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DOI
10.1016/j.cie.2018.02.039

Publication date
2019

Document Version
Accepted author manuscript

Published in
Computers and Industrial Engineering

Optimization models for high-speed train unit routing problems

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Abstract: Train unit routing problem determines the number of train units needed to carry out involved trips, which is a significant part of railway operation cost. In this paper, we focus on high-speed train unit routing problems, in which maintenance resource constraints both on time and distance are taken into account. Based on a connection network, this paper first proposes a general train unit routing model. Then, the general model is specialized to meet the circulation and maintenance conditions of high-speed train units in China, which is based on a special connection network with a two-day time horizon. A strategy is proposed to reduce the scale of the connection network, which improves the model's solvability. Furthermore, an extension on multi-depot train unit routing problem is discussed. Finally, numerical experiments based on the real data of Chinese high-speed railway are carried out to verify the effectiveness and efficiency of the proposed mode and method.

Keywords: High-speed railway; Train unit routing problem; Maintenance; Integer programming

1 Introduction

By the end of 2016, the total length of high-speed railway in China has been over 19,000km, and everyday there were more than 1,800 high-speed train units with different types carrying out about 3,500 trips. It is expected that by 2020, the total length of high-speed railway will be over 30,000km, and the number of train units in service will be 3,000. As is known, an efficient train unit circulation requires a small number of train units to carry out trips, and therefore the cost of high-speed railway operation can be significantly reduced, considering the high price of one train unit (about 6 million dollars).

For high-speed railway, train unit circulation problem can be roughly divided into two problems, namely *train unit routing* and *train unit assignment*. Train unit routing problem is in the tactical planning phase (see Maróti G., 2006), where *generic routings* that cover all involved trips are obtained. Note that these generic routings do not assign individual train units to trips explicitly, but serve as a reference in train unit assignment. In detail, train unit assignment problem is solved to assign individual train units to the generic routings, which is in the operational phase. In this paper, we are particularly interested in train unit routing problem, and hence train unit assignment problem and corresponding restrictions are out of the scope. Obviously, the more the number of generic routings is, the more train units are needed. In this sense, train unit routing problem determines the cost of railway operation.

Currently, there are seven types of high-speed train units and 5 levels of maintenance in China (illustrated in Table1). Note that some maintenances have two resource constraints, namely elapsed time and travel distance. For example, a CRH1 train unit must undergo a first level maintenance every 48 hours or 4000 km, which comes first. In addition, each train unit belongs to a depot, and the first level maintenance must be performed in its depot.

Table1. Maintenance Resource Constraints of Different Train Unit Types

<table>
<tr><th></th><th>1st</th><th>2nd</th><th>3rd</th><th>4th</th><th>5th</th></tr>
<tr><td>CRH1</td><td rowspan="3">48 h/
4000 km</td><td>*</td><td>1.2M km</td><td>2.4M km</td><td>4.8M km</td></tr>
<tr><td>CRH2
CRH380A</td><td>*</td><td>0.3M km/
1 year</td><td>1.2M km/
3 year</td><td>2.4M km /
6 year</td></tr>
<tr><td>CRH3
CRH380B
CRH380C</td><td>*</td><td>1.2M km</td><td>2.4M km</td><td>4.8M km</td></tr>
<tr><td>CRH5</td><td>48 h/
5000 km</td><td>*</td><td>1.2M km</td><td>2.4M km</td><td>4.8M km</td></tr>
</table>

*The second level maintenance has not published uniform requirements

In high-speed train unit routing problem, only the first level maintenance (daily check) is considered. Specifically, the total travel distance of a generic train unit routing cannot be longer than the maximum travel distance resource of the first level maintenance, and the total elapsed time cannot be longer than the maximum travel time resource as well. In other words, a generic train unit routing is between two consecutive first level maintenances.

The contributions of this paper are threefold. First, a general single-depot model for train unit routing problem with two resource constraints is proposed. In almost all existing literature, only one resource constraint is considered. The formulation of the general single-depot model is based on a connection network, which conveniently represents the circulation of train units. An arc in a connection network is either a maintenance arc or a non-maintenance arc, due to the duration and location attributes of the arc. On a non-maintenance arc, a train unit cannot undergo maintenance. Via the connection network, the restrictions on turnaround and maintenance can be easily expressed by the adjacency of nodes.

Second, considering the characteristics of train unit routing problem in Chinese high-speed network, a special model with two-day horizon for train unit routing problem is developed, in which the time resource constraints can be removed. As a result, the scale of the model is significantly reduced. Furthermore, a reduction of the number of arcs is carried out via introducing upper and lower bounds of the durations of arcs, which are obtained from practical experiences. Besides, the connection network of single-depot is extended to the case of multiple-depot, based on which a model for train unit routing problem with multiple-depot is formulated.

Third, based on the real data in Chinese high-speed railway network, several sets of experiments are carried out to test the effectiveness and efficiency of all the proposed models. The results show that after introducing upper and lower bounds of the durations of arcs in connection networks, the computation times of all models are significantly reduced, with no deterioration in the optimal objective values. Besides, if the number of involved trips in a day is less than 200, the computation times of the special model are all less than 8 mins. Since train unit routing problem is at the tactical level of railway optimization, the computation time of the special model is acceptable.

The remainder of this paper is organized as follows. Section 2 provides a review of the literature relating to the train unit routing problem. Section 3 presents the construction of a cyclic connection network, based on which a general single-depot model for train unit routing problem with two resource constraints is formulated. In Section 4, a special model for train unit problem in Chinese high-speed railway network is formulated. In Section 5, we extend the special model to train unit routing problems with multiple depots. Section 6 verifies the effectiveness and efficiency of the models via carrying out numerical experiments based on the real data of Chinese high-speed railway. In Section 7, we draw some conclusions for this paper.

2 Literature Review

As mentioned in the above section, train unit routing problem is the tactical part of train unit circulation problem, which is also known as *rolling stock planning/scheduling/rostering* problem. Train unit circulation may vary from country to country, due to management regulations and maintenance rules. In some countries, train unit circulation is not necessarily divided into train unit routing and train unit assignment. For this reason, in this section, we review some relevant literature on train unit circulation problem, instead of train unit routing problem.

In most existing literature, maintenance was not considered, and the authors mainly focused on optimizing the composition of trains to satisfy seat demand and reduce operating costs. As one of the earliest literature on this problem, Schrijver (1993) aimed to minimize the number of train units of different types for periodic trips in the Netherlands, under the requirement that the passengers' seat demand must be satisfied. The framework of this paper, namely an integer linear programming (ILP) model on a directed time-space graph, has been widely used in the research. Abbink et al. (2004) presented an integer programming model to assign train units to trips, the objective of which was to minimize the seat shortages during the morning peak hours. The problem was solved by CPLEX, and experiments on the real-life data from NS Reizigers were carried out. Considering (un)coupling of train units, Alfieri et al. (2006) focused on minimizing the numbers of train units of different types needed on a railway corridor. A solution approach based on an integer multi-commodity flow model was proposed, and applied to a real-life case of NS Reizigers. Peeters and Kroon (2008) formulated a model to determine an optimal daily assignment of train units on a railway corridor in the Netherlands. In their model, the changes of train composition at origin and destination stations were taken into account, and a branch-and-price algorithm was developed to solve the problem. Fioole et al. (2006) extended the model in Peeters and Kroon (2008) by considering underway combining and splitting of trains. A complex mixed integer linear programming model was constructed, which was solved by CPLEX. Cacchiani et al. (2013) described the train unit circulation problem as a multi-commodity flow model, the objective of which was to minimize the number of train units needed. The model was solved by a Lagrangian-relaxation-based heuristics, in which a local search algorithm was used to obtain feasible solutions.

In recent years, more and more literature on train unit circulation problem considered the restrictions of maintenance. Maróti et al. (2005, 2006, and 2007) considered train unit circulation problems in the Netherlands, in which the maintenance resource constraint was travel distance. In their publications, each train unit was scheduled to carry out as many trips as possible before maintenance, and if a train unit was in maintenance, an urgent train unit was scheduled to replace it. Maróti et al. (2005, 2006, and 2007) proposed two integer-programming models for the problem, namely an interchange model and a transition model. The former was designed to consider as many details as possible, while the latter was designed to simplify formulation of the problem with less input data. Hong et al. (2009) studied the train unit circulation problem in Korean high-speed railway, which was described as an Eulerian walk problem. The problem was solved by a proposed two-stage heuristic approach, which first found a routing without considering maintenance requirements. A heuristic was then employed to incrementally increase the number of train units to meet the maintenance requirements. Cacchiani et al. (2010) proposed a heuristic based on column generation to minimize the number of train units used. In their extended model, maintenance constraints were considered, which required that all train units must undergo at least one maintenance during a weekly schedule. Borndörfer et al. (2014) constructed a mixed-integer programming model on a hyper-graph for train unit circulation, which considered several requirements, such as train composition, maintenance constraints, infrastructure capacities, and regularity aspects. A heuristic based column generation and local search was developed to solve the model. Giacco et al. (2014) formulated a train unit circulation problem as a minimal cost Hamiltonian cycle problem in a graph, in which service parings, empty runs and short-term maintenances were all taken into account. The problem was solved by CPLEX, and the method was tested on real-world scenarios from an Italian railway company. Given generic routings for train units, Lai et al. (2015) investigated a train unit assignment problem, in which the second level of maintenance were considered. An exact optimization model was first proposed, and then a hybrid heuristic process was developed to improve solution quality and efficiency.

Besides train units circulation problems, some literature focused on dealing with the efficient circulation of locomotives. Ziarati et al. [17] proposed an integer linear programming model for the problem of assigning locomotives to trains. The model was solved by a branch-and-cut approach, and was tested on the actual data from Canadian National Railway Company. Cordeau et al. (2000) described a decomposition method for the simultaneous assignment of locomotives and cars for passenger trips. In a subsequent paper, Cordeau et al. (2001) extended their model by adding the maintenance requirement of locomotives. For a similar problem, Lingaya et al. (2002) described a heuristic based on branch-and-bound method, in which the linear relaxations were solved by column generation.

Note that in most existing literatures, either maintenance is not considered at all or only one maintenance resource constraint is taken into account. In our paper, we consider both travel distance resource constraints and elapsed time resource constraint, which is more in line with the practice. Besides, in Chinese high-speed railway, train unit circulation problem is divided into train unit routing problem and train unit assignment problem, which makes it different from most railway systems. This paper particularly proposes a special model on the train unit routing problem, which fulfills the characteristics of Chinese high-speed railway.

3 A General Train Unit Routing Model with Two Resource Constraints

3.1 Problem description and assumptions

In some high-speed railway systems, such as in China, train unit types and compositions are predetermined before making train unit routing plans. In this paper, we simply assume that there is only one type of train unit with the same composition.

Based on a given timetable, the dispatchers make a train unit routing plan. It is required that all the trips in the timetable must be covered by one and only one train unit routing. Note that a trip is characterized by the visited stations with corresponding arrival and departure times. Since the composition of a train is not changed in the intermediate stations, we only need to pay attention to the origin station and destination station of a trip. When a train unit arrives at the destination station of a trip, we say that the train unit finishes the trip. If the train unit is going to carry out another trip, the turnaround constraint must be satisfied. Specifically, the latter trip's origin station must be the former trip's destination station, and the time interval between the former trip's arrival time and the latter trip's arrival time must be greater than the minimum turnaround time. It should be pointed out that the minimum turnaround time depends on the necessary alighting and boarding times, and shunting time from the arrival track of the former trip to departure track of the latter trip.

As mentioned previously, each train unit is affiliated to a depot, and the first level maintenance must be performed in the depot. Note that a depot is usually located near a major station, and the travel time from the station to the depot for maintenance and the travel time from the depot to the station after maintenance should be considered. Usually, the duration of the first level maintenance ranges from three to four hours.

In order to describe the problem more clearly, Fig.1 presents a small railway network with 6 trips every day, namely G_1, G_2, G_3, G_4, G_5 and G_6. Note that S_1, S_2 and S_3 are origin and destination stations, and S_2 is the only station equipped with a depot, to which all train units in the network are affiliated. For simplicity, the same timetable is repeated every day. A possible solution to this routing problem is "$G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_1 \rightarrow G_5 \rightarrow G_6$". Note that the first three trips of this routing are in "day 1", and the last three trips are in "day 2". Specifically, in the morning of "day 1", station S_2 assigns a train unit that has just been maintained to trip G_2, starting the routing; in the evening of "day 2", this train unit finishes trip G_6 and returns to S_2 for maintenance in the depot, which indicates the end of the routing. Note that a train unit cannot carry out trip G_4 after carrying out trip G_5, since the time interval between the arrival of trip G_4 and the departure of trip G_5 is too short to turn around.

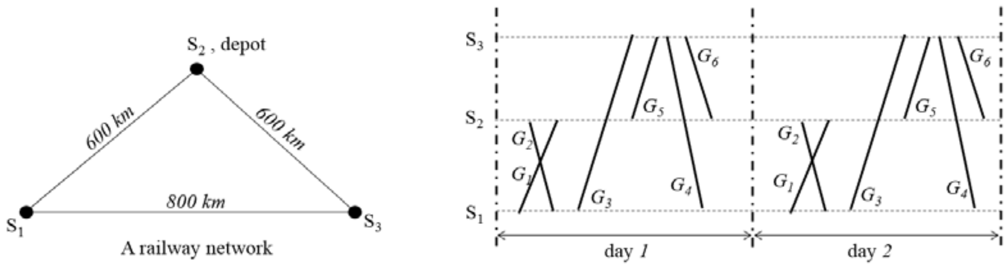

Fig.1. A small railway network with 6 trips every day

A solution to the train unit routing problem consists of several routings, and each trip in the considered time horizon is covered by (contained in) one and only one routing. For example, to cover all the trips in Fig.1, we need two routings like "$G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_1 \rightarrow G_5 \rightarrow G_6$", the first one starting in "day 1" and the second one starting in "day 2". As a result, it is easy to verify that two train units are needed in the depot in station S_2.

Obviously, there are more than one feasible solution to the problem in Fig.1. In this paper, we aim to find a solution with minimum total turnaround time. Besides containing all the trips, these routings also contain necessary maintenances, and turnaround times. The less the total turnaround time a train unit routing contains, the higher the usage of the train unit is, which further indicates that fewer train units are needed. Therefore, we use the total turnaround time of the train unit routings as the objective function.

It should be pointed out that the turnaround constraints are difficult to formulate, which may lead to a large number of constraints if improper formulation approach is used. In the next section, we develop a connection network to intuitively present the routings of train units, in which the turnaround constraints are expressed by the connection of nodes.

In this paper, we first consider the case that only one depot is in the high-speed railway network. Then, we further extend the train unit routing model to the case of multiple depots. In order to construct the optimization model, we now summarize the assumptions made in this paper.

(1) There is only one type of train unit, and there is no coupling or splitting of train units.

(2) The timetable is periodic with a period length of one day. In other words, the same timetable is repeated every day.

(3) Only the first level maintenance is considered in this paper. The corresponding travel distance resource is 4000 km and the elapsed time resource is 48 hours, whichever comes first.

(4) Each train unit is affiliated to a depot, and it undergoes the first maintenance in the depot it is affiliated to.

(5) A solution of the model consists of generic routings, without assigning individual train units to the trips. In other words, train unit assignment is not considered in this paper.

3.2 Underlying connection network

Before formulating models, we first construct a connection network to describe the circulation of train units. In the connection network, each trip is represented by a node. If the turnaround constraints are satisfied between two trips, there is a direct arc connecting the corresponding nodes. As a result, turnaround constraints are expressed by the adjacency of nodes in the connection network. The detailed construction of a connection network is shown as follows:

Step 1: Construct nodes. Node *i*, which corresponds to trip *i*, has six attributes: s_i^o, s_i^d, t_i^o, t_i^d, l_i and t_i, representing the origin station, destination station, departure time from the origin station, arrival time at the destination station, distance and travel time, respectively. The set of nodes is denoted by *V*.

Step 2: Construct arcs. For arbitrary nodes $i, j \in V$, if $s_i^d = s_j^o$, then connect them by a directed arc *(i, j)*, which means that the origin station of trip *j* is just the destination of trip *i*. The minimum turnaround time from the track of trip *i* to the track of trip *j* in station s_i^d is denoted by σ_{ij}. If $t_j^o - t_i^d \geq \sigma_{ij}$, the duration of arc *(i, j)* is set by $t_{ij} = t_j^o - t_i^d$, and arc *(i, j)* is referred to as *day arc*, since it connects trip *i* and *j* in the same day; otherwise, set $t_{ij} = t_j^o - t_i^d + 1440$, and arc *(i, j)* is referred to as *night arc*, since it connects trip *i* in the first day and trip *j* in the second day. The set of arcs is denoted by *A*.

Step 3: Classify arc set. Arc *(i, j)* has another attribute θ_{ij}, which indicates whether the first level maintenance can be performed on arc *(i, j)*. If station s_i^d is equipped with a depot and the duration t_{ij} is greater than necessary maintenance time ω_{ij}, then set $\theta_{ij} = 2$, which means that the first level maintenance can be performed on arc *(i, j)*, and arc *(i, j)* is referred to as a *maintenance arc*. If and destination station s_i^d is not equipped with a depot or $t_{ij} < \omega_{ij}$, set $\theta_{ij} = 1$, and arc *(i, j)* is referred to as a *non-maintenance arc*. The set of non-maintenance arcs is denoted by A_c, and the set of maintenance arcs is denoted by A_m. Obviously, we have $A = A_c \cup A_m$.

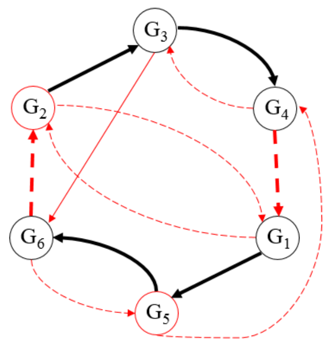

Fig.2. The connection network of the trips in Fig.1

In Step 3, the necessary maintenance time ω_{ij} is the sum of the duration of the first level maintenance, the travel time from station s_i^d to the depot, and the travel time from the depot to station s_i^d. According to the above construction steps, the connection network of the trips in Fig.1 is depicted in Fig.2, in which the nodes represent corresponding trips, and the arcs represent the turnaround between these trips. In detail, the solid arcs are day arcs, the dashed arcs are night arcs, the red arcs are maintenance arcs, the black arcs are non-maintenance arcs, and the red nodes indicate that the origin stations of the corresponding trips are equipped with depot. Routing "$G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_1 \rightarrow G_5 \rightarrow G_6$" is illustrated by the bold arcs. Note that a train unit routing is actually a circuit in a connection network, starting and ending at a node, the origin station of which is equipped with a depot. In this sense, in Fig.2, only node G_2 and G_5 can be the starting and ending nodes of a routing. For example, node G_5 is the starting and ending node of routing "$G_5 \rightarrow G_6$".

It is unlikely that an optimal solution contains a day arc with a very long turnaround time. For this reason, when constructing arcs, we may set an upper bound σ_{upper}^{day} on the day arcs, which indicates that the turnaround time cannot be too long. In detail, for nodes $i, j \in V$, if $s_i^o = s_j^d$, $t_j^o - t_i^d \geq \sigma_{ij}$, and $t_j^o - t_i^d \geq \sigma_{upper}^{day}$, then node *i, j* cannot be connected due to the upper bound σ_{upper}^{day}. Setting proper upper bounds, the number of arcs in the connection network may be largely reduced. Note that the value of σ_{upper}^{day} is obtained from practical experiences, which is usually set to be 3-6 hours. Similarly, we can also set upper bounds on night arcs.

3.3 Formulation of the general train unit routing model with single depot

Based on the connection network constructed in the above section, we formulate a general model for the train unit routing problem with single depot. The notations used in the model are introduced as follows:

A_i^o Set of arcs leaving node *i*.

A_i^d Set of arcs entering node *i*.

T_m^1 The resource of elapsed time for the first level maintenance, *i.e.*, T_m^1 =2880 min (two days).

L_m^1 The resource of travel distance for the first level maintenance, *i.e.*, L_m^1 =4000 or 5000 km.

M A sufficiently large positive integer.

The decision variables in the model mainly consist of x_{ij}, y_{ij}, a_i, and b_i, which are explained as follows:

x_{ij} Binary decision variable, which indicates whether arc (*i*, *j*) is contained in the path corresponding to a train unit routing. If it is, x_{ij} =1; otherwise, x_{ij} = 0.

y_{ij} Binary decision variable, which indicates whether a first level maintenance is performed on arc (*i*, *j*). If it is, y_{ij} =1; otherwise, y_{ij} =0.

a_i Assistant decision variable, which records the cumulative elapsed time of a train unit after carrying out trip *i*. It is the sum of the travel times and turnaround times of all involved trips.

b_i Assistant decision variable, which records the cumulative travel distance of a train unit after carrying out trip *i*.

Now, the general model for the train unit routing problem can be formulated, which is an integer linear programming model, *i.e.,*

$$\min z = \sum_{(i,j)\in A} t_{ij} \cdot x_{ij} \tag{1}$$

$$s.t. \quad \sum_{(i,j)\in A_i^o} x_{ij} = 1 \qquad \forall i \in V \tag{2}$$

$$\sum_{(i,j)\in A_i^o} x_{ij} = \sum_{(j,i)\in A_i^d} x_{ji} \qquad \forall i \in V \tag{3}$$

$$a_i \le T_m^1 \qquad \forall i \in V \tag{4}$$

$$a_j \le t_j + M*(1-y_{ij}) \qquad \forall (i,j) \in A_m \tag{5}$$

$$a_j \ge t_j - M*(1-y_{ij}) \qquad \forall (i,j) \in A_m \tag{6}$$

$$a_j \le a_i + t_j + t_{ij} + M*(1-x_{ij}+y_{ij}) \qquad \forall (i,j) \in A_m \tag{7}$$

$$a_j \ge a_i + t_j + t_{ij} - M*(1-x_{ij}+y_{ij}) \qquad \forall (i,j) \in A_m \tag{8}$$

$$a_j \le a_i + t_j + t_{ij} + M*(1-x_{ij}) \qquad \forall (i,j) \in A_c \tag{9}$$

$$a_j \ge a_i + t_j + t_{ij} - M*(1-x_{ij}) \qquad \forall (i,j) \in A_c \tag{10}$$

$$b_i \le L_m^1 \qquad \forall i \in V \tag{11}$$

$$b_j \le l_j + M*(1-y_{ij}) \qquad \forall (i,j) \in A_m \tag{12}$$

$$b_j \ge l_j - M*(1-y_{ij}) \qquad \forall (i,j) \in A_m \tag{13}$$

$$b_j \le b_i + l_j + M*(1-x_{ij}+y_{ij}) \qquad \forall (i,j) \in A_m \tag{14}$$

$$b_j \ge b_i + l_j - M*(1-x_{ij}+y_{ij}) \qquad \forall (i,j) \in A_m \tag{15}$$

$$b_j \le b_i + l_j + M*(1-x_{ij}) \qquad \forall (i,j) \in A_c \tag{16}$$

$$b_j \ge b_i + l_j - M*(1-x_{ij}) \qquad \forall (i,j) \in A_c \tag{17}$$

$$x_{ij} \ge y_{ij} \qquad \forall (i,j) \in A_m \tag{18}$$

$$x_{ij} \in \{0,1\} \qquad \forall (i,j) \in A \tag{19}$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in A_m \tag{20}$$

The objective function (1) is the total turnaround time of the train unit routings over the decision horizon. If the objective function is added by the total travel time of the trips, we obtain *the* *total circulation time* of train units needed, which is just an integer multiple of 1440, i.e., N*1440. Note that "N" is just the number of train units needed. For example, in Fig.2, the circulation time of feasible solution "$G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_1 \rightarrow G_5 \rightarrow G_6$" is 2880 mins, which indicates that 2 train units are needed. Since the total travel time of the trips is a constant, the difference between the objective values of two feasible solutions is also an integer multiple of 1440. For example, if we have two feasible solutions, namely SOL_1 and SOL_2, which need 6 and 7 train units to carry out the trips respectively, then the difference between the corresponding objective values is 1440, i.e., z_2-z_1=(7-6)*1440. In this sense, minimizing the total turnaround time is equivalent to minimizing the number of train units needed.

Constraint (2) ensures that each trip is covered by one and only one train unit routing. Constraint (3) is the conservation constraint, which ensures that every train unit routing forms a closed cycle. Constraint (4) ensures that at any node, the cumulative elapsed time is less than T_m^1. Constraints (5) to (8) calculate the cumulative elapsed time on maintenance arc (i, j). In detail, for each arc (i, j), if x_{ij} =1 and y_{ij} =0, which means that there is no maintenance on arc (i, j), then the travel time t_j and turnaround time t_{ij} are added to cumulative elapsed time a_i; if x_{ij} =1 and y_{ij} =1, which means that the first level maintenance is performed on arc (i, j), then the value of a_j is equal to t_j. Constraints (9) and (10) calculate the cumulative elapsed time on non-maintenance arc (i, j), which means that only when x_{ij} =1, turnaround time t_{ij} is added. Constraints (11) to (17) are similar to constraints (4) to (10); they are constraints of the cumulative travel distance. Constraint (18) expresses the relationship between decision variables x_{ij} and y_{ij}. Specifically, only when arc (i, j) is selected, *i.e.*, x_{ij} =1, a maintenance may be performed on it, *i.e.*, $y_{ij} \in \{0, 1\}$; when arc (i, j) is not selected, *i.e.*, x_{ij} =0, no maintenance can be performed on it, *i.e.*, y_{ij} =0. Constraints (19) and (20) describe the binary character of the decision variables.

4 A Special Train Unit Routing Model for Chinese High-speed Railway

In China, high-speed railway infrastructures undergo daily check from 0:00 to 6:00, and there is no high-speed trip during this time. Recall that a train unit returns to its depot for maintenance, and the first level maintenance is performed in the night. After maintenance, the train unit leaves the depot and starts to carry out trips in the next day. Besides, the elapsed time resource constraint of the first level maintenance is 2 days. Based on these characteristics, we can modify the connection network in Section 3.2 and construct a special connection network with a two-day horizon. Circulating in a two-day connection network, the time resource constraint is always satisfied, which indicates that we can remove it when formulating models.

Similar as in the general connection network, there are two types of turnaround arcs in a special connection network, namely day arcs and night arcs. The upper bounds of turnaround time on day arcs and night arcs can also be considered when constructing a special connection network.

4.1 Connection network with a two-day horizon

A two-day acyclic connection network is constructed as follows:

Step 1: Construct nodes. Trips in the timetable are represented by nodes as in Section 3.1, which constitute the node set of the first day. The node set of the second day is the copy of the node set in the first day, and the corresponding departure and arrival times are increased by 1440 min (exactly one day). The set of nodes representing trips is denoted by V_n. In addition, construct a pair of dummy nodes (*o* and *d*) to represent the depot. Define $V = V_n \cup \{o, d\}$.

Step 2: Construct trip arcs between nodes in V_n. For a pair of nodes $i, j \in V_n$, if $s_i^d = s_j^o$, $\sigma_{ij} \le t_j^o - t_i^d \le \sigma_{upper}^{day}$, $t_j^o, t_i^d < 1440$ or $t_j^o, t_i^d > 1440$, connect nodes *i* and *j* by directed arc (i, j) with duration $t_{ij}^{day} = t_j^o - t_i^d$, which is referred to as a *day arc*. For a pair of nodes $i, j \in V_n$, if $s_i^d = s_j^o$, $t_i^d < 1440$, $t_j^o > 1440$ and $\sigma_{lower}^{night} \le t_j^o - t_i^d \le \sigma_{upper}^{night}$, connect nodes *i* and *j* by directed arc (i, j) with duration $t_{ij}^{night} = t_j^o - t_i^d$, which is referred to as a *night arc*. Note that σ_{ij} and σ_{upper}^{day} (σ_{lower}^{night} and σ_{upper}^{night}) respectively represent the lower and upper bounds of turnaround time in the day (night). In practice, σ_{ij} depends on the arrival track of trip *i* and departure track of trip *j*, and σ_{lower}^{night} is usually equal to 360 min (6 h). Obviously, the smaller values σ_{upper}^{day} and σ_{upper}^{night} take, the fewer the directed arcs there is in the connection network.

Step 3: Construct depot arcs, which connect nodes in V_n and dummy nodes. First, construct origin depot arcs. For $i \in V_n$, if node *i* is on the first day and the depot is located near its origin station s_i^o, add a directed arc (*o, i*) with turnaround time $t_{oi} = t_i^o$. Second, construct destination depot arcs. For $i \in V_n$, if the depot is located near station s_i^d, add a directed arc (*i, d*). If *i* is a node on the first day, the turnaround time is $t_{id} = 1440 - t_i^d$; otherwise, the turnaround time is $t_{id} = 2880 - t_i^d$. Denote the set of all arcs as *A*, the set of day and night arcs as A_n, the set of origin depot arcs as A_o, and the set of destination depot arcs as A_d. Obviously, $A = A_n \cup A_o \cup A_d$.

Step 4: Remove all the nodes that have no in-arc or out-arc, and finally obtain a connection network with two-day horizon.

The connection network with a two-day time horizon for the trips in Fig.1 is depicted in Fig.3, in which the solid arcs are the day arcs, the dashed arcs are the night arcs and the dash-dot arcs are the depot arcs. As in Fig.2, the feasible routing "$G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_1 \rightarrow G_5 \rightarrow G_6$" is also illustrated by the bold arcs, *i.e.,* "$o \rightarrow G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow G_1' \rightarrow G_5' \rightarrow G_6' \rightarrow d$" in Fig.3. It should be pointed out that since the bounds of turnaround time on the arcs are set, the number of arcs is significantly reduced. According to Step 4, nodes G_1 and G_4' and the arcs connecting them are removed, because they do not have either in-arc or out-arc. In Fig. 3, nodes G_1 and G_4' are illustrated by dashed circles.

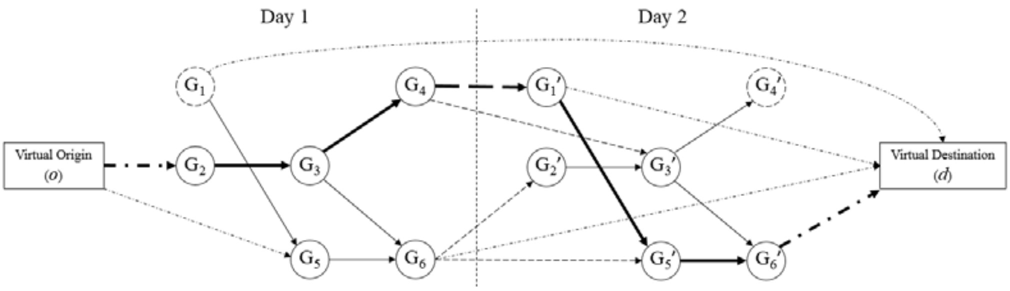

Fig.3. An acyclic connection network with a two-day time horizon

It is easy to verify that there exists a one-one mapping between a train unit routing and a path from the origin node to the destination node. Since the first level maintenance is performed at the destination node, the time span of a path is less than 2 days, which indicates that the time resource constraint is always satisfied if the formulation is based on the connection network in Fig.3. For this reason, we can simplify the train unit routing model formulated in Section 3.3 by removing the time resource constraints, which is discussed in detail in the following section.

4.2 Special Model

According to the above discussion, we can remove the time resource constraints in the general train unit routing model in Section 3.3, that is, constraints (4)–(10) and constraints (12)–(15), (18) and (20). In order to formulate the special model, another two notations are needed, *i.e.*,

Q Index set of trips in the given timetable.

q Index of a trip, *i.e.*, $q \in Q$.

Note that there are two nodes in the two-day connection network corresponding to each trip in the given timetable. For example, trip G1 in the timetable corresponds to nodes G1 and G1' in Fig.3. In a feasible solution, it is required that nodes G1 and G1' cannot be contained in the same path from node *o* to node *d*. The same applies to other trips. For these reasons, we formulate the special model as follows:

$$\min z = \sum_{(i,j)\in A} t_{ij} \cdot x_{ij} \tag{21}$$

$$s.t. \quad \sum_{\varepsilon(i)=q} \left(\sum_{(i,j)\in A_i^o} x_{ij} \right) = 1 \qquad \forall q \in Q \tag{22}$$

$$\sum_{(i,j)\in A_i^o} x_{ij} = \sum_{(j,i)\in A_i^d} x_{ji} \qquad \forall i \in V_n \tag{23}$$

$$b_i \le L_m^1 \qquad \forall i \in V_n \tag{24}$$

$$b_0 = 0 \tag{25}$$

$$b_j \le b_i + l_j + M*(1-x_{ij}) \qquad \forall (i,j) \in A_n \cup A_o \tag{26}$$

$$b_j \ge b_i + l_j - M*(1-x_{ij}) \qquad \forall (i,j) \in A_n \cup A_o \tag{27}$$

Note that Q is the index set of trips in the given timetable, and q is the index of a trip. We define a function ε_i from node set V_n to index set Q. For example, if the index of trip G1 is 1, which corresponds to nodes G1 and G1' in Fig.3, then we have $\varepsilon(G1) = \varepsilon(G1') = 1$. As a result, constraint (22) ensures that node G1 or node G1' must be covered by one and only one train unit routing, but these two nodes cannot be contained in the train unit routing at the same time. Constraint (23) is the flow conservation constraint. Constraint (24) ensures that at any node, the cumulative travel distance is less than L_m^1. Constraint (25) sets the value of the cumulative travel distance at the origin depot node (b_o) to be zero. Constraints (26) and (27) calculate the cumulative travel distance after carrying out trip *j*.

According to the definition of depot arcs, it is easy to find that if the total travel time of the trips is added to the objective function, we obtain the total circulation time of train units needed, i.e., N*1440, where "N" is just the number of train units needed. For example, in Fig.3, "$o \to G_2 \to G_3 \to G_4 \to G_1' \to G_5' \to G_6' \to d$" is a feasible solution. Obviously, the total circulation time of this solution is 2880 mins, which indicates that 2 train units are needed. An alternative solution is "$o \to G_2 \to G_3 \to G_4 \to G_1' \to d$" and "$o \to G_5 \to G_6 \to d$", and the total circulation time is 4320 mins, which indicates that 3 trains are needed.

5 Extension to Multi-Depot Train Unit Routing

In this section, we extend the train unit routing model with single depot to the case of multiple depots. Along high-speed railway lines with long distance, such as Beijing–Shanghai (1318 km) and Beijing– Guangzhou (2298 km), multiple depots are used, and train units are affiliated to these depots, where the first level maintenance is performed. Besides, in a high-speed railway network, there are usually more than one depots. For example, in the Yangtze Delta railway network, train units are affiliated to 5 depots. In this situation, we must solve the train unit routing problem with multiple depots. Therefore, we extend our model to the multi-depot train unit routing problem.

Given the timetable and the location of depots, the set of which is denoted by K, we construct an acyclic connection network in a similar way with that in Section 4.1. Note that there are several pairs of dummy nodes representing the multiple depots, *i.e.*, (o_k, d_k) for $k \in K$. Denote the set of o_k as V_o, and the set of d_k as V_d. Then, we have $V = V_n \cup V_o \cup V_d$.

Assume that in Fig.1, station S_1 is also equipped with a depot, which means that there are two depots in the railway network. Fig.4 depicts the corresponding two-day connection network for the considered trips. In Fig.4, there are two pairs of dummy nodes, *i.e.*, (o_1, d_1) and (o_2, d_2), and a path from o_1 to d_1 or from o_2 to d_2 is a feasible train unit routing. For example, the blue arcs represent feasible routing "$o_1 \to G_1 \to G_5 \to G_6 \to G_2' \to G_3' \to G_4' \to d_1$", and the red arcs represent feasible routing "$o_2 \to G_2 \to G_3 \to G_4 \to G_1' \to G_5' \to G_6' \to d_2$". However, path "$o_2 \to G_2 \to G_3 \to G_6 \to G_2' \to d_1$" is not a feasible train unit routing because o_2 and d_1 represent different depots.

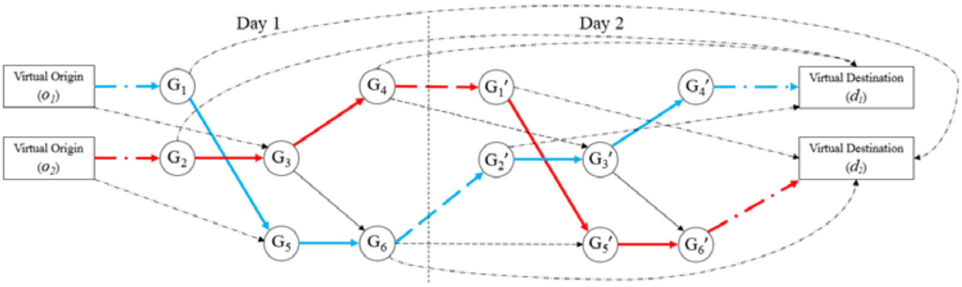

Fig.4. An Example of a Multi-Depot Acyclic Connection network

For multi-depot train unit routing problem, decision variables are defined as follows:

x_{ij}^{k} Binary decision variable, which indicates whether arc (i, j) is contained in the path corresponding to a train unit routing from depot k ($k \in K$). If it is contained, x_{ij}^{k} =1; otherwise, x_{ij}^{k} = 0.

b_{i}^{k} Assistant decision variable for train unit routing starting from depot k, which records the cumulative travel distance after a train unit carrying out trip i.

Then, the train unit routing model with multi-depot is formulated as follows:

$$\min z = \sum_{(i,j)\in A} t_{ij}^{k} \cdot x_{ij}^{k} \tag{28}$$

$$s.t. \quad \sum_{k\in K}\left(\sum_{\varepsilon(i)=q}\left(\sum_{(i,j)\in A_i^o} x_{ij}^{k}\right)\right)=1 \qquad \forall q \in Q \tag{29}$$

$$\sum_{(i,j)\in A_i^o} x_{ij}^{k} = \sum_{(j,i)\in A_i^d} x_{ji}^{k} \qquad \forall i \in V_n, \forall k \in K \tag{30}$$

$$b_{i}^{k} \le L_m^1 \qquad \forall i \in V_n, \forall k \in K \tag{31}$$

$$b_{0}^{k} = 0 \qquad \forall i \in V_o, \forall k \in K \tag{32}$$

$$b_{j}^{k} \le b_{i}^{k} + l_j + M*(1-x_{ij}^{k}) \qquad \forall (i,j) \in A_n \cup A_o \tag{33}$$

$$b_{j}^{k} \ge b_{i}^{k} + l_j - M*(1-x_{ij}^{k}) \qquad \forall (i,j) \in A_n \cup A_o \tag{34}$$

$$x_{ij}^{k} \in \{0,1\} \qquad \forall (i,j) \in A \tag{35}$$

Note that a node in the two-day connection network may be covered by different train unit routings from different depots. For example, in Fig.4, node G_3 and node o_1 is connected by an arc, which means that node G_3 can also be covered by a train unit routing from depot 1, such as "$o_1 \to G_3 \to G_4 \to d_1$". Constraint (29) ensures that one and only one of the nodes corresponding to a trip must be covered one and only one train unit routing, which is from one depot. Other constraints are similar to those in Section 4.2.

6 Numerical Experiments

To demonstrate the performance of the proposed train unit routing methods, we carry out numerical experiments based on the real data of four high-speed railway depots in China, which serve train units on the network consisting of Shanghai–Nanjing, Shanghai–Hangzhou, Beijing–

Shanghai and Beijing–Guangzhou high-speed railway lines, respectively. For simplicity, these four depots (railway lines) are referred to as “A”, “B”, “C” and “D”. In addition, two virtual high-speed railway network (“Va” and “Vb”) with multiple depots are built to further verify the effectiveness of our models. Without special directions, the travel distance resource of the first level maintenance is set to be 4000km, *i.e.*, $L_m^1 = 4000$, and the duration of a maintenance is set to be 3 hours.

The numerical experiments are performed on a personal computer with a 2.0 GHz CPU and 4.0 GB RAM, and all the programming is implemented in IBM ILOG CPLEX 12.3. Note that all the problems are solved to optimality with a gap of 0.01%.

6.1 Experiments on the general model

We first apply the general model (1)-(20) to solve the train unit routing problems, in which the elapsed time resource is 48 hours. Note that according to the general model, maintenance can be carried out both on day arcs and night arcs. Table 2 shows the parameters and experiment results of the cases. Note that the name of a case consists of three parts, namely, railway line, model and experiment number. For example, “A-G-1” means that it is the 1st experiment on depot “A”, where the general model (G) is used.

Column 2-5 of Table 2 list the parameters of a case, where “trips”, “nodes”, “t-arcs” and “m-arcs” respectively indicate the number of involved trips, the number of nodes, turnaround arcs and maintenance arcs in the corresponding connection network. Column 6 reports the number of train units needed to carry out all trips, column 7 reports the optimal objective values, and the last column, which is named “CT”, lists the computation times.

From Table 2, it is easy to find that the general model is efficient when the number of trips is small, for example the first three cases. When the number of trips increases, however, the computation time increases drastically. As case D-G-1 shows, when the number of trips is 188, the optimal solution cannot be obtained within 1800s.

Table 2: The computation results when the general model is used

Case	Numbers				Train units	Obj. [min]	CT [s]
	trips	nodes	t-arcs	m-arcs			
A-G-1	34	34	290	16	7	5847	3.88

A-G-2	34	34	290	16	6	4407	0.79
B-G-1	45	45	878	400	21	14626	0.79
C-G-1	94	94	3722	1372	15	11701	46.48
D-G-1	188	188	7538	2946	---	---	>1800

Note: In Case A-G-2, we relax the distance limit of the first level of maintenance from 4000 km to 4100 km according to practical train unit routing.

In practice, it is allowed that the travel distance resource L_m^1 can be relaxed in a reasonable range (10%), which indicates that L_m^1 takes value in [4000, 4400]. In case A-G-2, we set $L_m^1 = 4100$ km. Compared with case A-G-1, the total turnaround time of case A-G-2 decreases from 5847 minutes to 4407 minutes. Note that 5847-4407=1440, which means that in case A-G-2 the number of train units needed is reduced by 1. As column "Train unit" shows, the number of train units decreases from 7 to 6, which is a significant reduction. This also indicates that by appropriately relaxing L_m^1, the operation costs may be reduced largely. Besides, the computation time is also reduced from 3.98s to 0.73s.

Note that in the experiments in Table 2, there is no restriction on the bounds of turnaround times. Via setting the upper and lower bounds of turnaround times, the number of arcs in the connection network decreases largely, and accordingly the computation time may be reduced. Table 3 lists the results of the experiments when the upper bounds are set to be different values, i.e., $\sigma_{upper}^{day} \in \{180, 240, 360\}$ and $\sigma_{upper}^{night} \in \{720, 1440\}$. Since there is no high-speed trip from 0:00 to 6:00 in the morning, we set $\sigma_{lower}^{night} = 360$ for all experiments. In column "t-arc" and "m-arc", we can see that the corresponding values reduce a lot in Table 3, which indicates that we are dealing with a smaller model after setting the bounds of turnaround times.

Compared with the last three experiments in Table 2, the corresponding optimal values of objectives in Table 3 are not changed, however, the computation times all decrease. Especially, for depot "D", we cannot obtain an optimal solution without setting the bounds of turnaround times within 1800s, while after setting the bounds, an optimal solution can be obtained in less than 500s. Besides, a small reduction on σ_{upper}^{day} may lead to a significant reduction on computation time. For example, we set $\sigma_{upper}^{day}=240$ in case D-G-2, and $\sigma_{upper}^{day}=180$ in case D-G-3. As a result, the computation time decreases from 459.07s to 144.23s, while the objective value keeps unchanged.

Table 3: The computation results of the general model with bounds of turnaround times

Case	Numbers				σ [min]			Train units	Obj. [min]	CT [s]
	trips	nodes	t-arcs	m-arcs	σ_{upper}^{day}	σ_{lower}^{night}	σ_{upper}^{night}			
B-G-3	45	45	662	400	240	360	1440	21	14626	0.62
C-G-2	94	94	2372	1003	360	360	1440	15	11701	3.72
C-G-3	94	94	695	135	240	360	720	15	11701	1.55
D-G-2	188	188	2778	537	240	360	720	30	23402	462.77
D-G-3	188	188	2383	345	180	360	720	30	23402	150.54

6.2 Experiments on the special model

In this section, we carry out experiments on the special model for Chinese high-speed railway. Recall that the main difference between the special model and general model is that the first level of maintenance is required to be performed only on the night arcs. As a result, the connection network for a general model is modified to a connection network with a two-day horizon, and accordingly the model is also changed.

Table 4 lists the computation results. Since the special model is used, the middle letter of case name is "S". Note that the first level maintenance is performed in the night in the special model, so there is no need to distinguish maintenance arcs and non-maintenance arcs in the connection network. Recall that trips are represented by nodes in a connection network. As Table 4 shows, the number of nodes in the connection network with a two-day horizon is about twice of the number of trips every day. This is because two dummy nodes are added in Step 1 and a few of nodes are removed in Step 4. It should be pointed out that Step 4 is not always performed when constructing a connection network. For example, in case A-S-1, C-S-1, C-S-2, the numbers of nodes are all equal to "2 + 2×number of trips", which indicates that no node is removed. However, in case B-S-1, D-S-1 and D-S-1, the numbers of nodes are all equal to "2 + 2×number of trips - 2", which indicates that two nodes are removed in Step 4.

Compared with the experiments in Table 3, the number of train units needed in each case is unchanged in Table 4, however, the objective values increase. The main reason is that in the cases of Table 3, the first level of maintenance can be performed on day arcs and night arcs, while in the cases of Table 4, the maintenance can only be performed in the night, which makes the feasible region smaller than that in Table 3. As a result, the objective values in Table 4 are greater than the corresponding values in Table 3.

Table 4: The computation results of the special model with bounds of turnaround time

Case	Numbers			σ [min]			Train units	Obj. [min]	CT [s]
	trips	nodes	t-arcs	σ_{upper}^{day}	σ_{lower}^{night}	σ_{upper}^{night}			
A-S-1	34	70	140	240	360	720	7	6786	0.52
B-S-1	45	90	870	240	360	1440	21	15874	0.93
C-S-1	94	190	3161	360	360	1440	15	13899	1.83
C-S-2	94	190	1292	240	360	720	15	13899	1.01
D-S-1	188	376	4942	240	360	720	30	27800	204.81
D-S-2	188	376	4152	180	360	720	30	27800	117.05

Note that under the same condition of σ, the computation time in Table 4 decreases. This is because that in the special model, the time resource constraints are removed, which makes the model less difficult. We can see that the computation times are all less than 4 mins, which is acceptable in practice. Besides, for most depots, the numbers of train units of one type are not greater than 30, which means that the applicability of the special model is good. However, as the number of trips increases or multi-depot cases are considered, it is likely that the computation time increases drastically, which is shown in the next section.

6.3 Multi-depot cases

In this section, we discuss the cases with multiple depots. For all the cases in this section, only the special model is applied. Note that “Va” and “Vb” in the case names represent the virtual railway network with 4 depots and the virtual railway network with 5 depots, respectively.

It should be noted that although the number of trips in “Va” is less than half of that in “Vb”, the objective values of “Va” are only slightly less than “Vb”. Moreover, the number of train units needed in “Va” is even greater than in “Vb”. The reason is that “Va” mainly consists of long-haul trips, while “Vb” mainly consists of short-haul trips. A train unit in “Vb” may carry out several trips every day, while a train unit in “Va” can only carry out one or two trips every day. This further indicates that the solution of train unit routing problem largely depends on the characteristics of the involved trips.

Table 5: The computational results of cases with multi-depot

Case	Numbers				σ [min]			Train units	Obj. [min]	CT [s]
	depots	trips	nodes	t-arcs	σ_{upper}^{day}	σ_{lower}^{night}	σ_{upper}^{night}			
Va-S-1	4	117	242	11221	240	360	1440	41	32357	1.33
Va-S-2	4	117	242	6077	240	360	720	41	32360	0.84
Vb-S-1	5	242	494	51336	240	360	1440	40	33519	980.3
Vb-S-2	5	242	494	23291	240	360	720	40	33519	755.3

As is shown in Table 5, for the cases of “Va”, the computation time is very short; for the cases of “Vb”, the computation time is much longer. Although the computation time is still acceptable, it reveals that the proposed model is weak at dealing with large-scale cases.

7 Conclusions

In this paper, we focus on a train unit routing problem with maintenance resource constraints both on time and distance. We first propose a connection network to describe the turnaround between trips, based on which a general train unit routing model is formulated. Considering the features of high-speed train unit circulation in China, a special model is further constructed with less constraints. In order to improve the solvability, we propose a heuristic method to reduce the number of arcs in the connection arcs by setting the upper bounds of turnaround time. Besides, we extend the model to the case of multiple depots. We carry out a series of numerical experiments based on the real data in Chinese high-speed railway network to text our models. The computation result verifies the effectiveness of our models, and shows that in most cases, the computation time is acceptable. Besides, the computation result also shows the advantage of the special model, which requires a much less computation time than the general model.

In the future, the research can be extended in the following three aspects. First, train unit assignment problem can be integrated in the model. Although train unit routing problem is at a tactical level while train unit assignment problem is at an operational level, it is possible to consider them simultaneously and obtain a more optimal solution. If train unit routing problem is integrated, the second level of maintenance should be also considered. As a result, the model will be much more complex. Second, in this paper we assume that there is only one type of train unit, and there is neither coupling nor uncoupling. In some countries, coupling/uncoupling of train units is quite frequent. In our future model, we will take into account more types of train units and the composition of train units. Third, some sophisticated algorithm should be developed to solve train unit routing problems with large scales. In this paper, we actually use big-M method to deal with the constraints related to maintenance, and the experiments we carried out are of middle scales. In some literature, Lagrange-relaxation-based heuristic algorithms and column-generation-based heuristic algorithms were developed. It is challenge to develop an efficient heuristic algorithm to solve train unit routing problems.

References

Abbink E., Berg B.V.D., Kroon L., Salomon M., 2004. "Allocation of railway rolling stock for passenger trains". In: *Transportation Science*, vol.38, issue 1, pp. 33-41

Alfieri A., Groot R., Kroon L., Schrijver A., 2006. "Efficient circulation of railway rolling stock". In: *Transportation Science*, vol.40, issue 3, pp.378-391

Borndörfer, R., Reuther, M., Schlechte, T., Waas, K., Weider, S., 2015. "Integrated optimization of rolling stock rotations for intercity railways". In: *Transportation Science*, in press.

Brucker J., Hurink T., Rolfes T., 1998. "Routing of railway carriages: a case study". In: *Memorandum 1498*, Department of Applied Mathematics, University of Twente, Enschede.

Cacchiani V., Caprara A., Toth P., 2010. "Solving a real-world train-unit assignment problem". In: *Mathematic Programming*, vol. 124, pp. 207-231.

Cacchiani V., Caprara A., Toth P., 2013. "A Lagrangian heuristic for a train-unit assignment problem". In: *Discrete Applied Mathematics*, vol.161, pp.1707-1718.

Cordeau J.F., Soumis F., Desrosiers J.A., 2000. "Benders decomposition approach for the locomotive and car assignment problem". In: *Transportation Science*, vol.34, pp.133-149.

Cordeau J.F., Soumis F., Desrosiers J.A., 2001. "Simultaneous assignment of locomotives and cars to passenger trains". In: *Operations Research*, vol.49, pp.531-548.

Fioole P.J, Kroon L., Maróti G., Schrijver A., 2005. "A rolling stock circulation model for combining and splitting of passenger trains". In: *European Journal of Operational Research*, vol.174, pp.1281-1297.

Giacco, G. L., D'Ariano, A., Pacciarelli, D., 2014. "Rolling stock rostering optimization under maintenance constraints". In: *Journal of Intelligent Transportation Systems*, vol.18, issue 1, pp. 95-105.

Hong S.P., Kim K.M., Lee K., et al., 2009. "A pragmatic algorithm for the train unit routing: the case of Korea high-speed railway". In: *Omega*, vol. 37, issue. 3, pp.637-645.

Lai, Y. C., Fan, D. C., Huang, K. L., 2015. "Optimizing rolling stock assignment and maintenance plan for passenger railway operations". In: *Computers & Industrial Engineering,* vol.85, pp.284-295.

Lingaya N., Cordeau J.F., Desaulniers G., Desrosiers J., Soumis F., 2002. "Operational car assignment at VIA rail Canada". In: *Transportation Research Part B*, vol.36, pp.755–778.

Marc P., Kroon L., 2008. "Circulation of railway rolling stock a branch-and-price approach". In: *Computers & Operational Research*, vol.35, pp.538-556.

Maróti G., Kroon L., 2005. "Maintenance routing for train units: the transition model". In: *Transportation Science*, vol.39, issue 4, pp.518-525.

Maróti G., 2006. "Operations research models for railway rolling stock planning". PhD. Technische Universiteit Eindhoven.

Maróti G., Kroon L., 2007. "Maintenance routing for train units: The interchange model". In: *Computers & Operations Research*, vol.34, pp.1121-1140.

Schrijver A., 1993. "Minimum circulation of railway stock". In: *CWI Quarterly*, vol.6, pp.205-221.