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Nonlinear Repetitive Control for Mitigating Noise Amplification*

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Abstract— **Repetitive control can lead to high performance by attenuating periodic disturbances completely, yet it may amplify non-periodic disturbances. The aim of this paper is to achieve both fast learning and low errors in repetitive control. To this end, a nonlinear learning filter is introduced that distinguishes between periodic and non-periodic errors based on their typical amplitude characteristics to adapt the extent to which they are included in the repetitive controller. Convergence conditions for the nonlinear repetitive control system are derived by casting the resulting closed-loop as a discrete-time convergent system. Simulation results of the proposed approach demonstrate fast learning and small errors.**

I. Introduction

Repetitive control (RC) leads to high performance for systems with periodic disturbances, since it can attenuate these disturbances completely. RC is based on the internal model principle [1] and employs a periodic signal generator to fully reject periodic signals. While periodic disturbances are attenuated, non-periodic disturbances, including measurement noise, may be amplified by RC [2], [3]. To achieve high performance, RC should attenuate periodic disturbances fast without amplifying non-periodic disturbances, yet in traditional RC there is a trade-off between these requirements due to the waterbed effect. Essentially, the repetitive controller modifies the closed-loop system through a modifying sensitivity, which contains notches, i.e., local suppression, at the frequency of the periodic disturbance and its higher harmonics, and consequently exceeds $0\,\mathrm{dB}$ at other frequencies. Thus, non-periodic disturbances or disturbances with different periods are amplified. A similar effect is observed in the related field of iterative learning control (ILC) [4].

Existing linear time-invariant (LTI) methods to reduce the amplification of non-periodic disturbances in RC typically lead to a reduced learning speed and imperfect attenuation of periodic disturbances. Repetitive control often uses a low-pass filter to ensure robust stability. Lowering the cut-off frequency of this filter reduces not only the amplification of non-periodic disturbances, but also the attenuation of periodic disturbances, and is therefore undesirable. Reducing the learning gain reduces the amplification of non-periodic disturbances, at the cost of the learning speed [5] and the robustness against period variations. High-order RC [6], [7] induces similar trade-offs between noise and period robustness. In [8], an optimal model-based repetitive controller is designed in conjunction with an observer-state feedback controller, which allows for more extensive trade-offs yet retains the fundamental limitations. Kalman filters may reduce the amplification of non-periodic disturbances in RC, but the performance depends completely on models of the system and the periodic disturbance [2].

Since LTI RC approaches cannot overcome the fundamental limitation of the Bode sensitivity integral and as such always result in trade-offs, the use of a nonlinear repetitive controller is considered. To mitigate non-periodic disturbances, these are distinguished from periodic disturbances through their different amplitude. Essentially, this can be viewed as an adapting controller, where the gain is adapted depending on whether a periodic disturbance is detected. This leads to a nonlinear RC implementation, which does not suffer from the performance limitations in LTI systems, including the aforementioned Bode sensitivity integral.

The nonlinear repetitive controller relates to the idea of variable gain feedback controllers that are constructed through nonlinear filters, including [9], [10]. A similar idea has been applied to the related field of ILC in [11]–[13], where a deadzone nonlinearity is included in the learning filter in lifted or frequency-domain ILC. The main difference between ILC and RC is that ILC assumes resets between iterations, while RC assumes continuous operation. ILC does not influence the stability of the closed-loop system, even if the iteration-domain system can become unstable. Therefore, stability results for nonlinear ILC cannot be applied directly to nonlinear repetitive control. Existing stability analyses for nonlinear RC, see, e.g., [14]–[16], consider RC applied to continuous-time nonlinear systems, and cannot be applied directly to the discrete-time linear system with a nonlinear repetitive controller considered in this paper.

Although significant steps have been taken to optimize the performance of RC algorithms, the trade-off between fast attenuation of periodic disturbances and limiting amplification of non-periodic disturbances hampers performance improvements in the presence of non-periodic disturbances. The aim of this paper is to remove this trade-off by introducing a nonlinear learning filter in repetitive control, which exploits distinct amplitude characteristics of periodic and non-periodic disturbances to apply varying learning gains. The contribution consists of the following elements.

- The influence of RC on periodic and non-periodic disturbances is analyzed (Section III).
- A nonlinear RC framework is introduced that achieves fast attenuation of periodic disturbances while mitigating the

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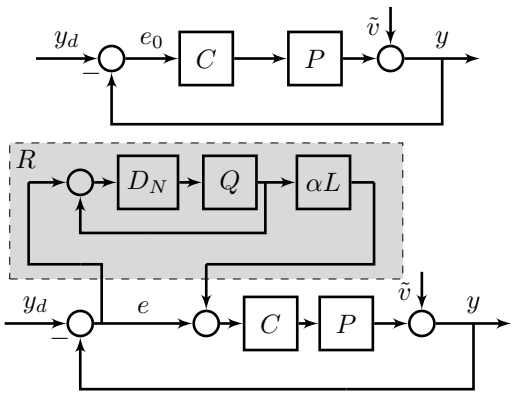

Fig. 1: Standard control scheme (top) and repetitive control scheme (bottom).

amplification of non-periodic disturbances (Section IV).

- Conditions for the convergence of the nonlinear RC system are developed, enabling design using identified frequency response functions of the linear system (Section V).
- The approach is illustrated in simulations (Section VI).

Notation: For a vector x and a matrix P, $\|x\|_P = \sqrt{x^{\mathsf{T}} P x}$ denotes the weighted 2-norm. The spectral radius of a matrix is denoted by $\rho(P)$. The $\mathcal{H}_\infty$-norm is denoted by $\|G\|_\infty = \sup_{\omega\in[0,2\pi)} |G(e^{i\omega})|$ for a real-rational, causal and stable transfer function $G \in \mathcal{RH}_\infty$. The sets of real, natural, and integer numbers are denoted by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$, respectively.

II. Problem formulation

Repetitive control aims to attenuate unknown exogenous periodic disturbances by learning a model of these disturbances that is included in the feedback controller in view of the internal model principle [1]. Consider a SISO LTI system P in feedback with a controller C as shown in Fig. 1 (top). The error e_0 of the original closed-loop system is given by

$$e_0 = \underbrace{(I + PC)^{-1}}_{S}(y_d - \tilde{v}), \tag{1}$$

with reference y_d and measurement noise $\tilde{v}$. A repetitive controller R is implemented as shown in Fig. 1 (bottom), such that the error e is given by

$$e = (1 + PC(1 + R))^{-1}(y_d - \tilde{v}) = \underbrace{(1 + TR)^{-1}}_{S_R} e_0, \tag{2}$$

with $T = PC(1 + PC)^{-1}$, see, e.g., [17] for a derivation. The repetitive controller R is given by

$$R(z) = \frac{\alpha L(z) z^{-N} Q(z)}{1 - z^{-N} Q(z)}, \tag{3}$$

with z^{-N} the z-domain representation of the delay operator D_N, i.e., $D_N(z) = z^{-N}$, which acts as a buffer storing the error signal of the previous repetition. Delay $N \in \mathbb{N}$ corresponds to a disturbance frequency $f_d = f_s/N$ for sampling frequency f_s. The learning gain is denoted by $\alpha \in (0, 1]$. Filters $Q(z) \in \mathcal{R}$ and $L(z) \in \mathcal{R}$ can have finite preview, i.e., be non-causal, as long as $R(z) \in \mathcal{RH}_\infty$ is causal which is ensured by embedding their preview in z^{-N}. Typically, L is chosen to approximate T^{-1}, for example using ZPTEC [18], and Q is a zero-phase low-pass filter. The resulting modifying sensitivity S_R is given by

$$S_R(z) = \frac{1 - z^{-N} Q(z)}{1 - z^{-N} Q(z)(1 - T(z)\alpha L(z))}. \tag{4}$$

A sufficient condition for the stability of the closed-loop system (2) is given by the following lemma [2].

Lemma 1. *Assume that S, T, Q and L are stable. Then the closed-loop repetitive control system (2) is stable for all $N \in \mathbb{N}$ if*

$$|Q(e^{i\omega})(1 - T(e^{i\omega})\alpha L(e^{i\omega}))| < 1 \quad \forall \omega \in [0, 2\pi). \tag{5}$$

Stability of S and T can be ensured through the feedback controller C, and Q and L are stable by design. Repetitive control can attenuate periodic exogenous disturbances with period N perfectly. For many control applications, only part of the exogenous error signal is periodic, for example when the reference $y_d(k) = y_d(k + N)$ is periodic but a noise term $\tilde{v}(k)$ is non-periodic. These non-periodic disturbances are amplified by traditional repetitive control due to the waterbed effect [2], as is further illustrated in Section III. The amplification may be reduced by reducing the learning gain, at the cost of reducing the learning speed.

The aim of this paper is to achieve fast attenuation of periodic disturbances while mitigating the amplification of non-periodic disturbances. It is assumed that these disturbances have distinct amplitude characteristics, which is exploited by a nonlinear learning filter that applies a high learning gain to high-amplitude periodic disturbances to ensure fast attenuation, and a low learning gain to low-amplitude non-periodic disturbances to avoid amplification.

III. Amplification and attenuation of disturbances in repetitive control

In this section, the attenuation and amplification of periodic and non-periodic disturbances in RC is analyzed. The error (2) of a stable RC system depends on S_R. The influence of Q and α on S_R is illustrated in the Bode magnitude diagram in Fig. 2, leading to the following observations:

- For $\alpha = 1$, the magnitude of the repetitive controller is $\ll 1$ at frequency f_d and its multiples, but due to the waterbed effect $|S_R| > 1$ at other frequencies.
- Reducing α if $Q \neq 1$ reduces the width and depth of the notches, and as a result the magnitude of S_R at other frequencies is increased less.
- Decreasing the cut-off frequency of the robustness filter Q reduces the number of multiples of the frequency f_d included in S_R, and it reduces the depth of the notches.

The influence of S_R on the settling time, in terms of the number of repetitions needed for the periodic disturbance to be attenuated, and on the converged errors is threefold. First of all, deeper notches result in faster attenuation of disturbances at the corresponding frequency, and therefore a faster settling time. Secondly, due to the waterbed effect

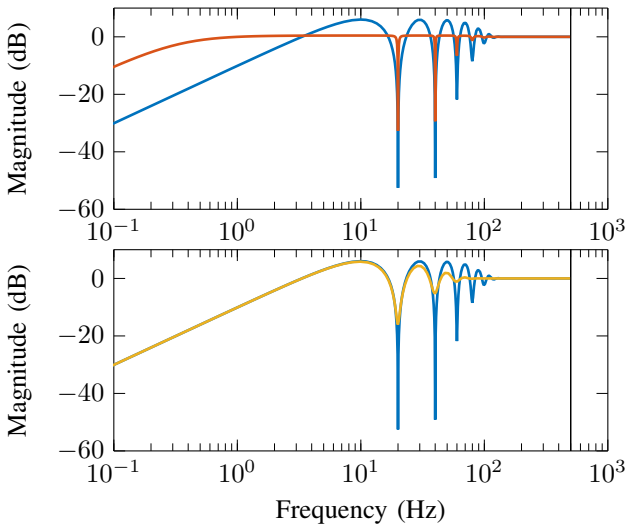

Fig. 2: Modifying sensitivity S_R for $L \approx T^{-1}$, $N = 50$ and $f_s = 1000\,\text{Hz}$. Top: $\alpha = 1$ (—) and $\alpha = 0.1$ (—) for a Q-filter with a cut-off frequency of 100 Hz. Bottom: Q-filters with cut-off frequencies 100 (—) and 50 Hz (—) for $\alpha = 1$.

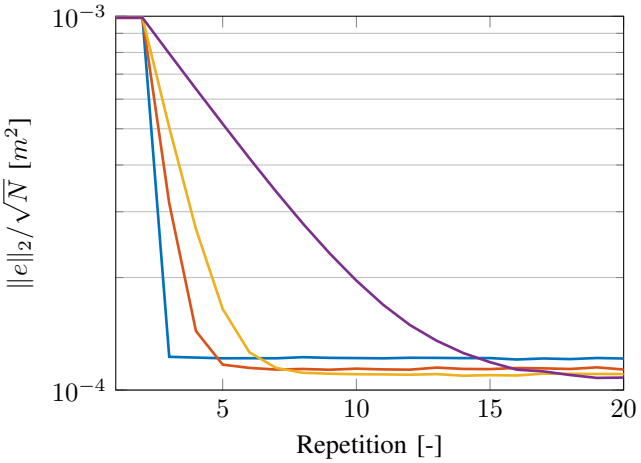

Fig. 3: Error norm reduction over repetitions for repetitive control (activated at repetition 2) for $\alpha = 1$(—), 0.7 (—), 0.5 (—) and 0.2 (—), averaged over 20 simulations. Reducing α reduces the converged error at the cost of convergence speed.

disturbances at frequencies other than f_d and its multiples may be amplified, and the amplification is stronger if the notches are wider and/or deeper. Thirdly, for wider notches more frequencies around f_d are suppressed, which is useful because in practice N may not be known exactly.

Based on these effects, it follows that the robustness filter Q is not suitable for reducing the amplification of non-periodic disturbances, since it also strongly influences the attenuation of periodic disturbances in the same frequency range. Regarding the learning gain, reducing α reduces the amplification of non-periodic disturbances significantly, but at the cost of reducing the width of the notches, which is undesirable if f_d is not known exactly, and at the cost of learning speed as illustrated in Fig. 3.

IV. Nonlinear Repetitive Control

In this section, the repetitive controller is extended by a deadzone nonlinearity, which adapts the learning gain based on the distinct amplitude characteristics of disturbances to achieve both fast attenuation of periodic disturbances and limited amplification of non-periodic disturbances.

The deadzone nonlinearity φ is implemented as shown in Fig. 4. The input e of the nonlinearity is written as a function of its output signal $e_\varphi = \varphi(e)$ and the disturbances contained in e_0, see (1), as follows.

$$e = T_R e_\varphi + S_R e_0, \tag{6}$$

with modifying sensitivity S_R defined according to (4), and T_R being the complementary sensitivity of the linear RC loop, given by

$$T_R = (1 + TR)^{-1} TR. \tag{7}$$

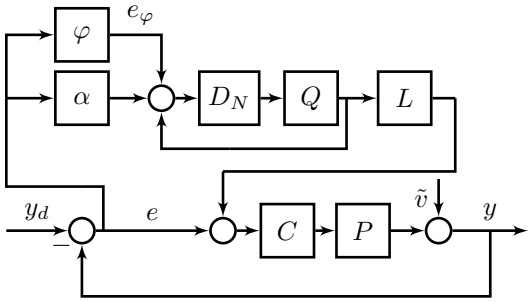

Fig. 4: Nonlinear repetitive control scheme.

The output of the deadzone nonlinearity $\varphi(e(k))$ is given by

$$\varphi(e(k)) = \begin{cases} 0, & \text{if } |e(k)| \le \delta \\ (\gamma - \frac{\gamma\delta}{|e(k)|})e(k), & \text{if } |e(k)| > \delta. \end{cases} \tag{8}$$

The deadzone is a static nonlinearity that satisfies the incremental sector condition

$$0 \le \frac{\varphi(a) - \varphi(b)}{a - b} \le \gamma \tag{9}$$

for any two scalars a and b. This property enables the convergence analysis of the nonlinear repetitive control system in the next section. In this paper, a deadzone nonlinearity is used to distinguish between periodic and non-periodic disturbances based on their amplitude characteristics, but the framework allows the use of any static nonlinearity that satisfies an incremental sector condition.

The main idea of extending RC with a deadzone is that only error values that are above the threshold δ are stored in the buffer, where δ is chosen such that $\varphi(e_{np}) \approx 0$ for non-periodic disturbances e_{np}. To determine a suitable deadzone width δ, a measurement without RC that contains several repetitions of the periodic disturbance is used. The measured

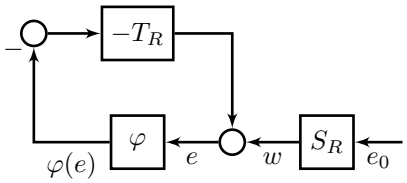

Fig. 5: Nonlinear repetitive control as a Lur'e system.

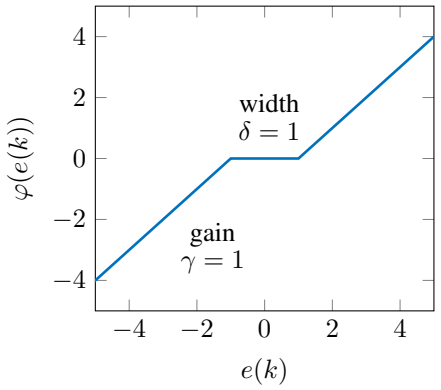

Fig. 6: Deadzone nonlinearity φ according to (8) with width $\delta = 1$ and gain $\gamma = 1$.

error is divided in n_r repetitions with length N, given by

$$e_j = \begin{bmatrix} e((j-1)N+1) & \dots & e(jN) \end{bmatrix}^{\mathsf{T}} = e_p + e_{np,j} \quad (10)$$

with e_p the periodic disturbance and $e_{np,j}$ the realization of the non-periodic disturbances for repetition j. An estimate of e_p is given by the sample mean of these repetitions, i.e.,

$$\hat{e}_p = \frac{1}{n_r} \sum_{j=1}^{n_r} e_j, \qquad \hat{e}_{np,j} = \hat{e}_p - e_j, \quad (11)$$

with $\hat{e}_{np,j}$ the estimates of the realizations of the non-periodic disturbances. These estimates are used to determine a suitable deadzone width δ, as illustrated in Section VI.

V. Convergence of Nonlinear Repetitive Control

To analyze the existence and global exponential stability of the steady-state solution of the nonlinear repetitive control system (6), it is rewritten in state-space form as a cascade of a linear system S_R and a Lur'e system formed by the linear system $-T_R$ with static nonlinearity φ in feedback, see Fig. 5. The dynamics of the cascaded system are given by

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}\right\} -T_R \quad (12a)$$

$$\left.\begin{aligned} u(k) &= -\varphi(y(k) + w(k)) \\ n(k+1) &= An(k) + Be_0(k) \\ w(k) &= Cn(k) + e_0(k) \end{aligned}\right\} S_R \quad (12b)$$

The matrices A, B and C form a minimal state-space realization of $-T_R$, such that (A, B) is controllable and (A, C) is observable. Note that since $S_R = 1 - T_R$, (12a) and (12b) have the same A, B and C-matrices.

Next, conditions under which the nonlinear repetitive control system is *convergent* are developed. Convergence for a discrete-time system of the form

$$x(k+1) = f(x(k), k), \quad (13)$$

with state $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n$ and discrete-time variable $k \in \mathbb{Z}$, is defined as follows [19, Definition 1].

Definition 2. *System (13) is called exponentially convergent if*

- *there exists a unique solution $\bar{x}(k)$ that is defined and bounded on $\mathbb{Z}$ (from $-\infty$ to $+\infty$),*
- *$\bar{x}(k)$ is globally exponentially stable, i.e., there exists $c > 0$ and $0 < \lambda < 1$ such that $|x(k) - \bar{x}(k)| \leq c\lambda^{k-k_0}|x(k_0) - \bar{x}(k_0)|$ for all $k \geq k_0$.*

Solution $\bar{x}(k)$ is called a steady-state solution. It follows from this definition that any solution of a convergent system converges to this steady-state solution, irrespective of the initial condition. The time-dependency of (13) is typically due to an input $w(k)$, and if the input is periodic then the steady-state solution of the convergent system is also periodic with the same period [19]. The convergence property is an extension of the stability properties of asymptotically stable linear systems excited by external inputs, and since for traditional (linear) RC asymptotic stability is desired [2], for nonlinear RC convergence is shown. For the convergence of a nonlinear system, the following holds [19, Theorem 1].

Lemma 3. *Consider system (13) with a Lipschitz continuous [20, Chapter 3] right-hand side satisfying*

$$\|f(x_1, k) - f(x_2, k)\|_P \leq \lambda \|x_1 - x_2\|_P, \quad \forall x_1, x_2 \in \mathbb{R}^n, k \in \mathbb{Z} \quad (14)$$

$$\sup_{k \in \mathbb{Z}} \|f(0, k)\|_P < +\infty, \quad (15)$$

for some matrix $P = P^{\mathsf{T}} \succ 0$ and number $\lambda \in (0, 1)$. Then system (13) is exponentially convergent.

For the cascade of two exponentially convergent systems, the following lemma holds.

Lemma 4. *Consider the cascaded system*

$$\begin{cases} x(k+1) = f(x(k), w(k), k) \\ w(k+1) = g(w(k), k). \end{cases} \quad (16)$$

Suppose the x-subsystem meets the conditions in Lemma 3 for any $w(k)$ bounded on $\mathbb{Z}$ and is therefore exponentially convergent, and the w-subsystem is exponentially convergent. In addition, $f(x, w, k)$ is globally Lipschitz with respect to w with Lipschitz constant K, i.e.,

$$\|f(x, w_1, k) - f(x, w_2, k)\| \leq K\|w_1 - w_2\| \forall x, k. \quad (17)$$

Then system (16) is exponentially convergent.

A proof of Lemma 4 is given in Appendix A. Applying Lemma 3 and 4 to the nonlinear repetitive control system leads to the following convergence theorem.

Theorem 5. *Let (12a) be a minimal realization of the linear system $-T_R$ with (A, B) controllable and (A, C) observable. The system (12) is exponentially convergent for any input $e_0(k)$ bounded on $\mathbb{Z}$ if the following conditions are met:*

a) $\rho(A) < 1$.

b) φ satisfies (9) for a certain γ.

c) The following small-gain condition holds:

$$\sup_{\omega\in[0,2\pi)} |T_R(e^{i\omega})| < \frac{1}{\gamma}. \tag{18}$$

Proof. Consider system (12) as a cascade of (12a) and (12b). The idea is to establish exponential convergence as a cascade of two exponentially convergent systems. First, condition a) ensures that system (12b) is exponentially convergent for any bounded input $e_0(k)$, as a linear exponentially stable system. Secondly, the incremental sector condition in b) ensures that the right-hand side of (12a) is globally Lipschitz in w, uniformly in x, i.e., that (17) holds. Thirdly, conditions a)-c) ensure that the nonlinear Lur'e system (12a) with $w(k)$ as inputs is convergent. This follows from the proof of [13, Theorem 9], which shows that a sufficient condition for the convergence of the nonlinear Lur'e system (12a) is the existence of a matrix $P = P^\mathsf{T} \succ 0$ such that the strict inequality

$$\begin{bmatrix}\Delta x\\ \Delta\varphi\end{bmatrix}^\mathsf{T} \begin{bmatrix}A^\mathsf{T}PA + C^\mathsf{T}C - P & -A^\mathsf{T}PB\\ -B^\mathsf{T}PA & B^\mathsf{T}PB - \frac{1}{\gamma^2}\end{bmatrix} \begin{bmatrix}\Delta x\\ \Delta\varphi\end{bmatrix} < 0,\ \forall x_1, x_2 \in \mathbb{R}^n, k \in \mathbb{Z}, \tag{19}$$

holds, with $\Delta x = x_1 - x_2$ and $\Delta\varphi = \varphi(Cx_1 + w(k)) - \varphi(Cx_2 + w(k))$. By the Kalman-Szegö lemma, see, e.g., [21, Lemma 17], a matrix $P = P^\mathsf{T} \succ 0$ for which (19) holds exists if and only if

$$\|-T_R\|_\infty = \|T_R\|_\infty = \sup_{\omega\in[0,2\pi)} |T_R(e^{i\omega})| < \frac{1}{\gamma}, \tag{20}$$

which is ensured by condition c). It follows that if conditions a)-c) are satisfied, then system (12a) is exponentially convergent for any bounded $w(k)$ by Lemma 4. □

Condition a) of Theorem 5 can be satisfied through the design of L and Q. Stability of the linear system is verified using identified frequency response functions of the system, and Q can be designed explicitly to ensure robustness against model uncertainty in L. Similarly, Condition c), which can be satisfied by choosing suitable values for α and γ, can be verified using a measured frequency response of the system.

VI. Printer Example

In this section, nonlinear repetitive control is applied to a simulated printer system, the carriage of which performs a periodic motion. The printer is modeled as a discrete-time non-collocated two-mass-spring-damper system, and the schematic representation and Bode diagram are shown in Fig. 7. The system is sampled at $f_s = 1000$ Hz and the reference y_d leads to a disturbance that is periodic with $N = 2000$ such that $f_d = 0.5$ Hz. The non-periodic disturbance comes from zero-mean Gaussian white output noise $\tilde{v}$ with a variance

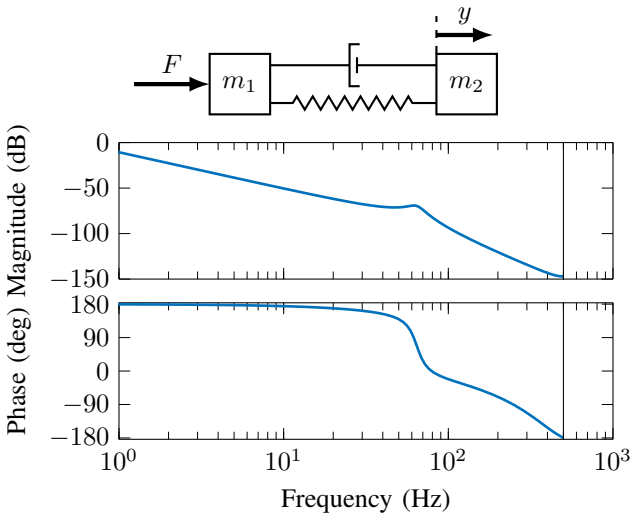

Fig. 7: Schematic representation (top) and Bode diagram (bottom) of the printing system.

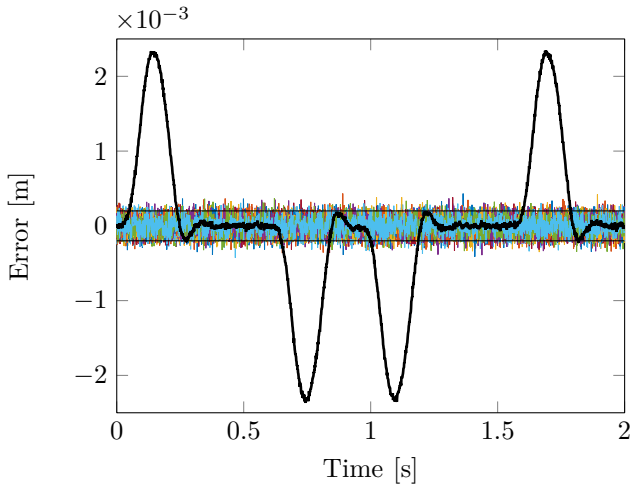

Fig. 8: Estimates of the periodic disturbance $\hat{e}_p$ (—) and 20 realizations of the non-periodic disturbances $\hat{e}_{np}$. Most noise is contained within the interval $[-2\times10^{-4}, 2\times10^{-4}]$, indicated by black lines.

of 10^{-8}. The learning filter L is constructed using ZPETC and the robustness filter Q is a 30th order zero-phase low-pass FIR filter [2] with a cut-off frequency of 200 Hz. Each simulation is repeated 20 times and averaged.

Based on the approach in Section IV, which is illustrated in Fig. 8, the deadzone width is set to $\delta = 2 \times 10^{-4}$. In Fig. 9 the error 2-norm over repetitions for nonlinear RC with $\gamma = 1$, $\alpha = 0.1$ is compared to linear RC with $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0.2$. It is shown that nonlinear RC reaches low errors without reducing the convergence speed.

VII. Conclusion

Using nonlinear filters in repetitive control enables adaptive adjusting of the learning gain, allowing highly accurate tracking while mitigating the amplification of non-periodic disturbances. This leads to a different and less stringent

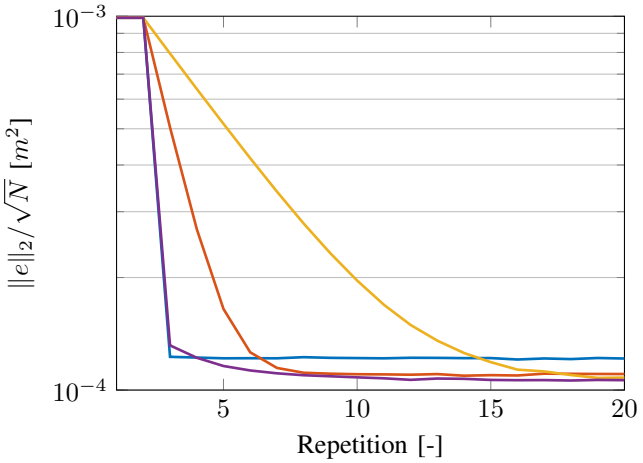

Fig. 9: Error norm reduction over repetitions for repetitive control, averaged over 20 simulations. Compared to linear RC for $\alpha = 1$(—), 0.5 (—), and 0.2 (—), nonlinear RC with $\gamma = 1$, $\alpha = 0.1$ and $\delta = 2 \times 10^{-4}$ (—) achieves faster learning and lower errors. RC is activated at repetition 2.

trade-off compared to LTI designs, which are limited by the traditional Bode sensitivity integral. The approach achieves both fast learning and small converged errors through a deadzone nonlinearity, which applies variable gains based on the amplitude characteristics of the disturbances. Conditions for the convergence of the discrete-time nonlinear RC system are developed. Simulations illustrate that the approach achieves fast learning and small errors, and future research will involve experimental implementation of the approach.

Appendix A

In this appendix the proof of Lemma 4 is given.

Proof of Lemma 4. For the cascaded system (16), define

$$F(z(k),k) := \begin{cases} x(k+1) = f(x(k),w(k),k) \\ w(k+1) = g(w(k),k), \end{cases} \tag{21}$$

with $z = \begin{bmatrix} x & w \end{bmatrix}^{\mathsf{T}}$. Let

$$\|f(x_1,w,k) - f(x_2,w,k)\|_P \leq \lambda_1 \|x_1 - x_2\|_P \tag{22}$$

$$\|g(w_1,k) - g(w_2,k)\|_Q \leq \lambda_2 \|w_1 - w_2\|_Q, \tag{23}$$

for some matrices $P = P^{\mathsf{T}} > 0$ and $Q = Q^{\mathsf{T}} > 0$ and numbers $\lambda_1, \lambda_2 \in (0,1)$, and $\mathbf{P} := \mathrm{diag}(P, \alpha Q)$ for some $\alpha > 0$. For system (16), it holds that

$$\begin{aligned} \|F(z_1,k) - F(z_2,k)\|_{\mathbf{P}} &= \left\| \begin{matrix} f(x_1,w_1,k) - f(x_2,w_2,k) \\ g(w_1,k) - g(w_2,k) \end{matrix} \right\|_{\mathbf{P}} \\ &= \|f(x_1,w_1,k) - f(x_2,w_2,k)\|_P + \|g(w_1,k) - g(w_2,k)\|_{\alpha Q} \\ &\leq \|f(x_1,w_1,k) - f(x_1,w_2,k)\|_P + \|g(w_1,k) - g(w_2,k)\|_{\alpha Q} \\ &\quad + \|f(x_1,w_2,k) - f(x_2,w_2,k)\|_P \qquad (24) \\ &\leq \lambda_1 \|x_1 - x_2\|_P + \lambda_2 \|w_1 - w_2\|_{\alpha Q} + K \|w_1 - w_2\|_P, \end{aligned}$$

with K the Lipschitz constant for f in (17). In inequality (24), $\alpha > 0$ can be chosen such that $L\|w_1 - w_2\|_P \leq \lambda_3 \|w_1 - w_2\|_{\alpha Q}$, where $\lambda_3 > 0$ is small enough to satisfy $\lambda_2 + \lambda_3 < 1$. Thus, we obtain from (24):

$$\begin{aligned} &\|F(z_1,k) - F(z_2,k)\|_{\mathbf{P}} \qquad (25) \\ &\quad \leq \lambda_1 \|x_1 - x_2\|_P + (\lambda_2 + \lambda_3) \|w_1 - w_2\|_{\alpha Q} \\ &\quad \leq \max\{\lambda_1, (\lambda_2 + \lambda_3)\} \|z_1 - z_2\|_{\mathbf{P}}, \end{aligned}$$

with $\max\{\lambda_1, (\lambda_2 + \lambda_3)\} < 1$. It follows that the cascaded system is exponentially convergent by Lemma 3. □

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