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Sturm-Liouville Boundary Value Problem for a Sea-Breeze Flow

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Abstract. We present recent results in study of a mathematical model of the Sea-Breeze flow, arising from a general model of the 'morning glory' phenomena. Based on analysis of the Dirichlet spectrum of a corresponding Sturm-Liouville problem and application of the Fredholm alternative, we establish conditions of existence/uniqueness of solutions to the given problem.

Mathematics Subject Classification. 37N10, 34B24, 34B05.

Keywords. Sea-Breeze flow, Sturm-Liouville problem, Dirichlet spectrum, Fredholm alternative, Legendre equations.

1. Introduction

Sea breeze is a atmospheric flow that develops due to a strong temperature contrast between the land and sea surfaces [17]. This flow is caused by the heating of the boundary layer over land, that results in the movement of low-level air from the sea to land (*Sea-Breeze*) with a return flow called the *return current* (see Fig. 1).

A typical Sea-Breeze flow is about 300 - 1000m thick (with the strongest wind speed at 50-200m hight), whereas the thickness of the return current is about twice that big, leaving for the total vertical dimension of the Sea-Breeze circulation about 1 to 3 km [17]. The mass of air transported by the Sea-Breeze and by the return current are almost the same.

The Sea-Breeze flow strongly depends on the season, latitude where it blows and time of the day. For example, in tropical and subtropical coastal regions it is a regular phenomenon throughout the year, whereas at higher latitudes it is observed during the spring and summer periods (for more details about the Sea-Breeze flows we refer the reader to [7,15–17]).

In this paper we analyze a mathematical model of the Sea-Breeze flow in the Gulf of Carpentaria region (North Australia), that was recently derived from the equations characterizing the *"morning glory"* phenomena (for more information we refer to [6,8,11]); see Fig. 2.

A remarkable feature of the Sea-Breeze flow in this particular region, what also makes it so interesting to study, is its inland penetration for a distance of about 350km, that is significantly larger than the average distance, typical for these flows (see [3,15]). This is caused by several factors, under which are the intensity of heating, relative dryness of the land surface (except the summer monsoon periods) and a small value of the Coriolis parameter at these latitudes [7].

The mathematical model of the Sea-Breeze flow in the North Australian region is written in the form of the Dirichet boundary value problem, solvability of which we are analysing in this paper. Based on the Sturm-Liouville (S-L) theory [2,12,13] and the Fredholm alternative [5,14] we make conclusions about existence and uniqueness of solutions to the original problem.

The presented here results are new and serve as a natural extension to the analyses in [8]. It helps us to better understand the Sea-Breeze flows in a broader setting, than studied before. Moreover, these results

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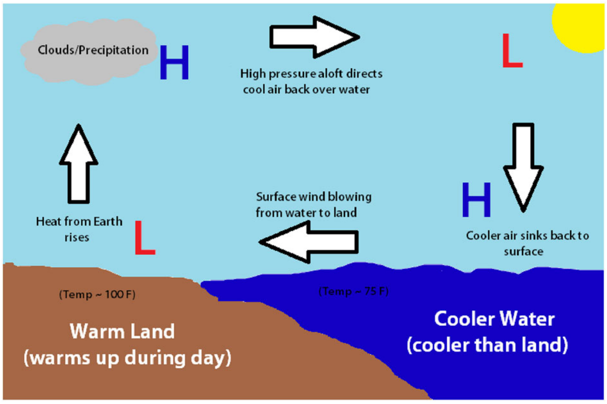

Fig. 1. The sea breeze formation, https://brianmejia.wordpress.com/2011/06/01/what-is-a-sea-breeze-shower

Fig. 2. Cloud "Morning Glory" in the Gulf of Carpentaria, Australia, https://www.meteorologiaenred.com/en/the-cloud-morning-glory-impressive-meteorological-phenomenon.html

complement the earlier research line in analysis of the S-L problems for ocean flows and atmospheric boundary-layer dynamics [9,10,13,14].

2. Mathematical Model of the Breeze-like Flow

In [8] Constantin, Johnson derived a non-dimensional model of the breeze-like flow, that in terms of a horizontal velocity $V_0(z, \Phi)$ reads as:

$$\rho_0 \sigma S V_0 + \frac{1}{R_e} \frac{\partial}{\partial z} \left(m(z) \frac{\partial V_0}{\partial z} \right) = - \left\{ \cos^2(\alpha) + \frac{\sin^2(\alpha)}{C} \right\} K(z, \Phi). \tag{1}$$

Here

- z corresponds to the thickness of the flow;
- $\rho_0(z)$ is the density function;
- S, α, C and $\sigma = \frac{2(\sin^2 \alpha + C \cos^2 \alpha)}{(1-C) \sin \alpha \cos \alpha}$, are the characteristics of the flow in a specific region;
- $R_e \approx 10^5$ is the Reynolds number;
- $m(z)$ is the viscosity function;
- Φ is a parameter, corresponding to the direction of the flow propagation and
- $K(z, \Phi)$ is the forcing term in the model.

Together with the equation (1) one can consider the physically relevant boundary conditions:

$V_0(0, \Phi) = 0$ – corresponding to the no-slip condition at the surface of the Earth,
$V_0(z_0, \Phi) = 0$ – standing for the level where the temperature inversion occurs.

In the Gulf of Carpentaria region, that is under our focus in this paper, one can determine the following physically relevant parameter values [8]:

$$C \approx 0.97, \quad S \approx -0.24, \quad \sigma \approx 133 \quad \text{and} \quad \alpha = \frac{5\pi}{4}.$$

Then the differential equation (1) is simplified to

$$\beta V_0 - \frac{\partial}{\partial s}\left(\hat{m}(s)\frac{\partial V_0}{\partial s}\right) = k_0(s, \Phi), \quad 0 < s < 1, \tag{2}$$

with

$$\beta = -\sigma S R_e > 0, \quad s = \frac{1}{\int_0^{z_0} \rho_0(\xi) d\xi} \int_0^z \rho_0(\xi) d\xi,$$

$$\hat{m}(s) = \frac{\rho_0(z) m(z)}{\left(\int_0^{z_0} \rho_0(\xi) d\xi\right)^2},$$

$$k_0(z, \Phi) = Re\left\{\cos^2(\alpha) + \frac{\sin^2(\alpha)}{C}\right\} \frac{K(z, \Phi)}{\rho_0(z)}, \tag{3}$$

where functions $\rho_0(z)$ and $K(z, \Phi)$ are assumed to be continuous (with Φ being a parameter), while $m(z)$ is continuously differentiable.

Additionally, for every fixed Φ one can write the physically relevant constraints of the flow in the form of the homogeneous Dirichlet boundary conditions:

$$V_0(0) = V_0(1) = 0. \tag{4}$$

In the following section we provide the solvability analysis to the problem (2), (4) using the spectral theory tools.

3. Sturm-Liouville Problem for a Sea-Breeze Flow Model

Let us associate with the BVP (2), (4) the correspondent S-L problem of the form:

$$\beta V_0 - \frac{\partial}{\partial s}\left(\hat{m}(s)\frac{\partial V_0}{\partial s}\right) = \lambda V_0(s), \quad 0 < s < 1, \tag{5}$$

$$V_0(0) = V_0(1) = 0, \tag{4}$$

where $\hat{m}(s)$ does not change its sign.

In [8] the authors showed that if

$$\mathcal{S} = \beta - \frac{\partial}{\partial s}\left(\hat{m}(s)\frac{\partial}{\partial s}\right)$$

is a self-adjoint unbounded linear operator acting in $L^2(0, 1)$ with domain the Sobolev space

$$H_2^0(0, 1) = \{V \in H^2(0, 1) : V(0) = V(1) = 0\},$$

then $\mathcal{S}$ has a discrete spectrum with simple eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ accumulating at infinity. Moreover, it was proved that $\lambda_1 > \beta$ and thus, for $\beta > 0$ there are no zero eigenvalues and the BVP (2), (4) has a unique solution for all functions $k_0(s, \Phi)$.

However, in some settings β admits a negative sign and thus, there exists such N that $\lambda_N = 0$, which due to the Fredholm alternative leads to certain restrictions on the right hand-side of (2). This case was studied in [8] under assumption that the function $\hat{m}(s)$ in the differential equation (2) is constant: $\hat{m}(s) \equiv$

m_0. This leads to a sequence of eigenvalues $\lambda_k = \beta + m_0\pi^2k^2$ with the correspondent eigenfunctions $f_n(s) = \sqrt{2}\sin(k\pi s)$, for $k \geq 1$. One can easily see that for a particular value of β the sequence of eigenvalues λ_k admits a zero value. This case occurs, for example, in the Calgary region where the sea-breeze-like flow is landlocked.

In this section we extend the spectral analysis presented in [8] by incorporating additional nonlinear profiles of $\hat{m}(s)$ in the differential equation (2).

3.1. Equation with Constant Coefficients

Assume that function $m(z)$ in the relation (3) is given by

$$m(z) = \frac{\left(\int_0^{z_0} \rho_0(\xi)d\xi\right)^2 (as+b)^2}{\rho_0(z)},$$

where $as + b > 0$, $\forall s \in (0,1)$, $a \neq 0$, $b \neq 0$ and $a + b \neq 0$. Then $\hat{m}(s) = (as+b)^2$ and the differential equation (2) can be written as

$$-(as+b)^2V_0''(s) - 2a(as+b)V_0'(s) + \beta V_0(s) = k_0(s,\Phi), \quad 0 < s < 1. \tag{6}$$

Consider now the correspondent Sturm-Liouville BVP

$$-(as+b)^2V_0''(s) - 2a(as+b)V_0'(s) + \beta V_0(s) = \lambda V_0(s), \quad 0 < s < 1, \tag{7}$$

$$V_0(0) = V_0(1) = 0, \tag{4}$$

and let us find the Dirichlet spectrum for this problem.

Note, that the differential equation (7) is the Legendre equation admitting its reduction to the equation with constant coefficients.

By introducing an **Ansatz** $x = \ln|as+b|$ we derive that

$$V_0'(s) = \frac{a}{as+b}V_0'(x), \quad V_0''(s) = \frac{a^2}{(as+b)^2}(V_0''(x) - V_0'(x)),$$

Then the equation (7) can be written as

$$a^2V_0''(x) + a^2V_0'(x) + (\lambda - \beta)V_0(x) = 0. \tag{8}$$

From the characteristic equation

$$a^2\mu^2 + a^2\mu + (\lambda - \beta) = 0,$$

corresponding to (8) we obtain that

$$\mu_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{a^2 - 4(\lambda - \beta)}}{2a}. \tag{9}$$

Below we provide a detailed analysis of the Sturm-Liouville problem (7), (4) for different values of λ.

Case I. Assume, $\lambda = \beta$. Then $\mu_{1,2} = 0, -1$ and

$$V_0(x) = c_1 + c_2e^{-x}$$

or, in terms of the s variable,

$$V_0(s) = c_1 + c_2\frac{1}{as+b}. \tag{10}$$

Substitution of (10) into the boundary conditions (4) leads to

$$V_0(s) \equiv 0$$

and thus, $\lambda = \beta$ cannot be an eigenvalue of the S-L problem (7), (4).

Case II. Assume, $D = \sqrt{a^2 - 4(\lambda - \beta)} > 0$. Then

$$V_0(x) = c_1e^{\mu_1 x} + c_2e^{\mu_2 x}$$

or

$$V_0(s) = c_1(as+b)^{\mu_1} + c_2(as+b)^{\mu_2}, \tag{11}$$

where $\mu_{1,2}$ are defined by (9).

The only possibility for $V_0(s)$ in (11) to satisfy the Dirichlet boundary constraints (4) is for $c_1 = c_2 = 0$. Thus, $V_0(s) \equiv 0$ and there are no eigenvalues for $\sqrt{a^2 - 4(\lambda - \beta)} > 0$.

Case III. Assume, $D = \sqrt{a^2 - 4(\lambda - \beta)} < 0$. Then

$$\mu_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{4(\lambda-\beta) - a^2}}{2a}.$$

Let us denote by $\nu = Im(\mu_{1,2})$. Then the general solution of (8) will be written as

$$V_0(x) = e^{-x/2}\Big\{c_1\cos(\nu x) + c_1\sin(\nu x)\Big\} \tag{12}$$

or

$$V_0(s) = \frac{1}{\sqrt{as+b}}\Big\{c_1\cos(\nu\ln|as+b|) + c_2\sin(\nu\ln|as+b|)\Big\}. \tag{13}$$

From the boundary conditions (4) we get a linear homogeneous algebraic system in terms of c_1, c_2 variables:

$$\begin{cases} c_1\cos(\nu\ln|b|) + c_2\sin(\nu\ln|b|) = 0, \\ c_1\cos(\nu\ln|a+b|) + c_2\sin(\nu\ln|a+b|) = 0. \end{cases} \tag{14}$$

Now, system (14) and thus, the S-L problem, will have a non-trivial solution iff

$$\Delta = \Big|\begin{matrix} \cos(\nu\ln|b|) & \sin(\nu\ln|b|) \\ \cos(\nu\ln|a+b|) & \sin(\nu\ln|a+b|) \end{matrix}\Big| = 0. \tag{15}$$

Let's find such values of λ, for which (15) holds. We get:

$$\cos(\nu\ln|b|)\sin(\nu\ln|a+b|) - \sin(\nu\ln|b|)\cos(\nu\ln|a+b|) = \sin\Big(\nu\ln|a+b| - \nu\ln|b|\Big) = 0.$$

From the last expression we get:

$$\sin\left(\nu\ln|\frac{a+b}{b}|\right) = 0$$

and thus,

$$\nu = \frac{\pi n}{\ln|\frac{a+b}{b}|}, \quad n = \mathbb{Z}_0.$$

Since $\nu = \frac{\sqrt{4(\lambda-\beta)-a^2}}{2a}$, we conclude that the Dirichlet spectrum is defined by

$$\lambda_n = \frac{1}{4}\left(\frac{(2a\pi n)^2}{\ln^2|\frac{a+b}{b}|} + a^2\right) + \beta, \quad n = \mathbb{N}_0. \tag{16}$$

The eigenfunctions in this case are given by:

$$\begin{aligned} V_{0n}^{(1)}(s) &= \frac{1}{\sqrt{as+b}}\cos\left(\frac{\pi n\ln|as+b|}{\ln|\frac{a+b}{b}|}\right), \\ V_{0n}^{(2)}(s) &= \frac{1}{\sqrt{as+b}}\sin\left(\frac{\pi n\ln|as+b|}{\ln|\frac{a+b}{b}|}\right), \end{aligned} \qquad n \in \mathbb{N}_0. \tag{17}$$

From (16) follows that for all $\beta \geq 0$ all eigenvalues $\lambda_n > 0$ and accumulate at $+\infty$. Thus, zero is not an eigenvalue of the S-L BVP (7), (4), and by the Fredholm alternative the original BVP (6), (4) has a unique solution [5].

However, if $\beta < 0$, then there might exist such N, for which $\lambda_N = 0$. Then, the BVP (6), (4) has a unique solution iff

$$< k_0(s, \Phi), f(s) >= 0,$$

where

$$f(s) = c_1 V_{0n}^{(1)}(s) + c_2 V_{0n}^{(2)}(s).$$

3.2. Legendre Equation

Consider the Sturm-Liouville problem (5), (4) and let us introduce an Ansatz:

$$x = 2s - 1, \quad x \in (-1, 1). \tag{18}$$

Then (5), (4) is written as

$$-4(\hat{m}(x)V_0'(x))' + \beta V_0(x) = \lambda V_0. \tag{19}$$

$$V_0(-1) = V_0(1) = 0, \tag{20}$$

where we assume that function $\hat{m}(x)($ is given by a relation:

$$\hat{m}(x) = 1 - x^2. \tag{21}$$

In this case the model equation (19) has the form

$$((1 - x^2)V_0'(x))' + \frac{1}{4}(\lambda - \beta)V_0(x) = 0 \tag{22}$$

and is coupled with the boundary constraints (20).

Based on the analysis of the S-L BVPs for Legendre equations (see [2]), it is easy to check that

$$\lambda_n = \beta + 4n(n + 1), \quad n \in \mathbb{N}_0$$

are the eigenvalues of the problem (22), (20) with the corresponding eigenfunctions being the Legendre orthogonal polynomials [1,4]:

$$P_n(x) = \frac{(-1)^n}{2^n n!} D^n[(1 - x^2)^n], \quad n \in \mathbb{N}_0.$$

Additionally, if $\beta > 0$ then all $\lambda_n > 0$ and thus, due to the Fredholm alternative, the original BVP (2), (4) under substitutions (18), (21) has a unique solution. On the other hand, if $\beta \leq 0$ (in particular for $\beta = -4n(n + 1)$), then there exists such N that $\lambda_N = 0$, and we cannot guarantee uniqueness in this case. For a solution to be unique, function $k_0(z, \Phi)$ should satisfy the orthogonality condition:

$$< k_0(z, \Phi), P_n(z) > 0, \quad \forall n \in \mathbb{N}_0.$$

Note, that this analysis can be extended towards more complex forms of the viscosity function $\hat{m}(s)$ that leads to S-L problems, for which their discrete spectrum can be explicitly found.

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Compliance with ethical standards

Conflicts of interest There is no conflict of interest to declare.

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