particle systems having self-duality (SEP, SIP and IRW), the "classical" dualities can all be derived from the same intertwining, which in turn is derived from consistency. Consistency roughly means that the time evolution commutes with the operation of randomly selecting a given number of particles out of the system. Equivalently, up to permutations, it implies that in a system of n particles, the k particle evolution is coinciding with the evolution of k particles out of these n particles, i.e., the effect of the interactions with the other $n - k$ particles is "wiped out". This is a remarkable property, trivially valid for independent particles, but also for interacting systems with special symmetries, such as the SEP and SIP.

The consistency property appeared (under a slightly different form) in the literature on stochastic flows [121], [149] including e.g. interacting Brownian motions, the Brownian web, and the Howitt-Warren flow. It also played a crucial role in the analysis of the KMP model ([107]). Therefore, the consistency property seems the natural starting point for establishing self-dualities for conservative particle systems in a general state space. Because we want to consider evolution of configurations of particles, we are naturally led to the context of point processes ([120]).

8.1.2 Summary of Main Results of the chapter

The main contributions of this chapter are summarized below.

1. We introduce a new framework in which self-duality type relations, more precisely self-intertwining relations, with respect to polynomials can be formulated for particle systems evolving on a general Borel space, thus also on $\mathbb{R}^d$. This framework also provides a new approach to self-duality.
2. We provide necessary and sufficient conditions to have self-intertwining relations with generalized falling factorial polynomials as intertwiners. In particular, we provide new self-intertwining results for systems such as independent and interacting Brownian motions. Moreover, from this new approach, the known self-duality functions for classical conservative interacting particle systems (i.e., SEP, IRW, SIP and the inhomogeneous

version of these processes) are recovered. Our approach is thus unifying and avoids the need of ad hoc computations for each system when proving duality.

3. We prove that, assuming reversibility for the particle system, the Gram-Schmidt orthogonalization procedure is a symmetry for the particle dynamics of a consistent process. As a consequence, orthogonalizing the previously introduced falling factorial polynomial self-intertwinings, we show orthogonal self-intertwinings in the same context of consistent particle systems on general state spaces. In doing so, we also show some properties of generalized orthogonal polynomials which are of independent interest. Again, our new machinery allows to recover all the known orthogonal duality functions for classical interacting particle systems.

4. We introduce and study a new process in the continuum, called *generalized symmetric inclusion process*, for which all our self-intertwining results apply. It turns out that the reversible measures of the generalized inclusion process are the distributions of the so-called Pascal point processes. We prove that generalized Meixner polynomials are self-intertwiners for the generalized symmetric inclusion process and some properties of these orthogonal polynomials are derived in a novel and simple way.

These self-intertwining results open doors to many potential future applications to the study of properties of particles systems in general state spaces, including characterization of the stationary measures and their attractors (see, e.g., [126, Chapter 8]), hydrodynamic limits (see, e.g., [47, Chapter 2]) and fluctuations (see, e.g., [11]), and boundary driven non-equilibrium systems (see, e.g., [107, 75]).

8.1.3 Organization of the chapter

This chapter is organized as follows. In Section 8.2 we introduce the general setting and the class of Markov processes under consideration. We then state the two main theorems, the two self-intertwining results where the intertwiners are respectively, generalized falling factorial and orthogonal polynomials. We also provide the proof of some properties of the generalized orthogonal polynomials. In Section 8.3 we list some examples of known processes which satisfy the assumptions of our main theorems. In particular we show how the known self-duality relations for exclusion and inclusion process follow from our general results. In Section 8.4 we introduce and study a continuum version of the inclusion process. In particular we identify its reversible distribution, we show that it satisfies the assumptions of the two intertwiner results, and finally we exhibit the relation between the generalized orthogonal polynomials and the Meixner polynomials.

8.2 Self-Intertwining Relations

In this section, we start by introducing the setting and the class of processes that we consider, namely the consistent and conservative Markov processes. Then, in Section 8.2.2, we introduce the generalized falling factorial polynomials and we state and prove our first main result, a self-intertwining relation. In Section 8.2.3, after providing the construction of generalized orthogonal polynomials, we state and prove a second self-intertwining relation.

8.2.1 Setting and Consistent Markov Processes

Throughout this article we investigate Markov processes whose state space consists of configurations of non-labelled particles in some general measurable space $(E, \mathcal{E})$. To avoid the technical difficulties associated with infinitely many particles (for example, a rigorous construction of interacting dynamics), we consider configurations of finitely many particles only.

We follow modern point process notation in modelling such configurations as finite counting measures on $(E, \mathcal{E})$. Thus, let $\mathbf{N}_{<\infty}$ be the space of finite counting measures, i.e., the space of finite measures that assign values in $\mathbb{N}_0$ to every set $B \in \mathcal{E}$. The space is equipped with the σ-algebra $\mathcal{N}_{<\infty}$ generated by the counting variables $\mathbf{N}_{<\infty} \ni \eta \mapsto \eta(B)$, $B \in \mathcal{E}$. Assumptions on $(E, \mathcal{E})$ are needed to ensure that every counting measure is a sum of Dirac measures, therefore we assume throughout the article that $(E, \mathcal{E})$ is a Borel space (see [120, Definition 6.1]). The reader may think of a Polish space or $\mathbb{R}^d$. It is well-known (see, e.g., [120, Chapter 6] or [100, Section 1.1]) that for a Borel space, every finite counting measure $\eta \in \mathbf{N}_{<\infty}$ is either zero or of the form $\eta = \delta_{x_1} + \cdots + \delta_{x_n}$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. In particular, the total mass $\eta(E)$ corresponds to the total number of particles.

For our purpose, a Markov process with state space $\mathbf{N}_{<\infty}$ is a collection $(\Omega, \mathfrak{F}, (\eta_t)_{t\geq 0}, (\mathbb{P}_\eta)_{\eta\in\mathbf{N}_{<\infty}})$, where $(\Omega, \mathfrak{F})$ is a measurable space, $\eta_t : (\Omega, \mathfrak{F}) \to (\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$ is a measurable map for all $t \geq 0$ and for $\eta \in \mathbf{N}_{<\infty}$, $\mathbb{P}_\eta$ are probability measures on $(\Omega, \mathfrak{F})$ such that $\mathbb{P}_\eta(\eta_0 = \eta) = 1$. The Markov property is implicitly assumed to be satisfied with respect to the natural filtration $\mathfrak{F}_t := \sigma(\eta_s,\ 0 \leq s \leq t)$.

We focus on a special class of Markov processes, which has been considered in [37, 121, 107, 149], namely consistent Markov processes. Intuitively speaking, consistency refers to the fact that the removal of a particle uniformly at random commutes with the time-evolution of the process. In order to precisely define the concept of consistent Markov process we introduce the lowering operator

$$\mathcal{A}f(\eta) := \int f(\eta - \delta_x)\eta(\mathrm{d}x), \quad \eta \in \mathbf{N}_{<\infty}$$

acting on functions $f \in \mathcal{G}$, where $\mathcal{G}$ denotes the set of measurable functions $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ such that the restriction of f to every n-particle sector $\mathbf{N}_n := \{\eta \in \mathbf{N}_{<\infty} : \eta(E) = n\}$ is bounded. Note $\mathcal{A}$ is well-defined and that $\mathcal{A}f \in \mathcal{G}$ for $f \in \mathcal{G}$.

Definition 8.2.1 (Consistent Markov process)**.** *Let $(\eta_t)_{t\geq 0}$ be a Markov process on $\mathbf{N}_{<\infty}$ with Markov semigroup $(P_t)_{t\geq 0}$. The process $(\eta_t)_{t\geq 0}$ said to be consistent if for all $t \geq 0$ and bounded and measurable function $f : \mathbf{N}_{<\infty} \to \mathbb{R}$*

$$P_t\mathcal{A}f(\eta) = \mathcal{A}P_tf(\eta), \quad \eta \in \mathbf{N}_{<\infty}. \tag{8.2.1}$$

Notice that (8.2.1) can be written as

$$\mathbb{E}_\eta\left(\int f(\eta_t - \delta_x)\eta_t(\mathrm{d}x)\right) = \int \mathbb{E}_{\eta-\delta_x}(f(\eta_t))\eta(\mathrm{d}x),$$

where on the left hand-side we first evolve the system and after we remove uniformly at random a particle, while on the right hand-side we first remove uniformly at random a particle from the initial configuration and then we let evolve the process from the new initial state. We refer to [37, Theorem 2.7 and Theorem 3.2] for further characterizations of consistency in terms of the infinitesimal generator L, namely $L\mathcal{A} = \mathcal{A}L$, and to Section 8.3 and 8.4 for some examples of consistent Markov processes.

For our results we need the following set of assumptions.

Assumption 8.2.1. *We assume that $(\eta_t)_{t\geq 0}$ is a Markov process on $\mathbf{N}_{<\infty}$ with Markov semigroup $(P_t)_{t\geq 0}$, such that*

1. *it is consistent;*
2. *it is conservative, i.e. if $\eta_0 \in \mathbf{N}_{<\infty}$ then $\eta_t(E) = \eta_0(E)$ for all $t \geq 0$.*

Notice that Assumption 8.2.1 (2) yields $P_tf \in \mathcal{G}$ for all $f \in \mathcal{G}$ and thus, by Assumption 8.2.1 (1), we obtain $P_t\mathcal{A}f(\eta) = \mathcal{A}P_tf(\eta)$ for $f \in \mathcal{G}$ and $\eta \in \mathbf{N}_{<\infty}$.

Let us briefly explain how consistency as defined in Definition 8.2.1 relates to a stronger form of consistency reminiscent of Kolmogorov's consistency theorem. Often the process $(\eta_t)_{t\geq 0}$ comes from a process for *labelled* particles, as is the case for the independent random walkers in Section 3.2.1. *Strong consistency*, called *compatibility* by Le Jan and Raimond [121, Definition 1.1], roughly means that time evolution and removal of any *deterministic* particle commute—there is no need to choose the particle to be removed uniformly at random.

Precisely, suppose that for each $n \in \mathbb{N}$, we are given a transition function $(p_t^{[n]})_{t\geq 0}$ on $(E^n, \mathcal{E}^n)$ that preserves permutation invariance. Then one can define a transition function $(P_t)_{t\geq 0}$ on $(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$ by $P_t(0, B) = \mathbf{1}_B(0)$ and

$$P_t(\delta_{x_1} + \cdots + \delta_{x_n}, B) = p_t^{[n]}(x_1, \ldots, x_n; \iota_n^{-1}(B)), \quad (x_1, x_2, \ldots, x_n) \in E^n, B \in \mathcal{N}_{<\infty} \tag{8.2.2}$$

where $\iota_n : E^n \to \mathbf{N}_{<\infty}$ is the map given by $\iota_n(x_1, \ldots, x_n) = \delta_{x_1} + \cdots + \delta_{x_n}$.

Definition 8.2.2. *The family $(p_t^{[n]})_{t\geq 0}$ is* strongly consistent *if for all $n \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, and $(x_1, \ldots, x_n) \in E^n$, the image of the measure $\mathcal{E}^n \ni B \mapsto p_t^{[n]}(x_1, \ldots, x_n; B)$ under the map from E^n to E^{n-1} that consists of omission of x_i is equal to the measure $\mathcal{E}^{n-1} \ni B \mapsto p_t^{[n-1]}(x_1, \ldots, \widehat{x_i}, \ldots, x_n; B)$, where $\widehat{x_i}$ means omission of the variable x_i.*

An elementary but important observation is that strong consistency of the family $(p_t^{[n]})_{t\geq 0}$ implies consistency of $(P_t)_{t\geq 0}$ in the sense of Definition 8.2.1. The observation yields a whole class of consistent processes, see Section 8.3.3.

Theorem 8.2.5 uses both $(P_t)_{t\geq 0}$ and a semigroup $(p_t^{[n]})_{t\geq 0}$ for labelled particles. As we wish to use the semigroup $(P_t)_{t\geq 0}$ as our starting point, let us mention that (8.2.2) implies

$$(P_tf)(\delta_{x_1} + \cdots + \delta_{x_n}) = (p_t^{[n]}f_n)(x_1, \ldots, x_n) \tag{8.2.3}$$

whenever $f_n = f \circ \iota_n$ and $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ is measurable and non-negative or bounded. This determines the action of $(p_t^{[n]})_{t\geq 0}$ on the space $\mathcal{F}_n$ of bounded, measurable, permutation invariant functions f_n uniquely. Therefore, given a conservative semigroup $(P_t)_{t\geq 0}$ on $\mathbf{N}_{<\infty}$ we may take (8.2.3) as the *definition* of an associated semigroup on the space of bounded permutation invariant functions $\mathcal{F}_n$. For $n = 0$, we set $\mathcal{F}_0 := \mathbb{R}$ and let $p_t^{[0]}$ be the identity operator on $\mathbb{R}$, for all $t \geq 0$.

8.2.2 Generalized Falling Factorial Polynomials

Let $\eta = \sum_{i=1}^m \delta_{x_i} \in \mathbf{N}$, $n \in \mathbb{N}$, and recall (see (3.2.4) above) that $\eta^{(n)}$ denotes the n-th factorial measure of η, i.e.

$$\eta^{(n)} := \sum_{1\leq i_1,\dots,i_n\leq m}^{\neq} \delta_{(x_{i_1},\dots,x_{i_n})},$$

where $\eta = 0$ yields $\eta^{(r)} = 0$.

Definition 8.2.3. *For $n \in \mathbb{N}$ and measurable $f_n : E^n \to \mathbb{R}$ we define the associated generalized falling factorial polynomial as follows*

$$J_n(f_n, \eta) := \int f_n(x_1, \dots, x_n)\eta^{(n)}(\mathrm{d}(x_1, \dots, x_n)), \quad \eta \in \mathbf{N}_{<\infty}.$$

For $n = 0$ and $f_0 \in \mathbb{R}$ we set $J_0(f_0, \eta) := \int f_0 \,\mathrm{d}\eta^{(0)} := f_0$.

In particular, we have $J_n(f_n, \cdot\,) \in \mathcal{G}$ for $f_n \in \mathcal{F}_n$.

Remark 8.2.4. *The fact that J_n generalizes falling factorial polynomials becomes evident when considering $f_n = \mathbf{1}_{B_1}^{\otimes d_1} \otimes \dots \otimes \mathbf{1}_{B_N}^{\otimes d_N}$ for pairwise disjoint sets $B_1, \dots, B_N \in \mathcal{E}$, $N \in \mathbb{N}$ and $d_1 \dots, d_N \in \mathbb{N}_0$, $d_1 + \dots + d_N =: n$. Indeed, it follows from the definition of the factorial measure that*

$$J_n(\mathbf{1}_{B_1}^{\otimes d_1} \otimes \dots \otimes \mathbf{1}_{B_N}^{\otimes d_N}, \eta) = (\eta(B_1))_{d_1} \cdots (\eta(B_N))_{d_N}, \quad \eta \in \mathbf{N}_{<\infty} \tag{8.2.4}$$

where $(a)_k := a(a-1)\cdots(a-k+1)$, $a \in \mathbb{R}$, $k \in \mathbb{N}$, $(a)_0 := 1$, denotes the falling factorial. Equation (8.2.4) *will be used in Section 8.3 below to recover known self-duality functions for particle systems on finite set from the abstract Theorem 8.2.5. We refer to [72] for further properties of the generalized falling factorial polynomials.*

Our first main result is an intertwining relation between the Markov semigroup $(P_t)_{t\geq 0}$ and $(p_t^{[n]})_{t\geq 0}$, with the generalized falling factorial polynomials J_n defined above as intertwiner. Thus, we view the result as a generalization of the self-duality relations for interacting particle systems on a finite set where the self-duality functions consist in weighted falling factorial moments of the occupation variables (see, e.g., [126, Theorem 1.1, p.363], (2.5.5) above and Section 8.3.1 below) .

Theorem 8.2.5 (Self-intertwining relation)**.** *Let $(\eta_t)_{t\geq 0}$ be a Markov process satisfying Assumption 8.2.1. We then have*

$$P_t J_n(f_n, \cdot\,)(\eta) = J_n(p_t^{[n]} f_n, \eta), \quad \eta \in \mathbf{N}_{<\infty} \tag{8.2.5}$$

for each $f_n \in \mathcal{F}_n$, $n \in \mathbb{N}_0$ and $t \geq 0$.

Proof. Let us define the lowering operator $\mathcal{A}_{r-1,r}$ acting on functions $f_{r-1} \in \mathcal{F}_{r-1}$ as

$$\mathcal{A}_{r-1,r} f_{r-1}(x_1, \dots, x_r) := \sum_{k=1}^r f_{r-1}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_r)$$

for $x_1, \dots, x_r \in E$ and $r \geq 2$ and $\mathcal{A}_{0,1} f_0 := f_0 \mathbf{1}$, $f_0 \in \mathbb{R}$ for $r = 1$. We then have, as a direct consequence of consistency of $(\eta_t)_{t\geq 0}$, that $p_t^{[r]} \mathcal{A}_{r-1,r} f_{r-1} = \mathcal{A}_{r-1,r} p_t^{[r-1]} f_{r-1}$, $r \in \mathbb{N}$. Denoting for all $r \geq n \geq 0$,

$$\mathcal{A}_{n,r} f_n := \begin{cases} \mathcal{A}_{n,n+1} \cdots \mathcal{A}_{r-1,r} f_n & r > n \\ f_n & n = r \end{cases},$$

for all $f_n \in \mathcal{F}_n$, one obtains, by induction, that

$$p_t^{[r]} \mathcal{A}_{n,r} f_n = \mathcal{A}_{n,r} p_t^{[n]} f_n.$$

The proof is concluded by noticing that for all $n \leq r$, $x_1, \dots, x_r \in E$,

$$J_n(f_n, \delta_{x_1} + \dots + \delta_{x_r}) = \frac{n!}{(r-n)!} \mathcal{A}_{n,r} f_n(x_1, \dots, x_r).$$

□

Remark 8.2.6. *A close look at the proof reveals that the relation in Theorem 8.2.5 is in fact an equivalence: A conservative process is consistent if and only if the self-intertwining relation* (8.2.5) *holds true for all n, f_n, t. The equivalence is closely related to Theorem 4.3 in [37] in the discrete setting.*

Theorem 8.2.5 can be rephrased in a number of ways. The first rephrasing is in terms of kernels and justifies the denomination *intertwining*. Let $\Lambda_n : \mathbf{N}_{<\infty} \times \mathcal{E}^n \to \mathbb{R}_+$ be the kernel given by $\Lambda_n(\eta, B) := \eta^{(n)}(B) = J_n(\mathbf{1}_B, \eta)$. Then, $P_t\Lambda_n = \Lambda_n p_t^{[n]}$ meaning that

$$\int P_t(\eta, \mathrm{d}\xi)\Lambda_n(\xi, B) = \int \Lambda_n(\eta, \mathrm{d}x)p_t^{[n]}(x, B)$$

for all $\eta \in \mathbf{N}_{<\infty}$ and all permutation invariant sets $B \in \mathcal{E}^n$. Hence, the kernel $\Lambda_n(\eta, B) = J_n(\mathbf{1}_B, \eta)$ intertwines the semigroups (P_t) and $(p_t^{[n]})$. The second rephrasing uses the semi-group (P_t) only, which makes the "self" in self-intertwining spring to the eye. Set

$$\mathcal{K}(f, \eta) := f(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f(\delta_{x_1} + \ldots + \delta_{x_n})\eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$$

for measurable bounded $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ and $\eta \in \mathbf{N}_{<\infty}$. Note that the integral vanishes for $n > \eta(E)$ and $\mathcal{K}(f, \cdot) \in \mathcal{G}$ for $f \in \mathcal{G}$. The function $\mathcal{K}(f, \cdot)$ is also known as K-transform of f (cf. [123]) and by linearity, it follows from (8.2.3) and (8.2.5) that $\mathcal{K}$ intertwines $(P_t)_{t\geq 0}$ with itself, i.e.,

$$P_t\mathcal{K}(f, \cdot)(\eta) = \mathcal{K}(P_t f, \eta). \tag{8.2.6}$$

for $f \in \mathcal{G}, \eta \in \mathbf{N}_{<\infty}$. For free Kawasaki dynamics, which is a special case of independent particles, this result is in fact known (see [110, Section 3.2]).

In terms of expectations, the self-intertwining relation becomes

$$\mathbb{E}_\eta\Big[\int f(\delta_{x_1} + \ldots + \delta_{x_n})\eta_t^{(n)}(\mathrm{d}(x_1, \ldots, x_n))\Big] = \int \mathbb{E}_{\delta_{x_1}+\ldots+\delta_{x_n}}[f(\eta_t)]\,\eta^{(n)}(\mathrm{d}(x_1, \ldots, x_n))$$

for measurable, bounded $f : \mathbf{N}_n \to \mathbb{R}$, $n \in \mathbb{N}_0$ and $t \geq 0$.

To conclude we note a corollary on the time-evolution of correlation functions and explain the relation with Proposition 3.2.2.

Corollary 8.2.7. *Under the assumptions of Theorem 8.2.5, the following holds true for every initial condition $\eta \in \mathbf{N}_{<\infty}$. Let $\alpha_n^t(B) := \mathbb{E}_\eta[\eta_t^{(n)}(B)]$ be the n-th factorial moment measure of the process $(\eta_t)_{t\geq 0}$ started in η. Then*

$$\alpha_n^t(B) = \int \alpha_n^0(\mathrm{d}x)p_t^{[n]}(x, B)$$

for all $n \in \mathbb{N}$, $t \geq 0$, and permutation-invariant sets $B \in \mathcal{E}^n$.

Of course for deterministic initial condition η the time-zero factorial moment measure is just $\alpha_n^0 = \eta^{(n)}$, but in the form given above the relation generalizes to random initial conditions.

Proof. We have

$$\alpha_n^t(B) = \mathbb{E}_\eta\Big[J_n(\mathbf{1}_B, \eta_t)\Big] = J_n(p_t^{[n]}\mathbf{1}_B, \eta) = \int \eta^{(n)}(\mathrm{d}x)(p_t^{[n]}\mathbf{1}_B)(x) = \int \alpha_n^0(\mathrm{d}x)p_t^{[n]}(x, B). \qquad \square$$

A generalized version of Proposition 3.2.2 is recovered under the additional condition that for some σ-finite measure λ on E and each $n \in \mathbb{N}$, there exists a measurable function $u_t^{[n]} : E \times E \to \mathbb{R}_+$ with $u_t^{[n]}(x, y) = u_t^{[n]}(y, x)$ on $E \times E$ and

$$p_t^{[n]}(x, B) = \int_B u_t^{[n]}(x, y)\lambda^{\otimes n}(\mathrm{d}y) \tag{8.2.7}$$

for all $t > 0$, $x \in E^n$, and permutation invariant set $B \in \mathcal{E}^n$. This assumption shares similarites with the notion of duality from probabilistic potential theory, see Blumenthal and Getoor [26, Chapter VI]; we emphasize that the latter notion of (self-)duality with respect to a measure is stronger than reversibility of the measure. The additional condition is satisfied for example by independent reversible diffusions. Corollary 8.2.7, (8.2.7), and the symmetry of $u_t^{[n]}$ yield

$$\mathbb{E}_\eta[\eta_t^{(n)}(B)] = \int_B\Big(\int_{E^n} u_t^{[n]}(y, x)\eta^{(n)}(\mathrm{d}x)\Big)\lambda^{\otimes n}(\mathrm{d}y). \tag{8.2.8}$$

This relation generalizes Proposition 3.2.2.

8.2.3 Generalized Orthogonal Polynomials

In this section we generalize the orthogonal self-duality relation introduced in Section 3.2.1 to the class of Markov processes on $\mathbf{N}_{<\infty}$ satisfying Assumption 8.2.1. More precisely, assuming that there exists a reversible measure ρ, we show another self-intertwining relation where the intertwiner satisfies an orthogonality relation with respect to this measure. The intertwiner is a so-called *generalized orthogonal polynomial*, a well studied object in the infinite dimensional analysis literature (see, e.g., [151], [160] and [128]). We thus start by constructing the generalized orthogonal polynomials, following closely [128].

Let ρ be a probability measure on $(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty})$. We use the shorthand $L^2(\rho) := L^2(\mathbf{N}_{<\infty}, \mathcal{N}_{<\infty}, \rho)$. Through the rest of the section we assume that all moments of the total number of particles are finite.

Assumption 8.2.2. *Assume* $\int \eta(E)^n \rho(\mathrm{d}\eta) < \infty$ *for all* $n \in \mathbb{N}$.

Assumption 8.2.2 implies that every map $\eta \mapsto \eta^{\otimes n}(f_n) = \int f_n \mathrm{d}\eta^{\otimes n}$, with $f_n : E^n \to \mathbb{R}$ a bounded measurable function, is in $L^2(\rho)$.

Orthogonal polynomials in a single real variable can be constructed by an orthogonalization procedure. This definition extends to the infinite-dimensional setting: generalized orthogonal polynomals are defined by taking an orthogonal projection onto a proper subspace of generalized polynomials, see [128] and references therein. We thus define the space $\mathcal{P}_n$ of generalized polynomials (with bounded coefficients) of degree less or equal than $n \in \mathbb{N}_0$ as the set of linear combinations of maps $\eta \mapsto \int f_k \mathrm{d}\eta^{\otimes k}$, $k \le n$, with bounded measurable $f_k : E^k \to \mathbb{R}_+$, with the convention $\eta^{\otimes 0}(f_0) := f_0 \in \mathbb{R}$. Thus the set $\mathcal{P}_0$ consists of the constant functions. We refer to the functions f_k as coefficients.

Assumption 2 guarantees that every polynomial is square-integrable, i.e., $\mathcal{P}_n$ is a subspace of $L^2(\rho)$. In general it is not closed, we write $\overline{\mathcal{P}_n}$ for its closure in $L^2(\rho)$. The linear space $\mathcal{P}_n$ and its closure have the same orthogonal complement $\mathcal{P}_n^{\perp} = \overline{\mathcal{P}_n}^{\perp}$ in $L^2(\rho)$.

The next definition is equivalent to a definition from [128, Section 5].

Definition 8.2.8 (Generalized orthogonal polynomials). *For* $n \in \mathbb{N}$ *and* $f_n : E^n \to \mathbb{R}$ *a bounded measurable function we define the associated generalized orthogonal polynomial as follows*

$$I_n(f_n, \cdot) := \textit{orthogonal projection of } (\eta \mapsto \eta^{\otimes n}(f_n)) \textit{ onto } \overline{\mathcal{P}_{n-1}}^{\perp}.$$

Equivalently,

$$I_n(f_n, \eta) = \eta^{\otimes n}(f_n) - Q(\eta)$$

with $Q \in \overline{\mathcal{P}_{n-1}}$ the orthogonal projection of $\eta \mapsto \eta^{\otimes n}(f_n)$ onto $\overline{\mathcal{P}_{n-1}}$. Notice that $I_n(f_n, \eta)$ is only defined up to ρ-null sets.

Remark 8.2.9 (Wick dots and multiple stochastic integrals). *In the literature (see, e.g., [128, Section 5]) the generalized orthogonal polynomial* $I_n(f_n, \eta)$ *is often denoted by* $: \eta^{\otimes n}(f_n) :$ *("Wick dots"). When* ρ *is the distribution of a Poisson point process with intensity measure* λ*, the generalized orthogonal polynomial is given by a multiple stochastic integral with respect to the compensated Poisson measure* $\eta - \lambda$ *(see the references provided at the end of Section 3.2.1), hence the notation* $I_n(f_n, \eta)$*. The notation has the advantage of being analogous to the one used for the self-intertwiner* J_n *in Section 8.2.2, which is why we keep it.*

Remark 8.2.10 (Orthogonality relation). *It follows from the definition that*

$$\int I_n(f_n, \cdot) I_m(g_m, \cdot)\, \mathrm{d}\rho = 0$$

for $n \neq m$*. Moreover* $f_n \mapsto I_n(f_n, \cdot)$ *extends to a unitary operator on the space of permutation invariant functions that are square integrable with respect to some measure* λ_n *(see, e.g., [128, Corollary 5.2] for further details). When* ρ *is the distribution of a Poisson process with intensity measure* λ*, the measure* λ_n *is the product* $\lambda_n = \lambda^{\otimes n}$*, but in general the measure* λ_n *is more complicated.*

Remark 8.2.11 (Chaos decompositions and Lévy white noise). *Generalized orthogonal polynomials appear naturally in the study of non-Gaussian white noise [16, 17], they are used to prove chaos decompositions. The relation between polynomial chaos and chaos decompositions in terms of multiple stochastic integrals with respect to* power jump martingales *[140] is investigated in detail [128]. Chaos decompositions play a role in the study of Lévy white noise and stochastic differential equations driven by Lévy white noise [58, 127, 137].*

We complement the definition of the generalized orthogonal polynomials by two propositions on their properties. The first proposition says that the orthogonal polynomials can also be obtained by an orthgonal projection of the generalized falling factorial polynomials $\eta \mapsto J_n(f_n, \eta)$ instead of $\eta \mapsto \eta^{\otimes}(f_n)$. This observation plays an important role in the proof of Theorem 8.2.15.

Proposition 8.2.12. *The following identities hold*

$$\mathcal{P}_n = \left\{ \eta \mapsto \sum_{k=0}^{n} J_k(f_k, \eta) : f_k \in \mathcal{F}_k, k \in \{0, \dots, n\}, n \in \mathbb{N}_0 \right\}, \tag{8.2.9}$$

$$I_n(f_n, \cdot\,) = \textit{orthogonal projection of } J_n(f_n, \cdot\,) \textit{ onto } \overline{\mathcal{P}_{n-1}}^{\perp}, \quad f_n \in \mathcal{F}_n. \tag{8.2.10}$$

We note that (8.2.9) is a direct consequence of the fact that $J_k(f_k, \cdot\,)$ can be written as linear combination of integrals with respect to the product measure of degree $\le k$ and vice versa, see [72, Eq. (3.1)-(3.3)]. We provide a complete proof of the above proposition in Section 8.2.4.

The second proposition applies under an additional assumption of complete independence. A finite point process ζ is *completely independent* (or *completely orthogonal*) [120] if the counting variables $\zeta(A_1), \dots, \zeta(A_m)$ associated with pairwise disjoint regions $A_1, \dots, A_m \in \mathcal{E}$, $m \in \mathbb{N}$, are independent. Complete independence implies a factorization property of generalized orthogonal polynomials with disjointly supported coefficients.

Proposition 8.2.13. *Suppose that ρ is the distribution of some finite completely independent point process. Let $N \ge 2$, $A_1, \dots, A_N \in \mathcal{E}$ pairwise disjont, and $d_1, \dots, d_N \in \mathbb{N}_0$. Further let $f_i : E^{d_i} \to \mathbb{R}$, $i = 1, \dots, N$ be bounded measurable functions that vanish on $E^{d_i} \setminus A_i^{d_i}$. Set $n := d_1 + \dots + d_N$. Then*

$$I_n(f_1 \otimes \ldots \otimes f_n, \eta) = I_{d_1}(f_1, \eta) \cdots I_{d_n}(f_n, \eta) \tag{8.2.11}$$

for ρ-almost all $\eta \in \mathbf{N}_{<\infty}$.

The proposition is proven in Section 8.2.4. For special cases of measures ρ that give rise to orthogonal polynomials of Meixner's type, a similar factorization property is found, for example, in [129, Lemma 3.1]. Our proposition instead holds true for all distributions of completely independent point processes.

Remark 8.2.14. *A particularly relevant case is when f_i is the indicator of $A_i^{d_i}$. Then Proposition 8.2.13 says that the orthogonalized version of $\eta \mapsto \prod_{i=1}^{n} \eta(A_i)^{d_i}$ is equal to the product of the orthogonalized versions of $\eta \mapsto \eta(A_i)^{d_i}$. When ρ is the distribution of a Poisson or Pascal point process (see Sections 8.3 and 8.4 below), the orthogonalized version of $\eta(A_i)^{d_i}$ is in fact a univariate orthogonal polynomial in the variable $\eta(A_i) \in \mathbb{N}_0$ and we obtain a product of univariate orthogonal polynomials, see* (8.3.5) *and* (8.4.3). *In general, however, the orthogonalized version of $\eta(A_i)^{d_i}$ need not be a univariate polynomial.*

We now state the second main theorem of this section, which is the analogue of Theorem 8.2.5 but where the self-intertwiner is the generalized orthogonal polynomial introduced above.

Theorem 8.2.15 (Self-intertwining relation)**.** *Let $(\eta_t)_{t\ge 0}$ be a Markov process on $\mathbf{N}_{<\infty}$ that satisfies Assumption 8.2.1, i.e. it is consistent and conservative. Let ρ be a reversible probability measure for $(\eta_t)_{t\ge0}$ that satisfies Assumption 8.2.2. Then,*

$$P_t I_n(f_n, \cdot)(\eta) = I_n(p_t^{[n]} f_n, \eta) \tag{8.2.12}$$

for ρ-almost all $\eta \in \mathbf{N}_{<\infty}$, all $t \ge 0$, and all $f_n \in \mathcal{F}_n$.

Proof. To lighten notation, we drop the second variable $I_n(f_n, \cdot\,)$ and write $I_n(f_n)$ when we refer to the function in $L^2(\rho)$, similarly for $J_n(f_n)$. Let Π_{n-1} be the orthogonal projection in $L^2(\rho)$ onto $\overline{\mathcal{P}_{n-1}}$, and id the identity operator in $L^2(\rho)$. By Proposition 8.2.12,

$$I_n(f_n) = (\mathrm{id} - \Pi_{n-1}) J_n(f_n).$$

The theorem follows once we know that the semigroup P_t commutes with the projection Π_{n-1} i.e.

$$P_t \Pi_{n-1} = \Pi_{n-1} P_t \tag{8.2.13}$$

since then, (8.2.12) is obtained as follows

$$P_t I_n(f_n) = P_t(\mathrm{id} - \Pi_{n-1}) J_n(f_n) = (\mathrm{id} - \Pi_{n-1}) P_t J_n(f_n) = (\mathrm{id} - \Pi_{n-1}) J_n(p_t^{[n]} f_n) = I_n(p_t^{[n]} f_n)$$

where we used Proposition 8.2.12 in the first and the fourth equality and Theorem 8.2.5 in the third equality.

Let $k \leq n-1$ and let us recall the characterization of $\mathcal{P}_n$ given in Proposition 8.2.12. Using Theorem 8.2.5 combined with the fact that $p_t^{[k]} f_k \in \mathcal{F}_k$ for all $f_k \in \mathcal{F}_k$, we have that $P_t J_k(f_k, \cdot) = J_k(p_t^{[k]} f_k, \cdot) \in \mathcal{P}_{n-1}$. Thus, for all $t \geq 0$ and $n \in \mathbb{N}_0$, $P_t \mathcal{P}_{n-1} \subset \mathcal{P}_n$ and by the boundedness of P_t on $L^2(\rho)$ we obtain

$$P_t \overline{\mathcal{P}_{n-1}} \subset \overline{\mathcal{P}_{n-1}}. \tag{8.2.14}$$

The operator P_t is self-adjoint in $L^2(\rho)$ because of the reversibility of ρ. It is a general fact that a bounded self-adjoint operator that leaves a closed vector space invariant commutes with the orthogonal projection onto that space. Let us check this fact for our concrete operators and spaces. For $f \in \overline{\mathcal{P}_{n-1}}^{\perp}$, by the self-adjointness of P_t on $L^2(\rho)$ and (8.2.14), we have, for all $g \in \overline{\mathcal{P}_n}$, that $\int (P_t f) g \mathrm{d}\rho = \int f (P_t g) \mathrm{d}\rho = 0$ and thus

$$P_t \overline{\mathcal{P}_{n-1}}^{\perp} \subset \overline{\mathcal{P}_{n-1}}^{\perp}. \tag{8.2.15}$$

We then have, using (8.2.14), (8.2.15) and $f - \Pi_{n-1} f \in \overline{\mathcal{P}_{n-1}}^{\perp}$ that, for all $f \in L^2(\rho)$,

$$\Pi_{n-1} P_t f = \Pi_{n-1} P_t \Pi_{n-1} f + \Pi_{n-1} P_t (f - \Pi_{n-1} f) = P_t \Pi_{n-1} f.$$

This completes the proof of (8.2.13) and the proof of the theorem. □

8.2.4 Properties of Generalized Orthogonal Polynomials. Proof of Propositions 8.2.12 and 8.2.13

This section is devoted to the proof of Propositions 8.2.12 and 8.2.13.

Orthogonalization of Generalized Falling Factorial Polynomials Proposition 8.2.12 follows from explicit formulas that link factorial measures $\eta^{(n)}$ and product measure $\eta^{\otimes n}$. These relations are similar to relations between moments and factorial moments of integer-valued random variables with Stirling numbers, see [46, Chapter 5]. A systematic treatment in terms of Stirling operators is found in [72].

Proof of (8.2.9). In order to show that $\mathcal{P}_n$ is the linear hull of generalized falling factorials $J_k(f_k, \eta)$, $k \leq n$, it is enough to check that every monomial $\eta \mapsto \eta^{\otimes n}(f)$ is a linear combination of falling factorials of degree $k \leq n$ and vice-versa.

Let $\eta = \delta_{x_1} + \cdots + \delta_{x_\kappa} \in \mathbf{N}_{<\infty}$ and $f : E^n \to \mathbb{R}$ a bounded measurable function. Then

$$\eta^{\otimes n}(f_n) = \sum_{1 \leq i_1, \ldots, i_n \leq \kappa} f_n(x_{i_1}, \ldots, x_{i_n}).$$

Every multi-index $(i_1, \ldots, i_n)$ on the right side gives rise to a set partition σ of $\{1, \ldots, n\}$ in which k and ℓ belong to the same block if and only if $i_k = i_\ell$. Denote by Σ_n the set of partitions of $\{1, \ldots, n\}$. For $\sigma \in \Sigma_n$, let $|\sigma|$ be the number of blocks of the set partition. Further let $(f_n)_\sigma : E^{|\sigma|} \to \mathbb{R}$ be the function obtained from f_n by identifying, in order of occurrence, those arguments which belong to the same block of σ. Grouping multi-indices $(i_1, \ldots, i_n)$ that give rise to the same partition σ, we find

$$\int f_n \mathrm{d}\eta^{\otimes n} = \sum_{\sigma \in \Sigma_n} \int (f_n)_\sigma \, \mathrm{d}\eta^{(|\sigma|)}$$

(compare [46, Exercise 5.4.5]) and conclude that $\eta^{\otimes n}(f_n)$ is a linear combination of generalized falling factorials of degrees $|\sigma| \leq n$.

Conversely,

$$\int f_n \mathrm{d}\eta^{(n)} = \sum_{\sigma \in \Sigma_n} (-1)^{n-|\sigma|} \int (f_n)_\sigma \, \mathrm{d}\eta^{\otimes |\sigma|} \tag{8.2.16}$$

hence the falling factorial of degree n on the left side is a linear combination of monomials $\eta^{\otimes k}(g_k)$ of degree $k \leq n$. □

Proof of (8.2.10). For $n = 0$ the identities $J_0(f_0, \eta) = f_0 = \eta^{\otimes 0}(f_0)$ yield (8.2.10). For $n \in \mathbb{N}$, we notice that (8.2.16) implies

$$\int f_n \mathrm{d}\eta^{(n)} = \int f_n \mathrm{d}\eta^{\otimes n} + Q(\eta)$$

for some $Q \in \mathcal{P}_{n-1}$, given by a sum over set partitions with a number of blocks $|\sigma| \leq n-1$. It follows that $\eta \mapsto J_n(f_n, \eta)$ and $\eta \mapsto \eta^{\otimes n}(f_n)$ have the same orthogonal projections onto $(\mathcal{P}_{n-1})^{\perp}$. □

Factorization Property of Generalized Orthogonal Polynomials In order to exploit the complete independence, it is helpful to check that if $f : E^n \to \mathbb{R}$ is supported in A^n, then $I_n(f,\eta)$ depends only on what happens inside A. We show a bit more. Let $\mathcal{P}_n(A) \subset \mathcal{P}_n$ be the space of linear combinations of maps $\eta \mapsto \eta^{\otimes k}(f_k)$, $k \leq n$, with bounded measurable $f_k : E \to \mathbb{R}$ vanishing on $E^k \setminus A^k$. Notice that every function $F \in \mathcal{P}_n(A)$ depends only on the restriction η_A, defined by $\eta_A(B) := \eta(A \cap B)$.

Lemma 8.2.16. *Let $d \in \mathbb{N}$, $A \in \mathcal{E}$, and $f : E^d \to \mathbb{R}$ a bounded measurable function that vanishes outside $E^d \setminus A^d$. Then there exists a map $Q \in \overline{\mathcal{P}_{d-1}(A)}$ such that $I_d(f,\eta) = \eta^{\otimes d}(f) - Q(\eta)$ for ρ-almost all $\eta \in \mathbf{N}_{<\infty}$.*

Proof. Let Q be the orthogonal projection of $\eta \mapsto \eta^{\otimes d}(f)$ onto $\overline{\mathcal{P}_{d-1}(A)}$. Then $Q \in \overline{\mathcal{P}_{d-1}(A)}$ and the difference $F(\eta) := \eta^{\otimes d}(f) - Q(\eta)$ is orthogonal to $\overline{\mathcal{P}_{d-1}(A)}$. We exploit the complete independence to show that F is actually orthogonal to the bigger space $\overline{\mathcal{P}_{d-1}}$.

Let $n \in \{1,\dots,d-1\}$. If $C \in \mathcal{E}^n$ is of the form $C_1 \times C_2$ with $C_i \in \mathcal{E}^{s_i}$ where $s_1, s_2 \in \mathbb{N}_0$ and $C_1 \subset A$, $C_2 \subset A^c$, then $\eta^{\otimes n}(C) = \eta^{\otimes s_1}(C_1)\eta^{\otimes s_2}(C_2)$ and by the complete independence (notice $F(\eta) = F(\eta_A)$)

$$\int F(\eta)\eta^{\otimes n}(C)\rho(\mathrm{d}\eta) = \Big(\int F(\eta)\eta^{\otimes s_1}(C_1)\rho(\mathrm{d}\eta)\Big)\Big(\int \eta^{\otimes s_2}(C_2)\rho(\mathrm{d}\eta)\Big).$$

The first integral on the right side vanishes because of $C_1 \subset A^{s_1}$, $s_1 \leq d-1$, and $F \perp \overline{\mathcal{P}_{d-1}(A)}$. Therefore F is orthogonal to $\eta \mapsto \eta^{\otimes n}(C)$.

More generally, every set $C \in \mathcal{E}^n$ is the disjoint union of Cartesian products $C_1 \times \cdots \times C_n$ in which every C_i is either contained in A or in A^c. Taking linear combinations and exploiting that $\eta^{\otimes n}(g)$ does not change if we permute variables in g, we find that F is orthogonal to $\eta^{\otimes n}(C)$ for all $C \in \mathcal{E}^n$ and then, by the usual measure-theoretic arguments, to all maps $\eta \mapsto \eta^{\otimes n}(g)$, $g : E^n \to \mathbb{R}$ bounded and measurable. The map F is also orthogonal to all constant functions because every constant function is in $\overline{\mathcal{P}_{d-1}(A)}$.

Hence, taking linear combinations of maps $\eta^{\otimes n}(f_n)$, $n \in \{0,\dots,d-1\}$, we see that F is orthogonal to the space $\overline{\mathcal{P}_{d-1}}$. As $F(\eta) = \eta^{\otimes n}(f) - Q(\eta)$ with $Q \in \overline{\mathcal{P}_{d-1}}$, it follows that $I_n(f,\eta) = F(\eta)$ for ρ-almost all η. □

When evaluating the product of two generalized orthgonal polynomials $I_n(f,\eta)$ using Lemma 8.2.16, it is important to know that the product of two polynomials is again a polynomial.

Lemma 8.2.17. *Let A and B be two disjoint measurable subsets of E and $m,n \in \mathbb{N}_0$. Pick $F \in \overline{\mathcal{P}_m(A)}$ and $G \in \overline{\mathcal{P}_n(B)}$. Then FG is in $\overline{\mathcal{P}_{m+n}(A \cup B)}$.*

Proof. Write $\|\cdot\|$ for the $L^2(\rho)$-norm. Let $(F_k)_{k\in\mathbb{N}}$ and $(G_k)_{k\in\mathbb{N}}$ be sequences in $\mathcal{P}_m(A)$ and $\mathcal{P}_n(B)$, respectively, with $\|F - F_k\| \to 0$ and $\|G - G_k\| \to 0$. We have $F_k(\eta) = F_k(\eta_A)$ for all k and η hence $F(\eta) = F(\eta_A)$ for ρ-almost all η. Similarly G_k and G depend on η_B only. The triangle inequality and the complete independence yield

$$\begin{aligned}\|FG - F_kG_k\| &\leq \|(F - F_k)G\| + \|F_k(G - G_k)\|\\ &= \|F - F_k\|\,\|G\| + \|F_k\|\,\|G - G_k\| \to 0.\end{aligned}$$

As each product F_kG_k is in $\mathcal{P}_{m+n}(A \cup B)$, the limit FG is in the closure $\overline{\mathcal{P}_{m+n}(A \cup B)}$. □

Proof of Proposition 8.2.13. It is enough to treat the case $N = 2$; the general case follows by an induction over N. Let A_1 and A_2 be two disjoint measurable subsets in $\mathcal{E}$. Let d_1, d_2 be two integers and $f_i : E^{d_i} \to \mathbb{R}$, $i = 1,2$ be two bounded measurable functions that vanish outside $A_1^{d_1}$ and $A_2^{d_2}$ respectively. By Lemma 8.2.16, there exist maps $Q_i \in \overline{\mathcal{P}_{d_i-1}(A_i)}$, $i = 1,2$, such that

$$I_{d_1}(f_1,\eta) = \eta^{\otimes d_1}(f_1) - Q_1(\eta), \quad I_{d_2}(f_2,\eta) = \eta^{\otimes d_2}(f_2) - Q_2(\eta)$$

for ρ-almost all η. Therefore by Lemma 8.2.17, we have

$$I_{d_1}(f_1,\eta)I_{d_2}(f_2,\eta) = \eta^{\otimes d_1}(f_1)\eta^{\otimes d_2}(f_2) - Q(\eta) \tag{8.2.17}$$

with $Q \in \overline{\mathcal{P}}_{d_1+d_2-1}$. Let $s_1, s_2 \in \mathbb{N}_0$ and $C_1 \in \mathcal{E}^{s_1}$, $C_2 \in \mathcal{E}^{s_2}$ with $s_1 + s_2 \leq d_1 + d_2 - 1$ and $C_1 \subset A_1^{s_1}$, $C_2 \subset (A_1^c)^{s_2}$. Then, by the complete independence,

$$\int I_{d_1}(f_1,\eta)I_{d_2}(f_2,\eta)\,\eta^{\otimes(s_1+s_2)}(C_1 \times C_2)\rho(\mathrm{d}\eta) = \prod_{i=1}^{2}\int I_{d_i}(f_i,\eta)\,\eta^{\otimes s_i}(C_i)\rho(\mathrm{d}\eta).$$

We must have $s_1 \leq d_1 - 1$ or $s_2 \leq d_2 - 1$, therefore at least one of the integrals on the right side vanishes and the product $I_{d_1}(f_1, \eta)I_{d_2}(f_2, \eta)$ is orthogonal to $\eta^{\otimes n}(C)$. We conclude with an argument similar to the proof of Lemma 8.2.16 that $I_{d_1}(f_1, \eta)I_{d_2}(f_2, \eta)$ is in fact orthogonal to $\overline{\mathcal{P}}_{d_1+d_2-1}$. It follows that the product is equal to $I_{d_1+d_2}(f_1 \otimes f_2, \eta)$ for ρ-almost all η. □

8.3 Examples

In this section we provide some examples of known consistent and conservative Markov processes, i.e. of processes satisfying Assumption 8.2.1. Moreover, we also provide the reversible distribution of those processes, when known, and we specify when the assumptions of Theorem 8.2.15 are also satisfied. In particular, we recover known self-duality functions of systems of particles hopping on a finite set. In the next section, we introduce a new process, which generalizes the inclusion process (see, e.g., [84] where the SIP is introduced) for which both main theorems apply.

Before doing that, we recall the definition of the Charlier and Meixner polynomials, see e.g. [108], which are polynomials orthogonal with respect to the Poisson and negative binomial distribution. Differently from the usual definition in the literature, we normalize orthogonal polynomials to be monic where a polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is called monic if $a_n = 1$. These sequences of orthogonal polynomials can be expressed by using the generalized hypergeometric function given by

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) := \sum_{k=0}^{\infty} \frac{(a_1)^{(k)} \cdots (a_p)^{(k)}}{(b_1)^{(k)} \cdots (b_q)^{(k)}} \frac{z^k}{k!} $$

for $a_1, \ldots, a_p, b_1, \ldots, b_q, z \in \mathbb{R}$, $p, q \in \mathbb{N}$, where we remind the reader that $(a)^{(0)} := 1$ and $(a)^{(k)} := a(a+1)\cdots(a+k-1)$ denotes the rising factorial (also called Pochhammer symbol). Similarly, we recall the falling factorial defined by $(a)_k := a(a-1)\cdots(a-k+1)$, $(a)_0 := 1$.

1. The monic Charlier polynomials are given by

$$ \mathscr{C}_n(x; \alpha) := (-\alpha)^n \, {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -\frac{1}{\alpha}\right) = \sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{n-k} (x)_k, \quad x \in \mathbb{N}_0 $$

for $n \in \mathbb{N}_0$ and $\alpha > 0$ and they satisfy the orthogonality relation

$$ \sum_{\ell=0}^{\infty} \mathscr{C}_n(\ell; \alpha)\mathscr{C}_m(\ell; \alpha)\text{Poisson}(\alpha)(\{\ell\}) = \mathbf{1}_{\{n=m\}}\alpha^n n! $$

for $n, m \in \mathbb{N}_0$, i.e., $\mathscr{C}_n(\,\cdot\,; \alpha)$ are orthogonal polynomials with respect to the Poisson distribution $\text{Poisson}(\alpha)(\{\ell\}) = e^{-\alpha}\frac{\alpha^\ell}{\ell!}$. $\ell \in \mathbb{N}_0$.

2. The monic Meixner polynomials are given by

$$ \mathscr{M}_n(x; a; p) := (a)^{(n)} \left(1 - \frac{1}{p}\right)^{-n} {}_2F_1\left(\begin{matrix} -x, -n \\ a \end{matrix} \middle| 1 - \frac{1}{p}\right) = \sum_{k=0}^{n} \binom{n}{k} \left(1 - \frac{1}{p}\right)^{k-n} (a+k)^{(n-k)} (x)_k, \quad x \in \mathbb{N}_0 $$

for $n \in \mathbb{N}_0$, $a > 0$, $p \in (0, 1)$ and they satisfy the orthogonality relation

$$ \sum_{\ell=0}^{\infty} \mathscr{M}_n(\ell; a; p)\mathscr{M}_m(\ell; a; p)\text{Negative-Binomial}(a, p)(\{\ell\}) = \mathbf{1}_{\{n=m\}} \frac{p^n n! (a)^{(n)}}{(1-p)^{2n}} \tag{8.3.1} $$

for $n, m \in \mathbb{N}_0$, i.e., $(\mathscr{M}_n(\,\cdot\,; a; p))_{n \in \mathbb{N}_0}$ are orthogonal polynomials with respect to the generalized negative binomial distribution

$$ \text{Negative-Binomial}(a, p)(\{\ell\}) = (a)^{(\ell)} \frac{p^\ell}{\ell!} (1-p)^a, \quad \ell \in \mathbb{N}_0. $$

8.3.1 Interacting Particle Systems on a Finite Set

Let E be a non-empty finite set and identify $\xi \in \mathbf{N}_{<\infty}$ with $(\xi_k)_{k \in E} := (\xi(\{x\}))_{x \in E} \in \mathbb{N}_0^E$. Let η be a Markov process on $\mathbf{N}_{<\infty}$ satisfying Assumption 8.2.1 and ρ be a reversible probability measure satisfying Assumption 8.2.2. We then have that $D^{\text{cheap}}(\xi,\eta) := \frac{\mathbf{1}_{\{\eta=\xi\}}}{\rho(\{\xi\})}$, for $\eta,\xi \in \mathbf{N}_{<\infty}$ is the so-called cheap or trivial self-duality function ([37, Eq. (4.2)]). In this section we recover well-known self-duality functions of systems of particles hopping on a finite set by applying the intertwiners J_n and I_n to the cheap duality function. Note that

$$D_n^{\text{cheap}}(\xi,x) := D^{\text{cheap}}\left(\xi, \sum_{k=1}^{n} \delta_{x_k}\right), \quad \xi \in \mathbf{N}_{<\infty}, x = (x_1,\dots,x_n) \in E^n, n \in \mathbb{N},$$

is a duality functions for $(P_t)_{t\geq 0}$ and the n-particle semigroup $(p_t^{[n]})_{t\geq 0}$, i.e., $P_t D_n^{\text{cheap}}(\,\cdot\,,x)(\xi) = p_t^{[n]} D_n^{\text{cheap}}(\xi,\,\cdot\,)(x)$ for each $\xi \in \mathbb{N}_0^E$, $x \in E^n$, $n \in \mathbb{N}$. Putting $D_0^{\text{cheap}}(\xi,) := D^{\text{cheap}}(\xi,0)$ yields $P_t D_0^{\text{cheap}}(\,\cdot\,,)(\xi) = p_t^{[0]} D_0^{\text{cheap}}(\xi,)$.

It is well-known that applying an intertwiner to a duality function, for instance $D_n^{\text{cheap}}(\xi,x)$, yields again a self-duality function, see e.g. [34, Theorem 2.5] or [84, Remark 2.7].

Proposition 8.3.1. *Let $\rho = \bigotimes_{k\in E} \rho_k$ where ρ_k are probability measures on $\mathbb{N}_0$ satisfying $\rho_k(\{\ell\}) > 0$ for each $\ell \in \mathbb{N}_0$. Consider for each ρ_k the sequence of monic orthogonal polynomials $(\mathscr{P}_n(\,\cdot\,,\rho_k))_{n\in\mathbb{N}_0}$. Then,*

1. *applying J_n to D_n^{cheap} yields*

$$\mathfrak{D}_n^{\text{cl}}(\xi,\eta) := \frac{1}{n!} J_n(D_n^{\text{cheap}}(\xi,\,\cdot\,),\eta) = \mathbf{1}_{\{\xi(E)=n\}} \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!} (\eta_x)_{\xi_x}, \quad n \in \mathbb{N}_0, \xi,\eta \in \mathbf{N}_{<\infty};$$

2. *applying I_n to D_n^{cheap} yields*

$$\mathfrak{D}_n^{\text{ort}}(\xi,\eta) := \frac{1}{n!} I_n(D_n^{\text{cheap}}(\xi,\,\cdot\,),\eta) = \mathbf{1}_{\{\xi(E)=n\}} \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!} \mathscr{P}_{\xi_x}(\eta_x,\rho_x), \quad n \in \mathbb{N}_0, \xi,\eta \in \mathbf{N}_{<\infty}.$$

As a consequence, Theorem 8.2.5 and Theorem 8.2.15 yield that $\mathfrak{D}_n^{\text{cl}}$ and $\mathfrak{D}_n^{\text{ort}}$ satisfy (3.2.1), i.e., they are self-duality functions for $(P_t)_{t\geq 0}$ for each $n \geq \mathbb{N}_0$. Moreover, summing over n in $\mathfrak{D}_n^{\text{cl}}$ and $\mathfrak{D}_n^{\text{ort}}$, we obtain the self-duality functions

$$\mathfrak{D}^{\text{cl}}(\xi,\eta) := \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!} (\eta_x)_{\xi_x}, \quad \xi,\eta \in \mathbf{N}_{<\infty}, \tag{8.3.2}$$

$$\mathfrak{D}^{\text{ort}}(\xi,\eta) := \prod_{x\in E} \frac{1}{\rho_x(\{\xi_x\})\xi_x!} \mathscr{P}_{\xi_x}(\eta_x,\rho_x), \quad \xi,\eta \in \mathbf{N}_{<\infty}. \tag{8.3.3}$$

Proof. Without loss of generality, let $E = \{1,\dots,N\}$ and fix $\xi \in \mathbb{N}_0^N$, $n \in \mathbb{N}$.

1. Note that

$$\mathbf{1}_{\{\xi=\delta_{x_1}+\ldots+\delta_{x_n}\}} = \mathbf{1}_{\{\xi(E)=n\}} \frac{n!}{\xi_1!\cdots\xi_N!} \mathbf{1}_{\{1\}}^{\otimes\xi_1} \widetilde{\otimes\cdots\otimes} \mathbf{1}_{\{N\}}^{\otimes\xi_N}(x_1,\dots,x_n), \quad x_1,\dots,x_n \in E \tag{8.3.4}$$

where $\mathbf{1}_{\{1\}}^{\otimes\xi_1} \widetilde{\otimes\cdots\otimes} \mathbf{1}_{\{N\}}^{\otimes\xi_N}$ denotes the symmetrization of $\mathbf{1}_{\{1\}}^{\otimes\xi_1} \otimes\cdots\otimes \mathbf{1}_{\{N\}}^{\otimes\xi_N}$. Hence, using (8.2.4), we obtain

$$\begin{aligned}\frac{1}{n!} J_n(D_n^{\text{cheap}}(\xi,\,\cdot\,),\eta) &= \frac{\mathbf{1}_{\{\xi(E)=n\}}}{\rho(\{\xi\})\xi_1!\cdots\xi_N!} \int \mathbf{1}_{\{1\}}^{\otimes\xi_1} \otimes\cdots\otimes \mathbf{1}_{\{N\}}^{\otimes\xi_N} \,\mathrm{d}\eta^{(n)} \\ &= \mathbf{1}_{\{\xi(E)=n\}} \prod_{x=1}^{N} \frac{1}{\rho_x(\{\xi_x\})\xi_x!} (\eta_x)_{\xi_x}.\end{aligned}$$

for each $\xi \in \mathbb{N}_0^N$.

2. Let $\mathbf{P}_n := \overline{\mathcal{P}_n} \cap \overline{\mathcal{P}_{n-1}}^{\perp}$. By the orthogonal decomposition

$$\mathbf{P}_n = \bigoplus_{d_1+\ldots+d_N=n} \text{span}\{\mathscr{P}_{d_1}(\,\cdot\,,\rho_1) \otimes\cdots\otimes \mathscr{P}_{d_N}(\,\cdot\,,\rho_N)\}$$

we obtain that the projection of $\mathbb{N}_0^N \ni \eta \mapsto \int \mathbf{1}_{\{1\}}^{\otimes\xi_1} \cdots \mathbf{1}_{\{N\}}^{\otimes\xi_N} \,\mathrm{d}\eta^{\otimes n} = \eta_1^{\xi_1}\cdots\eta_N^{\xi_N}$ onto $\mathbf{P}_n$ is equal to $\eta \mapsto \mathscr{P}_{\xi_1}(\eta_1,\rho_1)\cdots\mathscr{P}_{\xi_N}(\eta_1,\rho_N)$. Therefore, using (8.3.4)

$$\frac{1}{n!} I_n(D_n^{\text{cheap}}(\xi,\,\cdot\,),\eta) = \frac{\mathbf{1}_{\{\xi(E)=n\}}}{\rho(\{\xi\})\xi_1!\cdots\xi_N!} I_n(\mathbf{1}_{\{1\}}^{\otimes\xi_1} \otimes\cdots\otimes \mathbf{1}_{\{N\}}^{\otimes\xi_N},\eta)$$

$$= \mathbf{1}_{\{\xi(E)=n\}} \prod_{x=1}^{N} \frac{1}{\rho_x(\{\xi_x\})\xi_x!} \mathscr{P}_{\xi_x}(\eta_x, \rho_x)$$

for each $\eta \in \mathbb{N}_0^N$.

□

We consider three prominent examples of consistent and conservative Markov processes on $\mathbb{N}_0^E$. For a characterization of consistent particle system on countable E we refer to [37, Theorem 3.3]. Let $|E| \geq 2$, $c = \{c_{\{x,y\}}, x, y \in E\}$ be a set of symmetric and non-negative conductances, such that (E, c) is connected and let $(\alpha_y)_{y\in E} \subset \mathbb{N}$. Then, for $\sigma \in \{-1, 0, 1\}$, the Markov process with infinitesimal generator acting on functions $f : \mathbf{N}_{<\infty} \to \mathbb{R}$ as

$$Lf(\eta) = \sum_{x,y\in E} c_{\{x,y\}} \left(f(\eta - \delta_y + \delta_x) - f(\eta)\right)(\alpha_y + \sigma\eta(\{y\}))\eta(\{x\}), \qquad \eta \in \mathbf{N}_{<\infty}$$

is a consistent and conservative process. In particular, for $\sigma = -1$, we obtain the inhomogeneous partial exclusion process (SEP) (see, e.g., [74, Eq. (1.3)]), for $\sigma = 0$ a system of independent random walkers (IRW) and for $\sigma = 1$ the inhomogeneous inclusion process SIP (see, e.g., [**?** , Eq. (2.2)]).

By a simple detailed balance computation one can show that, for those processes, there exists a one parameter family $\{\rho_\theta,\ \theta \in \Theta\}$ with $\Theta = (0, 1]$ for $\sigma = -1$ and $\Theta = (0, \infty)$ for $\sigma \in \{0, 1\}$ of reversible measures, namely (cf. [**?** , Eq. (3.1)]) $\rho_\theta := \bigotimes_{x\in E} \rho_{x,\theta}$ with

$$\rho_{x,\theta} = \begin{cases} \text{Binomial}(\alpha_x, \theta) & \text{if } \sigma = -1 \\ \text{Poisson}(\alpha_x\theta) & \text{if } \sigma = 0 \\ \text{Negative-Binomial}\left(\alpha_x, \frac{\theta}{1+\theta}\right) & \text{if } \sigma = 1\ . \end{cases}$$

Using that $\rho_{x,\theta}(\{n\}) = \frac{w_x(n)}{z_{x,\theta}} \left(\frac{\theta}{1+\sigma\theta}\right)^n \frac{1}{n!}$ where

$$w_x(n) := \begin{cases} (\alpha_x)_n & \text{if } \sigma = -1 \\ \alpha_x^n & \text{if } \sigma = 0 \\ (\alpha_x)^{(n)} & \text{if } \sigma = 1 \end{cases} \quad \text{and } z_{x,\theta} := \begin{cases} (1-\theta)^{-\alpha_x} & \text{if } \sigma = -1 \\ e^{\alpha_x\theta} & \text{if } \sigma = 0 \\ (1+\theta)^{\alpha_x} & \text{if } \sigma = 1 \end{cases}$$

in (8.3.2) we obtain

$$\mathfrak{D}^{\text{cl}}(\xi, \eta) = \left(\frac{\theta}{1+\sigma\theta}\right)^{-\xi(E)} \left(\prod_{x\in E} z_{x,\theta}\right) \prod_{x\in E} \frac{(\eta_x)_{\xi_x}}{w_x(\xi_x)}$$

which are the classical duality functions for $(\eta_t)_{t\geq 0}$ (see, e.g., [76, Eq. (2.16)]). Notice that, due to Assumption 8.2.1 (2), the term $\left(\frac{\theta}{1+\sigma\theta}\right)^{-\xi(E)} \left(\prod_{k\in E} z_{k,\theta}\right)$ is constant in time and, thus, it does not play any role in the duality relation.

For these systems, the self-dualities provided by (8.3.3) coincide (up to a multiplicative constant depending on the total number of particles which is a conserved quantity) to the orthogonal dualities studied in [146], [78] and [76] which are given by product of Charlier polynomials for $\sigma = 0$, products of Meixner polynomials for $\sigma = -1$ and products of Krawtchouk polynomials (see, e.g., [108, Eq. (9.11.1)]) for $\sigma = -1$. Indeed, considering, for instance, the system of independent random walks, the self-duality function of (8.3.3) turns into

$$\begin{aligned} \mathfrak{D}^{\text{ort}}(\xi, \eta) &= \prod_{k\in E} \frac{1}{\rho_k(\{\xi_k\})\xi_k!} \mathscr{C}_{\xi_k}(\eta_k, \alpha_k) \\ &= \prod_{k\in E} \frac{1}{e^{-\alpha_k}\alpha_k^{\xi_k}} (-\alpha_k)^{\xi_k} {}_2F_0\left(\left.\begin{matrix} -\xi_k, -\eta_k \\ - \end{matrix}\right| -\frac{1}{\alpha_k}\right) \\ &= e^{\alpha(E)}(-1)^{\xi(E)} \prod_{k\in E} {}_2F_0\left(\left.\begin{matrix} -\xi_k, -\eta_k \\ - \end{matrix}\right| -\frac{1}{\alpha_k}\right) \end{aligned}$$

coinciding with the orthogonal self-dualities given in literature mentioned above. The same holds also for the exclusion and the inclusion process.

8.3.2 Independent Markov Processes

Every system of independent Markov processes (e.g. the free Kawasaki dynamics [110], independent Brownian motions) is consistent and conservative. For independent particles, our theorems results allow us to recover known results on intertwining with Lenard's K-transform and multiple stochastic integrals, see [110, 158] and the references therein. Our contribution is the proof that these intertwining relations correspond exactly to classical and orthogonal dualities for independent random walkers on lattices from [47, Proposition 2.9.4] and [78, Theorem 4].

Let $(p_t)_{t\geq 0}$ be a Markov transition function on $(E, \mathcal{E})$. The transition function for n independent labelled particles with one-particle evolution governed by $(p_t)_{t\geq 0}$ has transition function $p_t^{\otimes n}$ uniquely determined by

$$p_t^{\otimes n}(x_1, \dots, x_n; A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} p_t(x_i; A_i) \quad x_1, \dots, x_n \in E,\ A_1, \dots, A_n \in \mathcal{E}.$$

The family of transition functions $(p_t^{\otimes n})_{t\geq 0}$, $n \in \mathbb{N}$ is strongly consistent and therefore the associated conservative transition function $(P_t)_{t\geq 0}$ (see (8.2.2)) is consistent.

Hence, Theorem 8.2.5 applied to the process $(\eta_t)_{t\geq 0}$ with transition function $(P_t)_{t\geq 0}$ yields the self-intertwining relation $P_t J_n(f_n, \cdot)(\eta) = J_n(p_t^{[n]} f_n, \eta)$ or more concretely,

$$\mathbb{E}_\eta \Big[\int f_n \mathrm{d}\eta_t^{(n)} \Big] = \int (p_t^{\otimes n} f_n) \mathrm{d}\eta.$$

The relation holds true for all $t \geq 0$, all initial values $\eta \in \mathbf{N}_{<\infty}$, and all $f_n \in \mathcal{F}_n$. As noted in (8.2.6), it implies that Lenard's K-transform and the semigroup $(P_t)_{t\geq 0}$ commute. Hence, for free Kawasaki dynamics, we recover a relation from [110, Section 3.2].

If we find a σ-finite reversible measure λ for the one-particle dynamics $(p_t)_{t\geq 0}$, then the distribution of a Poisson process with intensity measure λ, denoted by π_λ, is reversible for $(\eta_t)_{t\geq 0}$. This property is a version of Doob's Theorem (cf. [47, Theorem 2.9.5]) and of the displacement theorem (cf. [105]). Moreover, $\lambda^{\otimes n}$ is reversible for $(p_t^{\otimes n})_{t\geq 0}$. For finite λ, the assumptions of Theorem 8.2.15 are satisfied and the self-intertwining relation $P_t I_n(f_n, \cdot)(\eta) = I_n(p_t^{\otimes n} f_n, \eta)$ holds for π_λ-almost all $\eta \in \mathbf{N}_{<\infty}$, all $f_n \in \mathcal{F}_n$ and all $t \geq 0$.

The construction of the generalized orthogonal polynomial with respect to the Poisson point process is standard and it is well-known that the orthogonality relation

$$\int I_n(f_n, \cdot) I_m(g_m, \cdot)\, \mathrm{d}\pi_\lambda = \mathbf{1}_{\{n=m\}} n! \int f_n g_m \mathrm{d}\lambda^{\otimes n}$$

holds for bounded $f_n \in \mathcal{F}_n, g_m \in \mathcal{F}_m$, $n, m \in \mathbb{N}_0$, with $\int f_0 g_0 \mathrm{d}\lambda^{\otimes 0} := f_0 g_0$, and they generalize the Charlier polynomial in the following sense (see, e.g., [119, Eq. (3.3)],

$$I_n(\mathbf{1}_{B_1}^{\otimes d_1} \otimes \cdots \otimes \mathbf{1}_{B_N}^{\otimes d_N}, \eta) = \prod_{k=1}^{N} \mathscr{C}_{d_k}(\eta(B_k); \lambda(B_k)) \tag{8.3.5}$$

for π_λ-almost all $\eta \in \mathbf{N}_{<\infty}$, $d_1 + \ldots + d_N = n$ and all pairwise disjoint $B_1, \dots, B_N \in \mathcal{E}$. Yet another viewpoint is that $I_n(f_n, \cdot)$ are multiple stochastic integrals with respect to the compensated Poisson measure $\eta - \lambda$. The reader interested in the relation between the generalized orthogonal polynomials $I_n(f_n, \cdot)$ and multiple Wiener-Itô integrals, chaos decompositions, and Fock spaces is referred to [118], [136] and [128].

In the language of multiple stochastic integrals, the intertwining relation from Theorem 8.2.15 says that applying the semigroup to the n-fold integral of f_n is the same as the n-fold integral of $p_t^{\otimes n} f_n$.

8.3.3 The Howitt–Warren Flow and a Consistent Family of Sticky Brownian Motions

As noted in Section 8.2.1, every strongly consistent family $(p_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$, of transition functions induces a consistent semigroup $(P_t)_{t\geq 0}$. Strongly consistent families have been studied in the context of stochastic flows: Le Jan and Raimond [121] investigate a one-to-one correspondence between strong consistency families and stochastic flows of kernels.

A particular and well studied case is the Howitt–Warren flow. It is a stochastic flow of kernels whose n point motions is given by a family of n interacting Brownian motions that interact, roughly, by sticking together for a while when they meet. The interacting diffusions can be constructed, for example, as solutions to a martingale problem [97]. Theorem 8.2.5 applies to the semigroup $(P_t)_{t\geq 0}$ induced by the strongly consistent family of transitions functions $(p_t^{[n]})_{t\geq 0}$, $n \in \mathbb{N}$, for n sticky Brownian motions.

The dynamics of sticky Brownian motion depends on a choice of parameters and includes diffusions with a drift. For zero drift and a special choice of parameters, Brockington and Warren [31] prove, using a Bethe ansatz, an explicit formula for transition probabilities and the reversibility of the n-point motions with respect to some explicit measure $m_\theta^{(n)}$. They work on the Weyl chambers $\bar{W}^n := \{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\}$ and show that the transition function is of the form $p_t^{[n]}(x, \mathrm{d}y) = u_t^{(n)}(x, y) m_\theta^{(n)}(\mathrm{d}y)$ for some symmetric function $u_t^{(n)}(x, y) = u_t^{(n)}(y, x)$. With this the self-intertwining relation from Theorem 8.2.5 can be rewritten as

$$\mathbb{E}_\eta \left[\int_{\bar{W}^r} f_r(y) \eta^{(r)}(\mathrm{d}y) \right] = \int_{\bar{W}^r} f_r(y) \left[\int_{\bar{W}^r} u_t^{(r)}(y, x) \eta^{(r)}(\mathrm{d}x) \right] m_\theta^{(r)}(\mathrm{d}y). \tag{8.3.6}$$

Thus we obtain an identity analogous to (3.2.7) and (8.2.8).

As the reversible measures $m_\theta^{(n)}$ from [31] have infinite total mass, it is not possible to construct from them a reversible measure supported on configurations of finitely many particles and Theorem 8.2.15 on orthogonal intertwining relations is not applicable. We leave the study of the orthogonal self-intertwining relation for the system of sticky Brownian motions for future research.

For other examples of strongly consistent families, beyond sticky Brownian motions, we refer to [121] and [149].

8.4 Generalized Symmetric Inclusion Process

As an example of interacting system of particles jumping on a general Borel space $(E, \mathcal{E})$, we introduce here a new process which is a natural extension of the SIP. Coherently with the setting of this chapter, we consider the finite particle case only. Extension to the infinite particle case is not part of the scope of the present work and it is left for future research.

The SIP on countable sets was introduced in [83] as a dual process of a model of heat conduction, which shares some features with the well-studied KMP model (see [107]). The process also appears, with a different interpretation, in mathematical population genetics. Indeed, in [33, Section 5], it is proved that the generator of the SIP coincide with the generator of an instance of the Moran model, which is dual to the Wright-Fisher diffusion process. Moreover, the scaling limit of the Moran model is the celebrated Fleming-Viot superprocess (see [60] and references therein) which has been studied using duality as well.

8.4.1 Introducing the gSIP

Let α be a finite, non-zero measure on E and $c : E \times E \to \mathbb{R}_+$ a bounded symmetric function with $c(x, x) = 0$ for all $x \in E$. The generalized symmetric inclusion process (gSIP) is the process with formal generator

$$Lf(\eta) = \iint (f(\eta - \delta_x + \delta_y) - f(\eta)) c(x, y)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x). \tag{8.4.1}$$

It is a continuous-time jump process with jump kernel

$$Q(\eta, B) = \iint \mathbf{1}_B(\eta - \delta_x + \delta_y) c(x, y)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x) \tag{8.4.2}$$

and it can be viewed, when $E = \mathbb{R}^d$, as a particular case of a Kawasaki dynamics (see, e.g., [109]). Notice that $Q(\eta, E) < \infty$ for finite measures α and finite configurations $\eta \in \mathbf{N}_{<\infty}$. Accordingly the process $(\eta_t)_{t\geq 0}$ can be constructed with the usual jump-hold construction and the semigroup $(P_t)_{t\geq 0}$ is the minimal solution of the backward Kolmogorov equation, see Feller [69].

The process is non-explosive because the particle number is conserved and $\sup\{Q(\eta, E) : \eta(E) = n\} < \infty$ for every particle number $n \in \mathbb{N}_0$. Therefore the minimal solution $(P_t)_{t\geq 0}$ is a Markov semigroup ($P_t(\eta, E) = 1$ rather than ≤ 1) and it is in fact the unique solution of the backward Kolmogorov equation.

Remark 8.4.1. *1. As we will see later, the gSIP $(\eta_t)_{t\geq 0}$ has the following connection to the well-known SIP of particles hopping on a finite set. Let $A_1, \ldots, A_m \in \mathcal{E}$, $m \in \mathbb{N}$ be a partition of E and let c be constant on $A_i \times A_j$ and equal to d_{ij} for each i, j. Then, the process $(\eta_t(A_1), \ldots, \eta_t(A_m))$ starting at $\eta_0 \in \mathbf{N}_{<\infty}$ behaves like a SIP on the finite set $\{1, \ldots, m\}$ with initial configuration $(\eta_0(A_1), \ldots, \eta_0(A_m))$ and transition rates d_{ij}.*

2. *Notice that a direct generalization of the Exclusion process analogous to the gSIP, would not be meaningful in general, because the probability to jump on already occupied points is zero whenever the jumping kernel of the single particle is not atomic. Thus an exclusion rule miming the one in the discrete setting cannot be modelled in the continuum.*

3. *The dynamics can be described informally as follows. Starting from an initial configuration $\eta_0 = \eta$ with $n = \eta(E)$ points $x_1, \dots, x_n$, set*

$$q_{i0} := \int c(x_i, y)\alpha(\mathrm{d}y), \quad q_{ij} := c(x_i, x_j), \quad z_i := \sum_{j=0}^{n} q_{ij}, \quad z := \sum_{i=1}^{n} z_i$$

and do the following:

(a) *Wait for an exponential time with parameter $Q(\eta, E) = z$.*

(b) *When time is up, choose one out of the n points $x_1, \dots, x_n$ of η. The point x_i is chosen with probability z_i/z. Move the chosen point $x = x_i$ to a new location y:*

- *With probability q_{ij}/z_i, the new location y is equal to $y = x_j$.*
- *With probability q_{i0}/z_i, the new location y is chosen according to the probability measure $\alpha(E)^{-1}\alpha(\mathrm{d}y)$.*

Then, repeat. The resulting process $(\eta_t)_{t\geq 0}$ and the associated semigroup $(P_t)_{t\geq 0}$, given by the minimal solution to the backward Kolmogorov equation, is in fact the unique solution.

8.4.2 Reversibility and Intertwiners for the gSIP

Fix $p \in (0, 1)$. A *Pascal point process* with parameters p and α is a point process with the following properties:

1. If $B_1, \dots, B_m \in \mathcal{E}$ are disjoint then $\zeta(B_1), \dots, \zeta(B_m)$ are independent.
2. For every $B \in \mathcal{E}$, the distribution of $\zeta(B)$ is given by a negative binomial law:

$$\mathbb{P}(\zeta(B) = k) = (1 - p)^{\alpha(B)}\alpha(B)(\alpha(B) + 1)\cdots(\alpha(B) + k - 1)\,\frac{p^k}{k!}, \quad k \in \mathbb{N}_0.$$

For $k = 0$, the equation is to be read as $\mathbb{P}(\zeta(B) = 0) = (1 - p)^{\alpha(B)}$.

The *Pascal distribution* is the distribution of a Pascal point process and it is a direct generalization of the product measure of negative binomial distributions that is reversible for SIP. Indeed, the measure $\otimes_{x\in E}$Negative-Binomial(α_x, p), $\alpha_x > 0$ can be seen as a Pascal distribution. Property (1) follows immediately whereas (2) follows from the fact that if $n_x \sim$ Negative-Binomial(α_x, p) and $n_y \sim$ Negative-Binomial(α_y, p), with n_x and n_y independent for $x \neq y \in E$, then $n_x + n_y \sim$ Negative-Binomial$(\alpha_x + \alpha_y, p)$.

Theorem 8.4.2. *Let α be a finite measure on E. Then*

1. *the generalized symmetric inclusion process with formal generator* (8.4.1) *is a consistent Markov process and thus the intertwining relation* (8.2.5) *with generalized falling factorials holds;*
2. *for every $p \in (0, 1)$, the Pascal distribution ρ with parameters α and p is reversible and thus, the intertwining relation* (8.2.12) *with generalized orthogonal polynomials holds.*

Notice that we have a family of reversible measures, indexed by $p \in (0, 1)$, moreover the reversible Pascal distributions do not depend on the function $c(x, y)$ in the dynamics.

Theorem 8.4.2(ii) is complemented by a concrete relation of the abstract generalized orthogonal polynomials $I_n(f_n, \cdot)$ with the univariate Meixner polynomials defined in Section 8.3.1. Generalized orthogonal polynomials of Meixner's type have been studied intensely in the context of non-Gaussian white noise [16, 17]. Connections with quantum probability and representations of *-Lie algebras and current algebras are investigated in [2, 1].

The following proposition is a variant of Lemma 3.1 in [129]. We give a self-contained proof in Section 8.4.4 that does not use the machinery of Jacobi fields or distribution theory.

Proposition 8.4.3. *The intertwiner I_n is related to the Meixner polynomials via*

$$I_n(\mathbf{1}_{B_1}^{\otimes d_1} \otimes \cdots \otimes \mathbf{1}_{B_N}^{\otimes d_N}, \eta) = \prod_{k=1}^{N} \mathscr{M}_{d_k}\left(\eta(B_k); \alpha(B_k); p\right). \tag{8.4.3}$$

for ρ-almost all $\eta \in \mathbf{N}_{<\infty}$ and all pairwise disjoint $B_1, \dots, B_N \in \mathcal{E}$, $n \in \mathbb{N}_0$, $d_1, \dots, d_N$, $N \in \mathbb{N}$ with $d_1 + \dots + d_N = n$.

We define a measure λ_n on E^n that replaces the product measure $\lambda^{\otimes n}$ in the Poisson-Charlier case. Let Σ_n be the collection of set partitions of $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$ and $g : E^n \to \mathbb{R}$, let $|\sigma|$ be the number of blocks of the partition σ and $g_\sigma : E^{|\sigma|} \to \mathbb{R}$ the function obtained by identifying, in order of occurrence, those arguments that belong to the same block of σ. Define

$$\lambda_n(B) = \sum_{\sigma \in \Sigma_n} \left(\prod_{A \in \sigma} (|A| - 1)! \right) \int (\mathbf{1}_B)_\sigma \, \mathrm{d}\alpha^{\otimes |\sigma|}, \quad B \in \mathcal{E}^n. \tag{8.4.4}$$

For example $\lambda_1 = \alpha$ and

$$\lambda_2(B) = \iint \mathbf{1}_B(x, y) \alpha(\mathrm{d}x) \alpha(\mathrm{d}y) + \int \mathbf{1}_B(x, x) \alpha(\mathrm{d}x)$$

for all $B \in \mathcal{E}^2$. Further set $\int f_0 g_0 \, \mathrm{d}\lambda_0 := f_0 g_0$ for $f_0, g_0 \in \mathcal{F}_0 = \mathbb{R}$.

The following proposition generalizes the univariate orthogonality relation (8.3.1). It is similar to Corollary 5.2 in [129], we provide a self-contained proof in Section 8.4.4.

Proposition 8.4.4. *The following orthogonality relations holds*

$$\int I_n(f_n, \cdot) I_m(g_m, \cdot) \, \mathrm{d}\rho = \mathbf{1}_{\{n=m\}} \frac{p^n n!}{(1-p)^{2n}} \int f_n g_m \, \mathrm{d}\lambda_n \tag{8.4.5}$$

for $f_n \in \mathcal{F}_n$, $g_m \in \mathcal{F}_m$, $n, m \in \mathbb{N}_0$.

Remark 8.4.5 (Sequential construction of λ_n)**.** *For $n \in \mathbb{N}$, define a kernel $k_{n,n+1} : E^n \times \mathcal{E}^{n+1} \to \mathbb{R}_+$ by*

$$k_{n,n+1}(x_1, \dots, x_n; B) = \int \mathbf{1}_B(x_1, \dots, x_n, y) \alpha(\mathrm{d}y) + \sum_{i=1}^n \mathbf{1}_B(x_1, \dots, x_n, x_i).$$

Then $\lambda_{n+1} = \lambda_n k_{n,n+1}$ meaning that $\lambda_{n+1}(B) = \int_{E^n} \lambda_n(\mathrm{d}x) k_{n,n+1}(x, B)$ for all $B \in \mathcal{E}^{n+1}$. Thus λ_n is formed by adding points one by one; at each step, a new point either joins a pile of existing points or is placed at a new location y. This relation on the one hand connects to the very definition of the dynamics of the gSIP and on the other hand is reminiscent of the Chinese restaurant process used in sequential constructions for random partitions [144, Chapter 3]. Notice that, upon normalization by the total mass of λ_n, (8.4.4) *gives rise to a probability measure on the set Σ_n of partitions, related to the Ewens sampling formula.*

8.4.3 Proof of Theorem 8.4.2

Here we prove Theorem 8.4.2. In addition, we remind the reader of an explicit description of the Pascal process as a compound Poisson process.

Consistency (Proof of Theorem 8.4.2(i)) We start by proving that the gSIP is consistent (see Definition 8.2.1). Since we consider the finite particle case only, it is enough to check the commutation property in Definition 8.2.1 for the generator instead of the semigroup, i.e., $\mathcal{A}Lf(\eta) = L\mathcal{A}f(\eta)$ for all $f \in \mathcal{G}$ and $\eta \in \mathbf{N}_{<\infty}$. Indeed, decompose the generator as $L = L_1 + L_2$ with

$$L_1 f(\eta) := \iint \Big(f(\eta - \delta_x + \delta_y) - f(\eta) \Big) c(x, y) \alpha(\mathrm{d}y) \eta(\mathrm{d}x)$$

and

$$L_2 f(\eta) := \iint \Big(f(\eta - \delta_x + \delta_y) - f(\eta) \Big) c(x, y) \eta(\mathrm{d}y) \eta(\mathrm{d}x).$$

Notice that L_1 is the generator of a system of independent Markov processes, namely, independent random walkers with transition kernel given by $c(x, y)\alpha(\mathrm{d}y)$. Thus, it is straightforward to check that $\mathcal{A}L_1 f(\eta) = L_1 \mathcal{A}f(\eta)$. It remains to show that

$$\mathcal{A}L_2 f(\eta) = L_2 \mathcal{A}f(\eta). \tag{8.4.6}$$

First, we compute

$$L_2 \mathcal{A}f(\eta)$$

$$= \iiint f(\eta - \delta_x + \delta_y - \delta_z)\eta(\mathrm{d}z)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x) - \iint f(\eta - 2\delta_x + \delta_y)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x)$$
$$+ \iint f(\eta - \delta_x)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x) - \iiint f(\eta - \delta_z)\eta(\mathrm{d}z)c(x,y)\eta(\mathrm{d}y)\eta(\mathrm{d}x)$$

second,

$$\begin{aligned}\mathcal{A}L_2 f(\eta) &= \iiint \left(f(\eta - \delta_z - \delta_x + \delta_y) - f(\eta - \delta_z)\right) c(x,y)(\eta - \delta_z)(\mathrm{d}y)(\eta - \delta_z)(\mathrm{d}x)\eta(\mathrm{d}z) \\ &= L_2\mathcal{A}f(\eta) \\ &- \iint \left(f(\eta - \delta_x) - f(\eta - \delta_z)\right) c(x,z)\eta(\mathrm{d}x)\eta(\mathrm{d}z) + \int \left(f(\eta - \delta_z) - f(\eta - \delta_z)\right) c(z,z)\eta(\mathrm{d}z).\end{aligned}$$

Because the last two integrals above are both 0, we obtain (8.4.6) and the proof is concluded. □

Explicit Representation of the Pascal Process. The Pascal process, also called negative binomial process, is a well-known point process (cf. [154], [112]). For the reader's convenience we recall the construction of that process.

Fix $p \in (0,1)$ and a finite measure α. Note that the Pascal point process has the structure of a measure-valued Lévy process, since $\zeta(A_1), \ldots, \zeta(A_n)$ are independent for pairwise disjoint $A_1, \ldots, A_n \in \mathcal{E}$ and the distribution of $\zeta(A)$ only depends on $\alpha(A)$, $A \in \mathcal{E}$. For more details, see [104, 105, 100],

More precisely, the Pascal process can be constructed in the following way, compound Poisson process (see [120, Chapter 15]), i.e.,

$$\zeta(A) := \int_{A\times\mathbb{N}} y\,\xi(\mathrm{d}(x,y)), A \in \mathcal{E}$$

where ξ is a Poisson point process on $E \times \mathbb{N}$ with intensity measure $\lambda := \alpha \otimes \nu$ where the Lévy measure is given by $\nu := \sum_{n=1}^{\infty} \frac{p^n}{n}\delta_n$. It can readily checked that the Laplace functional is given by

$$L_\zeta(f) := \mathbb{E}\left[e^{-\int f\,\mathrm{d}\zeta}\right] = \exp\left(\int \left(e^{-yf(x)} - 1\right)\lambda(\mathrm{d}(x,y))\right) = \exp\left(-\int \Phi(f(x))\alpha(\mathrm{d}x)\right) \tag{8.4.7}$$

with $\Phi(y) := \log\left(\frac{1-pe^{-y}}{1-p}\right)$, $y \geq 0$. Equation (8.4.7) implies for $A \in \mathcal{E}$

$$\mathbb{E}\left(e^{-\zeta(A)s}\right) = \exp\left(-\int \Phi(s\mathbf{1}_A(x))\alpha(\mathrm{d}x)\right) = \exp\left(-\alpha(A)\Phi(s)\right) = \left(\frac{1-p}{1-pe^{-s}}\right)^{\alpha(A)}, \quad s > 0$$

which is the Laplace transform of a negative binomial distributed random variable with parameters $\alpha(A)$ and p. Moreover, (8.4.7) implies the independence of $\zeta(A_1), \ldots, \zeta(A_n)$ immediately.

Reversible Measure (Proof of Theorem 8.4.2(ii)). Let $Q_c = Q$ be the jump kernel from (8.4.2). It is enough to check the detailed balance relation

$$\rho \otimes Q_c(\mathscr{A} \times \mathscr{B}) = \rho \otimes Q_c(\mathscr{B} \times \mathscr{A}) \quad \mathscr{A}, \mathscr{B} \in \mathcal{N}_{<\infty}, \tag{8.4.8}$$

where

$$\rho \otimes Q_c(\mathscr{A} \times \mathscr{B}) = \int_{\mathscr{A}} \rho(\mathrm{d}\eta) \iint \mathbf{1}_{\mathscr{B}}(\eta - \delta_x + \delta_y)c(x,y)(\alpha + \eta)(\mathrm{d}y)\eta(\mathrm{d}x).$$

The proof idea is that for particularly simple choices of $c(x,y)$ and $\mathscr{A}, \mathscr{B}$, the relation (8.4.8) boils down to a detailed balance relation for a discrete inclusion process.

We start with some preliminary observations. First, it is enough to prove (8.4.8) for functions c of the form

$$c(x,y) = \sum_{i,j=1}^{r} d_{ij}\mathbf{1}_{C_i}(x)\mathbf{1}_{C_j}(y) \tag{8.4.9}$$

with $r \in \mathbb{N}$, symmetric non-negative weights $d_{ij} = d_{ji} \geq 0$, and sets $A_1, \ldots, A_r \in \mathcal{E}$. Indeed, the set $\mathcal{M}$ of non-negative measurable functions $f : E \times E \to \mathbb{R}_+$ for which the symmetrized function $c(x,y) := \frac{1}{2}(f(x,y) +$

$f(y,x))$ satisfies (8.4.8) is closed under pointwise monotone limits. If (8.4.8) holds true for all functions c of the form (8.4.9), then $\mathcal{M}$ contains all indicators $\mathbf{1}_{A\times B}$, $A, B \in \mathcal{E}$. The monotone class theorem then implies that $\mathcal{M}$ contains all bounded non-negative measurable functions.

Second, by the π-λ theorem, it is enough to check (8.4.8) for sets of the form

$$\mathscr{A} = \bigcap_{j=1}^{k} \{\eta \in \mathbf{N}_{<\infty} : \eta(A_j) = m_j\}, \quad \mathscr{B} = \bigcap_{j=1}^{\ell} \{\eta \in \mathbf{N}_{<\infty} : \eta(B_j) = n_j\} \tag{8.4.10}$$

with $k, \ell \in \mathbb{N}$, $A_i, B_j \in \mathcal{E}$, and $m_i, n_j \in \mathbb{N}_0$.

Third, for the relation (8.4.8) to hold true for all $c(x,y)$ of the form (8.4.9) and all sets $\mathscr{A}$, $\mathscr{B}$ of the form (8.4.10), it is enough to consider the situation where $r = k = \ell$, $A_i = B_i = C_i$, and the sets $A_1, \dots, A_r$ are pairwise disjoint, as the general case follows by taking linear combinations.

In the situation of the last paragraph, we compute, for $\eta \in \mathscr{A}$, and assuming all diagonal elements d_{ii} vanish,

$$\begin{aligned} Q_c(\eta, \mathscr{B}) &= \sum_{i,j=1}^{r} d_{ij} \int_{A_i} \left(\int_{A_j} \mathbf{1}_{\mathscr{B}}(\eta - \delta_x + \delta_y)(\alpha + \eta)(\mathrm{d}y) \right) \eta(\mathrm{d}x) \\ &= \sum_{i,j=1}^{r} d_{ij} \eta(A_i)(\alpha(A_j) + \eta(A_j)) \mathbf{1}_{\{\eta(A_i)-1=n_i\}} \mathbf{1}_{\{\eta(A_j)+1=n_j\}} \prod_{s\notin\{i,j\}} \mathbf{1}_{\{\eta(A_s)=n_s\}} \\ &= \sum_{\substack{1\le i,j\le r:\\ i\neq j}} d_{ij} m_i (\alpha(A_j) + m_j) \delta_{m_i-1,n_i} \delta_{m_j+1,n_j} \prod_{s\notin\{i,j\}} \delta_{m_s,n_s}, \end{aligned} \tag{8.4.11}$$

and we recognize the transition rates of the SIP with state space $\mathbb{N}_0^r$. For non-zero diagonal elements, we need to add

$$\sum_{i=1}^{r} d_{ii} m_i (\alpha(A_i) + m_i) \prod_{s=1}^{r} \delta_{m_i,n_i}. \tag{8.4.12}$$

We denote the sum of (8.4.11) and (8.4.12) $q(m,n)$. Notice that for non-zero d_{ii} we may have $q(m,m) > 0$.

Abbreviate $\alpha(A_j) =: \alpha_j$. For $j \in \{1, \dots, r\}$ and $m_j \in \mathbb{N}_0$, set

$$\pi_j(m_j) := \rho(\{\eta : \eta(A_j) = m_j\}) = (1-p)^{\alpha_j} \alpha_j(\alpha_j + 1) \cdots (\alpha_j + m_j - 1) \frac{p^{m_j}}{m_j!}.$$

Further set $\pi(m) = \pi_1(m_1) \cdots \pi_r(m_r)$. Then

$$\rho \otimes Q_c(\mathscr{A}, \mathscr{B}) = \pi(m) q(m,n).$$

A similar computation shows $\rho \otimes Q_c(\mathscr{B}, \mathscr{A}) = \pi(n) q(n,m)$. The symmetry relation (8.4.8) now reads $\pi(m)q(m,n) = \pi(n)q(n,m)$ which is the detailed balance relation for the SIP. □

8.4.4 Properties of Generalized Meixner Polynomials. Proof of Propositions 8.4.3 and 8.4.4

Let $p \in (0,1)$. Note that the generating function of monic Meixner polynomials, given by (see, e.g., [108])

$$e_t(x,a) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{M}_n(x; a; p) = \left(\frac{1-p+t}{1-p+tp} \right)^x \left(\frac{1-p}{1-p+tp} \right)^a, \quad t, a > 0, x \in \mathbb{N}_0,$$

satisfies $e_t(x+y, a+b) = e_t(x,a) e_t(y,b)$ for each $t > 0$, $x, y \in \mathbb{N}_0$, $a, b > 0$. As a consequence, we get the convolution property (see, e.g., [5])

$$\mathscr{M}_n(x+y; a+b; p) = \sum_{k=0}^{n} \binom{n}{k} \mathscr{M}_k(x; a; p) \mathscr{M}_{n-k}(y; b; p). \tag{8.4.13}$$

Proof of Proposition 8.4.3. By the factorization property from Proposition 8.2.13 it is enough to show

$$I_1(\mathbf{1}_{A^d}, \eta) = \mathscr{M}_d(\eta(A); \alpha(A); p) \tag{8.4.14}$$

for all $d \in \mathbb{N}$ and $A \in \mathcal{E}$. As we have chosen our univariate Meixner polynomials $\mathscr{M}_d$ to have leading coefficient one, we know that $\mathscr{M}_d(\eta(A); \alpha(A); p)$ is equal to $\eta(A)^d$ plus some polynomial in $\eta(A)$ of degree $\leq d-1$. Therefore (8.4.14) follows once we know that the map $\eta \mapsto \mathscr{M}_d(\eta(A); \alpha(A); p)$ is orthogonal to the space $\mathcal{P}_{d-1}$. We shall see that this identity follows from the convolution property (8.4.13) and the complete independence.

We check first that $\eta \mapsto \mathscr{M}_d(\eta(A); \alpha(A); p)$ is orthogonal in $L^2(\rho)$ to all maps $\eta \mapsto \eta^{\otimes m}(C)$, for every $m \leq d-1$ and $C \in \mathcal{E}^m$ with $C \subset A^m$. When $C = A^m$, we are looking at two univariate polynomials in the variable $x = \eta(A)$ and the orthogonality relation follows from the orthogonality of the univariate Meixner polynomials $x \mapsto \mathscr{M}_d(\eta(A); \alpha(A); p)$ to the monomial $x \mapsto x^m$. The orthogonality to constant functions ($m = 0$) follows from univariate orthogonality as well.

Next consider the case $C = C_1^{d_1} \times \cdots \times C_N^{d_N}$ with $N \in \mathbb{N}$, $d_1, \ldots, d_N \in \mathbb{N}$ with $d_1 + \cdots + d_N \leq d-1$ and pairwise disjoint measurable sets $C_i \subset A$. Suppose first that $C_1 \cup \cdots \cup C_N = A$. We use the convolution property (8.4.13) and the complete independence of the Pascal point process to find

$$\int \mathscr{M}_d(\eta(A); \alpha(A); p)\eta^{\otimes m}(C)\rho(\mathrm{d}\eta) = \sum_{k_1+\cdots+k_N=m} \binom{m}{k_1, \ldots, k_N} \prod_{i=1}^N \int \mathscr{M}_{k_i}(\eta(C_i); \alpha(C_i); p)\eta^{\otimes d_i}(C_i)\rho(\mathrm{d}\eta). \quad (8.4.15)$$

In each summand, we must have $d_i < k_i$ for at least one $i \in \{1, \ldots, N\}$ and therefore by the orthogonality of univariate Meixner polynomials, at least one of the integrals on the right side above vanish. As a consequence,

$$\int \mathscr{M}_d(\eta(A); \alpha(A); p)\eta^{\otimes m}(C)\rho(\mathrm{d}\eta) = 0 \quad (8.4.16)$$

This holds true as well when each C_i is contained in A and $C_{N+1} := A \setminus (C_1 \cup \cdots \cup C_N)$ is non-empty. In that case we use a similar decomposition but now the sum on the right side of (8.4.15) is over $(k_1, \ldots, k_{N+1})$ and the product has an additional factor $\int \mathscr{M}_{k_{N+1}}(\eta(C_{N+1}); \alpha(C_{N+1}); p)\rho(\mathrm{d}\eta)$.

Every Cartesian product $C = D_1 \times \cdots \times D_m$ contained in A^m is a disjoint union of finitely many Cartesian products in which any two distinct factors are either distinct or equal. Therefore, by linearity, the orthogonality relation (8.4.16) extends to all such sets. The functional monotone class theorem yields the orthogonality of the generalized Meixner polynomial to all maps of the form $\eta \mapsto \eta^{\otimes m}(f_m)$ with bounded measurable $f_m : E^m \to \mathbb{R}$ supported in A^m and then, by linearity, the orthogonality to all linear combinations of such maps.

In the notation of Lemma 8.2.16 below, we have checked the orthogonality of $\mathscr{M}_d(\eta(A); \alpha(A); p)$ to $\mathcal{P}_{d-1}(A)$. Using complete independence and arguments similar to the proof of Lemma 8.2.16, we conclude that the Meixner polynomial is in fact orthogonal to $\mathcal{P}_{d-1}$. This completes the proof of the proposition. □

Proof of Proposition 8.4.4. The orthogonality of $I_n(f_n, \cdot)$ and $I_m(g_m, \cdot)$ for $m \neq n$ is an immediate consequence of the definition of generalized orthogonal polynomials, it does not use any properties of the Pascal distribution ρ. Thus we need only treat the case $m = n$.

Using linearity and the monotone class theorem as in the proof of Proposition 8.4.3, one finds that it suffices to show the orthogonality relation for functions $\tilde{f}_n, \tilde{g}_n$ that are symmetrized versions of indicator functions $f_n, g_n : E^n \to \mathbb{R}$ of the form

$$f_n = \mathbf{1}_{B_1}^{\otimes d_1} \otimes \cdots \otimes \mathbf{1}_{B_N}^{\otimes d_N}, \quad g_n = \mathbf{1}_{B_1}^{\otimes d'_1} \otimes \cdots \otimes \mathbf{1}_{B_N}^{\otimes d'_N}$$

with $B_1, \ldots, B_N \in \mathcal{E}$ disjoint, and $\sum_{i=1}^N d_i = \sum_{i=1}^N d'_i = n$. The symmetrization of f_n is given by

$$\tilde{f}_n(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

the symmetrization $\tilde{g}_n$ of g_n is defined in a similar way. Notice that $I_n(\tilde{f}_n, \eta) = I_n(f_n, \eta)$ and $I_n(\tilde{g}_n, \eta) = I_n(g_n, \eta)$ but in general $\int \tilde{f}_n \tilde{g}_n \mathrm{d}\lambda_n \neq \int f_n g_n \mathrm{d}\lambda_n$.

Proposition 8.4.3, the complete independence, and the orthogonality relation (8.3.1) for univariate Meixner polynomials yield

$$\int I_n(\tilde{f}_n, \eta)\, I_n(\tilde{g}_n, \eta)\, \rho(\mathrm{d}\eta) = \prod_{i=1}^N \mathbf{1}_{\{d_i = d'_i\}} \frac{d_i!\, p^{d_i}}{(1-p)^{2d_i}} \left(\alpha(B_i)\right)^{(d_i)}. \quad (8.4.17)$$

If $d_i \neq d_i'$ for at least one i, then the right side is zero, moreover $\tilde{f}_n\, \tilde{g}_n$ vanishes identically. Hence in that case

$$\int I_n(\tilde{f}_n, \eta)\, I_n(\tilde{g}_n, \eta)\, \rho(\mathrm{d}\eta) = 0 = \int \tilde{f}_n \tilde{g}_n \mathrm{d}\lambda_n.$$

and the required equality holds true.

If $d_i = d_i'$ for all i, then $f_n = g_n$ on E^n. By the definition of λ_n, we have

$$\int f_n^2\, \mathrm{d}\lambda_n = \lambda_n(B_1^{d_1} \times \cdots \times B_n^{d_n}) = \prod_{i=1}^{N} (\alpha(B_i))^{(d_i)}$$

hence (8.4.17) gives

$$\int (I_n(\tilde{f}_n, \eta))^2\, \rho(\mathrm{d}\eta) = \Big(\prod_{i=1}^{N} d_i!\Big) \int f_n^2 \mathrm{d}\lambda_n. \tag{8.4.18}$$

Next we check that the product of factorials on the right side disappears when f_n is replaced by the symmetrized function $\tilde{f}_n$. For $\sigma \in \mathfrak{S}_n$ and $x = (x_1, \dots, x_n) \in E^n$, let $x_\sigma := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Then, by the permutation invariance of the measure λ_n, we have

$$\int \tilde{f}_n^2\, \mathrm{d}\lambda_n = \frac{1}{n!^2} \sum_{\sigma,\tau \in \mathfrak{S}_n} \int f_n(x_\sigma) f_n(x_\tau) \lambda_n(\mathrm{d}x) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \int f_n(x_\pi) f_n(x) \lambda_n(\mathrm{d}x).$$

Because of the disjointness of the sets B_i, the product $f_n(x_\pi) f_n(x)$ vanishes unless π leaves the sets $\{1, \dots, d_1\}$, $\{d_1{+}1, \dots, d_1{+}d_2{-}1\}$ etc. invariant, and in the latter case $f_n(x_\pi) f_n(x) = f_n(x)^2$. The number of relevant permutations is equal to $d_1! \cdots d_N!$. As a consequence,

$$\int \tilde{f}_n^2\, \mathrm{d}\lambda_n = \frac{1}{n!} \Big(\prod_{i=1}^{N} d_i!\Big) \int f_n^2 \mathrm{d}\lambda_n.$$

By (8.4.18), we get

$$\int (I_n(\tilde{f}_n, \eta))^2 \rho(\mathrm{d}\eta) = \frac{n!\, p^n}{(1-p)^{2n}} \int \tilde{f}_n^2\, \mathrm{d}\lambda_n$$

which is the required equality (remember $\tilde{f}_n = \tilde{g}_n$). □

Chapter 9

Boundary driven Markov gas: duality and scaling limits

9.1 Introduction

9.1.1 Background and motivation

As explained in Chapter 1, boundary driven systems are important in the study of non-equilibrium steady states [21, 54, 122]. In the context of interacting particle systems on finite graphs, boundary driving means that one adds reservoirs at the boundaries, where particles can enter and leave the system. This is usually modeled via birth and death processes, where the birth and death rates are chosen in a manner adapted to the system. This means that the stationary measure of the reservoirs is a marginal of the stationary measure of the system. The simplest setting is a one-dimensional chain, where the action of the reservoirs is modeled by letting particles enter and leave the system at left and right end. The stationary distribution of such non-equilibrium systems and its macroscopic properties (e.g. the density profile, the current and their large deviations) are then the usual objects of study.

In the *discrete setting* of finite graphs, boundary driven systems of independent particles (and more generally zero-range processes) have a special status, because the non-equilibrium steady states are inhomogeneous product measures, in the case of independent particles product of Poisson distributions. For one dimensional systems, the parameters of these product measures then interpolate linearly between the densities λ_L and λ_R of the left reservoir and right reservoir. For a class of particle systems (including independent particles), one has the property of *duality* [47], which allows to express the n-point time-dependent correlation functions in terms of the evolution of n (dual) particles. In the discrete setting, these dual particles evolve on a larger system, where absorbing extra sites have been added, representing the action of reservoirs of the original system. Duality has been an essential tool to study detailed properties of different boundary driven systems such as the so-called KMP model (see [107]), the Exclusion process and the Inclusion process (see [75], [76], [79], [80]). See[32] for an account of dualities in the discrete boundary driven setting. Given the broad applicability of duality there is the need to extend it to continuum systems.

In Chapter 8 we started the study of self-duality beyond the discrete setting, i.e., self-duality of general independent Markov processes evolving as point configurations, which is the analogue of particle configurations in the discrete setting. There, self-duality turned out to be a general property of the evolution of the n-th factorial moment measure, which can be expressed via the evolution of n (dual) particles. In Chapter 8 we considered the setting of closed systems with a conserved number of particles. The goal of this chapter will be to initiate the analysis of duality for *boundary-driven systems in the continuum*, starting from the case of independent particles. We believe that the framework we build here can be used as well for boundary driven interacting particle systems in the continuum, but we leave this for future research.

The problem of modeling reservoirs in the continuum is more involved than in the discrete setting. Indeed, as explained in the introduction of the thesis (see Section 3.2.2), the addition of reservoirs in the continuum for a systems of independent Brownian motions on $(0,1)$ does not make sense. In [20] the *boundary-driven Brownian gas* on $[0,1]$ has been defined as a system of independent Brownian motions absorbed at 0 and at 1, to which is super–imposed an independent Poisson point process which adds particles on $(0,1)$ with well-chosen intensity.

The creation of particles no longer takes place at the boundaries only, rather particles are created everywhere in $(0, 1)$ with an intensity that guarantees the prescribed densities of the reservoirs. The authors in [20] then proceed by proving that this process is Markov.

In this chapter we establish in the setting of the boundary driven Brownian gas, the kind of duality results proved in [32, 76] for discrete boundary driven systems. To do this, we use the set-up introduced in Chapter 8 for closed systems in the continuum and extend it to the boundary driven Brownian gas. In particular we show that the time-dependent n-th factorial moment measures of this system can be written in terms of n dual Brownians, absorbed at the boundaries. Next, a second aim is to generalize this duality to the abstract setting of general boundary driven systems of independent particles in the continuum. For this we will need to generalize the construction of Bertini and Posta [20] first to systems of independent diffusion processes evolving on regular domain $\mathfrak{D} \subset \mathbb{R}^d$ and second to systems of general independent Markov processes which are allowed to jump and which thus can leave $\mathfrak{D}$ without hitting its boundary. As a by-product of such general construction and our duality relations two results will follow. We shall prove that in the discrete setting of a one-dimensional chain, modelling the reservoirs as: i) birth and death processes at the boundaries or ii) by a Poissonian addition of particles everywhere, are indeed equivalent processes. Furthermore the boundary driven Brownian gas (in the continuum) arises as the diffusive scaling limit of the model with birth and death processes (in the discrete) when the intensities are also scaled with the system size.

9.1.2 Duality results for Independent Random Walks

For readers convenience we recall the standard dualities of independent particles in the discrete setting, both in the case of closed and open systems.

Closed systems. Let us consider a system of simple independent random walks, namely the Markov process $\{\eta_t,\ t \geq 0\}$ with $\eta_t = \{\eta_t(x)\}_{x\in\mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$ where

$$\eta_t(x) := \text{number of particles at } x \text{ at time } t \geq 0$$

whose generator acts on bounded and local functions $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ as

$$(Lf)(\eta) = \sum_{\|x-y\|=1} \frac{1}{2}\Big[\eta(x)(f(\eta + \delta_x - \delta_y) - f(\eta)) + \eta(y)(f(\eta + \delta_y - \delta_x) - f(\eta))\Big]. \tag{9.1.1}$$

Here the sum is restricted to nearest neighbour sites and $\eta + \delta_x - \delta_y$ denotes the configuration where a particle has been moved from x to y in the configuration η. We then have that $\{\eta_t,\ t \geq 0\}$ is self-dual with self-duality function given by

$$D^{\mathrm{cl}}(\xi, \eta) = \prod_{x\in\mathbb{Z}^d} d(\xi(x), \eta(x)) \tag{9.1.2}$$

for $\xi,\ \eta \in \mathbb{N}^{\mathbb{Z}^d}$ with single-site self-duality function given by

$$d(k, n) = (n)_k \mathbf{1}_{\{k\leq n\}} := \frac{n!}{(n-k)!}\mathbf{1}_{\{k\leq n\}}. \tag{9.1.3}$$

If we denote by $\mathbb{E}^{\mathrm{IRW}}_\eta$ the expectation w.r.t. the law of the process evolving according to the generator given in (9.1.1) and starting from $\eta \in \mathbb{N}^{\mathbb{Z}^d}$, the self-duality relation is then expressed in the following way: for any $\eta,\ \xi \in \mathbb{N}^{\mathbb{Z}^d}$ and for any $t \geq 0$,

$$\mathbb{E}^{\mathrm{IRW}}_\xi[D^{\mathrm{cl}}(\xi_t, \eta)] = \mathbb{E}^{\mathrm{IRW}}_\eta[D^{\mathrm{cl}}(\xi, \eta_t)]. \tag{9.1.4}$$

The self-duality functions given in (9.1.2), that we refer to as *classical self-dualities* are products of falling factorial polynomials and they have been used to prove the hydrodynamic limit (see [47]) and in the previous chapter they have been generalized to the context of systems of independent particles evolving in the continuum (e.g., $\mathbb{R}^d$, and more generally in Borel spaces).

Open systems. Let us further consider a system of simple independent random walks on a finite chain $V_N := \{1, \ldots, N\}$ where the boundary points $\{1,\ N\}$ are in contact with reservoirs with intensity parameters $\lambda_L,\ \lambda_R \in (0, \infty)$. Namely, we consider the Markov process $\{\zeta_t,\ t \geq 0\}$ with state space $\mathbb{N}^{V_N}$ and whose generator acts on functions $f : \mathbb{N}^{V_N} \to \mathbb{R}$ as

$$L_{\mathrm{res}}f(\zeta) = L_{\mathrm{bulk}}f(\zeta) + L_{\mathrm{left}}f(\zeta) + L_{\mathrm{right}}f(\zeta) \tag{9.1.5}$$

where L_{bulk} denotes the generator of continuous-time symmetric independent random walkers jumping with rate $\frac{1}{2}$ over the edges $(i, i+1)$, $i \in \{1, \dots, N-1\}$ and where $L_{\text{left}}, L_{\text{right}}$ denote the boundary generators, modelling the contact with the reservoirs, which are given by

$$L_{\text{left}} f(\zeta) = \zeta(1)[f(\zeta - \delta_1) - f(\zeta)] + \lambda_L [f(\zeta + \delta_1) - f(\zeta)]$$

and

$$L_{\text{right}} f(\zeta) = \zeta(N)[f(\zeta - \delta_N) - f(\zeta)] + \lambda_R [f(\zeta + \delta_N) - f(\zeta)].$$

These generators describe the exit and entrance of particles via the reservoirs at left and right boundaries of the chain. Each particle can leave the system through the right or left end at rate 1, and at rate λ_L (resp. λ_R) particles enter the system at the left (resp. right) end. In the following we shall call the process ζ_t the *"reservoir process with parameters λ_L, λ_R"*.

In[32] the authors proved that the reservoir process with parameters λ_L, λ_R is dual to a system of independent random walkers on the lattice $\{0, \dots, N+1\}$ with absorbing boundaries. In the dual process the absorbing sites 0 and $N+1$ replace the reservoirs of the original process. With abuse of notation we shall use, for the dual process, the name $\{\xi_t, \ t \geq 0\}$ as in the previous paragraph, although now, in the boundary-driven context, the dual has absorbing boundary sites. The duality function D^{λ_L, λ_R} can be written as

$$D^{\lambda_L, \lambda_R}(\xi, \zeta) = \lambda_L^{\xi(0)} \lambda_R^{\xi(N+1)} D^{\text{cl}}(\xi, \zeta) \tag{9.1.6}$$

where $\xi \in \mathbb{N}^{\{0, \dots, N+1\}}$, $\zeta \in \mathbb{N}^{\{1, \dots, N\}}$ and $D^{\text{cl}}(\cdot, \cdot)$ is given in (9.1.2) but now the product is over V_N and not over $\mathbb{Z}^d$, namely

$$D^{\text{cl}}(\xi, \zeta) = \prod_{i=1}^{N} d(\xi(i), \zeta(i))$$

with $d(k, n) = \frac{n!}{(n-k)!} \mathbf{1}_{k \leq n}$. Let us denote by $\mathbb{E}_\zeta^{\text{res}}$ the expectation in the reservoir process with parameters λ_L, λ_R starting from $\zeta \in \mathbb{N}^{\{1, \dots, N\}}$. Moreover we denote by $\mathbb{E}_\xi^{\text{abs}}$ the expectation in the dual process starting from an initial configuration $\xi \in \mathbb{N}^{\{0, \dots, N+1\}}$. Then we have the following duality result: for any $\zeta \in \mathbb{N}^{V_N}$, $\xi \in \mathbb{N}^{\{0, \dots, N+1\}}$ and $t \geq 0$

$$\mathbb{E}_\zeta^{\text{res}} \left[D^{\lambda_L, \lambda_R}(\xi, \zeta_t) \right] = \mathbb{E}_\xi^{\text{abs}} \left[D^{\lambda_L, \lambda_R}(\xi_t, \zeta) \right] \tag{9.1.7}$$

or equivalently

$$\mathbb{E}_\zeta^{\text{res}} \left[\lambda_L^{\xi(0)} \lambda_R^{\xi(N+1)} D^{\text{cl}}(\xi, \zeta_t) \right] = \mathbb{E}_\xi^{\text{abs}} \left[\lambda_L^{\xi_t(0)} \lambda_R^{\xi_t(N+1)} D^{\text{cl}}(\xi_t, \zeta) \right]. \tag{9.1.8}$$

The main aim of this chapter is to extend the above duality result to general systems of boundary-driven independent particles. The random walk dynamics of each particle will be replaced by a generic Markov process. As a consequence we shall consider boundary driven systems of independent particles evolving not necessarily on the lattice, rather on generic regular domains $\mathfrak{D} \subset \mathbb{R}^d$, $d \geq 1$.

9.1.3 Outline

The rest of the chapter is organized as follows. In Section 2 we introduce basic notations. As a preliminary step, in Section 3 we present duality results for closed systems of independent particles in the continuum. First we recall a self-duality result from Chapter 8. Second, we prove a duality result, where the dual system is deterministic and follows the backward Kolmorogov equation associated to the single particle; we then use this duality result to provide a simple proof of the Doob's theorem. Section 4 contains the main result of this chapter regarding *boundary driven systems*. We start by recalling the definition of the boundary driven Brownian gas on $[0, 1]$, introduced in [20]. We then generalize this construction to general independent diffusion processes moving on regular domains $\mathfrak{D} \subset \mathbb{R}^d$ and finally to general independent Markov processes which can make jumps and thus can exit $\mathfrak{D}$ without hitting its boundary. For those systems we formulate, with increasing generality, the duality results in Theorems 9.4.1, 9.4.2 and 9.4.6, and in particular we use Theorem 9.4.2 to characterize the unique invariant measure of the systems. In Section 5, we use the duality result to show that the boundary-driven Brownian gas introduced in [20] is the scaling limit of the reservoir process of independent random walks with generator (9.1.5). Namely, we prove that the latter equals in distribution the 'boundary driven random walk gas' and that, when the parameters are scaled as λ_L/N, λ_R/N, it converges on the diffusive scale to the boundary driven Brownian gas with parameters λ_L, λ_R. Finally, in Section 6, orthogonal dualities are treated, extending to the continuum results from [76].

9.2 Setting and Notations

We will work in the context of independent particles moving in a state space E, which is assumed to be a Polish space, equipped with its Borel σ-algebra $\mathcal{E}$. In the relevant examples, $E = \mathbb{R}^d$, or E is a closed subset of $\mathbb{R}^d$ with regular boundary, or in the discrete setting $E = \mathbb{Z}^d$ or a finite graph. However, for the general duality results which we state here, there is no need to restrict to the finite dimensional setting.

9.2.1 Labeled independent particles

A single particle is moving as a Markov process $\{X_t : t \geq 0\}$ on E. A finite number of (labeled) independent particles is the process $\mathcal{X}_t = (X_t(1), \ldots, X_t(\mathbf{N}))$ arising from joining $\mathbf{N}$ independent copies of $\{X_t : t \geq 0\}$, possibly starting from different initial locations $X_0(i) = x_i \in E$. We denote by $\mathbb{E}_{x_1,\ldots,x_\mathbf{N}}$ the expectation of $\{\mathcal{X}_t, t \geq 0\}$ starting from $(x_1, \ldots, x_\mathbf{N})$, by S_t the semigroup of the Markov process $\{X_t : t \geq 0\}$, defined via $S_t f(x) = \mathbb{E}_x f(X_t)$, and by $S_t^{\otimes \mathbf{N}}$ the associated semigroup of $\mathbf{N}$ independent copies of $\{X_t : t \geq 0\}$. By independence we have

$$S_t^{\otimes \mathbf{N}} \prod_{i=1}^{\mathbf{N}} f_i(x_i) = \prod_{i=1}^{\mathbf{N}} \mathbb{E}_{x_i} \left[f_i(X_t(i)) \right] = \prod_{i=1}^{\mathbf{N}} S_t f_i(x_i).$$

We denote by S_t^* the dual semigroup working on measures μ (on $(E, \mathcal{E})$), defined via

$$\int f dS_t^* \mu = \int S_t f d\mu. \tag{9.2.1}$$

We remind the reader that we call a σ-finite measure m on E *reversible* if

$$\int_E S_t f \, g \, \mathrm{d}m = \int_E f \, S_t g \, \mathrm{d}m$$

for any $f, g \in L^2(E, m)$ and $t > 0$. Moreover, we say that the Markov process $\{X_t, \, t \geq 0\}$ is *strongly reversible* if there exists a reversible σ-finite measure m such that the transition probability measure is absolutely continuous w.r.t. m, i.e., there exists a transition density

$$\mathfrak{p}_t : E \times E \to [0, \infty)$$

such that, for all $t > 0$,

$$S_t f(x) = \int f(y) \mathfrak{p}_t(x, y) m(dy) = \int f(y) \mathfrak{p}_t(y, x) m(dy) \tag{9.2.2}$$

where the symmetry $\mathfrak{p}_t(x, y) = \mathfrak{p}_t(y, x)$ follows from the assumed reversibility of m. Relevant examples to keep in mind are i) Brownian motion, where m is the Lebesgue measure; ii) symmetric random walk, where m is the counting measure; iii) the Ornstein Uhlenbeck process, where m is the Gaussian measure.

9.2.2 Point configurations

As in the previous chapter, it is convenient for our purposes to describe the motion of independent particles modulo permutation, i.e. via configurations. More precisely, the initial configuration associated to $\mathbf{N}$ labeled particle positions $(x_1, \ldots, x_\mathbf{N}) \in E^\mathbf{N}$ is defined as

$$\eta = \sum_{i=1}^{\mathbf{N}} \delta_{x_i}, \tag{9.2.3}$$

which is viewed as a point configuration on E. The configuration at time t is then defined as

$$\eta_t = \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)} \tag{9.2.4}$$

where $X_0(i) = x_i$. Notice that by the fact that the independent particles are indistinguishable, $\{\eta_t, t \geq 0\}$ is a Markov process on the space of point configurations with total mass $\mathbf{N}$. More generally, if we have a point configuration on E, with potentially infinitely many particles, i.e., $\eta = \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$ where we now also allow $\mathbf{N} = \infty$, then we define the configuration at time $t > 0$ as in (9.2.4). In case we work with infinitely many particles, we have to assume that the initial configuration is such that no explosions take place, i.e., such that at any time $t > 0$, the configuration $\eta_t = \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}$ is a well-defined point configuration. In this chapter, however, in order to avoid technicalities, we will restrict to systems with finitely many particles. We denote by $\mathbb{E}_\eta$ the expectation in the configuration process $\{\eta_t, t \geq 0\}$.

For a configuration η we recall that its associated *n-th factorial measure* is defined by

$$\eta^{(n)} := \sum_{1 \leq i_1, \ldots, i_n \leq \mathbf{N}}^{\neq} \delta_{(x_{i_1}, \ldots, x_{i_n})} \tag{9.2.5}$$

where the superscript $\neq$ means that the summation is over n mutually distinct indexes $i_1, \ldots, i_n$ taken from $\{1, \ldots, \mathbf{N}\}$, with $\mathbf{N} = \eta(E)$. The measure $\eta^{(n)}$ is a point-measure on E^n. Intuitively speaking, $\eta^{(n)}$ corresponds to un-normalized sampling of n different particles out of the configuration η and takes the name *factorial* from the following identity: for any $B \in \mathcal{E}$

$$\eta^{(n)}(B^n) = (\eta(B))_n$$

with $(m)_n := m(m-1)\cdots(m-n+1)$ denoting the n-th falling factorial.

An important object of study is the expectation $\mathbb{E}[\eta^{(n)}]$ that is called the *n-th factorial moment measure*. Here $\mathbb{E}$ refers to the average w.r.t. the randomness of the distribution of points in η. We have that $\mathbb{E}[\eta^{(n)}]$ is a measure on E^n, and, in particular,

$$\mathbb{E}[\eta^{(n)}(B^n)] = \mathbb{E}[(\eta(B))_n]$$

provides the n-th factorial moment of the number of points of $B \in \mathcal{E}$. An important special case is when the points in η are distributed according to a Poisson point process with intensity measure λ: it is well know (see, e.g., [120, (4.11)]) that in this case one has

$$\mathbb{E}[\eta^{(n)}] = \lambda^{\otimes n} \tag{9.2.6}$$

which is a particular instance of the Mecke's equation.

In the next sections we will study, by duality, the expectation of the n-th factorial measure of the configuration at time t, i.e. $\mathbb{E}_\eta[\eta_t^{(n)}]$, which will be called *the n-th factorial moment measure at time t*.

9.3 General duality results for independent particles

In this section we review some known duality results for closed (i.e., without reservoirs) systems of independent particles: namely self-duality and duality w.r.t. deterministic systems.

9.3.1 Intertwining and self-duality

We now re-state an intertwining and a self-duality result for independent particles taken from Chapter 8 of which we provide here an alternative proof which relies on generating functions. This generating function approach is well suited to study boundary driven systems in Section 9.4 below. As already mentioned, in order to avoid technicalities, the results below are stated for finitely many particles. However, whenever the infinitely many particle limit is well-defined, by passing to this limit, the result extends immediately to the infinite case.

Theorem 9.3.1. *Let η be a finite point configuration as defined in* (9.2.3). *Assume that the particles evolve independently according to the Markov process $\{X_t : t \geq 0\}$.*

a) (Intertwining) The following identity holds

$$\mathbb{E}_\eta[\eta_t^{(n)}] = (S_t^{\otimes n})^* \eta^{(n)}, \tag{9.3.1}$$

where S_t^* is the dual semigroup defined in (9.2.1).

b) (Self-duality) If $\{X_t : t \geq 0\}$ is strongly reversible with reversible measure m then one can express the density of n-th factorial moment measure $\mathbb{E}_\eta[\eta_t^{(n)}]$ w.r.t. $m^{\otimes n}$ via

$$\frac{\mathrm{d}\mathbb{E}_\eta[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(z_1, \ldots, z_n) = \int \prod_{i=1}^n \mathfrak{p}_t(z_i, y_i) \eta^{(n)}(\mathrm{d}(y_1, \ldots, y_n)), \tag{9.3.2}$$

where $\mathfrak{p}_t(\cdot, \cdot)$ is the transition density defined in (9.2.2).

Proof. We start from the following identity from [120, Lemma 4.11], for a general finite random point configuration. Let $u : E \to (0, 1)$ then

$$\exp\left(\int \log(1-u(z))\eta(\mathrm{d}z)\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u^{\otimes n}(z_1, \dots, z_n)\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)). \tag{9.3.3}$$

We can now use this identity to prove (9.3.1). Let us adopt the abbreviation $u_t(z) = S_t u(z) = \mathbb{E}_z[u(X_t)]$. Using the independence of the processes $X_t(i)$, $i \in \{1, \dots, N\}$, we compute

$$\begin{aligned}
&\mathbb{E}_\eta\left[\exp\left(\int \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right] = \mathbb{E}_{x_1,\dots,x_N}\left[\prod_i (1-u(X_t(i)))\right] \\
&= \prod_i \mathbb{E}_{x_i}\left[1-u(X_t(i))\right] = \prod_i (1-u_t(x_i)) = \exp\left(\int \log(1-u_t(z))\eta(\mathrm{d}z)\right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u_t^{\otimes n}(z_1, \dots, z_n)\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u^{\otimes n}(z_1, \dots, z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)),
\end{aligned} \tag{9.3.4}$$

where we used (9.3.3) in the fourth identity. On the other hand, using (9.3.3) once more, we have

$$\mathbb{E}_\eta\left[\exp\left(\int \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right] = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u^{\otimes n}(z_1, \dots, z_n)\mathbb{E}_\eta[\eta_t^{(n)}](\mathrm{d}(z_1, \dots, z_n)) \tag{9.3.5}$$

and therefore, from (9.3.5) and (9.3.4) we conclude

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u^{\otimes n}(z_1, \dots, z_n)\mathbb{E}_\eta[\eta_t^{(n)}](\mathrm{d}(z_1, \dots, z_n)) \\
= \; &1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int u^{\otimes n}(z_1, \dots, z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)).
\end{aligned} \tag{9.3.6}$$

Because this holds for all u, identifying term by term in the above series and using a standard density argument for symmetric functions (linear combinations of functions of the form $u(z_1)u(z_2)\dots u(z_n)$ are dense in the set of symmetric functions), we obtain (9.3.1).

If in addition we assume strong reversibility, we then have, for any $f : E^n \to \mathbb{R}$ bounded,

$$\begin{aligned}
&\int f(z_1, \dots, z_n)\mathbb{E}_\eta[\eta_t^{(n)}](\mathrm{d}(z_1, \dots, z_n)) = \int f(z_1, \dots, z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)) \\
&= \int (S_t^{\otimes n} f)(z_1, \dots, z_n)\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)) \\
&= \int \left(\int f(y_1, \dots, y_n) \prod_{i=1}^{n} \mathfrak{p}_t(z_i, y_i) m^{\otimes n}(\mathrm{d}(y_1, \dots, y_n))\right)\eta^{(n)}(\mathrm{d}(z_1, \dots, z_n)) \\
&= \int \left(\int f(z_1, \dots, z_n) \prod_{i=1}^{n} \mathfrak{p}_t(z_i, y_i)\eta^{(n)}(\mathrm{d}(y_1, \dots, y_n))\right) m^{\otimes n}(\mathrm{d}(z_1, \dots, z_n)),
\end{aligned}$$

from which (9.3.2) follows. □

Remark 9.3.2. As noticed in [73, Remark 2.3(iii)], for the system of independent random walks on $\mathbb{Z}^d$ with generator given in (9.1.1), (9.3.2) is equivalent to the classic self-duality relation (9.1.4). Indeed for a singleton $(z_1, \dots, z_n)$, $z_i \in \mathbb{Z}^d$, we have the relation (see [73, Lemma 2.1])

$$\eta^{(n)}(\{(z_1, \dots, z_n)\}) = D^{\mathrm{cl}}(\delta_{z_1} + \dots + \delta_{z_n}, \eta) \tag{9.3.7}$$

with D^{cl} defined in (9.1.2) and thus

$$\mathbb{E}_\eta[\eta_t^{(n)}](\{(z_1, \dots, z_n)\}) = \mathbb{E}_\eta^{\mathrm{IRW}}[D^{\mathrm{cl}}(\delta_{z_1} + \dots + \delta_{z_n}, \eta_t)]$$

and

$$\int \prod_{i=1}^{n} \mathfrak{p}_t(z_i, y_i)\, \eta^{(n)}(\mathrm{d}(y_1, \ldots, y_n)) = \mathbb{E}^{\mathrm{IRW}}_{\delta_{z_1}+\ldots+\delta_{z_n}}[D^{\mathrm{cl}}(\xi_t, \eta)],$$

where ξ_t denotes the configuration of independent random walks at time t starting from $\xi_0 = \delta_{z_1} + \ldots + \delta_{z_n}$.

9.3.2 Duality w.r.t. the associated deterministic system

The so-called "associated deterministic system" is a dynamical system on functions $f : E \to \mathbb{R}$ which follows the flow of the Kolmogorov backwards equation of the Markov process $\{X_t,\ t \geq 0\}$. More precisely for $f : E \to \mathbb{R}$ we define $f_t(x) = S_t f(x) = \mathbb{E}_x[f(X_t)]$. This flow f_t is the solution of the system of ODE given by

$$\frac{df_t(x)}{dt} = \mathcal{L} f_t(x), \tag{9.3.8}$$

with $\mathcal{L}$ being the Markov generator associated to $\{S_t, t \geq 0\}$. Notice that, by the Markov semigroup property, $f_t > 0$ when $f > 0$. For $f : E \to (0, \infty)$ and a labeled process $\{\mathcal{X}_t, t \geq 0\} = \{(X_t(1), \ldots, X_t(\mathbf{N})), t \geq 0\}$ initialised from $\mathcal{X} = (x_1, \ldots, x_{\mathbf{N}})$, we define the function

$$\mathcal{D}(f, \mathcal{X}_t) = \prod_{i=1}^{\mathbf{N}} f(X_t(i)) \tag{9.3.9}$$

or, alternatively, in terms of the point configuration process $\{\eta_t, t \geq 0\}$ we set

$$D(f, \eta_t) := e^{\int \log f \mathrm{d}\eta_t}.$$

Duality between the configuration process and the deterministic system is then formulated as follows.

Theorem 9.3.3. *The process $\{\mathcal{X}_t\ :\ t \geq 0\}$ is dual to the deterministic evolution on functions $f : E \to (0, \infty)$ defined via $f_t(x) = S_t f(x) = \mathbb{E}_x[f(X_t)]$, with duality function $\mathcal{D}(f, \mathcal{X}) = \prod_{i=1}^{\mathbf{N}} f(X(i))$, i.e.,*

$$\mathbb{E}_{\mathcal{X}}\left[\mathcal{D}(f, \mathcal{X}_t)\right] = \mathcal{D}(f_t, \mathcal{X}), \tag{9.3.10}$$

or, equivalently, in terms of the point configuration process

$$\mathbb{E}_\eta\left[D(f, \eta_t)\right] = D(f_t, \eta). \tag{9.3.11}$$

Proof. The proof is straightforward, indeed by the independence of the particles and by the definition of f_t, we have

$$\mathbb{E}_{\mathcal{X}}\left[\prod_{i=1}^{\mathbf{N}} f(X_t(i))\right] = \prod_{i=1}^{\mathbf{N}} \mathbb{E}_{\mathcal{X}}\left[f(X_t(i))\right] = \prod_{i=1}^{\mathbf{N}} f_t(x_i).$$

□

Doob's theorem

Let us now consider the connection between the duality result of Section 9.3.2 with the time evolution of Poisson point processes. It is well-known that independent Markovian particle evolutions preserve Poisson processes: we refer to this result as Doob's theorem but it can also be viewed as a consequence of the random displacement theorem (see, e.g., [120]).

We briefly recall the definition of a Poisson point process. For a function $\rho : E \to [0, \infty)$ and a σ-finite measure m on $(E, \mathcal{E})$ the Poisson point process with intensity measure $\rho(z)m(dz)$ is defined as the random point configuration $\eta = \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that

1. For every $\omega \in \Omega$, the map $\mathcal{E} \ni A \to \eta(\omega, A)$ is a $\mathbb{N}$-valued measure on the σ-algebra $\mathcal{E}$.
2. For $A_1, \ldots A_n \in \mathcal{E}$, n disjoint measurable subsets of E, $\{\eta(A_i), i = 1, \ldots, n\}$ are independent Poisson random variables with parameter $m_i = \int_{A_i} \rho(z)m(\mathrm{d}z)$.

See [120] for background on Poisson point processes. We denote by $\mathcal{P}_\rho$ the probability in the Poisson point process with intensity measure $\rho(z)m(\mathrm{d}z)$ on the space of point configurations.

We recall the reader that a Poisson point process is uniquely characterized by its Laplace functional, i.e., by

$$\int \left(e^{\int_E f(z)\eta(\mathrm{d}z)}\right)\mathcal{P}_\rho(\mathrm{d}\eta) = e^{\int (e^{f(z)}-1)\rho(z)m(\mathrm{d}z)} \tag{9.3.12}$$

for all f for which the integral $\int (e^{f(z)}-1)\rho(z)m(\mathrm{d}z)$ is finite.

We denote by $\mathbb{E}_{\mathcal{P}_\rho}$ the expectation of the process of independent particles moving according to the Markovian dynamics corresponding to the semigroup S_t whose associated point configuration is initially distributed as $\mathcal{P}_\rho$. The following result then proves Doob's theorem via the duality (9.3.10).

Theorem 9.3.4. *Let $\{X_t, t \geq 0\} = \{(X_t(1), \ldots, X_t(\mathbf{N})), t \geq 0\}$ be a system of independent particles initialized at time zero from a Poisson point configuration with intensity measure $\rho(z)m(\mathrm{d}z)$, where m is a reversible measure of the Markov process $\{X_t : t \geq 0\}$. Then the distribution of the $\mathbf{N}$ particles at time $t \geq 0$, namely the random point configuration $\sum_i \delta_{X_t(i)}$, is a Poisson point configuration with intensity measure $\rho_t(z)m(dz)$, where*

$$\rho_t(z) = \mathbb{E}_z[\rho(X_t)] = S_t\rho(z).$$

More generally, if m is a stationary measure of $\{X_t : t \geq 0\}$, the Poisson point process is mapped to a Poisson point process with intensity measure

$$\rho_t = S_t^*\rho,$$

where S_t^* denotes the adjoint semigroup of S_t.

Proof. Using (9.3.11) and (9.3.12), we obtain

$$\begin{aligned}
\int \mathbb{E}_\eta\left[e^{\int \log f(z)\eta_t(\mathrm{d}z)}\right]\mathcal{P}_\rho(\mathrm{d}\eta) &= \int \mathbb{E}_\eta[D(f,\eta_t)]\mathcal{P}_\rho(\mathrm{d}\eta) \\
&= \int e^{\int \log f_t(z)\eta(\mathrm{d}z)}\mathcal{P}_\rho(\mathrm{d}\eta) \\
&= e^{\langle (f_t-1),\rho\rangle_{L^2(m)}} \\
&= e^{\langle (f-1),S_t^*\rho\rangle_{L^2(m)}}
\end{aligned} \tag{9.3.13}$$

From this we infer that η_t is again a Poisson point process with intensity $\rho_t(z)m(\mathrm{d}z)$ where $\rho_t(z) = \mathbb{E}_z[\rho(X(t))]$ if S_t is self adjoint in $L^2(m)$ and $\rho_t(z) = S_t^*\rho(z)$ in the general case. □

Corollary 9.3.5. *In the setting of Theorem 9.3.4, the Poisson point processes with intensity measure $\rho \cdot m(dz)$ parametrized by a constant $\rho > 0$ are reversible for $\{X_t, t \geq 0\}$. More generally, if m is a stationary measure of $\{X_t : t \geq 0\}$, the Poisson point process is stationary if and only if*

$$S_t^*\rho = \rho.$$

Proof. When ρ is constant we have, using (9.3.11) and (9.3.12),

$$\begin{aligned}
&\mathbb{E}_{\mathcal{P}_\rho}\left(\mathbb{E}_\eta\left[e^{\int \log f \mathrm{d}\eta_t}\right]e^{\int \log g \mathrm{d}\eta}\right) = \mathbb{E}_{\mathcal{P}_\rho}\left(e^{\int \log S_t f \mathrm{d}\eta}e^{\int \log g \mathrm{d}\eta}\right) \\
&= \mathbb{E}_{\mathcal{P}_\rho}\left(e^{\int \log((S_t f)g)\mathrm{d}\eta}\right) = e^{\rho\int ((S_t f)g-1)\mathrm{d}m}
\end{aligned}$$

and using the self-adjointness of S_t we obtain

$$e^{\rho\int ((S_t f)g-1)\mathrm{d}m} = e^{\rho\int ((S_t g)f-1)\mathrm{d}m} = \mathbb{E}_{\mathcal{P}_\rho}\left(\mathbb{E}_\eta\left[e^{\int \log g \mathrm{d}\eta_t}\right]e^{\int \log f \mathrm{d}\eta}\right)$$

which implies reversibility of $\mathcal{P}_\rho$.

The second statement follows immediately from Theorem 9.3.4. □

9.4 Duality for boundary driven systems of independent particles

In this section we will present a duality result for boundary driven systems of independent particles which generalizes previous results of that type obtained only in the discrete setting[32], [76]. In Section 9.4.1, we recall the definition of the *boundary driven Brownian gas* recently introduced in [20] and we state a duality result in this context (see Theorem 9.4.1 below). In Section 9.4.2 we consider more general systems of independent diffusion processes on regular domains $\mathfrak{D} \subset \mathbb{R}^d$. We prove first an intertwining result and secondly a duality result under an extra assumption on the transition probabilities of a single particle. In Section 9.4.3 we introduce a further generalization of the construction of Bertini and Posta in [20], namely boundary driven Markov processes with jumps, which can exit the domain without hitting its boundary.

9.4.1 The boundary driven Brownian gas on $[0, 1]$: definition and duality

Let $E = [0, 1]$ and denote by $\{W_t, t \geq 0\}$ a standard Brownian motion absorbed upon hitting 0 or 1. Let us denote by τ_0, τ_1 the hitting times of 0, resp. 1, of $\{W_t, t \geq 0\}$. We denote by $\mathbb{P}_x^{\text{abs}}$ and by S_t respectively the distribution of the trajectories of $\{W_t, t \geq 0\}$ starting from $x \in [0, 1]$ and the semigroup of the process. It is well known that the transition probability $p_t(\cdot, \cdot) : [0, 1] \times \mathcal{B}([0, 1]) \to [0, 1]$ of the absorbed Brownian motion satisfies

$$p_t(x, \mathrm{d}y) = \mathfrak{p}_t(x, y)\,\mathrm{d}y \qquad \forall x, y \in (0, 1) \tag{9.4.1}$$

with $\mathfrak{p}_t(x, y) = \mathfrak{p}_t(y, x)$ a symmetric function referred as transition density (see, e.g., [30, p. 122] for an explicit formula of $\mathfrak{p}_t(x, y)$). With a slight abuse of notation we denote by $\mathfrak{p}_t(x, 0)$ (respectively $\mathfrak{p}_t(x, 1)$) the probability, starting from $x \in [0, 1]$, of being absorbed at 0 (resp. at 1) by the time $t \geq 0$. We then have, for any $x \in (0, 1)$,

$$\int_0^1 \mathfrak{p}_t(x, y)\,\mathrm{d}y + \mathfrak{p}_t(x, 0) + \mathfrak{p}_t(x, 1) = 1 \tag{9.4.2}$$

and for any $f : [0, 1] \to \mathbb{R}$ bounded

$$S_t f(x) = \int_0^1 \mathfrak{p}_t(x, y) f(y)\,\mathrm{d}y + f(0)\mathfrak{p}_t(x, 0) + f(1)\mathfrak{p}_t(x, 1).$$

For $\xi := \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, $x_i \in (0, 1)$ and $\mathbf{N} \in \mathbb{N}$, we then consider the point configuration (on $[0, 1]$) valued Markov process given by

$$\begin{cases} \xi_t := \sum_{i=1}^{\mathbf{N}} \delta_{W_t(i)} \,, \\ \xi_0 = \xi \end{cases}$$

where $\{W_t(i)\}_{t\geq 0}$ are independent copies of $\{W_t\}_{t\geq 0}$ such that $W_0(i) = x_i$ for any $i \in [\mathbf{N}]$. The transition function $P_t(\xi, \cdot)$ of the process $\{\xi_t, t \geq 0\}$ is then given by the image of $\otimes_{i=1}^{\mathbf{N}} p_t(x_i, \cdot)$ under the mapping $(x_i)_{i=1}^{\mathbf{N}} \to \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$. For $\boldsymbol{x} = (x_1, \ldots, x_{\mathbf{N}}) \in (0, 1)^{\mathbf{N}}$, we denote by $\mathbb{E}_{\boldsymbol{x}}^{\text{abs}}$ the expectation in the process $\{\xi_t, t \geq 0\}$ starting from $\xi_0 = \sum_{i=0}^{N} \delta_{x_i}$. Finally, let Θ_t be a Poisson point configuration on $(0, 1)$ with time dependent intensity $\lambda_t(dx)$ given by

$$\lambda_t(\mathrm{d}x) = \lambda(t, x)\,\mathrm{d}x \tag{9.4.3}$$

and

$$\begin{aligned} \lambda(t, x) &= \lambda_L \mathbb{P}_x^{\text{abs}}(\tau_0 \leq t) + \lambda_R \mathbb{P}_x^{\text{abs}}(\tau_1 \leq t) \\ &= \lambda_L \mathfrak{p}_t(x, 0) + \lambda_R \mathfrak{p}_t(x, 1) \end{aligned} \tag{9.4.4}$$

for some $\lambda = (\lambda_L, \lambda_R) \in \mathbb{R}_+^2$. Moreover $\{\xi_t\}_{t\geq 0}$ and $\{\Theta_t\}_{t\geq 0}$ are independent. The process $\{\Theta_t\}_{t\geq 0}$, by adding particles in the bulk $(0, 1)$, models in turn the effect of the reservoirs at 0 and 1 (cf. [20, (2.1), (2.2)]). The *boundary driven Brownian gas* is then defined, for any $t > 0$, by

$$\eta_t = \xi_t\big|_{(0,1)} + \Theta_t \tag{9.4.5}$$

viewed as a point configuration on $(0, 1)$ and such that $\eta_0 = \xi_0\big|_{(0,1)}$. Here $\xi_t\big|_{(0,1)}$ denotes the restriction of the point configuration ξ_t to $(0, 1)$.

The motivation for this definition can be found in [20]. In Section 9.5.2 below we will show how the boundary driven Brownian gas arises as a scaling limit of the reservoir process on a chain $\{1, \ldots, N\}$ defined in (9.1.5).

Let us recall that $\eta^{(n)}$ denotes the n-th factorial measure corresponding to the initial configuration $\eta_0 = \eta$ made of $\mathbf{N}$ particles, and $\eta_t^{(n)}$ denotes the n-th factorial measure corresponding to η_t, i.e. the configuration at time t with $\mathbf{N}_t$ particles. We denote by $\mathbb{E}_\eta^\lambda$ the expectation in the process defined via (9.4.5) initialized from η. We will use the following abbbreviations: $\boldsymbol{x} = (x_1, \dots, x_n)$, for $I = (i_1, \dots, i_k) \subset \{1, \dots, n\}$ we put $\boldsymbol{x}_I = (x_{i_1}, \dots, x_{i_k})$ and we write $[k]$ for $\{1, \dots, k\}$. We shall also use the following shorthand for the transition density in the rest of the chapter

$$\mathfrak{p}_t^{(n)}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^n \mathfrak{p}_t(x_i, y_i). \tag{9.4.6}$$

For the boundary driven Brownian gas the following duality result holds, where the dual process is a system of independent absorbed Brownian motions.

Theorem 9.4.1. *For the boundary driven Brownian gas $\{\eta_t, t \geq 0\}$, the n-th factorial moment measure at time $t > 0$ is absolutely continuous w.r.t. $m^{\otimes n}$, with m denoting the Lebesgue measure on $(0,1)$ with the following density:*

$$\frac{\mathrm{d}\mathbb{E}_\eta^\lambda[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(\boldsymbol{z}) = \sum_{I \subset [n]} \mathbb{E}_{\boldsymbol{z}_I}^{\mathrm{abs}} \left[\lambda_L^{\xi_t(\{0\})} \lambda_R^{\xi_t(\{1\})} \mathbf{1}_{\{\xi_t(\{0,1\}) = |I|\}} \right] \int_{(0,1)^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(\boldsymbol{z}_{[n]\setminus I}, \boldsymbol{y}) \eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}). \tag{9.4.7}$$

This result can be read as a duality relation in the spirit of (9.3.2): in order to know the n-th order factorial moment measure at time $t > 0$, one has to follow (not more than) n dual particles. However, due to the presence of reservoirs, we have factors

$$\mathbb{E}_{\boldsymbol{z}_I}^{\mathrm{abs}} \left[\lambda_L^{\xi_t(\{0\})} \lambda_R^{\xi_t(\{1\})} \mathbf{1}_{\xi_t(\{0,1\}) = |I|} \right]$$

which can be considered as corresponding to $|I|$ "absorbed" dual particles. This result has to be compared with the discrete setting, namely (9.1.8), where an analogous term multiplying the product of falling factorial polynomials appears in the duality function and the process with reservoirs is dual to an absorbing process with two extra sites associated to the reservoirs (see [32], [76] and [75]).

In the next subsection we state and prove a more general version of Theorem 9.4.1, which applies to independent diffusions on regular domains $\mathfrak{D} \subset \mathbb{R}^d$ and includes also an intertwining result. (9.4.7) for the boundary driven Brownian gas on $(0,1)$ will then follow as a particular case of Theorem 9.4.2.

9.4.2 Boundary driven diffusion processes: definition and duality

Let $\mathfrak{D}$ be a regular domain of $\mathbb{R}^d$, where by regular domain we mean an open, simply connected and bounded subset $\mathfrak{D} \subset \mathbb{R}^d$ such that its boundary $\partial\mathfrak{D}$ is Lipschitz. Let $\{Y_t, t \geq 0\}$ be the diffusion process on $\mathbb{R}^d$ with generator

$$\mathcal{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} \tag{9.4.8}$$

with regular coefficient functions $a_{i,j}$, b_j and with $a = (a_{i,j})$ symmetric, non-degenerate and positive definite. We then denote by $\{X_t, t \geq 0\}$ the Markov process on $\bar{\mathfrak{D}} = \mathfrak{D} \cup \partial\mathfrak{D}$ with semigroup $\{S_t, t \geq 0\}$, evolving as $\{Y_t, t \geq 0\}$ on $\mathfrak{D}$ and absorbed upon hitting $\partial\mathfrak{D}$. More specifically the regularity assumptions on the coefficients are that $\frac{\partial^2 a_{i,j}}{\partial_{x_i} \partial_{x_\ell}}$ and $\frac{\partial b_i}{\partial_{x_j}}$ are locally uniformly Hölder continuous on $\mathfrak{D} \cup \partial\mathfrak{D}$ (see, e.g., [103]). Denote by $\mathbb{P}_x^{\mathrm{abs}}$ (resp. $\mathbb{E}_x^{\mathrm{abs}}$) the distribution (resp. the expectation) of the trajectories of $\{X_t, t \geq 0\}$ starting from $x \in \mathfrak{D}$. We then assume that

$$\mathbb{P}_x^{\mathrm{abs}}(\tau_{\partial\mathfrak{D}} < \infty) = 1, \quad \forall x \in \mathfrak{D} \tag{9.4.9}$$

where $\tau_{\partial\mathfrak{D}}$ denotes the hitting time of $\partial\mathfrak{D}$.

For $\xi := \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, $x_i \in \mathfrak{D}$, we consider the point configuration (on $\bar{\mathfrak{D}}$) valued Markov process given by

$$\begin{cases} \xi_t := \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}, \\ \xi_0 = \xi \end{cases}$$

where $\{X_t(i)\}_{t\geq 0}$ are independent copies of $\{X_t\}_{t\geq 0}$ such that $X_0(i) = x_i$ for any $i \in [\mathbf{N}]$. For $\boldsymbol{x} = (x_1, \dots, x_{\mathbf{N}}) \in \mathfrak{D}^{\mathbf{N}}$, we denote by $\mathbb{E}_{\boldsymbol{x}}^{\mathrm{abs}}$ the expectation in the process $\{\xi_t, t \geq 0\}$ starting from $\xi_0 = \sum_{i=0}^N \delta_{x_i}$.

Let now $\lambda : \partial\mathfrak{D} \to \mathbb{R}_+$ be a bounded measurable function giving the reservoir intensity at any $x \in \partial\mathfrak{D}$. If λ satisfies the just mentioned assumptions it is said to be regular. Finally we define Θ_t the Poisson point process on $\mathfrak{D}$ with time dependent intensity $\lambda_t(\mathrm{d}x)$ given by

$$\lambda_t(\mathrm{d}x) = \left(\int_{\partial\mathfrak{D}} \lambda(y)\,\mathbb{P}_x(\tau_{\partial\mathfrak{D}} \le t, X_{\tau_{\partial\mathfrak{D}}} \in \mathrm{d}y)\right)\mu(\mathrm{d}x), \tag{9.4.10}$$

for a finite measure μ on $\mathfrak{D}$. The *boundary driven diffusion gas in the domain* $\mathfrak{D}$ *with reservoir intensity* λ *and a priori measure* μ, denoted by $\{\eta_t, t \ge 0\}$, is then given, for any $t \ge 0$, by

$$\begin{cases} \eta_t = \xi_t\big|_{\mathfrak{D}} + \Theta_t\,, \\ \eta_0 = \xi_0 = \sum_{i=1}^N \delta_{x_i},\ x_i \in \mathfrak{D} \end{cases} \tag{9.4.11}$$

viewed as a point configuration on $\mathfrak{D}$, where $\xi_t\big|_{\mathfrak{D}}$ is the restriction of ξ_t to $\mathfrak{D}$.

We denote by $\mathbb{E}^{\lambda}_{\eta}$ the expectation in the process defined via (9.4.11) initialized from η. Following the strategy of [20], it can be shown that $\{\eta_t,\ t \ge 0\}$ is a Markov process when the transition probability $p_t(x, \mathrm{d}y)$ of the process $\{X_t, t \ge 0\}$ satisfies

$$p_t(x, \mathrm{d}y)m(\mathrm{d}x) = p_t(y, \mathrm{d}x)m(\mathrm{d}y) \qquad \text{on} \qquad \mathfrak{D} \times \mathfrak{D} \tag{9.4.12}$$

for a finite measure m and for the reservoir intensity in (9.4.10) we choose $\mu = m$. We refer to Section 9.7 below for further details.

We are now ready to state the main results of this section, namely a general intertwining relation for the factorial moment measure at time $t > 0$ of the boundary driven diffusion processes on a d-dimensional regular domain $\mathfrak{D}$ and a duality result, under an extra symmetry assumption on the transition probability of $\{X_t, t \ge 0\}$ (see (9.4.14) below), generalizing Theorem 9.4.1.

Theorem 9.4.2. *Let* $\{\eta_t, t \ge 0\}$ *be the boundary driven diffusion gas defined in* (9.4.11). *Then for all* $n \in \mathbb{N}$ *and* $t \ge 0$*, it holds:*

a) for all bounded, measurable and permutation invariant $f : \mathfrak{D}^n \to \mathbb{R}$

$$\mathbb{E}^{\lambda}_{\eta}\left[\int_{\mathfrak{D}^n} f(z)\eta_t^{(n)}(\mathrm{d}z)\right] = \sum_{k=0}^{n}\binom{n}{k}\int_{\mathfrak{D}^n} f(z)\lambda_t^{\otimes k}(\mathrm{d}z_{[k]}) \otimes (S_t^{\otimes n-k})^{*}\eta^{(n-k)}(\mathrm{d}z_{[n]\setminus[k]}); \tag{9.4.13}$$

b) assume further that the transition probability of $\{X_t, t \ge 0\}$ *satisfies*

$$p_t(x, \mathrm{d}y) = \mathfrak{p}_t(x, y)m(\mathrm{d}y) \tag{9.4.14}$$

for a symmetric function $\mathfrak{p}_t(x, y)$ *and a finite measure* m *on* $\mathfrak{D}$*. Then, choosing* $\mu = m$*, the following holds*

$$\frac{\mathrm{d}\mathbb{E}^{\lambda}_{\eta}[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(z) = \sum_{I \subset [n]} \mathbb{E}^{\mathrm{abs}}_{z_I}\left[e^{\int_{\partial\mathfrak{D}} \log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D}) = |I|\}}\right]\int_{\mathfrak{D}^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, \boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}) \tag{9.4.15}$$

Remark 9.4.3. *i) Notice that, if* $\lambda(x) \in \{\lambda_L, \lambda_R\}$ *for any* $x \in \partial\mathfrak{D}$ *and it is regular (as defined above), setting* $\partial\mathfrak{D}_L = \{x \in \partial\mathfrak{D} :\ \lambda(x) = \lambda_L\}$*, we then have*

$$\frac{\mathrm{d}\mathbb{E}^{\lambda}_{\eta}[\eta^{(n)}]}{\mathrm{d}m^{\otimes n}}(z) = \sum_{I \subset [n]} \mathbb{E}^{\mathrm{abs}}_{z_I}\left[\lambda_L^{\xi_t(\partial\mathfrak{D}_L)}\lambda_R^{\xi_t(\partial\mathfrak{D}\setminus\partial\mathfrak{D}_L)}\mathbf{1}_{\{\xi_t(\partial E) = |I|\}}\right]\int_{\mathfrak{D}^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, \boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}), \tag{9.4.16}$$

which is the multidimensional analogue of (9.4.7) *when there are two possible values for the reservoir intensity.*

ii) The multidimensional Brownian motion satisfies (9.4.14) *with* m *given by the Lebesgue measure (see, e.g. [14, Theorem 4.4]): thus the multidimensional boundary driven Brownian gas satisfies* (9.4.15).

iii) In one dimension, all diffusion processes satisfy (9.4.14) *(see, e.g. [30, pag.13]). In particular, consider the diffusion process on* $\mathbb{R}$ *with generator*

$$\mathcal{L}f(y) = \frac{1}{2}\,\sigma^2(y)\,\frac{\mathrm{d}^2 f}{\mathrm{d}y^2}(y) + b(y)\,\frac{\mathrm{d}f}{\mathrm{d}y}(y) \tag{9.4.17}$$

where the drift b and the diffusivity σ are continuous functions and with $\sigma^2(x) \geq \delta > 0$ for each $x \in (0,1)$. Then (9.4.14) *holds with m given by*

$$m(\mathrm{d}x) = \frac{1}{\sigma^2(x)} \exp\left(2\int_{x_0}^{x} \frac{b(y)}{\sigma^2(y)}\,\mathrm{d}y\right)\mathrm{d}x\,, \tag{9.4.18}$$

for an arbitrary $x_0 \in (0,1)$ (see, e.g. [30, pag.17]).

iv) We refer to [103] for conditions on the coefficients $a_{i,j}$ and b_i in (9.4.8) *ensuring that* (9.4.14) *holds.*

Proof. Coherently with what we have done in Section 9.3.1, we provide a proof relying on generating functions which uses the identity (9.3.3). Let $u : \mathfrak{D} \to \mathbb{R}$ bounded and measurable. By the independence of Θ_t and ξ_t we have

$$\begin{aligned} &\mathbb{E}_\eta^\lambda\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right] \qquad (9.4.19)\\ = \;& \mathbb{E}_{\mathcal{P}_{\lambda_t}}\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\Theta_t(\mathrm{d}z)\right)\right]\mathbb{E}_\eta^{\mathrm{abs}}\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\xi_t\big|_{\mathfrak{D}}(\mathrm{d}z)\right)\right], \end{aligned}$$

where $\mathbb{E}_{\mathcal{P}_{\lambda_t}}$ denotes the expectation in the Poisson point process Θ_t. Notice in particular that $\mathbb{E}_\eta\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\xi_t\big|_{\mathfrak{D}}(\mathrm{d}z)\right)\right] = \mathbb{E}_\eta\left[\exp\left(\int_{\mathfrak{D}} \log(1-u(z))\xi_t(\mathrm{d}z)\right)\right]$ and that for $\{\xi_t, t \geq 0\}$ Theorem 9.3.1 applies.

Using (9.3.3) combined with (9.2.6) and (9.3.1), we obtain

$$\begin{aligned} &\mathbb{E}_\eta^\lambda\left[\exp\left(\int \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right] \qquad (9.4.20)\\ = \;& \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\ldots,z_n)\lambda_t^{\otimes n}(\mathrm{d}(z_1\ldots z_n))\right)\\ &\times\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\ldots,z_n)(S_t^{\otimes n})^*\eta^{(n)}(\mathrm{d}(z_1,\ldots,z_n))\right)\\ = \;& 1 + \sum_{k,l} \frac{(-1)^{k+l}}{k!\,l!}\int u^{\otimes(k+l)}(z_1,\ldots,z_{k+l})\,\lambda_t^{\otimes k}(\mathrm{d}(z_1,\ldots,z_k)) \otimes (S_t^{\otimes l})^*\eta^{(l)}(\mathrm{d}(z_{k+1},\ldots,z_{k+l})). \end{aligned}$$

On the other hand we have

$$\mathbb{E}_\eta^\lambda\left[\exp\left(\int \log(1-u(z))\eta_t(\mathrm{d}z)\right)\right] = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\int u^{\otimes n}(z_1,\ldots,z_n)\mathbb{E}_\eta^\lambda[\eta_t^{(n)}](\mathrm{d}(z_1,\ldots,z_n)).$$

Then, via identification of the terms with n-fold tensor product of u in the last expression in (9.4.20) and the right hand side of the above identity, we obtain the following equality for all $n \in \mathbb{N}$:

$$\begin{aligned} &\int u^{\otimes n}(z_1,\ldots,z_n)\mathbb{E}_\eta^\lambda[\eta_t^{(n)}](\mathrm{d}(z_1,\ldots,z_n))\\ = \;& \sum_{k=0}^{n}\binom{n}{k}\int u^{\otimes n}(z_1,\ldots,z_n)\lambda_t^{\otimes k}(\mathrm{d}(z_1,\ldots,z_k)) \otimes (S_t^{\otimes(n-k)})^*\eta^{(n-k)}(\mathrm{d}(z_{k+1},\ldots,z_n)) \qquad (9.4.21) \end{aligned}$$

Via the above mentioned density argument of linear combinations of $u^{\otimes n}$ this implies (9.4.13).

Recalling the definition of $\lambda_t(\mathrm{d}z)$ we have that

$$\begin{aligned} \lambda_t^{\otimes k}(\mathrm{d}(z_1,\ldots,z_k)) &= \left(\prod_{i=1}^{k}\int_{\partial\mathfrak{D}} \lambda(u)\mathbb{P}_{z_i}^{\mathrm{abs}}(\tau_{\partial\mathfrak{D}} \leq t, X_{\tau_{\partial\mathfrak{D}}} \in \mathrm{d}u)\right)\mu^{\otimes k}(\mathrm{d}(z_1,\ldots,z_k))\\ &= \mathbb{E}_{z_{[k]}}^{\mathrm{abs}}\left[e^{\int_{\partial\mathfrak{D}} \log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}}\right]\mu^{\otimes k}(\mathrm{d}(z_1,\ldots,z_k)). \end{aligned}$$

If now we assume that (9.4.14) holds and choosing $\mu = m$, we obtain, integrating a bounded and permutation invariant function $f_n : \mathfrak{D}^n \to \mathbb{R}$

$$\mathbb{E}_\eta^\lambda\left[\int_{\mathfrak{D}^n} f_n(z_1,\ldots,z_n)\eta_t^{(n)}(\mathrm{d}(z_1,\ldots,z_n))\right]$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}^{\mathrm{abs}}_{z_{[k]}} \left[e^{\int_{\partial\mathfrak{D}} \log(\lambda) \mathrm{d}\xi_t} \mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}} \right] \int_{\mathfrak{D}^n} f_n(z) m^{\otimes k}(\mathrm{d}z_{[k]}) \otimes (S_t^{\otimes(n-k)})^* \eta^{(n-k)}(\mathrm{d}z_{[n]\setminus[k]}) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}^{\mathrm{abs}}_{z_{[k]}} \left[e^{\int_{\partial\mathfrak{D}} \log(\lambda) \mathrm{d}\xi_t} \mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}} \right] \\
&\quad \times \int_{\mathfrak{D}^n} \left(\int_{\mathfrak{D}^{n-k}} f_n(z_{[k]}, \boldsymbol{y}) \mathfrak{p}_t^{(n-k)}(z_{[n]\setminus[k]}, \boldsymbol{y}) m^{\otimes n-k}(\mathrm{d}\boldsymbol{y}) \right) m^{\otimes k}(\mathrm{d}z_{[k]}) \otimes \eta^{(n-k)}(\mathrm{d}z_{[n]\setminus[k]})
\end{aligned}
$$

where we recall that $\mathfrak{p}_t^{(r)}((v_1, \ldots, v_r), (u_1, \ldots, u_r)) = \prod_i^r \mathfrak{p}_t(v_i, u_i)$. Exchanging the integrals and using the the symmetry of the functions $\mathfrak{p}_t(\cdot, \cdot)$ leads to

$$
\begin{aligned}
&\mathbb{E}^{\lambda}_{\eta} \left[\int_{\mathfrak{D}^n} f_n(z_1, \ldots, z_n) \eta_t^{(n)}(\mathrm{d}(z_1, \ldots, z_n)) \right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \int_{\mathfrak{D}^n} m^{\otimes k}(\mathrm{d}z_{[k]}) \otimes m^{\otimes n-k}(\mathrm{d}\boldsymbol{y}) f_n(z_{[k]}, \boldsymbol{y}) \\
&\quad \times \left(\mathbb{E}^{\mathrm{abs}}_{z_{[k]}} \left[e^{\int_{\partial\mathfrak{D}} \log(\lambda) \mathrm{d}\xi_t} \mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}} \right] \int_{\mathfrak{D}^{n-k}} \mathfrak{p}_t^{(n-k)}(\boldsymbol{y}, z_{[n]\setminus[k]}) \, \eta^{(n-k)}(\mathrm{d}z_{[n]\setminus[k]}) \right)
\end{aligned}
$$

which, upon renaming the variables, can be rewritten as

$$
\begin{aligned}
\mathbb{E}^{\lambda}_{\eta} \left[\int_{\mathfrak{D}^n} f_n \mathrm{d}\eta_t^{(n)} \right] = \int_{\mathfrak{D}^n} f_n(z) \Bigg(&\sum_{k=0}^{n} \binom{n}{k} \mathbb{E}^{\mathrm{abs}}_{z_{[k]}} \left[e^{\int_{\partial\mathfrak{D}} \log(\lambda) \mathrm{d}\xi_t} \mathbf{1}_{\{\xi_t(\partial\mathfrak{D})=k\}} \right] \\
&\times \int_{\mathfrak{D}^{n-k}} \mathfrak{p}_t^{(n-k)}(z_{[n]\setminus[k]}, \boldsymbol{y}) \, \eta^{(n-k)}(\mathrm{d}\boldsymbol{y}) \Bigg) m^{\otimes n}(\mathrm{d}z).
\end{aligned}
$$

By taking the symmetrization of the above expression in brackets in the right hand side we obtain (9.4.15) and the proof is concluded . □

We conclude this section by looking at the evolution of a Poisson distributed particle cloud and by using duality to show the existence and the uniqueness of the stationary distribution for the system of boundary driven independent particles. Let ρ be a finite measure on $\mathfrak{D}$ and denote by $\mathcal{P}_\rho$ the Poisson point configuration with intensity ρ.

Theorem 9.4.4. *Let $\{\eta_t, t \geq 0\}$ be the boundary driven diffusion gas in the domain $\mathfrak{D}$ defined in* (9.4.11) *and let $\mu(dx)$ be the finite measure on $\mathfrak{D}$ appearing in* (9.4.10).

i) If η_0 is distributed according to $\mathcal{P}_\rho$, then η_t is the restriction to $\mathfrak{D}$ of the Poisson process on $\bar{\mathfrak{D}}$ with intensity

$$\rho_t = S_t^* \rho + \lambda_t \tag{9.4.22}$$

with λ_t defined in (9.4.3).

ii) Assume further (9.4.12) *and take $\mu = m$. Then, the unique stationary measure for the boundary driven diffusion process is given by the distribution of a Poisson point process with intensity*

$$\lambda_\infty(\mathrm{d}x) = h(x) m(\mathrm{d}x)$$

where

$$h(x) = \lambda(\infty, x) = \int_{\partial\mathfrak{D}} \lambda(u) \mathbb{P}^{\mathrm{abs}}_x (X_{\tau_{\partial\mathfrak{D}}} \in \mathrm{d}u).$$

Moreover, for any initial configuration η, the distribution of η_t converges weakly as $t \to \infty$ to the distribution of the Poisson point process with intensity $\lambda_\infty(\mathrm{d}x) = h(x)m(\mathrm{d}x)$.

Remark 9.4.5. *Notice that, when λ is a continuous function, $h : \mathfrak{D} \to \mathbb{R}$ given in Theorem 9.4.4(ii) is the solution of the following Dirichlet problem*

$$
\begin{cases}
\mathcal{L}h = 0 & \text{in } \mathfrak{D} \\
h = \lambda & \text{on } \partial\mathfrak{D}
\end{cases}
\tag{9.4.23}
$$

where $\mathcal{L}$ is the generator given in (9.4.17). *In particular, for the one-dimensional boundary driven Brownian gas, $\mathcal{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and*

$$\lambda_\infty(x) = \lambda_L(1 - x) + \lambda_R x.$$

Proof. The evolution of $\mathcal{P}_\rho$ under independent copies of absorbed particles $\{X_t, t \geq 0\}$ is equal to a Poisson point process with intensity $S_t^*\rho$ by the Doob's theorem (Theorem 9.3.4). Therefore (9.4.22) follows from the fact that the independent sum of two Poisson point processes is a Poisson point process with intensity measure the sum of the intensity measures. Further, notice that for every finite measure μ on $\mathfrak{D}$, we have

$$(S_t^{\otimes n})^* \mu^{\otimes n} \longrightarrow 0$$

as $t \to \infty$ because eventually all the mass from μ will be absorbed at the boundary $\partial\mathfrak{D}$. Therefore, by taking the limit $t \to \infty$ in (9.4.13), only the term $k = n$ survives and thus, the n-th factorial moment measures converge to $\lambda_\infty^{\otimes n}(\mathrm{d}\boldsymbol{x}) = \left(\prod_{i=1}^n h(x_i)\right)\mu^{\otimes n}(\mathrm{d}\boldsymbol{x})$ with

$$h(x) = \lim_{t\to\infty} \lambda(t, x) = \int_{\partial\mathfrak{D}} \lambda(u)\mathbb{P}_x^{\mathrm{abs}}(X_{\tau_{\partial\mathfrak{D}}} \in \mathrm{d}u).$$

This shows that the limiting distribution of η_t is indeed Poisson with intensity measure λ_∞. Since (9.4.12) implies that $\{\eta_t, t \geq 0\}$ is Markov, we conclude that the distribution of the Poisson point process with intensity λ_∞ is the unique stationary measure. □

9.4.3 Boundary driven Markov gas

In this section we provide another extension of the construction of Bertini and Posta [20] for systems of particles that can make jumps and thus, they do not necessarily hit the boundary when exiting a regular domain. Therefore, instead of associating a reservoir parameter function λ to the boundary of the domain only, we need to associate it rather to the complement of the domain. We therefore consider particles that evolve on a regular domain and are absorbed upon hitting a point in the complement of this domain. The examples that we have in mind are jump Markov processes (see, e.g., [110, Eq. 4]) with generator given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} a(x-y)\,[f(y) - f(x)]\;\mathrm{d}y\,, \tag{9.4.24}$$

with $a(-x) = a(x)$ and $f : \mathbb{R}^d \to \mathbb{R}$ a Borel measurable function with compact support (a system of particles evolving accordingly to (9.4.24) is then called *free Kawasaki dynamics*) and standard rotationally symmetric α stable processes (see, e.g., [41]) with generator given by

$$\mathcal{L} = \Delta^{\alpha/2} \tag{9.4.25}$$

for $\alpha \in (0, 2)$.

Let $\{Y_t, t \geq 0\}$ be a strong Markov process on $\mathbb{R}^d$. Let $\mathfrak{D}$ be regular domain of $\mathbb{R}^d$ (see Section 9.4.2) and define $\mathfrak{D}^{\mathrm{ext}} := \mathbb{R}^d \setminus \mathfrak{D}$.

Let $\{X_t, t \geq 0\}$ be the Markov process on $\mathbb{R}^d$ with semigroup $\{S_t, t \geq 0\}$ which evolves as $\{Y_t, t \geq 0\}$ on $\mathfrak{D}$ and is absorbed upon hitting $\mathfrak{D}^{\mathrm{ext}}$. We denote by $\mathbb{P}_x^{\mathrm{abs}}$ the distribution of the trajectories of $\{X_t, t \geq 0\}$ starting from $x \in \mathfrak{D}$, by $\tau_{\mathfrak{D}^{\mathrm{ext}}}$ the hitting time of the set $\mathfrak{D}^{\mathrm{ext}}$. We assume $\mathbb{P}_x^{\mathrm{abs}}(\tau_{\mathfrak{D}^{\mathrm{ext}}} < \infty) = 1$.

Let now $\lambda : \mathfrak{D}^{\mathrm{ext}} \to \mathbb{R}_+$ be a bounded measurable function giving the reservoir intensity at any $x \in \mathfrak{D}^{\mathrm{ext}}$ and let $\mu(dx)$ be a finite measure on $\mathfrak{D}$. We then define the point configuration (on $\mathbb{R}^d$) valued process $\{\xi_t, t \geq 0\}$ arising from independent copies of the absorbed Markov process $\{X_t, t \geq 0\}$ starting from $\xi_0 = \sum_i \delta_{x_i}$, $x_i \in \mathfrak{D}$. I.e., for $\xi := \sum_{i=1}^{\mathbf{N}} \delta_{x_i}$, $x_i \in \mathfrak{D}$, we define

$$\begin{cases} \xi_t := \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}\,, \\ \xi_0 = \xi \end{cases}$$

where $\{X_t(i)\}_{t\geq 0}$ are independent copies of $\{X_t\}_{t\geq 0}$ such that $X_0(i) = x_i$ for any $i \in [\mathbf{N}]$. For $\boldsymbol{x} = (x_1, \ldots, x_{\mathbf{N}}) \in \mathfrak{D}^{\mathbf{N}}$, we denote by $\mathbb{E}_{\boldsymbol{x}}^{\mathrm{abs}}$ the expectation in the process $\{\xi_t, t \geq 0\}$ starting from $\xi_0 = \sum_{i=1}^N \delta_{x_i}$. Finally we define Θ_t, a Poisson point process on $\mathfrak{D}$ independent of ξ_t and with time dependent intensity $\lambda_t(\mathrm{d}x)$ given by

$$\lambda_t(\mathrm{d}x) = \left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(y)\,\mathbb{P}_x^{\mathrm{abs}}(\tau_{\mathfrak{D}^{\mathrm{ext}}} \leq t, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}y)\right)\mu(\mathrm{d}x), \tag{9.4.26}$$

which is supposed to be finite. The *boundary driven Markov gas in the domain* $\mathfrak{D}$ *with reservoir intensity* λ, denoted by $\{\eta_t, t \geq 0\}$, is then given, for any $t \geq 0$, by

$$\eta_t = \xi_t\big|_{\mathfrak{D}} + \Theta_t, \tag{9.4.27}$$

viewed as a point configuration on $\mathfrak{D}$. We denote by $\mathbb{E}^\lambda_\eta$ the expectation in the process defined via (9.4.27) initialized from η.

Also in this context, $\{\eta_t,\ t \geq 0\}$ is a Markov process when the transition probability $p_t(x, \mathrm{d}y)$ of the process $\{X_t, t \geq 0\}$ satisfies

$$p_t(x, \mathrm{d}y)m(\mathrm{d}x) = p_t(y, \mathrm{d}x)m(\mathrm{d}y) \qquad \text{on} \qquad \mathfrak{D} \times \mathfrak{D} \tag{9.4.28}$$

for a finite measure m and for the reservoir intensity in (9.4.10) we choose $\mu = m$ (see Section 9.7 below).

We then have the following result generalizing Theorem 9.4.2. We omit the proof being a straightforward adaptation of the proof of Theorem 9.4.2.

Theorem 9.4.6. *Let $\{\eta_t, t \geq 0\}$ be boundary driven Markov gas defined in* (9.4.27). *Then for all $n \in \mathbb{N}$ and $t \geq 0$, it holds:*

a) for all bounded, measurable and permutation invariant $f : \mathfrak{D}^n \to \mathbb{R}$

$$\mathbb{E}^\lambda_\eta\left[\int_{\mathfrak{D}^n} f(z)\eta_t^{(n)}(\mathrm{d}z)\right] = \sum_{k=0}^{n}\binom{n}{k}\int_{\mathfrak{D}^n} f(z)\lambda_t^{\otimes k}(\mathrm{d}z_{[k]}) \otimes (S_t^{\otimes n-k})^*\eta^{(n-k)}(\mathrm{d}z_{[n]\setminus[k]}); \tag{9.4.29}$$

b) assume further that

$$p_t(x, \mathrm{d}y) = \mathfrak{p}_t(x, y)m(\mathrm{d}y) \tag{9.4.30}$$

for a symmetric function $\mathfrak{p}_t(x, y)$ and a finite measure m on $\mathfrak{D}$. Then, choosing $\mu = m$, the following holds

$$\frac{\mathrm{d}\mathbb{E}^\lambda_\eta[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(z) = \sum_{I\subset[n]}\mathbb{E}^{\mathrm{abs}}_{z_I}\left[e^{\int_{\mathfrak{D}^{\mathrm{ext}}}\log(\lambda)\mathrm{d}\xi_t}\mathbf{1}_{\{\xi_t(\mathfrak{D}^{\mathrm{ext}})=|I|\}}\right]\int_{\mathfrak{D}^{n-|I|}}\mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, \boldsymbol{y})\eta^{(n-|I|)}(\mathrm{d}\boldsymbol{y}) \tag{9.4.31}$$

Remark 9.4.7 (Examples). *i) The process $\{Y_t, t \geq 0\}$ with generator given in* (9.4.24) *is reversible with respect to the Lebesgue measure (see, e.g., [110, Remark 2.7]) but* (9.4.30) *is not satisfied since each particle has a positive probability to stay in the initial position during any time interval* $[0, t]$.

ii) A spherically symmetric α-stable processes on $\mathbb{R}^d$ with generator given in (9.4.25) *is strongly reversible w.r. to the Lebesgue measure (see, e.g. [41, Eq. 4.4]) and* (9.4.30) *is fulfilled.*

9.5 The discrete case

In this section we consider the discrete analogue of the boundary driven Brownian gas. Here by "discrete" we mean that the space on which the particles evolve is a lattice and the independent Brownians are replaced by independent random walks. Our first aim will be to show that such a process is equal (in distribution) to the *reservoirs process* $\{\zeta_t,\ t \geq 0\}$ defined via the generator in (9.1.5). We will then show how the boundary driven Brownian gas arises as a scaling limit of $\{\zeta_t,\ t \geq 0\}$.

9.5.1 On the equivalence of two definitions of boundary driven independent random walks

We consider the boundary driven Markov gas as explained in Section 9.4.3, where the process $\{Y_t, t \geq 0\}$ is chosen to be the rate $\frac{1}{2}$ symmetric nearest neighbor random walk jumping on the integers and domain $\mathfrak{D} = V_N = \{1, \dots, N\}$ with boundary $\{0, N + 1\}$. The restriction to the nearest neighbor case is for simplicity only. The generalization to independent walkers with generic jump rates $c(x, y)$, $x, y \in \mathbb{Z}$, absorbed upon leaving V_N is straightforward and so is the extension to more general graphs. Let $\widetilde{V}_N := \{0, \dots, N + 1\} = V_N \cup \{0, N + 1\}$ and $\{X_t, t \geq 0\}$ be the process evolving as $\{Y_t, t \geq 0\}$ on V_N and absorbed when hitting 0 or $N + 1$. Notice that in this context of nearest neighbor random walks, $\mathfrak{D}^{\mathrm{ext}}$ reduces to $\{0, N + 1\}$. We start the process from an initial configuration $\eta \in \mathbb{N}^{V_N}$, viewed as a point configuration on V_N, i.e. $\eta = \sum_i \delta_{x_i}$, where $x_i \in V_N$ are the initial positions of the particles. We define its time evolution as follows:

$$\eta_t = \xi_t\big|_{V_N} + \Theta_t. \tag{9.5.1}$$

Here ξ_t is the point configuration on $\tilde{V}_N$ at time t arising from $\xi_0 = \eta$ when all the particles in η evolve as independent copies of the process X_t defined above. For $z = (z_1, \dots, z_n) \in V_N^n$, we denote by $\mathbb{E}^{\mathrm{abs}}_z$ the expectation in the process $\{\xi_t, t \geq 0\}$ started from $\sum_{i=1}^n \delta_{z_i}$. Further, for $\lambda = (\lambda_L, \lambda_R) \in \mathbb{R}_+^2$, Θ_t is a Poisson point process on V_N with intensity defined by

$$\lambda_t(\mathrm{d}x) = \left(\lambda_L \mathbb{P}_x^{\mathrm{RW}}(X_t = 0) + \lambda_R \mathbb{P}_x^{\mathrm{RW}}(X_t = N+1)\right) m(\mathrm{d}x) \tag{9.5.2}$$

with $m(\mathrm{d}x)$ denoting the counting measure and $\mathbb{P}_x^{\mathrm{RW}}$ the path-space measure of the absorbed random walk $\{X_t, t \geq 0\}$. Thus, $\{\Theta_t(\{x\}), x \in V_N\}$ are independent random variables which are Poisson distributed with parameter $\lambda_t(x)$. The process defined in (9.5.1) is the discrete analogue of the process defined in (9.4.5) and from Theorem (9.4.6) applied to this context we have that

$$\frac{\mathrm{d}\mathbb{E}_\eta^\lambda[\eta_t^{(n)}]}{\mathrm{d}m^{\otimes n}}(z) = \sum_{I \subset [n]} \mathbb{E}_{z_I}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \int_{V_N^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, y)\eta^{(n-|I|)}(\mathrm{d}y), \tag{9.5.3}$$

where $\mathfrak{p}_t^{(n)}(z, y) = \prod_{i=1}^n \mathbb{P}_{z_i}^{\mathrm{RW}}(X_t = y)$ and $\mathbb{E}_\eta^\lambda$ denotes the expectation in the process $\{\eta_t, t \geq 0\}$ starting from η.

Let us now compare the process η_t with the process ζ_t, the reservoir process with parameters λ_L, λ_R introduced in Section 9.1.2.

Theorem 9.5.1. *Let $\eta \in \mathbb{N}^{V_N}$. Then $\{\zeta_t, t \geq 0\}$, denoting the reservoir process with parameters λ_L, λ_R and generator given in* (9.1.5) *started from η, and $\{\eta_t, t \geq 0\}$, denoting the boundary driven Markov gas defined in* (9.5.1) *started from η, are equal in distribution.*

Notice that in the statement of the Theorem we are implicitly identifying the point configuration η_t with the vector $(\eta_t(\{x\}))_{x \in V_N}$ of occupation variables.

Proof. In order to prove the result we will make use of the duality relations (9.1.7) and (9.5.3).

Indeed, it suffices to show that for all $\xi = \sum_{i=1}^n \delta_{z_i}$, $z_i \in \mathbb{N}^{V_N}$ one has for all η and $t \geq 0$

$$\mathbb{E}_\eta^{\mathrm{res}}\left[\prod_{x=1}^N d(\xi(\{x\}), \zeta_t(x))\right] = \mathbb{E}_\eta^\lambda\left[\prod_{x=1}^N d(\xi(\{x\}), \eta_t(\{x\}))\right]. \tag{9.5.4}$$

By (9.1.6) and (9.1.7) we have

$$\mathbb{E}_\eta^{\mathrm{res}}\left[\prod_{x=1}^N d(\xi(\{x\}), \zeta_t(x))\right] = \mathbb{E}_\xi^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x=1}^N d\left(\xi_t(\{x\}), \zeta(x)\right)\right].$$

On the other hand, by (9.3.7), we have

$$\mathbb{E}_\eta^\lambda\left[\prod_{x=1}^N d(\xi(\{x\}), \eta_t(\{x\}))\right] = \mathbb{E}_\eta^\lambda[\eta_t^{(n)}(\{(z_1, \ldots, z_n)\})]$$

and by (9.5.3)

$$\mathbb{E}_\eta^\lambda\left[\eta_t^{(n)}(\{z_1, \ldots, z_n\})\right] \tag{9.5.5}$$

$$= \sum_{I \subset [n]} \mathbb{E}_{z_I}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \int_{V_N^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, y)\eta^{(n-|I|)}(\mathrm{d}y). \tag{9.5.6}$$

It thus remains to show that

$$\sum_{I \subset [n]} \mathbb{E}_{z_I}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right] \int_{V_N^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, y)\eta^{(n-|I|)}(\mathrm{d}y) = \mathbb{E}_\xi^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x \in V_N} d\left(\xi_t(\{x\}), \eta(\{x\})\right)\right].$$

Notice that, for any $I \subset [n]$, by (9.3.7), we have

$$\int_{V_N^{n-|I|}} \mathfrak{p}_t^{(n-|I|)}(z_{[n]\setminus I}, y)\eta^{(n-|I|)}(\mathrm{d}y) = \mathbb{E}_{z_{[n]\setminus I}}^{\mathrm{abs}}\left[\prod_{x=1}^N d(\xi_t(\{x\}), \eta(\{x\}))\right] \tag{9.5.7}$$

We thus have,

$$
\begin{aligned}
&\mathbb{E}^{\lambda}_{\eta}\left[\eta^{(n)}_t(\{z_1,\dots,z_n\})\right]\\
&= \sum_{I\subset[n]} \mathbb{E}^{\text{abs}}_{z_I}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{1\})}\mathbf{1}_{\{\xi_t(\{0,N+1\})=|I|\}}\right]\mathbb{E}^{\text{abs}}_{z_{[n]\setminus I}}\left[\prod_{x=1}^{N} d(\xi_t(\{x\}),\eta(\{x\}))\right]\\
&= \mathbb{E}^{\text{abs}}_{z}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x=1}^{N} d(\xi_t(\{x\}),\eta(\{x\}))\right],
\end{aligned}
$$

where we used (9.5.5) and (9.5.7) in the first equality and the independence of particles in the second equality. □

9.5.2 Scaling limit

In this section we show how the process of independent random walkers with reservoirs λ_L and λ_R, when appropriately rescaled in space and time, and with rescaling of the reservoirs intensities, converges to the boundary driven Brownian gas. We start with the following lemma.

Lemma 9.5.2. *Consider $\{\Theta^{(N)}\}_{N\geq 1}$ a sequence of Poisson point processes on $(0,1)$ with the intensity measures*

$$\lambda^{(N)}(\mathrm{d}x) = \left(\frac{1}{N}\sum_{i=1}^{N} a_N(\tfrac{i}{N})\delta_{i/N}\right)(\mathrm{d}x) \tag{9.5.8}$$

with $a_N : \{\frac{1}{N},\dots,\frac{N-1}{N},1\} \to \mathbb{R}_+$. Assume furthermore that whenever $i/N \to x \in [0,1]$ then also

$$a_N(\tfrac{i}{N}) \to \alpha(x) \tag{9.5.9}$$

where $\alpha : [0,1] \to \mathbb{R}$ is a smooth function. Then as $N\to\infty$, $\Theta^{(N)}$ converges to the Poisson point process Θ with intensity $\alpha(x)\mathrm{d}x$.

Proof. Because sequences of Poisson point processes converge when the sequences of their intensity measures converge, it suffices to prove that (9.5.8) converges weakly to $\alpha(x)\mathrm{d}x$ as $N\to\infty$. Let $f : [0,1]\to\mathbb{R}$ continuous, then

$$\int f(x)\lambda^{(N)}(\mathrm{d}x) = \frac{1}{N}\sum_{i=1}^{N} f(\tfrac{i}{N})a_N(\tfrac{i}{N})$$

By the condition on $a_N(\frac{i}{N})$, this sum converges to the Riemann integral $\int_0^1 f(x)\alpha(x)dx$. □

We then have the following result.

Theorem 9.5.3. *Consider the reservoir process $\{\zeta_{N,t}, t\geq 0\}$ on the chain $\{1,\dots,N\}$, with reservoirs parameters $\frac{\lambda_L}{N}, \frac{\lambda_R}{N}$ and generator given in (9.1.5). Define $\eta_{N,t}(\mathrm{d}x)$ the point configuration on $[0,1]$ via*

$$\mathcal{Z}_{N,t}(\mathrm{d}x) = \left(\sum_{i=1}^{N}\zeta_{N,tN^2}(i)\delta_{i/N}\right)(\mathrm{d}x) \tag{9.5.10}$$

Assume that at time $t=0$,

$$\mathcal{Z}_{N,0} = \sum_{i=1}^{\mathbf{N}} \delta_{x_i^{(N)}/N} \tag{9.5.11}$$

where $x_i^{(N)}/N \to x_i \in (0,1)$ for all $i=1,\dots,\mathbf{N}$.

Then as $N\to\infty$ the process $\{\mathcal{Z}_{N,t}(\mathrm{d}x), t\geq 0\}$ converges (in the sense of convergence of finite dimensional distributions) to the boundary driven Brownian gas with parameters λ_L,λ_R, started at the configuration $\sum_{i=1}^{\mathbf{N}}\delta_{x_i}$.

Proof. As a consequence of Theorem 9.5.1, the reservoir process $\zeta_{N,t}$ equals (in distribution) the boundary driven Markov gas $\eta_{N,t}$ obtained as a sum of the configuration arising from letting the particles initially in the system evolve according to independent random walkers absorbed at 0 and $N+1$, and adding an independent Poisson point process on V_N with intensity

$$\lambda_t(i) = \tfrac{\lambda_L}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_t = 0) + \tfrac{\lambda_R}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_t = N+1) \tag{9.5.12}$$

where $\{X_t, t \geq 0\}$ denotes the random walk on V_N absorbed at the boundary $\{0, N+1\}$. Therefore after diffusive rescaling of space and time, the intensity of the Poisson point process on $(0,1)$ modelling the reservoirs effect becomes

$$\lambda_t^{(N)}(\mathrm{d}x) = \sum_{i=1}^{N} \Big(\tfrac{\lambda_L}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = 0) + \tfrac{\lambda_R}{N}\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = N+1) \Big)\, \delta_{i/N}(dx). \tag{9.5.13}$$

Because the absorbed random walk X_{tN^2}/N converges weakly, as $N \to \infty$, to the Brownian motion on $[0,1]$ absorbed at the boundaries, we can apply Lemma 9.5.2 with

$$a_N(\tfrac{i}{N}) = \lambda_L\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = 0) + \lambda_R\, \mathbb{P}_i^{\mathrm{RW}}(X_{tN^2} = N+1)$$

which converges, in the sense given by (9.5.9), to

$$\alpha(x) = \lambda_L\, \mathbb{P}_x^{\mathrm{abs}}(\tau_L \leq t) + \lambda_R\, \mathbb{P}_x^{\mathrm{abs}}(\tau_R \leq t) \tag{9.5.14}$$

where τ_L, τ_R are the hitting times of 0, resp. 1, and $\mathbb{P}_x^{\mathrm{abs}}$ the path space measure of Brownian motion started from x and absorbed whenever hitting $0, 1$. Therefore, the Poisson point processes (9.5.13) converge to the Poisson point processes with intensity (9.5.14). Clearly, by the weak convergence of the absorbed random walk X_{tN^2}/N to the absorbed Brownian motion, also the point configuration corresponding to the time evolution of the independent walkers initially in the system converge to the point configuration arising by letting the particles initially in the system evolve according to independent absorbed Brownian motions. Because the evolution of the particles initially in the system and the added Poisson process are independent, in the scaling limit, we obtain the sum of the evolution of the particles initially in the system and an independent Poisson point process with intensity (9.5.14), which is the boundary driven Brownian gas with reservoir parameters λ_L, λ_R. □

9.6 Orthogonal dualities

In order to have a complete analogy with the duality theory for independent random walks on a finite chain with reservoirs, we now investigate orthogonal dualities for the boundary driven Brownian gas.

9.6.1 Known orthogonal dualities

Closed discrete systems. Orthogonal self-duality functions are well known for the system of simple symmetric independent random walkers on $\mathbb{Z}^d$ described in Section 9.1.2 (see [78], [146]). More precisely, for any $\theta > 0$, the factorized functions given by

$$D_\theta^{\mathrm{or}}(\xi, \eta) = \prod_{x \in \mathbb{Z}^d} C_{\xi(x)}(\eta(x), \theta) \tag{9.6.1}$$

where $C_k(n, \theta)$ are the Charlier polynomials defined as

$$C_k(n, \theta) = \sum_{\ell=0}^{k} \binom{k}{\ell} (-\theta)^{k-\ell} (n)_\ell \tag{9.6.2}$$

($(n)_\ell$ denotes the ℓ-th falling factorial) are self-duality functions for the Markov process $\{\eta_t,\ t \geq 0\}$ with generator given in (9.1.1). The dualities in (9.6.1) satisfy the following orthogonality relation w.r.t. the measure $\mu_\theta^{\mathrm{rev}} = \otimes_{x \in \mathbb{Z}^d} \mathrm{Poisson}(\theta)$ which is reversible for $\{\eta_t,\ t \geq 0\}$: for any $\xi,\ \xi' \in \mathbb{N}^{\mathbb{Z}^d}$

$$\int D_\theta^{\mathrm{or}}(\xi, \eta) D_\theta^{\mathrm{or}}(\xi', \eta) \mathrm{d}\mu_\theta^{\mathrm{rev}}(\eta) = \mathbf{1}_{\{\xi = \xi'\}} \frac{\xi!}{\theta^{|\xi|}}$$

where $\xi! := \prod_{x \in \mathbb{Z}^d} \xi(x)!$ and $|\xi| = \sum_{x \in \mathbb{Z}^d} \xi(x)$.

Notice that the relation between orthogonal and classical dualities is given by (see [76, Remark 4.2])

$$D_\theta^{\mathrm{or}}(\xi, \eta) = \sum_{\xi' \leq \xi} (-\theta)^{|\xi| - |\xi'|} \binom{\xi}{\xi'} D^{\mathrm{cl}}(\xi', \eta) = \sum_{I \subset [n]} (-\theta)^{n - |I|} D^{\mathrm{cl}}\left(\sum_{i \in I} \delta_{y_i}, \eta \right), \tag{9.6.3}$$

where $\xi' \leq \xi$ means that $\xi'(x) \leq \xi(x)$ for any $x \in \mathbb{Z}^d$ and $\binom{\xi}{\xi'} := \prod_{x \in \mathbb{Z}^d} \binom{\xi(x)}{\xi'(x)}$.

Open discrete systems. Let us now reconsider the reservoir process with parameters λ_L, λ_R defined in Section 9.1.2. In Chapter 6, we proved that the following functions, for $\theta > 0$,

$$D^{\text{or}}_{\text{res},\theta}(\xi,\zeta) = (\lambda_L - \theta)^{\xi(0)} \mathscr{D}^{\text{or}}_{\theta}(\xi,\zeta)(\lambda_R - \theta)^{\xi(1)} \tag{9.6.4}$$

with

$$D^{\text{or}}_{\theta}(\xi,\zeta) = \prod_{x\in\widetilde{V}_N} C_{\xi(x)}(\zeta(x),\theta)$$

are duality functions between $\{\zeta_t, t \geq 0\}$ the Markov process on $V_N = \{1,\dots,N\}$ with generator given in (9.1.5) and $\{\xi_t,\ t\geq 0\}$ the system of random walkers on $\widetilde{V}_N = \{0,\dots,N+1\}$ absorbed at $\{0, N+1\}$. Notice that the orthogonality relation is w.r.t. $\mu_\theta = \otimes_{x\in V_N}\text{Poisson}(\theta)$ which is not stationary for the reservoir process with general parameters λ_L, λ_R, but it is reversible for the reservoir process with parameters $\lambda_L = \lambda_R = \theta$, the last case referred as the reservoir process in *equilibrium*.

Closed systems in the continuum. Generalizations of orthogonal self-dualities for systems considered in Section 9.3.1, namely closed systems of independent Markov processes on general Polish spaces E, has been studied in Chapter 8. More precisely, let $\eta_t = \sum_{i=1}^{\mathbf{N}} \delta_{X_t(i)}$ with $\{X_t(i),\ t\geq 0\}$ independent copies of a Markov process on E started from x_i, strongly reversible w.r.t. to a measure m. Then, the measure defined for any $t\geq 0$, $n\in\mathbb{N}$ and $\theta > 0$ as

$$\eta_t^{(n),\theta}(\mathrm{d}z) := \sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}z_I)\, m^{\otimes^{n-|I|}}(\mathrm{d}z_{[n]\setminus I}) \tag{9.6.5}$$

satisfies the following duality relation (see Proposition 3.2.5)

$$\frac{\mathrm{d}\mathbb{E}^{\lambda}_{\eta}[\eta_t^{(n),\theta}]}{\mathrm{d}m^{\otimes n}}(z_1,\dots,z_n) = \int \prod_{i=1}^{n} \mathfrak{p}_t(z_i,y_i)\eta^{(n),\theta}(\mathrm{d}(y_1,\dots,y_n))$$

and generalizes the orthogonal self-dualities given in (9.6.1) in the following sense:

i) let

$$\mathbf{1}_{\boldsymbol{B}}(z_1,\dots,z_n) := \left(\mathbf{1}_{B_1}^{\otimes d_1}\otimes\dots\otimes\mathbf{1}_{B_K}^{d_K}\right)(z_1,\dots,z_n)$$

for $\boldsymbol{B} = \{B_1,\dots,B_K\}$ a family of mutually disjoint sets in E and $\{d_1,\dots,d_K\}$ such $\sum_{i=1}^K d_i = n$, then

$$\int \mathbf{1}_{\boldsymbol{B}}(z_1,\dots,z_n)\,\eta^{(n),\theta}(d(z_1,\dots,z_n)) = \prod_{\ell=1}^{K}(-\theta m(B_\ell))^{d_\ell}\ C_{d_\ell}(\eta(B_\ell);\theta m(B_\ell)) \tag{9.6.6}$$

with $C_k(n,x)$ being the Charlier polynomials defined above.

ii) If we denote by $\mathcal{P}_{\theta m}$ the distribution of a Poisson point process with intensity measure θm, then, the following orthogonal relation holds

$$\mathbb{E}_{\mathcal{P}_{\theta m}}\left[\left(\int f_n \mathrm{d}\zeta^{(n),\theta}\right)\left(\int g_{n'}\mathrm{d}\zeta^{(n'),\theta}\right)\right] = \mathbf{1}_{\{n=n'\}}\cdot n! \int f_n\, g_n\ \mathrm{d}(\theta m)^{\otimes n} \tag{9.6.7}$$

for $\zeta\sim\mathcal{P}_{\theta m}$ and $f_n : E^n\to\mathbb{R}$, $g_{n'} : E^{n'}\to\mathbb{R}$ bounded and permutation invariant functions.

We refer to [120] for a proof of the two above facts.

The aim of the next section is to generalize the orthogonal dualities for the reservoir system given in (9.6.4) in the context of the boundary driven Brownian gas on $(0,1)$.

9.6.2 Orthogonal dualities for the boundary driven Brownian gas

Let us now consider the boundary driven Brownian gas on $(0,1)$ with parameters λ_L and λ_R

$$\eta_t = \xi_t + \Theta_t$$

defined in Section 9.4.1. We have previously proved that the factorial measure $\eta_t^{(n)}$ is the right object to study in order to have a duality result for boundary driven system. Inspired by the relation highlighted in the previous subsection between classical and orthogonal dualities we now study for any $n \in \mathbb{N}$ and $\theta > 0$

$$\eta_t^{(n),\theta}(\mathrm{d}z) := \sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}z_I)\, m^{\otimes^{n-|I|}}(\mathrm{d}z_{[n]\setminus I}), \tag{9.6.8}$$

viewed as a measure on $(0,1)^n$. Here $m(\mathrm{d}z)$ is the Lebesgue measure on $(0,1)$ and the orthogonality properties (9.6.6) and (9.6.7) hold for (9.6.8) for, respectively, $\boldsymbol{B} = \{B_1,\ldots,B_K\}$ a family of mutually disjoint sets in $(0,1)$ with $\{d_1,\ldots,d_K\}$ such $\sum_{i=1}^K d_i = n$, and bounded and permutation invariant functions $f_n : (0,1)^n \to \mathbb{R}$, $g_{n'} : (0,1)^{n'} \to \mathbb{R}$.

Notice that the orthogonality relations holds true w.r.t. the intensity measure of the Poisson point process whose distribution is reversible for the boundary driven Brownian gas *in equilibrium*, namely with $\lambda_L = \lambda_R = \theta$.

Moreover, since we will integrate the above defined measure $\eta^{(n),\theta}$ against $\mathfrak{p}_t^{(n)}(\cdot,\cdot) : [0,1]^n \times [0,1]^n \to \mathbb{R}$, i.e. a function defined on $\{0,1\}$ as well, we extend $\eta^{(n),\theta}$ in the following way: we define $\bar{m}(dz) = m(dz) + \delta_0(dz) + \delta_1(dz)$ and we denote

$$\eta_t^{[n],\theta}(\mathrm{d}z) := \sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}z_I)\, \bar{m}^{\otimes^{n-|I|}}(\mathrm{d}z_{[n]\setminus I}) \tag{9.6.9}$$

whenever integrated against functions being non zero also at the boundary $[0,1]$. Notice that $\int_{[0,1]} \mathfrak{p}_t(x,y)\bar{m}(\mathrm{d}y) = 1$ for any $x \in [0,1]$ and that we used the brackets $[\cdot]$ in the upper index of $\eta_t^{[n],\theta}$ to emphasize the difference with $\eta_t^{(n),\theta}$.

We then have the following theorem, providing orthogonal dualities between the boundary driven Brownian gas and the system of independent Brownian motions on $[0,1]$ absorbed at the boundaries.

Theorem 9.6.1. *For the boundary driven Brownian gas, the expectation of the measure given in* (9.6.8) *at time* $t \geq 0$ *is absolutely continuous w.r.t.* $m^{\otimes n}$ *with the following density:*

$$\frac{\mathrm{d}\mathbb{E}_\eta^\lambda[\eta_t^{(n),\theta}]}{\mathrm{d}m^{\otimes n}}(z) = \sum_{J\subset[n]} \mathbb{E}_{z_J}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \int_{E^{n-|J|}} \mathfrak{p}_t^{(n-|J|)}(z_{[n]\setminus J}, \boldsymbol{y})\eta^{[n-|J|],\theta}(\mathrm{d}\boldsymbol{y}). \tag{9.6.10}$$

Proof. Using (9.6.8) and (9.4.7) we have

$$\begin{aligned}
\mathbb{E}_\eta^\lambda[\eta_t^{(n),\theta}](\mathrm{d}z) &= \mathbb{E}_\eta^\lambda\left[\sum_{I\subset[n]} (-\theta)^{n-|I|}\, \eta_t^{(|I|)}(\mathrm{d}z_I)\, m^{\otimes^{n-|I|}}(\mathrm{d}z_{[n]\setminus I})\right] \\
&= \sum_{I\subset[n]} (-\theta)^{n-|I|}\mathbb{E}_\eta^\lambda\left[\eta_t^{(|I|)}(dz_I)\right] m^{\otimes^{n-|I|}}(\mathrm{d}z_{[n]\setminus I}) \\
&= \sum_{I\subset[n]} (-\theta)^{n-|I|}\Bigg(\sum_{J\subset I} \mathbb{E}_{z_J}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad\qquad \times \int_{E^{|I|-|J|}} \mathfrak{p}_t^{(|I|-|J|)}(z_{I\setminus J}, \boldsymbol{y})\, \eta^{(|I|-|J|)}(\mathrm{d}\boldsymbol{y})\Bigg) m^{\otimes^{n-|J|}}(\mathrm{d}z_{[n]\setminus J})
\end{aligned}$$

and by exchanging the order of the summation in the last expression above we obtain

$$\begin{aligned}
&\mathbb{E}_\eta^\lambda[\eta_t^{(n),\theta}](\mathrm{d}z) \\
&= \sum_{J\subset[n]} \mathbb{E}_{z_J}^{\mathrm{abs}}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right] \\
&\qquad \times \left(\sum_{I\subset[n]\setminus J} (-\theta)^{n-|I|-|J|} \int_{E^{|I|}} \mathfrak{p}_t^{(|I|)}(z_I, \boldsymbol{y})\, \eta^{(|I|)}(\mathrm{d}\boldsymbol{y})\right) m^{\otimes^{n-|J|}}(\mathrm{d}z_{[n]\setminus J})
\end{aligned}$$

We conclude by noticing that

$$\sum_{I\subset[n]\setminus J} (-\theta)^{n-|I|-|J|} \int_{E^{|I|}} \mathfrak{p}_t^{(|I|)}(z_I, \boldsymbol{y})\, \eta^{(|I|)}(\mathrm{d}\boldsymbol{y}) = \int_{E^{n-|J|}} \mathfrak{p}^{(n-|J|)}(z_{[n]\setminus J}, \boldsymbol{y})\, \eta^{[n-|J|],\theta}(\mathrm{d}\boldsymbol{y}). \tag{9.6.11}$$

which can be proved using (9.6.9). □

Notice that the same result holds for any boundary driven system of strongly reversible Markov processes as the ones treated in Section 9.4.2 and for the discrete system defined in (9.5.1).

We thus conclude the section by showing that indeed Theorem 9.6.1 generalizes the duality relation for the discrete system w.r.t. the orthogonal dualities given in (9.6.4). Notice, indeed, that we have, from (9.6.8) and (9.6.4),

$$\mathbb{E}^{\lambda}_{\eta}\left[\eta_t^{(n),\theta}(\{z_1,\dots,z_n\})\right] = \mathbb{E}^{\lambda}_{\eta}\left[D^{\mathrm{or}}_{\theta}(\xi,\eta_t)\right].$$

It thus remains to prove the following.

Proposition 9.6.2. *Let η_t denote the process defined in* (9.5.1)*. Then for all $n \in \mathbb{N}$ and $z_1,\dots,z_n \in V_N$, denoting $\sum_{i=1}^n \delta_{z_i} = \xi$, we have*

$$\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{z_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\int \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\,\eta^{(n-|J|),\theta}(d\boldsymbol{y}) = \mathbb{E}^{\mathrm{abs}}_{\xi}\left[D^{\mathrm{or}}_{\mathrm{res},\theta}(\xi_t,\eta)\right]. \tag{9.6.12}$$

Proof. By the definition of $\eta^{(n-|J|),\theta}$ we obtain

$$\begin{aligned}
&\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{z_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\int \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\,\eta^{(n-|J|),\theta}(d\boldsymbol{y})\\
&=\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{z_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\left(\sum_{I\subset[n]\setminus J}(-\theta)^{n-|J|-|I|}\int_{E^{|I|}}\mathfrak{p}_t^{(|I|)}(\boldsymbol{z}_I,\boldsymbol{y})\,\eta^{(|I|)}(d\boldsymbol{y})\right)\\
&=\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{z_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\left(\sum_{I\subset[n]\setminus J}(-\theta)^{n-|J|-|I|}\,\mathbb{E}^{\mathrm{abs}}_{z_I}\left[\prod_{x=1}^N d(\xi_t(\{x\}),\eta(\{x\}))\right]\right)\\
&=\sum_{J\subset[n]}\sum_{I\subset[n]\setminus J}(-\theta)^{n-|J|-|I|}\,\mathbb{E}^{\mathrm{abs}}_{z_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\mathbb{E}^{\mathrm{abs}}_{z_I}\left[\prod_{x=1}^N d(\xi_t(\{x\}),\eta(\{x\}))\right]\\
&=\sum_{U\subset[n]}(-\theta)^{n-|U|}\,\mathbb{E}^{\mathrm{abs}}_{z_U}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\prod_{x=1}^N d(\xi_t(\{x\}),\eta(\{x\}))\right],
\end{aligned}\tag{9.6.13}$$

where in the last line we used the independence of the particles. Combining (9.6.13), (9.3.7) and (9.1.6) we get

$$\begin{aligned}
&\sum_{J\subset[n]} \mathbb{E}^{\mathrm{abs}}_{z_J}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}\mathbf{1}_{\{\xi_t(\partial E)=|J|\}}\right]\int \mathfrak{p}_t^{(n-|J|)}(\boldsymbol{z}_{[n]\setminus J},\boldsymbol{y})\,\eta^{(n-|J|),\theta}(d\boldsymbol{y})\\
&=\sum_{U\subset[n]}(-\theta)^{n-|U|}\,\mathbb{E}^{\mathrm{abs}}_{z_U}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}D^{\mathrm{cl}}(\xi_t,\eta)\right]=\sum_{U\subset[n]}(-\theta)^{n-|U|}\,\mathbb{E}^{\mathrm{abs}}_{z_U}\left[D^{\lambda_L,\lambda_R}(\xi_t,\eta)\right].
\end{aligned}$$

We then have, using again the independence of particles,

$$\begin{aligned}
&\sum_{U\subset[n]}(-\theta)^{n-|U|}\,\mathbb{E}^{\mathrm{abs}}_{z_U}\left[\lambda_L^{\xi_t(\{0\})}\lambda_R^{\xi_t(\{N+1\})}D^{\mathrm{cl}}(\xi_t,\eta)\right]\\
&=\mathbb{E}^{\mathrm{abs}}_{z}\left[\sum_{\xi'\le\xi_t}\binom{\xi_t}{\xi'}(-\theta)^{n-\xi'(\bar V_N)}\,\lambda_L^{\xi'(\{0\})}\,\lambda_R^{\xi'(\{N+1\})}\,D^{\mathrm{cl}}\left(\xi'\big|_{V_N},\eta\right)\right]\\
&=\mathbb{E}^{\mathrm{abs}}_{z}\left[\left(\sum_{\ell=0}^{\xi_t(\{0\})}\binom{\xi_t(\{0\})}{\ell}(-\theta)^{\xi_t(\{0\})-\ell}\,\lambda_L^{\ell}\right)\left(\sum_{\xi'\le\xi_t|_{V_N}}\binom{\xi_t|_{V_N}}{\xi'}(-\theta)^{\xi_t(V_N)-\xi'(V_N)}\,\mathscr{D}^{\mathrm{cl}}(\xi',\eta)\right)\right.\\
&\qquad\qquad\left.\times\left(\sum_{r=0}^{\xi_t(\{N+1\})}\binom{\xi_t(\{N\})}{r}(-\theta)^{\xi_t(\{N\})-r}\,\lambda_R^{r}\right)\right]\\
&=\mathbb{E}^{\mathrm{abs}}_{\xi}\left[(\lambda_L-\theta)^{\xi_t(\{0\})}(\lambda_R-\theta)^{\xi_t(\{N+1\})}D^{\mathrm{or}}_{\theta}\left(\xi_t\big|_{V_N},\eta\right)\right]
\end{aligned}$$

where the third identity follows from (9.6.3). The proof is concluded by noticing that

$$(\lambda_L-\theta)^{\xi_t(\{0\})}(\lambda_R-\theta)^{\xi_t(\{N+1\})}D^{\mathrm{or}}_{\theta}\left(\xi_t\big|_{V_N},\eta\right) = D^{\mathrm{or}}_{\mathrm{res},\theta}(\xi_t,\eta).$$

□

9.7 Markov property of the boundary driven independent particles

Theorem 9.7.1. *Assume that*

$$p_t(x, \mathrm{d}y)m(\mathrm{d}x) = p_t(y, \mathrm{d}x)m(\mathrm{d}y) \qquad on \qquad \mathfrak{D} \times \mathfrak{D} \tag{9.7.1}$$

for some finite measure $m(\mathrm{d}x)$ *on* $\mathfrak{D}$. *Denote by* $\mathcal{P}_{\lambda_t}$ *the law of a Poisson point process with intensity measure given in* (9.4.26) *with the measure* m *in place of* μ *and let* $P_t^{\mathrm{res}} : \Omega \times \mathcal{B}(\Omega) \to [0, 1]$, $t \geq 0$ *defined by*

$$P_t^{\mathrm{res}}(\eta, B) := \int_{\Theta+\xi\in B} \mathcal{P}_{\lambda_t}(\mathrm{d}\Theta)P_t(\eta, \mathrm{d}\xi), \tag{9.7.2}$$

where P_t *denotes the semigroup of the process* $\{\xi_t,\ t \geq 0\}$. *Then, the family* P_t^{res}, $t \geq 0$ *is a time homogeneous transition function on* $(\Omega, \mathcal{B}(\Omega))$ *and there exists a Markov family with transition function* P_t^{res}.

Proof. We need to show that P_t^{res} satisfies the Chapman-Kolmogorv equation, which, due to [20, Lemma A.3], boils down to check that for any continuous function ψ with compact support strictly contained in $\mathfrak{D}$ and for any $s, t > 0$

$$\int P_{s+t}^{\mathrm{res}}(\eta, \mathrm{d}\bar{\eta})e^{i\int\psi\mathrm{d}\bar{\eta}} = \int\int P_s^{\mathrm{res}}(\eta, \mathrm{d}\zeta)P_t^{\mathrm{res}}(\zeta, \mathrm{d}\bar{\eta})e^{i\int\psi\mathrm{d}\bar{\eta}}.$$

By the definition of P_{t+s}^{res} and using (9.3.12), we have that the left hand side is equal to

$$\exp\left\{\int(e^{i\psi} - 1)\mathrm{d}\lambda_{t+s}\right\}\int P_{s+t}(\eta, \mathrm{d}\xi)e^{i\int\psi\mathrm{d}\xi}.$$

On the other hand, for the right hand side we have,

$$\begin{aligned}
&\int\int P_s^{\mathrm{res}}(\eta, \mathrm{d}\zeta)P_t^{\mathrm{res}}(\zeta, \mathrm{d}\bar{\eta})e^{i\int\psi\mathrm{d}\bar{\eta}} \\
&= \int\mathcal{P}_{\lambda_s}(\mathrm{d}\Theta_1)\int P_s(\eta, \mathrm{d}\xi_1)\int\mathcal{P}_{\lambda_t}(\mathrm{d}\Theta_2)\int P_t(\Theta_1 + \xi_1, \mathrm{d}\xi_2)e^{i\int\psi\mathrm{d}(\xi_2+\Theta_2)} \\
&= \exp\left\{\int(e^{i\psi} - 1)\mathrm{d}\lambda_t\right\}\int\mathcal{P}_{\lambda_s}(\mathrm{d}\Theta_1)\int P_s(\eta, \mathrm{d}\xi_1)\int P_t(\Theta_1 + \xi_1, \mathrm{d}\xi_2)e^{i\int\psi\mathrm{d}\xi_2} \\
&= \exp\left\{\int(e^{i\psi} - 1)\mathrm{d}\lambda_t\right\} \\
&\qquad\times\int\mathcal{P}_{\lambda_s}(\mathrm{d}\Theta_1)\int P_s(\eta, \mathrm{d}\xi_1)\int P_t(\Theta_1, \mathrm{d}\xi_{2,1})e^{i\int\psi\mathrm{d}\xi_{2,1}}\int P_t(\xi_1, \mathrm{d}\xi_{2,2})e^{i\int\psi\mathrm{d}\xi_{2,2}} \\
&= \exp\left\{\int(e^{i\psi} - 1)\mathrm{d}\lambda_t\right\}\int\mathcal{P}_{\lambda_s}(\mathrm{d}\Theta_1)\int P_t(\Theta_1, \mathrm{d}\xi_{2,1})e^{i\int\psi\mathrm{d}\xi_{2,1}}\int P_{t+s}(\eta, \mathrm{d}\xi)e^{i\int\psi\mathrm{d}\xi}
\end{aligned}$$

where we used the definition of P_t^{res} first, the independence of the particles after and finally the Champan-Kolmogorov equation for P_t. Thus, it remains to show that

$$\int\mathcal{P}_{\lambda_s}(\mathrm{d}\Theta_1)\int P_t(\Theta_1, \mathrm{d}\xi_{2,1})e^{i\int\psi\mathrm{d}\xi_{2,1}} = \exp\left\{\int(e^{i\psi} - 1)\mathrm{d}\lambda_{t+s} - \int(e^{i\psi} - 1)\mathrm{d}\lambda_t\right\}. \tag{9.7.3}$$

By the independence of the particles and (9.3.12) follows that

$$\int\mathcal{P}_{\lambda_s}(\mathrm{d}\Theta_1)\int P_t(\Theta_1, \mathrm{d}\xi_{2,1})e^{i\int\psi\mathrm{d}\xi_{2,1}} = \exp\left\{\int S_t(e^{i\psi} - 1)(x)\lambda_s(\mathrm{d}x)\right\},$$

where S_t denotes the semigroup of the absorbed Markov process upon hitting $\mathfrak{D}^{\mathrm{ext}}$ and which is given by

$$S_t f(x) = \int_{\mathfrak{D}} p_t(x, \mathrm{d}y)f(y) + \int_{\mathfrak{D}^{\mathrm{ext}}} f(z)\mathbb{P}_x(\tau_{\mathfrak{D}^{\mathrm{ext}}} \leq t,\ X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)$$

for any $f : \mathfrak{D}^* \to \mathbb{R}$ bounded function. Being ψ zero at $\mathfrak{D}^{\mathrm{ext}}$ we have that

$$\int S_t(e^{i\psi} - 1)(x)\lambda_s(\mathrm{d}x) = \int\left(\int p_t(x, \mathrm{d}y)(e^{i\psi(y)} - 1)\right)\lambda_s(\mathrm{d}x)$$

and thus, (9.7.3) is given if one proves that

$$\int p_t(x, \mathrm{d}y)\lambda_s(\mathrm{d}x) = \lambda_{t+s}(\mathrm{d}y) - \lambda_t(\mathrm{d}y). \tag{9.7.4}$$

Using the definition of λ_s given in (9.4.26), we have that the left hand side of (9.7.4) is equal to

$$\left[\int p_t(x, \mathrm{d}y)\left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z)\,\mathbb{P}_x(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le s,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)\right) m(\mathrm{d}x)\right].$$

Using the strong Markov property of the absorbed Markov process, we have

$$\begin{aligned}
\lambda_{t+s}(\mathrm{d}y) &= \left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z)\mathbb{P}_y(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le t+s,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)\right) m(\mathrm{d}y)\\
&= \left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z)\mathbb{P}_y(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le t,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)\right) m(\mathrm{d}y)\\
&\qquad + \left[\int p_t(y, \mathrm{d}x)\left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z)\mathbb{P}_x(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le s,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)\right) m(\mathrm{d}y)\right]\\
&= \lambda_t(\mathrm{d}y) + \left[\int p_t(x, \mathrm{d}y)\left(\int_{\mathfrak{D}^{\mathrm{ext}}} \lambda(z)\mathbb{P}_x(\tau_{\mathfrak{D}^{\mathrm{ext}}} \le s,\, X_{\tau_{\mathfrak{D}^{\mathrm{ext}}}} \in \mathrm{d}z)\right) m(\mathrm{d}x)\right]
\end{aligned}$$

where in the last identity we used the condition (9.7.1). Thus (9.7.4) follows, concluding the proof. □

Bibliography

[1] L. Accardi and A. Boukas. Quantum probability, renormalization and infinite-dimensional*-Lie algebras. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 5:Paper 056, 2009.

[2] L. Accardi, U. Franz, and M. Skeide. Renormalized Squares of White Noise and Other Non-Gaussian Noises as Lévy Processes on Real Lie Algebras. *Communications in Mathematical Physics*, 228(1):123–150, 2002.

[3] E. Aifantis. A new interpretation of diffusion in high-diffusivity paths - a continuum approach. *Acta Metallurgica*, 27:683–691, 1979.

[4] E. Aifantis and J. Hill. On the theory of diffusion in media with double diffusivity i. basic mathematical results. *The Quarterly Journal of Mechanics and Applied Mathematics*, 33:1–21, 1980.

[5] W. Al-Salam and T. Chihara. Convolutions of orthonormal polynomials. *SIAM Journal on Mathematical Analysis*, 7(1):16–28, 1976.

[6] G. Amir, C. Bahadoran, and E. Busani, O. Saada. Invariant measures for multilane exclusion process. *arXiv preprint arXiv:2105.12974*, 2021.

[7] S. Andres, M. T. Barlow, J.-D. Deuschel, and B. M. Hambly. Invariance principle for the random conductance model. *Probability Theory and Related Fields*, 156(3-4):535–580, 2013.

[8] S. Andres, J.-D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. *The Annals of Probability*, 43(4):1866–1891, 2015.

[9] S. Andres, J.-D. Deuschel, and M. Slowik. Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probability Theory and Related Fields*, 164(3-4):931–977, 2016.

[10] S. Andres, A. Chiarini, J.-D. Deuschel, and M. Slowik. Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *The Annals of Probability*, 46(1):302–336, 2018.

[11] M. Ayala, G. Carinci, and F. Redig. Higher order fluctuation fields and orthogonal duality polynomials. *Electronic Journal of Probability*, 26:1–35, 2021.

[12] C. Bahadoran, H. Guiol, K. Ravishankar, and E. Saada. Euler hydrodynamics for attractive particle systems in random environment. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 50(2):403–424, 2014.

[13] M. T. Barlow and J.-D. Deuschel. Invariance principle for the random conductance model with unbounded conductances. *The Annals of Probability*, 38(1):234–276, 2010.

[14] R. Bass. *Probability techniques in analysis*. Springer-Verlag, 1995.

[15] P. Bella and M. Schäffner. Quenched invariance principle for random walks among random degenerate conductances. *The Annals of Probability*, 48(1):296–316, 2020.

[16] Y. Berezansky. Infinite-dimensional non-gaussian analysis and generalized shift operators. *Functional Analysis and Its Applications*, 30(4):61–65, 1996.

[17] Y. Berezansky. Pascal measure on generalized functions and the corresponding generalized Meixner polynomials. *Methods of Functional Analysis and Topology*, 8(1):1–13, 2002.

[18] N. Berger and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probability Theory Related Fields*, 137(1-2):83–120, 2007.

[19] D. Bernstein and W. So. Some explicit formulas for the matrix exponential. *IEEE Trans. Autom. Control*, 38(8):1228–1232, 1993.

[20] L. Bertini and G. Posta. Boundary driven Brownian gas. *ALEA, Latin American Journal of Probability and Mathematical Statistics*, 16:361—-388, 2019.

[21] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim. Macroscopic fluctuation theory. *Reviews of Modern Physics*, 87(2):593–636, 2015.

[22] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1968.

[23] M. Biskup. Recent progress on the random conductance model. *Probability Surveys*, 8:294–373, 2011.

[24] M. Biskup. An invariance principle for one-dimensional random walks among dynamical random conductances. *Electronic Journal of Probability*, 24:29 pp., 2019.

[25] J. Blath and N. Kurt. Population genetic models of dormancy. In *Probabilistic Structures in Evolution (eds. E. Baake and A. Wakolbinger)*, pages 247–265, 2021.

[26] R. Blumenthal and R. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York-London, 1968.

[27] T. Bodineau and M. Lagouge. Large deviations of the empirical currents for a boundary-driven reaction diffusion model. *The Annals of Applied Probability*, 22:2282–2319, 2012.

[28] V. Bogachev. *Measure Theory*, volume 1,2. Springer Berlin Heidelberg, 2007.

[29] G. Boldrighini, A. De Masi, and A. Pellegrinotti. Nonequilibrium fluctuations in particle systems modelling reaction-diffusion equation. *Stochastic Processes and their Applications*, 42:1–30, 1992.

[30] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion Facts and Formulae*. Second Edition, Springer,, 2002.

[31] D. Brockington and J. Warren. The Bethe Ansatz for Sticky Brownian Motions. Preprint. arXiv:2104.06482, 2021.

[32] G. Carinci, C. Giardinà, C. Giberti, and F. Redig. Duality for stochastic models of transport. *Journal of Statistical Physics*, 152(4):657–697, 2013.

[33] G. Carinci, C. Giardinà, C. Giberti, and F. Redig. Dualities in population genetics: A fresh look with new dualities. *Stochastic Processes and their Applications*, 125(3):941–969, 2015.

[34] G. Carinci, C. Franceschini, C. Giardinà, W. Groenevelt, and F. Redig. Orthogonal dualities of Markov processes and unitary symmetries. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 15:Paper 053, 2019.

[35] G. Carinci, C. Giardinà, and F. Redig. *Duality for Markov Processes: a Lie Algebraic Approach*. monograph in progress, 2020.

[36] G. Carinci, S. Floreani, C. Giardinà, and F. Redig. Boundary driven markov gas: duality and scaling limits. *arXiv preprint arXiv:2112.12698*, 2021.

[37] G. Carinci, C. Giardinà, and F. Redig. Consistent particle systems and duality. *Electronic Journal of Probability*, 26:1–31, 2021.

[38] E. A. Carlen, S. Kusuoka, and D. W. Stroock. Upper bounds for symmetric markov transition functions. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 23(S2):245–287, 1987.

[39] M. Cavalcanti, V. Cavalcanti, and L. Tebou. Stabilization of the wave equation with localized compensating frictional and kelvin-voigt dissipating mechanism. *Electronic Journal of Differential Equations*, 83:1–18, 2017.

[40] J. P. Chen and F. Sau. Higher order hydrodynamics and equilibrium fluctuations of interacting particle systems. *Markov Processes and Related Fields*, 27(3):339–380, 2021.

[41] Z. Q. Chen, A. A. Meerschaert, and E. Nane. Space–time fractional diffusion on bounded domains. *Journal of Mathematical Analysis and Applications*, 393:479–488, 2012.

[42] Z.-Q. Chen, D. A. Croydon, and T. Kumagai. Quenched invariance principles for random walks and elliptic diffusions in random media with boundary. *The Annals of Probability*, 43(4):1594–1642, 2015.

[43] J. Cividini, M. D., and H. Posch. Driven tracer with absolute negative mobility. *Journal of Physics A: Mathematical and Theoretical*, 51:085001, 2018.

[44] G. C. Colangeli, M., C. Giberti, and C. Vernia. Non-equilibrium two dimensional ising model with stationary uphill diffusion. *Physical Review E*, 96:052137, 2017.

[45] M. Colangeli, A. De Masi, and E. Presutti. Microscopic models for uphill diffusion. *Journal of Physics A: Mathematical and Theoretical*, 50:435002, 2017.

[46] D. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. I: Elementary theory and methods*. Probability and its Applications. Springer, New York, second edition, 2003.

[47] A. De Masi and E. Presutti. *Mathematical methods for hydrodynamic limits*. Lecture Notes in Mathematics. Springer, Berlin, 1991.

[48] A. De Masi, P. A. Ferrari, and J. L. Lebowitz. Rigorous derivation of reaction-diffusion equations with fluctuations. *Physical Review Letters*, 55:1947–1949, 1985.

[49] A. De Masi, P. A. Ferrari, and J. L. Lebowitz. Reaction-diffusion equations for interacting particle systems. *Journal of Statistical Physics*, 44:589–644, 1986.

[50] A. De Masi, P. A. Ferrari, S. Goldstein, and W. D. Wick. An invariance principle for reversible Markov processes. Applications to random motions in random environments. *Journal of Statistical Physics*, 55 (3-4):787–855, 1989.

[51] A. De Masi, A. Merola, and E. Presutti. Reservoirs, fick law and the darken effect. *Journal of Mathematical Physics*, 62:073301, 2021.

[52] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Revista Matemática Iberoamericana*, pages 181–232, 1999.

[53] T. Demaerel and C. Maes. Active processes in one dimension. *Physical Review E*, 97:032604, 2018.

[54] B. Derrida. Non-equilibrium steady states: fluctuations and large deviations of the density and of the current. *Journal of Statistical Mechanics: Theory and Experiment*, 2007.

[55] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier. Exact solution of a 1d asymmetric exclusion model using a matrix formulation. *Journal of Physics A: Mathematical and Theoretical*, 26:1493–1517, 1993.

[56] B. Derrida, J. L. Lebowitz, and E. Speer. Large deviation of the density profile in the steady state of the open symmetric simple exclusion process. *Journal of Statistical Physics*, 107:599–634, 2002.

[57] B. Derrida, J. Lebowitz, and E. Speer. Entropy of open lattice systems. *Journal of Statistical Physics*, 126 (4–5):1083–1108, 2007.

[58] G. Di Nunno, B. Oksendal, and F. Proske. White noise analysis for lèvy processes. *Journal of Functional Analysis*, 206(1):109–148, 2004.

[59] A. E. Continuum basis for diffusion in regions with multiple diffusivity. *Journal of Applied Physics*, 50: 1334, 1979.

[60] A. Etheridge. *An Introduction to Superprocesses*. AMS, University Lecture Series, 2000.

[61] A. Etheridge. Evolution in fluctuating populations. In *In Mathematical statistical physics*. 2006.

[62] S. Ethier and T. Kurtz. *Markov Processes, Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics, 1986.

[63] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.

[64] A. Faggionato. Bulk diffusion of 1D exclusion process with bond disorder. *Markov Processes and Related Fields*, (13):519–542, 2007.

[65] A. Faggionato. Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit. *Electronic Journal of Probability*, 13:2217–2247, 2008.

[66] A. Faggionato. Hydrodynamic limit of zero range processes among random conductances on the supercritical percolation cluster. *Electronic Journal of Probability*, 15:259–291, 2010.

[67] A. Faggionato. Stochastic homogenization in amorphous media and applications to exclusion processes. *arXiv:1903.07311*, 2020.

[68] A. Faggionato, M. Jara, and C. Landim. Hydrodynamic behavior of 1D subdiffusive exclusion processes with random conductances. *Probability Theory and Related Fields*, 144(3-4):633–667, July 2009.

[69] W. Feller. *An Introduction to Probability Theory and Its Applications, Vol. 2*, volume Volume 2. Wiley, 2 edition, 1971.

[70] P. Ferrari and J. Martin. Martin multiclass processes, dual points and M/M/1 queues. *Markov Processes and Related Fields*, 12:273–299, 2006.

[71] P. Ferrari and J. Martin. Stationary distributions of multi-type totally asymmetric exclusion processes. *The Annals of Probability*, 35:807–832, 2007.

[72] D. Finkelshtein, Y. Kondratiev, E. Lytvynov, and M. Oliveira. Stirling operators in spatial combinatorics. *Journal of Functional Analysis*, 45(Paper 109285), 2021.

[73] S. Floreani, S. Jansen, R. Redig, and S. Wagner. Intertwining and duality for consistent markov processes. Preprint. arxiv:2112.11885, 2021.

[74] S. Floreani, F. Redig, and F. Sau. Hydrodynamics for the partial exclusion process in random environment. *Stochastic Processes and their Applications*, 142:124–158, 2021.

[75] S. Floreani, C. Giardinà, F. den Hollander, S. Nandan, and F. Redig. Switching interacting particle systems: scaling limits, uphill diffusion and boundary layer. *Journal of Statistical Physics*, 186(33), 2022.

[76] S. Floreani, F. Redig, and F. Sau. Orthogonal polynomial duality of boundary driven particle systems and non-equilibrium correlations. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 58(1):220–247, 2022.

[77] E. Fodor and C. Marchetti. The statistical physics of active matter: From self-catalytic colloids to living cells. *Physica A: Stat. Mech. Appl.*, 504:106–120, 2018.

[78] C. Franceschini and C. Giardinà. Stochastic duality and orthogonal polynomials. In *Sojourns in Probability Theory and Statistical Physics-III*, pages 187–214. Springer, 2019.

[79] C. Franceschini, P. Gonçalves, and F. Sau. Symmetric Inclusion Process with slow boundary: hydrodynamics and hydrostatics. *Bernoulli*, 28(2):1340–1381, 2022.

[80] R. Frassek and C. Giardinà. Exact solution of an integrable non-equilibrium particle system. Preprint. arXiv:2107.01720, 2021.

[81] M. Friedlin. *Functional Integration and Partial Differential Equations, Annals of Mathematics Studies*, volume 109. Princeton University Press, 2011.

[82] N. Gantert, E. Nestoridi, and D. Schmid. Mixing times for the simple exclusion process with open boundaries. *arXiv preprint arXiv:2003.03781*, 2020.

[83] C. Giardinà, J. Kurchan, and F. Redig. Duality and exact correlations for a model of heat conduction. *Journal of Mathematical Physics*, 48(3):033301, 2007.

[84] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi. Duality and Hidden Symmetries in Interacting Particle Systems. *Journal of Statistical Physics*, 135:25–55, 2009.

[85] C. Giardinà, F. Redig, and K. Vafayi. Correlation inequalities for interacting particle systems with duality. *Journal of Statistical Physics*, 141(2):242–263, 2010.

[86] T. Gilbert. Heat conduction and the nonequilibrium stationary states of stochastic energy exchange processes. *Journal of Statistical Mechanics: Theory and Experiment*, 8(27):083205, 2017.

[87] P. Gonçalves and M. Jara. Scaling Limits for Gradient Systems in Random Environment. *Journal of Statistical Physics*, 131(4):691–716, 2008.

[88] P. Gonçalves, M. Jara, O. Menezes, and A. Neumann. Non-equilibrium and stationary fluctuations for the ssep with slow boundary. *Stochastic Processes and their Applications*, 7:4326–4357, 2020.

[89] A. Greven, F. den Hollander, and M. Oomen. Spatial populations with seed-bank: well-posedness, duality and equilibrium. *Electronic Journal of Probability*, 27:1–88, 2022.

[90] W. Groenevelt. Orthogonal Stochastic Duality Functions from Lie Algebra Representations. *Journal of Statistical Physics*, 174(1):97–119, 2019.

[91] R. Großmann, F. Peruani, and B. M. Diffusion properties of active particles with directional reversal. *New Journal of Physics*, 18:043009, 2016.

[92] I. S. Helland. Central limit theorems for martingales with discrete or continuous time. *Scandinavian Journal of Statistics*, pages 79–94, 1982.

[93] J. Hill. A discrete random walk model for diffusivity in media with double diffusivity. *Journal of the Australian Mathematical Society*, 22:58–74, 1980.

[94] J. Hill. On the solution of reaction-diffusion equations. *IMA Journal of Applied Mathematics*, 27:177–194, 1981.

[95] J. Hill and E. Aifantis. On the theory of diffusion in media with double diffusivity ii. boundary-value problems. *The Quarterly Journal of Mechanics and Applied Mathematics*, 33:23–42, 1980.

[96] R. A. Holley and D. W. Stroock. Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. *Publications of the Research Institute for Mathematical Sciences*, 14(3):741–788, 1978.

[97] C. Howitt and J. Warren. Consistent families of Brownian motions and stochastic flows of kernels. *The Annals of Probability*, 37(4):1237–1272, 2009.

[98] M. Jara. Hydrodynamic Limit of the Exclusion Process in Inhomogeneous Media. In M. M. Peixoto, A. A. Pinto, and D. A. Rand, editors, *Dynamics, Games and Science II*, pages 449–465, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[99] M. D. Jara and C. Landim. Quenched non-equilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder. *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, 44(2):341–361, 2008.

[100] O. Kallenberg. *Random Measures, Theory and Applications*. Springer, 2017.

[101] G. Kallianpur and J. Xiong. Stochastic differential equations in infinite dimensional spaces. *Lecture Notes-Monograph Series*, 26:iii–342, 1995.

[102] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus, Graduate Texts in Mathematics*, volume 113. Springer-Verlag, New York, 1991.

[103] J. Kent. Time-reversible diffusions. *Advances in Applied Probability*, 10:819–835, 1978.

[104] J. Kingman. Completely random measures. *Pacific Journal of Mathematics*, 21(1):59 – 78, 1967.

[105] J. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York, 1993.

[106] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*, volume 320 of *Grundlehren der mathematischen Wissenschaften*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[107] C. Kipnis, C. Marchioro, and E. Presutti. Heat flow in an exactly solvable model. *Journal of Statistical Physics*, 27(1):65–74, 1982.

[108] R. Koekoek, P. Lesky, and R. Swarttouw. *Hypergeometric Orthogonal Polynomials and their q-Analogues*. Springer, Heidelberg, 2010.

[109] Y. Kondratiev, E. Lytvynov, and M. Röckner. Equilibrium Kawasaki Dynamics Of Continuous Particle Systems. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 10(02):185–209, 2007.

[110] Y. Kondratiev, T. Kuna, M. Oliveira, J. da Silva, and L. Streit. Hydrodynamic limits for the free Kawasaki dynamics of continuous particle systems. Preprint. arXiv:0912.1312, 2009.

[111] M. Kourbane-Houssene, C. Erignoux, T. Bodineau, and J. Tailleur. Exact hydrodynamic description of active lattice gases. *Physical Review Letters*, 120:268003, 2018.

[112] T. Kozubowski and K. Podgórski. Distributional properties of the negative binomial Lévy process. *Probability and Mathematical Statistics*, 29(1):43–71, 2009.

[113] R. Krishna. Uphill diffusion in multicomponent mixtures. *Chemical Society Reviews*, 44:2812–2836, 2015.

[114] A. Kràmli, N. Simànyi, and D. Szàsz. Random walks with internal degrees of freedom. *Probability Theory and Releted Fields*, 72:603–617, 1986.

[115] J. Kuan. Probability distributions of multi-species q-tazrp and asep as double cosets of parabolic subgroups. *Annales Henri Poincarè*, 20:1149–1173, 2019.

[116] T. G. Kurtz. Semigroups of conditioned shifts and approximation of markov processes. *The Annals of Probability*, 4(3):618–642, 1975.

[117] C. Landim, A. Milanès, and S. Olla. Stationary and nonequilibrium fluctuations in boundary driven exclusion processes. *Markov Processes and Related Fields*, 14:165–184, 2008.

[118] G. Last. Stochastic analysis for Poisson processes. In G. Peccati and M. Reitzner, editors, *Stochastic analysis for Poisson point processes. Malliavin calculus, Wiener-Itô chaos expansions and stochastic geometry*, volume 7 of *Bocconi Springer Ser.*, pages 1–36. Bocconi Univ. Press, 2016.

[119] G. Last and M. Penrose. Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probability Theory and Related Fields*, 150(3):663–690, 2011.

[120] G. Last and M. Penrose. *Lectures on the Poisson Process*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2017.

[121] Y. Le Jan and O. Raimond. Flows, coalescence and noise. *The Annals of Probability*, 32(2):1247 – 1315, 2004.

[122] J. Lebowitz, E. Presutti, and H. Spohn. Microscopic models of hydrodynamic behavior. *Journal of Statistical Physicsa*, 51(5):841–862, 1998.

[123] A. Lenard. Correlation functions and the uniqueness of the state in classical statistical mechanics. *Communications in Mathematical Physics*, 30(1):35 – 44, 1973.

[124] J. Lennon, F. den Hollander, M. Wilke Berenguer, and J. Blath. Principle of seed banks and the emergence of complexity from dormancy. *Nature Communications*, 12(4807), 2021.

[125] T. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[126] T. Liggett. *Interacting particle systems*. Classics in mathematics. Springer, Berlin, 2 edition, 2005.

[127] A. Løkka and F. Proske. Infinite dimensional analysis of pure jump Lévy processes on the Poisson space. *Mathematica Scandinavica*, 98(2):237–261, 2006.

[128] E. Lytvynov. Orthogonal Decompositions For Lévy Processes With An Application To The Gamma, Pascal, And Meixner Processes. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, pages 73–102, 2003.

[129] E. Lytvynov. Polynomials of Meixner's type in infinite dimensions—Jacobi fields and orthogonality measures. *Journal of Functional Analysis*, 200(1):118–149, 2003.

[130] P. Lévy. *Processus stochastiques et mouvement Brownien.* Gauthier-Villars, Paris, 1948.

[131] K. Malakar, V. Jemseena, A. Kundu, K. K Kumar, S. Sabhapandit, S. Majumdar, S. Redner, and A. Dhar. Steady state, relaxation and first-passage properties of a run-and-tumble particle in one-dimension. *Journal of Statistical Mechanics: Theory and Experiment*, 4:043215, 2018.

[132] F. Martinelli and A. Faggionato. Hydrodynamic limit of a disordered lattice gas. *Probability Theory and Related Fields*, 127:535–608, 2003.

[133] P. Mathieu. Quenched Invariance Principles for Random Walks with Random Conductances. *Journal of Statistical Physics*, 130(5):1025–1046, 2008.

[134] P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 463(2085):2287–2307, 2007.

[135] S. McInerney, E. J. Carr, and M. J. Simpson. Parameterising continuum models of heat transfer in heterogeneous living skin using experimental data. *International Journal of Heat and Mass Transfer*, 128:964–975, 2019.

[136] P. Meyer. *Quantum Probability for Probabilists*, volume 1538 of *Lecture Notes in Mathematics*. Springer, Berlin, 2 edition, 1995.

[137] T. Meyer-Brandis. Differential equations driven by Lévy white noise in spaces of Hilbert space-valued stochastic distributions. *Stochastics. An International Journal of Probability and Stochastic Processes*, 80 (4):371–396, 2008.

[138] I. Mitoma. Tightness of probabilities on $C([0,1];\mathscr{S}')$ and $D([0,1];\mathscr{S}')$. *The Annals of Probability*, 11(4): 989–999, 1983.

[139] K. Nagy. Symmetric random walk in random environment in one dimension. *Periodica Mathematica Hungarica*, 45(1-2), 2002.

[140] D. Nualart and W. Schoutens. Chaotic and predictable representations for Lévy processes. *Stochastic Processes and their Applications*, 90(1):109–122, 2000.

[141] P. Nàndori. Local equilibrium in inhomogeneous stochastic models of heat transport. *Journal of Statistical Physics*, 164(2):410–437, 2016.

[142] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences 44, Springer-Verlag, New York, 1983.

[143] P. Pietzonka, K. Kleinbeck, and U. Seifert. Extreme fluctuations of active brownian motion. *New Journal of Physics*, 18:052001, 2016.

[144] J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer, Berlin, 2006.

[145] G. D. Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, 2nd edition, 2014.

[146] F. Redig and F. Sau. Factorized Duality, Stationary Product Measures and Generating Functions. *Journal of Statistical Physics*, 172(4):980–1008, 2018.

[147] F. Redig, E. Saada, and F. Sau. Symmetric simple exclusion process in dynamic environment: hydrodynamics. *Electronic Journal of Probability*, 25(47):Paper No. 138, 2020.

[148] R. Rhodes. Stochastic homogenization of reflected stochastic differential equations. *Electronic Journal of Probability*, 15:989–1023, 2010.

[149] E. Schertzer, R. Sun, and J. Swart. The Brownian web, the Brownian net, and their universality. In *Advances in disordered systems, random processes and some applications*, pages 270–368. Cambridge Univ. Press, Cambridge, 2017.

[150] K. Schmüdgen. *The moment problem*. Springer Berlin Heidelberg, New York, 2017.

[151] W. Schoutens. *Stochastic processes and orthogonal polynomials*, volume 146 of *Lecture Notes in Statistics*. Springer, New York, 2000.

[152] G. Schütz and S. Sandow. Non-Abelian symmetries of stochastic processes: Derivation of correlation functions for random-vertex models and disordered-interacting-particle systems. *Physical Review E*, 49(4): 2726–2741, 1994.

[153] T. Seppäläinen. *Translation Invariant Exclusion Processes*. monograph in progress, 2016.

[154] R. Serfozo. Point processes. In *Stochastic models*, volume 2 of *Handbooks in operations research and management science*, pages 1–93. North-Holland, Amsterdam, 1990.

[155] V. Sidoravicius and A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probability Theory Related Fields*, 129(2):219–244, 2004.

[156] F. Spitzer. Interaction of markov processes. *Advances in Mathematics*, 5:246–290, 1970.

[157] D. W. Stroock and W. Zheng. Markov Chain Approximations to Symmetric Diffusions. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 33(5):619–649, 1997.

[158] D. Surgailis. On multiple Poisson stochastic integrals and associated Markov semigroups. *Probability and Mathematical Statistics*, 3(2):217–239, 1984.

[159] A. A. Tsambali, A. Konstantinidi, and E. Aifantis. Modeling double diffusion in soils and materialsy. *Journal of the Mechanical Behavior of Materials*, 20182003, 2018.

[160] A. Yablonski. The Calculus of Variations for Processes with Independent Increments. *Rocky Mountain Journal of Mathematics*, 38(2):669 – 701, 2008.

[161] Z. C. Yin, G. Properties of solutions of stochastic differential equations with continuous state dependent switching. *Journal of Differential Equations*, 249:2409–2439, 2010.

Summary

Interacting particle systems (IPS) is a subfield of probability theory that provided a fruitful framework in which several questions of physical interests have been answered with mathematical rigor. An interacting particle system is a stochastic system consisting of a very large number of particles interacting with each other. The class of IPS considered in this manuscript is the one of systems satisfying stochastic duality. Stochastic duality is a useful tool in probability theory which allows to study a Markov process (the one that interests you) via another Markov process, called dual process, which is hopefully easier to be studied. The connection between the two processes is established via a function, the so-called duality function, which takes as input configurations of both processes. In the context of IPS, one of the typical simplifications provided by stochastic duality is that a system with an infinite number of particles can be studied via a finite number of particles (the simplification *from many to few*).

In this thesis, we extend the theory and the applications of stochastic duality in the following two contexts:

i) evolution of particles in space inhomogeneous settings and more precisely, processes in random environment and processes in a multi-layer system;

ii) evolutions of particles in the continuum.

IPS in random environment

We introduce a new random environment for the exclusion process in $\mathbb{Z}^d$ obtained by assigning a maximal occupancy α_x to each site $x \in \mathbb{Z}^d$. This maximal occupancy is allowed to randomly vary among sites, and partial exclusion occurs. We refer to *random environment* as the collection $\boldsymbol{\alpha} = \{\alpha_x,\ x \in \mathbb{Z}^d\}$ and we denote the partial exclusion process in the random environment $\boldsymbol{\alpha}$ by SEP($\boldsymbol{\alpha}$). We show that, under the assumption of ergodicity under translation and uniform ellipticity of the environment, for almost every realization of $\boldsymbol{\alpha}$, the path-space hydrodynamic limit of SEP($\boldsymbol{\alpha}$) is a deterministic diffusion equation with a non-degenerate diffusion matrix not depending on the realization of the environment. To this purpose, first we show that SEP($\boldsymbol{\alpha}$) satisfies self-duality. Second, by employing the technology developed for the random conductance model, we prove a homogenization result in the form of an arbitrary starting point quenched invariance principle for a single particle in the same environment, which is a result of independent interest. More precisely, we prove that for almost every realization of $\boldsymbol{\alpha}$, for all $T > 0$, for any macroscopic point $u \in \mathbb{R}^d$ and for any sequence of points $\{x_N\}_{N\in\mathbb{N}} \subseteq \mathbb{Z}^d$ such that $\frac{x_N}{N} \to u$ as $N \to \infty$, the laws of the diffusively rescaled random walk in the environment $\boldsymbol{\alpha}$ started from $\frac{x_N}{N}$, converge weakly to the law of the Brownian motion started from $u \in \mathbb{R}^d$ and with a non-degenerate covariance matrix Σ. Σ is deterministic and does not depend of the realization of the environment $\boldsymbol{\alpha}$. Finally, the self-duality property of the partial exclusion process allows us to transfer this homogenization result to the particle system.

We then consider symmetric partial exclusion and inclusion processes in a general graph V in contact with reservoirs, where we allow for both edge disorder $\boldsymbol{\omega} = \{\omega_{\{x,y\}}\ :\ x, y \in V\}$ and site disorder $\boldsymbol{\alpha} = \{\alpha_x\ :\ x \in V\}$. This disorder may be thought as a realization of a random environment. We extend the classical dualities to this context and then we derive new orthogonal polynomial dualities. From the classical dualities, we derive the uniqueness of the non-equilibrium steady state and obtain correlation inequalities. Starting from the orthogonal polynomial dualities, we show universal properties of n-point correlation functions in the non-equilibrium steady state for systems with at most two different reservoir parameters θ_L and θ_R, such as a chain with reservoirs at left and right ends. Namely, denoting by μ_{θ_L,θ_R} the unique non-equilibrium steady state, we prove that for all distinct $x_1, \ldots, x_n$ in V

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\prod_{i=1}^{n}\left(\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_{x_i}\right)\right] = (\theta_L - \theta_R)^n\, \psi(\delta_{x_1} + \cdots + \delta_{x_n})\,,$$

where $\psi(\delta_{x_1} + \cdots + \delta_{x_n}) \in \mathbb{R}$ does not depend on neither θ_L nor θ_R.

IPS in a multi-layer system

We considers three classes of interacting particle systems on $\mathbb{Z}$: independent random walks, the exclusion process and the inclusion process. Particles are allowed to switch their jump rate (the rate identifies the *type* of particle) between 1 (*fast particles*) and $\epsilon \in [0,1]$ (*slow particles*). The switch between the two jump rates happens at rate $\gamma \in (0,\infty)$. In the exclusion process, the interaction is such that each site can be occupied by at most one particle of each type. In the inclusion process, the interaction takes places between particles of the same type at different sites and between particles of different type at the same site. We derive the macroscopic limit equations for the three systems, obtained after scaling space by N^{-1}, time by N^2, the switching rate by N^{-2}, and letting $N \to \infty$. The limit equations for the macroscopic densities associated to the fast and slow particles is the well-studied double diffusivity model, i.e.

$$\begin{cases} \partial_t \rho_0 = \Delta\rho_0 + \Upsilon(\rho_1 - \rho_0) \\ \partial_t \rho_1 = \epsilon\Delta\rho_1 + \Upsilon(\rho_0 - \rho_1) \end{cases}$$

where ρ_i, $i \in \{0,1\}$, are the macroscopic densities of the two types of particles, and $\Upsilon \in (0,\infty)$ is the scaled switching rate. The above system was introduced in the 70s to model polycrystal diffusion (more generally, diffusion in inhomogeneous porous media) and dislocation pipe diffusion, with the goal of overcoming the restrictions imposed by Fick's law. Non-Fick behaviour is immediate from the fact that the total density $\rho = \rho_0 + \rho_1$ does not satisfy the classical diffusion equation, but the *thermal telegrapher equation*, i.e.

$$\partial_t (\partial_t \rho + 2\Upsilon\rho) = -\epsilon\Delta(\Delta\rho) + (1+\epsilon)\Delta(\partial_t\rho + \Upsilon\rho).$$

Additional motivations to study multi–layer IPS comes from population genetics, where individuals can be either active or dormant (see, e.g., [124]), and from models of interacting active random walks with an internal state that changes randomly (e.g. activity, internal energy) and that determines their diffusion rate and or drift (see, e.g. [114]).

We provide a discussion on the solution of the double diffusivity model, thereby connecting mathematical literature applied to material science and financial mathematics.

In order to investigate the microscopic out–of–equilibrium properties,we analyse the system on $[N] = \{1,\dots,N\}$, adding boundary reservoirs at sites 1 and N of fast and slow particles, respectively. Inside $[N]$, particles move as before, but now they are injected and absorbed at sites 1 and N with prescribed rates that depend on the particle type. We compute the steady-state density profile and the steady-state current. This leads to two interesting phenomena. The first phenomenon is *uphill diffusion*, i.e., in a well-defined parameter regime, the current can go against the particle density gradient: when the total density of particles at the left end is higher than at the right end, the current can still go from right to left. The second phenomenon is *boundary-layer behaviour*: in the limit as $\epsilon \downarrow 0$, in the macroscopic stationary profile the densities in the top and bottom layer are equal, which for unequal boundary conditions in the top and bottom layer results in a *discontinuity* in the stationary profile. Corresponding to this jump in the macroscopic system, we identify a boundary layer of size $\sqrt{\epsilon}\,\log(1/\epsilon)$ in the microscopic system where the densities are unequal. The quantification of the *size* of this boundary layer is an interesting corollary of the exact macroscopic stationary profile that we obtain from the microscopic system via duality.

Systems of particles evolving in the continuum

Using the language of point process theory, we introduce a new framework in which self-duality type relations, more precisely self-intertwining relations, with respect to polynomials can be formulated for particle systems evolving on a general Borel space, thus also on $\mathbb{R}^d$. This framework also provides a new approach to self-duality. We thus first provide necessary and sufficient conditions to have self-intertwining relations with generalized falling factorial polynomials as intertwiners. In particular, we provide new self-intertwining results for systems such as independent and interacting Brownian motions. Moreover, from this new approach, the known self-duality functions for classical conservative interacting particle systems are recovered. Our approach is thus unifying the previous self-duality results for conservative systems and avoids the need of ad hoc computations for each system when proving duality. Second, we prove that, assuming reversibility for the particle system, the Gram-Schmidt orthogonalization procedure is a symmetry for the particle dynamics of a consistent process. As a consequence, orthogonalizing the previously introduced falling factorial polynomial self-intertwinings, we show orthogonal self-intertwinings in the same context of consistent particle systems on general state spaces. In doing so, we also show some properties of generalized orthogonal polynomials which are of independent interest. We also introduce and study a new process in the continuum, called *generalized symmetric inclusion process*, for which all our self-intertwining results apply.

Finally, inspired by the recent work of Bertini and Posta [20], who introduced the boundary driven Brownian gas on $[0, 1]$, we study stochastic duality for boundary driven systems of independent particles in a general setting. The *boundary-driven Brownian gas* on $[0, 1]$ is defined (see [20]) as a system of independent Brownian motions absorbed at 0 and at 1, to which an independent Poisson point process is super–imposed, which adds particles on $(0, 1)$ with well-chosen intensity. For our aim, we need to generalize the construction of Bertini and Posta [20] first to systems of independent diffusion processes evolving on regular domain $\mathfrak{D} \subset \mathbb{R}^d$ and second to systems of general independent Markov processes which are allowed to jump and which thus can leave $\mathfrak{D}$ without hitting its boundary. We then prove duality for such systems with a dual process that is absorbed at the boundaries, thereby creating a general framework that unifies dualities for boundary driven systems in the discrete and continuum setting. In particular, we show that the time-dependent n-th factorial moment measures of the boundary driven system can be written in terms of n dual particles, absorbed at the boundaries. We use duality first to show that from any initial condition the systems evolve to the unique invariant measure, which is a Poisson point process with intensity the solution of a Dirichlet problem. Second, we prove that in the discrete setting of a one-dimensional chain, modelling the reservoirs as: i) birth and death processes at the boundaries or ii) by a Poissonian addition of particles everywhere, are indeed equivalent processes. To conclude, we show that the boundary driven Brownian gas (in the continuum) arises as the diffusive scaling limit of the model with birth and death processes (in the discrete) when the intensities are also scaled with the system size.

Samenvatting

Interacterende deeltjessystemen (IDS) is een onderdeel van de kansrekening dat een rijk kader heeft verschaft waarin verscheidene vragen, afkomstig uit natuurkundig belang, zijn beantwoord met wiskundige nauwkeurigheid. Een interacterend deeltjessysteem is een stochastisch systeem bestaande uit een groot aantal deeltjes die interactie op elkaar uitoefenen. De IDS die in dit manuscript worden beschouwd zijn de systemen die voldoen aan stochastische dualiteit. Stochastische dualiteit is een handig hulpmiddel in de kansrekening waarmee een Markov process (degene die jou interesseert) bestudeerd kan worden via een ander Markov process, het duale process genoemd, die hopelijk gemakkelijker bestudeerd kan worden. De verbinding tussen de twee processen wordt gevormd door een functie, de zogenaamde dualiteit functie, waarin configuraties van beide processen ingevoerd worden. Een typische vereenvoudiging die geleverd wordt door stochastische dualiteit in de context van IDS, is dat een systeem met oneindig veel deeltjes bestudeerd kan worden via eindig veel deeltjes (de vereenvoudiging *van veel naar weinig*).

In dit proefschrift breiden we de theorie en toepassingen van stochastische dualiteit uit in de volgende twee contexten:

i) evolutie van deeltjes in plaats-inhomogene zettingen en, preciezer, processen in willekeurige omgeving en processen in een meerlaags systeem.

ii) evolutie van deeltjes in het continuüm.

IDS in willekeurige omgeving

We introduceren een nieuwe willekeurige omgeving voor het exclusie process in $\mathbb{Z}^d$ verkregen door een maximale bezetting α_x toe te wijzen aan elke plek $x \in \mathbb{Z}^d$. Deze maximale bezetting mag willekeurig variëren, wat leidt tot partiële exclusie. We refereren naar *willekeurige omgeving* als de collectie $\boldsymbol{\alpha} := \{\alpha_x : x \in \mathbb{Z}^d\}$ en we noteren het partiële exclusie process in de willekeurige omgeving $\boldsymbol{\alpha}$ als SEP($\boldsymbol{\alpha}$). We laten zien dat, onder de aanname van ergodiciteit onder translatie en uniforme ellipticiteit van de omgeving, voor bijna alle $\boldsymbol{\alpha}$, de hydrodynamische limiet in de padenruimte van SEP($\boldsymbol{\alpha}$) een deterministische diffusievergelijking is met een niet-gedegenereerde diffusie matrix, welke niet afhangt van de realisatie van de omgeving. Hiervoor laten we eerst zien dat SEP($\boldsymbol{\alpha}$) voldoet aan zelf-dualiteit. Daarna, door middel van een techniek gebruikt voor het willekeurige geleidingsmodel, bewijzen we het homogenisatie resultaat in de vorm van willekeurig beginpunt gedoofd invariantie principe voor een enkel deeltje in dezelfde omgeving, wat een resultaat is van afzonderlijke interesse. Exacter, we bewijzen dat voor bijne alle realisaties $\boldsymbol{\alpha}$, voor alle $T > 0$, voor elk macroscopisch punt $u \in \mathbb{R}^d$ en voor elke rij punten $\{x_N\}_{N\in\mathbb{N}} \subset \mathbb{Z}^d$ zodat $\frac{x_N}{N} \to u$ wanneer $N \to \infty$, de verdelingen van de diffuus geschaalde toevalswandeling in de omgeving $\boldsymbol{\alpha}$ startend van $\frac{x_N}{N}$ zwak convergeert naar de verdeling van de Brownse beweging startend van $u \in \mathbb{R}^d$ met niet-gedegenereerde covariantie matrix Σ. Σ is deterministisch en hangt niet af van de realisatie van de omgeving $\boldsymbol{\alpha}$. Tenslotte, met de zelf-dualiteits eigenschap van het partiële exclusie process kunnen we dit homogenisatie resultaat overdragen naar het deeltjessysteem.

Daarna beschouwen we symmetrische partiële exclusie en inclusie processen in een algemene graaf V in contact met reservoirs, waar we zowel onregelmatigheden aan de rand $\boldsymbol{\omega} = \{\omega_{\{x,y\}} : x, y \in V\}$ als aan de plaats $\boldsymbol{\alpha} = \{\alpha_x : x \in V\}$ beschouwen. Deze onregelmatigheden kunnen gezien worden als een realisatie van een willekeurige omgeving. We breiden de klassieke dualiteiten in deze context uit en daarna leiden we nieuwe dualiteiten af met orthogonale polynomen. Vanuit de klassieke dualiteiten leiden we de de uniciteit van de stabiele toestand uit evenwicht af en verkrijgen we correlatie ongelijkheden. Vanuit de dualiteiten met orthogonale polynomen laten we universele eigenschappen van n-punts correlatie functies in de stabiele toestand uit evenwicht zien voor systemen met uiterst twee verschillende reservoir parameters θ_L en θ_R, zoals een keten met reservoirs aan de twee uiteinden. Oftewel, als we μ_{θ_L,θ_R} noteren als de unieke stabiele toestand uit evenwicht, bewijzen we dat voor alle distinctieve $x_1, ..., x_n \in V$

$$\mathbb{E}_{\mu_{\theta_L,\theta_R}}\left[\prod_{i=1}^{n}\left(\frac{\eta(x_i)}{\alpha_{x_i}} - \bar{\theta}_{x_i}\right)\right] = (\theta_L - \theta_R)^n\, \psi(\delta_{x_1} + \cdots + \delta_{x_n})\,,$$

waar $\psi(\delta_{x_1} + \cdots + \delta_{x_n}) \in \mathbb{R}$ niet afhangt van zowel θ_L als θ_R.

IDS in een meerlaags systeem

We beschouwen drie klassen van interacterende deeltjessystemen op $\mathbb{Z}$: onafhankelijke toevalswandelingen, het exclusie process en het inclusie process. Deeltjes kunnen hun sprongsnelheden schakelen (de snelheid bepaalt het type deeltje) tussen 1 (*snelle deeltjes*) en $\epsilon \in [0, 1]$ (*langzame deeltjes*). De schakeling tussen sprongsnelheden gebeurt met snelheid $\gamma \in (0, \infty)$. In het exclusieprocess is de interactie zo dat elke plek enkel door één deeltje per type bezet kan worden. In het inclusieprocess vindt de interactie plaats tussen deeltjes van hetzelfde type op verschillende plekken en tussen verschillende typen op dezelfde plek. We herleiden de macroscopische limietvergelijkingen voor de drie systemen, verkregen na het herschalen van de ruimte met N^{-1}, de tijd met N^2, de schakelssnelheid met N^{-2} en daarna $N \to \infty$. De limietvergelijkingen van de macroscopische dichtheden geassocieerd met de snelle en langzame deeltjes is het goed bestudeerde dubbel diffisiviteits model, i.e.,

$$\begin{cases}\partial_t\rho_0 = \Delta\rho_0 + \Upsilon(\rho_1 - \rho_0)\\ \partial_t\rho_1 = \epsilon\Delta\rho_1 + \Upsilon(\rho_0 - \rho_1)\end{cases}$$

waar $\rho_i, i \in \{0, 1\}$ de macroscopische dichtheden zijn van de twee typen deeltjes, en $\Upsilon \in (0, \infty)$ de geschaalde schakelsnelheid is. Het systeem boven was geïntroduceerd in de jaren 70 om polykristal diffusie (algemener, diffusie in inhomogene poreuze media) en dislocatie pijp diffusie te modelleren, met als doel het overkomen van de restricties opgelegd door de wet van Fick. Non-Fick gedrag volgt direct uit het feit dat de totale dichtheid $\rho = \rho_0 + \rho_1$ niet de klassieke diffusievergelijking volgt, maar de *thermale telegraafvergelijking*, i.e.,

$$\partial_t(\partial_t\rho + 2\Upsilon\rho) = -\epsilon\Delta(\Delta\rho) + (1 + \epsilon)\Delta(\partial_t\rho + \Upsilon\rho)$$

Aanvullende motivaties om meerlaagse IDS te bestuderen komen van populatiegenetica, waar individuen ofwel actief of inactief kunnen zijn (zie, e.g., [124]), en van modellen van interacterende actieve toevalswandelingen met een interne toestand die willekeurig verandert (e.g. activiteit, interne energie) wat hun diffusiesnelheid en/of drijfkracht bepaald (zie, e.g., [114])

Wij voorzien een discussie over de oplossing van het dubbel diffusiviteitsmodel, waarbij we mathematische literatuur, toegepast op materiaalwetenschap, verbinden met financiële wiskunde.

Om de microscopische eigenschappen buiten evenwicht the onderzoeken, analyseren we het systeem op $[N] = \{1, ..., N\}$, met reservoirs voor snelle en langzame deetlejs aan de randen op de plekken 1 en N. Binnenin $[N]$ bewegen deeltjes zoals eerder, maar nu worden ze geinjecteerd en geabsorbeerd op de plekken 1 en N met voorgeschreven snelheden die afhangen van het type deeltje. We berekenen het dichtheidsprofiel en de stroming in stabiele toestand. Dit leidt tot twee interessante fenomenen. Het eerste fenomeen is bergopwaartse diffusie, i.e., in een goed gedefinieerde regime kan de stroming tegen de helling van de dichtheid van deeltjes ingaan: wanneer de totale dichtheid van deeltjes aan de linkerkant hoger is dan aan de rechterkant, dan kan de stroming toch nog van rechts naar links gaan. Het tweede fenomeen is rand-laags gedrag: in de limiet, wanneer $\epsilon \downarrow 0$, zijn de dichtheden van de bovenste en onderste laag in het macroscopische stationaire profiel gelijk, wat bij ongelijke randvoorwaarden in de bovenste en onderste laag resulteert in een discontinuïteit in het stationaire profiel. Corresponderend aan deze sprong in het macroscopische system identificeren we een randlaag van grootte $\sqrt{\epsilon}\log(1/\epsilon)$ in het microscopische systeem waar de dichtheden ongelijk zijn. De kwantificering van de grootte van deze randlaag is een interessante gevolgtrekking van het exacte macroscopische stationaire profiel dat we verkrijgen van het microscopische systeem via dualiteit.

Deeltjessystemen die evolueren in het continuüm

Gebruikmakend van de taal van de punt process theorie introduceren we een nieuw kader waarin zelf-dualiteits relaties, of preciezer zelf-verwevendheids relaties, met respect tot polynomen geformuleerd kunnen worden voor deeltjessystemen die evolueren op een algemene Borel ruimte, dus ook op $\mathbb{R}^d$. Dit kader biedt ook een nieuwe aanpak van zelf-dualiteit. Daarvoor geven we dus eerst benodigde en voldoende voorwaarden voor een zelf-verwevendheids relatie met algemene vallende faculteit polynomen als verwevendheid. We geven met name nieuwe zelf-verwevendheids resultaten voor systemen zoals onafhankelijke en interacterende Brownse bewegingen. Verder, vanuit deze nieuwe aanpak krijgen we bekende zelf-dualiteits functies voor klassieke conservatieve interacterende deeltjessystemen terug. Onze aanpak verenigd dus de vorige zelf-dualiteits resultaten voor conservatieve systemen en vermijdt de noodzaak van ad hoc berekeningen voor het bewijzen van dualiteit voor elk systeem. Ten tweede, we bewijzne dat, uitgaande van reversibiliteit van het deeltjessysteem, het Gram-Schmidt orthogonalisatie process een symmetrie is voor de dynamiek van deeltjes in een consistent process. Als gevolg, door de eerder geïntroduceerde vallende faculteit polynoom zelf-verwevendheid te orthogonaliseren, laten we orthogonale zelf-verwevendheid zien in dezelfde context van consistente deeltjessystemen op algemene toestandsruimtes. Hierdoor laten we ook wat eigenschappen van algemene orthogonale polynomen zien, wat van onafhankelijk belang is. We introduceren en bestuderen ook een nieuw process in het continuüm, genaamd het *veralgemeniseerde symmetrische inclusieprocess*, waarvoor onze zelf-verwevendheids resultaten van toepassing zijn.

Ten slotte, geïnspireerd door het recente werk van Bertini en Posta [20], die het rand gedreven Brownse gas op $[0, 1]$ hebben geïntroduceerd, bestuderen we stochastische dualiteit voor rand gedreven systemen van onafhankelijke deeltjes in een algemene zetting. Het *rand-gedreven Brownse gas* op $[0, 1]$ is gedefinieerd (zie [20]) als een systeem van onafhankelijke Brownse bewegingen geabsorbeerd door 0 en door 1, waarnaar een onafhankelijk Poisson punt process is gesuperponeerd wat deeltjes toevoegt op $(0, 1)$ met een goed gekozen intensiteit. Voor ons doel moeten we de constructie van Bertini en Posta [20] generaliseren, ten eerste naar systemen van onafhankelijke diffusieprocessen evoluerend op het reguliere domein $\mathfrak{D} \subset \mathbb{R}^d$ en ten tweede naar systemen van algemene onafhankelijke Markov processen die zijn toegestaan sprongen te maken en die $\mathfrak{D}$ dus kunnen verlaten zonder de randen te raken. Daarna bewijzen we dualiteit voor zulke systemen met een duaal process dat wordt geabsorbeerd door de randen, waardoor we een algemeen kader vormen wat dualiteit voor rand gedreven systemen in de discrete en continuüm zetting verenigd. In het bijzonder laten we zien dat de tijdsafhankelijke n-de faculteit-moment maten van het rand gedreven systeem geschreven kunnen worden in termen van n duale deeltjes, geabsorbeerd door de randen. We gebruiken dualiteit eerst om te laten zien dat de systemen vanaf elke beginvoorwaarde convergeren naar de unieke invariante maat, wat een Poisson punt process is met als intensiteit de oplossing van een Dirichlet probleem. Ten tweede bewijzen we dat, in de discrete zetting van een één-dimensionale ketting, de modellen waar we de reservoirs modelleren als: i) geboorte- en overlijdensprocessen aan de randen of ii) een Poissoniaanse toevoeging van deeltjes overal, zijn inderdaad equivalente processen. Tot slot laten we zien dat het rand gedreven Brownse gas (in het continuüm) voorkomt als de diffusieve schalingslimiet van het model met geboorte- en overlijdensprocessen (in het discrete geval) wanneer de intensiteiten ook geschaald zijn met dezelfde grootte.

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Curriculum Vitae

Simone Floreani was born in San Daniele del Friuli, in the province of Udine, Italy, on the 25th of October 1993. After completing the high-school studies at Liceo Scientifico Giovanni Marinelli, he moved to Milan to acquire the Bachelor in Mathematical Engineering at Politecnico di Milano.

He then obtained his Master in Mathematical Modelling for Engineering at the same university. He participated to the Erasmus program at Delft University of Technology, where he followed the course entitled *Interacting Particle systems*, taught by Frank Redig. He performed his Master thesis back in Italy under the supervision of professors Luigi Ambrosio (Scuola Normal Superiore, Pisa) and Sandro Salsa (Politecnico di Milano).

After achieving his Master degree and conducting a few months of research at the Delft Center for Systems and Control at TU Delft, in October 2018 Simone Floreani started his PhD in the applied probability group at TU Delft, under the supervision of Frank Redig, Frank den Hollander and Cristian Giardinà. His research was supported by the Netherlands Organisation for Scientific Research (NWO) through grant TOP1.17.019. During his PhD, he visited for 4 months CPHT (Institut Polytechnique de Paris), in the realm of Chaire d'Alembert (Paris-Saclay University), as guest of Jean- René Chazottes, and he participated to a 4-month junior trimester program entitled *Stochastic modelling in the life science: from evolution to medicine* at the Hausdorff research institute for mathematics in Bonn.

In November 2022, Simone Floreani will join the stochastic analysis group at the mathematical institute of University of Oxford as a postdoctoral fellow.

Publications

Published

1. **“Hydrodynamics for the partial exclusion process in random environment”**
   with F. Redig (TU Delft) and F. Sau (IST Austria)
   Published in: Stochastic Processes and their Applications, 142 (2021), 124–158.

2. **“Switching interacting particle systems: scaling limits, uphill diffusion and boundary layer”**
   with C. Giardinà (Modena Uni.), F. den Hollander (Leiden Uni.), S. Nandan (Leiden Uni.) and F. Redig (TU Delft)
   Published in: Journal of Statistical Physics, 186 (2022), Paper 33.

3. **“Orthogonal polynomial duality of boundary driven particle systems and non-equilibrium correlations”**
   with F. Redig (TU Delft) and F. Sau (IST Austria)
   Published in: Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, 58, 1 (2022), 220–247.

Submitted

1. **“Intertwining and Duality for Consistent Markov Processes”**
   with S. Jansen (LMU Munich), F. Redig (TU Delft) and S. Wagner (LMU Munich)

2. **“Boundary driven Markov gas: duality and scaling limits”**
   with G. Carinci (Modena Uni.), C. Giardinà (Modena Uni.) and F. Redig (TU Delft)