Vector Space Ramsey Numbers

by

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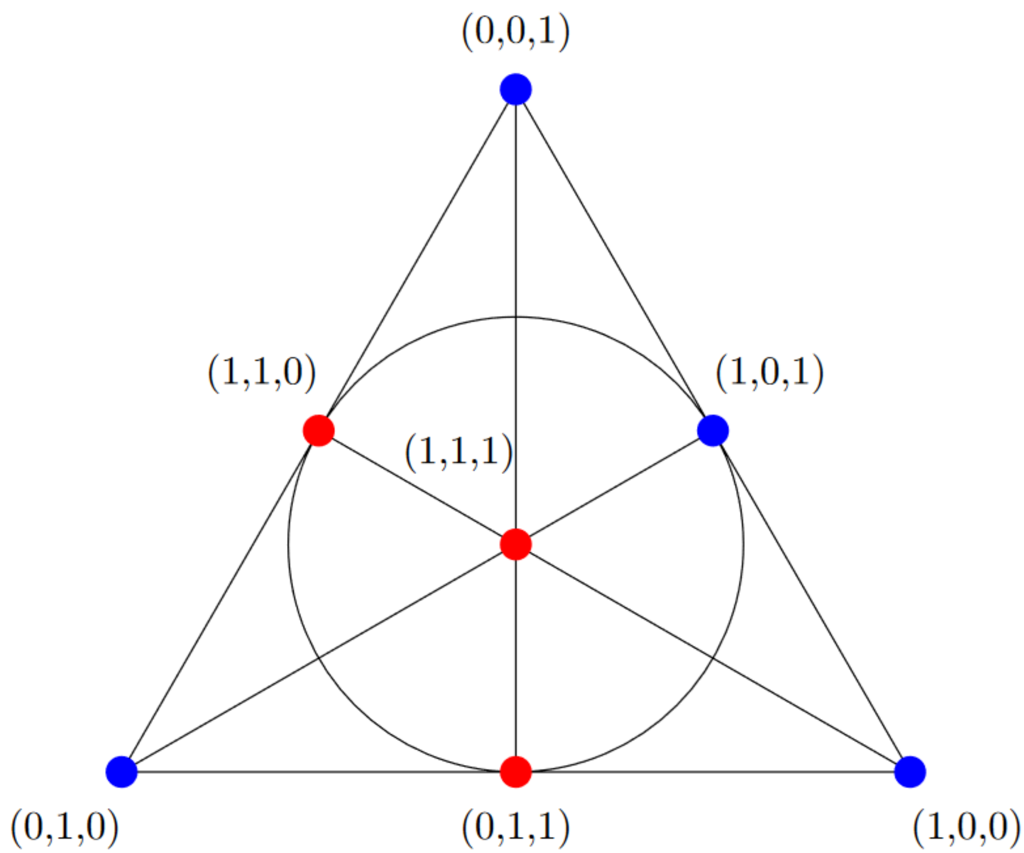

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Abstract

In this text, we begin by giving a definition of Vector Space Ramsey Numbers. It concerns colourings of t-dimensional subspaces of some vector space $\mathbb{F}_q^n$. We want to ensure that each colouring contains a monochromatic k-dimensional subspace. After proving that these numbers always exist, we continue with studying asymptotic bounds for these numbers. We study a selection of methods, such as through coding theory or using the probabilistic method, to obtain lower and upper bounds for vector space Ramsey numbers. Lastly, we introduce two methods to directly compute vector space Ramsey numbers. That being through an ILP formulation and through a SAT formulation.

Contents

Chapter 1

Introduction

Say that you invite 6 people to have a party with. Having chosen them a bit arbitrarily, you take some interest in the relations between your 6 guests. After doing this for a number of parties, you realise that in each group of 6 guests, there were always 3 guests that were all friends with each other or 3 where no 2 of them were friends. It may seem like coincidence, but you actually stumbled into the world of Ramsey numbers. To get to the mathematical way of look at these numbers, replace the guests with points in a graph and replace the relations with the colours red and blue. Then, a Ramsey number $R(s,t) = n$ implies that a complete graph of n points has a red clique of size s or a blue clique of size t for every red-blue colouring of the edges. With a red clique of size s we mean a complete subgraph of our graph whose edges are all coloured red. In particular, this is the smallest such n. In our party example, we see that $R(3,3) \leq 6$.

The notion of these 'classical' Ramsey numbers tells us that graphs of sufficient size always have some sort of structure to them. Though, graphs are scarcely the only notion to which this idea can be applied to. In this text, we study what Ramsey numbers look like when applied to vector spaces defined over finite fields. As will be explained in chapter 2, instead of colouring edges we shall be investigating colourings of the subspaces of some vector space. In the same chapter it is also shown that these Ramsey numbers always exist, showing that here too some sort of structure always arises.

As with classical Ramsey numbers, it can quickly be concluded that, save for low values, these vector space Ramsey numbers are incredibly difficult to compute. Thus, in chapter 3, we study some of the bounds of these Ramsey numbers. In particular, we use three methods to demonstrate a similar lower bound. We also look back to classical Ramsey numbers to find an upper bound. Then, we consider some of the recent upper bounds that have been given on more general forms of vector space Ramsey numbers. We still attempt to compute some values in chapter 4. We explore two methods to compute various Ramsey numbers, notably through an ILP and a SAT-problem formulation. Aside from values we also obtain some specific colourings to study. We then have a discussion in chapter 5 surrounding the explained computational and theoretical methods to compute these numbers, as well as the found numbers. That includes some recommendations of what could be researched further concerning vector space Ramsey numbers. We end with conclusions in chapter 6, collecting what we have discussed throughout this text.

Chapter 2

Theory

In this chapter, we shall lay out the basics of vector space Ramsey numbers. First, in section 2.1 an explanation of Ramsey Numbers shall be given. Then, in section 2.2 the proof that these numbers always exist is given.

2.1 Defining vector space Ramsey numbers

To define what these Ramsey Numbers are we first need to go over what Vector Spaces defined over Finite Fields are. For this, we first give a short introduction to finite fields as well as discuss what it means to define a Vector Space over one in subsection 2.1.1. This is followed up by colouring them in subsection 2.1.2. With all that knowledge, we then go on to define what vector space Ramsey numbers are in subsection 2.1.3.

2.1.1 Vector Spaces over Finite Fields

In this section we will discuss finite fields in a more practical manner, focusing on the fields we are generally interested in. For a more rigorous definition of fields, see Appendix A. In short, a field F is a ring such that $F\backslash\{0\}$ is an abelian group under multiplication.

The fields we are interested in are finite, containing $q = p^n$ elements, where p is prime. From here on out, we shall denote these fields as $\mathbb{F}_q$. To see exactly how these fields are constructed, refer to Appendix A. For the most part, we'll be dealing with $\mathbb{F}_2$, the binary field, consisting of the elements $\{0, 1\}$.

We can define vector spaces over finite fields. We denote that n-dimensional vector space defined over $\mathbb{F}_q$ as $\mathbb{F}_q^n$. This space consists of all vectors of length n whose elements belong to $\mathbb{F}_q$. To give a more rigorous set notation:

$$\mathbb{F}_q^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{F}_q \text{ for } i \in \{1, ..., n\}\} \tag{2.1}$$

Alternatively, you can see it as the span of the vectors $e_i = (0, ..., 0, 1, 0, ..., 0)$ (the vector where all values are 0 except for 1 at position i) for $i \in \{1, ..., n\}$, where the scalars for the linear combinations are in $\mathbb{F}_q$.

To get a feel for what these spaces look like, we look at the simple example of $\mathbb{F}_2^2$. So the 2-dimensional vector space defined over the binary field. Its vectors are of the form (x, y) where $x, y \in \mathbb{F}_2 = \{0, 1\}$. Taking the span approach, we have that

$$\mathbb{F}_2^2 = span(\{(0, 1), (1, 0)\}) = \{a \cdot (1, 0) + b \cdot (1, 0) | a, b \in \{0, 1\}\} \tag{2.2}$$

From here, it can be seen that $\mathbb{F}_2^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The number of elements in this space are interesting to note. We have vectors consisting of 2 elements, where each element can take one of two values. This results in $|\mathbb{F}_2^2| = 2^2 = 4$ vectors total. In general, we see that a vector space $\mathbb{F}_q^n$ has vectors of length n, where each element of the vector can take one of q values. Thus we have $|\mathbb{F}_q^n| = q^n$.

We are interested in the subspaces of these vector spaces too. Let's, for example, look at the 1-dimensional subspaces of $\mathbb{F}_2^2$. That is, the span of any non-zero vector of $\mathbb{F}_2^2$. Since there are only 2 scalars, 0 and 1, the subspace consists of two elements: the non-zero vector and the zero vector. Indeed, the 1-dimensional subspaces of $\mathbb{F}_2^2$ are $\{(0,0),(0,1)\}, \{(0,0),(1,0)\}$ and $\{(0,0),(1,1)\}$. We can similarly calculate the amount of elements for a k-dimensional subspace of $\mathbb{F}_q^n$ as q^k. Though what about the amount of subspaces? For that, we have the following theorem:

Theorem 1. *Let $\mathbb{F}_q^n$ be the n-dimensional vector space defined over $\mathbb{F}_q$ and $k \in \mathbb{N}$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^n-q)(q^n-q^2)...(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)...(q^k-q^{k-1})} \tag{2.3}$$

is the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

Proof. We prove the theorem essentially by counting the number of bases, or sets $\{v_1, .., v_k\}$ of k linearly independent vectors of $\mathbb{F}_q^n$. For the choice of v_1 we can take any non-zero vector of $\mathbb{F}_q^n$, which there are $q^n - 1$ of. For v_2, we can pick any non-zero vector of $\mathbb{F}_q^n$ so long as it is not in the span of v_1. We know that the span of v_1 contains $q^1 = q$ elements, so there are $q^n - q$ choices for v_2. In general, for $i \leq k$, we pick any non-zero vector that does not belong to the span of $\{v_1, ..., v_{i-1}\}$. That span contains q^{i-1} elements. So we have choice of $q^n - q^{i-1}$ for v_i. This all results in

$$(q^n-1)(q^n-q)(q^n-q^2)...(q^n-q^{k-1}) \tag{2.4}$$

possible sets of k linearly independent vectors. Of course, some of these sets form bases for the same k-dimensional subspace, so we need to divide this number by the amount of bases for a k-dimensional subspace. To get this amount, we simply fill in $n = k$ in equation 2.4. Thus, we see that the total number of k-dimensional subspaces of $\mathbb{F}_q^n$ is indeed

$$\frac{(q^n-1)(q^n-q)(q^n-q^2)...(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)...(q^k-q^{k-1})}$$

□

2.1.2 Colouring of Vector Spaces

Like with the Graph Theoretic Ramsey Numbers, we are interested in colourings. For the definition, we introduce two sets of notation. First, we have $[r] = \{1, ..., r\}$ for $r \in \mathbb{N}$. Secondly, $\begin{bmatrix} V \\ t \end{bmatrix}$ denotes the set of t-dimensional subspaces of a vector space V.

Definition 1. *Let $r, n, t \in \mathbb{N}$ with $t \leq n$ and let $\mathbb{F}_q^n$ be the n-dimensional vector space defined over $\mathbb{F}_q$. An r-colouring of the t-dimensional subspaces of $\mathbb{F}_q^n$ is a function $\chi : \begin{bmatrix} \mathbb{F}_q^n \\ t \end{bmatrix} \to [r]$.*

So, for vector spaces over finite fields, we assign colours to the t-dimensional subspaces of $\mathbb{F}_q^n$. Not the points of those subspaces, the subspaces themselves. To understand this better we once again look at $\mathbb{F}_2^2 = \{(0,0),(0,1),(1,0),(1,1)\}$. We know the 1-dimensional subspaces of this are $\{(0,0),(0,1)\}, \{(0,0),(1,0)\}$ and $\{(0,0),(1,1)\}$. We can represent these vectors and the colouring of these 1-dimensional subspaces visually as seen in figure 2.1.2. In the case of $\mathbb{F}_2^n$, since the 1-dimensional subspaces consist of the zero vector and then some non-zero vector, we sometimes informally refer to it as colourings of the non-zero vectors of $\mathbb{F}_2^n$.

Lastly, we define what it means for a vector space to be monochromatic.

Definition 2. *Let $r, n, t, k \in \mathbb{N}$ with $t \leq k \leq n$ and let $\mathbb{F}_q^n$ be the n-dimensional vector space defined over $\mathbb{F}_q$. Let χ be an r-colouring of the t-dimensional subspaces of $\mathbb{F}_q^n$. Then a k-dimensional subspace K of $\mathbb{F}_q^n$ is called monochromatic with colour $i \in [r]$ if for all $T \in \begin{bmatrix} K \\ t \end{bmatrix}$ we have that $\chi(T) = i$.*

That is, a k-dimensional subspace of $\mathbb{F}_q^n$ is monochromatic with colour $i \in [r]$ if all of its t-dimensional subspaces are coloured i. For example, if the three 1-dimensional subspaces of $\mathbb{F}_2^2$ are assigned the colour 0, then it it monochromatic with colour 0.

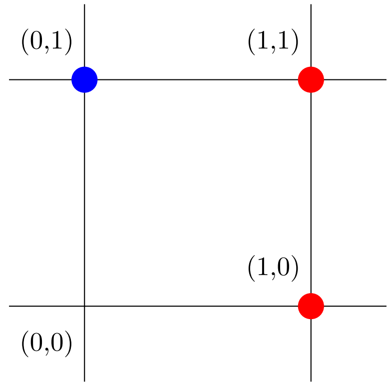

Figure 2.1: Visual representation of the colouring of the 1-spaces (informally written down as non-zero vectors)

2.1.3 Definition

With all of that defined and thought out, we can finally move on to the definition of vector space Ramsey numbers. As we are mostly interested in $\mathbb{F}_2^n$ and 2-colourings of 1-dimensional subspaces, we start with a simplified variant:

Definition 3 (Simplified vector space Ramsey numbers)**.** *Let $s, t, n \in \mathbb{N}$ with $s, t \leq n$. Then $R_2(s,t) = n$ implies the following property. Let $\mathbb{F}_2^n$ be the n-dimensional vector space defined over $\mathbb{F}_2$. Let $\chi : \begin{bmatrix} \mathbb{F}_2^n \\ 1 \end{bmatrix} \to \{0,1\}$ be an arbitrary 2-colouring of the 1-dimensional subspaces of $\mathbb{F}_2^n$. Then there is either an s-dimensional subspace of $\mathbb{F}_2^n$ that is monochromatic with colour 0 or a t-dimensional subspace of $\mathbb{F}_2^n$ that is monochromatic with colour 1. Moreover, There is a 2-colouring of $\mathbb{F}_2^n$ such that there is neither an s-dimensional subspace that is monochromatic with colour 0 or a t-dimensional subspace that is monochromatic with colour 1.*

Remark. *The 2 in $R_2(s,t)$ in this case implies that the vector space is defined over $\mathbb{F}_2$.*

Remark. *The last part essentially implies that n is the smallest number for which the first property holds.*

To explain what this definition encompasses, we shall look at the example of $R_2(2,2)$. For $n = 2$, we look at a 2-colouring χ of the non-zero vectors of $\mathbb{F}_2^2$. We note that the only 2-dimensional subspace of $\mathbb{F}_2^2$ is the vector space itself. Hence, figure 2.1.2 already serves as an example of a colouring where no 2-dimensional subspace is monochromatic. In fact, any non-monochromatic 2-colouring of $\mathbb{F}_2^2$ suffices. Since we can find colourings that don't have monochromatic 2-dimensional subspaces, we may conclude that $R_2(2,2) > 2$. We then look at the case that $n = 3$, so we get a 2-colouring χ of the non-zero vectors of $\mathbb{F}_2^3$. Though the vectors can once again be represented in a grid, this time 3-dimensional, we instead opt to represent the 7 non-zero vectors with the Fano plane, done so in figure 2.1.3. Here, the edges go through 3 points each, including the circle going through (1,1,0), (1,0,1) and (0,1,1). These edges represent the 2-dimensional subspaces of $\mathbb{F}_2^3$.

As can be seen, in this specific example of a colouring, we see that the subspace $\{(0,0,0),(0,0,1),$ $1,0,0),(1,0,1)\}$ is monochromatic with colour blue. These monochromatic subspaces need to occur in *every* colouring of course, though this turns out to hold. Though time-consuming, as there are $2^7 = 128$ distinct colourings, this can be verified manually. It will not be demonstrated here and will be left to the reader to think about. Regardless, once it's shown that each of the colourings indeed contains a monochromatic 2-dimensional subspace, we can conclude that $R_2(2,2) \leq 3$. With the prior result that $R_2(2,2) > 2$, we can further state that $R_2(2,2) = 3$.

Now, we state the more general definition of these Ramsey Numbers:

Definition 4 (Vector space Ramsey numbers)**.** *Let $q, t, r, n, k_1, ..., k_r \in \mathbb{N}$ with $t \leq k_i \leq n$ for $i \in [r]$ and $q = p^m$ for p prime and $m \in \mathbb{N}_{>0}$. Then $R_{q,t}(k_1, ..., k_r) = n$ implies the following property. Let $\mathbb{F}_q^n$ be the n-dimensional vector space defined over $\mathbb{F}_q$. Let $\chi : \begin{bmatrix} \mathbb{F}_q^n \\ t \end{bmatrix} \to [r]$ be an arbitrary r-colouring of*

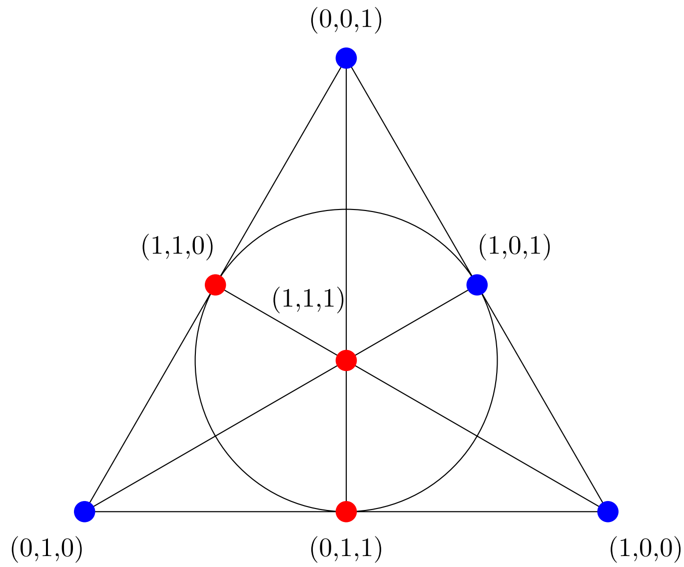

Figure 2.2: Representation of a colouring of $\mathbb{F}_2^3$ via the Fano plane.

the t-dimensional subspaces of $\mathbb{F}_q^n$. Then for some $i \in [r]$, there exists $K \in \begin{bmatrix}\mathbb{F}_q^n \\ k_i\end{bmatrix}$ that is monochromatic with colour i. Moreover, There is an r-colouring of $\mathbb{F}_2^{n-1}$ such that there is no k_i-dimensional subspace that is monochromatic with colour i.

Remark. *In general, we look at the case that $t = 1$. For those cases, we shall simply write the Ramsey Number as $R_q(k_1, ..., k_r)$*

In this definition, we see that there are 3 important generalisations. First, instead of just looking at 2 colours with corresponding subspaces of dimension s, t, we now look at r-colourings with subspaces of dimension $k_1, ..., k_r$. Secondly, instead of just focusing on the binary field, we increase our scope to any finite field $\mathbb{F}_{p^m}$. Lastly, we now consider the colourings of t-dimensional subspaces rather than just 1-dimensional subspaces.

So, for example, $R_{4,2}(5, 2, 3) = n$ implies the following. For any 3-colouring χ of $\mathbb{F}_4^n$, there is either a 5-dimensional subspace that is monochromatic with colour 1, a 2-dimensional subspace that is monochromatic with colour 2 or a 3-dimensional subspace that is monochromatic with colour 3. This n also is the smallest number for which this holds.

2.2 Existence of vector space Ramsey numbers

With Ramsey Numbers properly defined in section 2.1, the question can be asked whether they always exist. It turns out they do, and we shall present the proof for it in this section. The proof is largely based on the work by Spencer (1979) [1]. Though we only consider the case for vector spaces defined over $\mathbb{F}_q$, whereas Spencer takes a more general look at affine spaces (a translation of a vector space) defined over finite fields. It is also important to not that the existence of vector space Ramsey numbers can also be derived from the Graham-Rothschild theorem, from the work of Graham and Rothschild (1971)[2].

Before we proceed we restate some notation and introduce some more new notation for simplicity. We have $[r] = \{1, ..., r\}$ for $r \in \mathbb{N}$. Secondly, $\begin{bmatrix}V \\ t\end{bmatrix}$ denotes the set of t-dimensional subspaces of a vector space V. Finally, since we shall be referring to subspaces and their dimensions a lot, t-space shall refer to a vector space of dimension t. For example, T is a t-space of V means that T is a t-dimensional subspace of V. F shall also now generally refer to $\mathbb{F}_q$.

The proof is quite complex, so we shall begin with giving an overview of the steps in subsection 2.2.1 as well as the statement of the theorem. Then in subsection 2.2.2, we introduce the Hales-Jewett theorem, alongside a corollary about monochromatic tuples. Both of these will prove vital to proving the theorem. We proceed with defining what special vector spaces are in subsection 2.2.3, as well as showing that we can always guarantee finding one. Then, as we intend to use induction, we show it holds for the base case in subsection 2.2.4, followed up by the induction in subsection 2.2.5.

2.2.1 Outline of the proof

First, we shall give a statement of the theorem.

Theorem 2. *For all $t, r, k_1, ..., k_r \in \mathbb{N}$ with $t \leq k_i$ for $i \in [r]$ there exists $n = N^{(t)}(k_1, ..., k_r)$ with the following property. Let $\mathbb{F}_q^n$ be an n-space defined over $\mathbb{F}_q$ and $\chi : \begin{bmatrix} \mathbb{F}_q^n \\ t \end{bmatrix} \to [r]$ be an arbitrary r-colouring of the t-spaces of $\mathbb{F}_q^n$. then for some $i \in [r]$, there exists $K \in \begin{bmatrix} V \\ k_i \end{bmatrix}$ that is monochromatic with colour i.*

Remark. *Here, $n = N^{(t)}(k_1, ..., k_r)$ means that n is dependent on $t, k_1, ..., k_r$ somehow.*

As stated, the idea of the proof is to make use of strong induction. Notably, we do induction over both t and $\sum_{i=1}^{r} k_i$. First, we assume the existence of n for all $t' < t$ for any choice of $k_1, ..., k_r$. Then, for t, we assume n exists for all $\sum_{i=1}^{r} k_i' < \sum_{i=1}^{r} k_i$.

To make use of these assumptions, we want to find lower-dimensional subspaces of $\mathbb{F}_q^n$ that are 'special'. These will be defined and discussed properly in subsection 2.2.3, but we give a short idea here. Given is some subspace B of $\mathbb{F}_q^n$ equipped with a projection that 'reduces' its dimension by one. Then B is special if its 'transversal' t-spaces have their colouring determined by their projection. We prove a lemma that tells us that we can essentially always know such a special space B exists.

This proof, however, makes use of the Hales-Jewett theorem, which says that we can find monochromatic lines. So we begin with introducing that. Alongside it, we prove a corollary that helps us find monochromatic sets using Hales-Jewett. With that corollary, we show that we can indeed construct spaces in such a manner to guarantee a special space B.

With that, we finally move on to the main course, proving theorem 2. We start with the base case. That is, that n exists for $t = 0$ and all $\sum_{i=1}^{r} k_i' < \sum_{i=1}^{r} k_i$. Using Hales-Jewett and a convenient bijection from $\mathbb{F}_q^n$ to $(\mathbb{F}_q^k)m$ where $n = km$, we find a monochromatic line, whose inverse is then also monochromatic in $\mathbb{F}_q^n$. We then of course show that it is also a k-space of $\mathbb{F}_q^n$.

With the base case proven, we move onto the induction step. By cleverly choosing our n, we can find some special spaces of $\mathbb{F}_q^n$, as well as induce 2 different colourings on them based on the projections and the original colouring. By then finding a monochromatic subspace in the projection of one of the special spaces, we can find a desired monochromatic k_i-space of $\mathbb{F}_q^n$ using the induced colourings.

2.2.2 Hales-Jewett

The Hales-Jewett is an important results proven by Alfred W. Hales and Robert I. Jewett (1963) [3]. Here, we shall only give the statement of the theorem, and prove a corollary that is useful for our proof. Before we give the theorem, we first define what a line is. Say we have some finite set A. Let $p_i : A^n \to A$ denote the ith coordinate projection for $1 \leq i \leq n$. For example, $p_2((x, y, z)) = y$. Then, a set $L \in A^n$ with $|L| = |A|$ is called a combinatorial line if the following two things hold. Given some non-empty subset $I \subset [n]$ of the indexes, we have that p_i is is bijective $i \in I$ and that these various p_i are identical to one another. Additionally, for all $i \notin I$, the value of p_i is constant (though these values may differ per $i \notin I$). For an example of a combinatorial line, see figure 2.3. You see that p_1 and p_2 are bijective and that p_1 and p_2 identical, whereas p_3 is constant with value 2.

With that definition in mind, we state the Hales-Jewett theorem:

Theorem 3 (Hales-Jewett)**.** *For all l, c there exists $m = HJ(l : c)$ with the following property. Let $|A| = l$ and $\chi : A^m \to [c]$. Then there exists a monochromatic line $L \subset A^m$.*

Remark. *$m = HJ(l : c)$ simply implies that m is dependent on l and c somehow. $HJ(l : c)$ is also sometimes referred to as the 'Hales-Jewett function'.*

To relate this more directly to our problem, we can for example take $A = \mathbb{F}_q^n$. Then $l = |\mathbb{F}_q^n| = q^n$ and $\chi : (\mathbb{F}_q^n)^m \to [c]$. Here $(\mathbb{F}_q^n)^m$ contains vectors of length m where the element of each vector is a vector

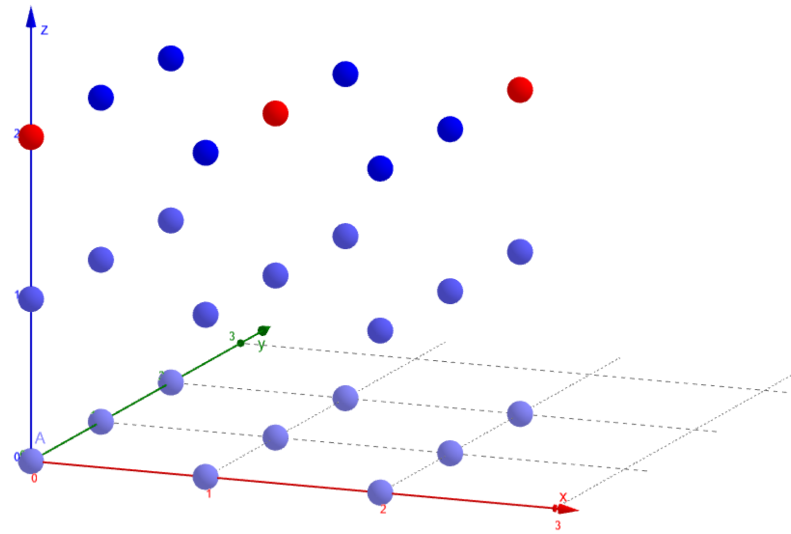

Figure 2.3: The red dots here form the combinatorial line $L = \{(0,0,2),(1,1,2),(2,2,2)\}$ in $\mathbb{F}_3^3$.

out of $\mathbb{F}_q^n$. Then we have a monochromatic line $L \in (\mathbb{F}_q^n)^m$. As we'll see in the base case, we can make good use of this with a nicely defined bijection.

Before we get to the base case, we want to prove that we can always find 'special' spaces. To prove that we can do so, the following corollary to Hales-Jewett plays an important role:

Corollary 1. *Let $m = HJ(|F|^{u+1} : c)$. Let χ be a c-coloring of the ordered (u+1)-tuples $(\vec{x}_0, ..., \vec{x}_u), \vec{x}_i \in F^m$. Then there exist parallel affine lines $L_0, ..., L_u \subset F^m$ so that $\{(\vec{x}_0, ..., \vec{x}_u) : \vec{x}_i \in L_i\}$ is monochromatic.*

Proof. We consider the bijection $\Phi : (F^m)^{u+1} \to (F^{u+1})^m$ which is defined as follows. Given a $(u+1)$-tuple $(\vec{x}_0, ..., \vec{x}_u)$, where $\vec{x}_i = (x_{i1}, ..., x_{im}), 0 \leq i \leq u$, we get an m-tuple $(\vec{y}_1, ..., \vec{y}_m), \vec{y}_j \in F^{u+1}$ with $\vec{y}_j = (x_{0j}, ..., x_{uj}), 1 \leq j \leq m$. Intuitively, you can think of it as 'swapping' the indexes of the coordinates and vectors. For example, $\vec{y}_1$ would consist of all the first coordinates all the vectors in the $(u+1)$-tuple.

Given some colouring $\chi : (F^m)^{u+1} \to [c]$, we can naturally induce a colouring on $(F^{u+1})^m$. Since Φ is a bijection, we define $\chi^{\Phi^{-1}} : (F^{u+1})^m \to [c]$ as $\chi^{\Phi^{-1}}((\vec{y}_1, ..., \vec{y}_m)) = \chi(\Phi^{-1}((\vec{x}_0, ..., \vec{x}_u)))$, for some m-tuple in F^{u+1} such that $\phi^{-1}((\vec{y}_1, ..., \vec{y}_m)) = (\vec{x}_0, ..., \vec{x}_u)$. Since $m - HJ(|F|^{u+1} : c)$, we make use of theorem 3 to deduce there is a combinatorial line $L \subset (F^{u+1})^m$ that is monochromatic under $\chi^{\Phi^{-1}}$. Hence, $\Phi^{-1}(L)$ is also monochromatic, in this case under χ. What remains to be shown is that $\Phi^{-1}(L)$ consists of parallel affine lines in F^m.

Let $L \subset (F^{u+1})^m$. This L is of the form $\{((x_{01}, ..., x_{u1}), ..., (x_{0m}, ..., x_{um})), (x_{1j}, ..., x_{uj}) \in F^{u+1}$ for $j \in [m]\}$. Since L is a combinatorial line, we have that for some $j \in [m]$ that the corresponding $(x_{0j}, ..., x_{uj})$ are all identical and bijective under p_j. For the other values of j, the corresponding $(x_{0j}, ..., x_{uj})$ are constant under p_j. Let $J \subset [m]$ denote all the indexes for which p_j is identical and bijective. Note that this means that for some $0 \leq i \leq u$, we have that x_{ij} is identical and bijective for all $j \in J$. For all $j \notin J$, we have that $(x_{0j}, ..., x_{uj})$ is constant.

We now turn our attention to $L' = \Phi^{-1}(L)$. This L' is of the form $\{((x_{10}, ..., x_{m0}), ..., (x_{1u}, ..., x_{mu})), (x_{1i}, ..., x_{mi}) \in F^m$ for $0 \leq i \leq u\}$. We fix some $0 \leq i \leq u$, thus looking at a specific $(x_{1i}, ..., x_{mi})$ of L'. As we noted earlier, we see that x_{ji} is identical and bijective for all $j \in J$. For $j \notin J$, we know that x_{ji} is constant. Hence, $(x_{1i}, ..., x_{mi})$ consists of a combination of identical and bijective coordinates as well as some constant coordinates. Thus, $L_i = \{(x_{1i}, ..., x_{mi}), x_{ji} \in F \text{ for } j \in [m]\}$ is an affine line. As this goes for any $0 \leq i \leq u$, we get a collection of parallel affine lines $L_0, ..., L_u$, as desired. $\square$

2.2.3 Special Vector Spaces

In this section we will show what special vector spaces are exactly, as well as prove a lemma that makes sure we can always find such a space. First, we define what vertical and transversal subspaces are. Let $u \in \mathbb{N}$ with $t \leq u < n$ and B be a $(u+1)$-space of $\mathbb{F}_q^n$. Let $p : B \to F^u$ be a surjective projection. For a t-space T of B, there are two cases under this projection. In one case, $p|_T : T \to F^u$ is injective.[1] That means that $p(T) \in \begin{bmatrix} F^u \\ t \end{bmatrix}$, in which case we call it transversal. In the second case, where the projection is not injective, we have that $p(T) \in \begin{bmatrix} F^u \\ t-1 \end{bmatrix}$ and we call T vertical. Note that a vertical space T is indeed $(t-1)$-dimensional, intuitively as p only 'reduces' B by one dimension as well.

With those 2 notions in mind, we can define special spaces. Given some colouring χ of the t-spaces of $\mathbb{F}_q^n$, a $(u+1)$-space B of $\mathbb{F}_q^n$ is called special with respect to the colouring χ and the above mentioned projection p if the following holds. If $T_1, T_2 \in \begin{bmatrix} B \\ t \end{bmatrix}$ are transversal and have that $p(T_1) = p(T_2)$, then $\chi(T_1) = \chi(T_2)$. That is to say, the colours of transversal t-spaces in B are essentially determined by their projection.

Of course, for the proof, we want to make sure that we can have such a special space B. That's what the following lemma is for:

Lemma 1. *For all $t, u, r \in \mathbb{N}$ with $t \leq u$, there exists $m = M^{(t)}(u : r)$ such that, given an r-coloring χ of the t-spaces of F^{u+m}, there exists a special (u+1)-space B.*

Proof. We shall in fact show that these spaces exist specifically using the projection $p : F^{u+m} \to F^u$ that simply takes the first u coordinates. We denote $v = \begin{bmatrix} u \\ t \end{bmatrix}_q$ as the number of t-spaces of F^u. We take $m = HJ(|F|^{u+1} : r^v$. Let $\chi : \begin{bmatrix} F^{u+m} \\ t \end{bmatrix} \to [r]$ be an r-colouring of the t-spaces of F^{u+m}.

We first intend to define an *affine* u-space $X \subset F^{u+m}$ based on a $(u+1)$-tuple of vectors in F^m and induce a colouring χ' on it. Then, using Hales-Jewett, we generate a $(u+1)$-space B which we show to be special by relating its t-spaces back to the prior defined X.

To get to this affine u-space X, we begin by defining $\vec{e}_0, \vec{e}_1 ..., \vec{e}_u \in F^u$ as $\vec{e}_0 = \vec{0}$ and $\vec{e}_i = (0, ..., 0, 1, 0, ..., 0)$ as the ith basis vector of F^u. Furthermore, let A_i be the set such that $p^-1(\vec{e}_i) \subset F^{u+m}$ for all $0 \leq i \leq u$. That is, A_i consists of all the vectors in F^{u+m} such that the first u coordinates correspond to $\vec{e}_i$.

Now, let $(\vec{x}_0, ..., \vec{x}_u)$ be a $(u+1)$-tuple with $\vec{x}_i \in F^m$ for all $0 \leq i \leq u$. We define $\vec{y}_i \in F^{u+m}$ by $\vec{y}_i - \vec{e}_i \vec{x}_i$. So, y_i is $\vec{x}_i$ with the ith basis vector of F^u in front of it. These y_i then generate a unique *affine* u-space $X \subset F^{u+m}$. Since the first u coordinates of the y_i are the basis vectors of F^u and the zero vector, we see that the projection $p|_X : X \to F^u$ is bijective.

Take some ordering of the t-spaces of F^u, denoted by $T_1, ..., T_v$. As $p|_X : X \to F^u$ is bijective, we know there is a unique T_i' such that $p(T_i') = T_i$ for each $i \in [v]$. We induce a (r^v)-colouring χ' of $(F^m)^{u+1}$, that is, on the $(u+1)$-tuples, as follows:

$$\chi'[(\vec{x}_0, ..., \vec{x}_u)] = (\chi(T_1'), ..., \chi(T_v'))$$

On the right, we see the colour is dictated by the colours of all the T_i'. As each T_i' for $i \in [v]$ is assigned one of r colours, we indeed get an (r^v)-colouring. Practically, this colouring means that two $(u+1)$-tuples are coloured the same if and only if the u-spaces generated by them as defined above are coloured identically.[2] Importantly, this means the colouring of these tuples are dictated by the projection p.

We now move on to construct a $(u+1)$-space B we intend to prove to be special. By how we defined m, we may make use of corollary 1 to obtain parallel affine lines $L_0, ..., L_u, L_i \in A_i$ for which all

[1] Here $p|_T$ refers to the projection restricted to the values of T.

[2] Of course, it is not actually the u-space that is coloured, but each t-space of the u-space can be uniquely identified with a t-space in F^{u+m}, which does have a colouring.

$(u+1)$-tuples $(\vec{x}_0, ..., \vec{x}_u), \vec{x}_i \in L_i$ receive the same colour under χ'. Since these lines are parallel, one can generate a $(u+1)$-space B using them. We claim B is special with respect to χ and $p|B$.

Let $T \in \begin{bmatrix} B \\ t \end{bmatrix}$ be a transverse t-space of B. For some $j \in [v]$, we have that $p(T) = T_j$ (Where T_j is the jth t-space of F^u, as per the prior ordering). To show B is special, we need to show that the colour of T, $\chi(T)$, depends entirely on its projection. Ergo, it needs to depend on j. We consider the transverse u-spaces of B. Intuitively, T must be contained in at least one of these. We fix such a u-space X. Since X is transverse, we know that for $0 \leq i \leq u$, the intersection $X \cap A_i$ must contain exactly 1 vector. If it contained more than 1, then the $p|_X$ would no longer be injective, thus not transverse. Thus, for each $0 \leq i \leq u$, we set $X \cap A_i = \{\vec{y}_i\}$. Thus, X can be seen as generated by $\vec{y}_0, ..., \vec{y}_u$, bringing us back to how we defined X earlier in this proof.

We know that T is the unique t-space of F^{u+m} such that $p(T) = T_j$ for some $j \in [v]$. Hence, $\chi(T)$ corresponds to the jth coordinate of $\chi'[(\vec{x}_0, ..., \vec{x}_u)] = (\chi(T'_1), ..., \chi(T'_j), ..., \chi(T'_v))$. By how we defined B, we know that χ' is constant on it. Hence, $\chi(T)$ depends solely on the value j, as desired. Thus, B is special.

□

2.2.4 The Base Case

Lemma 2. *Theorem 2 holds for the case $t = 0$.*

We shall actually prove this lemma for the specific case that $k_1, ..., k_r$ all assume the same value k. Note that it then also still proves it for the general case. To show why, say there is some $k_i < k$ for $i \in [r]$. Here we can specifically take k_i as less than k, since we're proving the lemma for an arbitrary value of k. If some k-space K of $\mathbb{F}_q^n$ is monochromatic with colour i, then we simply take a k_i-space of K, which is then still monochromatic with colour i.

Proof. Set $n = km$ where $m = HJ(|F|^k, r)$. We introduce the bijection $\Psi : \mathbb{F}_q^n \to (F^k)^m$, which is defined by grouping the coordinates of a vector $\vec{x} \in \mathbb{F}_q^n$ into m disjoint sets of k coordinates. For example, say we're working with $\mathbb{F}_2$. Let $k = 2$ and $m = 3$, giving $n = 6$. Take the vector $\vec{x} = (0, 0, 1, 0, 1, 1)$. Then $\Psi(\vec{x}) = ((0, 0), (1, 0), (1, 1))$.

Say we have some colouring $\chi : \mathbb{F}_q^n \to [r]$. Since Ψ is a bijection, we can easily induce a colouring $\chi' : (F^k)^m \to [r]$ by taking, for $\vec{x} \in (F^m)^k$, $\chi'(\vec{x}) = \chi(\Psi^{-1}(\vec{x}))$. Since we defined m with the Hales-Jewett function, we know by theorem 3 that there is a monochromatic combinatorial line $L \subset (F^k)^m$ under χ'. Then, $\Psi^{-1}(L) \subset \mathbb{F}_q^n$ is also monochromatic.

We claim that $\Psi^{-1}(L)$ is an *affine* k-space in $\mathbb{F}_q^n$. For this, we observe that L is of the form $\{((x_{11}, ..., x_1k), ..., (x_{m1}, ..., x_{mk})), (x_{i1}, ..., x_{ik}) \in F^k \text{ for } i \in [m]\}$. As L is a combinatorial line, we take $I \subset [m]$ to be the set of coordinates such that $(x_{i1}, ..., x_{ik})$ are identical and bijective under p_i for $i \in I$. Then for $i \notin I$, $(x_{i1}, ..., x_{ik})$ is constant. Note that for some choice of $j \in [k]$, that x_{ij} is also identical and bijective for all $i \in I$.

We now consider $L' = \Psi^{-1}(L)$. Put crudely, the 'order' of the coordinates is preserved under Ψ, so that we can see the form of L' as $\{(x_{11}, ..., x_{1k}, x_{21}, ..., x_{2k}, ..., x_{m1}, ..., x_{mk}), x_{ij} \in F \text{ for } i \in [m] \text{ and } j \in [m]\}$. For some $i \in I$, we know that all choices of $j \in [k]$ result in identical x_{ij}. For $i \notin I$, we get that for any choice of $j \in [m]$ that x_{ij} is some constant in F. Intuitively, as there are k distinct sets of identical x_{ij}, we see that L' defines an affine k-space. Since L was monochromatic under χ', we then have that $L' = \Psi^{-1}(L)$ is also monochromatic χ.

The theorem has now been proven for affine k-spaces. Though we want it to be proven for vector k-spaces. Since $t = 0$, we are just colouring the points of $\mathbb{F}_q^n$. Any such point $\vec{a} \in \mathbb{F}_q^n$ can be written as $\vec{a} = \vec{c} + \vec{w}$, where $\vec{w}$ is determined by $\vec{a}$. Thus, we induce the colouring $\chi''(\vec{a}) = \chi(\vec{w})$. So χ'' is still a colouring of the points of $\mathbb{F}_q^n$. As we demonstrated prior, we can then find an affine k-space B which is monochromatic under χ''. Then we can set $B = \vec{b} + V$ where V is some vector k-space determined by B. Then, for a point $\vec{v} \in \mathbb{F}_q^n$, we have $\chi(\vec{v}) = \chi''(\vec{b} + \vec{v})$, which is monochromatic. Hence, we have obtained a monochromatic vector k-space as desired. □

2.2.5 Induction

Now that the base case is proven, we move onto the induction step. For convenience, we restate the theorem here before proving it.

Theorem. *For all $t \geq 0, r, k_1, ..., k_r$ there exists $n = N(t, k_1, ..., k_r)$ with the following property. Let $\mathbb{F}_q^n$ be an n-space defined over $\mathbb{F}_q$ and $\chi : \begin{bmatrix} \mathbb{F}_q^n \\ t \end{bmatrix} \to [r]$ be an arbitrary r-colouring of the t-spaces of $\mathbb{F}_q^n$. then for some $i \in [r]$, there exists $K \in \begin{bmatrix} V \\ k_i \end{bmatrix}$ that is monochromatic with colour i.*

Proof. First, assume the existence of n for $t' < t$ for all values of $\sum_{i=1}^r k_i$. Then for t, assume the existence of n for all $\sum_{i=1}^r k_i' < \sum_{i=1}^r k_i$. Since we want to make use of special spaces, we are going to set the following values:

$$\begin{aligned} s &= \max_{1 \leq i \leq r} N^{(t)}(k_1, ..., k_i - 1, ..., k_r), \\ u &= N^{(t-1)}(s : r) \\ m &= M^{(t)}(u : r) \\ n &= u + m \end{aligned} \tag{2.5}$$

We show that based on these values, n has the desired property.[3] What exact role s, u and m play will become clear as the proof progresses. Let $\chi : \begin{bmatrix} F^{u+m} \\ t \end{bmatrix}$ be an arbitrary r-colouring of the t-spaces of F^{u+m}. Using lemma 1, we see that there is a special $(u+1)$-space B which we fix. Let $p : B \to F^u$ be the associated projection for this special space B. We induce a colouring $\chi' : \begin{bmatrix} F^u \\ t-1 \end{bmatrix} \to [r]$ as follows: for $T \in \begin{bmatrix} F^u \\ t-1 \end{bmatrix}$, $\chi'(T) = \chi(p^-1(T))$. That is, a $(t-1)$-space T of F^u is coloured by the same colour as its corresponding t-space $p^{-1}(T)$ of B. As $p(p^{-1}(T)) = T$ is a $(t-1)$-space, we note that $p^{-1}(T)$ is vertical.

By our assumption of n existing for $t' < t$ and our definition of u, we know there exists an s-space X of F^u that is monochromatic under χ'. Without loss of generality, we can take X to be monochromatic with colour 1 under χ'. Note that $p^{-1}(X)$ is vertical because X by definition contains vertical subspaces. For such a vertical subspace T, we have that the points in $p^{-1}(T)$ are not injective under the projection and are also in $p^{-1}(X)$. Hence, $p(p^-1(X))$ is also not injective, so $p^{-1}(X)$ is vertical. Additionally, $p^{-1}(X)$ is a special $(s+1)$-space with projection $p|_{p^{-1}(X)} : p^{-1}(X) \to F^s$, as it still uses the same projection as B, just restricted to the vectors of X. Lastly, we note that the vertical t-spaces of $p^{-1}(X)$ are coloured 1. As for a vertical t-space T of $p^{-1}(X)$, $\chi(T) = \chi'(p(T)) = 1$.

Now, we induce a colouring on the projection of the transversal t-spaces of $p^{-1}(X)$. We set $\chi'' : \begin{bmatrix} X \\ t \end{bmatrix} \to [r]$ by $\chi''(T) = \chi(T')$ where T' is a t-space of $p^{-1}(X)$ such that $p(T') = T$. We note that there may be multiple options for T'. However, since $p^{-1}(X)$ is special, we know that for two choices T_1', T_2' we have that $\chi(T_1') = \chi(T_2')$ since $p(T_1') = p(T_2') = T$. Essentially, χ'' applies the colouring of the transverse t-spaces of $p^{-1}(X)$ to the t-spaces of X.

By our definition of s, we know that $s \geq N^{(t)}(k_1 - 1, k_2, ..., k_r)$. Clearly, $k_1 - 1 + \sum_{i=2}^r k_i < \sum_{i=1}^r k_i$. So we know there exists $W \subset X$ that falls under one of 2 cases:

1. for $2 \leq j \leq r$, W is monochromatic with colour j under χ'' and has dimension k_j.
2. for $j = 1$, W is monochromatic with colour 1 under χ'' and has dimension $k_1 - 1$

We can essentially have the dimension of W be $k_j - 1$ for any j. However, we will make use of the fact that the vertical t-spaces of X are coloured 1 to 'fix' the lowered dimension.

We begin with case 1, we can find a transverse k_j-space $W' \subset p^{-1}(X)$ such that $p(W') = W$. Note we can indeed do this. Observe that $p^{-1}(W)$ is either k_j-dimensional or $(k_j + 1)$-dimensional. In the former case, $W' = p^{-1}(W)$ is transverse. In the latter case, we look at the basis of $p^{-1}(W)$. Under the projection, one of these two basis vectors must be projected to the same point. Pick one of those

[3] Note that this will not give us a specific value for n or some bound on it. As the 'functions' s, u and m are defined by merely state that they are dependent on those variables somehow. There is no indication in what way they are dependent on them.

basis vectors $\vec{w}_i$ and let W' be the k_j-space that has the same basis vectors as $p^{-1}(W)$ except for w_i. Then $p(W') = W$ still, so W' is transverse. Then $\left[{W \atop t}\right]$ is monochromatic with colour j under χ.

For case 2, set $W' = p^{-1}(W)$. Then W' is a vertical k_1-space of B. Let T be a t-space of W'. If T is transversal, $\chi(T) = \chi''(p(T)) = 1$ as $p(T)$ is a t-space of W. If T is vertical it is a vertical t-space of $p^{-1}(X)$, hence $\chi(T) = 1$. □

Chapter 3

Asymptotic Bounds

As mentioned, computing actual values of these vector space Ramsey numbers quickly becomes infeasible as the possible number of colourings and subspaces simply becomes too big. Hence, there is a strong interest in finding bounds on these numbers. The first three will focus on lower bounds. In fact, each method will uncover roughly the same lower bound for vector space Ramsey numbers. This is intended to show how different methods can be used to achieve these lower bounds.

In section 3.1, we investigate what the Probabilistic Method is and how it can be used to find a lower bound. Then, in section 3.2 we look to Code Theory to obtain a lower bound. For the third, in section 3.3, we look for a lower bound using Projective Spaces.

We also want to put emphasis on the difference between finding a lower bound and an upper bound. For a lower bound for Ramsey Numbers, all that is required is finding *one* colouring the does *not* give rise to a monochromatic subspace. For upper bounds, however, it needs to be shown that for *all* colourings there is such a monochromatic subspace. This requires different approaches, one of which is demonstrated in section 3.4. Here, an upper bound is found by making use of an upper bound in Classical Ramsey Numbers. After that, we discuss some other recent upper bounds of vector space Ramsey Numbers in subsection 3.5

All of these bounds concern themselves with vector space Ramsey numbers over $\mathbb{F}_2$. Hence, all the relevant definitions and theories shall also only be given in context to $\mathbb{F}_2$. Note some of it might apply more generally though.

3.1 Probabilistic Method

For this section, the work from Bishnoi et al. (2023)[4] was studied. The Probabilistic Method is a means of proving certain properties or structures exist that is applied to more than just vector space Ramsey numbers. The general idea is to show that the probability that a desired structure arises from some experiment is non-zero. This is of course much easier said than done. For the case of vector space Ramsey numbers, one can also see it as there being 'bad' events we want to avoid. A bad event in this case is a monochromatic k_i-space coloured i. Here, the experiment we do would be randomly assigning one of r colours to each 1-space. These events are of course not independent, as there is a lot of overlap in the k_i-spaces. Still, we can make use of this idea to obtain a lower bound for vector space Ramsey numbers, as shall be demonstrated in 3.1.1

3.1.1 Lower bound using the Probabilistic Method

In this section $R_2(2,t) > (\frac{3}{2} - o(1))t$ is the claim we intend to prove.[1] The proof of this was provided by Ravi (personal communication, 2024)[5]. To do this, we shall give a rough upper bound on the probability that we get a monochromatic 2-space coloured red or t-space coloured blue. Then we demonstrate that this upper bound is in fact strictly less than 1 for the desired value of n.

[1] $o(1)$ here essentially represents some function $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} f = 0$

Theorem 4. $R_2(2,t) > (\frac{3}{2} - o(1))t$

Proof. We colour the 1-spaces of $\mathbb{F}_2^n$ randomly and independently, such that a 1-space has probability p to be coloured red and $(1-p)$ to be coloured blue. Then, for a 2-space R of $\mathbb{F}_2^n$, the event that it is coloured mono-chromatically red shall be denoted by M_R, and its probability is $P(M_R) = p^3$. For a t-space B of $\mathbb{F}_2^n$, the even that it is coloured mono-chromatically blue is denoted by M_B, and its probability is $P(M_B) = (1-p)^{2^t-1}$. Note that $2^t - 1$ is the number of 1-spaces in a t-space (essentially, the number of non-zero vectors). Then, the probability that there is a monochromatic 2- or t-space is the probability that any of the M_R or M_B happen. These events are of course not independent, but we can upper bound this probability still by 'neglecting' the dependence.

Denoting the event that there is at least one monochromatic 2-space that is red or monochromatic t-space that is blue as M, we get:

$$P(M) \leq {\textstyle\begin{bmatrix} n \\ 2 \end{bmatrix}_2} p^3 + {\textstyle\begin{bmatrix} n \\ t \end{bmatrix}_2} (1-p)^{2^t-1}$$

If we can show that the right expression is strictly less than 1, then there is a non-zero chance that there isn't a monochromatic 2- or t-space. Thus, there must then be a colouring such that there is no monochromatic 2- or t-space. To work towards this, we set $n = \alpha t$, where α is some constant. We turn our attention to the second term of the inequality $\begin{bmatrix} n \\ t \end{bmatrix}_2 (1-p)^{2^t-1}$. We wish to find an upper bound for it. For $\begin{bmatrix} n \\ t \end{bmatrix}_2$, we observe the following:

$$\frac{(2^n-1)(2^n-2)...(2^n-2^{t-1})}{(2^t-1)(2^t-2)...(2^t-2^{t-1})} < 2^{nt-t^2}$$

It is easy to see that $1 - p < e^{-p}$. Combining these 2 results, we get

$$\begin{bmatrix} n \\ t \end{bmatrix}_2 (1-p)^{2^t-1} < 2^{nt-t^2} e^{-p(2^t-1)} < 2^{nt-t^2-p2^t+p}$$

If we get that exponent to roughly equal -1, then the second term of the inequality will be strictly less than $\frac{1}{2}$. Thus, rewriting $nt - t^2 - p2^t + p \approx -1$ givues us $p \approx \frac{nt-t^2+1}{2^t-1} = \frac{t^2(\alpha-1)+1}{2^t-1}$.

Using the earlier shown inequality, we have that $\begin{bmatrix} n \\ 2 \end{bmatrix}_2 < 2^{2n-4}$. By using the expression for p we just obtained, we get that the first term of inequality results in:

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_2 p^3 < 2^{2\alpha t-4}(\frac{t^2(\alpha-1)+1}{2^t-1})^3$$

As a last step, we take the limit of this expression.

$$\lim_{t\to\infty} 2^{2\alpha t-4}(\frac{t^2(\alpha-1)+1}{2^t-1})^3 = \lim_{t\to\infty} \frac{2^{2\alpha t}}{2^{3t}}$$

Taking $\alpha = \frac{3}{2} - \epsilon$, we see that the limit goes to 0. Hence, we obtain $P(M) < 1$ and $R_2(2,t) > n$ for $n = (\frac{3}{2} - o(1))t$. □

The notable thing that differentiates this method from some of the others we will study is that we do not obtain any sort of construction for an actual colouring. Instead, we've merely demonstrated that such a colouring *must* exist. For the purposes of a lower bound, it's not generally necessary to know what the colouring looks like. Thus, the probabilistic method proves a powerful tool to research these bounds.

3.2 Codes

A second method to finding a lower bound for vector space Ramsey numbers is through coding theory. In subsection 3.2.1, some relevant basics from Code Theory will be discussed as well as a theorem about the maximum dimension of a code with certain distance. This theorem is taken from Enomoto et al. (1987)[6]. Then in subsection 3.2.2 we discuss how that theorem can be used to obtain a lower bound for vector space Ramsey numbers. Though the thoeries from coding theory were known prior, the connection with vector space Ramsey numbers was discovered by A. Bishnoi (personal communications, 2024)

3.2.1 Codes with given distances

We call any arbitrary subset $C \subset \mathbb{F}_2^n$ a code of length n. A code is linear if it is a subspace of $\mathbb{F}_2^n$. So notably, the subspaces we consider when looking at Ramsey Numbers are linear codes.

Two important notions to define on the elements of these codes are the Hamming distance and weight. The Hamming distance between some vectors x and y, written as $d(x, y)$, is defined as the number of coordinates where these two vectors have different values. For example, let $x = (0, 1, 1, 0, 1)$ and $y = (0, 0, 1, 0, 0)$ then we see that $d(x, y) = 2$. The weight of a vector, written as $w(x)$ is the number of non-zero coordinates. So for our previously written x we see that $w(x) = 3$.

We also define the dual of a code $C^\perp$ as follows:

$$C^\perp = \{x \in \mathbb{F}_2^n : (x, y) = 0 \text{ for all } y \in C\} \tag{3.1}$$

Here, (x, y) refers to the scalar product of x and y defined as $(x, y) = \sum_{i=0}^{n} x_i y_i (\text{mod } 2)$. So, the dual is defined as all the vectors in $\mathbb{F}_2^n$ such that the scalar product with all vectors in the code C are zero. If C is linear, we have two interesting properties. Namely that $dim\ C + dim\ C^\perp = n$ and $C = (C^\perp)^\perp$.

Lastly, we define the function $l(n, D)$, which we are interested in studying. It is defined as follows:

$$l(n, D) = max\{dim\ C : C \subset \mathbb{F}_2^n,\ C \text{ is linear and } D(C) \subset D\} \tag{3.2}$$

Here, $D \subset \{1, 2, ..., n\}$ and $D(C)$ is the set of all distances of C. Thus, $l(n, D)$ is the maximum dimension of a linear code C so that it contains only distances in D. In particular, we are interested in the following special case:

$$l(n, \bar{t}) = l(n, \{1, 2, ..., n\} - \{t\}) \tag{3.3}$$

That is, the maximal dimension of a code C that does not contain two vectors with distance t. The aim specifically is to prove the following theorem:

Theorem 5.

$$\begin{aligned} l(4t, \overline{2t}) &= 2t \textit{ and} \\ l(n, \overline{2t}) &= 2t - 1 \textit{ for } 2t - 1 \leq n < 4t \end{aligned} \tag{3.4}$$

To prove this theorem, we want to make use of the following theorem:

Theorem 6.

$$l(4t, \overline{\{2t - 1, 2t\}}) = 2t - 1 \tag{3.5}$$

In turn, this theorem requires a lemma to be proven. Before that, we introduce the notion of binormal. A matrix is said to be in binormal form if it is k by n with $2k \leq n$ and the $(2i-1)^{th}$ and $(2i)^{th}$ columns differ in only the ith place for $i \in [k]$. So, the 1st and 2nd column differ in position 1, the 3rd and 4th in position 2, and so on. We then get the following lemma:

Lemma 3. *Suppose that M is a k by n matrix with binary entries with rank k. Furthermore, suppose $2k < n$, n is odd and every row of M is orthogonal to $\mathbf{1}$. Then M can be brought to binormal form by row operations and permutations of the columns.*

Furthermore, we require the following proposition:

Proposition 1. *If M is in binormal form and* $\mathbf{c}$ *is an arbitrary (0,1)-vector of length k. Then there exists a unique combination of k columns of M such that its sum is* $\mathbf{c}$*. Here, the ith column of this combination is either the $(2i-1)^{th}$ or the $(2i)^{th}$ column.*

The proof for the lemma and the proposition shall be omitted here, but can be found in the original work by Enomoto et al. (1987)[6]. We proceed with the proof of theorem 6.

Proof. Let $C \subset \mathbb{F}_2^{4t}$ be a linear code of dimension $2t$. We aim to obtain an upper bound of $l(4t, \overline{\{2t-1, 2t\}})$ if we show that C contains a vector of weight $2t-1$ or $2t$. We can then sum such a vector with the zero vector to obtain the distance of $2t-1$ or $2t$. We shall prove this by contradiction.

First, we claim we may assume that $\mathbf{1} \in C$. If it isn't, we can obtain a code of same dimension that does contain it as follows. Let $C_1 =< C, \mathbf{1} >$. That is to say, C_1 is the code generated by adding $\mathbf{1}$ to C. This code has dimension $2t+1$. It also has no vectors of weight $2t$ since $w(v + \mathbf{1} = 4t - w(v)$. Since we assume there is no $v \in C$ such that $w(v) = 2t$, we also have that $4t - w(v) \neq 2t$.

Next, we take a subcode C_0 of C_1 by only taking the vectors with even weight. This has either dimension $2t$ or $2t+1$[2]. In the former case, it contains $\mathbf{1}$, has no vectors of weight $2t-1$ or $2t$ and is of the desired dimension. In the latter case, one could simply take a $2t$ subcode of C_0 that contains $\mathbf{1}$ to achieve the same.

Now we consider the dual of C, $C^{\perp}$. Its dimension is $4t - 2t = 2t$. Let M be a generating matrix[3] of $C^{\perp}$. So M is a $2t$ by $4t$ matrix. We obtain M' by adding a column of zeros to M. Note M' is $2t$ by $4t+1$. Since $\mathbf{1} \in C$, $\mathbf{1}$ is orthogonal to every row of M'. Taking $k = 2t < 4t+1 = n$, we observe that M' then satisfies the conditions of lemma 3. Hence, we can write M' in binormal form.

Doing so, we make use of proposition 1 to obtain a combination of $2t$ columns of M' such that its sum is $\mathbf{0}$. If this combination includes the last column, we obtain $2t-1$ non-zero columns of M' that add up to zero. Else, we have $2t$ columns of M' that add up to zero. Take I as the set of indices corresponding to these columns. Then we define a vector $v = (v_1, ..., v_{4t})$ as follows:

$$v_i = \begin{cases} 1 \text{ if } i \in I \\ 0 \text{ if } i \notin I \end{cases}$$

Thus $Mv = \mathbf{0}$ as per the proposition. So $v \in (A^{\perp})^{\perp} = A$. Additionally, $w(v) = |I|$ which is either $2t$ or $2t-1$. This is a contradiction, as we assumed there to be no vectors of weight $2t$ or $2t-1$. Hence, it is proven that $l(4t, \overline{\{2t-1, 2t\}}) < 2t$.

For the lower bound, we consider the following code:

$$C = \{(v_1, ..., v_{4t}) : v_{2t-1} = ... = v_{4t}$$

That is, C is the code where $(2t-1)^{th}$ entry and all the ones after it have the same value. Clearly, the dimension of C is $2t-1$. Furthermore, if $v_{2t-1} = ... = v_{4t} = 0$, we see that $w(v) < 2t-1$. On the other hand, if $v_{2t-1} = ... = v_{4t} = 1$, then $w(v) > 2t+1$. Hence C excludes distances $2t-1$ and $2t$ as desired. Thus, $l(4t, \overline{\{2t-1, 2t\}}) \geq 2t-1$.

Combining the two results, we get that $l(4t, \overline{\{2t-1, 2t\}}) = 2t-1$. ☐

Finally, we prove theorem 5:

[2]This claim is not proven here but it is indeed true that either half or all of a linear code's vectors are of even weight. If it's half, the dimension of C_0 is $2t$. If it's all, $C_0 = C_1$ and its dimension remains $2t+1$.

[3]A generating matrix of a code C with dimension k and length n is a k by n matrix such that the vector $Mv = c$ with $v \in \mathbb{F}_2^k$ is in C. In other vectors, the rows of M form a basis for C

Proof. There are two parts to theorem 5. The case where $n = 4t$ and the case where $2t - 1 \leq n < 4t$. We first look at lower bounds for these two cases. For the latter case, we take the following code:

$$C = \{(v_1, ..., v_n) : v_{2t} = ... = v_n = 0\}$$

So, the last $n-2t+1$ entries are 0. Clearly, C is of dimension $2t-1$ as desired. Also, $w(v) < 2t$ for any $v \in C$. Thus, We get the lower bound of $l(n, \overline{2t}) \geq 2t - 1$. For the former case, we take $C' =< C, \mathbf{1} >$. That is, the code generated by adding $\mathbf{1}$ to the prior defined C. This is then a code of dimension $2t$. Also, as discussed in the proof for theorem 6, since C does not contain a word of weight $2t$, nor does C'. So we get the lower bound $l(4t, \overline{2t}) \geq 2t$.

For the upper bound for $l(4t, \overline{2t})$, we look at an arbitrary linear code C of $\mathbb{F}_2^4 t$ that does not contain words of weight $2t$. Let C_e be the subcode that consists of all the vectors of even weight of C. So $dim\ C_e \geq do,\ C - 1$. C_e also contains no vectors of $2t$. Additionally, since $2t - 1$ is odd C_e also contains no vectors of that weight. Hence, we may apply theorem 6 to get

$$dim\ C \leq 1 + dim\ C_e \leq 2t$$

Giving us the upper bound of $l(4t, \overline{2t}) \leq 2t - 1$ as desired.

For the upper bound of $l(n, \overline{2t})$ we look at a linear code C of $\mathbb{F}_2^{4t}$ where all the entries after the nth are 0 and that once agains contains no vectors of weight $2t$. Since some entries of vectors are always zero, we know that $\mathbf{1} \notin C$. Hence, we take $C' =< C, \mathbf{1} >$. This has $dim\ C' = dim\ C + 1$. It once again also contains no vectors of weight $2t$. Now, the case $l(4t, \overline{2t}) = 2t$ provides us an upper bound for the dimension of C'. This gives:

$$dim\ C' = dim\ C + 1 \leq 2t$$

Thus, $dim\ C \leq 2t - 1$ as desired. □

3.2.2 Lower bound using Codes

Now we proceed to observe how theorem 5 may help us find a lower bound for vector space Ramsey numbers. Particularly, we'll be looking at $R_2(2, 2t)$. We claim that $R_2(2, 2t) \geq 3t$ (For $t > 1$). To prove this, we take A to be all vectors of weight $2t$ of $\mathbb{F}_2^{3t-1}$. We note that all $2t$-spaces of $\mathbb{F}_2^{3t-1}$ intersect with A. This follows from theorem 5. Indeed, the theorem states the maximum dimension of a linear code (ergo a subspace) of $\mathbb{F}_2^{3t-1}$ that does not contain a vector of weight $2t$ is $2t - 1$. Hence, a subspace of dimension $2t$ must contain a vector of such weight, thus it intersects with A.

Importantly as well, if you take two vectors from A, their sum will not be of weight $2t$ so not be in A. Intuitively, this is because the two vectors will also overlap in more than t spaces because of their weight of $2t$. This means that A also does not contain any 2-spaces of $\mathbb{F}_2^{3t-1}$.

Thus, our colouring is as follows: A is coloured blue (or colour 1) and the complement of A is coloured red (or colour 2). Since A does not contain any 2-spaces of $\mathbb{F}^{3t-1}$, this colouring also does not give any monochromatically blue 2-spaces. Similarly, since all $2t$-spaces intersect with A, at least one non-zero vector of each of these subspaces is coloured blue. Hence, none of them are monochromatically coloured red. Thus, we have constructed a colouring of $\mathbb{F}_2^{3t-1}$ that does not contain any monochromatic 2- or $2t$-spaces. Hence, we have that $R_2(2, 2t) \geq 3t$.

3.3 Projective spaces

The third method to finding a lower bound for vector space Ramsey numbers is in spirit similar to the one in subsection 3.2. We make use of vectors with sufficiently high weight to achieve our colouring. However, here we approach that using Projective Spaces. The theorems used to find this lower bound are discussed in subsection 3.3.1. These are taken from the work by Lisoněk and Khatirinejad (2005)[7]. Then, in subsection 3.3.2, we discuss how a lower bound can be found using Projective Spaces. Similar as with the coding theory method, the connection to vector space Ramsey numbers was discovered by A. Bishnoi (personal communications, 2024).

3.3.1 Caps in Projective Spaces

Let PG(n,2) denote the n-dimensional projective space over $\mathbb{F}_2$. The points of PG(n,2) correspond to the non-zero vectors of $\mathbb{F}_2^{n+1}$. These points shall be denoted by $\sum_{n+1}$. A cap in PG(n,2) is a set $C \subseteq \sum_{n+1}$ such that the sum of any two points in C is not in C. A cap is called complete if it is not a proper subset of some other cap in the same space. For $m \geq 0$, an m-flat is defined as a subspace of PG(n,2) spanned by $m+1$ independent points.

We shall work with the same definition of the weight of a vector as in 3.2. We also define the notion of a support, which is the set of non-zero coordinates of a vector. That is, if we have a vector $x \in \mathbb{F}_2^n$, then the supper is a subset of $[n]$ coorresponding to the non-zero entries of x. Further, for $0 \leq i \leq n$, U_i^n denoted the set of all vectors of weight i. If there is no confusion about the dimension, we may also denote it simply as U_i. Furthermore, for two subsets X, Y of $\mathbb{F}_2^n$, their sum is defined as follows:

$$X \oplus Y := \{x + y : x \in X, y \in Y \text{ and } x \neq y\}$$

We will be constructing a complete cap. To do so, we require the following Lemma, which shall be stated without proof:

Lemma 4. *Let $n \geq r \geq s$. If $r = s$, let $e = 2$. If instead $r > s$, let $e = r - s$. Furthermore, if $r + s \leq n$, then let $f = r + s$. Otherwise, if $r + s \geq n$, let $f = n - (r + s)$. Then we have that*

$$U_r^n \oplus U_s^n = U_e^n \sqcup U_{e+2}^n \sqcup ... \sqcup U_f^n$$

[4]

With that in mind, we state the set we intend to prove is a complete cap:

Definition 5. *For $k \in \mathbb{N}_{>0}$ let*

$$\mathcal{L}_k = \bigsqcup_{i=2k+1}^{3k+1} U_i^{3k+1}$$

So, essentially, $\mathcal{L}$ encompasses all vectors of weight $2k+1$ and higher. With that, we prove the following:

Theorem 7. *The set $\mathcal{L}_k$ is a complete cap in PG(3k, 2) for all $k \in \mathbb{N}_{>0}$.*

Proof. To prove $\mathcal{L}$ is a cap, we need that the sum of no two distinct vectors in $\mathcal{L}_k$ is in $\mathcal{L}_k$ again. This can also be written as needing that $(\mathcal{L}_k \oplus \mathcal{L}_k) \cap \mathcal{L}_k = \emptyset$. That is, there is no overlap between $\mathcal{L}_k$ and the sum with itself. For it to be complete we require $(\mathcal{L}_k \oplus \mathcal{L}_k) \sqcup \mathcal{L}_k = \sum_{3k+1}$. That means that the union of $\mathcal{L}_k$ and the sum with itself needs to encompass all points of PG($3k$, 2). If it didn't, it would mean there are points that could be added to $\mathcal{L}_k$ still to increase the size of the cap. Combining these two, we get that $(\mathcal{L}_k \oplus \mathcal{L}_k)$ must be the complement of $\mathcal{L}_k$. This gives us that we must show the following holds:

$$(\mathcal{L}_k \oplus \mathcal{L}_k) = \bigsqcup_{i=1}^{2k} U_i^{3k+1} \tag{3.6}$$

Suppose that $x \in \bigsqcup_{i=1}^{2k} U_i^{3k+1}$. Then $1 \leq w(x) \leq 2k$. We claim that either $x \in (U_{2k+1} \oplus U_{2k+2})$ or $x \in (U_{2k+1} \oplus U_{2k+1})$. To prove the claim, we make use of lemma 4. We get that $(U_{2k+1} \oplus U_{2k+2}) = U_1 \sqcup U_3 \sqcup ... \sqcup U_{2k-1}$. That is, $(U_{2k+1} \oplus U_{2k+2})$ encompasses all the odd weights between 1 and $2k$. We also have $(U_{2k+1} \oplus U_{2k+1}) = U_2 \sqcup U_4 \sqcup ... \sqcup U_2k$. So $(U_{2k+1} \oplus U_{2k+1})$ encompasses all the even weights. Hence, if $w(x)$ is odd, it is in $(U_{2k+1} \oplus U_{2k+2})$. If $w(x)$ is even, it is in $(U_{2k+1} \oplus U_{2k+1})$. Thus, we see that $\bigsqcup_{i=1}^{2k} U_i^{3k+1} \subset (\mathcal{L}_k \oplus \mathcal{L}_k)$.

Now, suppose that $x \in (\mathcal{L}_k \oplus \mathcal{L}_k)$. So $x = y + z, y \in U_i, z \in U_j$ for some $2k+1 \leq i, j, \leq 3k+1$ with $y \neq z$. We want that $1 \leq wt(x) \leq 2k$. The lower bound follows form the fact that $y \neq z$. For the upper bound, we once again look to lemma 4. We try to maximise f. That will result in the biggest

[4] For clarity, $\sqcup$ here means a disjoint union of sets.

possible weight for any choice of i, j. It is relatively easy to see that $i = j = 2k + 1$ results in the largest f. Namely $f = 2(3k + 1) - 2(2k + 1) = 2k$. Hence we get that $wt(x) \leq f \leq 2k$. Thus, we have that $(\mathcal{L}_k \oplus \mathcal{L}_k) \subset \bigsqcup_{i=1}^{2k} U_i^{3k+1}$.

Combining these two results, we obtain the equality in equation 3.6 as desired. $\square$

Furthermore, we want to demonstrate that $\mathcal{L}$ also intersects with all $2k$-flats. To do this, we shall need an additional Lemma:

Lemma 5. *Let $r < n$ and $r, n \in \mathbb{N}_{>0}$. Let $S_1, ..., S_r$ be r non-empty distinct subsets of $[n]$. Then there exists a set $S' \subseteq [n]$ with $|S'| \geq n - r$ such that $|S' \cap S_i|$ is even for all $i \in [r]$.*

Proof. Proof Lacking for now, adding later $\square$

Using this Lemma, we shall now prove the following theorem:

Theorem 8. *For all $k \in \mathbb{N}_{>0}$, $\mathcal{L}_k$ intersects every $2k$-flat of PG(3k, 2)*

Proof. Let F be an arbitrary $2k$-flat of PH($3k$,2). We shall write F as the intersection of k hyperplanes of PG($3k$,2). Writing the hyperplanes as $H_i = \{x \in \sum_{3k+1} : (h_i, x) = 0\}$[5] for each $i \in [k]$, we can then write $F = \bigcap_{i=1}^{k} H_i$. Let $S_i \subseteq [3k + 1]$ be the support of h_i. Then the various S_i have k non-empty distinct subsets of $[3k + 1]$. This satisfies the conditions of 5. So, there exists a $S' \subseteq [3k + 1]$ with $|S'| \geq 3k + 1 - k = 2k + 1$ and $|S' \cap S_i$— is even for all $i \in [k]$.

Let $s' \in \mathbb{F}_2^{3k+1}$ such that its support is S'. For all $i \in [k]$, we have $|S' \cap S_i|$ even. That is, h_i and s' both take on the value 1 in an even amount of coordinates. Hence, (h_i, x) is the sum of an even amount of ones, meaning $(h_i, x) = 0$. Therefore, s' is in each H_i and then also in the $2k$-flat F. Additionally, since $|S'| \geq 2k + 1$, we have that $w(s') \geq 2k + 1$, hence we also have that $s' \in \mathcal{L}_k$.

Thus, we conclude that $F \cap \mathcal{L}_k \neq \emptyset$. $\square$

3.3.2 Lower bound using Projective Spaces

We proceed with showing how $\mathcal{L}_k$ may be used to derive a lower bound for vector space Ramsey numbers. We claim that, using this complete cap, we can show that $R_2(2, 2t + 1) > 3t + 1$. Thus we shall consider $\mathbb{F}_2^{3t+1}$. Note that $\mathbb{F}_2^{3t+1}/\{0\}$ is exactly the points of PG($3t$,2). Then, $\mathcal{L}_t$ is a complete cap that also intersects every $2t$-flat. By definition, a $2t$-flat with the zero vector added is a $(2t+1)$-space of $\mathbb{F}_2^{3t+1}$. So in a sense $\mathcal{L}_t$ intersects every $2t + 1$-space. Furthermore, $\mathcal{L}_t$ also does not contain all non-zero vectors of any 2-space as it is a cap. That is, any sum of two vectors in $\mathcal{L}_t$ is not in $\mathcal{L}_t$.

Hence, we proceed with the colouring similarly as we did in 3.2. $\mathcal{L}_t$ is coloured blue (or colour 1) and the complement of $\mathcal{L}_t$, that is all vectors of weight $2t$ or less, is coloured red (or colour 2). Since $\mathcal{L}_t$ does not contain any 2-spaces, the colouring does not give any monochromatically blue 2-spaces. Additionally, since $\mathcal{L}_t$ intersects all $2t + 1$-spaces, each $2t + 1$-space has at least one vector that is coloured blue. So there are no $2t + 1$-spaces that are coloured monochromatically red. Thus, we have a colouring of $\mathbb{F}_2^{3t+1}$ that avoids monochromatic 2- or $2t + 1$-spaces. Thus we conclude that $R_2(2, 2t + 1) > 3t + 1$.

3.4 Graph Ramsey Numbers

Lastly, we would also like to demonstrate an upper bound for vector space Ramsey numbers. Unlike the previous 3, where we only had to demonstrate there was at least one colouring that avoided monochromatic k_i-spaces, we now need to verify for *all* that there is some monochromatic k_i-space. Specifically, we shall achieve this by relating it to a bound of Classical Ramsey Numbers.

[5] (h_i, x) here refers to the scalar product defined in subsection 3.2

First, in subsection 3.4.1, we shall briefly discuss what Classical Ramsey Numbers are. Then, in subsection 3.4.2, we show an upper bound on $R(3;r)$. Lastly, in subsection 3.4.3, we demonstrate how the upper bound on $R(3;r)$ can be related to an upper bound for $R_2(2;r)$.[6]

3.4.1 Classical Ramsey Numbers

Classical Ramsey Numbers originate from Graph Theory. The example oft discussed is that of a group of 6 people. Imagine you represent the individuals with vertices in a graph. Then, the edges represent whether 2 people are friends or not. If they are, the edge is coloured red, blue otherwise.[7] You may for example, get the graph in figure 3.4.1.

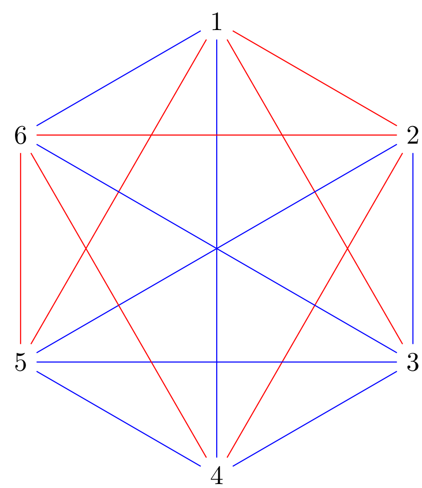

Figure 3.1: Graph representing 6 people and who is friends with who.

You can quickly see there are triangles of the same colour. For example, 2, 4 and 6 form such a 'clique' of 3. As it turns out, these monochromatic triangles will always form in a group of 6 people. This can be more formally stated and proven as follows:

Theorem 9. *In a 2-colouring of the edges of K_6, there will always be a monochromatic clique of size 3.*[8]

Proof. We may assume that vertex 1 of K_6 (for some ordering of the vertices) has 3 edges that are coloured red. After all, if there are 2 or less coloured red, there are at least 3 coloured blue. For the 3 vertices that are connected to these 3 red edges, there are 2 cases. If one pair of these 3 vertices is connected by a red coloured edge, then we have a red clique of size 3. If none of the 3 are connected by a red coloured edge and instead by blue coloured edges, then those 3 form a blue coloured clique. □

The question is whether this is the smallest size graph for which this is true. A quick colouring of K_5 demonstrated in figure 3.4.1 verifies this. That then brings us to the definition of a Classical Ramsey Number.

Definition 6 (Classical Ramsey Number)**.** *$R(k_1, ..., k_r) = n$ implies that n is the smallest such number with the following property. For any r-colouring of K_n, there is a $i \in [r]$ such that there is a clique of size k_i coloured i.*

This definition is quite a bit more general than the example. Notably, we can have Classical Ramsey Numbers for more than 2 colours and the sizes of the cliques for each colour need not be the same. Our example for 6 vertices would be written as $R(3, 3) = 6$.

[6] Though the notation is quite similar, note that the distinction is that $R_q(k_1, ..., k_r)$ refers to vector space Ramsey numbers over $\mathbb{F}_q$ whereas $R(k_1, ..., k_r)$, without the q, refers to Classical Ramsey Numbers.

[7] For the purposes of a mathematical example, friendship is a binary notion here.

[8] K_6 here refers to the complete graph of 6 vertices.

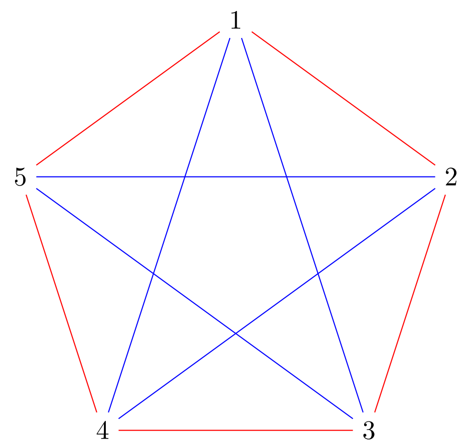

Figure 3.2: Colouring of K_5 that has no monochromatic cliques of size 3

3.4.2 Upper bound on $R(3; r)$

We shall be proving an upper bound for $R(3; r)$ (Which is $R(3, ..., 3)$ where 3 is repeated r times). Specifically, we intend to prove $R(3; r) \leq 3r!$. You can see $R(3; r)$ as the multi-colour generalisation of the case $R(3, 3) = 6$. In fact, the idea of the proof will be similar, we shall show that a vertex must have a certain amount of edges of the same colour, then showing there are monochromatic cliques somewhere among that vertex and the vertices connected to the monochromatic edges.

Theorem 10. $R(3; r) \leq 3r!$

Proof. We prove this by induction. The base case $R(3, 3) \leq 3(2!) = 6$ was already proven in subsection 3.4.1. For the induction step, we assume that $R(3; (r-1)) \leq 3(r-1)!$ holds. We consider an arbitrary r-colouring of the graph $G = K_{3r!}$. Let v be an arbitrary vertex of the graph. We claim that at least $\lceil \frac{3r!-1}{r} \rceil$ are of the same colour. For $r > 1$ we see that:

$$\frac{3r! - 1}{r} = \frac{3r!}{r} - \frac{1}{r} = 3(r-1)! - \frac{1}{r}$$

So, this gives us that $\lceil \frac{3r!-1}{r} \rceil = 3(r - 1!)$. If each colour appears less than $3(r-1)!$ times we see that at most $r(3(r-1)! - 1) = 3r! - r$ of the edges can be coloured. Thus, there must be a colour that appears at least $3(r-1)!$ times. Without loss of generality, say that is colour 1. let G' be the subgraph consisting of the vertices of the graph G which are the neighbours of v that are connected by an edge with colour 1. Then, if any pair of vertices in G' is coloured 1, we have a monochromatic clique of size 3 coloured 1. If none of them are coloured 1, then G' contains $r-1$ colours. Additionally, we know that $|V'| \geq 3(r-1)!$. By our assumption, we can then find a monochromatic clique of size 3 with colour i with $i \in \{2, ..., r\}$. Thus, we have that $R(3, 3) \leq 3r!$. □

3.4.3 Upper bound of $R_2(2; r)$

To see how this provides an upper bound for, we actually intend to aim the following lemma first:

Lemma 6. $R_2(2; r) > n \implies R(3; r) > 2^n - 1$

We can then take the contrapositive of this statement to obtain our upper bound. To prove this, we will need to make use of Schur's Theorem, which is as follows:

Theorem 11 (Schur's Theorem)**.** *Let $r \in \mathbb{N}_{>0}$. Then there exists a positive integer n such that for every partition of $[n]$ into r parts, there is a part containing integers x, y, z such that $x + y = z$.*

Importantly, we shall show that $R(3; r)$ is sufficient for this theorem. Once proven, we then know that this is a property that $R(3; r)$ satisfies in general. Thus, if we then have an n and a partition of $[n]$ into r parts that does not satisfy that property, we know that $R(3; r) > n$.

Proof. Given some $r \in \mathbb{N}_{>0}$, take $n = R(3; r)$. Let χ be an r-colouring of $[n]$.[9] Now consider the graph K_n, where the vertices have some ordering $\{v_1, ..., v_n\}$. We induce a colouring χ' on K_n as follows: the edge between v_i and v_j receive the same colour that $|i - j|$ received in the colouring of $[n]$.

By definition of Classical Ramsey Numbers, there is a monochromatic triangle somewhere in K_n. Say that the vertices belonging to that triangle have indexes $i > j > k$. Then we know that $i - j$, $j - k$ and $i - k$ received the same colour under χ. Thus, we set $x = i - j$, $y = j - k$ and $z = i - k$. Then we get $x + y = i - j + j - k = i - k = z$. □

We shall now prove lemma 6.

Proof. Let $r \in \mathbb{N}_{>0}$. Since $R_2(2; r) > n$, we know there is a colouring for $\mathbb{F}_2^n$ that has no monochromatic 2-spaces, we denote this colouring by χ. We note that for a vector in $\mathbb{F}_2^n$, one can see it as a number is base 2. For example, the vertex $(0, 1, 1, 0, 1)$ can be seen as $01101_2 = 13_{10}$. Thus, we order the non-zero vectors of $\mathbb{F}_2^n$ based on which binary number the vector 'represents'. For $i < j < k \leq 2^n - 1$ that satisfy $i + j = k$, we see that $v_i + v_j = v_k$. Important to note is that then $\{v_i, v_j, v_k, 0\}$ is a 2-space of $\mathbb{F}_2^n$.

We induce a colouring χ' on $[2^n - 1]$ simply by $\chi'(i) = \chi(v_i)$. For any $i < j < k \leq 2^n - 1$ such that $i + j = k$, we have that v_i, v_j, v_k are part of a 2-space of $\mathbb{F}_2^n$. Since χ has no monochromatic 2-space, we then know that v_i, v_j, v_k do not all have the same colour. Thus, under χ', i, j, k do not all have the same colour either. This means that $2^n - 1$ is not large enough to satisfy theorem 11. Since we've proven that $R(3; r)$ is large enough for it, it then must hold that $R(3; r) > 2^n - 1$. □

With that lower bound established, We take the contrapositive of lemma 6. We add one to the upper bound of $R(3; r)$ as well for convenience. This means we get $R(3; r) \leq 2^n \implies R_2(2; r) \leq n$. As per theorem 10, we know that $R(3; r) \leq 3r!$. It is pretty easy to see that $r^r \geq r!$. For $r = 3$ specifically, we see that $3(3!) = 18 < 27 = 3^3$. Hence, for $r \geq 3$, we have $3r! \leq r^r$. This bound is quite bad, but we shall rewrite it in a nice form for our purposes: $r^r = 2^{\log_2 r^r} = 2^{r \log_2 r}$. Thus, we see that $R(3; r) \leq 2^{r \log_2 r}$. Combining this with the contrapositive of lemma 6, we get that $R_2(2; r) \leq r \log_2 r$.

3.5 Other Upper Bounds

In this section we shall more briefly discuss some recent discoveries and improvements on upper bounds of vector space Ramsey numbers. Rather than providing a full proof, the idea of the proof shall be given in short. For the first few bounds that we shall be looking at, we have studied the work of Frederickson and Yeprenmyan (2023)[8]. Their work gives improved upper bounds on $R_2(2, t)$, $R_3(2, t)$ as well as delving into general bounds for $R_q(k_1, ..., k_r)$.

An earlier upper bound for the specific case of $R_2(t; r)$ was given by Taylor (1981)[9], by reducing the problem to the unions problem for finite sets. The bound is as follows:

Theorem 12 (Taylor). *The number $R_2(t; r)$ is at most a tower of height $2k(t - 1)$ of the form*

$$R_2(t; r) \leq k^{3^{k^{\cdot^{\cdot^{\cdot^{k}}}}}}$$

The work of Frederickson and Yepremyan sets out to improve this bound for $q \in \{2, 3\}$ specifically, as well as have it be applicable for general choices $t_1, ..., t_k$. Before this is discussed, some notions will be briefly introduced.

Most notably, instead of looking at subspaces, we concern ourselves with subsets of $\mathbb{F}_q^n$ that have either a linear or affine configuration. Linear means that on $A = \{x_1, ..., x_k\} \subseteq \mathbb{F}_q^n$ we have the equation of the form $\sum_{i=1}^{k} \lambda_i x_i = 0$, where $\lambda_1, ..., \lambda_k \in \mathbb{F}_q$. If we have $sum_{i=1}^{k} \lambda_i = 0$, we call it affine.

Though it isn't really mentioned in the following text, the subsets being Sidorenko play a vital role in a lot of the inequalities and arguments. Thus we define it still. An affine configuration $B \subset \mathbb{F}_q^n$ is C-weakly Sidorneko if, for any $A \subseteq \mathbb{F}_q^n$ of density α,

[9] An r-colouring is essentially partitioning something into r parts, we simply opt for the former.

$$\hom_{aff}(B, A) \geq \sigma^C N^{\text{rank}_{aff}(B)}$$

We simply call it Sidorenko if $C = |B|$. Here, $N = q^n$, $\hom_{aff}(B, A)$ represents the number of affine homomorphism $B \to A$ and $\text{rank}_{aff}(A)$ is the size of an affine basis of A.

For $A \subseteq \mathbb{F}_q^n$ to contain an affine copy of $B \subseteq \mathbb{F}_q^n$, we require there to be a non-degenerate affine homomorphism $B \to A$. This essentially meaning that both relations and non-relations are preserved. If, for some family of subsets $\mathcal{B} = \{B_i\}_{i \in I}$, we have that $A \subseteq \mathbb{F}_q^n$ contains no affine copy of any B_i. then A is called $\mathcal{B}$-free. The largest size of an affine subset of $\mathbb{F}_q^n$ such that it remains $\mathcal{B}$-free we denote by $\text{ex}_a ff(n, \mathcal{B})$ amd os called the n-th affine extremal number of $\mathcal{B}$.

We intend to prove the following bound:

Theorem 13. *There exists a constant $C_0 \approx 13.901$ such that for $\sigma_2 = 2$ and $\sigma_3 = C_0$ the following holds for $q \in \{2, 3\}$. For any $r \geq 2$ and any $t_r \geq ... \geq t_1 \geq 2$, $R_q(t_1, ..., t_r$ is at most a tower of height $\sum_{i=1}^{r-1}(t_i - 1) + 1$ of the form*

$$R_q(t_1, ..., t_k) \leq \sigma_q^{\sigma_q^{\cdot^{\cdot^{\cdot^{\sigma_q^{3t_k}}}}}}$$

The intent is to prove this with induction, where the base case is $R_q(s, t)$. For that, we have the following theorem:

Theorem 14. *For $q \in \{2, 3\}$, let $\sigma_2 = 2$ and $\sigma_3 = C_0 \approx 13.901$. For any $t \geq s \geq 2$, $R_q(s, t)$ is at most a tower of height s of the form*

$$R_q(s, t) \leq \sigma_q^{\sigma_q^{\cdot^{\cdot^{\cdot^{\sigma_q^{2t}}}}}}$$

This in turn, too, is proven using induction. Here, we fix t and the induction is done on s. We choose $n = t\sigma_q^r$ with $r = R_q(s-1, t)$. We essentially suppose there is a 2-colouring of $\mathbb{F}_q^n \backslash \{0\}$ that does not translate into a monochromatic s-space coloured 1 or a monochromatic t-space coloured 2. By then making use of bounds on $\text{ex}_a ff(n, \mathbb{F}_q^r$ and the size of the set of vectors coloured 1, we may conclude that that set contains an affine r-space. We then get a monochromatic $(s-1)$-space which we can use to obtain a monochroamtic s-space, bringing us to a contradiction.

This then gives is that $n = t\sigma_q^r$ is an upper bound, which through induction gives the desired upper bound.

With that base case proven, we can prove the general case of $R_q(k_1, ..., k_r)$. Here, use is made of the fact that:

$$R_q(t_1, ..., t_k) \leq R_q(t_1, ..., t_{k-2}, R_q(t_(k-1, tk))$$

That is, the last two colours are replaced with the Ramsey number of those 2 colours specifically. Through induction we then obtain the desired bound.

For $R_q(2, t)$, Nelson and Nomoto (2018)[10] observed that it was upper bounded by $m_2(t)$. Here, $m_2(t)$ refers to the minimum n such that for ever $A \subseteq \mathbb{F}_2^n$ of size at least 2^{n-t+1} we have that $A + A := \{x + y : x, y \in A\}$ contains a linear t-space. Once again, we prove this by assuming we have a colouring with no monochromatic 2-space or t-space. This being for $n = m_2(t)$. By lower bounding the size of the set of vectors receiving colour 1, we can find two vectors that are coloured 1 that are linearly independent. This would be a contradiction. as the span of these 2 would then form a 2-space that is coloured 1. With that contradiction, the desired bound of $R_q(2, t) \leq m_2(t)$ is obtained.

To now make use of this bound, an upper bound for $m_2(t)$ is needed. To obtain this, $m_2(t)$ is recontextualised as the minimum n such that $\text{ex}_{aff}(n, \mathcal{B}_2^t) < 2^{n-t+1}$ wjere $\mathcal{B}_q^t := \{B \subseteq \mathbb{F}_q^n : m \geq$

$1, \omega(B+B) \geq t$. That is, $\mathcal{B}$ consists of all subsets B of $\mathbb{F}_2^m$ such that the sumset $B+B$ contains a t-space.

For more general $m_q(t)$, it is the minimum n such that $\text{ex}_{aff}(\mathcal{B}_q^t) < q^{n-t+1}$.

Through some inequalities we can prove that $\text{ex}_{aff}(n, \mathcal{B}_2^t < 2^{n-\frac{(n-4k)}{6^k-4k}}$ and $\text{ex}_{aff}(n, \mathcal{B}_3^t) < 2^{n-\frac{n-t}{C_0^t-t}}$. By filling in $n_1 = (t-1)(6^k-4k)+4k$ and $n_2 = (t-1)(C_0^t-t)+t$ respectively, both get upper bounded by 2^{n-t+1}. This can then be used to show the following two bounds:

$$R_2(2,t) \leq m_2(t) \leq (t-1)(6^k-4k)+4k) = O(t6^{\frac{t}{4}})$$
$$R_3(2,t) \leq m_3(t) \leq (t-1)(C_0^t-t)+t = O(tC_0^t)$$

Lastly, we mention a bound given by Hunter and Pohoata (2023)[11]. For the 2-colour case, they proved:

Theorem 15. *For every integer $d \geq 1$, there exists an absolute constant $C > 0$ such that $R_2(2,d) \leq Cd^7$.*

Furthermore, they showed a generalisation for more colours:

Theorem 16. *Let $r \geq 2$ and let $R_2^{(r)}(2;d)$ denote the r-colour Ramsey number where the first $r-1$ entries are 2 and the r-th is d. Ergo $R_2(2,...,2,d)$. There exists an absolute constant $C > 0$ such that the following holds for every integer $d \geq 1$:*

$$R_2^{(r)}(2;d) \leq Cr(\log r + d)^7$$

Chapter 4

Computational Methods

Though vector space Ramsey numbers become practically impossible to compute after the first few numbers, we still take interest in computing those first few numbers. For this, we first made use of an ILP formulation of the problem, which we discuss in section 4.1. First by giving the formulation in subsection 4.1.1. Then, we also implemented it using SAT, discussed in section 4.2. First a small elaboration on what SAT is, is given in 4.2.1. Then, we give the formulation for the problem in SAT form subsection 4.2.2. With the two methods explained, we explore some of the computed values for these formulations in 4.3.

4.1 ILP

4.1.1 ILP Formulation

To explain the ILP formulation, we shall work with the example of $R_2(2,2)$, taking $n = 3$ as the dimension of our vector space. That means that we have vector space:

$$\mathbb{F}_2^3 = \{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$$

And the subspaces that we are colouring are 1-dimensional and contain the zero vector and a non-zero vector. In general, this formulation will be for $R_{q,t}(k_1,...,k_r)$.

Before explaining the ILP formulation, we begin with expressing the intent behind it. Rather than trying to formulate vector space Ramsey numbers as an ILP directly, we actually take the opposite approach. The ILP will look for a colouring of the 1-dimensional subspaces so that none of the k-dimensional subspaces are monochromatic. By then incrementing the dimension n, we will eventually run into a formulation that is not feasible. This then implies that for that n there is no colouring to avoid some monochromatic k-dimensional subspace, meaning that $R_{q,t}(k_1,...,k_r) = n$.[1]

To begin any ILP formulation, variables are required first. We define our variables $x_{i,j}$ as follows:

$$x_{i,j} = \begin{cases} 1, & \text{if subspace } i \text{ has colour } j \text{ assigned to it} \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

This goes for some ordering of the t-dimensional subspaces we are colouring. For example, say $i = 1$ represents the subspace $\{(0,0,0),(0,1,0)\}$. Then $x_{1,0} = 1$ implies that $\{(0,0,0),(0,1,0)\}$ is assigned the colour 1. We note we then do not also want that $x_{1,1} = 1$, as each subspace should only be assigned 1 colour. Thus, this gives rise to our first set of constraints:

$$\sum_{j=0}^{r} x_{i,j} = 1 \text{ for all } i \in [T] \tag{4.2}$$

[1] It actually implies $R_{q,t}(k_1,...,k_r) \leq n$. However, as we increment n by 1 each time we find a feasible solution, we know that $n-1$ is feasible. That implies $R_{q,t}(k_1,...,k_r) > n-1$. Combining those results gets us $R_{q,t}(k_1,...,k_r) = n$.

Here, T is the number of t-dimensional subspaces of $\mathbb{F}_q^n$. To see what this T looks like, refer to section 2.1.1. This constraint says that every t-dimensional subspace of $\mathbb{F}_q^n$ must be assigned exactly 1 colour. In our example subspace, that constraint will look like this:

$$x_{0,0} + x_{0,1} = 1$$

Now that our colourings behave correctly, we introduce the constraint that 'prevents' our k-dimensional subspaces from being monochromatic. Let I_S denote that set of indexes of the t-dimensional subspaces that belong to the k-dimensional subspace S and let T_S the number of t-dimensional subspaces of S. Then the constraints are of the form:

$$\sum_{i \in I_S} x_{i,j} \leq T - 1 \text{ for all } S \in \begin{bmatrix} \mathbb{F}_q^n \\ k_j \end{bmatrix} \text{ and } j \in [r] \tag{4.3}$$

This constraint states that for each j and each k_j-dimensional subspace S, at most $T-1$ of its subspaces may be coloured j. A subspace S would be monochromatic if all T of its subspaces were coloured j. Hence, this constraint 'prevents' subspaces from being monochromatic. Say we take the 2-dimensional subspace $S = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)$ of $\mathbb{F}_2^3$. Where the 3 1-dimensional $\{(0,0,0),(0,0,1)\},\{(0,0,0),(0,1,0)\}$ and $\{(0,0,0),(0,1,1)\}$ are assigned indexes 0, 1 and 4 respectively. Then the constraints looks like:

$$x_{0,0} + x_{1,0} + x_{4,0} \leq 2$$
$$x_{0,1} + x_{1,1} + x_{4,1} \leq 2$$

The last part of the ILP we have yet to describe is the objective function. As mentioned, the aim of this ILP is to check for feasibility, not finding an optimal solution. However, for the sake of studying the results, we might want to optimise for certain values. Most notably, we are interested in maximising for a certain colour. Hence, our objective function generally shall be of the form:

$$\max \sum_{i=0}^{T} x_{i,j} \text{ for a prior fixed } j \in [r] \tag{4.4}$$

That is, for some predetermined j, this objective function maximises the amount of subspaces that receive that colour. Say, for our example, we maximise for colour 1. Then the objective function would look like this:

$$\max x_{0,1} + x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} + x_{5,1} + x_{6,1} + x_{7,1}$$

Putting it all together our ILP is as follows:

$$\begin{aligned} \max \sum_{i=0}^{T} x_{i,j} & \qquad \text{for a prior fixed } j \in [r] \\ \text{s.t.} \sum_{i \in I_S} x_{i,j} \leq T - 1 & \qquad \text{for all } S \in \begin{bmatrix} \mathbb{F}_q^n \\ k_j \end{bmatrix} \text{ and } j \in [r] \\ \sum_{j=0}^{r} x_{i,j} = 1 & \qquad \text{for all } i \in [T] \end{aligned} \tag{4.5}$$

4.2 SAT

4.2.1 SAT Explanation

SAT is the Boolean Satisfiability Problem. That is, given some expression of Boolean variables (for example something like $x_1 \vee (x_2 \wedge \neg x_3)$) is there a set of variables that satisfies the expression. Given such an expression there exist solvers that compute whethere the expression is satisfiable. These solvers generally work off a specific form of SAT expressions, called 'Conjunctive Normal Form' (or CNF for short). A CNF is a 'conjunction' of clauses. Here, a clause is a series of 'OR' statements. For example $x_1 \vee x_2 \vee x_3$. Then, you take the conjunction of those clauses by stringing them together using 'AND' statements. An example of a CNF might be:

$$(x_1 \vee x_2) \wedge (x_1 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_4)$$

We wish to formulate our problem in this way. [More explanation of SAT needed?]

4.2.2 SAT Formulation

We base our SAT formulation on the way we formulated the ILP. As SAT focuses solely on feasibility, we need only find a way to transform the following two constraints into clauses:

$$\sum_{j=0}^{r} x_{i,j} = 1 \text{ for all } i \in [T]$$

$$\sum_{i \in I_S} x_{i,j} \leq T - 1 \text{ for all } S \in \begin{bmatrix} \mathbb{F}_q^n \\ k_j \end{bmatrix} \text{ and } j \in [r]$$

For the variables used in the SAT formulation, we essentially use the same. Except instead of binary values, $x_{i,j}$ now of course assume boolean values. Thus, it looks like this:

$$x_{i,j} = \begin{cases} \text{True, if subspace } i \text{ has colour } j \text{ assigned to it} \\ \text{False, otherwise} \end{cases} \tag{4.6}$$

For the first constraint, essentially getting the colours to behave 'correctly', we first look at an example where we have 2 colours. We claim that $(x_{i,0} \vee x_{i,1}) \wedge (\neg x_{i,0} \vee \neg x_{i,1})$ makes it so that subspace i has to assume exactly one colour. To verify this, we look at the truth table 4.1

$x_{i,0}$	$x_{i,1}$	$x_{i,0} \vee x_{i,1}$	$\neg x_{i,0} \vee \neg x_{i,1}$	$(x_{i,0} \vee x_{i,1}) \wedge (\neg x_{i,0} \vee \neg x_{i,1})$
T	T	T	F	F
F	T	T	T	T
T	F	T	T	T
F	F	F	T	F

Table 4.1: Truth table for $(x_{i,0} \vee x_{i,1}) \wedge (\neg x_{i,0} \vee \neg x_{i,1})$

Indeed we see that the expression is only true if only 1 of $x_{i,0}$ or $x_{i,1}$ is true. So for 2 colours, which is generally the colourings we are interested in, the clauses simply look like $(x_{i,0} \vee x_{i,1}) \wedge (\neg x_{i,0} \vee \neg x_{i,1})$ for each subspace i.

For more general colourings, . Note that the two clauses essentially 'block out' only the case that i is assigned all colours or no colours. Take 3 colours for example, if we try to generalise what we did before we would get $(x_{i,0} \vee x_{i,1} \vee x_{i,2}) \wedge (\neg x_{i,0} \vee \neg x_{i,1} \vee \neg x_{i,2})$. However, we see that for $x_{i,0} = x_{i,1} =$True and $x_{i,2}$=False that the expression is True. Of course, we do not want this, as i can then be assigned multiple colours. So, we can add a clause $\neg x_{i,0} \vee \neg x_{i,1}$ To prevent both of those being true at once. This needs to be done for all pairs of colours. Note that if any pair of colours is 'blocked' by this expression, then any bigger groups of colours can not be true either. Hence, the expression suffices as this:

$$(x_{i,0} \vee x_{i,1} \vee x_{i,2}) \wedge (\neg x_{i,0} \vee \neg x_{i,1}) \wedge (\neg x_{i,0} \vee \neg x_{i,2}) \wedge (\neg x_{i,1} \vee \neg x_{i,2}) \tag{4.7}$$

To verify this has the desired effect, we shall give the truth table of expression 4.7.

$x_{i,0}$	$x_{i,1}$	$x_{i,2}$	expression 4.7
T	T	T	F
F	T	T	F
T	F	T	F
F	F	T	T
T	T	F	F
F	T	F	T
T	F	F	T
F	F	F	F

Table 4.2: Truth table for expression 4.7

Thus the expression has the desired effect. We can generalise this for r-colours. Then the expression is the conjunction of $(\bigvee_{j=0}^{r} x_{i,j})$ and all $(\neg x_{i,j} \vee \neg x_{i,k})$ such that $j, k \in [r]$ and $j \neq k$.

Fortunately, the second constraint is easier to transform into such a SAT expression. Say once again that for some k-dimensional subspace S of $\mathbb{F}_q^n$ that I_S are the indexes of the t-dimensional subspaces of S. We want that not all t-dimensional subspaces of S receive the same colour. Say $I_S = \{1, 3, 8\}$ for some subspace S. Then we want for colour 0 for example:

$$\neg(x_{1,0} \wedge x_{3,0} \wedge x_{8,0}) \tag{4.8}$$

This we can easily rewrite to be:

$$(\neg x_{1,0} \vee \neg x_{3,0} \vee \neg x_{8,0}) \tag{4.9}$$

This clause is in the form we want, thus we can use it for the final expression. First, we give a generalisation. For some colour $j \in [r]$ and some k_j-dimensional subspace S of $\mathbb{F}_q^n$, the clause is:

$$\bigvee_{i \in I_S} \neg x_{i,j} \tag{4.10}$$

For our final expression, this needs to be done for every $j \in [r]$ and every k_j-dimensional subspace S. With that, our SAT formulation is complete.

4.3 Results

We now explore some of the values and solutions computed using the 2 methods explained above. In subsection 4.3.1 we look at the computed values for various vector space Ramsey numbers $R_{q,y}(k_1, ..., k_r)$. We begin with some prior known values, mostly of the form $R_2(2, t)$. Then move on to some other values, such as $R_3(2, 2)$. Then, in subsection 4.3.2, we take a look at some of the specific solutions for the case there are feasible colourings.

4.3.1 Values

Using the computational methods, we began with verifying some of prior known values for the vector space Ramsey numbers. Partially, this was to verify that the code functioned correctly. For example, we know that $R_2(2,2) = 3$. Indeed, for $n = 2$ in the ILP we obtain a feasible solution, but for $n = 3$ the ILP is infeasible. Similarly, the SAT for $n = 2$ gives us values for the variables that satisfy the expression, but for $n = 3$ it informs us it is not possible.

Using the ILP method, we see the confirmed values for $R_2(2,t)$ in table 4.3.

$R_2(2,t)$	n
2	3
3	5
4	6
5	≥ 8

Table 4.3: Values for $R_2(2,t)$, where the values on the left are values for t.

Additionally, some values for $R_2(2:r)$ (2 repeated r times) were verified. These can be found in table 4.4.

$R_2(2:r)$	n
2	3
3	5
4	6

Table 4.4: Values for $R_2(2:r)$, where the values on the left are values for t.

To conclude with these more 'conventional' values of vector space Ramsey numbers, we found that $R_2(3,3) \geq 7$ Apart from the number of colours and sizes of subspaces, there are of course also other factors that we can vary. For example, we have mostly focused of $\mathbb{F}_2$ so far, but we may also look at other fields. For example, we can see what Ramsey Numbers are like for $\mathbb{F}_3$. For that, the following results have been found:

$R_3(2,t)$	n
2	4
3	≥ 6

Table 4.5: Values for $R_3(2,t)$, where the values on the left are values for t.

Additionally, we also looked at an example for $\mathbb{F}_3$ with more than 2 colours, and found that $R_3(2,2,2) \geq 6$.

Aside from the field, we can also change the dimension of the subspaces we are colouring. Say we look at colourings of 2-spaces of $\mathbb{F}_2^n$. We were then able to compute that $R_{2,2}(3,3) \geq 10$.

4.3.2 Solutions

For the ILP, if there was a colouring such that the ILP was satisfied, a table was generated for that colouring. With 'satisfied' we of course mean that the colouring had no monochromatic k_i-space coloured i. For these tables specifically, we also set an actual objective function to optimise for the maximum amount of subspaces coloured 0 possible. To see how this table is made, refer to Appendix B.

This table consists of 3 columns. The first column indicates what index was assigned to the associated t-space. The 2nd column gives the basis for the t-space. Since we mostly work with $t = 1$, the 2nd column mostly contains one non-zero vector. Then, the 3rd column displays which colour the t-space received, represented by a number between 0 and r. For example, table 4.3.2 displays the colouring of the 1-spaces of $\mathbb{F}_2^3$ such that no 2-dimensional subspace is coloured 0 and no 3-dimensional subspace is coloured 1. Of course, this is a relatively trivial example, as long as all but one 1-space is coloured 1, the ILP is satisfied.

index	**basis**	**colour**
0	[(1, 0, 0)]	1
1	[(1, 0, 1)]	0
2	[(1, 1, 0)]	0
3	[(1, 1, 1)]	1
4	[(0, 1, 0)]	0
5	[(0, 1, 1)]	1
6	[(0, 0, 1)]	0

Table 4.6: Colouring that satisfies the ILP for $R_2(2, 3)$ with $n = 3$

As $R_2(2, 3) = 5$, we shall instead look at the case that $n = 4$, as displayed in 4.3.2. We proceed with a number of other examples of colourings. Though note that the sizes of the tables are kept small, even though they grow quite quickly for larger choices of n. Additionally, all of these use the same objective function, that being for optimising the colour 0.

index	**basis**	**colour**
0	[(1, 0, 0, 0)]	0
1	[(1, 0, 0, 1)]	1
2	[(1, 0, 1, 0)]	1
3	[(1, 0, 1, 1)]	1
4	[(1, 1, 0, 0)]	0
5	[(1, 1, 0, 1)]	1
6	[(1, 1, 1, 0)]	0
7	[(1, 1, 1, 1)]	0
8	[(0, 1, 0, 0)]	1
9	[(0, 1, 0, 1)]	0
10	[(0, 1, 1, 0)]	1
11	[(0, 1, 1, 1)]	1
12	[(0, 0, 1, 0)]	1
13	[(0, 0, 1, 1)]	1
14	[(0, 0, 0, 1)]	1

Table 4.7: Colouring that satisfies the ILP for $R_2(2, 3)$ with $n = 4$

index	basis	colour
0	[(1, 0, 0, 0)]	2
1	[(1, 0, 0, 1)]	2
2	[(1, 0, 1, 0)]	2
3	[(1, 0, 1, 1)]	0
4	[(1, 1, 0, 0)]	2
5	[(1, 1, 0, 1)]	1
6	[(1, 1, 1, 0)]	2
7	[(1, 1, 1, 1)]	3
8	[(0, 1, 0, 0)]	3
9	[(0, 1, 0, 1)]	3
10	[(0, 1, 1, 0)]	3
11	[(0, 1, 1, 1)]	0
12	[(0, 0, 1, 0)]	1
13	[(0, 0, 1, 1)]	1
14	[(0, 0, 0, 1)]	0

Table 4.8: Colouring that satisfies the ILP for $R_2(2:4)$ with $n = 4$

index	basis	colour	index	basis	colour
0	[(1, 0, 0, 0)]	1	20	[(1, 2, 0, 2)]	1
1	[(1, 0, 0, 1)]	0	21	[(1, 2, 1, 0)]	1
2	[(1, 0, 0, 2)]	0	22	[(1, 2, 1, 1)]	0
3	[(1, 0, 1, 0)]	1	23	[(1, 2, 1, 2)]	1
4	[(1, 0, 1, 1)]	0	24	[(1, 2, 2, 0)]	1
5	[(1, 0, 1, 2)]	0	25	[(1, 2, 2, 1)]	1
6	[(1, 0, 2, 0)]	0	26	[(1, 2, 2, 2)]	1
7	[(1, 0, 2, 1)]	0	27	[(0, 1, 0, 0)]	0
8	[(1, 0, 2, 2)]	0	28	[(0, 1, 0, 1)]	1
9	[(1, 1, 0, 0)]	0	29	[(0, 1, 0, 2)]	0
10	[(1, 1, 0, 1)]	1	30	[(0, 1, 1, 0)]	0
11	[(1, 1, 0, 2)]	1	31	[(0, 1, 1, 1)]	1
12	[(1, 1, 1, 0)]	0	32	[(0, 1, 1, 2)]	1
13	[(1, 1, 1, 1)]	1	33	[(0, 1, 2, 0)]	1
14	[(1, 1, 1, 2)]	1	34	[(0, 1, 2, 1)]	0
15	[(1, 1, 2, 0)]	1	35	[(0, 1, 2, 2)]	0
16	[(1, 1, 2, 1)]	0	36	[(0, 0, 1, 0)]	1
17	[(1, 1, 2, 2)]	1	37	[(0, 0, 1, 1)]	1
18	[(1, 2, 0, 0)]	1	38	[(0, 0, 1, 2)]	1
19	[(1, 2, 0, 1)]	1	39	[(0, 0, 0, 1)]	1

Table 4.9: Colouring that satisfies the ILP for $R_3(2, 3)$ with $n = 4$

index	basis	colour
0	[(1, 0, 0), (0, 1, 0)]	1
1	[(1, 0, 0), (0, 1, 1)]	0
2	[(1, 0, 1), (0, 1, 0)]	0
3	[(1, 0, 1), (0, 1, 1)]	1
4	[(1, 0, 0), (0, 0, 1)]	0
5	[(1, 1, 0), (0, 0, 1)]	0
6	[(0, 1, 0), (0, 0, 1)]	0

Table 4.10: Colouring that satisfies the ILP for $R_{2,2}(3, 3)$ with $n = 4$

Chapter 5

Discussion

In subsection 2.1.3, the Fano plane is shown to represent the colouring of the 1-spaces of $\mathbb{F}_2^3$, where the edges represent the 2-spaces. This is actually an example of a hypergraph, that being a graph where edges may connect any number of vertices together (including zero!). In general, colourings of subspaces $\mathbb{F}_q$ can be represented as hypergraphs, with the vertices representing the t-spaces and the edges the k_i-spaces. Unfortunately, there was no proper space or time found to explore the possible connections and relations between vector space Ramsey numbers and hypergraphs. In particular, the connection to the chromatic number of a hypergraph may be useful to study further.

It was not until quite late in the process that the possibility of using SAT to compute Ramsey Numbers. Thus, there was not much time to explore how much we could have benefited from this. Intuition would suggest that SAT, which only looks for satisfiability, would get values of vector space Ramsey numbers faster. There is perhaps also a chance for a more efficient SAT formulation. The current one is based on the ILP formulation, and there may be formulations that are more efficient. Although, one would still have to contend with rapidly increasing numbers of colourings and subspaces.

Additionally, if we want to study specific solutions for cases where it is feasible, the ILP still better suits our needs there. That is because we can optimise for specific colours or some other property that may be of interest.

For that matter, unfortunately there wasn't enough time to spend on analysing some of the actual colourings that were computed. It is very possible that studying colourings that do avoid monochromatic k_i-spaces may offer us insight into how vector space Ramsey numbers behave. As mentioned, the tables in subsection 4.3.2 optimise for as many spaces of colour 1 as possible.

There are of course other factors that could be optimised for that may produce insightful results. Though we do have to note the current form of the ILP makes some of those difficult. If you want to, for example, optimise based on hamming distance between basis vectors or something akin to that, that is not quite directly possible currently. It may be possible to model hamming weights or distances in the ILP as well and maximise or minimise for those somehow.

Chapter 6

Conclusion

In this text, we have introduced the notion of vector space Ramsey numbers. Where $R_{q,t}(k_1, ..., k_r) = n$ implies that n is that smallest such number that for any r-colouring χ of the t-spaces of $\mathbb{F}_q^n$, we must have an $i \in [r]$ such that k_i is monochromatic with colour i under χ. We've also demonstrated these numbers exist for any choice of $q, t, k_1, ..., k_r$. This proof required a number of things to be introduced like Hales-Jewett and special spaces. Using those tools and strong double induction, first on all $t' < t$ for any $\sum_{i=1}^{r} k_i$, and then for t and all $\sum_{i=1}^{r} k'_i < \sum_{i=1}^{r} k_i$, we were able to demonstrate these Ramsey Numbers indeed exist.

We then quickly concluded that these numbers are very difficult to compute because of the quickly increasing number of colourings and subspaces. Thus, to say anything about bigger values for the vector space Ramsey numbers, we studies some asymptotic bounds. In particular, we studied 3 methods to obtain a lower bound that is roughly the same, that being roughly $R_2(2, t) > \frac{3}{2}t$. The first one was the probabilistic method, where we demonstrated the chance that there was a monochromatic 2- or t-space was less than 1, meaning there was at least one colouring that didn't have these monochromatic subspaces.

The 2nd method was based on coding theory. We showed that the maximum dimension of a code that does not contain two vectors with distance t is $2t - 1$ for $\mathbb{F}_2^n$ where $2t - 1 \leq n < 4t$. Thus, if we fix $n = 3t - 1$, we know that any $2t$-space must contain a vector of weight $2t$. By then taking A to be all vectors of weight $2t$, it was easy to show that this contained no 2-spaces fully. By then assigning that the colour red and its complement the colour blue, we then obtained a colouring that had no monochromatic 2- or t-spaces. Hence, giving us $R_2(2, 2t) > 3t$.

Lastly, we made use of projective spaces. It was demonstrated that $\mathcal{L}_k$, consisting of all vectors of weight $2k + 1$ and higher, is a complete cap in PG($3k$,2) that intersects every $2k$-flat as well. In similar fashion to Code Theory, we then took $\mathcal{L}_k$ as our set and showed it both intersected all $2k$-spaces and contained no 2-space, resulting in a similar bound.

We also introduced one method to obtain an upper bound. For this, we went back to Classical Ramsey Numbers, showing that $R(3; r) \leq 3r!$. We then demonstrated that $R_2(2; t) > n \implies R(3; r) > 2^n - 1$ using Schur's Theorem. By then taking the contrapositive of this we were able to get our bound $R_2(2; r) \leq r \log_2 r$.

Though for high values, vector space Ramsey numbers are difficult to compute, we still have an interest in finding it for lower values. For this, we explored 2 methods to compute some values. First was by means of an Integer Linear Program. There were 2 sets of constraints, one making sure t-spaces were assigned at most 1 colour, and another making sure that no k_i space was monochromatic with colour i. Notably, we did not need to optimise for something in this ILP. The intent is to verify whether the ILP is feasible for certain values of n. If it is, then the vector space Ramsey number is bigger than n. If it isn't, then the Ramsey Number is less than or equal to n.

Based on the ILP formulation, we also made a formulation based on the Boolean Satisfiability Problem. As this is more directly about feasibility, in theory it should be quite a bit faster than the ILP formulation. This of course being a bit up to implementation.

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Appendix A

Fields

A.1 Definition of Fields

Fields are a special type of ring. So to define fields, we first need to define rings. In turn, to first define rings, we first look at what an abelian group is.

Definition 7 (Abelian Group)**.** *A set R is an abelian group under addition if it satisfies the following three condition:*

1. *(Associativity)* $(a+b)+c = a+(b+c)$ *for all* $a, b, c \in R$
2. *(Identity) There is an element* $0 \in R$ *such that* $a+0 = a$ *and* $0+a = a$ *for all* $a \in R$
3. *(Inverses) For all* $a \in R$*, there is an element* $-a \in R$ *such that* $a+(-a) = 0$ *and* $(-a)+a = 0$
4. *(Commutativity)* $a+b = b+a$ *for all* $a, b \in R$

We immediately proceed with the definition of a ring.

Definition 8 (Rings)**.** *A set R, equipped with two operations $+$ and $\cdot$ (addition and multiplication, is called a ring if it satisfies the following three conditions:*

1. *R is an abelian group under addition.*
2. *$\cdot$ (multiplication) is associative. So for all* $a, b, c \in R$ *we have* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. *The distributive law holds. That is, for all* $a, b, c \in R$ *we have that* $a \cdot (b+c) = a \cdot b + a \cdot c$ *and* $(a+b) \cdot c = a \cdot c + b \cdot c$

Remark. *In general, R does not contain multiplicative identity, nor does it have a multiplicative inverse for every element. It also is not necessarily commutative under multiplication.*

With this, we can properly define what a field is:

Definition 9 (Fields)**.** *A ring R is a field if $R \backslash \{0\}$ is an abelian group under multiplication.*

Remark. $\mathbb{Q}, \mathbb{R}$ *and* $\mathbb{C}$ *are examples of fields.*

We are mostly interested in fields with finite elements. For this, we introduce the notion of an integral domain

Definition 10 (Integral Domain)**.** *Say we have a ring R with the following properties:*

1. *R has a multiplicative identity. That is, there is an element* $1 \in$ *such that* $a \cdot 1 = a$ *and* $1 \cdot a = a$.
2. *R is commutative under multiplication. That i, for all* $a, b \in R$ *we have that* $a \cdot b = b \cdot a$

Then it is called an integral domain if $a \cdot b = 0$ *implies that* $a = 0$ *or* $b = 0$.

Remark. *By definition, all fields are integral domains.*

For finite fields, we look to the following theorem:

Theorem 17. *A finite integral domain is a field.*

With this in mind, it can be shown that the set $\mathbb{Z}/n\mathbb{Z}$, with addition and multiplication calculated according to modulo n, is an integral domain if n is prime. After all, if n is prime then there are no elements $a, b \in \mathbb{Z}/n\mathbb{Z}\backslash\{0\}$ such that $a \cdot b = 0$. Given that it is finite and an integral domain, theorem 17 then tells us $\mathbb{Z}/n\mathbb{Z}$ with n prime is also a finite field.

A.2 Construction of $\mathbb{F}_{q^n}$

Definition 11. *For a ring R we define the polynomial ring $R[x]$ over R as follows. $R[x]$ consists of the polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$. Where $ai \in R$ for $i \in [n]$. Addition and multiplication are as expected.*

Theorem 18. *If a polynomial ring $F[x]$ is defined over a field F, then it is an integral domain.*

Definition 12. *$g(x) \in F[x]$ divides $f(x) \in F[x]$ if there exists an $h(x) \in F[x]$ such that $f(x) = g(x)h(x)$. Then f is also called a multiple of g.*

Remark. *In general, dividing two polynomials $f, g \in F[x]$ gives two unique polynomials $q, r \in F[x]$ such that $f(x) = q(x)g(x) + r$, where q is known as the quotient and r is referred to as the rest.*

Definition 13. *Let F be a field and let $f, g, h \in F[x]$. We call f modulo h equal to r if r is the rest when dividing f by h, noted down as $r = fmodh$. We say f and g are equivalent under modulo h if they have the same rest after dividing, noted down as $f \equiv gmodh$.*

Definition 14. *A polynomial $p \in F[x]$ with degree≥ 1 is called irreducible in $F[x]$ if $p = bc$ with $b, c \in F[x]$ imples that either b or c is a constant polynomial.*

Remark. *Degree refers to the highest power x^i present in the polynomial.*

Theorem 19. *Let F be a field and let $h \in F[x]$. The quotient ring $F[x]/h$ consists of all polynomials of $F[x]$ with degree smaller than the degree of h. Calculating according to modulo h results in $F[x]/h$ being a commutative ring with a multiplicative identity.*

Remark. *For a field F with q elements and $h \in F[x]$ met degree $n \geq 0$, the number of elements in $F[x]/h$ is equal to q^n.*

Theorem 20. *Let F ve a field. Let $f \in F[x]$ be an irreducible polynomial. Then $F[x]/f$ is a field. Furthermore, if F is finite, then so is $F[x]/f$.*

Appendix B

Code

B.1 ILP Gen

The aim of this code is to generate an ILP that can be passed on to ILP solvers. It was coded in SageMath.

```
def subspace_count(q: int, n: int, k: int):
    res = q**n - 1
    for i in range(1, k):
        res *= (q**n - q**i)
    for i in range(k):
        res /= (q**k - q**i)
    return int(res)

def VSRN_ILP_Gen(n: int, K: list, q=2, y=1, c=0):
    #Here, n is the dimension of the vector space, K is a list of subspaces of
    #which we avoid monochromatic subspaces.
    #q is the size of the field, y the size of the subspaces we colour.
    #c is which colour we maxamise for.
    m = MixedIntegerLinearProgram() #create model
    x = m.new_variable(binary=True) #set-up variables.

    V = VectorSpace(GF(q),n) #Create F_2^n
    T = V.subspaces(dim=y) #Generator for all y-dim subspaces of V

    #Make a dictionary to associate the y-dim subspaces of V
    #with variables x, where i
    #corresponds to the subspace and j to the colour.
    mvars = {}
    i = 0
    for t in T:
        X = [x[i,j] for j in range(len(K))]
        #make a list of variables, where i is associated with the subspace
        #t and j the colour
        mvars[t] = X
        i+=1

    #Add constraints that makes sure that every y-dim subspace
    #is assigned at most 1 colour.
    for key in mvars:
        m.add_constraint(sum(mvars[key]) == 1)
```

```
    #Add constraints that ensure each k-dimensional subspace
    #is not monochromatic for all k in K
    for j in range(len(K)):
        S = V.subspaces(dim=K[j])
        for s in S:
            R = s.subspaces(dim=y)
            I = []
            for t in R:
                I.append(mvars[t][j])
            m.add_constraint(sum(I) <= subspace_count(q, K[j], y)-1)

    m.set_objective(sum([x[o,c] for o in range(i)]))
    return m, n, K, q, y, mvars

m, n, K, q, y, mvars = VSRN_ILP_Gen(3,[2,2])
m.write_lp('VSRN_ILP.lp')
with open('values.txt', 'w') as f:
    f.write(str(n) + ' ' + str(len(K)) + ' ' + str(q) + ' ' + str(y) +
            '\n' + str(K))

with open('mvars.txt', 'w') as f:
    for key in mvars.keys():
        l = [i for i in key.basis_matrix()]
        f.write(str(l) + '\n')
```

B.2 ILP Solve

This code solves the ILP generated by the code in section B.1. It also generates an excel file containing the solution. It was coded in Python.

```
import gurobipy as gp
import pandas as pd

def subspace_count(q: int, n: int, k: int):
    res = q**n - 1
    for i in range(1, k):
        res *= (q**n - q**i)
    for i in range(k):
        res /= (q**k - q**i)
    return int(res)

def hamming_distance(v1, v2):
    res = 0
    for i in range(len(v1)):
        if v1[i] != v2[i]:
            res += 1
    return res


m = gp.read("VSRN_ILP.lp")
m.setParam('SolutionLimit', 1)
m.optimize()
with open('values.txt', 'r') as f:
    temp = f.read().split('\n')
    v = temp[0].split(' ')
```

```
        n, r, q, y = int(v[0]), int(v[1]), int(v[2]), int(v[3])
        K = eval(temp[1])

    r = len(K)
    ss = subspace_count(q, n, y)

    opt_vals = {'index': [], 'basis': [], 'colour': []}
    with open('mvars.txt', 'r') as f:
        lines = f.read().split('\n')
        i = 0
        for line in lines:
            if line != '':
                v = eval(line)
                for j in range(r):
                    s = i*r + j + 1
                    z = m.getVarByName('z_%s' % s)
                    if int(z.X) == 1:
                        opt_vals['index'].append(i)
                        opt_vals['basis'].append(v)
                        opt_vals['colour'].append(j)
                i += 1

    df = pd.DataFrame(opt_vals)
    #File name depends on type of Ramsey number, R_q,y(s,t) or R_q,y(k:r)
    df.to_excel("R_q%s_y%s_s%s_t%s_n%s.xlsx" % (q,y,K[0],K[1],n), index=False)
    df.to_excel("R_q%s_y%s_k%s_r%s_n%s.xlsx" % (q,y,K[0],r,n), index=False)
```

B.3 SAT Gen

The aim of this code is to generate a solver using the boolean expressions outlined in subsection 4.2.2. It generates a text file that can be input into SAT solver such as Glucose. It was coded in SageMath.

```
from sage.sat.solvers.dimacs import DIMACS

def VSRN_SAT_Gen(n: int, K: list, q=2, y=1):
    solver = DIMACS(filename="VSRN_SAT.txt")

    r = len(K)

    V = VectorSpace(GF(q),n)
    T = V.subspaces(dim=y)

    mvars = {}
    i=1
    for t in T:
        X = [int(i*r+j) for j in range(r)]
        mvars[t] = X
        solver.add_clause(tuple(X))
        for j in range(r):
            for k in range(j+1,r):
                solver.add_clause((-X[j],-X[k]))
        i+=1

    for j in range(r):
        S = V.subspaces(dim=K[j])
        for s in S:
```

```
            R = s.subspaces(dim=y)
            clause = ()
            for t in R:
                clause += (-mvars[t][j],)
            solver.add_clause(clause)

    return solver

solver = VSRN_SAT_Gen(3, [2,2])

DIMACS.render_dimacs(solver.clauses(), "VSRN_SAT.txt", solver.nvars())
```