FORM-FINDING OF BRANCHING STRUCTURES SUPPORTING FREEFORM ARCHITECTURAL SURFACES

STRUCTURAL MECHANICS

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Building technology master thesis

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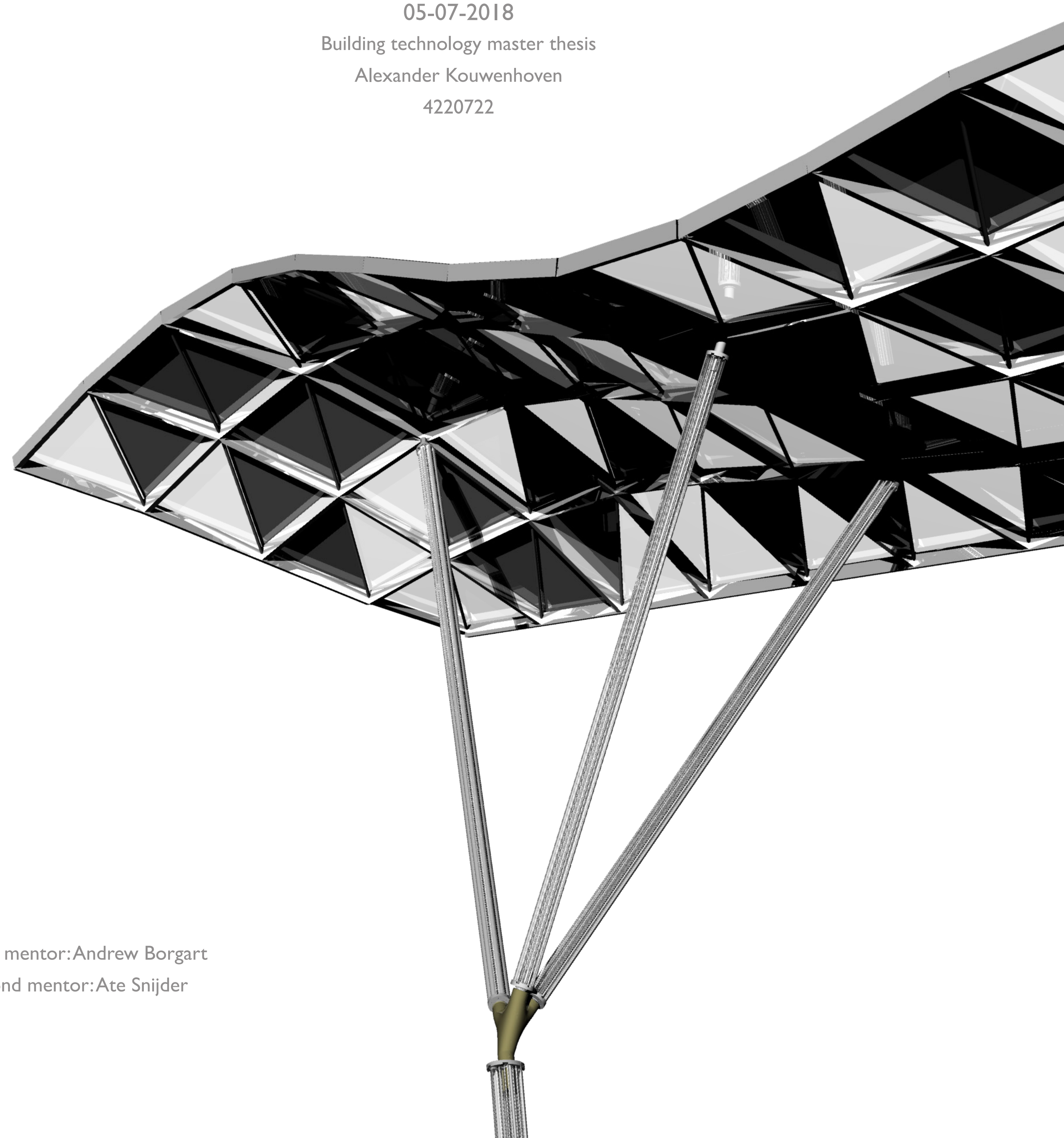

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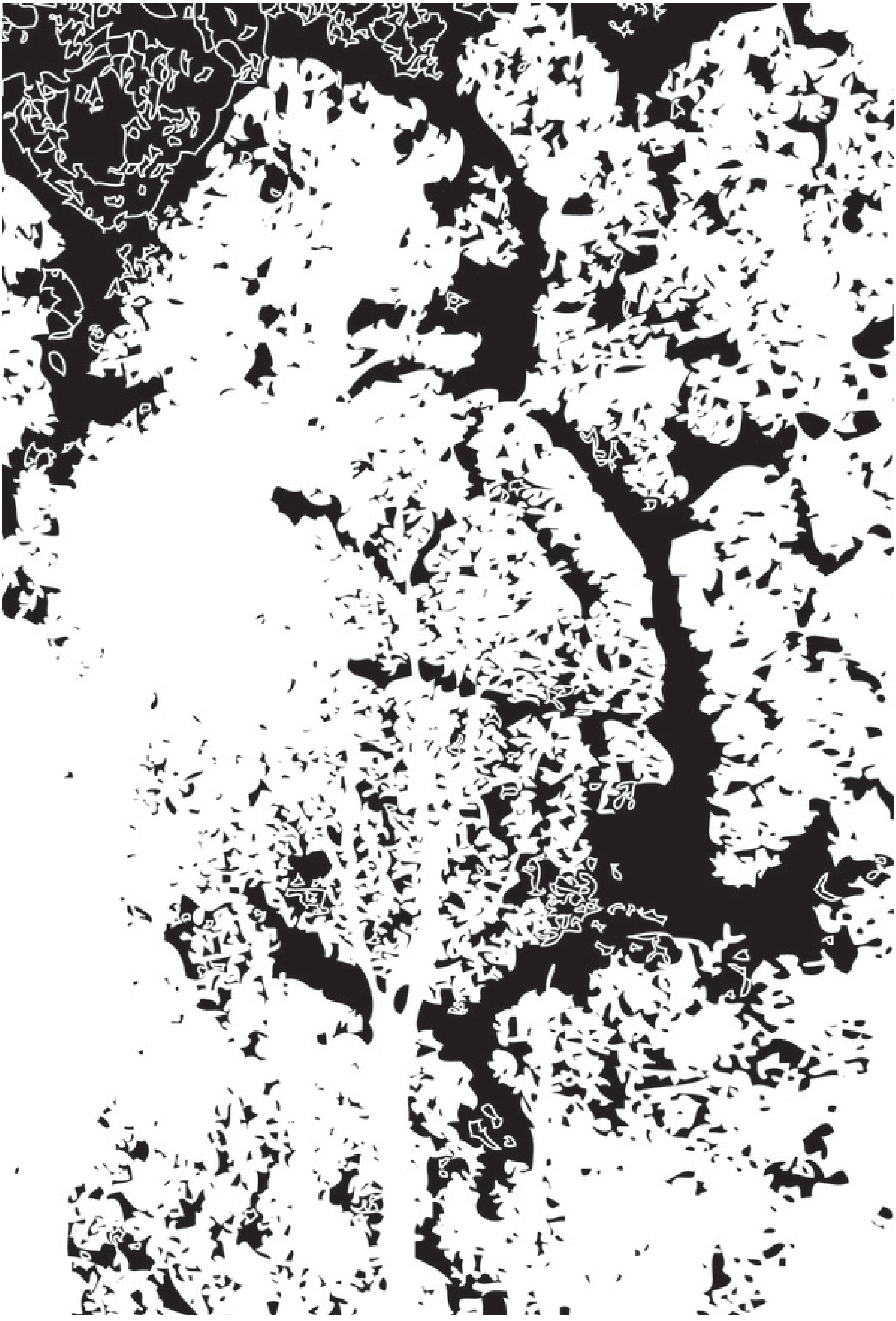
Introduction

'Form-finding of branching structures supporting freeform architectural surfaces' is the graduation thesis of the master track Building Technology at the University of Technology in Delft. This thesis is made by Alex kouwenhoven in the time-span of three-quarter of a year and tutored by Andrew Borgart and Ate Snijder.

With the increasing complexability of the built environment, the challenges for both the architect and the engineer are becoming bigger. Architects tend to make new shapes or free-forms and the engineers are challenged to make it buildable. A shape often used as a support of this free-form architecture is the tree-like column or branching column. A structure branching out and redirecting forces from a big roof surface to one single point.

However, the design of these complex columns is often done by the computer and optimizing software. There is a missing link in the field of structural mechanics and these types of columns, the knowledge of forces flowing through these complex three-dimensional structures.

In this thesis, a method of designing these complex structures is proposed. By an analytical approach, an efficient structural column can be made and multiple optimization strategies are proposed. Following these strategies, a design can be made. In the final chapters of the thesis, an example of a design is made using this strategy.



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1. *Research framework*

1.1. Background

The use of natural forms in architecture has been used for over decades for both esthetical for constructional purposes. One of these natural forms is the tree. For decades architects like Antoni Gaudí and Frei Otto have been researching these shapes and built several examples using tree-like structures. The main issue about these structures is finding the most reasonable form to solve the problem of an actual project. The branches of tree structures should be arranged properly. Then only tension or compression is present, resulting in a higher structural efficiency and less material use. (Zhao, Liang, & Liu, 2017). Research had been done by several methods of form-finding. In the beginning, this form-finding consisted of physical models and the usage of graphic statics (Sassone, 2014).

In the past decade, the form-finding methods have changed due to the usage of computers. Computer-aided Design (CAD) and Finite Element (FE) models are programmed to calculate and design 'optimized' branching structures. (Knippers, 2016) Problem with computer simulated solutions is that they often are based on an enormous amount of iterations, searching for the best possible solution or approximation. This method is time-consuming, and the designer or engineer loses the ability to truly understand what the effect of different forces on a structure is.

1.2. Problem statement

With the expansion of knowledge in the field of building and construction, architects and engineers try to reach the limits of what's possible. The result is wider span architecture or new challenging shapes. These new shapes leave the engineers with more complicated problems. Branching structures in three dimensions are often used to support these new shapes, but the structural behavior of the branches is challenging to predict. Finding statically equilibrium for a two-dimensional branching structure can be done, but a third dimension often creates statically indeterminate structures. Right now, there is no method of form-finding the optimal branching structure as a support of freeform architectural expressions other than using the computer to approach it. Therefore, filling the gap will result in a better understanding of the way that branching structures transfer loads. This will be done in the following way:

1.3. Research question

In this research I will answer the following question

"How can we design structurally efficient three-dimensional branching structures as a support for non-uniform roof surfaces."

To answer this question, the following set of sub-questions will be answered:

What are branching structures?
What is the structural behavior and advantages of branching structures?
What previous methods used for form-finding of branching structures?

After the analysis of the branching structures in specific, some research will be done to answer the following set of questions:

What is graphic statics and how can it be applied to branching structures?
How can we determine the best state of equilibrium?

To define the best state of equilibrium, research will be done to answer:

How can statically indeterminate structures be calculated?
What other methods of optimizing load structures are used?

1.4 Research goal

By combining the knowledge of structural behavior of branching structures and methods used for optimizing other type of structures I want to achieve the following during my graduation:

Create a form finding tool or design method for 3D branching structures.

With this final tool a design can be made for a specific location and shape. The boundary condition such as the location and shape will be determined in advance. By scripting this tool or method in grasshopper, a plug-in for Rhinoceros, and making it parametric, it can be used for multiple design inputs. The shape of the roof and support points will generate output in the form of an optimized structure. This optimized structure will form the starting point for a final design.

1.5 Methodology

In the first weeks, the graduation topic is being defined. By searching literature on the topic, It was possible to get a broader idea of the possibilities and shortcomings on the field of form-finding of branching structures. Trough physical model studies, computer models, and studies that were done by other researchers it will be possible to get a good description of the structural behavior of branching structures.

With this knowledge and more literature study, a new method of describing optimal branching structures can be written/derived. The method will be scripted in grasshopper, a powerful parametric design plugin for rhinoceros. At the same time, some hand calculations will be done to verify the script. Once the script is verified and proven to work a design can be made. While creating a case or a design problem, some criteria and boundary conditions will be set up. Together with the script, this will lead to a design. With this design, the original criteria will be checked, and if they match, a final design can be made.

By structurally analyzing the design, conclusions can be drawn whether the described method works. If it is considered correct, it can be used for designing branching structures supporting free-form architectural surfaces.

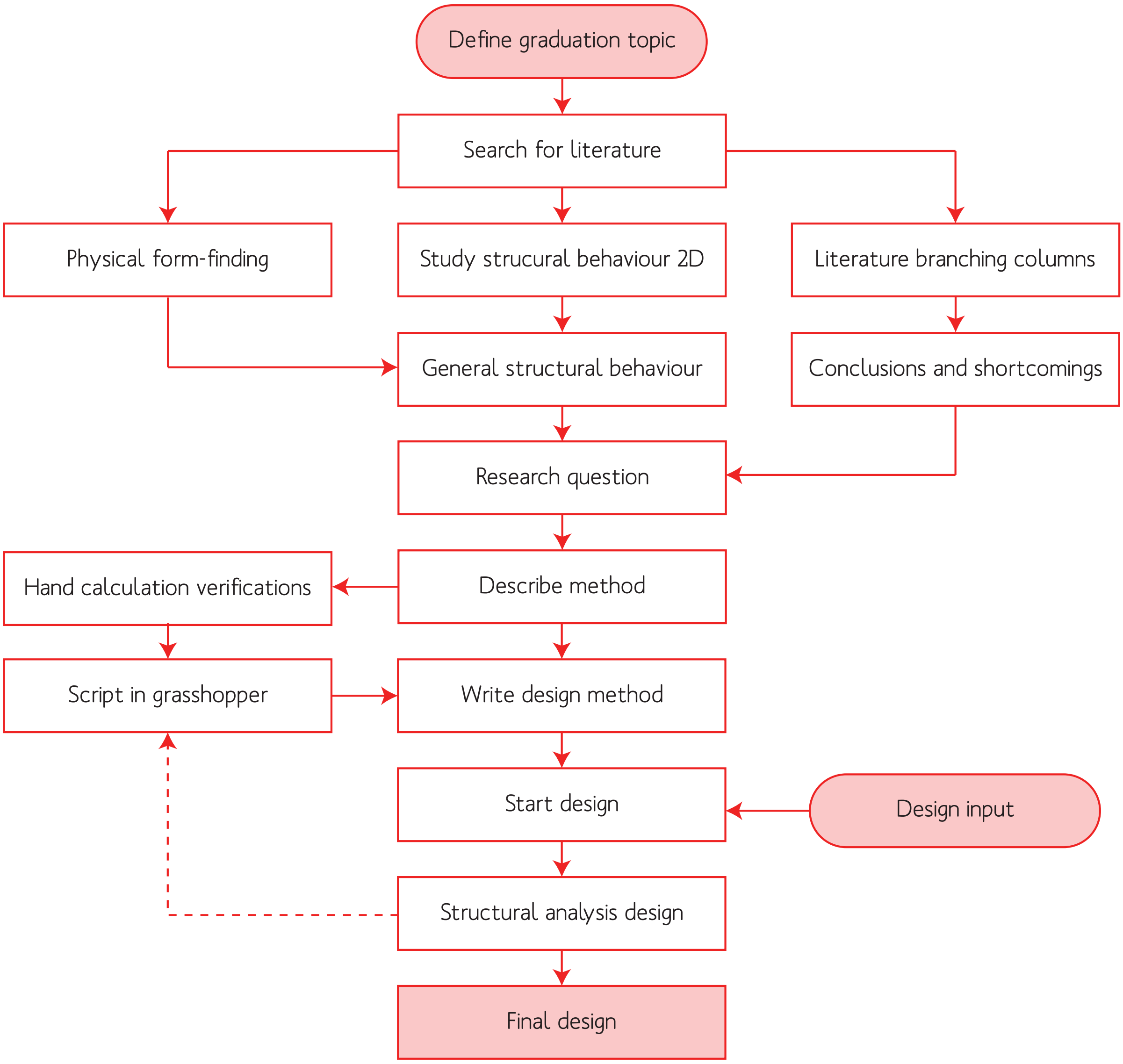


Figure 1 Methodology in scheme. (own image)

2. *Imitating nature*

2.1. Design by natural principles

Great architects such as Frei Otto and Antonio Gaudi are known for their imitation and use of natural forms and principles in buildings and constructions. Nature has been around for millions of years and has transformed all its organisms to be optimal and efficient. However, this optimization is due to the conditions in what place or conditions the organism lives. As a consequence, biological solutions are very rarely optimal but rather optimized for fulfilling multiple functions at the same time (Knippers, 2016). As an inspiration, researchers from many disciplines started to investigate natural forms, patterns, and repetitions. "You could look at nature as being like a catalog of products, and all of those have benefited from a 3.8 billion year research and development period. And given that level of investment, it makes sense to use it." Architect Michael Pawlyn once stated.

A detailed analysis of biological solutions often reveals that they can be interpreted as 'good enough' to allow survival and propagation in a particular ecological niche and to outcompete rival species by using a minimum of resources concerning materials and energy (evolutionary preference of 'cheap but good enough' solutions. (Knippers, 2016)

Like most modern architects, Frei Otto was preoccupied with the fundamental question of structure: how to achieve more with less —to construct greater spans with less material and energy. (Glaeser, 1972) To create these efficient structures knowledge of the behavior of forces is needed. Because there are different ways to organize structural elements within the same configuration or design, the structural efficiency is considered essential. However, figure 2 shows that different structural layouts affect how spaces or buildings are interpreted. Therefore, designing a structure is always a combination of both structural as aesthetic quality.

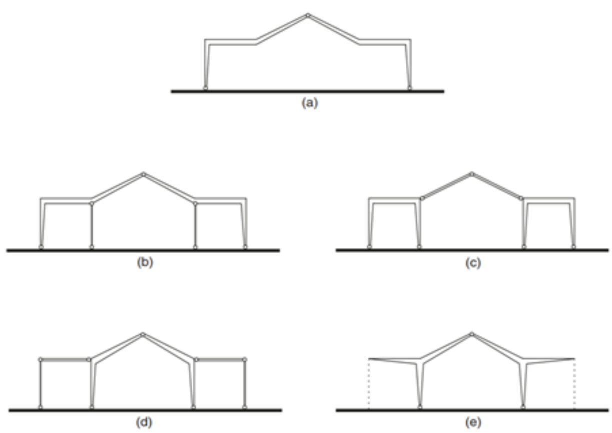
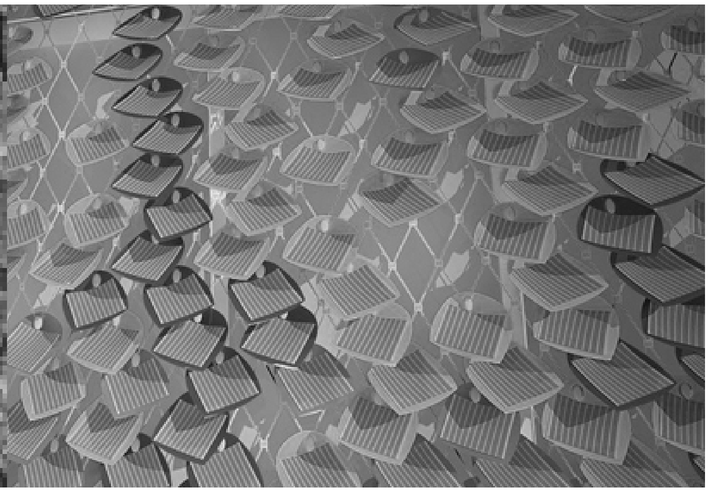
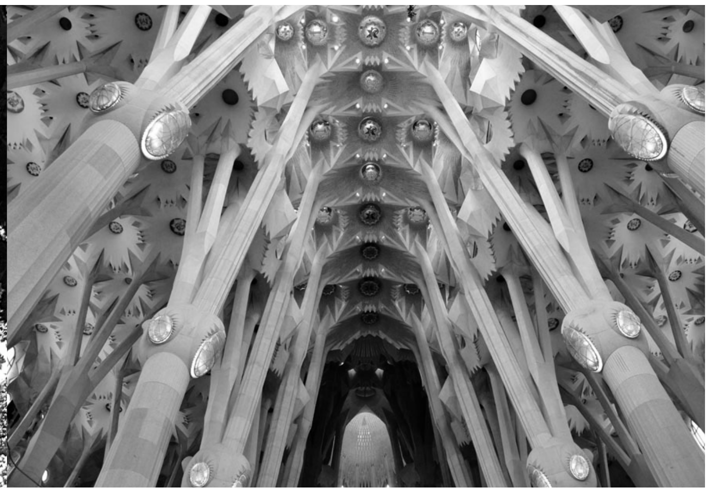


Figure 2 Different structural layouts affect how spaces are read. (Charleson, 2005)







3. *Branching structures*

The most inspiring feature of a natural tree undoubtedly is its capacity of carrying a large surface supported by a narrow element (trunk) trough fractal- like branching configuration.
(Rian & Sassone, 2014).

3.1 Introduction

A branching structure is based on this principle of a tree. Expanding from one point the structure branches out to multiple branches, expanding its reach. With a broader reach, a larger surface area can be supported. This is the considered the main advantage of a branching structure. The structure can transfer the loads on a large surface to one single column or point with high structural efficiency. Due to the dividing members a large span or area can be carried. This division of members results in a shorter span for the roof members and therefore to smaller structural members. Also, the length of each member is decreased reducing the buckling length.

Fractal geometry is based on multiple mathematical or reproduction rules. By repeating these rules or functions, a spatial structure is created as can be seen in figure 3. Each repetition is called an iteration. On the left the first iteration, following by a second and third.

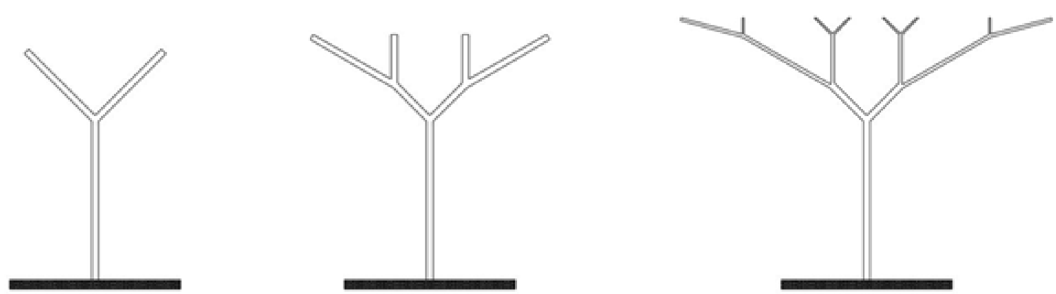


Figure 3 Three iterations of a branching structure (Rian & Sassone, 2014)

3.2 Historic overview

As can be seen in the introduction of natural principles in architecture, many principles are inspired by natural phenomena. The main topic of this thesis is the structural behavior of the branching, tree-like structure. In the early ages, the natural shapes in construction mainly were represented by sculptural shapes. A well-known example is the Corinthian columns in the

Roman period; these columns were decorated with floral decorations.

A few ages later the first example of a real branching like structure appeared in eastern of Asia. Wooden brackets were stacked in a way that each bracket cantilevers over the lower bracket. This way loads of a larger surface could be transferred to one point or column as can be seen in figure 4a.

The technique, called dougong, was founded a long period before the birth of Christ but gained its popularity during the Medieval period. Especially in the period of Song dynasty (960 AD to 1279 AD). (Rian & Sassone, 2014) It was mainly used in temples and other religious buildings. Later it became one of the essential ornaments of Asian temple architecture.

In Europe, the first examples of branching were born during the construction of cathedrals. To construct the high open ceilings a new type of construction was born, the fan vault. The vault is an abstract imitation of the form of a tree. The ceiling is formed by a structure branching from a column or wall, creating structural arches. Figure 4b shows a Gothic ribbed vault where the arches can be analyzed as 2D arches (Wolfe, 1921)

Between 1880 AD and 1920 AD a unique style of branching structures was created by Antonio Gaudi. Within his work, he was able to merge architectural form and structural efficiency by introducing the forms of trees and plants. Gaudi often said: 'There is no better structure than the trunk of a tree or a human skeleton' (Barallo and Sánchez-Beita, 2011). Especially in the Sagrada Familia Cathedral in Barcelona, where the structure was based on a hanging model using cables and branches, these natural preferences can be found.

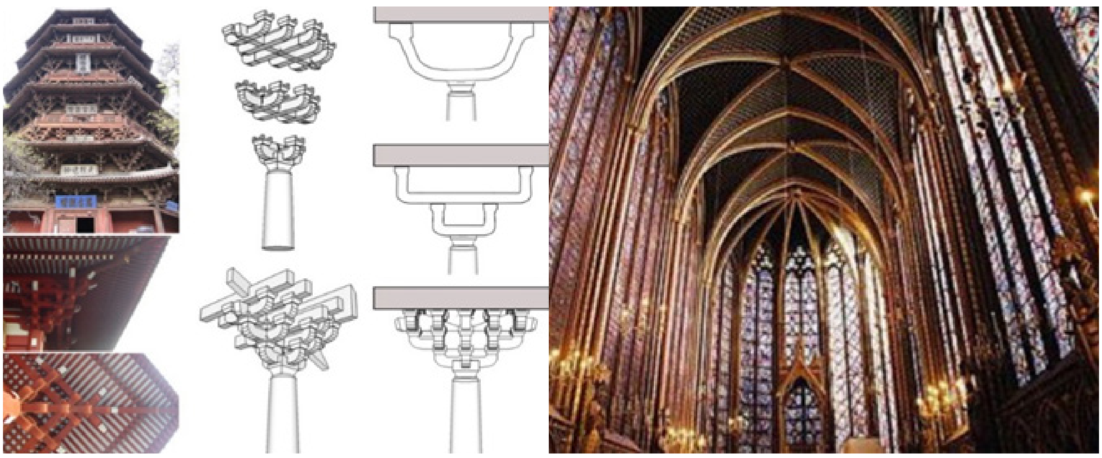


Figure 4 (a) dougong technique and (b) a gothic ribbed vault. (Rian & Sassone, 2014)

In the early 1960's another architect known for its experimenting with nature's forms, started to work with branching structures. Frei Otto, German architect, professor and founder of the Institute for Lightweight Structures (Verein zur fördering des leichtbaus e.V.), was intrigued by the branching of trees. As a result, Otto started to design and investigate hanging models of branching systems as can be seen in figure 5. Different configurations of architectural trees were developed and experimented with. In IL Publication 32 (1983) Otto describes branching structures as: "Verzweigungskonstruktionen können Druck- und Biegekrafte mit geringer eigener masse übertragen.". Branching structures can transfer compression- and bending forces with a low own mass. This embraces his ideas about lightweight and efficient structures.

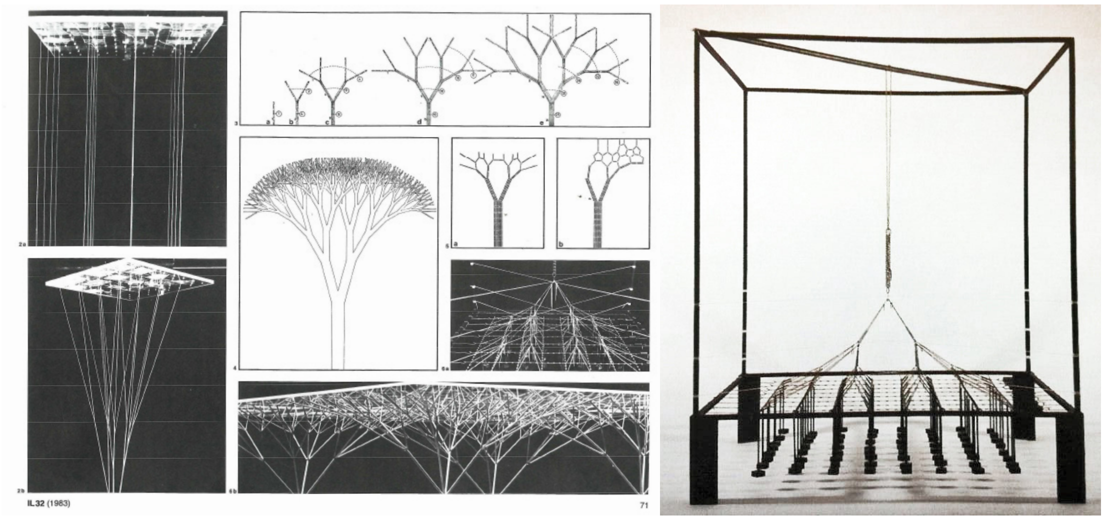


Figure 5 Studies and models of branching structures (IL Publications)

From this moment on, multiple architects and engineers started to implement branching structures to span large distances. Many examples can be found in large structures such as public centers, shopping malls, airports or railway stations. For example Stuttgart airport (a), The national gallery museum extension in Singapore (b), The tote restaurant in Mumbai (c) and Changsha south railway station(d).







KFC


3.3 Branching in freeform roofsurfaces

Recent developments in architecture and building technology have resulted in more challenging architecture and architectural forms. Architects are free to design almost every desired shape. Using double-curved roof structures and grid shells instead of flat roofs the structural challenges increase as well. In a few recent built examples branching structures are used to support these freeform roof structures. For instance, the Zlote Tarasy in Warschau (Figure #). Multiple branching columns are supporting the double curved blob which covers the shopping mall. These steel columns are placed in strategic places to reduce the roof member size. Also at the Westfield shopping mall (Figure #), nine columns are used to transfer the load of the waving roof to the ground, creating a forest of branching columns.

With this increasement of freeform shapes, the lacking knowledge of supporting it becomes visible. Either the members are hinged at both sides to force a moment free construction or the nodes and members are dimensioned extremely big.

3.4. Mechanical behaviour

The most inspiring feature of a natural tree undoubtedly is its capacity of carrying a large surface supported by a narrow element (trunk) trough fractal-like branching configuration. (Rian & Sassone, 2014). This is considered the main advantage of a tree and is copied into a branching structure. The structure is able to transfer the loads from a large surface to one single column or point in a both aesthetic as structural optimal form.

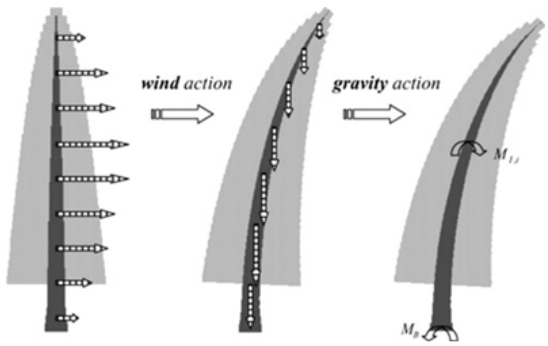


Figure 6 Behavior of a tree in the wind. (Rian & Sassone, 2014).

Branching structures, however, have a slightly different structural behavior compared to actual trees. A tree is able to reduce the loading by deformations and stream-lining of branches, twigs, and leaves. (Born et al., 2016) Also, as shown in figure 6, a tree can absorb bending moments due to the big spreading roots and flexible trunk. Both trees as building structures suffer from the same loading types such as static, snow and wind loading. The main difference is that these big deformations are not allowed in the building industries because of safety reasons. This leaves limited space for flexibility and asks for a different approach to designing the structure. In figure 7a a branching column is shown if it were to be designed like a tree, allowing the branches to move freely. A 2D comparison between a stiff structure and one with free moving branches is shown in figure 7b, only taken into account movement in two directions. The difference is the horizontal element securing all the branches to stay in place. This results in higher normal forces but reduces the total bending moment to zero.

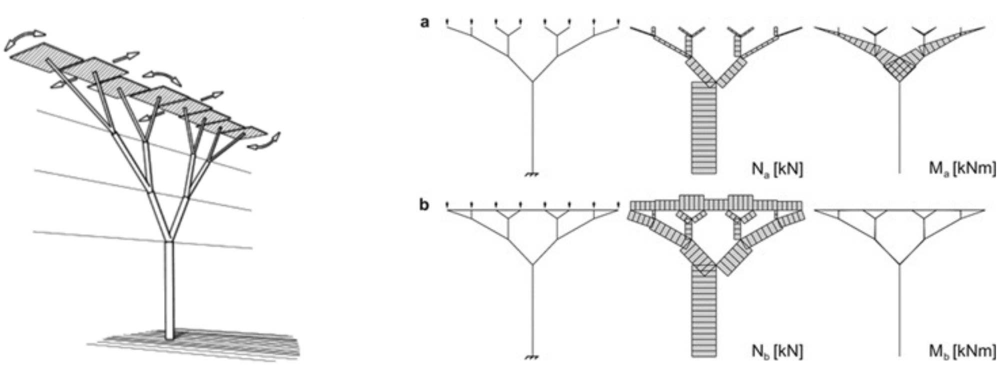


Figure 7 Free moving branches of a real tree (a) and comparison of structural behavior(b) (Zhao et al, 2017)

Because of the differences in structural behavior allowances, learning from this biological process is a complex task. Despite its challenges, biomimetic research and development represent a highly promising approach for optimizations in lightweight engineering. (Born et al., 2016) Each aspect of the tree must be studied and decided whether or not it can be applicated in the building industries.

The main problem with branching structures is to find the best shape to solve the problem of an actual project. The configuration of the structure influences the mechanical behavior and properties. By shaping the tree in an efficient way, it can reach high structural efficiency. This efficiency comes from the fact that spatial structures mainly suffer from tension or compression and no or tiny bending moments. Therefore, the branches of tree structures should also be arranged properly. Then only tension or compression is present and smaller cross-sections can be achieved. (Zhao et al., 2017) Figure 8 shows the influence of both compression and bending moment. On the left, a column is exposed to a normal force. Due to the force F, compression is present in the column. The compressive stress is shown below the column. In the middle, a bending moment is applied to the column. This bending moment causes the column to extend on one side of the column and shorten on the other side. Due to this elongation tension in one side and compression in the other side is resulted. Combining both compression and bending moment will result in a stress-pattern as shown on the right.

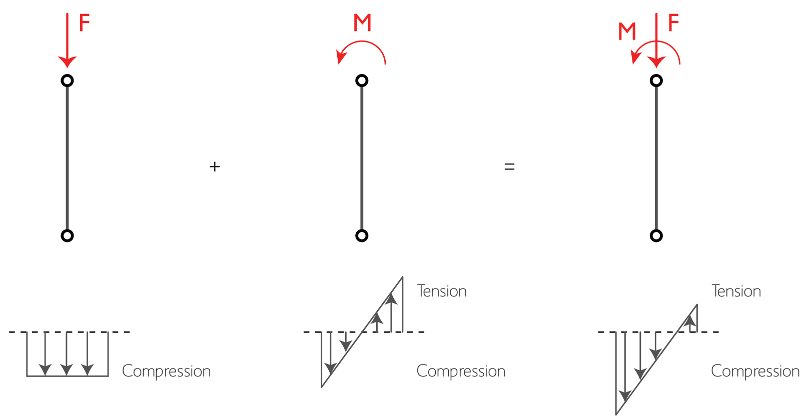


Figure 8 Structural behavior of normal forces and bending moments (own image)

The efficiency of the structure is a result of the ratio between bending and axial forces, which is caused by the shape of the structure. The process of finding the most reasonable shape is called form-finding. A process that has been around for years and can be done in several ways.

4. Form finding

4.1 physical form-finding

In order to find the ideal structure for a certain load case, physical form finding is a powerful method. One way of making a structural approximation for a load case is the use of tension based hanging models. Like Gaudi's famous hanging model of the Sagrada Familia (figure 9a) in Barcelona the model is hanged with small bags filled with sand. This sand represents the vertical loading on the building and shapes the network of strings. Because funicular models only suffer from tension, high structural efficiency is achieved. By reversing the model, all forces will be reversed and the most efficient structure for the compression load case is shaped.

Figure 9 The hanging model of the Sagrada Familia (a) and right frei otto's model study (b)

This principle of reversing tension becoming compression was the base of many other form finding model studies. In figure 9b, the famous hanging model of Frei Otto is shown. Otto used small weights to represent a uniform distributed load. A studie done by Kolodziejczyk uses wettened strings to see what the path of least resistance is. Since the flow of forces can be compared to the flow of liquids, this method shows resemblance to an optimal structure. Other IL (leichtbau institute) researchers show different models but use the same basic principles. Tension based models with elastic strings and movable nodes. Changing the length of the members and increasing or decreasing the internal forces.

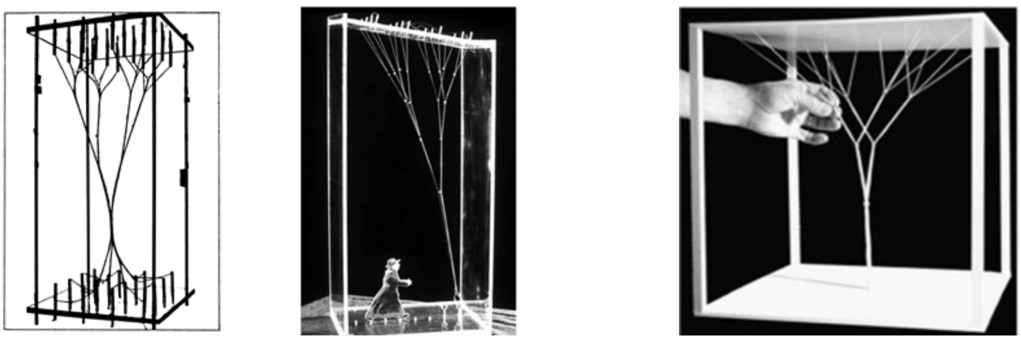


Figure 10 Different model studies of branching structures.

4.2 Numerical form-finding

In the past decade numerical form-finding method has drawn most of the researcher's attention. von Buelow (2007) Proposed a genetic algorithm for form finding of branching structures. Hunt (2009) designed a numerical method with the assumption that the connections were hinges. Zhang (2014) used sliding cable elements to simulate the pulling-only characteristics of the cable element. But the sliding cable element needs tedious programming work and is not suitable for common engineering designers. The form-finding analysis is the first step in the design of tree structures. But there is almost no applicable method that can resolve form-finding problem efficiently. (Zhao et al, 2017)

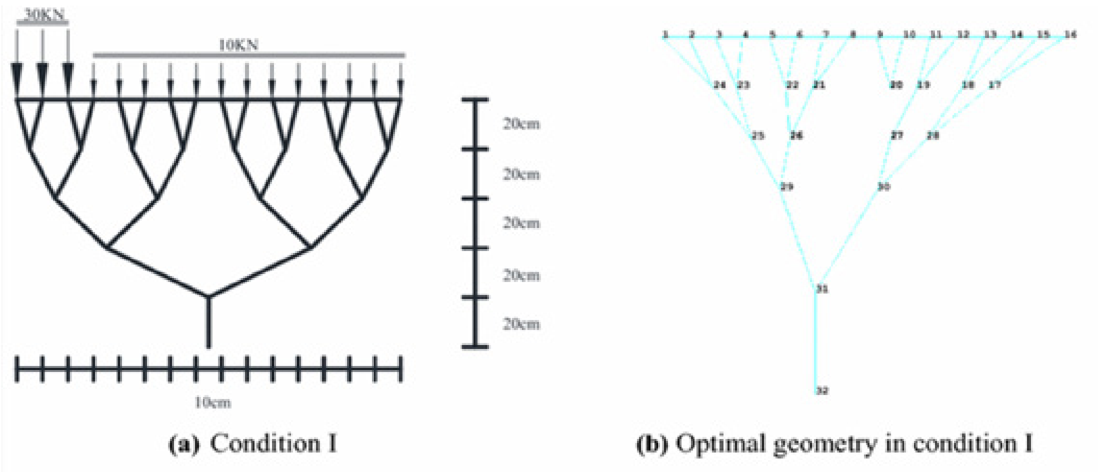


Figure 11 form-finding of non-uniform distributed loads using double element method. (Zhao et al., 2017)

The study done by Zhao et al. (2017) uses a double element method. Assuming that every component is built up from two elements, one with only bending stiffness and one without. Manipulating the bending stiffness let the beams act more as a thread. Outcomes are generated in 2D branching structures and axis-rotated 3D structures. Another study done by Peng (2016) uses topology optimization for the structure. Although the proposed method is not considered a smart method of finding the ideal structure, it has some interesting conclusions concerning the stiffness and shape of the roof element. The stiffer the roof element gets; the less branches are found in 'the optimal configuration'. This method however is nog preffered by eingineers because there is no understanding of the actual force paths. The computer removes material where the smalles stresses occour.

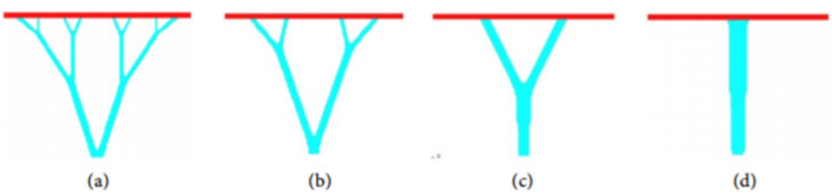


Figure 12 Increasing stiffness of the roof element leads to different optimal shapes (Peng, 2016)

4.3 Own form finding

To get a first feeling of the forces acting on a tension based model, a study model was made. By changing places of branching, different equilibria are made. multiple insights came with this model:

- The total length of each string changes by shifting the nodes.
- The branching pattern can be changed in multiple ways
- The internal forces (tension) increase when the length is decreased.
- There are infinite ways of making an equilibrium.
- The force in each member differs.

To get a better analytical insight into the structural behavior we will have to look at another way of researching structural mechanics: Graphic statics.

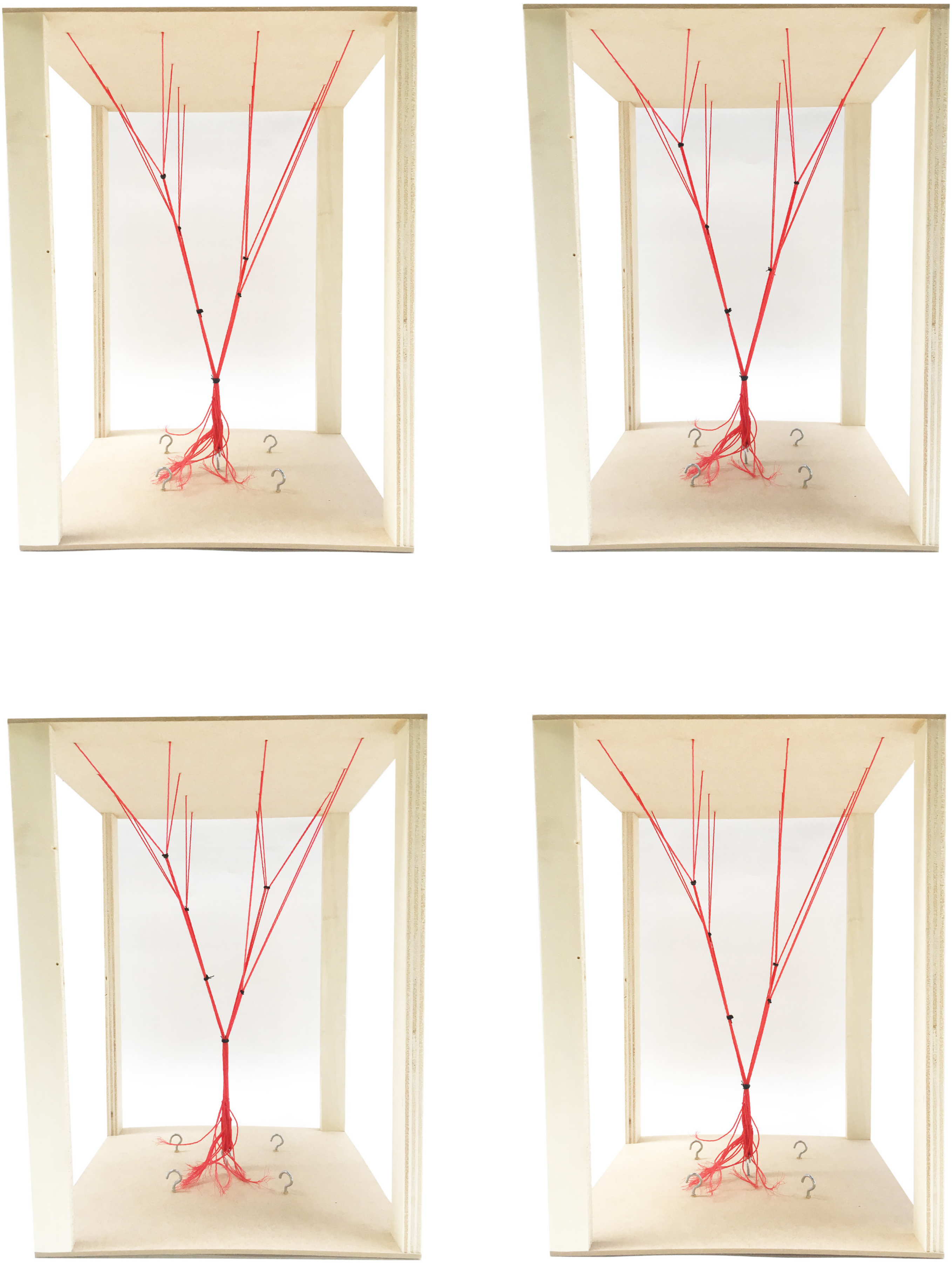


Figure 13 Different branching configurations (own work)

5. Graphic statics

5.1 Graphic statics

Graphic statics is a way to get insight into the forces acting on a structure graphically. Using force directions and polygons the equilibrium of a structural system is represented. To get a better understanding of the forces flowing through the branching structures, this will be studied in detail. First, the fundamental principle of graphic statics will be explained. In the second part a more detailed method which can be used in the analysis of branching structures.

5.2 Head tail method

The Head tail method is an easy way to check weather forces on a point are in equilibrium. If the directions of forces or members are known a force polygon can be drawn. The polygon resembles an equilibrium if it is a closed polygon. In figure 15a one force is known, and the other two directions are given. By plotting the direction of the forces, a closed polygon and thereby an equilibrium can be made.

If more than two forces are unknown, the system becomes statically indeterminate. This means that there are infinite options of closing the force polygon. Figure 14b shows multiple options of making the equilibrium, all corresponding with different force pairs. (Dool, 2012)

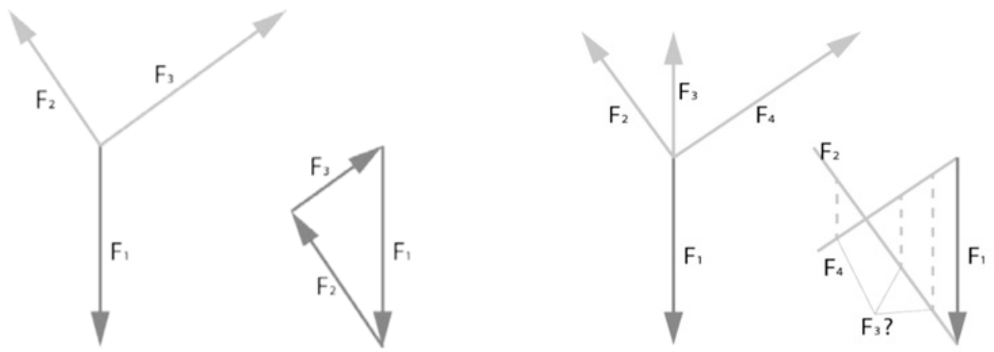


Figure 14 Head tail method (a) and a statically indeterminate configuration(b) (van Dijk, 2014)

5.3 Cable and arch forces

To understand the power of graphic statics we can look at multiple forces acting on a cable. In this cable, the only force acting is tension, so the forces are in the same direction as the cable. By combining the force polygons of the multiple points of the cable, a closed loop can be drawn, and the size and direction of the forces can be derived. In figure 16 an example is given.

If the direction of the external forces F1 and F2 flips, the direction of the internal forces flips as well. As a result, the members or strings only suffer from compression. Based on this principle hanging- and tension-Compression models are made. The shape of the cable is called the thrust line. This line represents the ideal path of forces for the given model. If members of a structure follow this path, they will only suffer normal forces and no shear force.

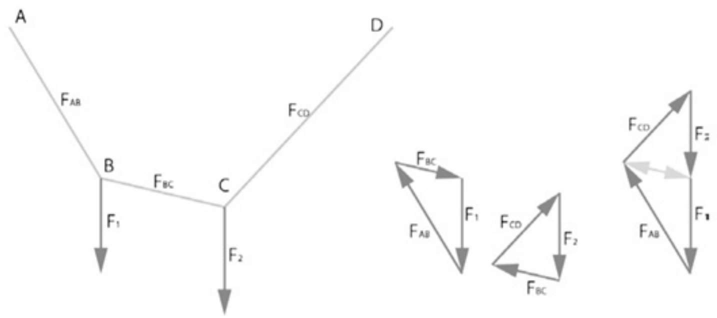


Figure 15 Multiple forces on a cable in equilibrium (Dool, 2012)

Bow's notation or a Cremona diagram is used to determine the size and direction of forces. With the slope of the lines from the polar coordinate to the start point and the end point of the forces, the thrust line of the beam can be drawn. (Dool, 2012) For different polar coordinates, there are different thrust lines to be made. This is a result of the varying angles of the Cremona diagram. Each representing an ideal flow of force pattern for the same load-combination. Higher internal forces are found in the lower thrust line but still an equilibrium for the same loading.

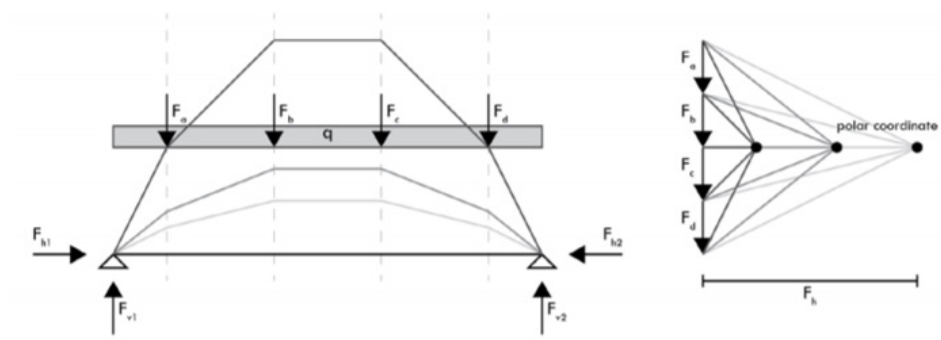


Figure 16 Different thrust lines for different polar coordinates (Dool, 2012)

5.4 Branching form and force diagrams

If a branching structure is statically determined it can be represented with a Cremona diagram. A design study done by Allen and Zalewsky (2012) shows an example of a Cremona diagram for a branching structure. In this case, the structure is inclined and has a point load of 7.5 K on each branch.

The Polygon shows the internal forces in each member of the structure. The letters resemble the area between different forces. For instance, between A and C there is a force of 15K. In the diagram, this is represented by the distance between a and c. The numbers left represent an area corresponding with the surrounding members. In the polygon, these numbers are represented by points. All the lines connecting that point are the forces acting on the members. For instance, the left branch of the bottom tier is represented by the curve between a and 4 in the Cremona diagram. The length of the curve represents the normal force in the branch. Also, the forces in each tier can be read. The line a-2-4-6-i represents the middle tier and a-4-i the bottom tier.

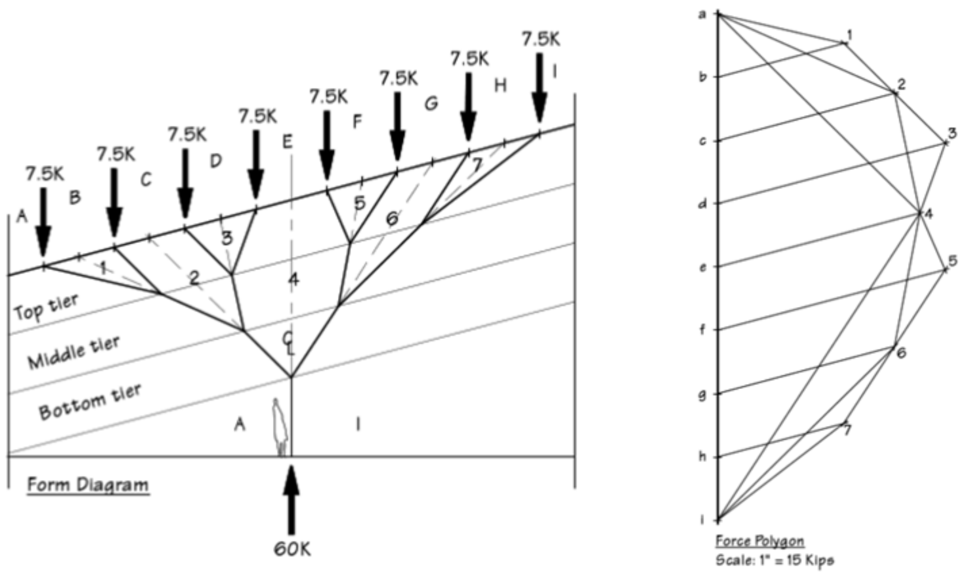


Figure 17 Form and Force (cremona) diagram of a branching structure (Allen & Zalewsky, 2012)

Like the example of the arch, each branching structure also has multiple solutions for equilibrium. In figure 17 an example of a branching structure is found. If the force polygon or Cremona is closed, the structure is a stable system, and all the members are only suffering from normal forces. To see what happens if the points in the structure changes I have done some studies with the Cremona diagrams.

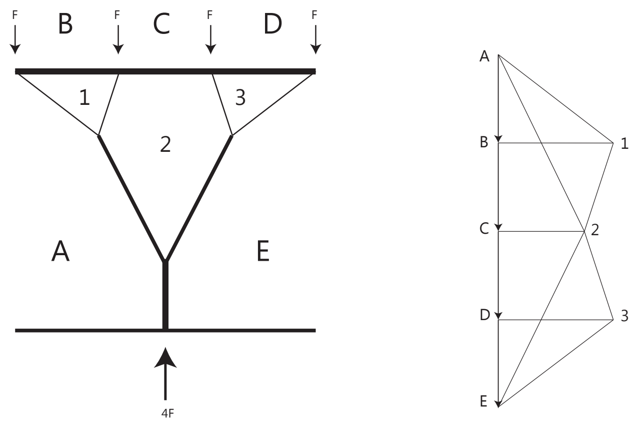


Figure 18 Branching column in equilibrium (own image)

In figure 18 the simple tree with two iterations is shown. The Cremona diagram shows the internal forces of the members. A-2-E represents the bottom tier and A-1-2-3-E the top tier members. The top members of the represented by the horizontal lines B-1, C-2, and D-3. Because the Cremona is a closed polygon, the tree is in equilibrium. If a point in the Cremona is moved, it influences all the connecting members or lines. This changes the shape of the tree to make another equilibrium. As can be seen in figure 19, point 2 is shifted horizontally. This influences all the members that surround area 2 in the drawing. Directions of the members and the corresponding force changes with this shift. The red lines show the new equilibrium made.

Conclusions that can be derived from this example:

- *If a point moves, it affects all the members of the space that it surrounds.*
- *There are multiple ways of making an equilibrium for the same load-case*

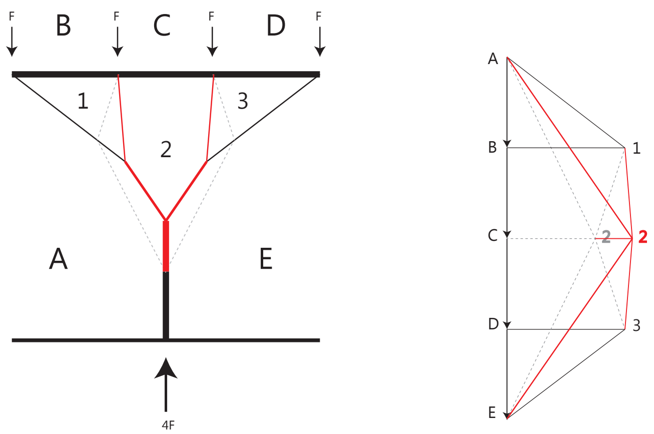


Figure 19 Moving point of branching (own image)

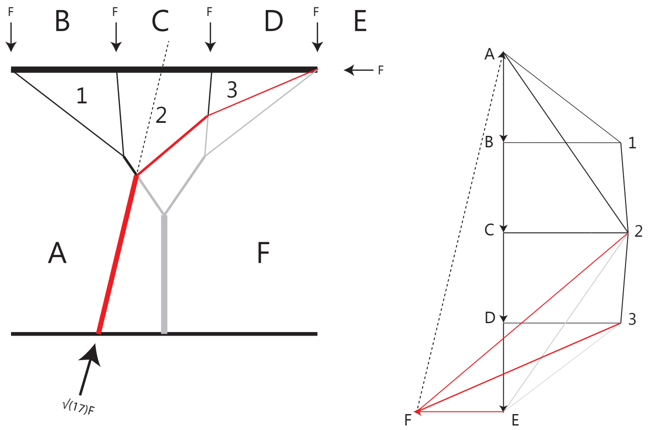


Figure 20 External force on branching structure (own image)

In the example shown in figure 20, an extra horizontal force is added to the structure. As a result, an extra force drawn in the Cremona diagram. Direction and size corresponding to the external force. The equilibrium is made by changing the direction and size of the trunk and the branches on the right. Because the reaction force must close the force polygon the dotted line shows the direction. All the lines that were connected with point E now are connected with point F, changing its direction and size. Conclusions that can be made from this example:

- The reaction force is the closing line of the force polygon. All external forces one after another show the direction and size.
- The change of loading changes the closing point and the lines connected with it.
- Like the previous example, there are different ways of making an equilibrium for the same external loads.

In figure 21 the roof of the structure is tilted. As a result, all the horizontal lines in the Cremona diagram are tilted as well. This leads to a new equilibrium with new points for 1,2 and 3. This shift changes almost all the member size and, or direction. The external forces and directions still stay the same.

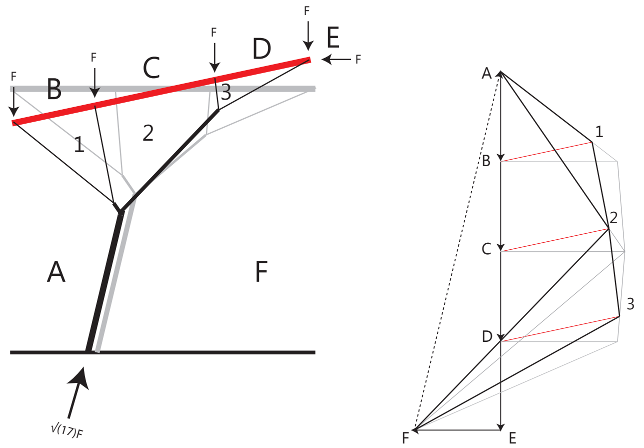


Figure 21 Tilting of the top surface. (own image)

In the end, we can conclude that the Cremona diagram is a useful tool to investigate the forces within a branching structure. The direct influence of changes in force, direction and configuration are easy to derive. Still, there are some questions to be answered. Because there are infinite ways to make an equilibrium we still can not derive what's the 'optimal branching pattern.' This is why we will have to look at the theory proposed by Borgart & Liem (2011). This theory uses the analysis of Complementary energy to determine the optimal thrust line for arches and shells. By rewriting the theory for branching structures, statically indeterminate structures and optimal branching patterns can be calculated.

6. Solving statically indeterminate structures

6.1 Complementary Energy

When a structural element is exposed to forces, the element will elongate or shorten. If a material behaves linear elastically, the stress level will behave linear proportional compared to the elongation. This is called Hooke's law. In Figure X the stress, - and the elongation, - are plotted. The area beneath the curve represents the strain energy, Ev, and the area above the curve shows the complementary energy, Ec. (Blaauwendraad, 2004)

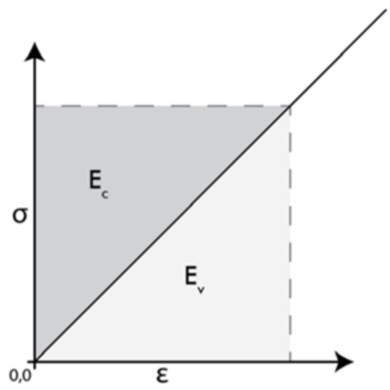


Figure 22 Elongation as a function of the stress (Blaauwendraad, 2004)

From the graph we can derive:

$$E_{compl} = \frac{1}{2}\sigma\varepsilon = E_v$$

Because the elongation can be described as:

$$\varepsilon = \frac{\sigma}{E}$$

And the stresses as:

$$\sigma = \frac{N}{A}$$

The complementary energy can be expressed as:

$$E_{compl} = \int_v \frac{1}{2}\frac{\sigma^2}{E}dV = \frac{1}{2}\frac{N^2}{EA}$$

therefore the complementary energy is given by:

$$E_{compl} = \frac{1}{2}\frac{\sigma^2}{E}$$

The complementary energy for a structural member with length l can be solved with:

$$E_{compl} = \frac{1}{2}\frac{\sigma^2}{E}l$$

For the total of the complementary energy of a structural system the following formula can be used:

$$E_{compl,total} = \sum_{i=1}^{n} F_i^2 l_i$$

To find the solution of the statically indeterminant problem the minimum complementary energy has to be found.

$$E_{compl,total} = \sum_{i=1}^{n} F_i^2 l_i = minimum$$

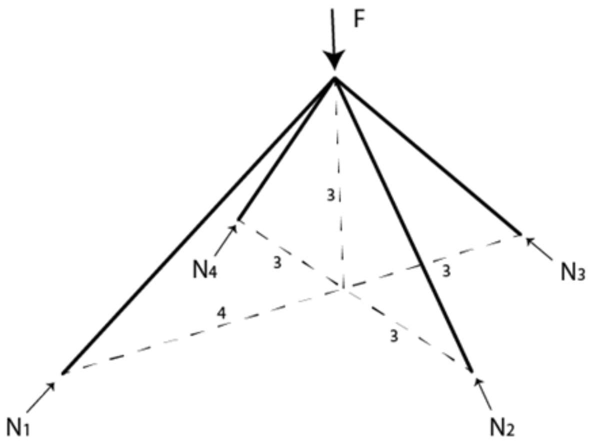


Figure 23 Statically indeterminate system (Own image)

6.2 Example

To solve a statically indeterminate system of elements the minimum complementary energy can be used. In this example, there are four members where one has a different length. On the structure, a force F is applied. With the three equilibrium conditions and the minimum complementary energy, the corresponding reaction forces can be found. These reaction forces correspondent to the normal forces of the individual members.

First equilibrium to be solved is the equilibrium of forces in the x-direction:

$$\sum F_x = 0$$
$$\frac{1}{2}\sqrt{2}\,N_3 = \frac{4}{5}N_1$$
$$N_3 = \frac{4}{5}\sqrt{2}\,N_1$$

Second is the equilibrium of forces in the y-direction:

$$\sum F_y = 0$$
$$N_2 = N_4$$

Third the equilibrium of forces in the y-direction:

$$\sum F_z = 0$$
$$\frac{3}{5}N_1 + \frac{1}{2}\sqrt{2}\,N_2 + \frac{1}{2}\sqrt{2}\,N_3 + \frac{1}{2}\sqrt{2}\,N_4 = F$$
$$6N_1 + 5\sqrt{2}\,N_2 + 5\sqrt{2}\,N_3 + 5\sqrt{2}\,N_4 = 10F$$

The fourth equation is the minimum complementary energy:

$$E_{compl,total} = \sum_{i=1}^{n} F_i^2 l_i = minimum$$
$$E_{compl,total} = N_1^2 l_1 + N_2^2 l_2 + N_3^2 l_3 + N_4^2 l_4$$
$$E_{compl,total} = 5N_1^2 + 3\sqrt{2}\,N_2^2 + 3\sqrt{2}\,N_3^2 + 3\sqrt{2}\,N_4^2$$

By computing equation (1) and (2) into (3), we can rewrite N2 as a function of N1 and F:

$$6N_1 + 5\sqrt{2}\,N_2 + 5\sqrt{2}\,N_3 + 5\sqrt{2}\,N_4 = 10F$$
$$6N_1 + 5\sqrt{2}\,N_2 + 5\sqrt{2}\left(\frac{4}{5}\sqrt{2}\,N_1\right) + 5\sqrt{2}\,N_2 = 10F$$
$$6N_1 + 5\sqrt{2}\,N_2 + 8N_1 + 5\sqrt{2}\,N_2 = 10F$$

Rewriting the equation gives:

$$N_2 = \frac{10F - 14N_1}{10\sqrt{2}}$$

By computing equation (1),(2) and (3) into (4), the complementary energy equation can be written as a function of N1 and F

$$E_{compl,total} = 5N_1^2 + 3\sqrt{2}\,\left(\frac{10F - 14N_1}{10\sqrt{2}}\right)^2 + 3\sqrt{2}\,\left(\frac{4}{5}\sqrt{2}\,N_1\right)^2 + 3\sqrt{2}\,\left(\frac{10F - 14N_1}{10\sqrt{2}}\right)^2$$

Using F=2 kN and rewriting the equation gives:

$$E^{compl,total} = 5N_1^2 + \frac{243}{25}\sqrt{2}\,N_1^2 - \frac{84}{5}\sqrt{2}\,N_1 + 12\sqrt{2}$$

To find the minimal complementary energy, the derivative of the equation has to be zero.

$$\frac{d}{dN_1}(EC) = \left(10 + \frac{486}{25}\sqrt{2}\right)N_1 - \frac{84}{5}\sqrt{2} = 0$$
$$\left(10 + \frac{486}{25}\sqrt{2}\right)N_1 = \frac{84}{5}\sqrt{2}$$
$$N_1 = 0.6337\ kN$$

Filling in equation (1), (2) and (3) gives the solution for the other reaction forces.

$$N_2 = 0.7867\ kN$$
$$N_3 = 0.7168\ kN$$
$$N_4 = 0.7867\ kN$$

This is the distribution of forces that will take place in the statically indeterminate structure and is verified in structural calculation programs such as Matrix frame and GSA.

If the structure is multiple statically indeterminate this method also can be applied. In Figure 24, the system is 2 times statically indeterminate. To solve this configuration, Andrew Borgart showed that the derivative over two unknowns could be taken. These so-called partial derivatives have to be equated to zero to find the minimal complementary energy for a multiple statically indeterminate structure

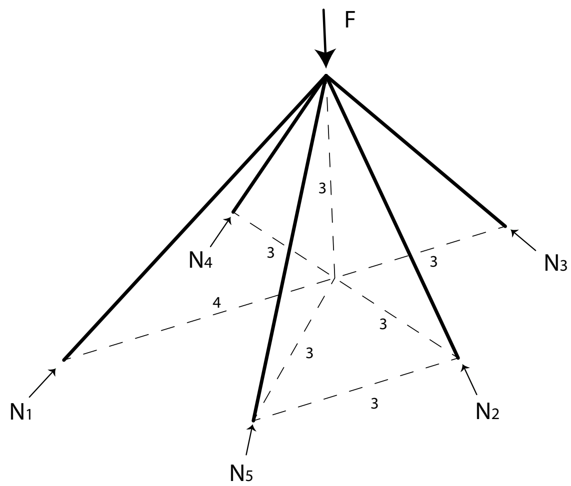


Figure 24 2 times statically indeterminate structure (Own image)

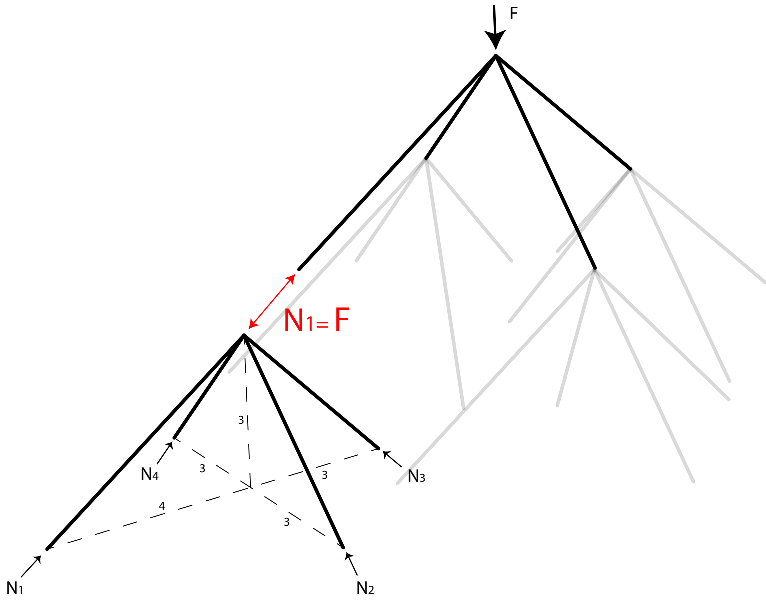


Figure 25 Multiple layer structure (Own image)

6.3 Multi-layered structure

As showed in the previous example, a statically indeterminate structure can be solved using minimalization of the complementary energy as a fourth equation. This is also possible for multi-layered structural systems. In this case, the reaction force in the top branch gives the input of force for the bottom branch.

Because the normal or resultant force has the same direction as the structural member, the proportions are equal. In the example, both the force and the member form a triangle between the x component, the z component and the diagonal. If the diagonal is set at value 1, the x component has a value of 0.8 or 4/5, and the z component has a value of 0.6 or 3/5.

The vector of the force can be described as a result of the proportions in x, y and z-direction and can be written as:

$$\vec{\mathbf{F}} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

In order to calculate multiple layers, this has to be included in the equilibria in all directions. The amount of unknown variables remains the same and using complementary energy will solve the statically indeterminate construction.

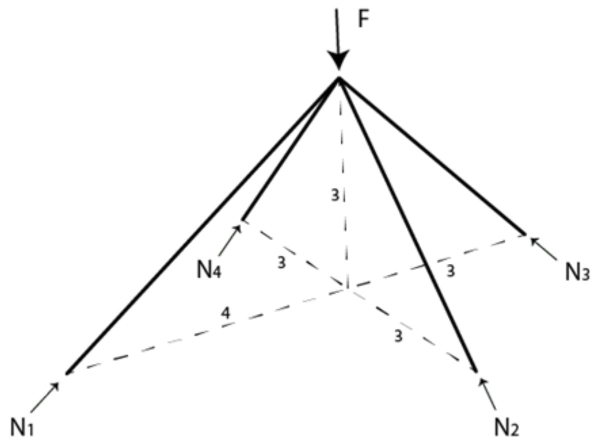


Figure 26 Statically indeterminate structure (Own image)

6.4 Force density

The force density method is commonly used as a form-finding tool. The method describes the relationship between forces and the length of a member. The force density method is based on the mathematical assumption that the ratio between the length and tension of each cable element is a constant value [Liem, 2011]. The force density is described as:

$$Force\ Densitiy = \sum \frac{N_i}{l_i}$$

In this example with four members we get:

$$Force\ Densitiy = \frac{N_1}{l_1} + \frac{N_2}{l_2} + \frac{N_3}{l_3} + \frac{N_4}{l_4}$$

From the equilibrium in z-direction we found:

$$\frac{3}{5}N_1 + \frac{1}{2}\sqrt{2}N_2 + \frac{1}{2}\sqrt{2}N_3 + \frac{1}{2}\sqrt{2}N_4 = F$$

If we multiply the Force density by 3, we get the same equation as the z-equilibrium shown above. This means that:

$$Force\ Density = \frac{N_1}{5} + \frac{N_2}{3\sqrt{2}} + \frac{N_3}{3\sqrt{2}} + \frac{N_4}{3\sqrt{2}} = \frac{1}{3}F$$

After testing this force density formula for multiple heights of z, the constant factor of the force density can be described as:

$$Force\ Density = \frac{1}{z}F$$

This relation means that the equilibrium can be scaled in the z-direction.

6.5 Reciprocal figures and optimal load path

In 1870 James Clerk Maxwell wrote the ground-breaking paper, "On Reciprocal Figures Frames and Diagrams of Forces" (Maxwell, 1870). In this paper, Maxwell discusses the relation between form and force diagrams. Starting with the concept of Rankine, explaining the principle of the equilibrium of polyhedral frames, he extended the work to show how specific spatial structures have reciprocal diagrams which represent the forces in the trusses. (Baker, 2013) This was the birth of the form-force or Cremona diagrams earlier described.

Figure 27 shows the relation between the form and force of a single statically indeterminate structure. The top left shows a branching structure with an external force F. Bottom left shows a possible equilibrium of this branching structure. On the right, the force diagram is rotated and also indicates stability. Force becomes length and vice versa. The length of the members left correspondents with the normal forces in the members on the right.

Based on this principle Liew, Pagonakis, Van Mele and Block (2018) described a load-path optimization tool. This tool is based on the minimal volume of the structure.

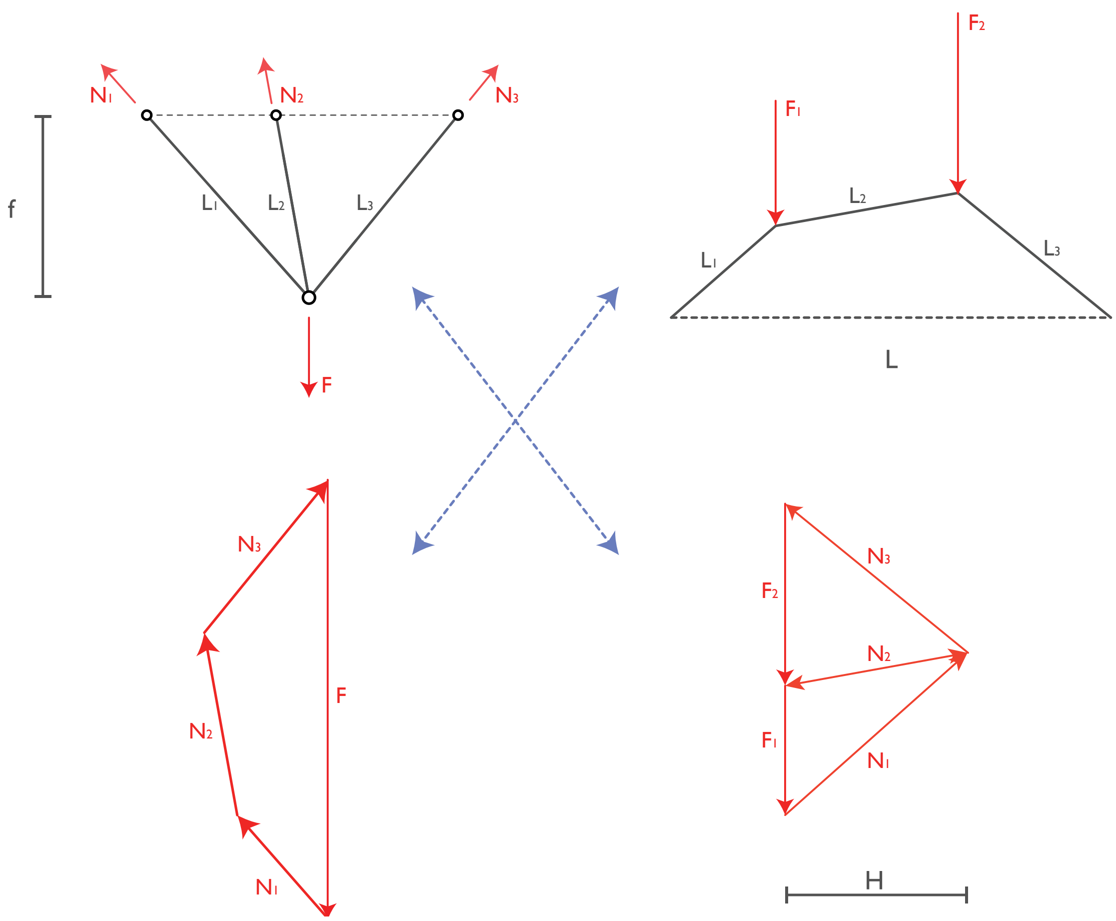


Figure 27 Reciprocal forces and lengths (Own image)

The internal forces are described as a load-path. By minimalizing the volume, the load path will be optimized and thereby the structure. Multiple studies have proven that this works for given stresses in each member such that:

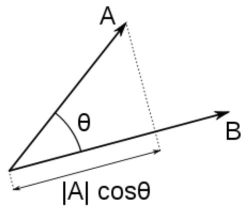
$$\min \sum V_i = \min \sum A_i l_i = \min \frac{1}{\sigma} \sum |F_i| l_i$$

Vi is the volume, Ai the cross-sectional area, li the member length and Fi, the axial force of the member. From Maxwell's theorem can be derived that:

$$\sum |F_i| l_i = \sum \vec{\mathbf{P}}_j \cdot \vec{r}_j$$

Pj*rj is the dot product of the external forces. The dot product represents the magnitude of the force along a distance vector. If A represents the force vector and B the direction vector, the dot product is:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |A| |B| \cos(\theta)$$



Splitting Maxwell's equation, we get the internal forces on the left side and the external forces on the right side:

$$(\sum F_i \cdot l_i)_{compression} + (\sum F_i \cdot l_i)_{tension} = (\sum \vec{\mathbf{P}} \cdot \vec{r})_{loads} + (\sum \vec{\mathbf{P}} \cdot \vec{r})_{reactions}$$

Because the construction only suffers from axial forces, either compression or tension, the formula can be rewritten. In this case, only compression is present. Also, the dot product can be written as such:

$$(\sum F_i \cdot l_i)_{compression} = |F| |r| \cos(\theta)_{loads} + |F| |r| \cos(\theta)_{reactions}$$

Using this equation, the optimal volume to height ratio can be found. This can be a powerful tool in the optimization of the branching columns.

7. Reversed problem

7.1 Actual design problem

In the previous examples, the approach was as if the tree was designed from top bottom to top. This is useful to describe the distribution of forces through a statically indeterminate structure. For the design of a branching structure supporting a freeform roof such as the Westfield mall in Londen (figure 28), the approach is different. In this case, we have a set of points where the branches are attached to the roof. On these points, a vertical force is applied as a result of the dead load of the roof and snow or wind loads. Because of the free-formed shape of the roof, these vertical forces differ on every point. This gives a unique set of points with corresponding unique forces.

Figure 28 Westfield shopping mall.

7.2 Simplification

To solve this design problem first a simplified model is made. From the first schematization, step-by-step the model will be extended to look more like the actual design problem. The final solution has to be a multilayered branching structure supporting a unique set of points.

The first step is to schematize for one layer. In principle, this is the same structure as the one described in the chapters concerning the complementary energy. Four members connected in the bottom point and horizontally fixed in the top points by the roof. One (vertical) force at the point where the members come together and a horizontal and a vertical force in the top.

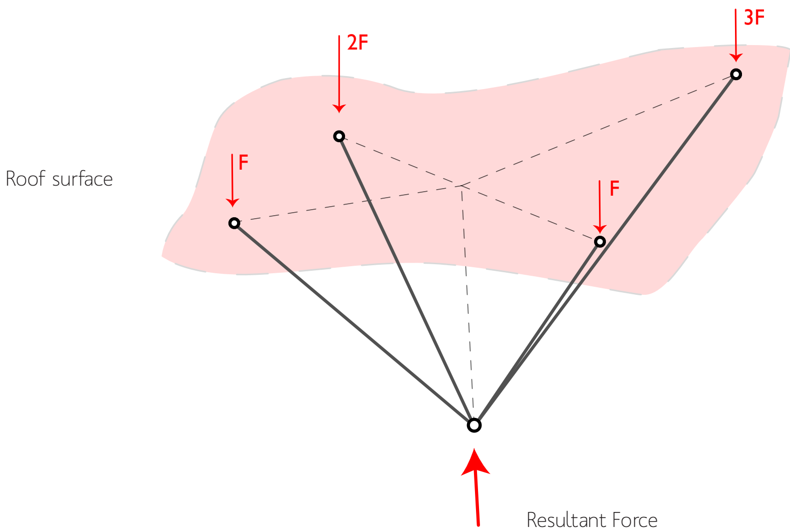


Figure 29 Reversed problem (Own image)

This first step is shown in figure 29. Four forces in a plane with acting on four members. The forces are unequally distributed, and the reaction force is the resultant of the four forces. To be stable structure, the condition is that the resultant force only has a vector in the z-direction and therefore is vertical.

To make this stable equilibrium, a point on the x,y plane must be found where the resultant force only has a vertical component. In this point, the resultant of all horizontal forces must be zero. Notice that there are horizontal forces due to the inclination of the members, but the sum in x and y-direction has to be zero.

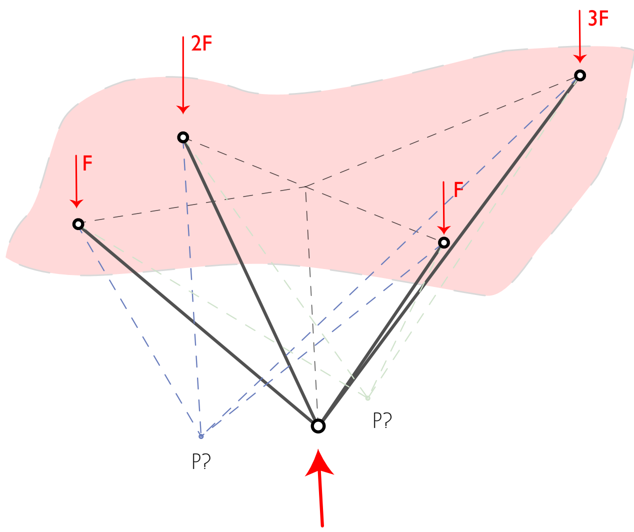


Figure 30 Finding the point of equilibrium (Own image

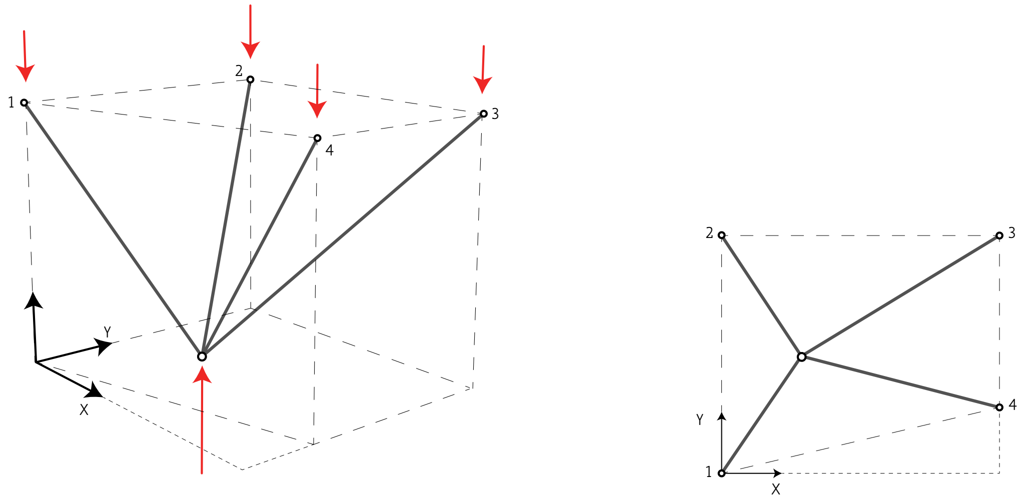


Figure 31 Vertical (in z-direction) equilibrium (Own image)

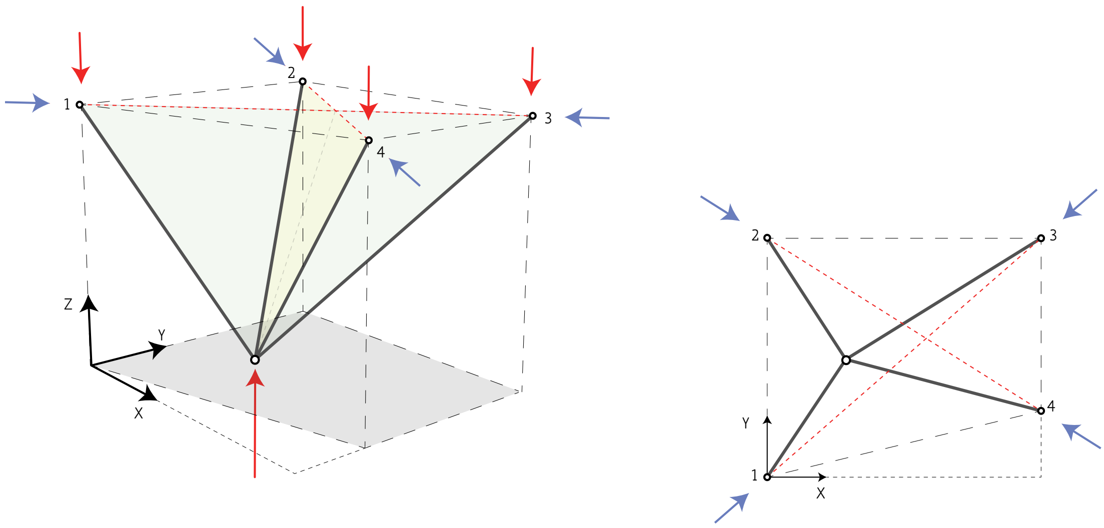


Figure 32 Horizontal forces due to inclination of members (Own image)

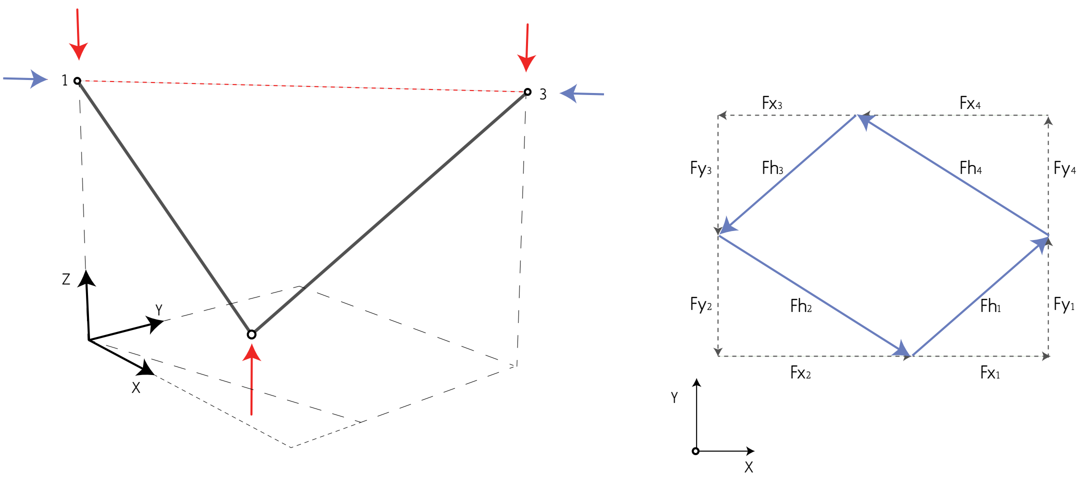


Figure 33 Equality of opposite forces and horizontal closed force polygon (Own image)

7.3 Searching the equilibrium

An equilibrium in the x,y plane has to be made. Therefore we can look at the problem from the top view. In Figure 31, on the left, the structure is drawn and on the right the top view. By analyzing the structure, two triangles can be found. One triangle between 1-point-3 and the other at 2-point-4. Figure 32 shows the triangles and the corresponding horizontal forces.

To make an equilibrium the x,y plane these horizontal forces have to be equal. By observing triangle 1-point-3, we can conclude that the horizontal force in 1 has to be the same magnitude as the horizontal force in 3.(Figure 33, left). If these forces are uneven, the triangle will tilt over. We can compare it to a game of tug of war. When both teams pull with the same amount of strength, the rope will stay stable in the middle. If one is more significant, the system will become unstable and starts to move.

Now we know that the horizontal forces have to be equal, we can look at the equilibrium from the top view again. Here we see that:

Fh1 = -Fh3 and Fh2 = -Fh4.
Also: Fx1 + Fx2 = -Fx3 + Fx4 and, Fy1 + Fy4 = -Fy2 + Fy3

This gives a better understanding of the positioning of the point. The positioning of the point must ensure that:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

The vertical forces on the structure are known, the corresponding horizontal force can be expressed in a factor of the vertical force. Due to the proportions, each member can be seen as a triangle. The difference in height between the start and endpoint is mentioned as dz, the distance in the x-direction is called dx, and the space between the y coordinates is called dy.

Maxwell's theorem about the reciprocity of the form and force diagrams describes that the force can be expressed as a factor of the length. The length is known by the geometry of the structure and all the other forces can be found using equality.

7.4 Proportions of form and force

With the knowledge of equality between form and forces, it is possible to describe all the forces in terms of the length. The most important is to notice that the vertical force corresponds with the difference in height. Since Fv is the only known force, the length dz can be used to scale the other forces.

Figure 34 shows the different lengths and the related internal or external forces. First, let's look at the normal force Ni in the member, the length is length li. These relate to each other in the same way that Fv and dz do so it can be described as:

$$\frac{F_{v,i}}{d_z} = \frac{N_i}{l_i}$$

By converting the equation, by multiplying both sides with li, the normal force Ni can be described as a factor of length and Fi,v:

$$N_i = \frac{l_i \cdot F_{v,i}}{d_z}$$

All the other forces can be described in the same way:

$$\frac{F_{v,i}}{d_z} = \frac{F_{h,i}}{d_i} \qquad \frac{F_{v,i}}{d_z} = \frac{F_{hx,i}}{d_x} \qquad \frac{F_{v,i}}{d_z} = \frac{F_{hy,i}}{d_y}$$

$$F_{h,i} = \frac{d_i \cdot F_{v,i}}{d_z} \qquad F_{hx,i} = \frac{d_x \cdot F_{v,i}}{d_z} \qquad F_{hy,i} = \frac{d_y \cdot F_{v,i}}{d_z}$$

With this knowledge, all forces can be calculated as a factor of Fv. Because a member always is a straight line between two points, this line or length can be written as the difference of those points:

$$l_i = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \qquad l_i = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Naturally, this same principle works for the forces:

$$N_i = \begin{pmatrix} F_{hx,i} \\ F_{hy,i} \\ F_{v,i} \end{pmatrix} \qquad N_i = \sqrt{(F_{hx,i})^2 + (F_{hy,i})^2 + (F_{v,i})^2}$$

In the design of the branching column, the aim is to find a construction with only normal forces and no shear forces. Because the normal force always has the same direction as the member (axial force), the resultant force is the normal force. This will play an essential role in the design of the final multilayered structure.

$$x_p = \frac{\sum(F_n \cdot x_n)}{\sum(F_n)} \qquad y_p = \frac{\sum(F_n \cdot y_n)}{\sum(F_n)}$$

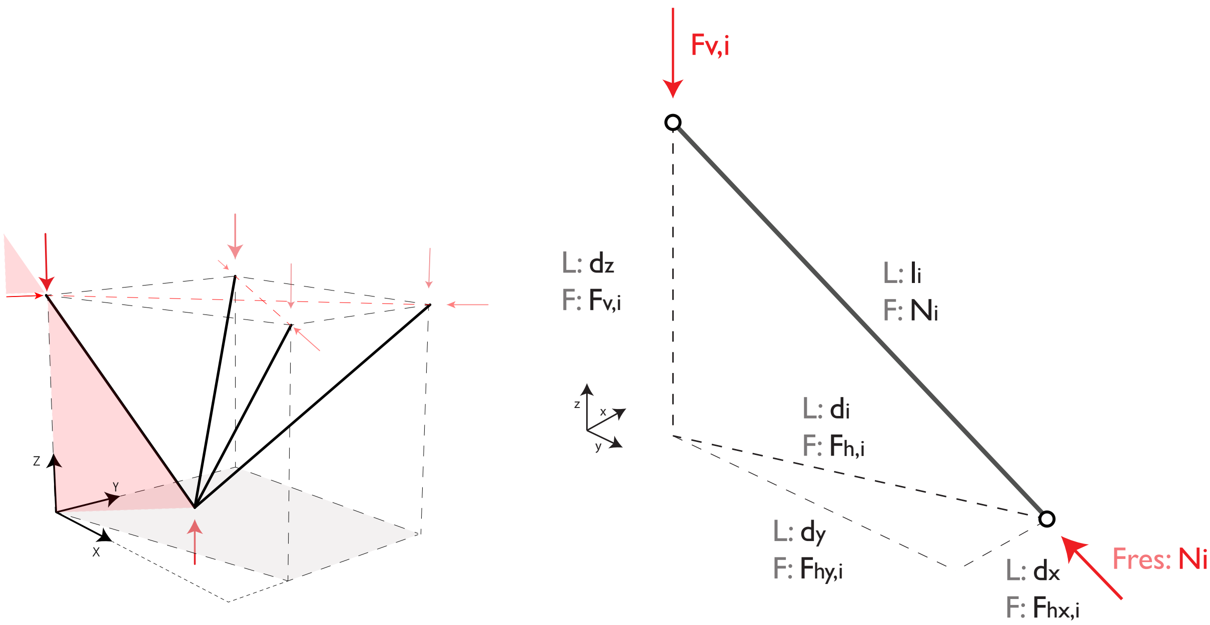


Figure 34 Proportions of force and length (Own image)

7.5 Multiple layers

The design of a branching column consists of multiple layers. Therefore the next step is to look at the second layer. The conditions stay the same. At the bottom support, the resultant force has to be vertical, and the top branches are loaded with unequal forces in a plane. As a result of the external forces F1 to F4, a resultant force is transferred to the next structural layer.

To create a structure which only suffers from axial forces, the resultant force of the top branch must follow the same direction as the next structural member. This second layer condition can be found in Figure 35. For each structural layer, the resultant of that layer should follow the next member resulting in a vertical member. Multiple branching points must be found where these conditions are respected.

Each point of branching has three unknowns, the x, y and z coordinate. By describing multiple mathematical equations, the number of unknowns can be decreased. Also in this phase of the problem, the proportions are critical.

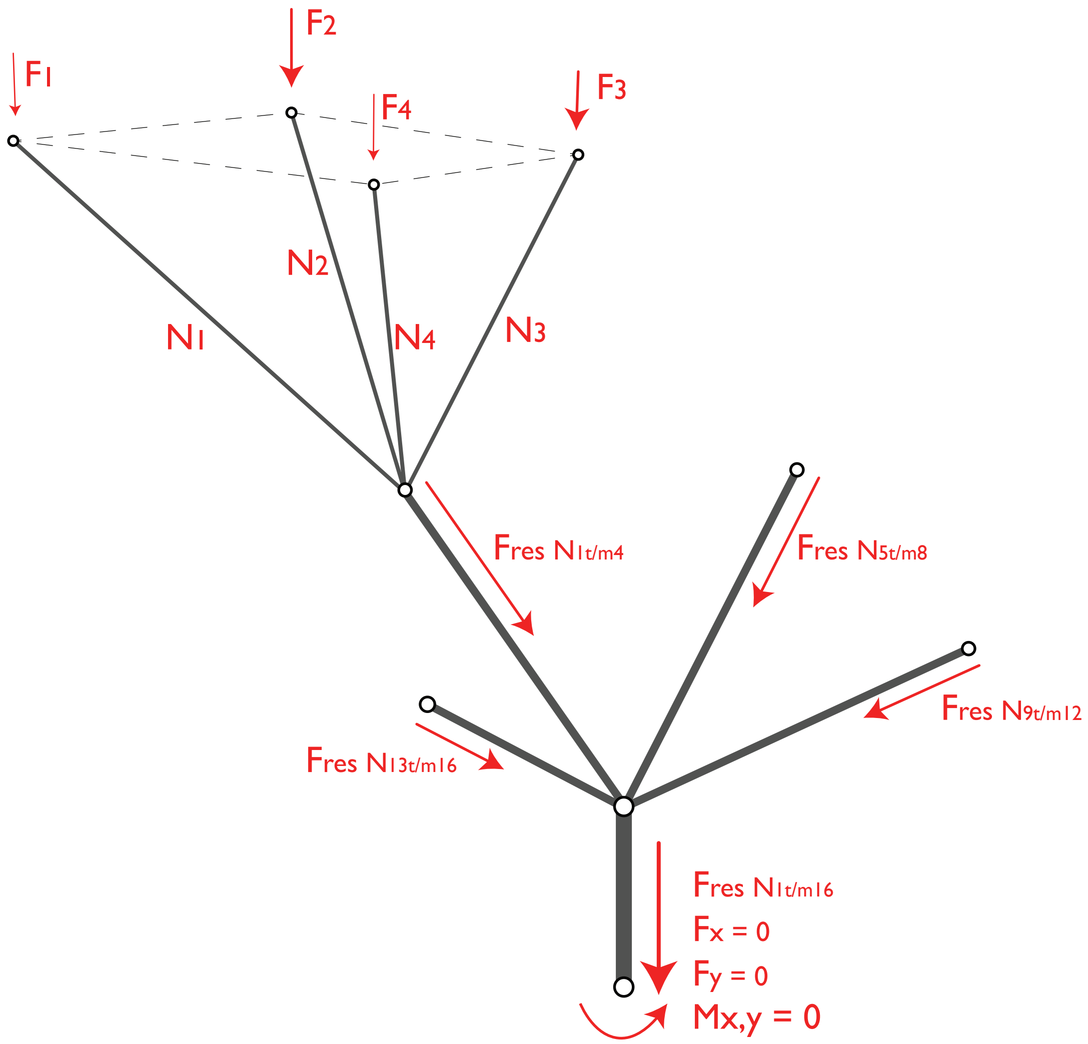


Figure 35 Solving multiple layer branching equilibrium (Own image)

The vertical force still is the only known force which is independent of the shape of the branching column. Also in the second branch, the vertical force will be used to describe the horizontal forces. In figure 36 the proportions can be found. With the same method used in chapter XXXX, the relations between force en distance can be derived:

$$\sum_{r=1}^{n} Fx_r = \frac{dx_{next\ member} \cdot \sum_{r=1}^{n} Fz_r}{dz_{next\ member}}$$

$$\sum_{r=1}^{n} Fy_r = \frac{dy_{next\ member} \cdot \sum_{r=1}^{n} Fz_r}{dz_{next\ member}}$$

If both these conditions are applied, the next member will only suffer from axial stresses. With the use of the complementary energy method and Maxwell's load path optimization the optimal it must be possible to find the most optimal branching point (x,y,z). If these points are found, the optimal construction can be found for a given problem.

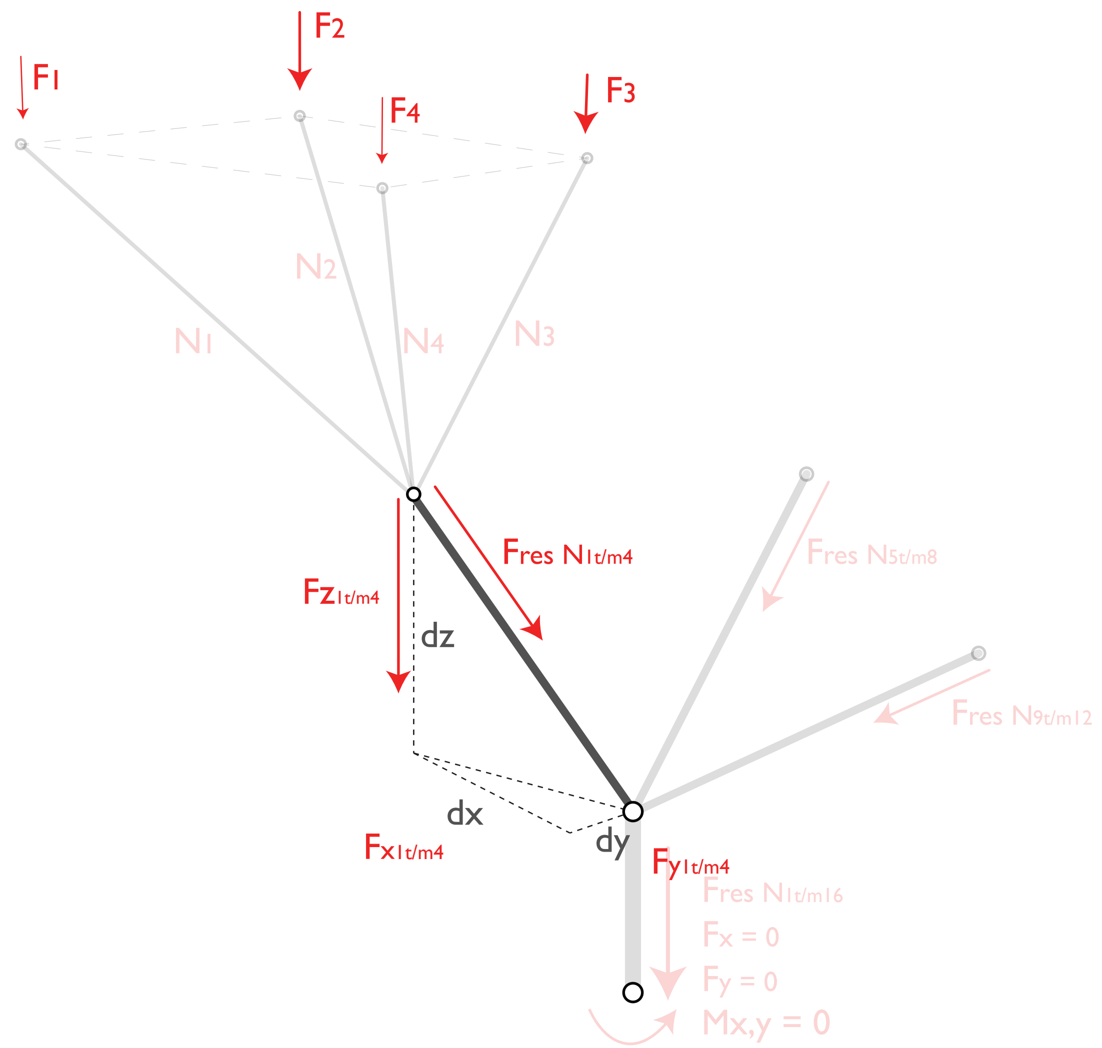


Figure 36 Proportions of second layer member (Own image)

7.6 Example

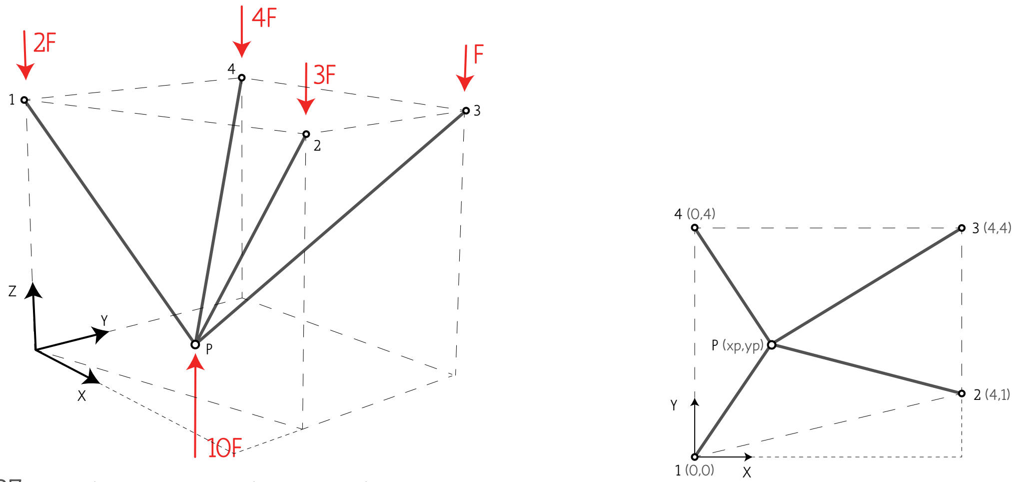


Figure 37 Example configuration (Own image)

Consider a one layer construction. The same as in the previous examples but with external loading. There acts a force of 2F, 3F, F, and 4F on respectively point 1,2,3 and 4. The resultant force in point p (xp, yp, zp) is 10F and has to be vertical.

In order to find the point where the horizontal forces are in equilibrium, the coordinates of the point in x and y-direction have to be found. The summation of the force times the distance in each point divided by the total force will give the point of stability.

$$x_p = \frac{\sum(F_n \cdot x_n)}{\sum(F_n)} \qquad y_p = \frac{\sum(F_n \cdot y_n)}{\sum(F_n)}$$

Filling in the equation gives:

$$x_p = \frac{(F_1 \cdot x_1 + F_2 \cdot x_2 + F_3 \cdot x_3 + F_4 \cdot x_4)}{(F_1 + F_2 + F_3 + F_4)}$$
$$x_p = \frac{(2 \cdot 0 + 3 \cdot 4 + 1 \cdot 4 + 4 \cdot 0)}{(2 + 3 + 1 + 4)}$$
$$x_p = \frac{8}{5}$$

$$y_p = \frac{(F_1 \cdot y_1 + F_2 \cdot y_2 + F_3 \cdot y_3 + F_4 \cdot y_4)}{(F_1 + F_2 + F_3 + F_4)}$$
$$y_p = \frac{(2 \cdot 0 + 3 \cdot 1 + 1 \cdot 4 + 4 \cdot 4)}{(2 + 1 + 1 + 3)}$$
$$y_p = \frac{23}{10}$$

The point p (8/5, 23/10, Zp) now only has one unknown, the distance in the z-direction. The method of force density showed that it is possible to scale the construction in the z-direction. To find the optimal z-coordinate of the point, Maxwell's equation of the minimal volume problem is used. In the previous chapter is derived that:

$$(\sum F_i \cdot l_i)_{compression} = (\sum \vec{\mathbf{P}} \cdot \vec{r})_{loads} + (\sum \vec{\mathbf{P}} \cdot \vec{r})_{reactions}$$

Within this configuration, the point chosen to calculate maxwell is of importance for the angle of the dot product. By choosing it's place wisely, the size of the formula can be decreased. In this case point, P is chosen. This point is chosen because the distance vector of the reactions is zero and the angle of the loads is zero. Therefore the dot product of the reaction is zero and the cosine of zero degrees is 1. The only thing left is the summation of the normal forces times the length of the members.

$$(\sum F_i \cdot l_i)_{compression} = |F||l|\cos(0)_{loads} + |F||0|\cos(\theta)_{reactions}$$

$$(\sum F_i \cdot l_i)_{compression} = |F||l|_{loads}$$

$$Maxwell = (\sum N_i \cdot l_i)_{compression}$$

With the x and y coordinates known, the only unknown variable is the height z. Filling in the equation with the proportions of length and force gives:

$$N_i = \frac{l_i \cdot F_{v,i}}{d_z} \qquad l_i = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$Maxwell = (N_1 \cdot l_1 + N_2 \cdot l_2 + N_3 \cdot l_3 + N_4 \cdot l_4)$$

$$Maxwell = \frac{100 \cdot dz^2 + 785}{50 \cdot dz} + \frac{3 \cdot (100 \cdot dz^2 + 745)}{100 \cdot dz} + \frac{100 \cdot dz^2 + 865}{100 \cdot dz} + \frac{100 \cdot dz^2 + 545}{25 \cdot dz}$$

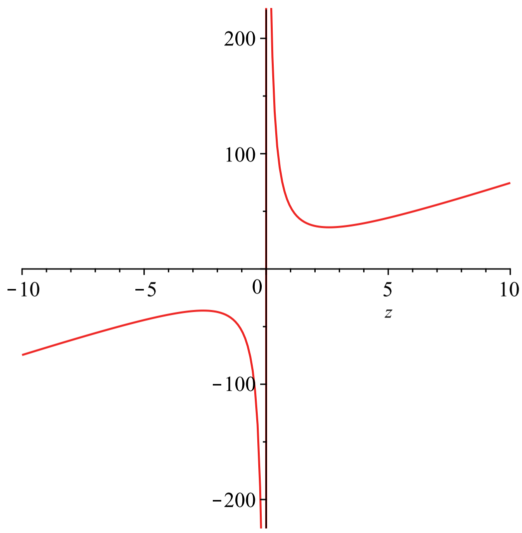
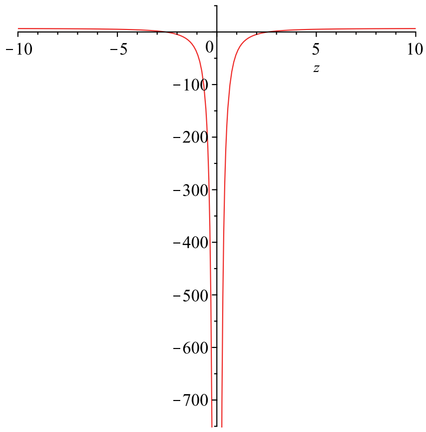


Figure 38 Plot of maxwell's equation (Own image)

Searching the local minimum is done by differentiation over dz and equate to zero.

$$20-\frac{100\cdot dz^2+785}{50\cdot dz^2}-\frac{3\cdot(100\cdot dz^2+745)}{100\cdot dz^2}-\frac{100\cdot dz^2+865}{100\cdot dz^2}-\frac{100\cdot dz^2+545}{25\cdot dz^2}=0$$



$$dz=\frac{\sqrt{685}}{10}$$

Solving the equation gives an optimal branching position, derived from the minimal volume theorem, of approximately z=2.62. In order to check whether this is also the correct in the other direction, the complementary energy method can be used. The same configuration but up-side-down is used. Four members with a force of 10 F vertical on point P. If this configuration gives the same results as the input, we can conclude that this is the optimal branching point.

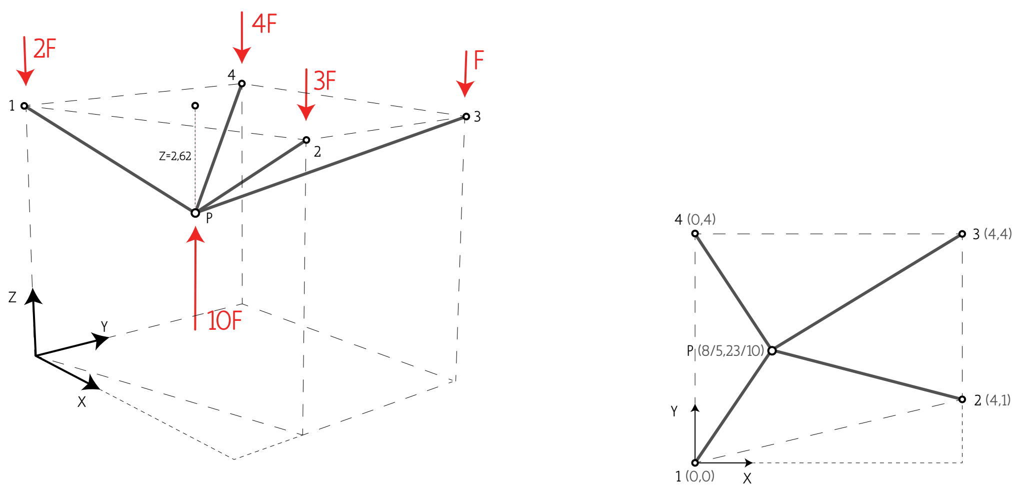


Figure 39 Optimal height branching point (Own image)

The geometry for calculating the minimum complementary energy:

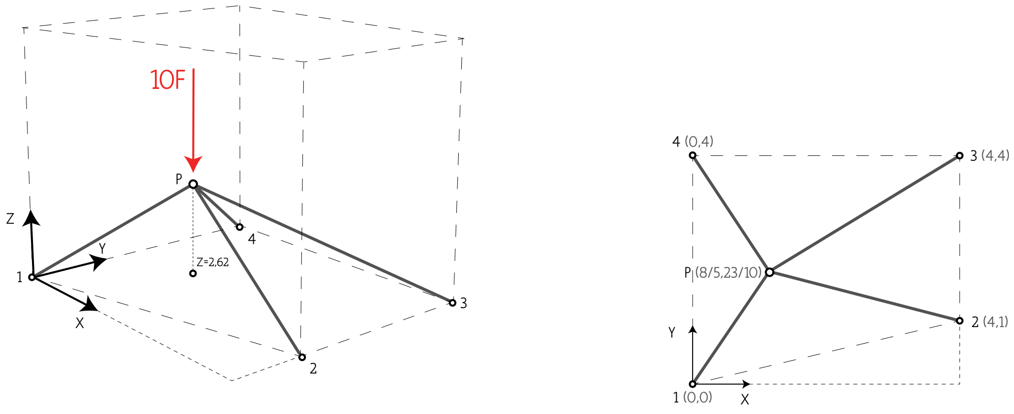


Figure 40 Reversed calculation branching point (Own image)

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$E_{compl,total} = \sum_{i=1}^{n} F_i^2 l_i = minimum$$

Just as the example the four conditions must be applied. Horizontal, vertical equilibrium and the least complementary energy has to be found. The same procedure as in chapter 6.2 has to be followed in order to decrease the amount of unknown variables to one. Differentiating the Energy equation over the last unknown and equate to zero gives the solution for the statically indeterminate structure. In Figure 41 the solution to this problem is given. Comparing this solution to the initial input of the problem, a discrepancy can be seen. In the input, the forces were 2F, 3F, F, and 4F. With the Complementary energy, the found vertical reaction points are 2,64F, 2.15F, 1.85F, and 3,36F. Both structures are in equilibrium, and both have the same vertical force component.

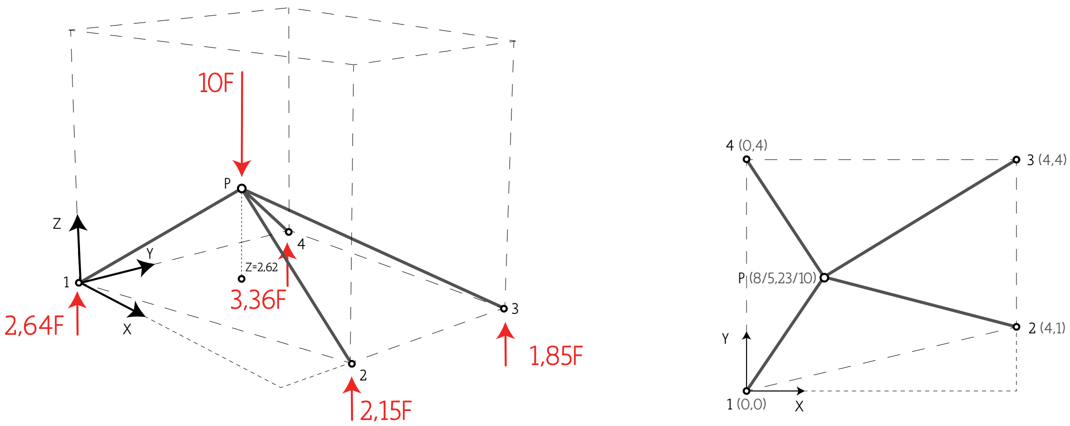


Figure 41 Solution of minimal complementary energy (Own image)

7.7 Conclusion reversing the structure

This discrepancy seems off in the first place. The theory about reciprocity of forces would suggest that the outcome has to be the same as the input. After all, the minimal volume solution is considered the optimal load path and the complementary energy solution is considered the least energy consuming distribution of forces. Both solutions should match.

If we look at the system again from another angle, the solutions make more sense. The flow of forces throughout a structure often is compared to the flow of water, choosing the path of the least resistance. In this case, changing the forces into water will give the solution of the discrepancy. In figure 42 a 2D version of the problem is shown. On the left, the structure derived from Maxwell's equation, the optimal load-path for a given load. On the right, the solution based on Complementary energy.

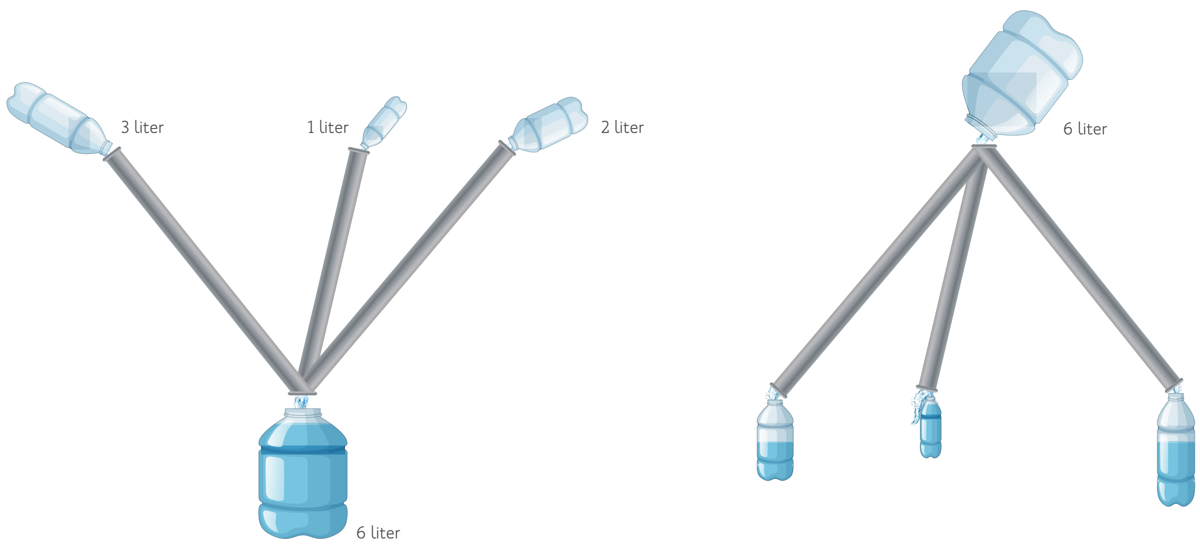


Figure 42 Problem of reversing the structure (Own image)

Looking at these images, it makes sense that there is a difference in the outcome. While the left gives a perfect stable solution for the given input, the right shows another distribution of water within the same configuration and the same total amount of water. This leads to the realization that:

The optimal configuration can be found for a given load case.
This load case, however, is not the most efficient distribution of forces within in the found configuration.

With this realization, the steps that need to be taken to solve the problem become clearer. To solve the compression only branching structures, the system does not need to be approached as a set of members. Instead of members, the branches need to be examined as working lines of forces. When the problem is examined as done before, the structure is analysed as if it would consist of members. The material properties are taken into account in the calculation of the elongation or shortening. However, the structure needs to be examined as being paths of forces redirected to one point.

Before, the model of frei otto was described as being the optimal branching pattern. For this distributed load case, this pattern was considered the best solution. Logically with the history of hanging models and reversibility of the forces in mind.

Looking at the model now, the realization comes that this is only the optimal solution due to the given properties. The strings of the funicular network have a certain thickness and length. Due to these properties, the model will be formed. For instance, if the strings were made of a more elastic material, the elongation would be larger. This means that this model only is formed this way because of the (design) decisions made earlier.

To create the optimal branching solution, the system needs to be examined as a combination of force paths, without any other properties.

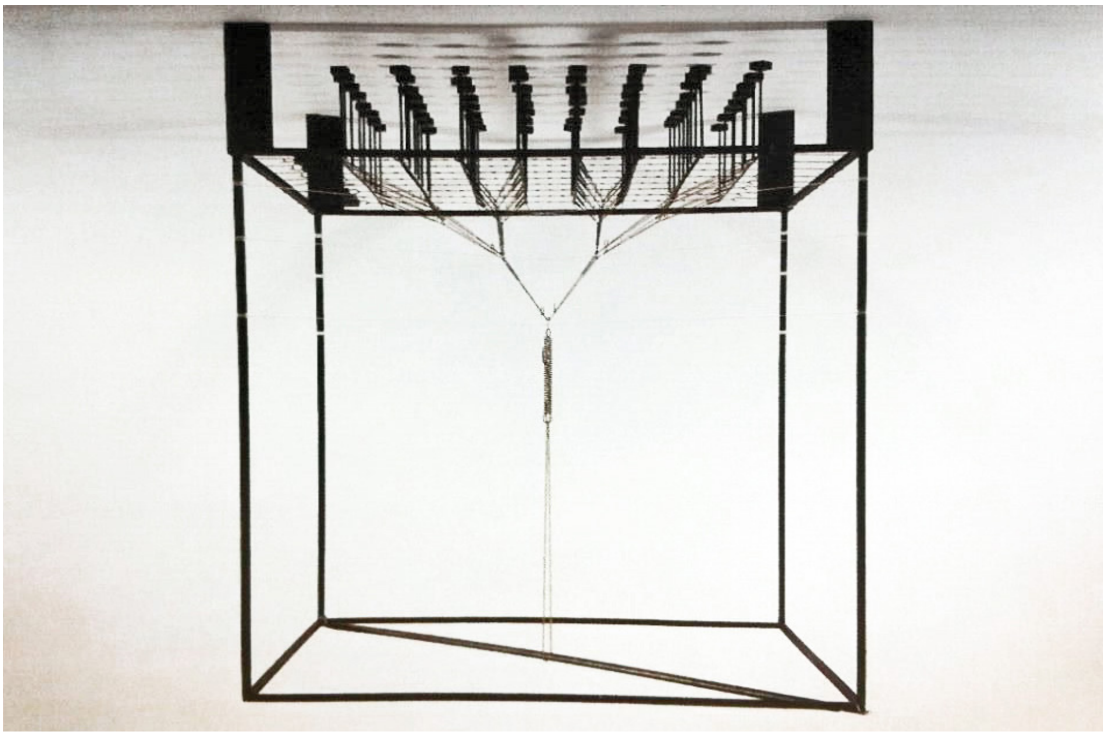


Figure 43 Reversed hanging model by Frei Otto

8. Working lines

8.1 2D horizontal

With the realization of the previous chapter, the design problem can be put into a different light. Not the least energy consuming solution has to be found to make a compression-only structure, but the working lines of the forces are of importance.

Scaling the problem back into a 2D structure with three members, the problem becomes manageable. Considering the structure given in Figure 44.

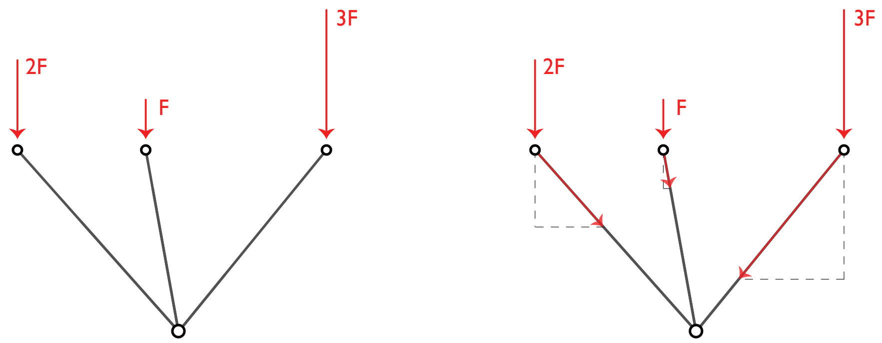


Figure 44 2D configuration and the resulting forces (Own image)

Three vertical forces are acting on the structure. As a result, the three members suffer from an axial force. Size of the normal force is a result of the angle, force size, and member length. On the right, the size and direction of the axial force are shown. In order to make a stable equilibrium, the head-tail method can be used. Placing the forces of the members in a row, the resulting force must close the polygon. This way, the reaction force makes an equilibrium in point p (Figure 45)

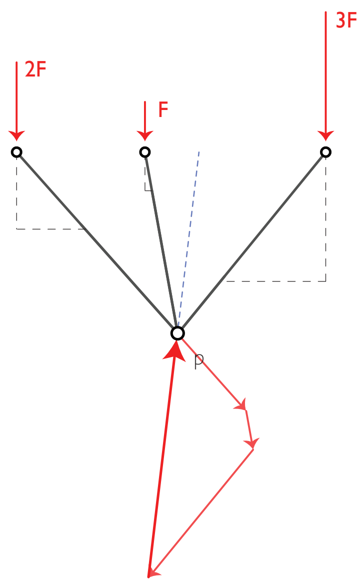


Figure 45 Closed force polygon, reaction force (Own image)

Changing point P changes the direction and size in all three members. This resultant force logically changes too. In Figure 46, the point has changed to a new position. Closing the polygon gives the resultant force. The forces are different from the previous example, but there is one common factor. The direction lines of the resulting forces both go through the same point. No matter where point P is placed, the resultant force goes through the same point.

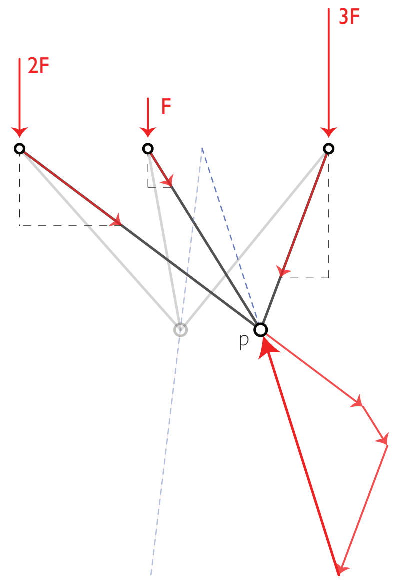


Figure 46 Moving the branchin point (Own image)

After analyzing this point, it appears to be the same point found when searching the x and y coordinates in the example. It turns out that this is the center of forces and can be found by:

$$x_p = \frac{\sum(F_n * x_n)}{\sum(F_n)}$$

That the center of forces is the same point as the point found earlier makes perfect sense. In the example, the search was for a point where there was only a vertical resultant force. When the point is placed vertically below the center of forces, the resultant will be vertical as well. In the example, it was possible to scale the construction in the z-direction. With the new insight, this means that the construction can be scaled along the working line.

By scaling the structure along the working line, the resultant force stays the same. In figure 47 a movement of the branching point along the line can be seen. As shown, the resultant stays the same. This because both the sum of the vertical forces and the angle relative to the center of forces remain equal. The axial forces in the three bars differ because the angle of the member changes but the resultant stays the same. A conclusion that can be drawn from this insight is that the points of branching can be scaled/ moved along the working line. Every point on the line gives the same resultant force, both in size and direction. Therefore, when a point is found with only axial forces in the system, the point can be moved without causing shear force/bending moments.

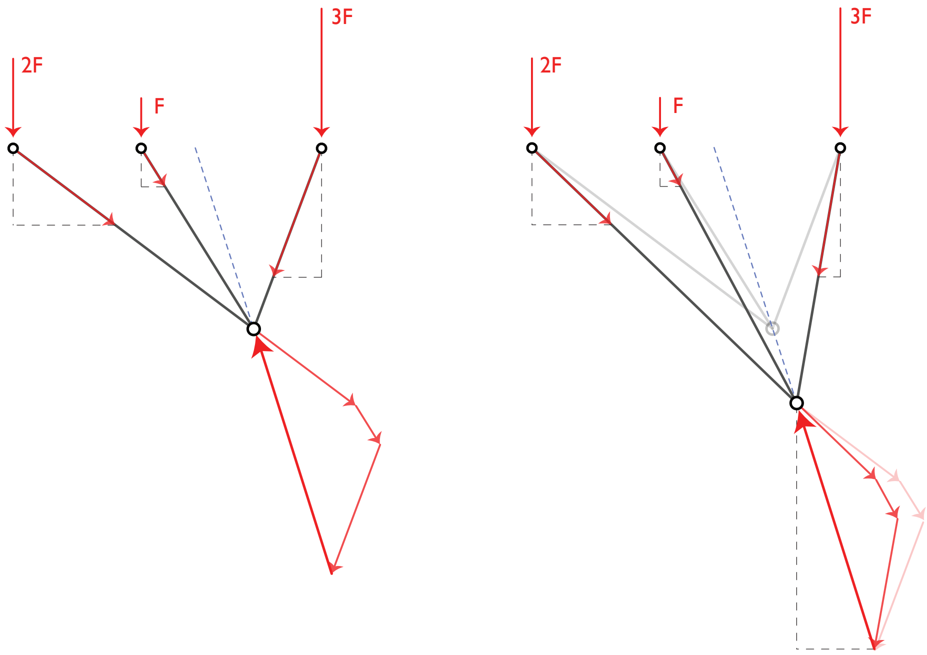


Figure 47 Scaling the branching point along working line (Own image)

Finding the branching point is the next step in solving the problem. To find the branching point, the next branching point or end of next member must be known. If this is the case, finding the point is possible. Because the resultant force follows the work line to the center of forces, the point has to be placed on the working line between the next point and the center of forces. (Figure 48) By placing the point, the resultant follows the same direction as the next member. This point meets the condition previously described:

$$\sum_{r=1}^{n} Fx_r = \frac{dx_{next\ member} \cdot \sum_{r=1}^{n} Fz_r}{dz_{next\ member}}$$

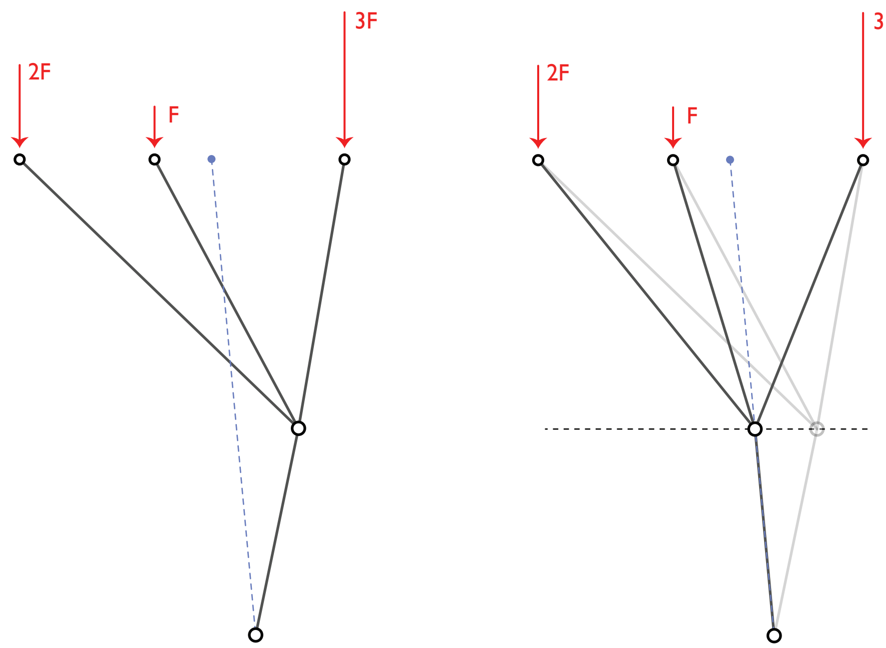


Figure 48 Move branching point to exclude bending moments (Own image)

8.2 3D horizontal

Now the branching points in 2D are found, it is time to look at 3D branching structures. The only difference is an extra dimension, y. The y-coordinates can be found in the same way as the x-coordinates. Combining the two coordinates, a working line in 3D can be made. In figure 50 an example is shown in 3D. Both structures only suffer from axial forces with a resultant vertical force.

Scaling the construction doesn't affect the way it works. The only change is: the horizontal forces in the top become bigger if the branching point is moved upwards. Due to the change of angle and the constant vertical force.

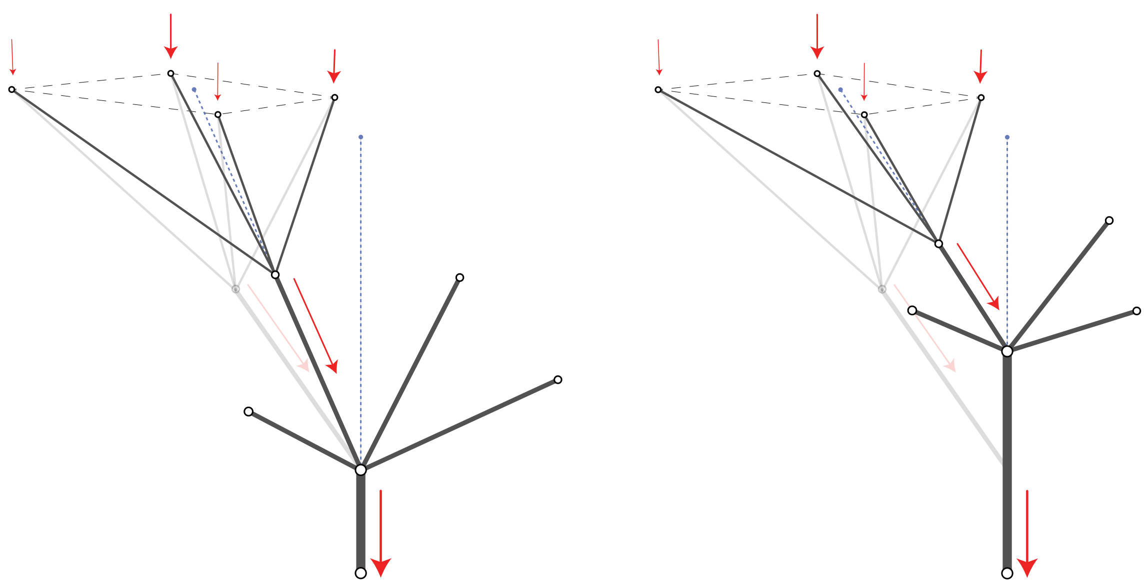


Figure 49 Scaling a 3D branching column along working lines (Own image)

8.3 2D non-linear structure

Until now, the upper points were situated on a horizontal line or in a plane. The goal of this thesis, however, is to find the optimal branching structure supporting freeform architectural surfaces. Freeform waving surfaces such as the roof of Westfield mall in London. In order to design compression-only branching columns for these kinds of roofs, the points have to be out of the plane. Firstly the two-dimensional branching structure will be studied. Figure 50 shows a simple branch with two points out of plane.

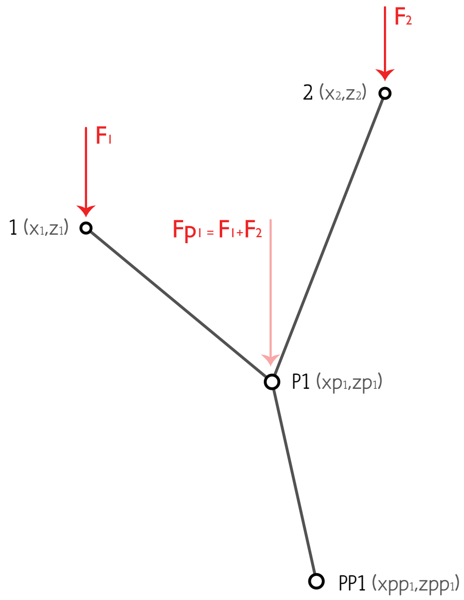


Figure 50 2D branch, non-lineair (Own image)

Also in this freeform branch, the conditions are the same. To ensure that there is no shear force in the construction the following has to be fulfilled:

$$\sum_{r=1}^{n} Fx_r = \frac{dx_{next\ member} \cdot \sum_{r=1}^{n} Fz_r}{dz_{next\ member}}$$

In this case, it means that:

$$\frac{(x_1 - xp_1) \cdot F_1}{zp_1 - z_1} + \frac{(x_2 - xp_1) \cdot F_2}{zp_1 - z_2} = \frac{(xp_1 - xpp_1) \cdot Fp_1}{zpp_1 - zp_1}$$

Within a structure, the points (1,2 and pp1) and forces (F1 and F2) are known. Therefore, the only unknown variables are xp1 and zp1. By rewriting the equation, the variable xp1 can be expressed as a function of zp1.

$$xp_1 = \frac{F_1 \cdot x_1 \cdot z_2 \cdot zp_1 - F_1 \cdot x_1 \cdot z_2 \cdot zpp_1 - F_1 \cdot x_1 \cdot zp_1^2 + F_1 \cdot x_1 \cdot zp_1 \cdot zpp_1 + F_2 \cdot x_2 \cdot z_1 \cdot zp_1 - F_2 \cdot x_2 \cdot z_1 \cdot zpp_1 - F_2 \cdot x_2 \cdot zp_1^2 + F_2 \cdot x_2 \cdot zp_1 \cdot zpp_1 + Fp_1 \cdot xpp_1 \cdot z_1 \cdot z_2 - Fp_1 \cdot xpp_1 \cdot z_1 \cdot zp_1 - Fp_1 \cdot xpp_1 \cdot z_2 \cdot zp_1 + Fp_1 \cdot xpp_1 \cdot zp_1^2}{F_1 \cdot z_2 \cdot zp_1 - F_1 \cdot z_2 \cdot zpp_1 - F_1 \cdot zp_1^2 + F_1 \cdot zp_1 \cdot zpp_1 + F_2 \cdot z_1 \cdot zp_1 - F_2 \cdot z_1 \cdot zpp_1 - F_2 \cdot zp_1^2 + F_2 \cdot zp_1 \cdot zpp_1 + Fp_1 \cdot z_1 \cdot z_2 - Fp_1 \cdot z_1 \cdot zp_1 - Fp_1 \cdot z_2 \cdot zp_1 + Fp_1 \cdot zp_1^2}$$

By filling in the equation, the size will decrease because of the large number of known variables. To make the formula more understandable, we can look at Figure 51.

Denominator	Numerator	z1	z2	zpp1	zp1
F1 * x1	F1		●		●
			●	●	
				●	●
					●●
F2 * x2	F2	●			●
		●		●	
				●	●
					●●
Fp1 * xpp1	Fp1	●			●
		●	●		
			●		●
					●●

Figure 51 Table of variables two branching points (Own image)

Each dot means a multiplication and each row represents a group of variables. For instance, the first row corresponds with: (F1 * x1 * z2 * zp1) in the denominator and (F1 * z2 * zp1) in the numerator.

When the construction has three points in the top branch, the number of rows per force doubles and each group of variables increases by one variable. This is shown in figure 52. Each point extra gives more and bigger sets of variables.

Denominator	Numerator	z1	z2	z3	zpp1	zp1
F1 * x1	F1		●	●	●	
				●	●	●
			●	●		●
			●		●	●
			●			●●
				●		●●
					●	●●
						●●●

Figure 52 Fragment of table of variables three branching points (Own image)

The amount of sets variables keeps on increasing with every point added to the construction. The quantity can be given by:

$$n^2 \cdot (n+1) \text{ times } (n+2) \text{ variables}$$

The amount of variables grows exponentially:

No. of points	1	2	3	4	5	6	7	8
Numerator	4 * 3 var.	12 * 4 var.	38 * 5 var.	80 * 6 var.	192* 7 var.	448 * 8 var.	1024 * 9 var.	2304 * 10 var.
Denominator	4 * 2 var.	12 * 3 var.	38 * 4 var.	80 * 5 var.	192 * 6 var.	448 * 7 var.	1024 * 8 var.	2304 * 9 var.

No. of points	9	10	11	12	13	14	15	16
Numerator	5120 * 11 var.	11264 * 12 var.	24576 * 13 var.	53248 * 14 var.	114688 * 15 var.	245760 * 16 var.	524288 * 17 var.	1114112 * 18 var.
Denominator	5120 * 10 var.	11264 * 11 var.	24576 * 12 var.	53248 * 13 var.	114688 * 14 var.	245760 * 15 var.	524288 * 16 var.	1114112 * 17 var.

Figure 53 Number of variables in describing the working line (Own image)

This means that the formula becomes extremely big if needed to calculate for instance a 16 point working line. However again, if the known variables are filled in, the size decreases to a maximum of n groups of variables with a constant and a power function of zp1.

8.4 Example

Now that the formula for the working line is known, it can be graphically represented. For instance a three-point structure. Three points out of a plane and the next branching point are known. With the equation 54 in the previous chapter, an expression of xp in terms of zp can be written. This gives a working line for the branching point (blue line Figure 55).

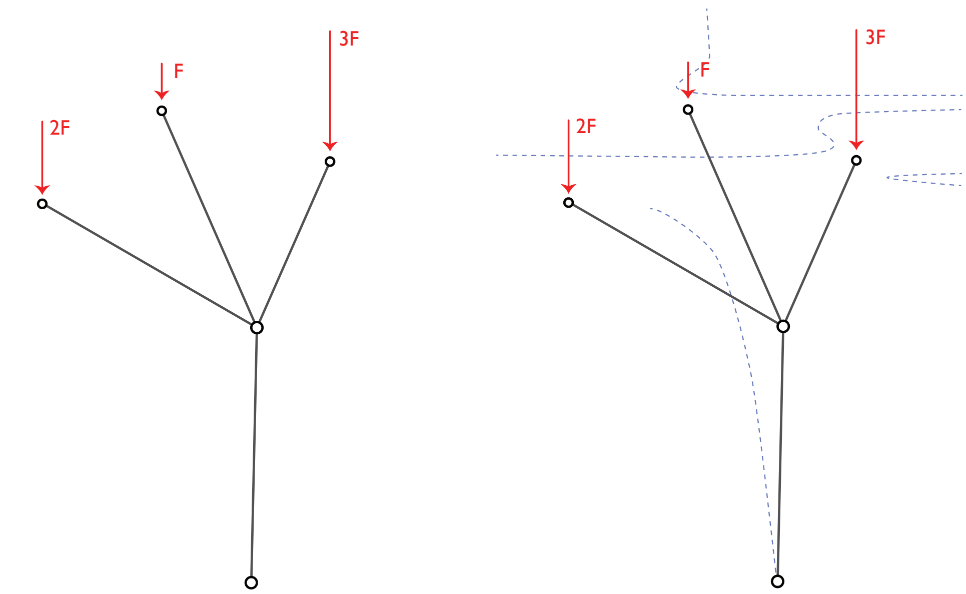


Figure 54 Plot of working line for a non lineair branch (Own image)

The same rules are followed for designing the structure as when the points were in a plain. When the branching point lies on the blue working line, the resultant force will be directed to the next point.

The working is a freeform line flowing between the points. In the example with points in a plane, the forces were scaled with the same magnitude. Now, the points are out of plain which leads to a different scaling of the forces. The further away the branching point is, the lesser the influence of the inequality of the points. Therefore, the working line shows a resemblance of a straight line far from the points of engagement. Closer to the points, the line starts to flow, and the points act as repellant points.

Because of the nature of the structure, the branching points will only be located beneath the upper points. Mainly following the (almost) straight curve.

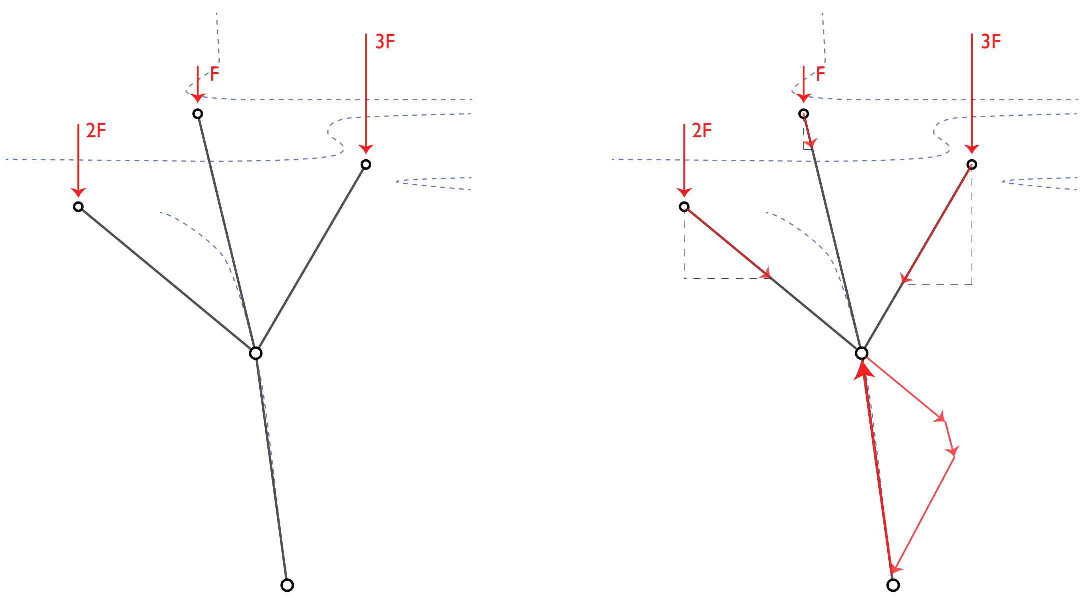


Figure 55 Moving the point to the working line, excluding bending moments (Own image)

8.5 3D non-linear structure

As seen in the previous chapter, a 2D working line can be plotted and described. This is also possible for a 3D structure. Combining the x and y coordinate will give a curve in three dimensions. An example can be found in Figure 56.

Combining the knowledge from the previous chapters will enable the possibility to design multi-points branching structures supporting freeform surfaces. In the design, there is only one rule: The branching point must be located on the working line. Along the curve, it can be scaled and moved and still only suffer from axial forces.

This line however still doesn't give 'the optimal solution'. Multiple possible branching points are given but how and where do we optimize the structure. This optimization process will be of significant influence for the outcome of the structure and will be treated in the next chapter.

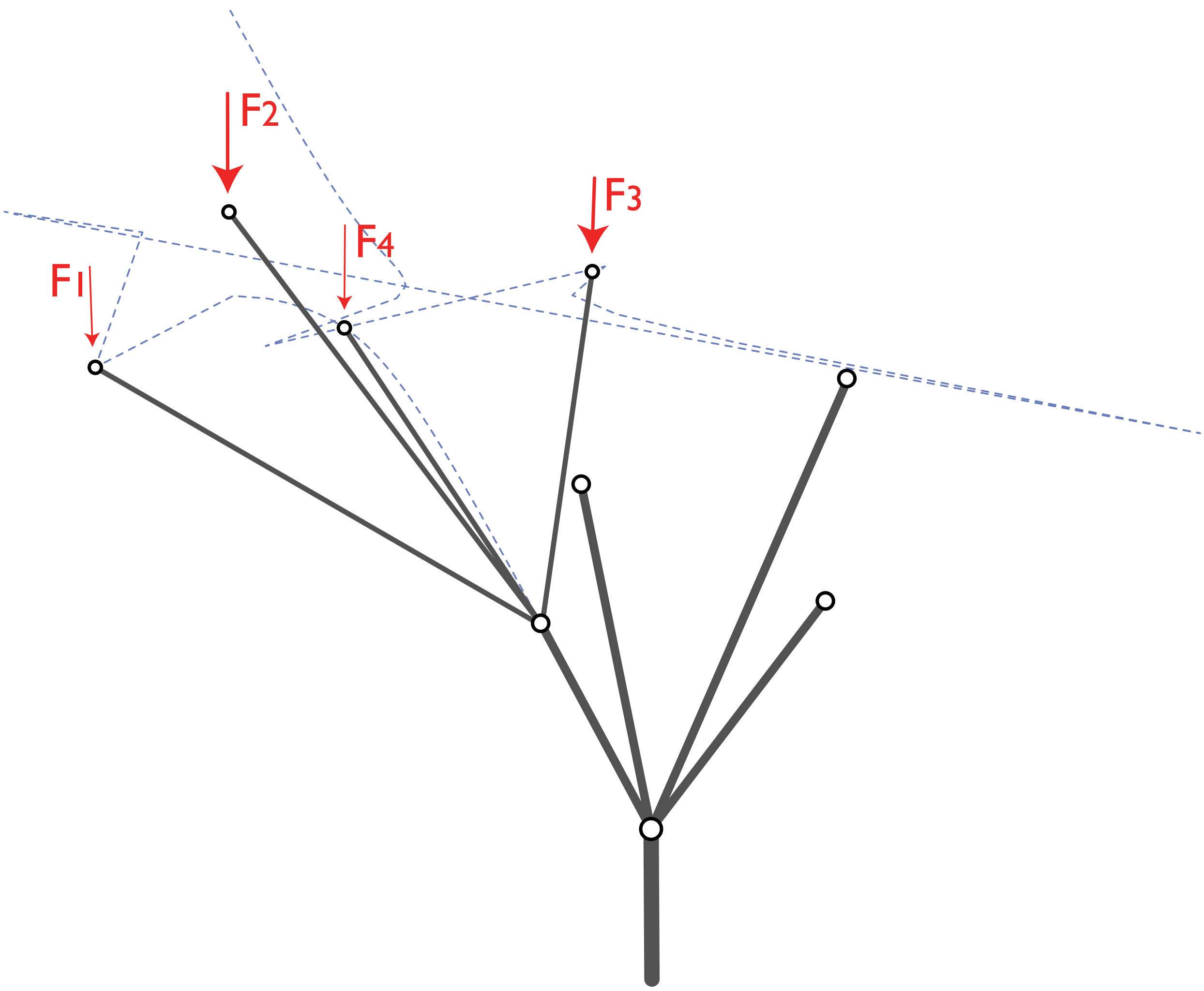


Figure 56 3D freeform points. Scaling along the working line (blue) (Own image)

9. Optimization of the branching column

9.1. Introduction

The possible branching points narrowed down to a single line for each set of points. Now the optimization of placing the branching along that curve is possible. For this optimization, there are multiple possibilities. At first, Maxwell's theorem will be tested. Secondly, the problem with buckling of the members will be described, following by multiple ways of optimizing the column. This leads to a method of describing the optimization for each design-problem separately.

9.2 Minimal volume (Maxwell)

The first option for optimizing the structure is looking at Maxwell's theorem. The same minimum volume equation is used as before. By minimalizing the sum of the normal forces times the length of the members, the optimal force path is described. In figure 57 the starting position is shown on the left. Minimalizing the volume gives the solution on the right. Straight paths from the forces to the (support) point. This solution is found due to multiple reasons. Firstly the endpoint is set. A straight line to the endpoint is indeed the path of least resistance or optimal force path. The force doesn't have to be redirected along multiple members to get to the destination point. Secondly, due to the low scaling point, the horizontal force is minimal. This results in a lower normal force in the members.

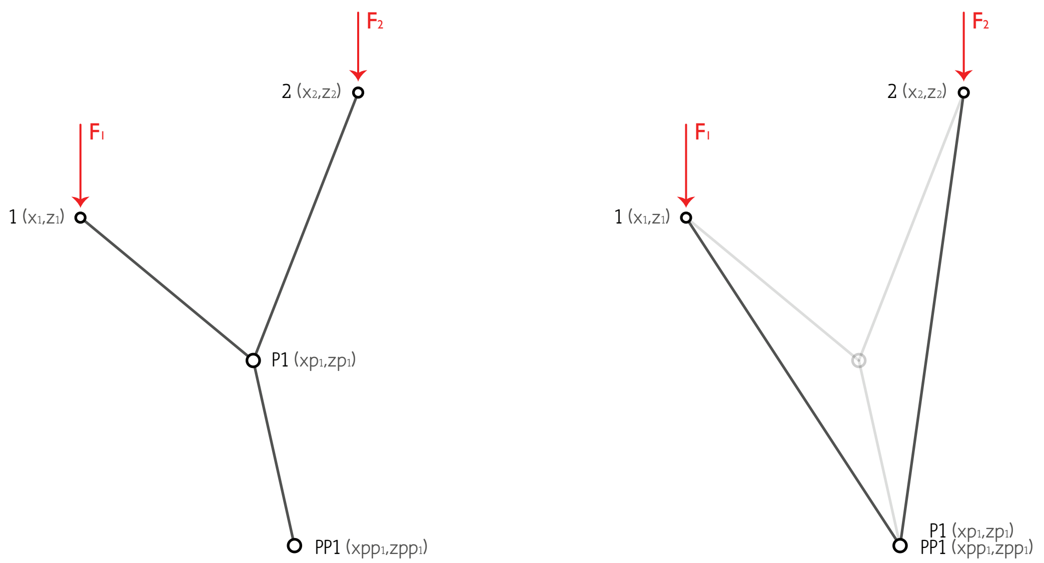


Figure 57 Optimisation by Maxwell's theorem (Own image)

9.3 Buckling

Translating the working lines into actual structural members introduces material and section properties. This gives more opportunities to the optimization of the branching column but also introduces some difficulties. The first difficulty is buckling. of members.

Due to the relative high axial forces and length of the members, buckling might be the main factor influencing the branching pattern. Welleman, Dolfing, and Hartman (2001) described a method of dimensioning members using the Euler buckling force and relative slenderness. The Euler buckling force is described as:

$$F_k = \frac{\pi^2 EI}{l_k^2}$$

The stresses due to the buckling force are given by:

$$\sigma_k = \frac{F_k}{A}$$

Therefore, the Euler buckling stress can be written as:

$$\sigma_k = \frac{\pi^2 \cdot E \cdot I}{A \cdot l_k^2}$$

Introducing I as the inertial radius, a quantity defined by the profile of the member, the stresses can be rewritten:

$$i = \sqrt{\frac{I}{A}} \qquad \sigma_k = \frac{\pi^2 \cdot E \cdot i^2}{l_k^2}$$

Introducing the slenderness of a member, lambda, gives:

$$\sigma_k = \frac{\pi^2 \cdot E}{\lambda^2} \qquad \lambda = \frac{l_k}{i} \qquad i = \sqrt{\frac{I}{A}}$$

The Euler buckling stress needs to be smaller than the yield strength (fy) of the material. If the buckling stress is higher, the yield strength will be normative. Otherwise, the maximum buckling stress will be of influence. Therefore:

$$f_y > \frac{\pi^2 \cdot E}{\lambda^2} \qquad \lambda > \sqrt{\frac{\pi^2 \cdot E}{f_y}}$$

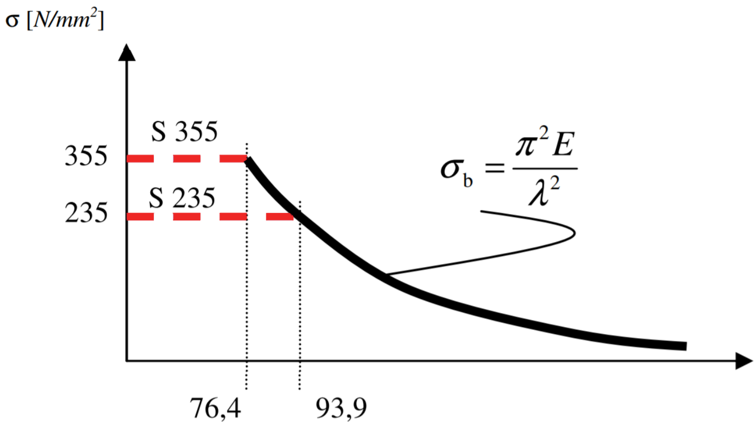


Figure 58 Slenderness (x-axis) and the maximum stress (y-axis)

For instance, taking steel S235. The E-modulus of steel is 2.1x105 N/mm2, and the yield strength is 235:

$$\lambda > \sqrt{\frac{\pi^2 \cdot 2.1 \cdot 10^6}{235}} = 93.9$$

The yield strength is leading if the slenderness is smaller than 93.9. Otherwise, the Euler buckling stress is decisive.

In the slenderness formula, a buckling length lk is given. The length is a result of the support conditions of the column. The multiple conditions can be found in Figure 59. For instance, if the column is hinged at both sides of the column, the effective length of the member is 1 (middle left). If the member is rotationally fixed in both sides, the effective length is 0,7. Setting up the right support conditions can reduce or enlarge the buckling length.

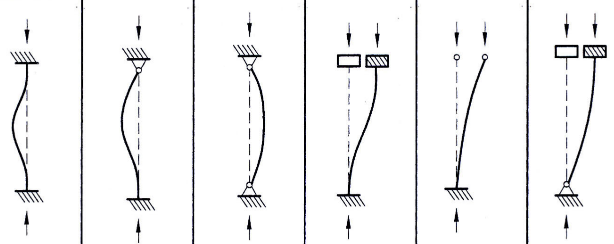
	Braced member			Sway member		
Buckled shape						
Effective length factor (k_e)	0.7	0.85	1.0	1.2	2.2	2.2
Symbols for end restraint conditions	= Rotation fixed, translation fixed = Rotation free, translation fixed			= Rotation fixed, translation fixed = Rotation free, translation fixed		

Figure 59 Multiple buckling lengths (Cade systems, nd)

9.4 Geometry

With the rules for dimensioning the members are due to buckling set, other determining factors are looked at.

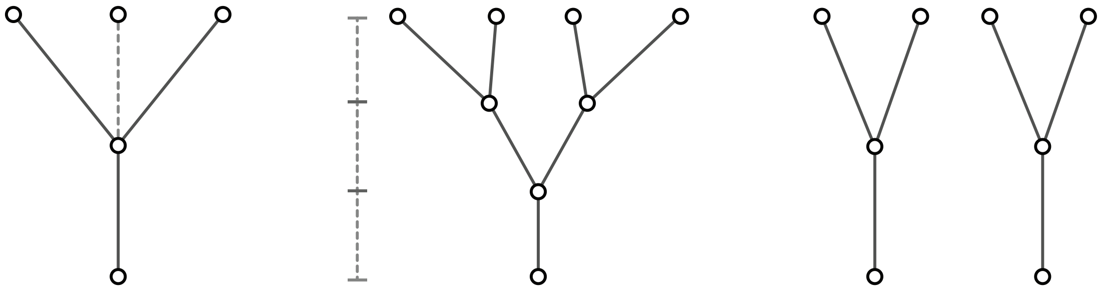


Figure 60 Design descisions. Branches per iteration, amount of iterations, amount of columns. (Own image)

Before any optimization can take place, three design decisions have to be made concerning the configuration of the tree. The first is the number of branches per iteration. Secondly, the number of iterations per tree and last the amount of trees. These three decisions will be based on the width and height of the roof. Based on these outcomes, the scaling lines can be drawn for further optimization.

These decisions are of influence of the structural behaviour of the branching column. The number of branches per iteration and amount of iterations can be seen in figure 62. This relation influences the amount of branches in total and in the top iteration.

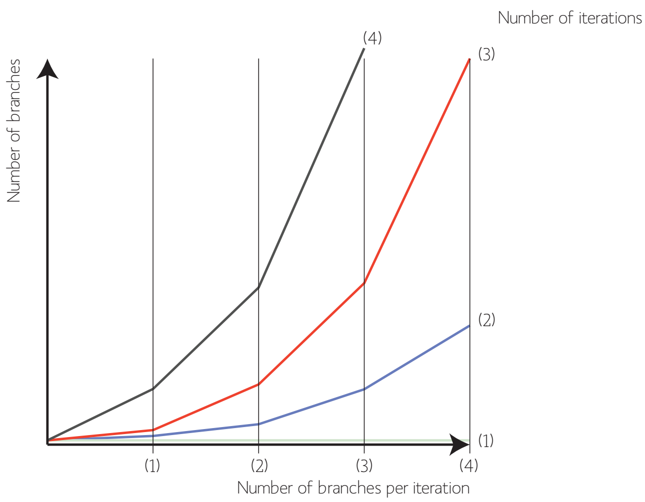


Figure 61 Number of branches per configuration (Own image)

With the number of top branches, the supported area decreases. This decreases the forces in each member and therefore the cross-sections of both the branching column and the roof members. In figure 62 the square meters of roof for each top branch are shown.

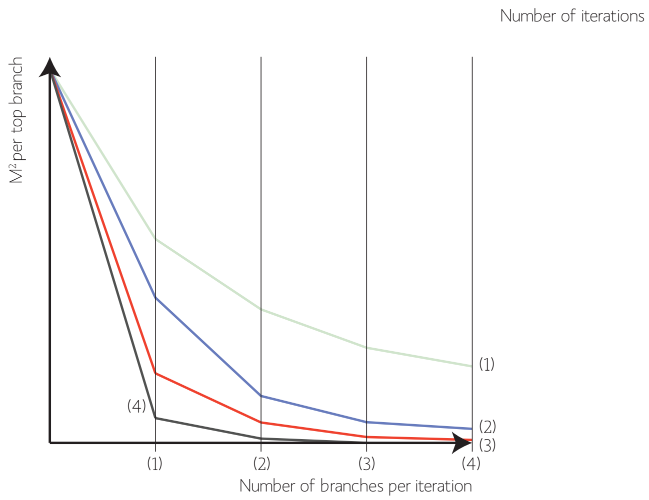


Figure 62 Area to support per configuration (Own image)

The total length of the column changes as well. With the introduction of more iterations, all points shift along the working lines. This changes the total length of the members. Each extra iteration changes this length but also the amount of members per iteration as can be seen in figure 63.

With these considerations, on forehand a design decision has to be made. Then, further optimization can take place.

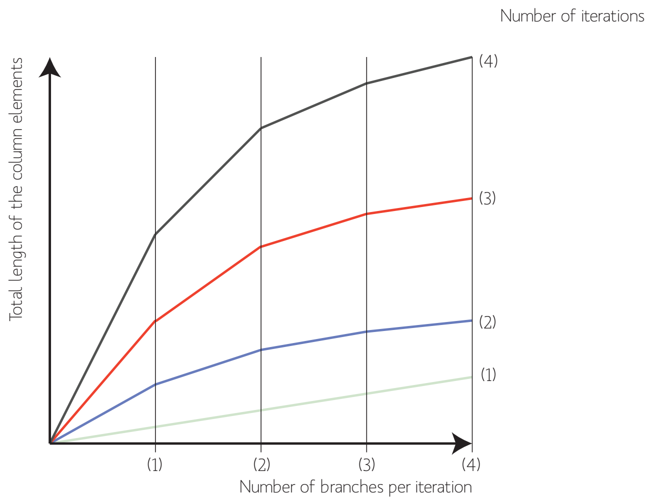


Figure 63 Total length of elements (Own image)

9.5 Optimization

There are multiple ways to optimize the branching form and will mostly give other outcomes. The first factor to optimize on is the total weight of the structure. Minimalizing the weight is done by minimizing the total volume of the material. The only difference with Maxwell's equation is the embedding of buckling into the calculation. The section will increase when the member becomes longer. The mass of each member is defined by the product of the area of the section, A, the length of the member, l, and the density of the material, p.

$$mass = A \cdot l \cdot \rho = \min$$

The second way to optimize the structure is to ensure that all the members have the same section. This simplifies the production process and reduces the costs. By decreasing the maximal difference in A within the structure this can be done.

$$\Delta A = (A_{\max} - A_{\min}) = \min$$

A third way to optimize is optimizing in terms of material use. Maximizing the lowest Euler stress within the column gives the maximal use of the material. Because the material's maximal stays the same, the percentage of the section used will increase.

$$lowest\ \sigma_b = \max$$

The fourth way to optimize the column is by searching the lowest costs. With a set of possible profiles given and the price per unit length, the price for the column can be calculated. By using more common profiles, the total price will decrease.

A fifth way to optimize is in reducing the total embodied energy. The embodied energy is the total energy needed to produce the elements, this includes, for instance, the transportation and fabrication. Reducing this energy reduces the environmental impact. Using the same method as the costs reducing method, this can be calculated. By calculating the embodied energy per profile per unit length.

The last optimization factor is the size of the external forces. Due to the vertical load on the branching column, a horizontal force in the surface is resulted. This force is higher when the top branching points are close to the surface. Scaling the points will influence the dimensioning of the members

Apart from all the optimization strategies, there are boundary conditions for a given design. For instance: the trunk or bottom branch has to be at least 4 meters high to prevent people from climbing in the structure. These conditions have to be decided by the designer.

Because there are multiple ways to optimize the structure, there isn't just one optimal structure. The designer will have to choose what factors are of importance for the design. This way a multi-criteria optimization can be used. As a result, a weighted structure is created. Optimal for this case by the needs of the particular design. An example can be seen in figure 64. Combining these optimization strategies into a parametric design tool, the wishes can be adjusted later. Each input resulting in a different structure.

Optimisation	Factor of importance
Minimal weight	4
Same profiles	2
Maximal material use	3
Minimal costs	1
Minimal Embodied Energy	5
Reduce forces roofstructure	2
Other design conditions	3

Figure 64 Multiple criteria optimization (Own image)

Now the optimization is explained and the structure can be adjusted it is time to look at the surface.

10. Freeform surface.

10.1 2D points of attachment

A branching structure supporting a freeform architectural surface. What exactly is a freeform architectural surface? In this thesis, this surface is a waving surface used as for instance a roof of a hall, passage or indoor garden. An architectural expression such as the Zlote tarasy in Warschau or the Westfield mall in London. These surfaces are double curved surfaces and supported by branching columns. To design a structural efficient structure, the right points of attachment must be found.

In figure 65 a 2D example of a freeform (single curved) beam is showed. On the beam acts a global gravity load as a result of for instance snow or dead load. Both sides are hinged connected. To see where the beam should be supported, the buckling line is showed as a dashed line. If the beam would have no stiffness it would hang as a cable between the two hinges. To remain the shape, a large moment of inertia is needed. As described in the section about arches and cables, the distance between the dashed line and the beam describes the bending moment. This bending moment causes the need for a large section.

If the beam is cut open in the middle the problem is shown. The left part is acting as a cable where the right part is acting as an arch. This arch has a reacting force pushing the arch in its position. It suffers from compression. In the left part, the cable, the reaction is the other way around. The reaction suffers from tension.

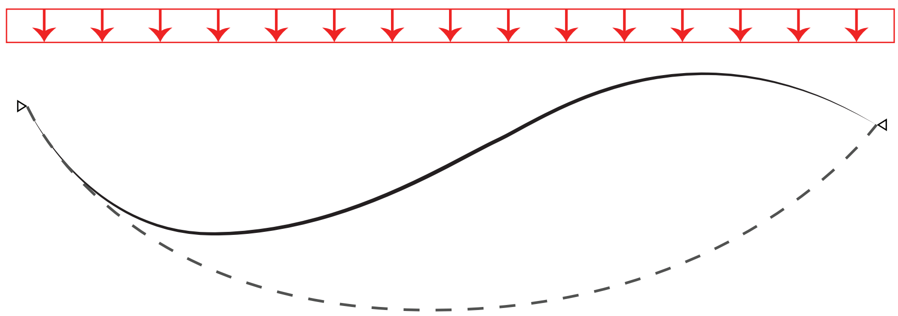


Figure 65 Buckling shape of freeform roof structure (Own image)

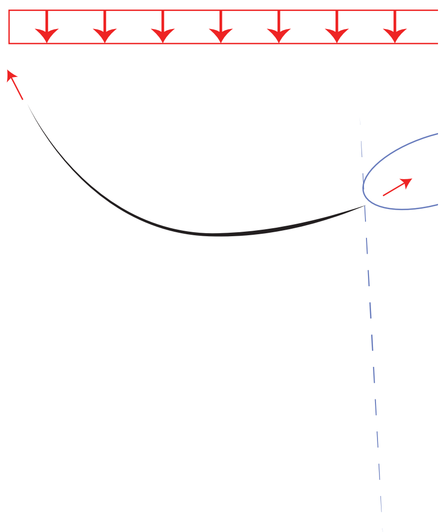
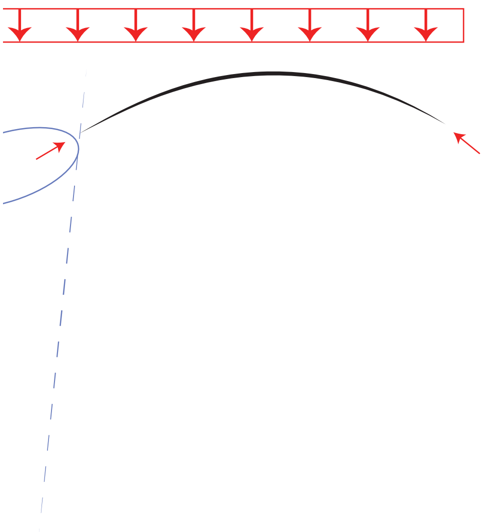


Figure 66 Internal forces after opening

Both the arch and the cable suffer from minimal bending moment because both shapes follow the parabola shape. This results in very thin profiles and only axial stresses. The problem, however, is the equilibrium in the middle. Both reaction forces have the same direction and no other members to make an equilibrium.

With this realization, the points of attachment are becoming clear. By supporting the beam in the middle, this equilibrium can be made. The structure must be in balance with the 'missing' equilibrium to be stable. An example of a structural solution is given on the right. A single branching column is placed.

The same principle works for a 3D double curved surface. To reduce the bending moments in the roof surface and therefore the size of the members of the roof, strategic points of attachment must be found for the roof. If these points are found, they can act as the attachment points of the branching column.



construction (Own image)

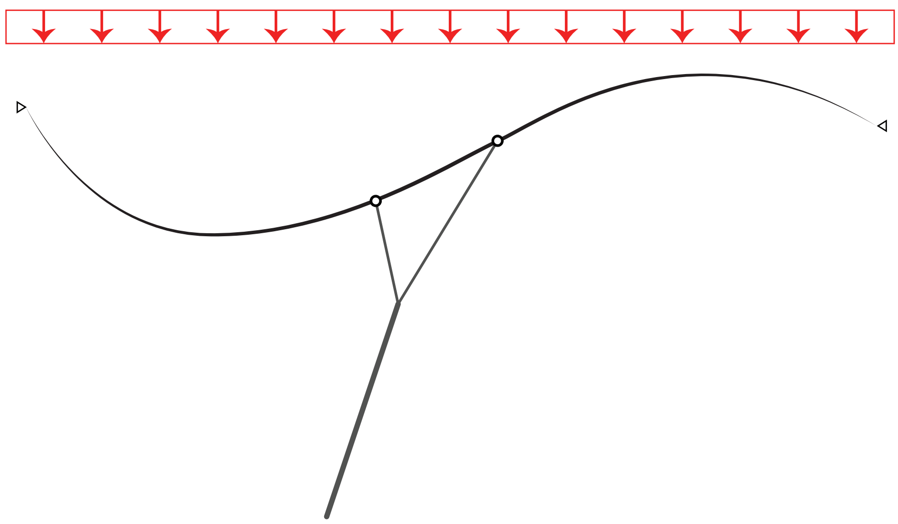


Figure 67 Branching column to reduce bending moments (Own image)

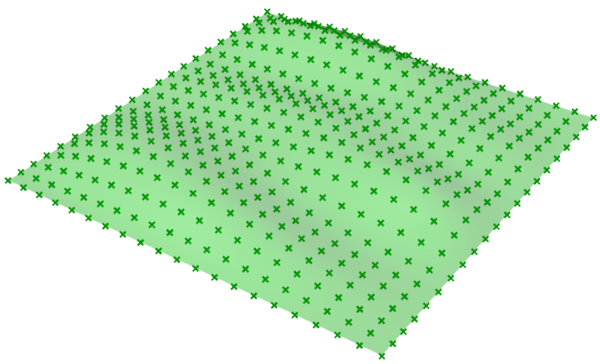
10.2 3D points of attachment

An example of a 3D double curved freeform surface is given in figure 69. This surface was created in Rhinoceros 5, a 3D modeling software for windows. By drawing multiple curves and 'loft' them this surface is formed. Other ways to model the surface are also possible, but in this case, the loft function is used. To find the branching attachment points, two strategies are applied. The first strategy is to look at the curvature of the surface just as the 2D example. The second way is to use parametric modeling software to approximate the best solution.

The first method is shown in Figure 69. Rhinoceros has a curvature analysis tool which is used in this case. In the 2D example, the points of attachment were placed in the transition from a cable to an arch. In 3D, these points are represented by the change of curvature. Between a concave and convex surface. With the tool, the surface is analyzed. The Gaussian curvature is the product of two principal curvatures. The mathematics behind the Gaussian curvature is explained in Lisle (1994), but for the analysis of the surface, the most important points are given in figure 69. K is the first principal curvature and H the second. If both curvatures are positive, the surface forms a pit. Is one of them negative, a peak is formed.

Analysing the surface gives insight into the way that the surface behaves. On the right of figure 68b, the curvature analysis of the surface is shown. The red color represents the peak parts, and the blue is the pit. Searching the points where the surface goes from a peak to a pit will give the best points of attachment. With the surface analysis, these area's are provided by the color green.

The second way to optimize the points of attachment is to minimize the deflection. To do this, again Rhino is used. This time the plugin Grasshopper is used to make a parametric model and analyze it with the structural analysis tool Karamba. The search for the minimal deflection is done by continually changing the support points and calculate the deflection. This is considered a somewhat improvident method because there is no logic involved.



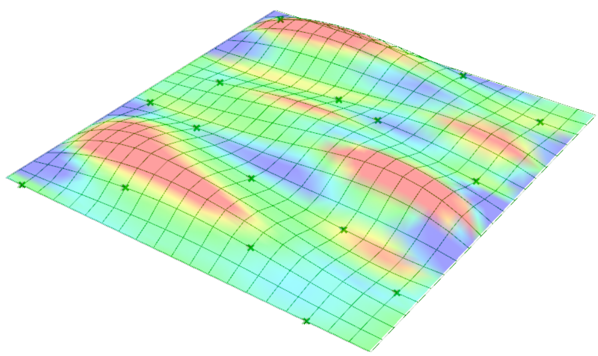


Figure 68 Optimization of surface attachment points,
(a) possible points (b) chosen supports (c) reaction forces (d) branching support column (Own image)

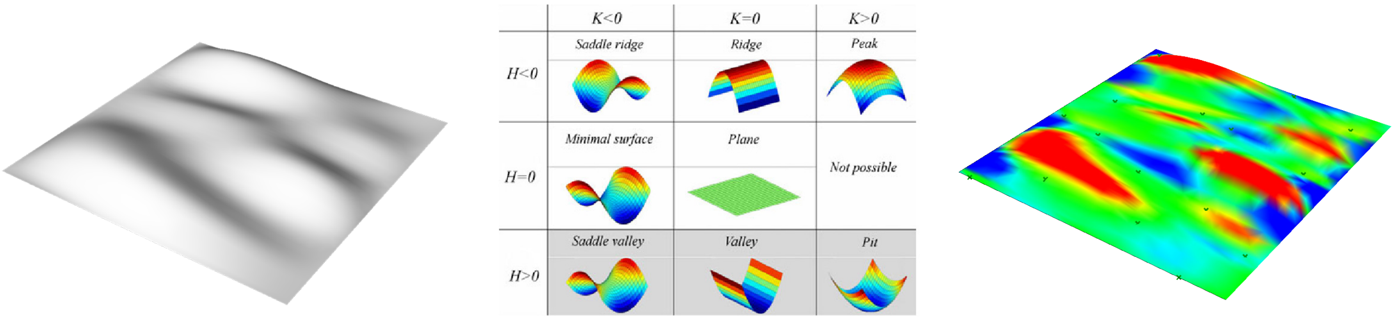
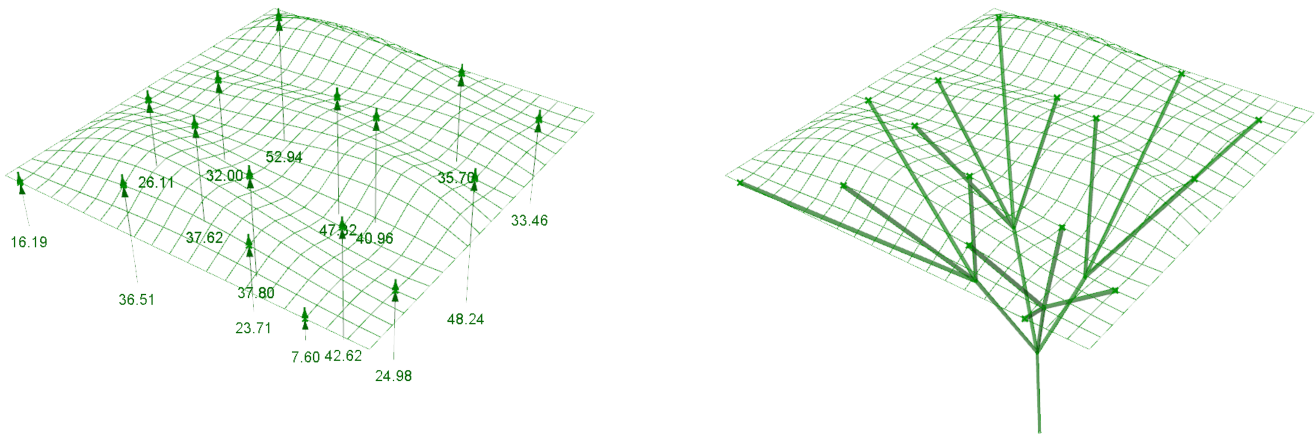


Figure 69 Surface curvature analysis

However, in this case, it is used to generate the surface attachment points quickly. These points and the corresponding reaction forces are later used to create the branching column. Optimizing the points can be part of a further research. For now, these points are used.

An example is given in figure 68. The original surface is divided into a grid. Each intersection represents a possible attachment point. In this case, the design is chosen of a 16 point 2 layer branching column. By optimizing the deflection, 16 points are generated. These points are cross-checked with the results of the curvature analysis and seem to match the green area's. Therefore, the assumption of points is considered correct and is used as input for the column.

Now that the points are known, the vertical forces can be calculated. Using the same structural analysis method as before, the vertical loads of the column are determined. This input is used to design the branching column as described before.



10.3 Horizontal roof forces

In section 7.3, the resultant horizontal forces due to the branching are explained. Because there is a resultant force between the two branches, an extra force is introduced. If the roof follows the resultant force, only normal forces are introduced to the roof. Due to the free-form character of the roof, the surface does not follow this line.

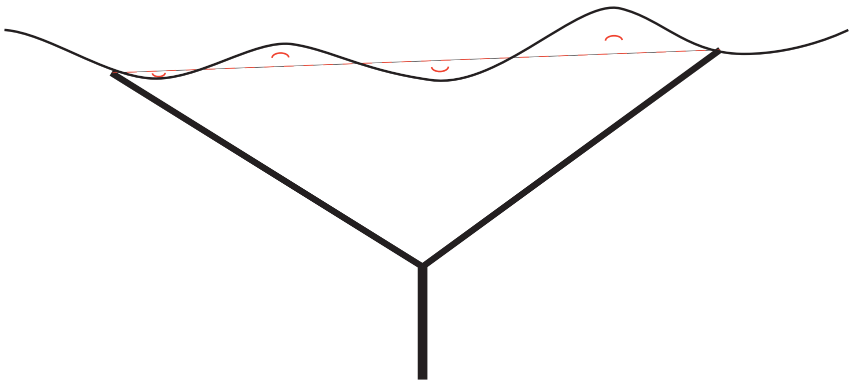


Figure 70 Surface curvature analysis

With the free-form, bending moments are introduced into the roof. The resultant line (dotted line in figure 70) represents the neutral line. If the roof members do not follow the neutral line, shear forces and therefore bending stresses are present. The placing of the attachment points, therefore, is not only influencing the surface's behavior due to the external forces. it also influences the roof's structural behavior due to the branching of the columns.

By introducing more elements or extra iterations, the supported roof length decreases. Therefore, the bending influence of the external forces reduce. Also, the normal forces in the top members are reduced. However, with the arrangement of figure 71, there are still bending stresses as a result of the branching.

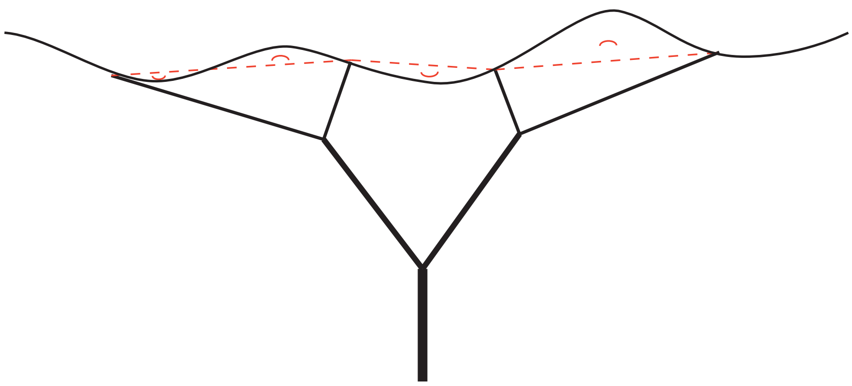


Figure 71 Surface curvature analysis

Introducing an extra branch in the first iteration, another configuration can be made. This configuration follows the waving surface. This way, the influence of the horizontal roof forces are minimal. This can be seen in figure 72. There are some bending moments present in the roof surface as the roof does not exactly follow the red dotted line but they are significantly smaller than the bending moments in figure 70 and 71.

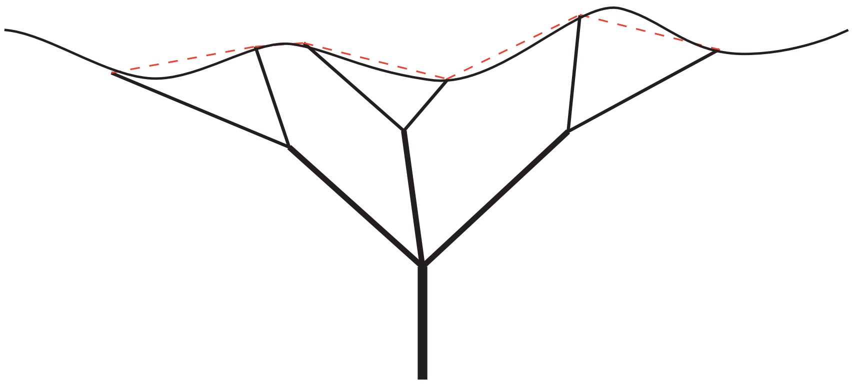


Figure 72 Surface curvature analysis

The placing of these points, however, is contradictory to the points found in the previous section. In the previous section, the points were placed at the transition from convex to concave. This placement is the optimal place to transfer the external loads within the roof surface. In this section. the optimal placing is at the top and bottom of the wave to resist the forces resulting from the branching column. In order to find the optimal points of support, both forces have to be taken into account.

11. Glass column

11.1 Glass construction

Now that the design of the column ensures that there is only compression in the members, a building material can be chosen. Steel is often used for branching columns because it is a homogeneous material. This means that the material has equal strength in both tension and compression. This is an advantage relative to other building materials such as wood and concrete. In these materials, one direction is stronger than the other.

Another recently developing building material is glass. Glass is considered as strong as steel in compression. Steel reaches a compressive strength of 355 Mpa where glass is tested to have a compressive strength of almost 500 Mpa. (Oikonomopoulou, van den Broek & Bristogianni, 2017) The tensile strength, however, lies far below the steel strength and depends on the way of producing the glass. Annealed glass reaches a tensile strength of around 20 Mpa and heat-strengthened glass reaches 40 Mpa. Therefore, glass is more and more used in (experimental) constructions.

The first structures consist of glass beams and columns (a). Multiple layers of glass were laminated into stronger profiles in order to resist the loads. Later, the glass brick was developed. The bricks were a TU Delft research project and implemented into multiple facades and other compression-based structures (b). Recently, a glass truss was built on the TU Delft campus as part of a PhD. Research project (c). The truss is built with bundles of glass rods. In the middle, a steel member has placed to pre-stress the bundles. This technique creates bundled cross-sections and can be adapted to the needed sectional area. (Figure 73)

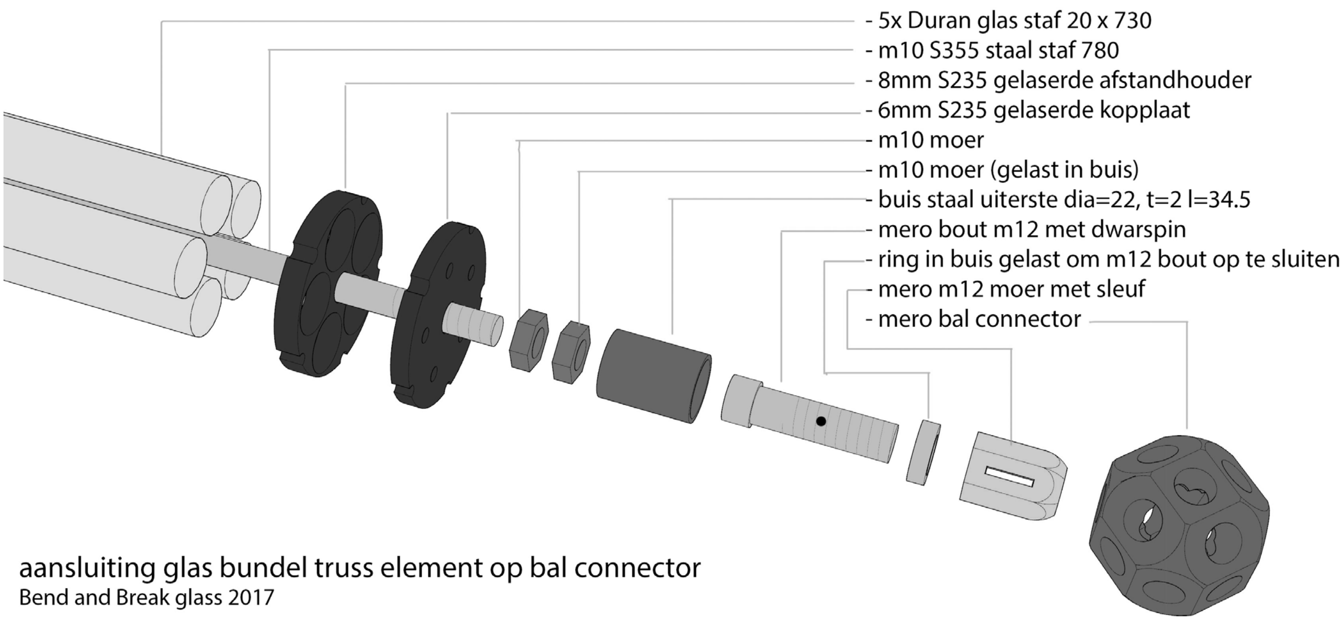
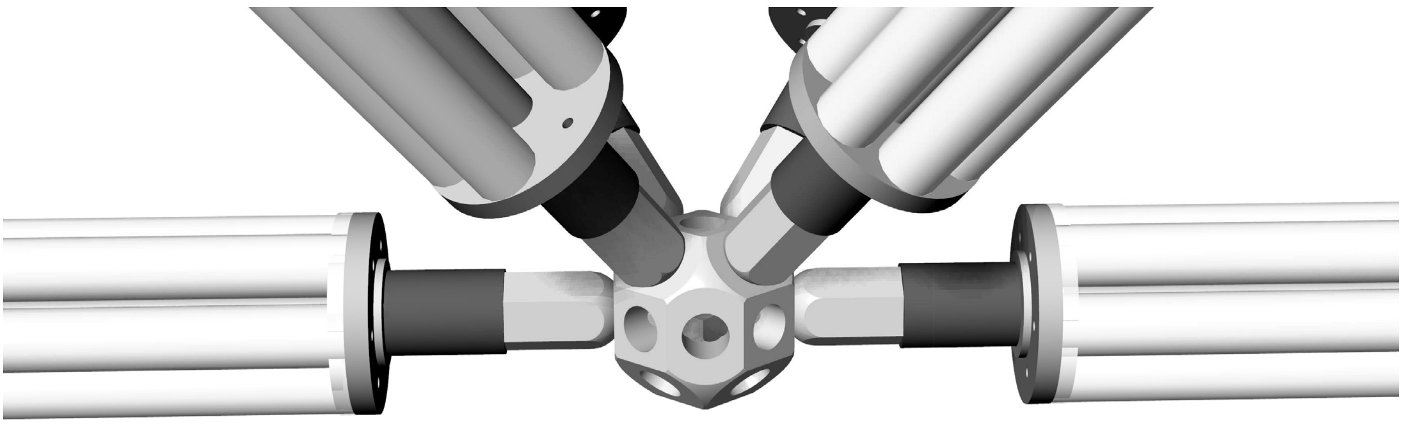


Figure 73 Bundled glass truss element (Bend and break, TU delft)


CHANEL



11.2 Glass columns

Currently, also research is done to make columns out of glass. There are five ways of making these glass columns according to Nijsse and ten Brincke (2014). The first is making a combined profile with flat sheets of glass. These look like the known steel profiles used in building construction. The second is a circular hollow column. This profile can be compared with for instance tubing steel profiles. A third way of producing a glass column is stacking multiple sheets or elements. This can be done both vertical as horizontal. A fourth option is the bundled column. This type of column was used in the glass truss bridge. The individual glass rods are held in place by steel end parts. The fifth type of column is a cast column. Recently more research has been done to create glass columns with interlocking cast glass components. These columns are currently being developed and tested. Students of the TU Delft have been involved in this research.

With this knowledge, the consideration can be made to make the branching column out of glass. The best options to use for a branching column is option 2 and 4. Mainly because in all the existing branching structures round profiles are used. The buckling resistance is even in all directions and the columns are aesthetically appealing. Currently, The fabrication of tubing profiles is limited to a diameter of 465 mm and a wall thickness of 7 mm. The rods are limited to a 30 mm diameter.

The moment of inertia (I) of a round tubing column can be calculated by:

$$I = \frac{\pi r^{outer^4}}{4} - \frac{\pi r_{inner}^{\ \ 4}}{4}$$

For a bundled column:

$$I' = I + Ad^2 = \frac{\pi r^4}{4} + \pi r^2 \cdot d^2$$

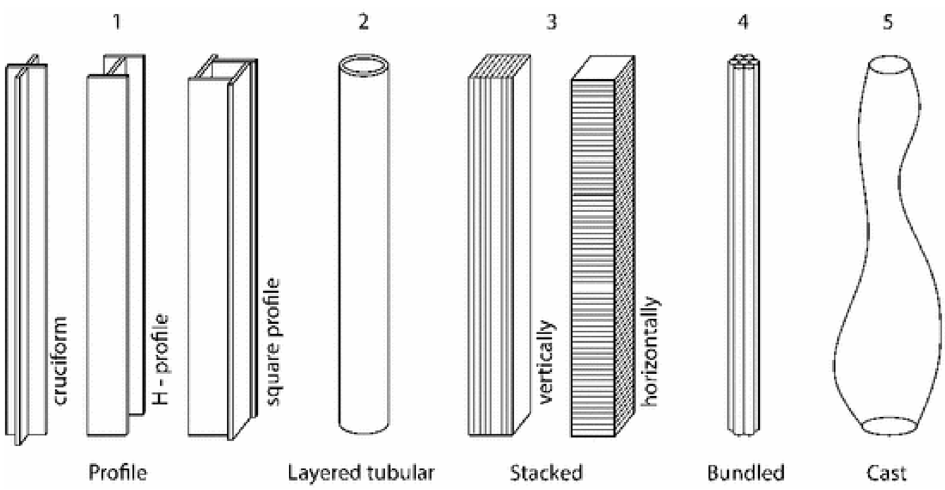


Figure 74 Multiple ways of making a glass column (Nijsse and ten Brincke, 2014)

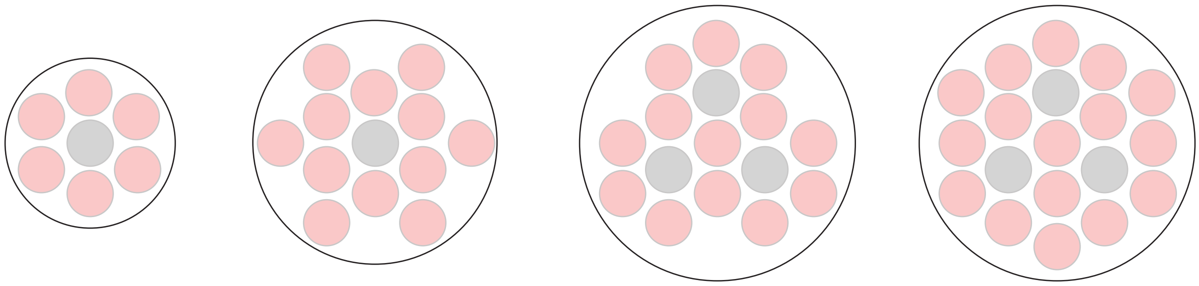


Figure 75 Possible bundled configurations (Own image)

The bundled column has a few advantages compared to the single hollow tubes. The first advantage is the customizability of the bundled column. As can be seen in Figure 75, There are multiple ways of configuring a bundled column. Each configuration again can be made with multiple diameters of rods. This results in a variety of moment of inertia's and each column can be optimized for the given design. The second advantage is the re-direction of forces over multiple members. If one member cracks or breaks, the others are still able to carry the load. Thirdly, the combined profiles can be prestressed using one or multiple steel rods/cables. With this prestressing unforeseen shear forces can be taken. Also, the steel rod's give more security because of the linear elastic material properties.

The hollow tubing again has two advantages. Visually, the column will be more transparent, and the columns will be lighter because less material is used to reach the same moment of inertia.

Standard product range
Rod

Diameter		Rod weight Length approx. 1,500 mm	Carton contents		Pallet load	
mm		g	Number of rods	Weight approx. kg	Number of boxes	Weight approx. kg
3	±0.13	24	529	12.5	44	550.0
4	±0.13	42	298	12.5	44	550.0
5	±0.13	66	183	12.0	44	528.0
6	±0.13	95	140	13.2	44	580.8
7	±0.13	129	98	12.6	44	554.4
8	±0.18	168	80	13.4	44	589.6
9	±0.18	213	63	13.4	44	589.6
10	±0.18	263	45	11.8	44	519.2
12	±0.18	378	35	13.2	44	580.8
14	±0.26	515	24	12.4	44	545.6
16	±0.26	672	20	13.4	36	482.4
18	±0.36	851	20	17.0	27	459.0
20	±0.36	1 050	16	16.8	27	453.6
22	±0.40	1 271	12	15.3	36	550.8
24	±0.40	1 512	12	18.2	27	491.4
26	±0.50	1 775	9	16.0	27	432.0
28	±0.70	2 059	9	18.5	27	499.5
30	±0.70	2 363	6	14.2	36	511.2

Standard length: approx. 1,500 mm

Standard product range
Tubing

Outer diameter		Wall thickness		Tube weight Length approx. 1,500 mm	Carton contents		Pallet load	
mm		mm		g	Number of tubes	Weight approx. kg	Number of cartons	Weight approx. kg
300	±3.70	5.0 7.0 9.0	±0.70 ±1.10 ±1.40	15 492 21 542 27 508	1 1 1	15.5 21.5 27.5	9 9 9	139.5 193.5 247.5
315	±3.80	7.0 9.0	±1.10 ±1.40	22 645 28 926	1 1	22.6 28.9	9 9	203.4 260.1
325	±4.00	9.0 10.0	±1.40 ±1.40	29 871 33 085	1 1	29.9 33.0	4 9	119.6 297.0
350	±4.00	5.0	±0.80	18 118	1	18.1	4	72.4
365	±4.50	7.0	±1.40	26 321	1	26.3	4	105.2
400	±5.00	6.0	±1.50	24 829	1	24.8	4	99.2
415	±5.00	7.0	±1.50	29 997	1	30.0	4	120.0
420	±5.00	9.5	±1.50	40 960	1	41.0	4	164.0
430	±5.00	6.0	±1.00	26 720	1	26.7	4	106.8
440	±5.00	7.0	±1.00	31 836	1	31.8	4	127.2
450	±5.00	7.0 8.0	±1.00 ±1.00	32 571 37 140	1 1	32.6 37.1	4 4	130.4 148.4
460	±5.50	8.5	±1.20	40 309	1	40.3	4	161.2
465	±6.00	7.0	±1.00	33 674	1	33.7	4	134.8

Standard length: approx. 1,500 mm

SCHOTT
glass made of ideas

Figure 76 Current production possibilities (Schott glass, nd)

To ensure the required moment of inertia against buckling, the individual members need to be calculated on their buckling length. To decrease the buckling length spacers have to be installed in multiple places along the length of the column. In Figure 77 an example is shown. This method, however, has not been tested yet but theoretically, each spacer should reduce the buckling length. By screwing the spacers onto the inner steel bar, they will remain in place.

These bundled columns are part of research currently done by Ate Snijder, the second mentor. Together with Ate, Rob Nijsse and Lennert van der Linden a design for a glass swing is made. A glass swing combining the latest technologies. 3D printed plastic nodes and glass members are combined into a design for the glassTec in Dusseldorf. Currently, multiple students of civil engineering are testing the 3D printed connections, and the members are being optimized.

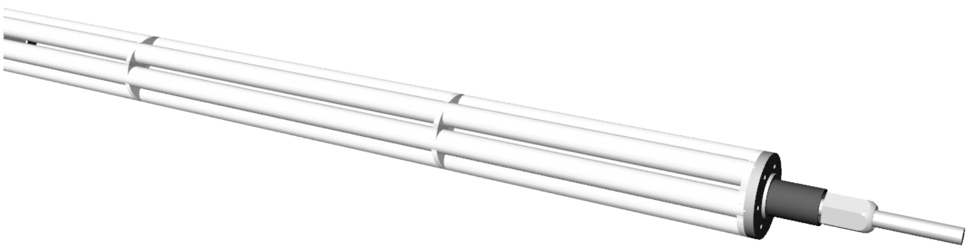


Figure 77 Multiple spacers along the member length to decrease buckling length (Own image)

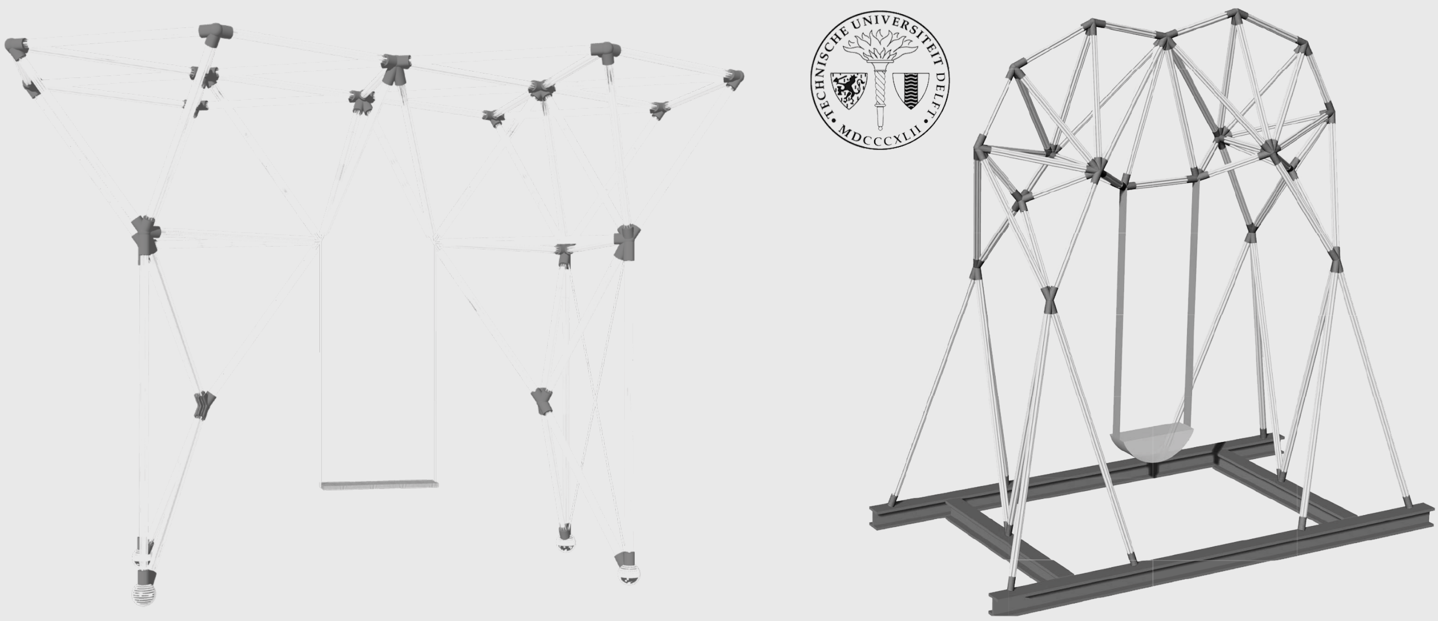


Figure 78 First and optimized design of the glass swing (Own image) and (Ate Snijder)

11.3 Glass swing

The Glass swing is a fun way to make people aware of the structural potential of glass. We all agree that glass is beautiful, but not everyone is convinced that it can also be safe. The glass and transparency group at the TU Delft has been researching how to design and engineer safe glass structures. The Glass Swing lets visitors of the GlasTec engage with the structure. They will experience in a straightforward way that the glass structure can carry the loads caused by a person on the swing. This swing exhibits some of the latest developments in structural engineering:

The structure for the glass swing is developed using Bidirectional Evolutionary Structural Optimisation (BESO). The cross-sections of the struts are then optimized for buckling length. The struts will consist of prestressed bundles developed by the Glass and Transparency group and students of the TU Delft. The bundles will be made of Schott glass rods. The strength of the glass bundle struts has been experimentally tested in the lab at the TU Delft. The nodes will be 3D printed from recyclable plastic. Their strength will be experimentally determined in the lab and modeled in FE analysis. Before the Glasstec, the entire structure will be built and tested in the Stevin lab, to ensure the structural safety of the swing at the Glasstec.
(Ate Snijder)

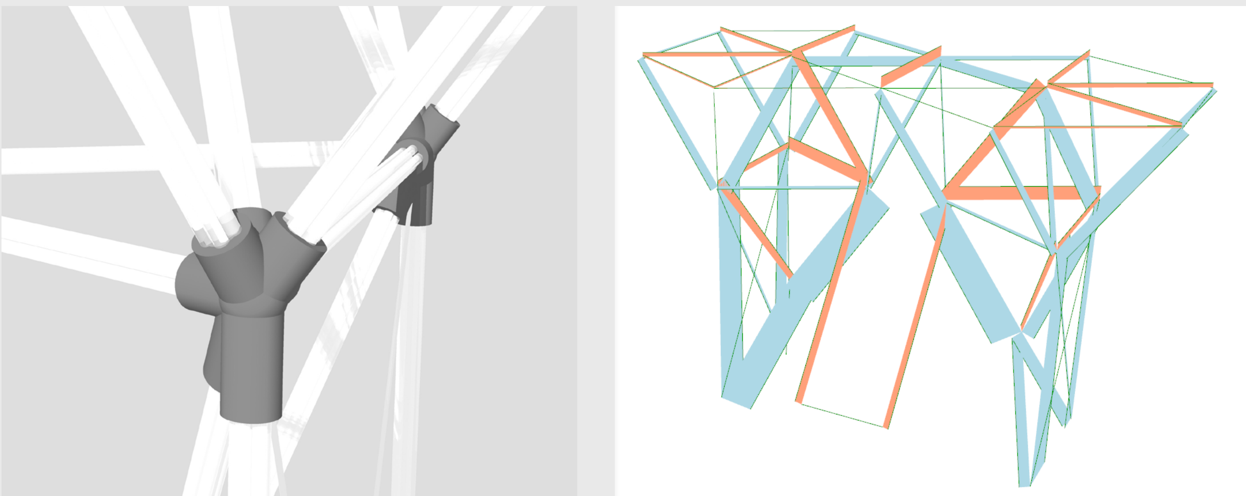


Figure 79 3D printed nodes and structural analysation (Own image)

12. The design

12.1 Design method

With the multiple aspects of the design highlighted, a method for design can be described. Following the steps of this method, a design can be made. This design will be based on multiple decisions and steps taken. If the steps are done correctly, a design of a freeform architectural surface supported by highly structural efficient branching columns is made

A diagram of this design strategy is described in figure 80. The design starts with a specific location. The location will determine the dimensions of the surface that has to be supported by the columns. Following the steps, a final design will be made. To show the design procedure, a design is made. Each step will be discussed and explained to show the method. In the end, a comparison is made between the construction and an common, alternative construction.

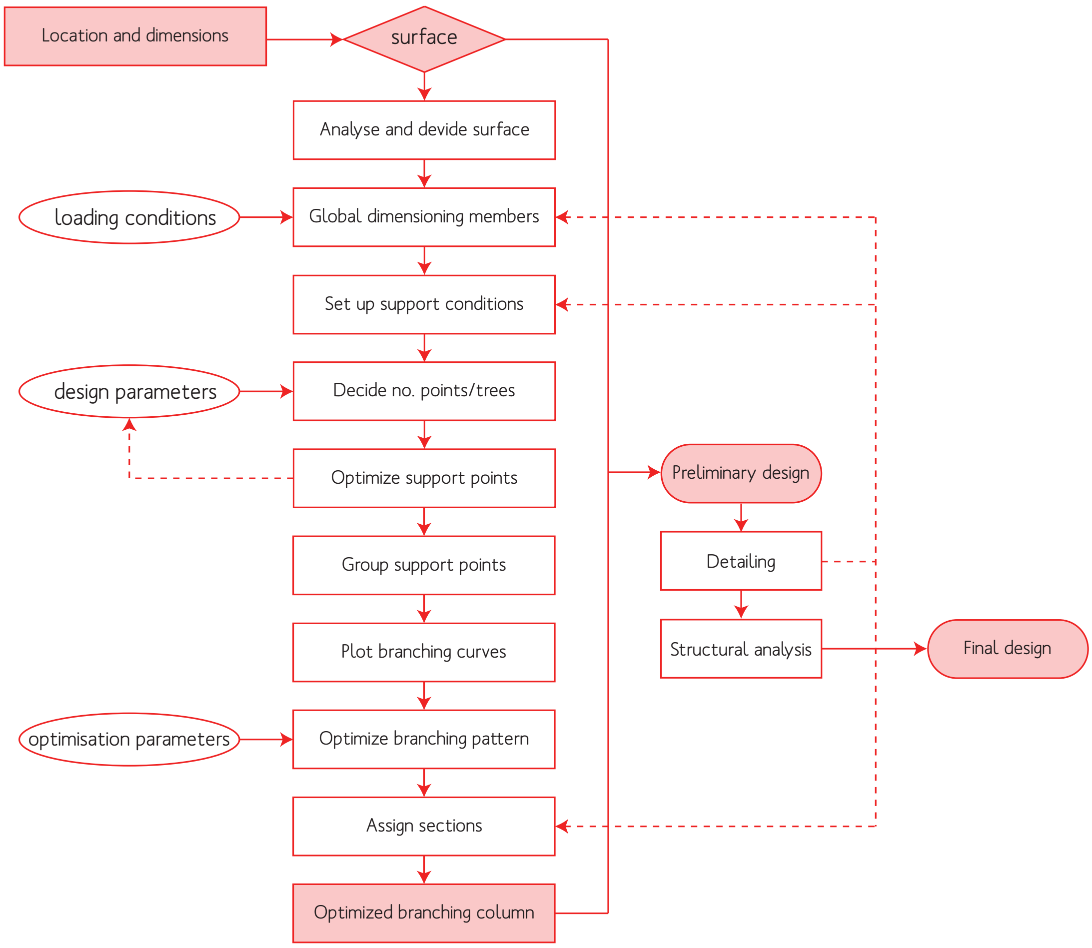


Figure 80 Design method derived from research (Own image)

12.2 The West hall; A new extension for BK city.

A new extension for BK city. Due to the growing demand for the widely orientated building technologist, a glass hall is being built on the west side of the building. This hall is situated on the existing parking square next to the western entrance of the building. The space will be used as a new product development area where prototypes can be made, and lectures can be given.

This fictional design assignment will be the start of my design. The hall will be a state of the art building extension, showcasing the field of building technology. The dimension of the new hall is 21 by 27 meter. This is slightly smaller than the orange hall which is 29 by 33 meter. The roof will be the same height as the second floor of the existing building, just as the orange and maquette hall. This height is 13 meter.

The existing building surrounds the new hall at three sides. At these sides, the roof will be horizontally connected to the building. This shows much resemblance to the orange hall, and therefore the hall will be used to compare with. The big difference is that the orange hall has a flat closed roof. The new extension will have a freeform, partially glass roof.

The first step in the design is shaping the roof. Now that we have the dimensions, the freeform can be shaped. In this case, a waving roof is chosen. Also, the decision is made to make four trees with both two iterations of four branches

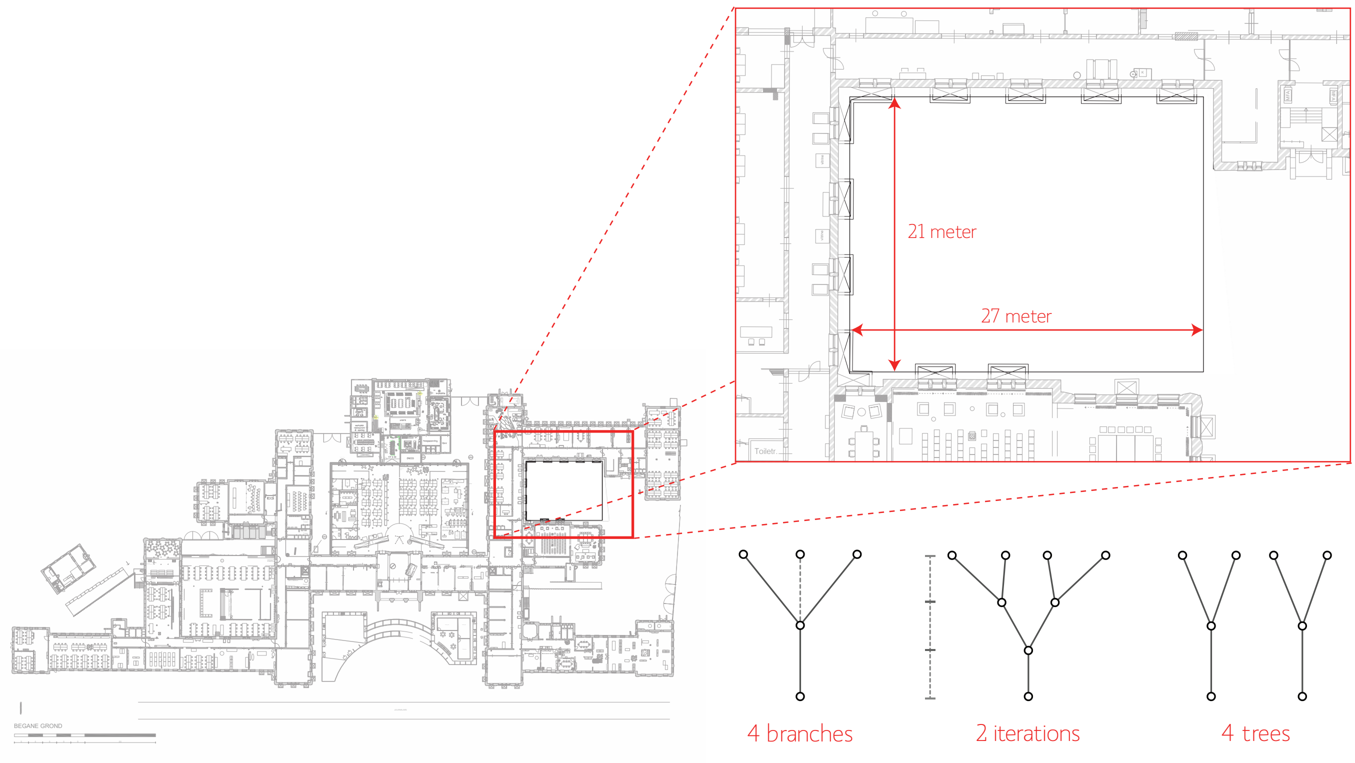
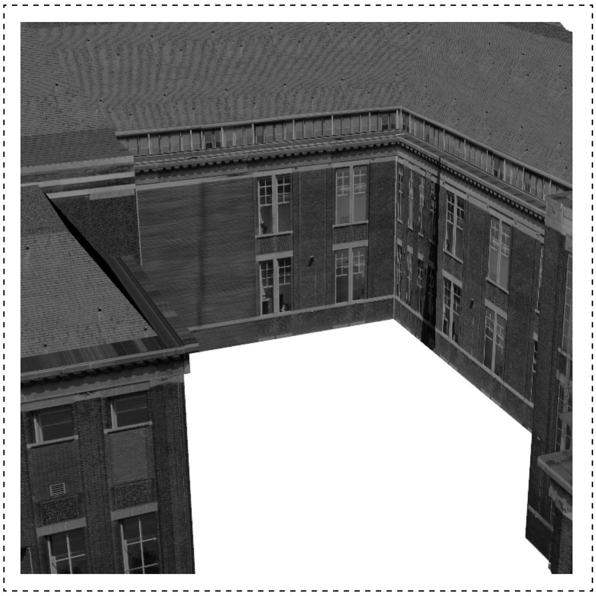
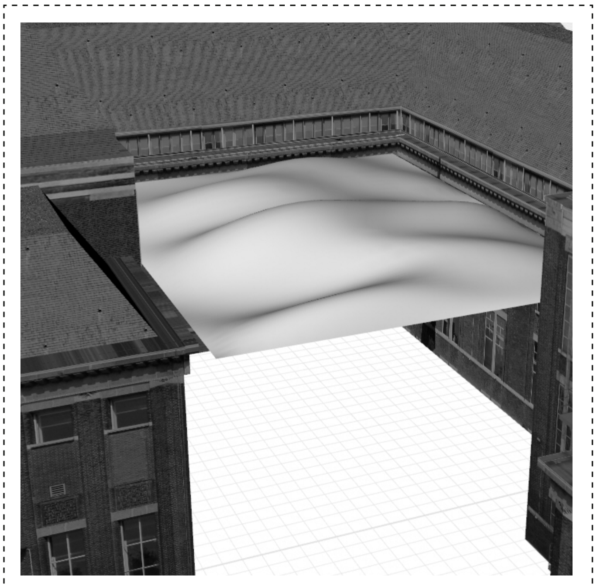


Figure 81 Design location and decisions (Own image)

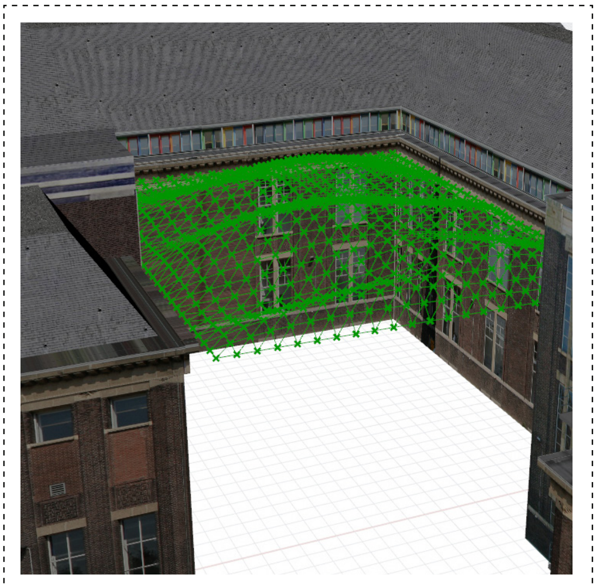
Now that the dimensions are known, the existing building can be modeled in 3D. In figure # the west side of the BK city building is shown. On the top floor, the Building technology ateliers are settled. The students will be able to look over the new building.

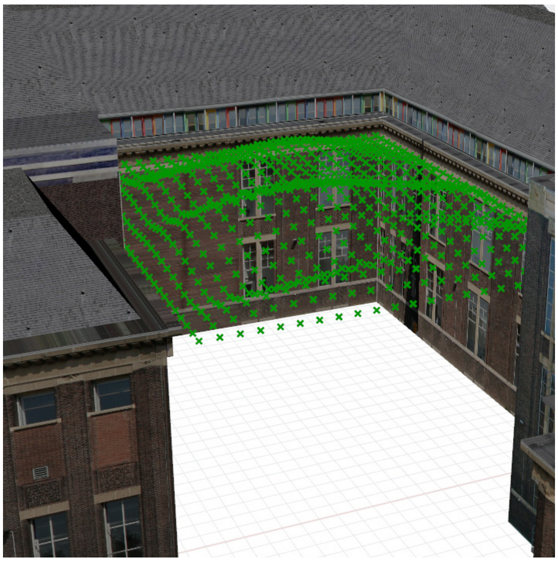


After shaping a few possible freeform roof shapes, the most appealing roof was chosen. This, of course, can be any shape desired by the designer. The selected roof has a waving, flowing character. By analyzing the surface, the curvature can be detected. Also, the flow of rainwater must be taken into account. The lowest points must be located at the sides. Otherwise, the rain will stay on the roof causing leakage.

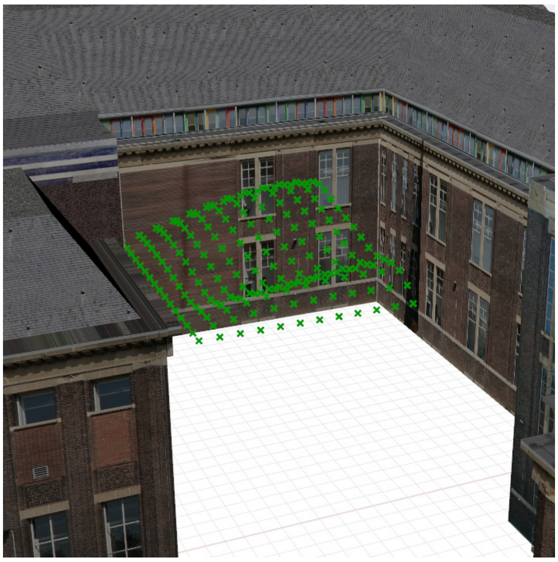


With the surface defined, the grid can be defined. For this roof, a division of 1 meter is chosen. In the process of dividing the surface, the member lengths are held as equal as possible. The curvature creates members of unequal lengths. The diagonals are used to ensure the strength of the grid.

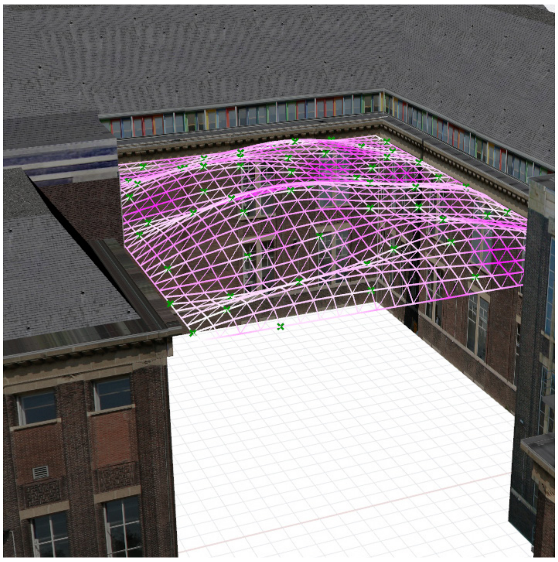




The vertexes of the grid are used as possible branch attachment points. In total 768 points are possible supports. In the optimization of the support points, all these points will be possible support points.

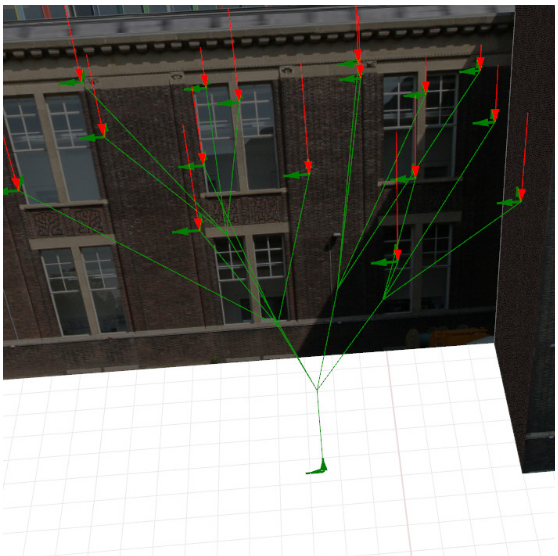


Because the decision is made to make a design with four trees, the surface is divided into four quarters. Each quarter has 192 attachment points. One tree has to be placed below this part of the surface. Because the tree has two iterations of four branches, each tree has 16 attachment points. In total, 64 branches are redirected into four trunks.

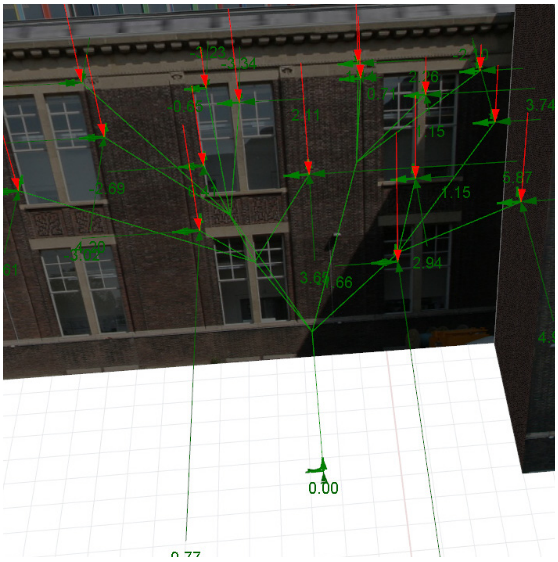


These points are used to examine the surface deflection. In the structural analysis, the points are used as vertical supports. With the parametric modeling software Grasshopper, the location is optimized. Using the deflection as a factor to optimize on. With this optimization, the location of the points is set. The corresponding reaction forces are used as input for the columns. All required input to design the column now is known.

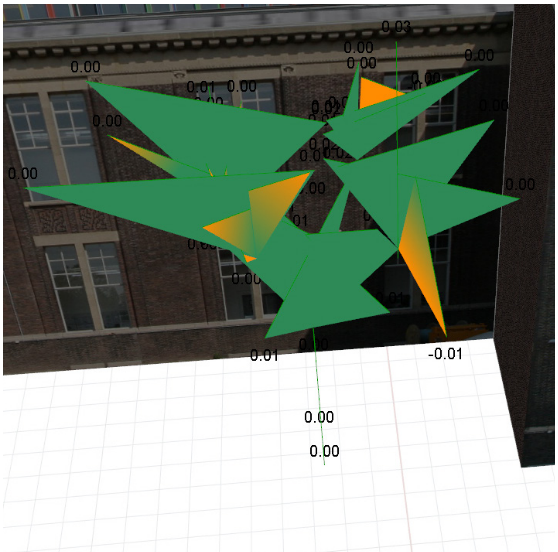
The forces and surface attachment points are imported into Grasshopper. With the use of the working lines found in this thesis, a branching column is generated. The height of the iterations now is random decided. This column is a moment free compression only column but is not yet considered as the optimal branching column.

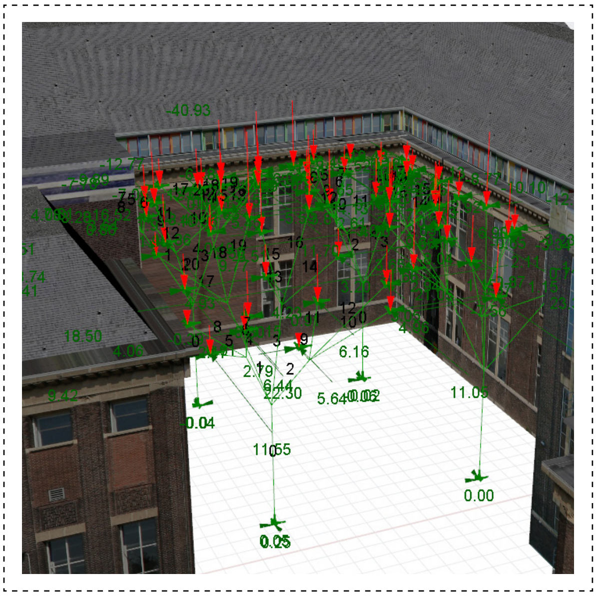


To optimize the structure, the optimization parameters must be decided. In this case, the column is optimized to have minimal weight. The branching points are scaled along the working lines, and the minimal total weight is optimized. Or as described previously, the new minimal volume, area times the length.

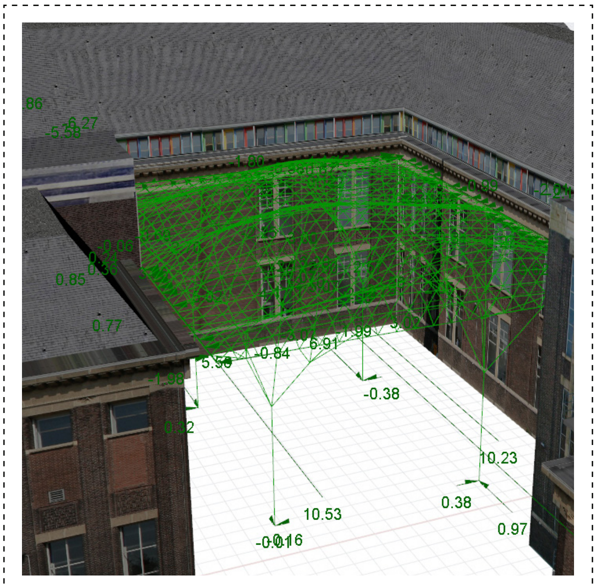


To check whether the branching has gone right, the bending moments can be checked with the structural analysis tool Karamba. Because Karamba has a slight discrepancy compared to other structural analysis tools, the column still suffers from a 0.01 KNm bending moment. One thing is essential to notice. Only the external forces are applied to the structure. The dead weight of the individual members is not taken into account.

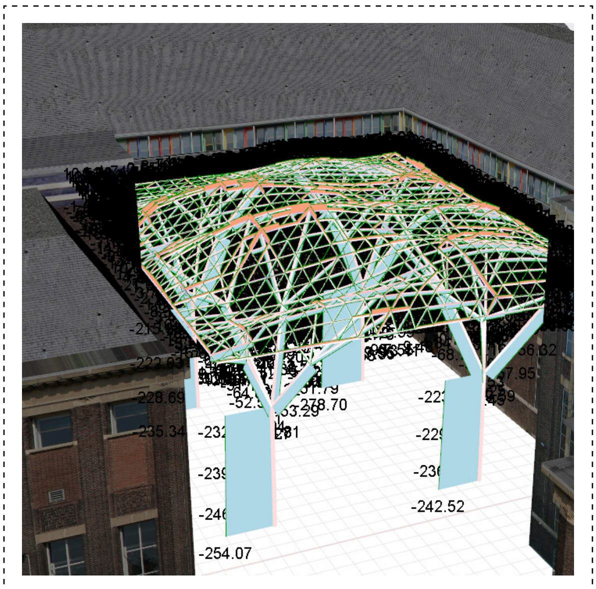




The same process is repeated for all four columns. This results in four columns without any bending moments. Because the forces are a representation of the forces acting on the surface, it can not be used as a total system.

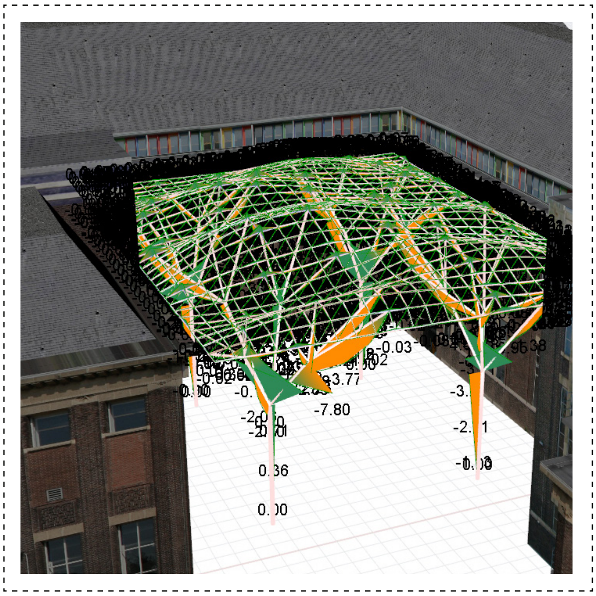


To review the total system (roof and columns), both grasshopper scripts are linked to each other. This way the deflection of the roof and elongation or shortening of the structural members will be taken in to account.

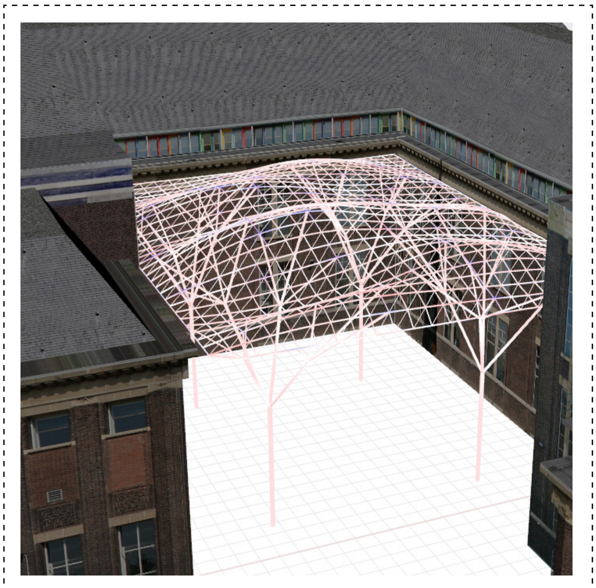


The total system now is analyzed on its structural behavior. Firstly, the normal forces are examined. The blue color represents the compression and orange means tension. The trees only suffer from compression. In the roof surface, both tension and compression are present. This calculation includes the dead load of all structural members.

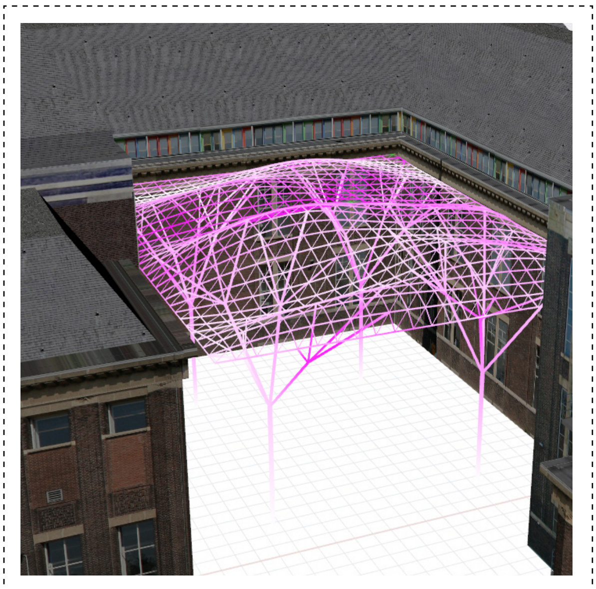
Now that the total system works and can be examined and the dead loads of the columns are taken into account as well, small bending moments in the members can be seen. In the figure, the bending moment around the local y-axis is showed. Also, the bending moment around the z-axis has to be studied. The torsional moment (around the x-axis) is almost zero. The bending moments can be reduced later by slightly shifting the points.

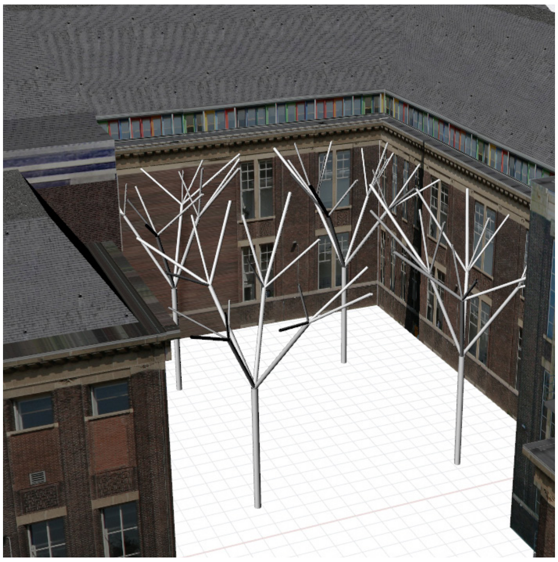


In grasshopper (Karamba), the material properties and sections of each individual member can be added. After calculating the required moment of inertia against buckling, the cross-sections can be assigned to the model. With this model, the utilization of the elements can be examined. With this analyze, the member's which are suffering the most can be addressed. If needed, cross sections can be adjusted to reduce stresses.

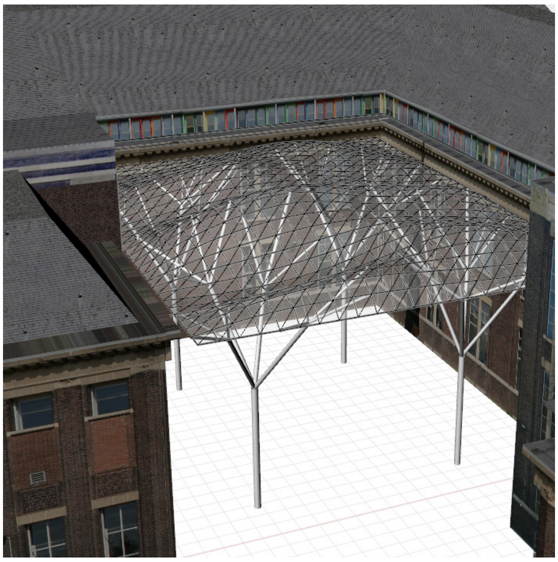


Another analyze that can be done is the total displacement. Purple represents the maximum displacement and white means almost no displacement. The maximum displacement will be a result of the maximum stresses due to bending moment.

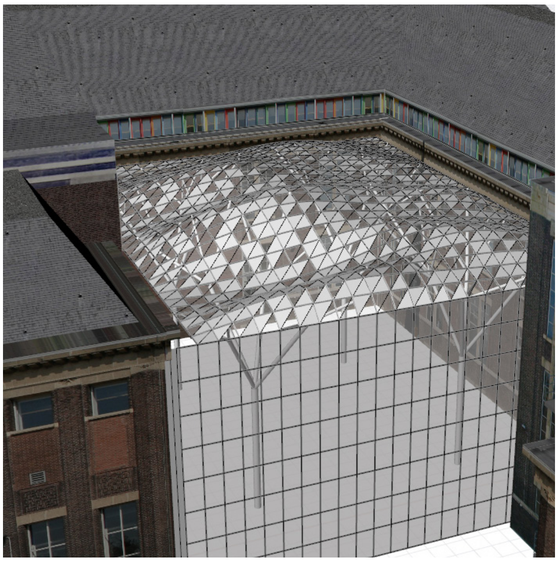




The branching columns are optimized, analyzed and dimensioned. Now they can be used for the design. The lower members have a bigger cross section due to the length and forces acting on the members. The cross sections used are based on round profiles. Later the combined glass bundles will be introduced.



The surface now can be added to the design. The division used before will be the used for the structural members of the roof. The connections are detailed later.



With a lovely pattern on the roof and a curtain wall closing the hall, the preliminary design is finished. The assumption of the member size is made, and the constructive and building details can be made.

The preliminary design is complete. Next step is to detail the design. In the phase before the preliminary design, some assumptions are made concerning the supports and the dimensions of the members. In the design phase, all nodes of the branching column were considered rigid except for the top nodes. The top nodes need to be rotational free in all axes. In the detailing, this means that the top joints are designed as a hinge or a ball joint. The branching points are detailed as solid connections. An example of the rigid connection can be seen in figure 82. This node can be a (steel) 3D printed node or steel cast node.

Also, the member size can be assigned. All members can be optimized individually to their required buckling moment of inertia. By calculating the weight of the individual sections, the weight of the members can be reconsidered and be used the final structural calculations. By repeating this process a few times, the members are optimized and the total model can be considered more accurate.

In the optimization process, the assumption is made that the horizontal forces are taken by the existing building. This connection needs to be a hinged connection. to ensure that there are no bending moments transferred to the existing building. By looking at the existing orange hall (Figure 83), the idea was formed to use pendulum rods. Like the orange hall, triangles are formed to transfer the wind loads to the building and ensuring stability. These connections can be represented as a hinged roll connection. It can move up and down and only transfer horizontal forces.

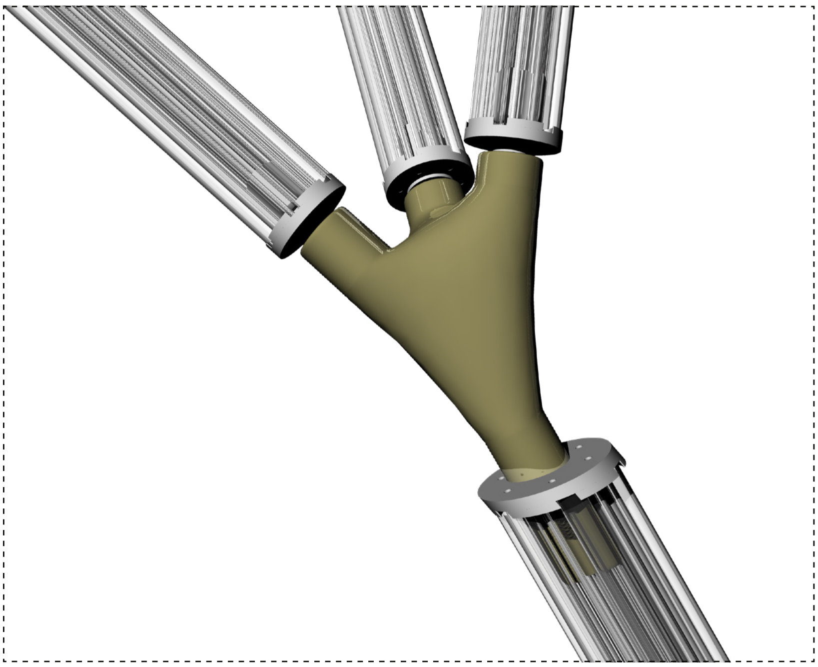


Figure 82 Detailed (3D printed) connection (Own image)

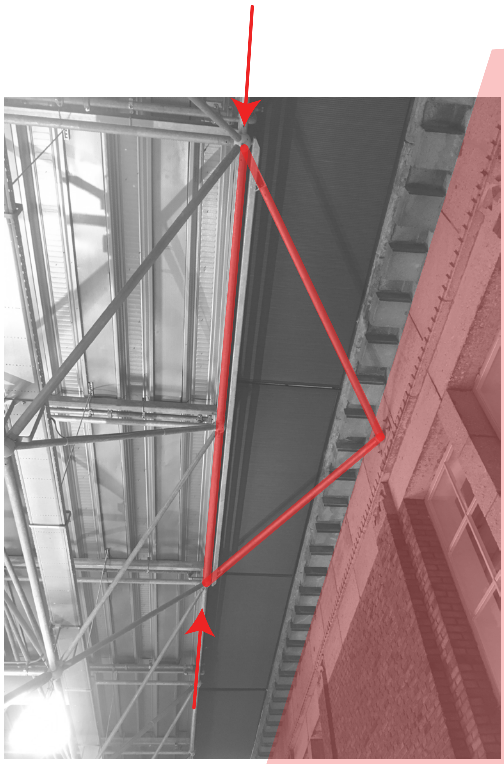


Figure 83 Stability in the orange hall (Own image)

The roof surface consists of welded rectangular profiles with diagonal rods. On top of the structure, glass and closed elements are placed. Just like the Zlote tarasy or the Westfield shopping mall, triangular elements are placed on top of the structure, The pattern is decided through a randomization script but can be adjusted to the needs of the space below. Also, different materials can be used to cover the roof.

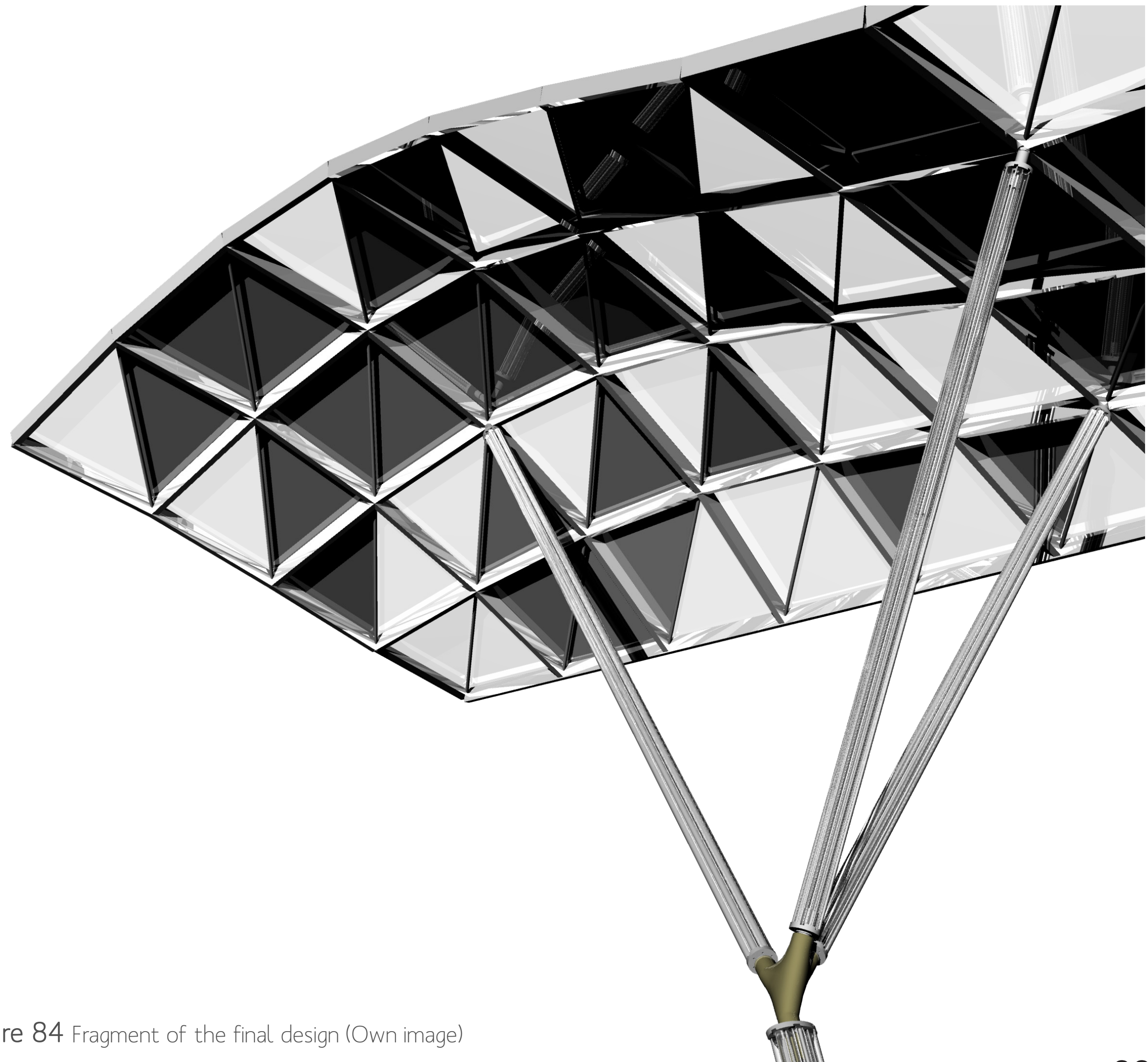


Figure 84 Fragment of the final design (Own image)

The structural concept of the glass extension can be found in figure 85. The columns support the roof structure but can only take the vertical forces. For the wind force, a steel truss is placed behind the curtain wall on the south-west facade. Other horizontal forces are taken by the existing building and the truss. In the section the size of the elements can be seen and the architectural space created.

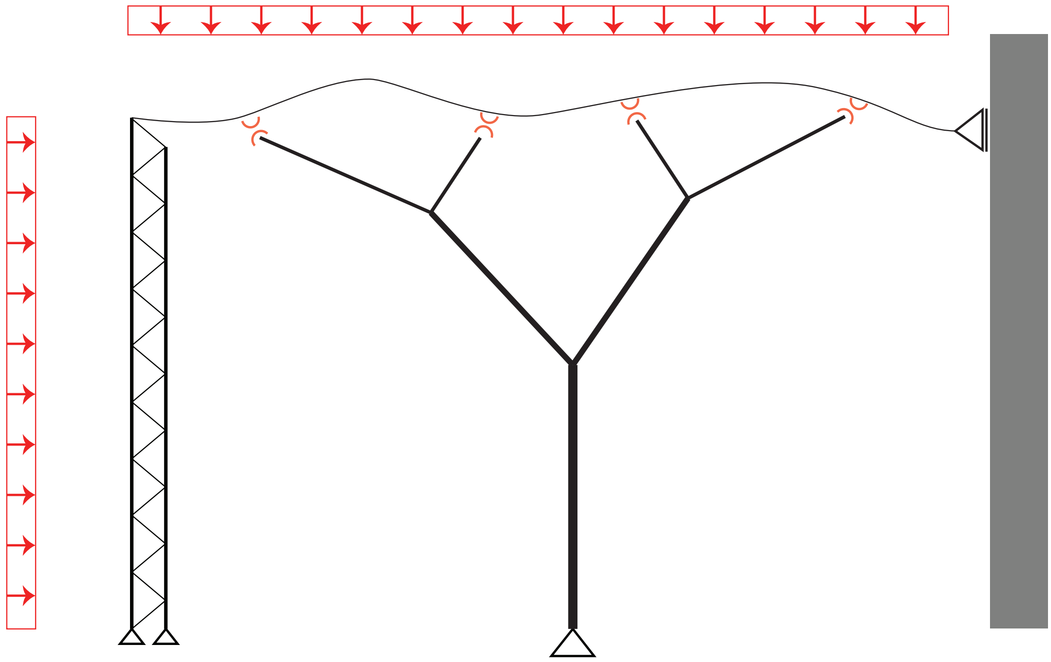
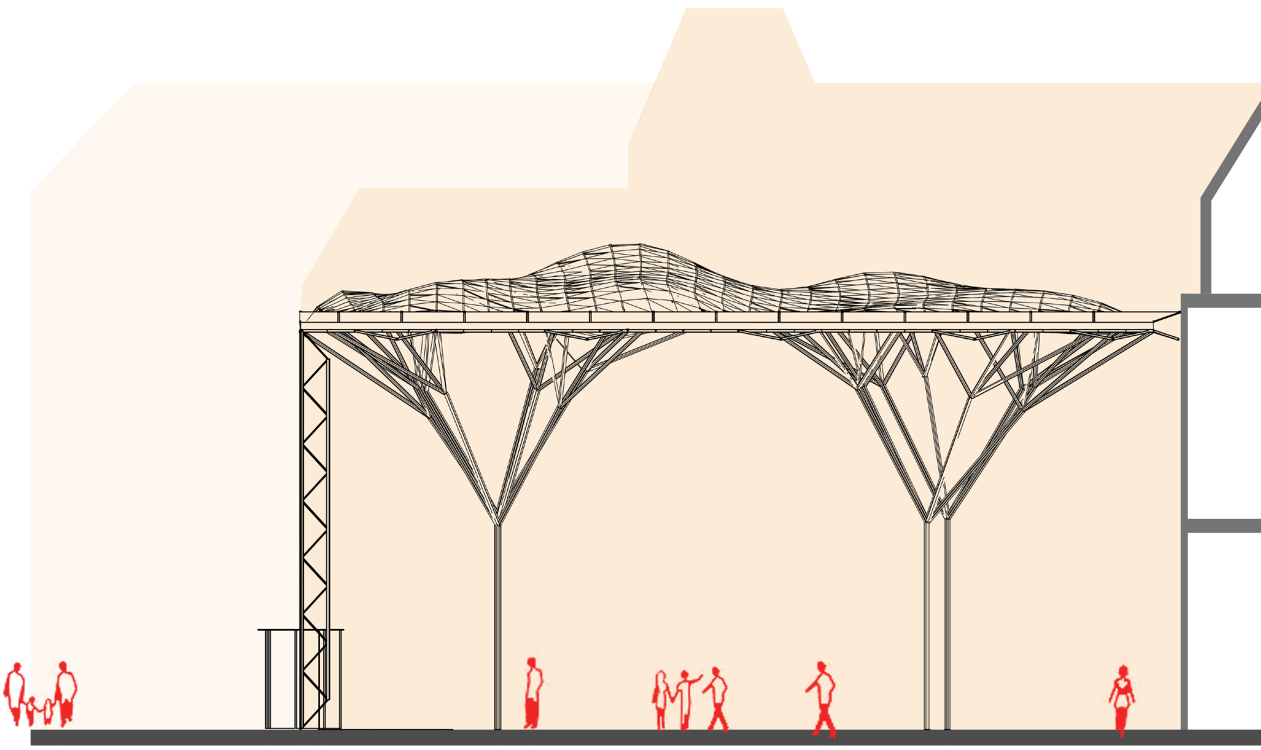


Figure 85 Structural scheme final design (Own image)



13. Comparisson

13.1 Branching VS spaceframe

A spaceframe is often used for large span roof surfaces. This is also the case in the other two extension halls of BK city, the maquette hall, and the orange hall. To compare the influence of the branching columns with the standardized system, a few factors have to be changed within the existing design. The roof construction will be changed to a freeform spaceframe, also, the material is changed to steel, just as the existing halls.

The orange hall consists of a spaceframe of 29 * 33 * 1.5 m (w*l*h). The spaceframe is held up by six steel columns. With a frame size of 3 times 3 meter, the total length of the roof members is 1909 meters and there are 162 similar nodes in the construction. Because the roof surface is straight, it has a structural advantage compared to the free form roof surface. If the roof would have been supported by branching columns, the span of the roof would become significantly smaller. Leading to lower bending moments and therefore a smaller cross-section. This lower cross-section would decrease the height of the roof and decrease the material needed.

However, the optimal height to width ratio of the internal members remains the same. Thus, the decreation of height leads to a smaller grid of the spaceframe, introducing more nodes and increase the total length of the roof members. In this case, the reducement of forces on the roof leads to smaller members but a higher total length and a lot more nodes. For a space frame, this is not a convenient thing to do.

Looking at the columns, the comparison is made between a branching column of 15 meters, supporting 10 times 10 meters (100 m2) of roof surface, and a single column of 15 meters supporting the same roof. Assuming a solid column, the branching column weighs 2072 kilo and the normal column weighs 2320 kilo. A difference of 11%.

Type	One column	Branching	Percentage
Weight	2320 kilo	2072 kilo	89%
Roof span	10 meters	2.5 meters	25%

Figure 86 Comparison between branching and normal column (Own image)

Another advantage of the branching column compared to the spaceframe is the ability to support free-form roof surfaces. Although this isn't efficient to do using a spaceframe because of all the individual nodes and member lengths, the flexibility of the roof's shape is way higher and the stiffness of the roof members is less important.

13.2 Branching VS gridshell

Another way of shaping freeform roof surfaces is to design it as a gridshell. The gridshell, however, can only have a shell-like shape. The advantage of a branching column supporting a freeform shape is the extra flexibility in shape. This results in a transition from convex to concave for instance instead of only a concave shape. The rest of the roof can work in the same way.

14. Results and discussion

14.1 Results

"How can we design structurally efficient three-dimensional branching structures as a support of non-uniform roof surfaces"

In this thesis, a method of designing a branching structure supporting a freeform structure is described. This method is based on literature researched and own research. By verifying the method in structural analysis programs and make it applicable to a design, proof of it's correctness is given.

The method describes a solution to the problem regarding unequal distribution of points and forces. These unequal distributed forces are a result of a freeform architectural surface and its optimized support points. With a given set of points and corresponding forces, working lines are described where the structure has the highest structural efficiency. Along these lines, all points can be chosen as branching points. This result in a structure with only axial forces and therefore a high structural efficiency.

By making optimization and design choices, the points of branching can be scaled along these lines. This leads to a structural optimized design for the given conditions. Each factor of optimization results in a different optimal design. Setting up a multi-criteria optimization will weigh all the factors resulting in a normalized design.

The result has led to more understanding of the structural behavior of branching columns. Also, a mathematical description of the optimal branching points is given. This way the structures can be designed without the use of computational form finding tools.

14.2 Discussion

Within the field of form-finding, the ideal structure is found for one given load case. In structural design, however, a structure must be able to withstand multiple load-cases. For the design of a branching column, this means that each load case has a different optimal branching pattern.

Also, this method only works when the new roof is attached to an existing building, or a ring with a large profile surrounds the roof. Due to the shape of the surface, horizontal forces need to be transferred to the sides. Also, The stability needs to be ensured by either the surrounding building or braces at the side.

15. Reflection

15.1 Process

The research approach worked out well in the end. Between P2 and P3, however, the progress made was minimal. A lot of research was done to find a method for optimizing the branching position. Mathematics was studied to solve variation calculations, and in more depth, the design of arches and shells was studied. All work to find the one formula that could describe the optimal points. Together with the mentors, we could not figure it out. Until, the realization of the reversing structure problem. This moment came after P3 which gave less time to integrate it into the final model.

Nevertheless, the results aimed for were reached. A method was described and the optimal branching pattern for branching structures supporting freeform architectural surfaces is described.

Design and research are interwoven in this thesis. The research question was formed by a shortcoming in the design method/description. This lead to a structural problem. By solving the research question and describe a possible method, a design can be realized. The output of the research therefore also is a design.

In the process, no moral/ethical issues were encountered.

15.2 Societal impact

The results are applicable in the field of design of complex structures. It gives insight into the structural behavior of complex branching structures. The project is innovative because it describes the forces acting on funicular structures.

The project contributes to sustainable development in two ways. The first way is the fact that the columns now can be considered optimal regarding material use. The required strength is reduced by placing the member in logical places and therefore, less material can be used. Also, It shows that it is possible to design such structures with glass elements. Glass is a natural resource which can be fully recycled.

There is no socio-cultural or ethical impact other than the improvement of understanding for the engineers and designers.

The project affects architecture because it gives an extra method designing structures to support new shapes or surfaces. This gives the designers more freedom and can result in more possible shapes.

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