The relativistic dynamics of labor economics

An economic engineering treatment

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Master of Science Thesis

Delft University of Technology

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Abstract

In this thesis, the analogy between the special theory of relativity and the dynamics of a laborer is developed in the context of labor economics. At the basis of this analogy stands an individual laborer who cannot supply more than 24 hours of labor in a day. This represents the theoretical limit to the flow of labor services (velocity). We argue this limit is analogous to the speed of light. The development of the analogy continues using hyperbolic functions independent of the demand frame of reference (frame of reference) and dependent on the degree of demand (rapidity) as well as the wage inelasticity (mass). This analogy describes the behavior of an individual laborer, detailing the quantity of labor services (position) and their flow, the wage (momentum) and the causation of changes in the flow of labor services (forces). These dynamics in labor economics are consistent with the theory of special relativity, demonstrating economic engineering principles.

Economic engineering is applied to model an individual laborer using the newly developed analogy. The laborer's supply curve shows that wage inelasticity does not change when a laborer performs more labor. Instead, the nonlinear supply curve is attributed to the difference in a laborer's perception of time (proper time). The perception of time depends on the flow of labor services of the observer, making it possible to observe the labor market from different perspectives, including those of companies and laborers. The laborer's perspective on their supply is visualized on the Poincaré disk, from which occupational compositions and job transitions can be analyzed.

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Preface

Writing this thesis was the final task before graduating, and a daunting one at that. It required me to let go of much of the control I thought I had over the planning of my life. Through this, it taught me the importance of enjoyment when deciding on which time commitments to make. Looking back, the changes in the direction of this thesis caused me the most stress, but learning about so many different topics is also what caused me the most joy. Therefore, I am excited to present my thesis, and with it start my new, unpredictable life.

I extend my heartfelt gratitude for the invaluable support I have received during the process of writing this thesis. First and foremost, I would like to express my deep appreciation to Dr. Ir. Mendel for being so readily available to provide feedback and engage in discussions at every step of the way. I am truly inspired by his exceptional mathematical intuition and expertise of physics, mathematics, and economics. Furthermore, he taught me to stray critical of the beliefs of people seemingly more accomplished, and encouraged me to trust my own judgment.

Secondly, I want to thank my dad for his unwavering support throughout the entirety of my university studies. I also want to thank my partner James for gifting me a tablet at the beginning of the process, which I used almost every day, for hours. Furthermore, I want to thank him, and my other friend groups for hearing me out about my thesis and making my studies in Delft very enjoyable, especially Jasper and Sachin whom I've known since the first week of my Bachelor, and Jingru whom I met the first week of this Masters and Hong Yang.

Furthermore, I would like to thank the economic engineering group, ir. Hutters, Gijs, Pien, Berend, Tjeerd and Jeroen for their feedback and suggestions following my presentations.

Brigitte de Graaf

Delft, November 17, 2023

“The value of a college education is not the learning of many facts but the training of the mind to think.”

— *Albert Einstein [20]*

Chapter 1

Introduction

In the field of labor economics, the objective is to comprehend the dynamics of the labor markets. Labor markets consist of laborers who perform labor, usually in exchange for a wage. The wage is determined based on demand and supply, influenced by social, cultural and political variables. Economists construct macroeconomic models of the economy, of which labor market models are a component. These labor market models are used by policymakers to develop policies regulating the labor market, such as those concerning unemployment rates and wage gaps. [6], [63]

A first attempt at developing an economic engineering labor market model was made by Huisman, who utilized a commodity analogy. As a result of this approach, he modeled the sale of laborers instead of their labor services. This is not an accurate representation of the labor market in the Netherlands for it is not legal to sell people [54]. The model furthermore does not consider that it is not theoretically possible for a single laborer to perform more than 24 hours of labor in a day. [31]

The goal of this thesis is therefore to develop an economic engineering analogy that describes the dynamics of a laborer. A laborer who is not considered to be a commodity, nor for whom it is possible to perform labor services at a rate faster than time. For the purpose of interpretation of the analogy, we furthermore have the objective to visualize this analogy, starting with the laborer's kinematics.

This goal marks a step towards the broader vision of working towards a macroeconomic economic engineering model that includes the modeling of individual laborers. Economic engineering offers the possibility of modeling the dynamics of individual laborers as part of macroeconomic models through agent-based modeling [34]. Whereas the CPB is currently not able to model this due to computational limitations [15], [24]. Modeling individual laborers has the potential to provide insights to policy analysts on topics considered important by contemporary governments [73]. For example, it allows assessing distributions of variables (such as unemployment) instead of only considering the mean, and it allows for assessing group differences (such as wage gaps) [51], [35], [28].

In chapter 2 the analogy of labor services to linear mechanics is first developed, in a similar manner as other analogies in economic engineering. An overview of this linear mechanical analogy is provided in Table B-1 in appendix B-1. The linear mechanical analogy is then extended to the rotational mechanics analogy for labor services in section 2-3, an overview of this analogy is provided in Table B-2 in appendix B-2.

The results of the complete analogy are compared to the expected results based on the current knowledge in labor economics in section 2-4-2. In this section, the contradictions between our Newtonian analog and labor economics are detailed, and the next steps in developing a correct analogy are determined.

For the reader already familiar with economic engineering and somewhat familiar with labor economics I suggest not reading the entirety of chapter 2. Instead, I would suggest only reading section 2-4-2 and the paragraph 'Inertia as inelasticity' in section 2-2. Because, the Newtonian analogy ultimately does not consider the theoretical limit to the flow of labor services and thus is not the analogy ultimately proposed.

In chapter 3 the knowledge of the special theory of relativity is used to develop the dynamics describing a laborer functioning in the context of labor economics. Developing the analogy analogous to the special theory of relativity allows for modeling the theoretical limit to the flow of labor services. This is because the flow of labor services is deemed analogous to the velocity, whose theoretical limitation is the speed of light. Throughout the development of the relativistic analogy, the symbols are chosen to be the same as their counterpart in special relativity, as much as reasonably allows, such that the developed formulas can just as easily be used by physicists as by economic engineers.

The mathematical structures, describing the laborer's quantity of labor over time, are set up in section 3-2, forming LS spacetime. From here, the dynamics of the laborer are developed. To determine the running labor cost, the stationary action principle is utilized in subsection 3-3-1. The running labor cost is then used to determine the wage in subsection 3-3-2. The obtained information is filled into the Legendre transform which is then used to define the laborer's surplus in subsection 3-3-3.

Furthermore, in section 3-4 the wage-surplus relationship is determined, analogous to the momentum-energy relationship in special relativity to describe the total surplus of the laborer. Lastly, the drivers of the laborer are identified through the wage vector, analogous to the 4-momentum vector in subsection 3-4-3.

In chapter 4 a method is proposed to visualize the flows and real flows of labor services. We do this by projecting different flows onto one hyperboloid which describes the flow space. In turn, this flow space is projected onto a Poincaré disk. Then, this disk is used to define job compositions and job transformations.

Finally, the conclusions and recommendations are presented in chapter 5. The extent to which the relativistic analogy describes the dynamics of labor economics is discussed. In addition, a reflection on what is achieved concerning visualizations of the analogy is provided.

Furthermore, the next steps in the development of the relativistic analogy are proposed.

In section 5-2, further recommendations are made on how the analogy developed in this thesis can be utilized to build macroeconomic models representing the labor market. It also elaborates on the potential of the analogy to provide insights concerning other service analogs in economic engineering. Finally, a suggestion for the continuation of the interpretation of descriptive geometry is provided.

Chapter 2

The Newtonian dynamics of labor services

2-1 Introduction

As a first step towards modeling the labor market through economic engineering principles, analogies relating engineering and labor economics are developed. The analogies developed interpret labor economics through the linear mechanical analogy and the rotational mechanical analogy, also referred to as the commodity analogy and the capital analogy. This set of analogs is referred to as the Newtonian analogy throughout this thesis. Afterward, a comparison is provided between the resulting kinematics of the new analogies and what economists describe to be the dynamics of labor services.

At the core of labor economics are labor services. The term *labor services* will be used throughout this thesis to describe labor that a laborer provides, in exchange for a wage.

The overview of the developed analogies is provided in Appendix B, the commodity analogy in Table B-1 and capital analogy in Table B-2. A brief overview of the economic engineering approach, and the reasoning for starting with developing a linear and rotational mechanical labor analog, are provided in Appendix A.

2-2 The linear mechanical labor analogy

In this section, labor economics will be interpreted analog to the linear mechanical analogy, in economic engineering named the commodity analogy [48].

The linear mechanical analogy is used to describe translational motions. It treats labor as a commodity, an object that can be bought, sold and stored. However, this is not the case for labor, as a laborer can either perform labor during a certain time frame or not. Labor can thus not be stored, nor can the laborer go back in time to do more labor during times the laborer did not initially work. Finally, it is not possible for a single laborer to work more than one hour within a single hour. Furthermore, a laborer themselves cannot legally be sold, they instead yield labor which, in turn, can be sold [54]. This makes a capital analogy more suitable, for it describes an economic situation in which goods offer value only through selling their yield. The capital analogy is described by the analogy to rotational mechanics, which is an extension of the linear mechanical analogy [48]. Therefore, regardless of the capital analogy describing the phenomena of labor more accurately, the commodity analogy still needs to be developed first.

The particle as laborer

Newtonian mechanics describes how an object behaves, whereas labor economics describes how a laborer behaves. In mechanics the object can be idealized as a point particle [53]. Similarly, in economic engineering a laborer can be idealized. The laborer decides how much labor it will perform based on where its supply and the demand for labor on the labor market intersect.

The behavior of the laborer can be visualized as in Figure 2-1. In here, the x-axis represents the quantity of labor that the laborer performs. The y-axis represents the wage rate. The blue line represents the quantity of labor the laborer is willing to supply for the wage rate associated with it. The red line represents the quantity of labor the demander, such as a company, is willing to employ when paying the associated wage rate. The wage rate is the pay per unit of time. The laborer's behavior in economics is thus visualized through a supply line. [49], [3], [6]

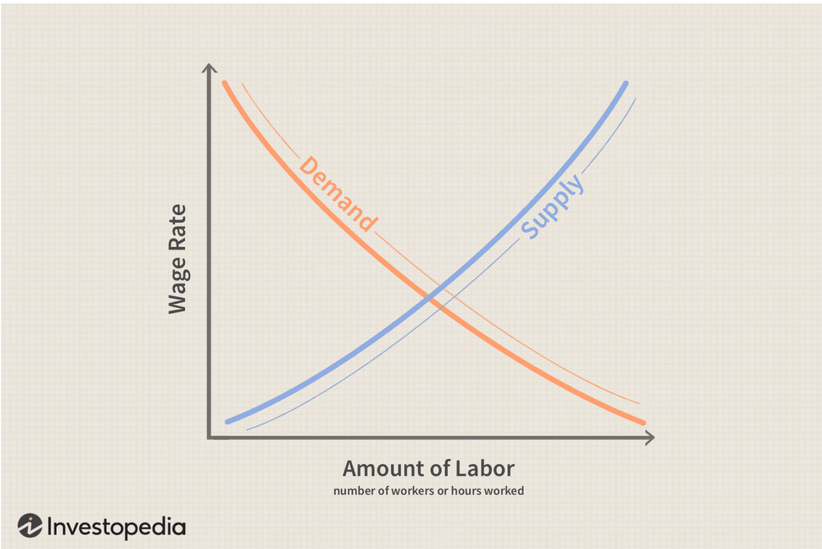


Figure 2-1: Supply and demand curve for labor [3]

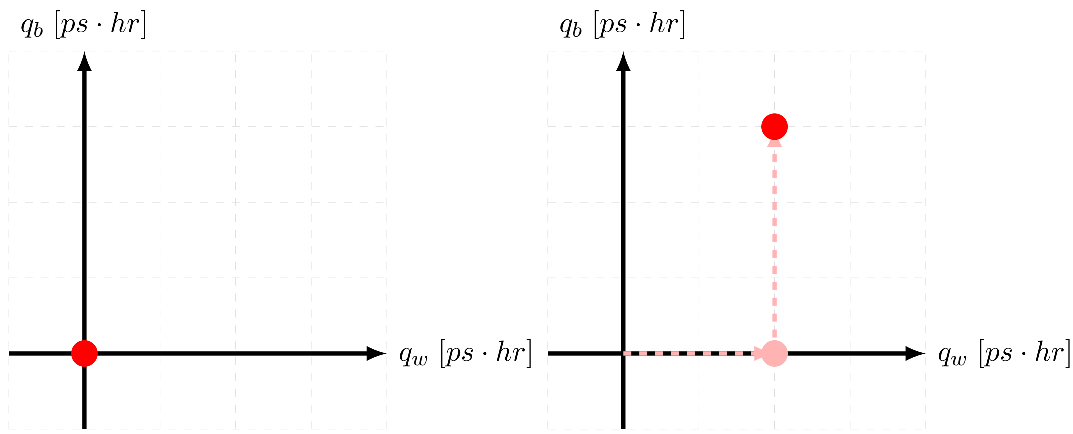
Kinematics to describe a laborers participation in the labor market

The next step towards developing the analogy is describing the laborers' movements in commodity space. The analogy to commodity space is physical space, as this is where the movements of the laborer will take place. The commodity space is an Euclidean space, just like physical space. However, it can differ in the numbers of dimensions. The dimension can represent the different types of labor which are defined based on the criteria deemed best fit for the desired application.

For example, when studying the wage gap between different genders, one dimension could be used to describe labor performed by men, a second dimension for labor performed by women, and a third for labor of non-binary people [21]. Or, when assessing the role of different types of labor in society the different dimensions could be white and blue-collar labor (office labor and manual labor) [30]. Commodity space for this case is thus $\mathbb{R}^n = \mathbb{R}^3$. For the number of dimensions of commodity space n is equal to the number of different types of labor assessed.

The position of the laborer in commodity space is defined by the time the laborer has performed the labor associated with the dimensions q, representing the *quantity of labor services*. Continuing with the last example, the commodity space can be visualized as in Figure 2-2. In Figure 2-2, the dimension of white-collar work is represented by q_w and that of blue-collar work by q_b. In Figure 2-2a the laborer has not performed labor yet. Once the laborer starts performing labor, the total number of hours performed by the laborer are tracked, and expressed in units $[ps \cdot hr]$.

For intuition, when a task takes 20 hours to complete, this task thus requires a single laborer 20 hours. It positions the laborer at a quantity of labor services of 20 when completing the task. However, when studying a group of 4 laborers instead of a single laborer, it is finished in 5 hours if they all work on it. This positions the group, again, at a quantity of labor services of 20 at the end of the task. Therefore, the unit of the dimension becomes the hours the persons performed labor, thus, q will be in $[person \cdot hours]$, in short, $[ps \cdot hr]$.



(a) Supply curve in economic engineering **(b)** Supply curve in economic engineering

Figure 2-2: Movement of a laborer through commodity space

Movement through commodity space takes place when the laborer performs labor. Movement is a change in quantity of labor services of the laborer. For example, when the laborer performs two hours of white-collar labor services the laborer will move over the light red arrow to position $(2, 0)$, visualized by the light red dot in Figure 2-2b. When the labor then performs an additional three hours of blue-collar labor services the laborer will find itself at position $(2, 3)$, as visualized by the red dot.

Whereas displacement in physical space takes place at a rate called the velocity, the change in quantity of labor takes place at a rate called the *flow of labor services*. Which will be denoted by $\dot{q}$ and is presented in equation (2-1). Similar to physical space, the rate of change is per unit of time. Therefore, the units of the flow of labor services are $[\frac{ps \cdot hr}{hr}] = [ps]$. For intuition, when asking someone how fast a task requiring 20 $ps \cdot hr$ of labor services is going, an example of a response is "we are working on it with 5 people".

$$\dot{q} = \frac{dq}{dt} \tag{2-1}$$

Next, we consider the interpretation of the change (over time) in the number of laborers performing labor. It is interpreted as the *additional hiring* and is described per equation (2-2). The variable used to represent the additional hiring is $\ddot{q}$ which is expressed in the units $[\frac{ps}{hr}]$. Unsurprisingly, this is analogous to the acceleration, for it is the time derivative of the velocity.

$$\ddot{q} = \frac{d\dot{q}}{dt} \tag{2-2}$$

The labor economics interpretation analogous to the kinematics of linear mechanics is presented in Table 2-1.

Labor economics	**Symbol**	**Unit**	**Linear mechanical analog**
Laborer			Particle
Quantity of labor services	q	$[ps \cdot hr]$	Position
Flow of labor services	$\dot{q}$	$[ps]$	Velocity
Additional hiring	$\ddot{q}$	$[\frac{ps}{hr}]$	Acceleration
Time	t	$[hr]$	Time

Table 2-1: Labor economics analogous to the kinematics of linear mechanics

Inertia as inelasticity

The equivalent of mass in the domain of labor economics is the resistance against a change in the flow of labor services. For the inertia of a point particle is the measure of resistance against a change in its velocity. And this inertia is called the mass in the case of linear mechanics. [50] To determine the interpretation of this in labor economics we consider what economists deem to be the factors in determining the flow of labor services. These factors are the wage rate and the inelasticity of demand or supply of the flow of labor services. This is visualized in Figure 2-3 where the demand concerns the demand of the company for a flow of labor services.

The higher the wage elasticity, the more elastic the demand or supply is considered to be.

And the more elastic the demand or supply is, the flatter the demand or supply line is. Thus, the economic engineering analogy is that an elastic (flexible) laborer will require relatively little change in wage to perform a different flow of labor services. [32],

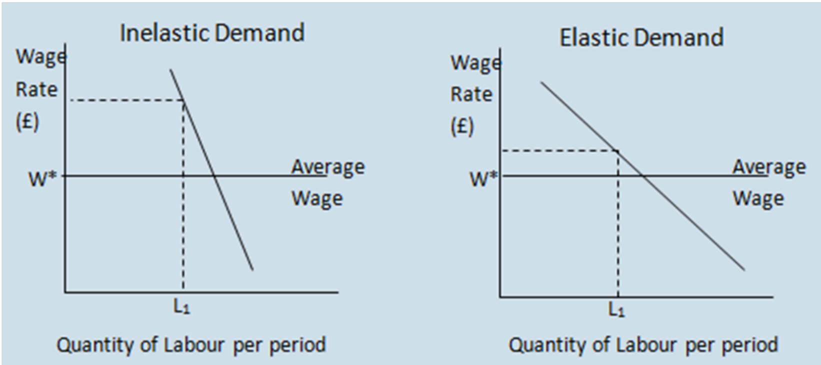


Figure 2-3: Inelastic and elasticity demand curve [59]

In physics, however, a relatively large mass changes velocity less compared to a smaller one, when the same force is applied [50]. Thus, whilst a large mass results in a small change in velocity, a large elasticity results in a large change in the flow of labor services. Therefore, the mass is considered to be the *wage inelasticity* m, with *wage elasticity* ϵ, as presented in Table 2-2.

It is important to note that the definition of inelasticity in economic context differs slightly from the economic engineering context. For in economics, the wage elasticity of supply is calculated based on the percentage change, as in equation (2-3), [71]. This formula is expressed in the economic engineering terminology used thus far.

$$\text{wage elasticity of supply} = \frac{\text{percentage change in flow of labor services supplied}}{\text{percentage change in wage}} \tag{2-3}$$

Whilst in the economic engineering context it is based on the absolute changes, as presented in equation (2-4) [49].

$$\text{wage elasticity of supply} = \frac{\text{change in flow of labor services supplied}}{\text{change in wage}} \tag{2-4}$$

From labor economics, it is known that a wage is expressed in a unit of money over the unit totaling the time spent, regardless of what its analogy is in physics [72]. Therefore the units are $[\frac{\$}{ps \cdot hr}]$ when taking \$ for the currency representing money. It can thus be derived that the units for the wage elasticity are $[\frac{ps^2 \cdot hr}{\$}]$, as in equation (2-6). Furthermore, the wage elasticity is the inverse of the wage inelasticity. Resulting in the units $[\frac{\$}{ps^2 \cdot hr}]$ for the wage inelasticity, as derived in equation (2-6).

$$[\text{wage inelasticity of supply}] = ([\text{wage elasticity of supply}])^{-1} \tag{2-5}$$

$$= (\frac{ps}{\frac{\$}{ps \cdot hr}})^{-1} = \frac{\$}{ps^2 \cdot hr} \tag{2-6}$$

Labor economics	Symbol	Units	Linear mechanical analog
Wage inelasticity	$m = \frac{1}{\epsilon}$	$\frac{\$}{ps^2 \cdot hr}$	Mass

Table 2-2: Labor economics analogous to the inertia

Dynamics to describe the laborers' incentives for participating in the labor market

Let us determine the interpretation of the momentum. So far, the m and $\dot{q}$ have been determined to be the wage inealsticity and the flow of labor services, which are the analogs to the mass m and velocity $\dot{q}$. It is furthermore known from physics that using these two variables the momentum is determined simply by multiplying them. [58] The equation to determine what is analogous to momentum becomes as presented in equation (2-7).

$$p = m\dot{q} \tag{2-7}$$

Multiplying the wage inelasticity with the change in flow of labor services results in the units $[\frac{\$}{ps \cdot hr}]$, which aligns with the expected units of wage as well as its interpretation. Just like in labor economics, when performing labor, $\dot{q} > 0$ ps, a wage needs to be provided $p > 0$ $\frac{\$}{ps \cdot hr}$. When no wage is provided $p = 0$ $\frac{\$}{ps \cdot hr}$, the laborer will not provide labor and therefore $\dot{q} = 0$ ps. Therefore, the momentum is analogous to the *wage*.

The time derivative of the momentum is the force [58]. Analogous to this, the change in wage is the incentive for changing the flow of labor services as described in equation (2-8). This incentive ultimately yields a change in the flow of labor services that the laborer wants to perform. Hence, the force is analogous to what is named the *want for labor*, which is expressed in the units $[\frac{\$}{ps \cdot hr^2}]$.

$$F = \frac{dp}{dt} \tag{2-8}$$

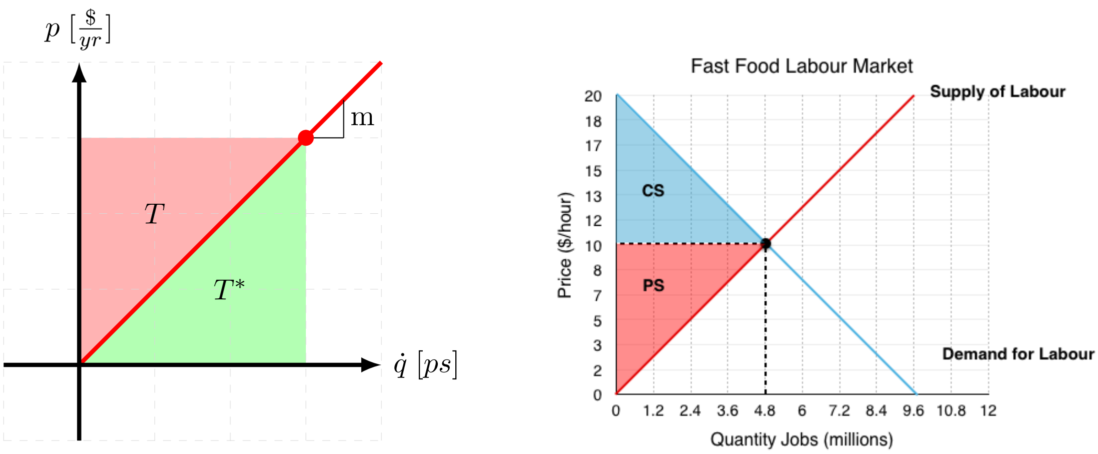
The dynamics of the labor commodity analogy are presented in Table 2-3.

Labor economic	Symbol	Unit	Linear mechanical analog
Wage	p	$[\frac{\$}{ps \cdot hr}]$	Momentum
Want for labor	F	$[\frac{\$}{ps \cdot hr^2}]$	Force

Table 2-3: Labor economics analogous to the dynamics of linear mechanics

Co-energy as labor cost and energy as labor surplus

To find the analogy to the kinetic (co-)energy the $p\dot{q}$-diagram is set up and presented in Figure 2-4a. The red line represents the wage p, forming the supply line for labor. On the x-axis the flow of labor services that the laborer is willing to provide in exchange for the associated wage p is presented on the y-axis. The space set up in this figure is thus the same space that economists have set up in Figure 2-1 and in Figure 2-4b.



(a) $p\dot{q}$-Diagram of the labor commodity analog

(b) Demand and supply in labor economics [33]

Figure 2-4: $p\dot{q}$-diagrams for labor services

When comparing the figures in Figure 2-4 the light red area T is the same area as PS. PS stands for the producer surplus (and CS for consumer surplus), in this case the producer is the laborer [33]. As can be seen, the energy T is analogous to the surplus obtained by the laborer. Therefore T is named the *labor surplus*.

T can be determined using equation (2-9), [37]. Using this equation, the units are determined to be $[\frac{\$}{hr}]$.

$$T = \int_0^p \dot{q}\ dp \tag{2-9}$$

Furthermore, the little green area forms the analogy to the kinetic co-energy T^*. T^* is determined through equation (2-10) [37] and thus has units $[\frac{\$}{hr}]$. In labor economics, a rational laborer is willing to supply labor once supplying that labor yields more than the value to the laborer of not performing the labor. [68], [29] The area $T*$ can thus be interpreted as the cost for the laborer and will be called the *labor cost*.

$$T^* = \int_0^{\dot{q}} p\ d\dot{q} \tag{2-10}$$

The analogy to the energy and co-energy is presented in Table 2-4.

Labor economics	Symbol	Unit	Linear mechanical analog
Labor cost	T^*	$[\frac{\$}{hr}]$	Co-Energy
Labor surplus	T	$[\frac{\$}{hr}]$	Energy

Table 2-4: Labor economics analogous to the (co-)energy

2-3 The rotational mechanical labor analogy

In this section, labor economics is interpreted analogous to the rotational mechanical analogy, called the capital analogy in economic engineering. In physics, it is used to describe motions that are rotational motions. The rotational mechanical analogy can be seen as an extension of the linear mechanical analogy and as a consequence of this, t, $\dot{q}$, $\ddot{q}$, m, p, F, T^* and T are still determined through the same formulas as presented in the previous section. The economic engineering capital analogy treats the 'capital' as something that yields value that in turn can be sold, instead of selling the actual capital.

The first step in developing the capital analogy is setting up the capital space. This space is visualized in Figure 2-5. Instead of all axes representing different types of labor, one axis, q_0, represents the *labor force*. It is the number of laborers that can yield an hour of labor and therefore is expressed in $[ps \cdot hr]$ as well as the other q axis. The other q axis, e.g. q_1, represents the *quantity of labor services* performed by the labor force q_0. These axis are the same axis as in commodity space. The capital space thus consists of the x-axis representing the labor force, q_0, with the labor it produces, q_1 on the y-axis.

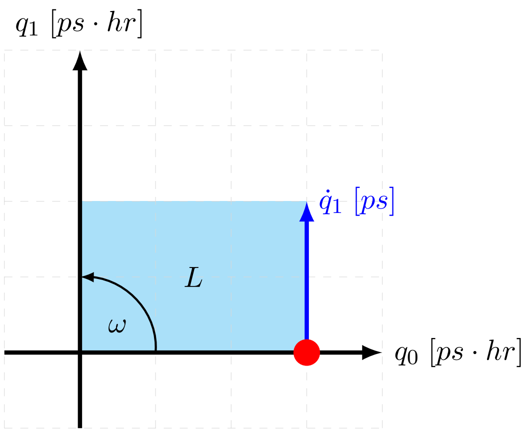


Figure 2-5: Capital space with a red marker depicting a laborer, the blue arrow depicting the flow of labor services $\dot{q}$ of this laborer, the light blue area the workforce's wage L and the productivity ω

The next step is determining the analogy to the angular velocity ω [25]. The assumption is made that the labor force is constant. As a result of this, the only velocity component is $\dot{q}_1$ which is perpendicular to the y-axis and has an arm of length q_1. Therefore, ω can be determined through equation (2-11) and is visualized by the curved arrow in Figure 2-5.

$$\omega = \frac{\dot{q}_1}{q_0} \tag{2-11}$$

ω is thus the ratio between the flow of labor services that are actually produced, and the number of people that can perform an hour of labor. Therefore ω is deemed the *productivity* and has the unit $[\frac{1}{hr}]$.

The time integral of the angular velocity becomes the angle [14]. The angle has no units, for the time integral of the productivity expresses the hours of labor services performed per available hours, resulting in the units $[\frac{hr}{hr}] = [-]$. As a result of this, the angle θ_h will be named the *working hours*.

On the other hand, the time derivative of the angular velocity becomes the instantaneous angular acceleration [14]. The time derivative of the productivity will be deemed the *change in productivity*, with symbol α and units $[\frac{1}{hr^2}]$.

Contrary to the inertia in the linear mechanical domain, the inertia in the rotational mechanical domain is not the mass. Instead, it is the moment of inertia and is represented by equation (2-12) in which the arm r is q_0, as can be seen in Figure 2-5. [36] I thus has the units $[\$ \cdot hr]$, and concerns the *full employment cost* [57].

$$I = m\ r^2 = m\ q_0^2 \tag{2-12}$$

Through the full employment cost, it is possible to calculate the total wage the labor force receives. Therefore it is called the *workforce's wage*. This can be calculated by multiplying the full employment cost, with how much labor each laborer on average performs, namely, the *productivity*. This is the light blue area depicted in Figure 2-5 This is expressed in equation (2-13) [36]. Through here it can be determined that the *workforce's wage* has the units [$].

$$L = I\omega \tag{2-13}$$

Lastly, the change in full employment cost is deemed the *wage inflation* [16]. For it is known that the time derivative of the angular momentum is the torque, equation (2-14) follows [36]. The wage inflation is denoted in here using the same symbol as its analog, τ_N and thus has the units $[\frac{\$}{hr}]$.

$$\tau_N = \frac{dL}{dt} = I\alpha \tag{2-14}$$

The results of developing the extension of the commodity analogy to the capital analogy for labor is presented in Table 2-5. Table 2-4.

2-4 The inconsistency between the kinematics of the Newtonian analogy and the dynamics of labor services

The results of the developed analogies are compared with what economists observe in the labor market. To compare these results, it is important to first understand the difference in terminology used by economists and engineers. Then, the terminology that is used throughout this thesis is presented, such that it is clear what is meant moving forward. The engineering terms are considered leading. After this, the difference in findings from our analogy of labor economics and economists' views are discussed, and the next step in the development of a labor service analogy is formulated.

Labor economics	Symbol	Unit	Rotational mechanical analog
Quantity of labor services	q_1	$[ps \cdot hr]$	Position
Labor force	q_0	$[ps \cdot hr]$	Arm
Working hours	θ_h	$[-]$	Angle
Productivity	ω	$[\frac{1}{hr}]$	Angular velocity
Change in productivity	α	$[\frac{1}{hr^2}]$	Angular acceleration
Full employment cost	I	$[\$ \cdot hr]$	Inertia
Workforce's wage	L	$[\$]$	Angular momentum
Wage inflation	τ_N	$[\frac{\$}{hr}]$	Torque

Table 2-5: Labor economics analogous to rotational mechanics

2-4-1 The difference in definitions of dynamics for economists and engineers

There is a difference between what economists describe as dynamics and what engineers call dynamics. For the purpose of this thesis being understandable for economists as well as engineers, the difference is explained. Throughout the remainder of this thesis, what economists consider mainly dynamics will be referred to as kinematics. Because, what economists consider 'dynamics' is what economic engineers consider 'kinematics', whilst there is no term in economics that describes what engineers call 'dynamics'.

The economists' definition of dynamics

Determining the labor dynamics is done by labor economists from IZA, "a nonprofit research institute and the leading international network in labor economics" [52], as follows:

> "Specifically, we take a microeconomic perspective by tracking movements of individuals across labor market states and discuss under which conditions these transition rates can summarize the whole dynamics of the labor market." [13]

As stated, they track the movements of laborers. In economic engineering, movement is defined through the position in commodity space q that represents the quantity of labor. Economists do not just track the displacements as part of 'movements' but also consider the time changes concerning these movements:

> " Measurement over time of changes in the activity status of individuals and of changes in jobs of employed persons" [67]

In economic engineering the activity status is represented by $\dot{q}$, the flow of labor services. Furthermore, the measurement over time of change in the activity of individuals is represented by $\ddot{q}$, the additional hiring. Besides q, $\dot{q}$ and $\ddot{q}$, the wage p is also measured. It is important to note that all these variables are measured and statistics are performed on them, resulting in correlations. These statistical correlations are thus what economists call dynamics, these relations not causally determined. [67], [6], [13]

The engineers' definition of kinematics

Engineers define kinematics as:

> "Kinematics is a study of motion without regard to the cause of the motion" [39]

Which thus concerns measuring the variables in physics analogous to q, $\dot{q}$, $\ddot{q}$ in economic engineering. In kinematics the forces resulting in the measured values are not considered, thus causality is not considered. Therefore kinematics is a more suitable description of what economists describe as 'dynamics'. [4]

The engineers' definition of dynamics

Engineers define dynamics as:

> " Dynamics is that branch of mechanics which deals with the motion of bodies under the action of forces" [36]

The difference between kinematics and dynamics is that dynamics concerns itself with not only the motion of laborers (bodies) but also the wants (forces) that cause motions, whereas kinematics does not consider the wants [39]. Economists measure the motions and the wage but do not consider the wants (forces) causing these motions. Therefore, dynamics is not a suitable description of what economists describe as 'dynamics', instead what they do is mainly kinematics.

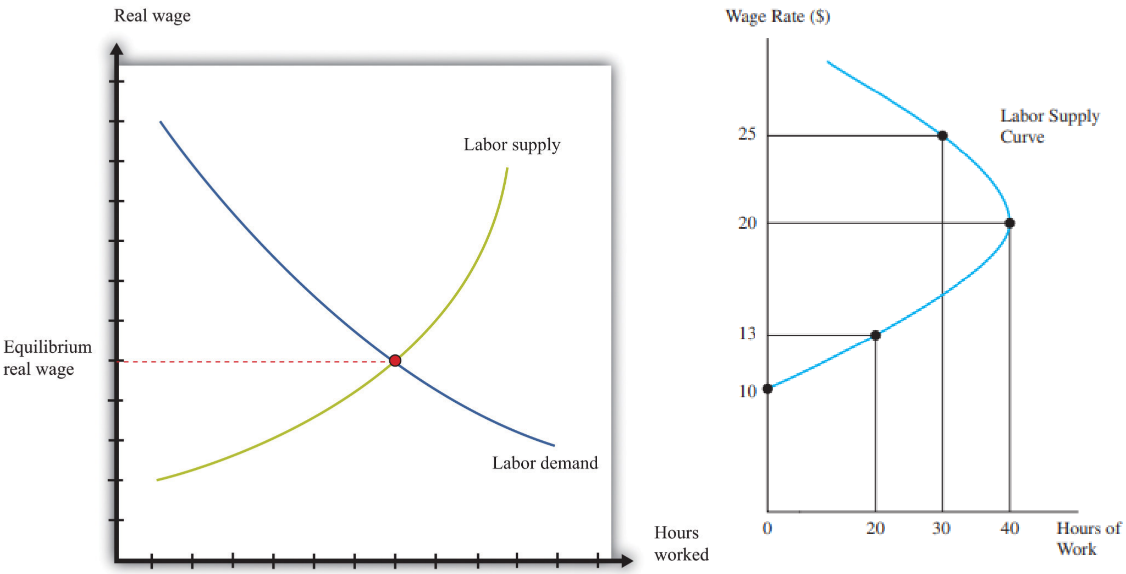
Defining the (engineering) dynamics of economic systems is an important contribution of economic engineering [49]. Therefore, the decision is made to follow the engineering terminology and refer to what economists call 'dynamics' as 'kinematics' instead, and continue to refer to what engineers consider 'dynamics' as 'dynamics'.

2-4-2 Inconsistencies between the flows of labor services

Let us compare the newly developed analogy to the kinematics of labor economics. For the variables defined in both domains, the results are expected to be similar, as they describe the same variables.

The labor supply curve as modeled by labor economists is not a linear curve. It is either a convex curve or a backwards bending curve, as can be seen in Figure 2-1, Figure 2-6a and Figure 2-6a. [6]

The result of plotting the labor supply curve for a single laborer through the commodity or capital analogy are shown in Figure 2-7. In here, the x-axis represents the flow of labor services expressed in $[\frac{ps \cdot hr}{dy}]$, in which dy stands for 'day', consisting of 24 hr. On this axis, a value of 8 $\frac{ps \cdot hr}{dy}$ is a regular work day. The y-axis represents the wage expressed in [\$] for a laborer with inelasticity $m = 1\ \frac{\$}{ps^2 \cdot hr}$. In this figure, two contradictory results are observed.



(a) Convex labor supply curve [38]

(b) Backwards-bending labor supply curve [6]

Figure 2-6: Supply curve and surplus in economics

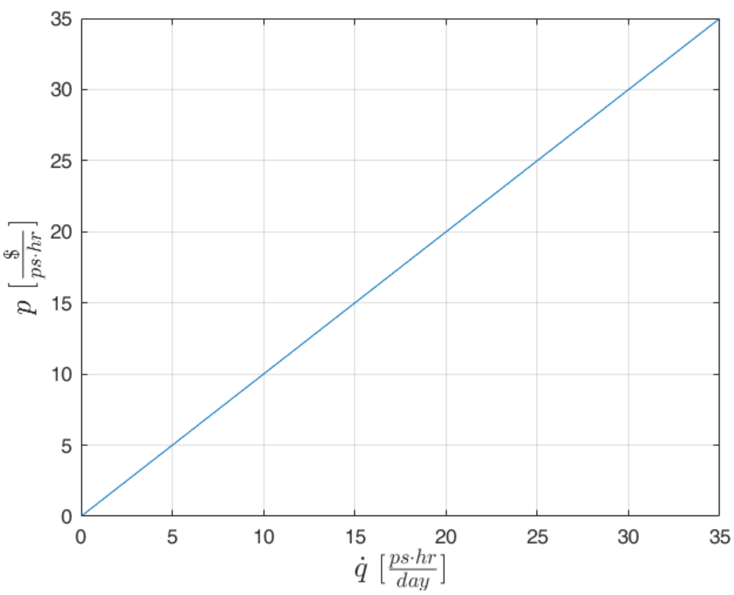


Figure 2-7: Supply curve for labor ($m = 1\ \frac{\$}{ps^2 \cdot hr}$) when modeled using the capital or commodity analog

The first contradictory result is that the supply curve is linear, and thus does not have a convex or backwards bending part for the positive flow axis. Consider the following scenarios. Scenario 1: when working a job of 6 hours a day, how much more would your employer need to pay you to work 9 hours a day instead? Scenario 2: when you are already working 9 hours a day, how much would your employer need to pay you to work 12 hours a day instead?

According to economists, your answer in scenario two is expected to be higher than that of scenario 1, thus resulting in an upwards sloping supply curve [10]. We did not find a more convincing argument supporting a linear supply line and thus concluded our linear supply line is still missing something fundamental.

The second observation is that a person is willing to perform more than 24 $\frac{ps \cdot hr}{dy}$ in exchange for a wage of more than $24 \frac{\$}{ps \cdot dy}$. However, regardless of how much a laborer wants to work more than 24 $ps \cdot hr$ in a day, there are not more than 24 hr in a day. Thus working more than $24 \frac{ps \cdot hr}{dy}$, or $1 \frac{ps \cdot hr}{hr}$ as a single laborer is not possible.

The inconsistency implies something is missing that neither the commodity nor capital analogy is able to describe. Besides interpreting existing analogs to labor economics, it is also an option to attempt to develop a completely new type of analogy. As per the hard theoretical limitations that labor services have, and the blatant disregard for these by the current analogies, it seems like this is the only option left. A new type of analogy furthermore holds the potential to be valuable for economic engineering in general, for many other supply curves are convex curves as well [5]. The next step is thus to develop a new type of analogy for labor services which accounts for the theoretical limitation to labor services.

Chapter 3

The relativistic dynamics of labor services

3-1 Introduction

The goal of this chapter is to present the development of the relativistic analogy for labor services. As established in the previous chapter, there is a need for a new type of economic engineering analogy that takes into account the theoretical limitations to the movement of the laborer.

We argue that the space describing the flow of labor services is hyperbolic by defining the mathematical structures that set up LS spacetime. From here, the cost-benefit analysis, analogous to the stationary action principle, is used to determine the running labor cost. This is, in turn, used to determine the wage analogous to the momentum, which allows us to formulate the Legendre transform. This transform is used to define the Hamiltonian, interpreted as the surplus of the laborer. The filled-in Legendre transform is transformed into an equivalence yielding the energy-momentum equation, or for labor economics, the surplus-wage equation. The components of this equation are in turn used to define and interpret the wage-vector and wants-vector, analogous to the four-momentum and force-vector.

The overview of the analogy developed in this chapter is presented in Table B-3 in appendix B-3. Furthermore, the MATLAB code used is provided in appendix F.

3-2 Describing a laborer in labor service spacetime

3-2-1 The structure of the space describing a laborer

This section builds up the geometrical structure of the space in which to model a laborer performing labor. Each different layer of the geometrical structure has its own mathematical structure. The reasons for these structures are argued and the consequences are presented.

Topological structure for observing the laborer

Let us start with the assumption that a continuous map of labor timesheets ξ exists. A *labor timesheet* corresponds to a *quantity of labor* q in units $[ps \cdot hr]$, performed at *time* t with unit $[hr]$. The values of these timesheets form the coordinates in *labor service (LS) spacetime*, which is visualized in Figure 3-1. The map is considered continuous because we observe that a laborer cannot skip ahead in time and can only perform labor over time. Thus nearby timesheets have nearby coordinates.

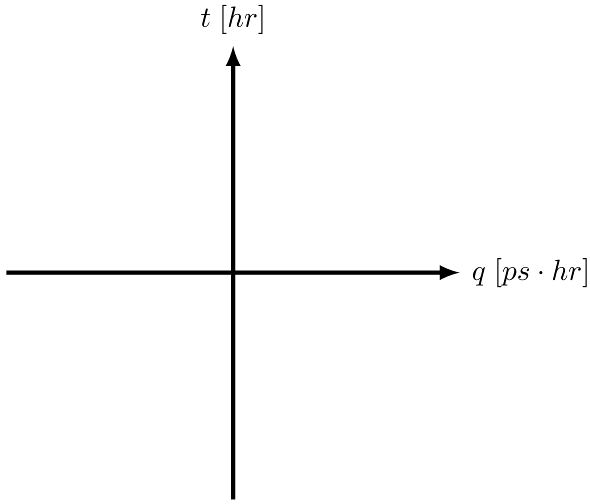


Figure 3-1: $\mathbb{R}^2$ LS spacetime

The dimensions of these continuous maps are dependent on the number of different types of labor that are chosen to be modeled $n \in \mathbb{Z}^+$, and time $+1$. Thus, $\xi :\rightarrow \mathbb{R}^{n+1}$. The case for $n > 1$ is discussed in more detail in Section 3-2-3. There is no theoretical limit to the value of n. All the timesheets associated with a laborer form a continuous line that represents a laborer's *career path*, visualized in blue in Figure 3-2.

Projective structure for observing career paths

Besides the topology of laborers and timesheets, there are other structures to be observed in labor economics. One of these structures is the subclass of laborers whose wants for labor and wants for leisure are in equilibrium. As a consequence, the laborer continues to perform labor services at the rate they have been performing the labor services at. They will be called *equilibrium laborers* and their career paths are asserted to be described by straight lines in $\mathbb{R}^{n+1}$. This results in the projective structure presented in Figure 3-3, whereas the laborer in Figure 3-2 does experience non-zero wants and is not in equilibrium at all points in their career.

The coordinates associated with equilibrium laborers are not unique. As long as a transformation $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ for map ξ preserves the projective structure, the new map ξ_{new} is a valid map.

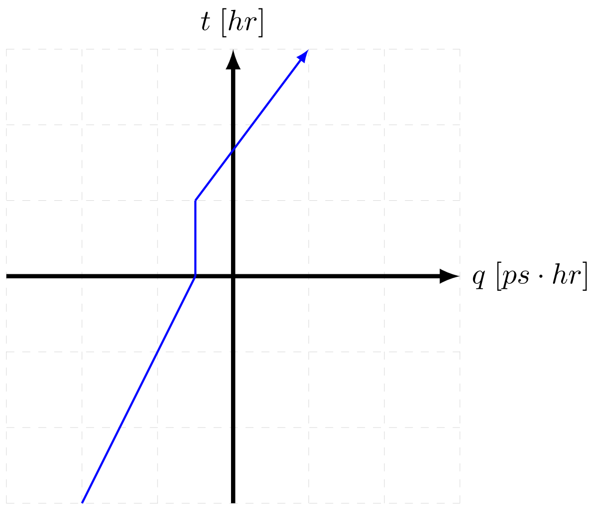


Figure 3-2: $\mathbb{R}^2$ LS spacetime with in it a career path (blue) of a laborer

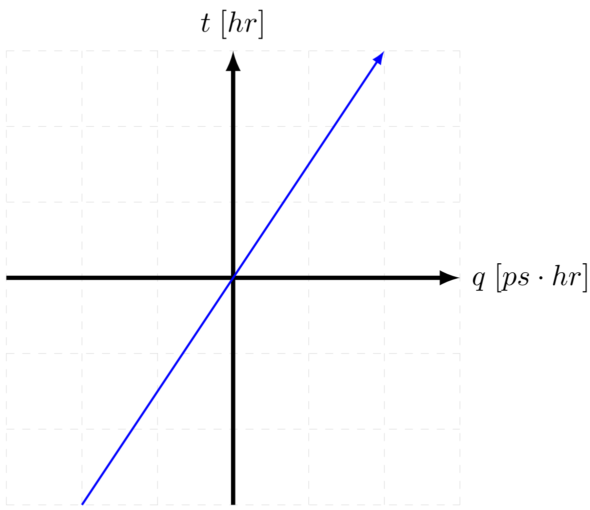


Figure 3-3: $\mathbb{R}^2$ LS spacetime with the career path (blue) of an equilibrium laborer

Conformal structure obtained from robots

As observed in the previous chapter, it is not theoretically possible for a laborer to perform labor services at a rate faster than time $\dot{q} = 1\frac{ps \cdot hr}{hr}$. With $\dot{q}$ defined as in equation (2-1).

Performing labor services continuously at this rate is only possible for *robots*, as laborers need breaks [22]. The career paths of a robot performing labor all the time, and a robot not performing anything are visualized in orange, in LS spacetime, in Figure 3-4.

Regardless of how much labor services laborer A is performing, when observing another laborer B, laborer B is not able to perform labor at a higher flow than the robot. Because demand frames can be compared through excess supply, the flow of demand is measured from one demand frame relative to the other. Thus a laborer assesses another laborer based on the difference in the flow of labor services they are performing. However, when observing a robot,

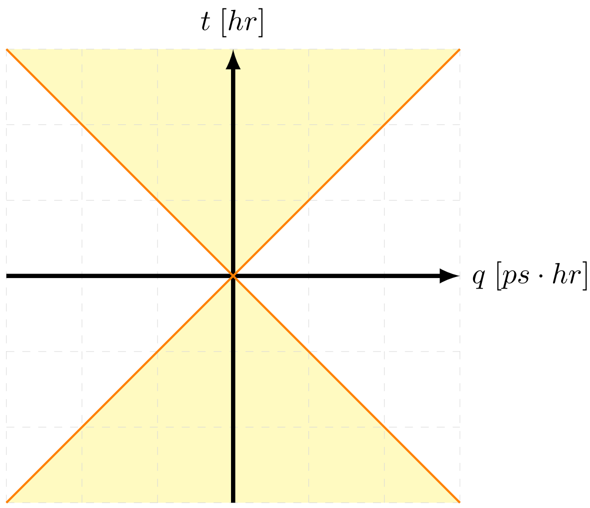


Figure 3-4: $\mathbb{R}^2$ LS spacetime with the career paths of robots (orange) and all the possible career paths for laborers (yellow)

both laborers will observe that the robot is performing labor services at the same rate that time passes, thus $\dot{q} = 24\ \frac{ps \cdot hr}{dy} = 1\ \frac{ps \cdot hr}{hr}$, which from now on will be denoted by c. Resulting in the second postulate:

Second postulate

All laborers assess a robot performing labor services at the rate of time, $c = 1\ [\frac{ps \cdot hr}{hr}]$

Because it is not possible to perform labor services faster than time, the career paths of laborers are restricted to stay within the cone set up by the career paths of robots, the complete set of career paths is depicted in yellow. The possible career paths of the robot are described by an angle that is dependent on the chosen units of the axis. In the case of choosing q in $ps \cdot hr$ and t in hr the angle of all possible career paths must stay within the range $[\frac{\pi}{4}, \frac{3\pi}{4}]$ for their positive t-valued part, and in the range $[\frac{5\pi}{4}, \frac{7\pi}{4}]$ for their negative t-valued part. The conformal structure of LS spacetime is thus a result of the special class of directions described by the maximum flow of labor services c theoretically possible, represented by the career path of a robot.

Affine structure to describe perceived time

The concept relevant to the structure we consider next of labor economics is clocks. Clocks assign values to the intervals along career paths. These values describe the time perceived from the perspective of the laborer. There are two assertions at the core of setting up the affine structure, which will be presented. An affine structure is a structure that conserves only properties related to parallelism and the ratio between lengths of parallel line segments, it thus does not preserve angles or distances. [9]

The first assertion is universality. The perception of time for each laborer is considered the same. The perception when performing labor can vary between laborers' performing labor at a different flow $\dot{q}$.

For example, consider doing something you enjoy for about an hour, this leisure time is expected to feel like it passes by faster compared to if you would perform labor during that hour. The rate of the perceived time when performing labor, compared to the time on the clock is, thus slower than when performing leisure, for this case. However, at some point, when you allocate a lot of time for leisure, it can get boring and time will also become perceived to pass slower. Implying that there is a ratio between labor and leisure that minimizes the perceived time. The assumption is made that when performing labor at the same flow the perceived time is experienced the same. Thus, the *perceived time* τ is the universal property for which the experience of a defined interval is the same for all laborers, whereas for the experience of a *time clock* (TC) interval t this is not necessarily the case.

Extending this observation leads to the perceived time passing slower for equilibrium laborers when performing higher rates of labor services or allocating more of their time to leisure. Thus, when converting the perceived time to clock time, the clock time can differ between laborers performing labor services at different rates. Regardless of clock time, a particular interval of perceived time thus describes an experience that is the same for all laborers.

This leads us to the second assertion, uniformity. In the natural affine structure, coordinates can be found such that the affine structure of $\mathbb{R}^{n+1}$ shows compatible readings between clocks of career paths of free laborers performing labor at the same rate (aka lines at a same angle). This can be visualized through a parallelogram of career paths, as presented in Figure 3-5. As can be seen in Figure 3-6, the perceived time between laborers not performing labor at the same rate is different.

Uniformity is thus not dependent on comparing a set of clocks to another clock, but by comparing four through the parallelogram, with the sides parallel to each other having the same flow of labor services $\dot{q}$. Thus, when the career paths are not parallel, they represent laborers performing labor at different rates, and uniformity no longer holds. Whereas if the career paths of the equilibrium laborers are parallel, and thus the laborers are performing labor at the same rate, uniformity holds.

Any representations of an equilibrium laborer in $\mathbb{R}^{n+1}$ that is compatible with this affine structure forms the basis for a type of *inertial frame of demand*, which has a constant flow of labor services $\dot{q}$.

That the flow of labor services of an inertial frame of demand is constant, is an important observation. From physics it is known that the equations of motion for different inertial reference frames are the same, but vary for acceleration reference frames. Based on previous work in economic engineering on developing analogies, the assumption is made that the scenario observed in physics extends to the economic engineering domain [49].

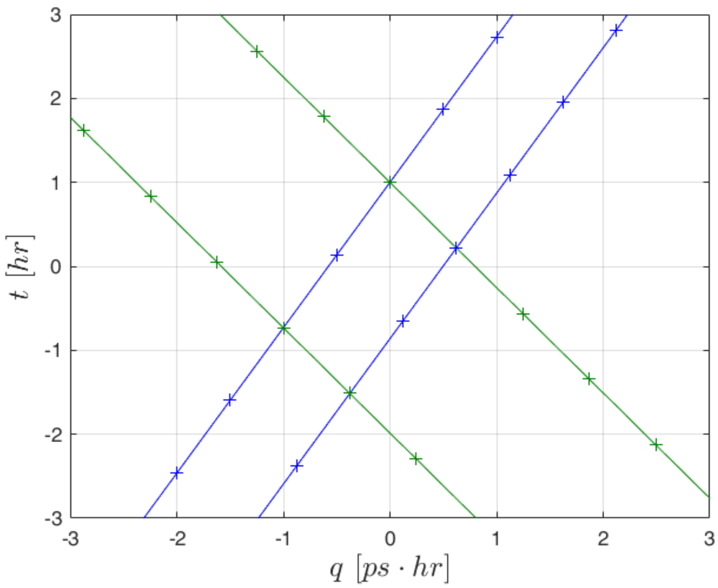


Figure 3-5: $\mathbb{R}^2$ LS spacetime with career paths of an equilibrium laborer, green lines are parallel to each other and the blue lines are parallel to each other, $+$ visualizes an interval of size 1 over the career paths

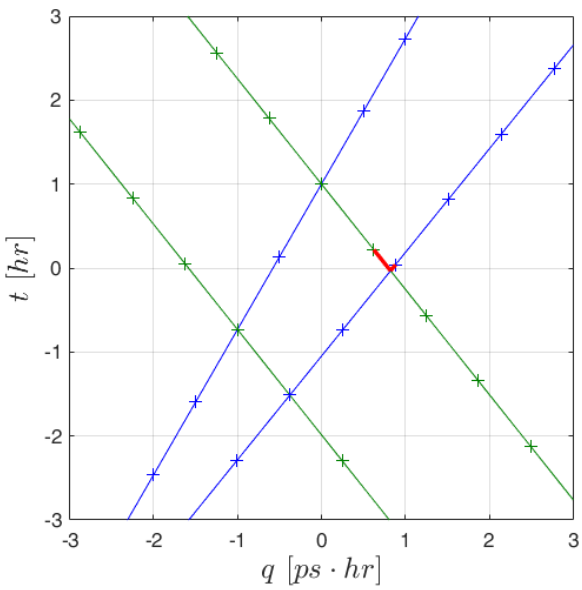


Figure 3-6: $\mathbb{R}^2$ LS spacetime with career paths of an equilibrium laborer, green lines are parallel to each other and the blue lines are not parallel to each other, $+$ visualizes an interval of size 1 over the career paths, in red the difference in perceived time between the blue lines

Resulting in the formulation of the first postulate:

First postulate

The same economic laws of demand will be valid for all inertial demand frames of reference for which equations of economic engineering hold good.

Obtaining the demand frame of reference by combining the different structures

Now we combine the conformal structure obtained from robots and the affine structure resulting from perceived time. For simplicity, the case in $\mathbb{R}^2$ is considered. Consider maps ξ in accordance with the affine structure. From these, select the ones describing robots, thus the ones at slope -1 and 1.

A laborer assessing a robot sees the robot performing labor services with the rate of time (career path with slope c), regardless of the laborer's flow, as per the second postulate. Therefore, the only possible result of a transformation of the demand frame of the career path of a robot is that the career path stretches or contracts. For a robot not performing any labor services (career path with slope $-c$), the same holds.

Due to the affine structure defined, the ratios of lengths of parallel line segments are kept constant. As a result, if one career path of a robot is stretched, the opposing career path must be squeezed with the same ratio. Consider a robot described by a whilst not performing any labor, and describe another robot that is performing labor by b. The transformation for the coordinates (a, b) is then described by equation (3-1) and visualized as in Figure 3-7.

$$(a, b) \mapsto (\epsilon a, \frac{1}{\epsilon} b) \tag{3-1}$$

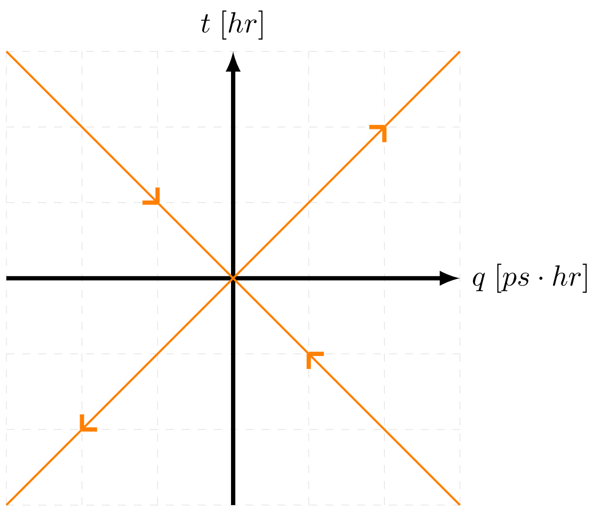


Figure 3-7: (Lorentz) transformation of the coordinates associated to robots

Which is the Lorentz transformation, for which a detailed derivation is provided in appendix C. For intuition, determine a line from the career path of the robot not performing labor to the career path of the one performing labor, and pick this line such that it is parallel to the q-axis. The middle of this line now crosses the t-axis. As we change to demand frames with different flows, the line will no longer be horizontal. The middle of this line (which is being stretched when viewing it from demand frames with different flows), then draws the hyperbola we find in Figure 3-10.

The Lorentz transformation is thus only dependent on the conformal and affine structures. Using the Lorentz transformation any career path of an equilibrium laborer can be made vertical. Our structure now provides a unique representation when units of the axis have been chosen. Resulting in the *demand frame of reference*. These frames of demand thus have a constant flow of labor services $\dot{q}$ and adhere to the conformal structure.

Even though the focus of this thesis is on defining the supply of labor, the reference frame is called a demand frame of reference. In economic engineering, the term "demand frame of reference" is considered analogous to reference frames. Furthermore, action=reaction is seen as analogous to demand=supply. Therefore, the decision is made to stay consistent with the economic engineering terminology. [49]

Perceived time in LS spacetime

Through the demand frame of reference, the perceived time of different laborers is now studied more precisely. Let us start with the career path of an equilibrium laborer that overlaps with the t axis. Note that the perceived time of the laborer only overlaps with the t axis when the laborer performs labor at a rate of $\dot{q} = 0 \ \frac{ps \cdot hr}{hr}$ relative to the demand frame in LS spacetime. This laborer is presented in Figure 3-8 in which the horizontal stripes indicate a difference in perceived time of $\tau = 1 \ hr$.

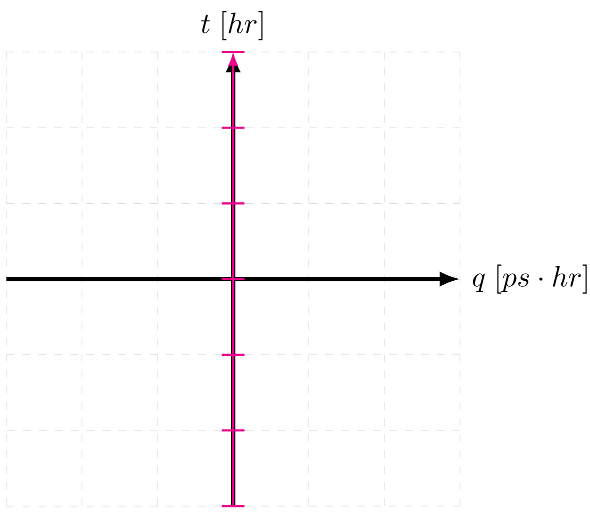


Figure 3-8: $\mathbb{R}^2$ LS spacetime with the career path (magenta) of an equilibrium laborer with a relative flow of $\dot{q} = 0 \ ps$, as viewed from demand frame A

Next, the career path is observed from a different demand frame. This demand frame is chosen to have a different (constant) flow of labor services. From this frame, the laborer is thus no longer at rest. The difference in the flow of labor services $\dot{q}$ between these frames is called the *excess supply*. Besides the difference in demand frames, the excess supply also describes the difference in flow between laborers. When it is positive, the laborer supplies labor. When it is negative the laborer demands labor. And when zero, there is no interaction between the laborers. The laborer is thus described in a relative manner, through a demand frame of reference.

The coordinates of the career path observed from the different demand frame are found by applying the Lorentz transformation and are depicted in Figure 3-9. The horizontal magenta stripes indicate, again, intervals of $\tau = 1\ hr$.

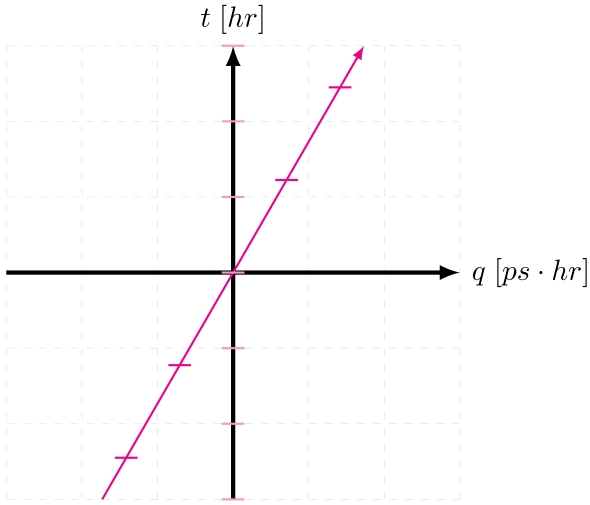


Figure 3-9: $\mathbb{R}^2$ LS spacetime with the career path (magenta) of an equilibrium laborer with a relative flow of $\dot{q} = 0\ ps$, as viewed from demand frame B with horizontal stripes (magenta) portraying one interval of perceived time

In figure Figure 3-9 it is visible that the unit interval of the perceived time (visualized by the horizontal magenta stripes) is not the same value of t as when looking at it from the demand frame in Figure 3-8 (visualized by the light horizontal magenta stripes). This result is in line with the example concerning perceived time in the paragraph about the affine structure.

The procedure can be performed for all different possible demand frames from which this career path can be observed. Through this, a set of points H is defined to represent the coordinates of all the different career paths after one unit interval of perceived time ($\Delta\tau = 1\ hr$). This set of points is described by a hyperbola described in equation (3-2) and presented in Figure 3-10. This hyperbola is called the *flow space*.

$$t^2 - q^2 = 1 \tag{3-2}$$

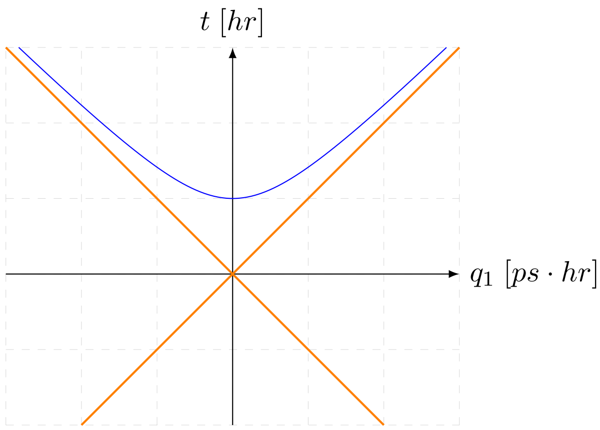


Figure 3-10: $\mathbb{R}^2$ LS spacetime with the flow space (blue) at $\tau = 1$ hr for a laborer at the origin

Here, it can be seen that the equilibrium laborers along different career paths thus have a slower perceived time when they are not performing labor at the same rate of the demand frame of labor. Which is contrary to the Newtonian analogy. [7]

3-2-2 Kinematics LS spacetime

Because the flow space in LS spacetime is hyperbolic, lengths are defined differently in LS spacetime. Defining lengths differently does not just affect the length but also the other variables. Therefore, the kinematics of LS spacetime are defined using a hyperbolic metric, with metric signature $[+-]$.

To define the lengths, first, the metric is chosen to be $[+-]$. Thus, a positive sign is used for the time dimension t, and a negative sign is used for the spatial dimension(s) q. This metric is also the metric used in equation (3-2), and is a matter of personal preference, for this metric is often used when treating special relativity. Using $[-+]$ leads to the same theory. [18], [43], [55]

The hyperbola described by equation (3-2) enables us to define a length s in LS spacetime. It describes the possible locations of the laborer after $\tau = 1$ hr when the laborer is at the origin when $\tau = 0$ hr. However, the length in LS spacetime is not necessarily equal to τ. It depends on the scaling of the spatial axis of LS spacetime by c, resulting in equation (3-3).

$$\Delta s = c\Delta\tau \tag{3-3}$$

The change in length can be visualized as in Figure 3-11. In which the red dot represents the laborer, and the blue hyperbola is the same hyperbola as in Figure 3-10.

Here, $c = 1$ $[\frac{ps \cdot hr}{hr}]$, the maximum flow of labor services from postulate 2. The hyperbola is a projection and the labor timesheets consist of homogeneous coordinates (q, t) in LS spacetime, for they are assumed to all have the same properties. Because of this, it is possible to make Δ infinitely small, resulting in equation (3-4). [8]

$$ds^2 = (cdt)^2 - dq^2 \tag{3-4}$$

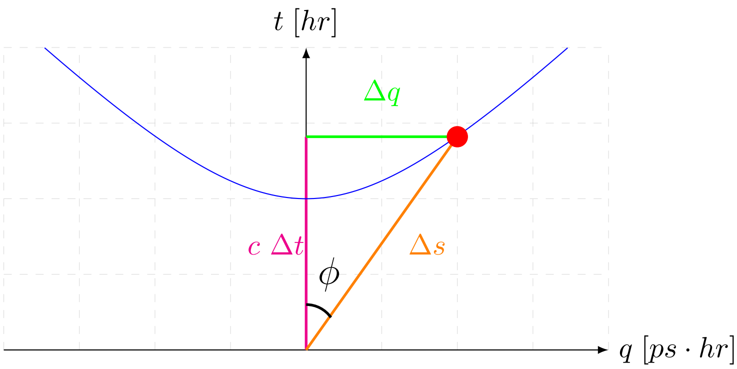


Figure 3-11: LS spacetime with a laborer (red), flow space (blue), visualizing the variables Δs, Δq, $c\ \Delta t$ and ϕ

In Figure 3-11 no negative values of the t-axis are shown. The focus is on the non-negative values, because the laborer, just like you, is not able to go back in time.

Because the demand frame is relative, it is possible for the laborer starting at the origin to move to a negative q value. The restriction concerning the q value is shown in Figure 3-4. A laborer who is performing labor services at the same rate of the demand frame will stay at $q = 0\ ps \cdot hr$ over time. If that laborer instead starts to perform labor services at a lower rate, it may look like the laborer is undoing labor from the perspective of the demand frame. However, this is not the correct interpretation. The laborer simply decreases their flow of labor.

For example, if the initial demand frame represents a full-time flow, and the laborer is at rest. It means that the laborer is a full-time laborer and will move over the t-axis. This laborer is the pink dot in Figure 3-12 with its trajectory in light pink. If a full-time laborer lowers its flow and the laborer, for example, becomes a part-time laborer, the laborer moves in the direction of the negative q-value. The part-time laborer is depicted by the orange dot, its prior movements are depicted in light orange.

For additional interpretation, it could also be seen as the laborer hiring one other laborer to perform (part of) the first laborer's flow of labor services.

From Figure 3-11, the average flow of labor services can be determined as in equation (3-5). In which ϕ represents the hyperbolic angle between $c\ \Delta t$ and Δs.

$$\dot{q} = \frac{\Delta q}{\Delta t} = c \tanh \phi \tag{3-5}$$

However, ϕ can also be seen as the hyperbolic angle between two different demand frames. The difference between the demand frames is the flow of labor services $\dot{q}$ with which they propagate through LS spacetime. Therefore ϕ is determined to be the *degree of demand*.

Because of the projective structure, Δ can be made infinitesimally small, allowing equation

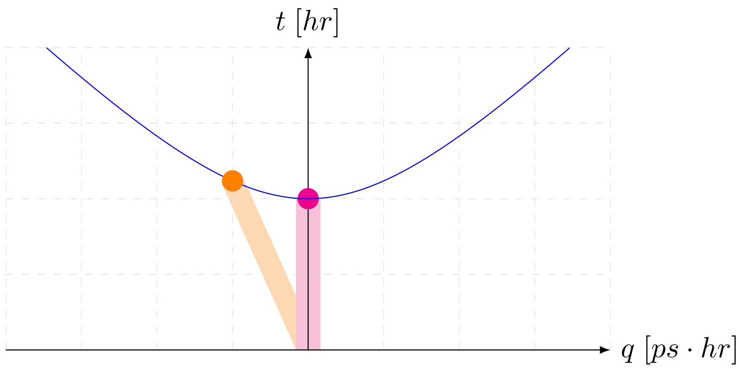


Figure 3-12: LS spacetime with a full-time laborer (red dot) with prior positions (light red), a part-time laborer (orange dot) with prior positions (light orange) and the flow space (blue)

(3-5) to be rewritten as the flow of labor services presented in equation (3-6).

$$\dot{q}(t) = c \tanh \phi = \frac{dq}{dt} \tag{3-6}$$

Furthermore, when transforming $d\tau$ to dt, as in Figure 3-13, using the Lorentz transformation, it is found that perceived time and TC are related through equation (3-7). Here, γ represents the Lorentz factor. As presented, the Lorentz factor can be expressed in hyperbolic functions. It shows that the perception of time of the laborer can be different than that of the demand frame it is viewed from.

$$d\tau = \frac{1}{\cosh \phi} dt = \frac{1}{\gamma} dt \tag{3-7}$$

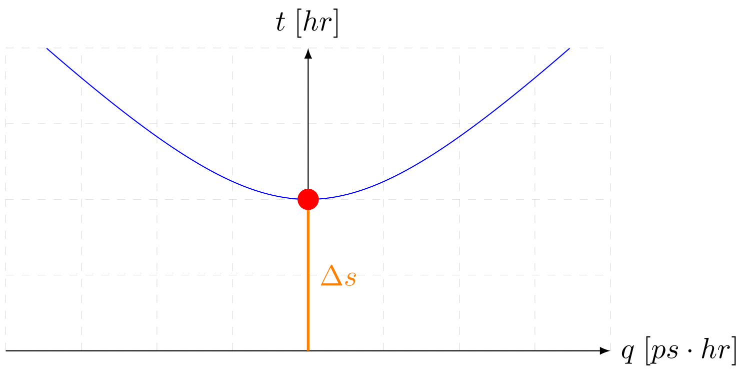


Figure 3-13: Lorentz transformation applied to Figure 3-11 to the demand frame from the perspective of the laborer (red), with the flow space (blue)

For example, if the demand frame A represents a company, and demand frame B represents a laborer performing labor services for that company. Then, the time of the company is the time in which the quantity of labor services performed is counted. This additional quantity of labor services a laborer accumulates during a shift is the duration between clocking-in and clocking-out. Therefore t is referred to as the time of the time clock (TC) in $[hr]$.

On the other hand, when the laborer has a flow of labor services, they will experience time differently compared to the clock time. Because, the more flow of labor services the person delivers, the slower their perceived time will go compared to the TC. Thus, the more you would work during a day, the slower it will feel like time is going.

The overview of the derived analog so far is presented in Table 3-1.

Labor economics	**Symbol**	**Unit**
Length in LS spacetime	ds	$[ps^2 \cdot hr^2]$
Quantity of labor services	q	$[ps \cdot hr]$
Time clock (TC)	t	$[hr]$
Flow of a robot	c	$[ps]$
Flow of labor services	$\dot{q}$	$[ps]$
Perceived time	τ	$[hr]$
Lorentz factor	γ	$[-]$
Degree of demand	ϕ	$[-]$

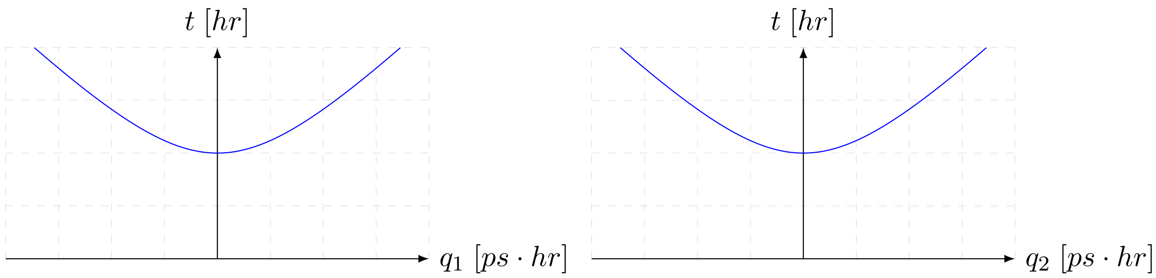
Table 3-1: Labor economics analogous to kinematics

3-2-3 More than two dimensions in LS spacetime

LS spacetime can consist of more than two dimensions. In this chapter, we, so far, focused on the case of one spatial dimension and one time dimension. However, the number of spatial dimensions can be increased indefinitely to describe different types of labor. For labor can be differentiated based on an indefinite number of characteristics, the number of dimensions that can theoretically be set up is also infinite. Contrary to this, we consider that the number of time dimensions is always 1. Here we follow the same argument as presented in physics [74]. Thus, the number of dimensions in LS spacetime can be determined using equation (3-8) and can theoretically be infinite.

$$\text{nr. of dimensions in LS spacetime} = \text{nr. of spatial dimensions} + 1 \tag{3-8}$$

Let us consider (again,) the case of manual and office labor. These form the two different dimensions, manual labor is represented by the dimension q_1 and office labor by q_2, both follow the units of q, thus $[ps \cdot hr]$. The planes at $q_2 = 0$ and $q_1 = 0$ are presented in Figure 3-14a and Figure 3-14b. Here, the blue hyperbola represents the possible positions the laborer can have when performing labor services at different flows for a perceived duration of one hour ($d\tau = 1\ hr$).



(a) LS spacetime for a laborer only performing manual labor and the flow space (blue)

(b) LS spacetime for a laborer only performing office labor and the flow space (blue)

Figure 3-14: Two planes in LS spacetime intersecting the origin and an entire axis of a spatial dimension

The next step is considering a job that requires the laborer to perform manual labor as well as office labor. For example by the ratio 2 : 1. This ratio is visualized in Figure 3-15 through the dark green dashed line. The red represents the possible positions in the spatial dimensions the laborer can have after a perceived duration of an hour. These possible positions are defined as R in equation (3-9). The blue area represents the possible quantity of labor services a laborer can obtain when it would perform labor at the different possible ratios of q_1 and q_2 and at different flows, with the dashed orange line representing the attainable values at maximum flow.

$$R = \{t \begin{bmatrix} \sin(\arctan(\frac{2}{1})) \\ \cos(\arctan(\frac{2}{1})) \end{bmatrix} \mid t \in [0, 1]\} \tag{3-9}$$

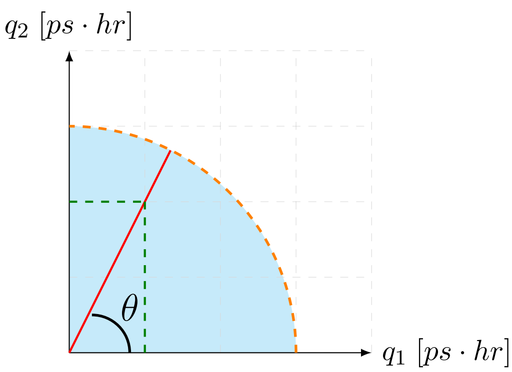


Figure 3-15: Possible positions in the spatial dimensions a laborer can obtain at $\tau = 1\ hr$ (light blue), the furthest positions (orange) and the red line depicting the possible positions for a laborer performing manual and office labor with a ratio 2:1 (green)

The blue dashed line is part of a circle. When also considering the buying of leisure of one other person, a full circle with radius c is obtained. This can be represented by the expression equation (3-10). The equation also shows the result of extending this theory to n spatial dimensions. Forming the basis for the maximum attainable positions in the case of n number of spatial dimensions after one hour of perceived time.

$$c = \sqrt{q_1^2 + q_2^2} \qquad \Rightarrow \qquad c = \sqrt{\sum_{i=1}^{n} q_i^2} \tag{3-10}$$

When applying this knowledge described in equation (3-11) a hyperboloid as visualized in Figure 3-16 for the three-dimensional case is obtained.

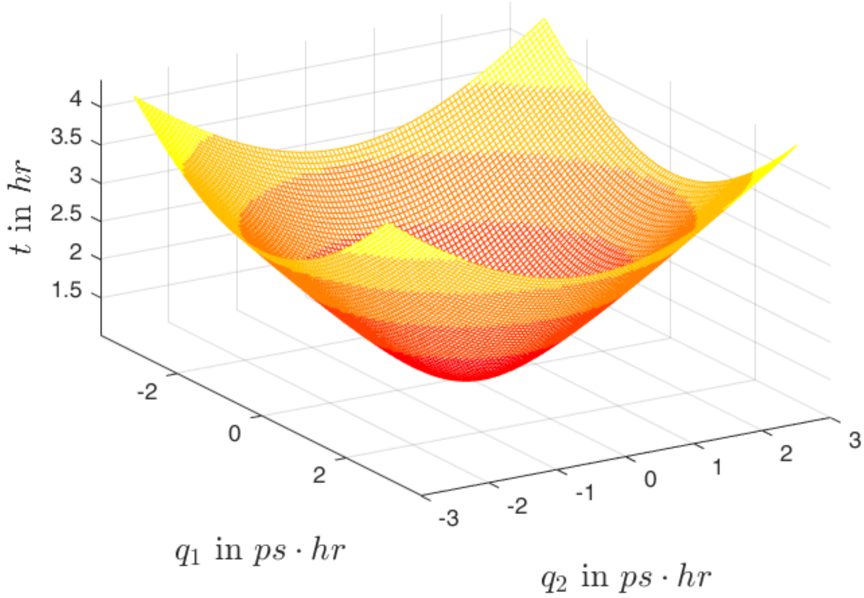


Figure 3-16: Flow space ($\tau = 1\ hr$) in 3D LS spacetime

$$ds^2 = (cd\tau)^2 - \sum_{i=1}^{n} q_i^2 \tag{3-11}$$

3-3 The cost-benefit analysis

In economic engineering analytical mechanics becomes advantageous when considering an economic system with a large number of agents, similar to physics for considering a large number of particles [42]. It furthermore allows building in holonomic constraints through the generalized coordinates. The theory presented is designed to model the labor market. The advantage of initially deriving the analog through Newtonian mechanics was that the intuition came easier. However, we think sufficient intuition has been created for the variables' connection with labor economics throughout chapter 2. Since the labor market contains many laborers, the decision is made to continue the derivation through analytical mechanics instead of Newtonian Mechanics. [45]

3-3-1 Determining laborer's cost using the stationary-action principle

In economic engineering, the stationary action principle is considered analog to the principle of minimum cost. With the action representing the periodic cost and the Lagrangian representing the running cost [48]. In the case of labor economics, the *periodic labor cost* can be defined as in equation (3-12) in which $\mathcal{L}$ represents the *running labor cost.*

$$S[q] = \int_{t_i}^{t_f} \mathcal{L}(q(t), \dot{q}(t), t)\ dt \tag{3-12}$$

The negative of the running labor cost $-\mathcal{L}$, in turn is considered the *laborer's running benefit.*

The action S is taken to be proportional to the perceived time τ. For it is the Lorentz invariant variable representing the length of the path through LS spacetime. Because this is an optimization problem, the right side of the equation can be multiplied by a constant β without changing the optimization results. Leading to equation (3-13).

$$S = \beta \int d\tau \tag{3-13}$$

The equation states that the period cost is proportional to the integral of the perceived time. This is to be expected, for the cost for a laborer is the value it would receive from leisuring rather than performing labor services.

As it is desirable for the period cost to be minimized, the term in equation (3-13) needs to be negative. For intuition; the more labor a laborer performs in a set amount of clock time (higher $\dot{q}$), the more tiresome a job is expected to become (lower τ), thus the higher the value of the counterpart, leisure (higher $\mathcal{L}$). And with this, implying a negative constant of proportionality between the action and the proper time.

When it comes to the value of the proportionality constant there are two further requirements. First, the proportionality constant needs to result in matching units between the left and right sides of the equation. Second, it needs to be based on constants available. The constants available are the wage inelasticity $[m] = [\frac{\$}{ps^2 \cdot hr}]$ and the robots flow of labor services $[c] = [ps]$. The wage inelasticity is known in the relativistic analogy for it can be determined in the same manner as done in the Newtonian analogy, described in Section 2-2. Since the action represents the running cost, the desired unit is $[S] = [\$]$. The combination of constants that is suitable as the proportionality constant is mc^2, for it yields the correct units for the action, as presented in equation (3-14).

$$[S] = [\$] = [\beta] \cdot [\tau] = [\frac{\$}{hr}] \cdot [hr] = [\frac{\$ \cdot ps^2}{hr \cdot ps^2}] \cdot [hr] = [m] \cdot [c^2] \cdot [\beta] \tag{3-14}$$

When combining the result of requiring the proportionality constant to be negative and contain the constants mc^2, equation (3-15) is formulated.

$$S = -mc^2 \int d\tau \tag{3-15}$$

The question may arise as to why $-mc^2$ and not for example $-2mc^2$. Theoretically, this would be possible too, for S is relative. The reason for choosing $-mc^2$ is that at a small scale, it is desirable that the values are equal to values obtained by the commodity analogy such that these can be compared. Appendix D shows this is the case for $\beta = -mc^2$.

When filling equation (3-7) into equation (3-15), equation (3-16) is found.

$$S = -mc^2 \int \frac{1}{\cosh(\phi)} dt \tag{3-16}$$

Using equation (3-12), the laborer cost is extracted from equation (3-16) resulting in equation (3-17). The constant mc^2 is added, such that the running cost is zero when the degree of demand is zero [12]. Adding a constant is possible because the value L is relative.

$$\mathcal{L}(\phi) = -mc^2 \frac{1}{\cosh \phi} + mc^2 = -mc^2 \operatorname{sech} \phi + mc^2 \tag{3-17}$$

L is visualized in Figure 3-17, for the case of $c = 1$ ps and various m. From here, two observations are made.

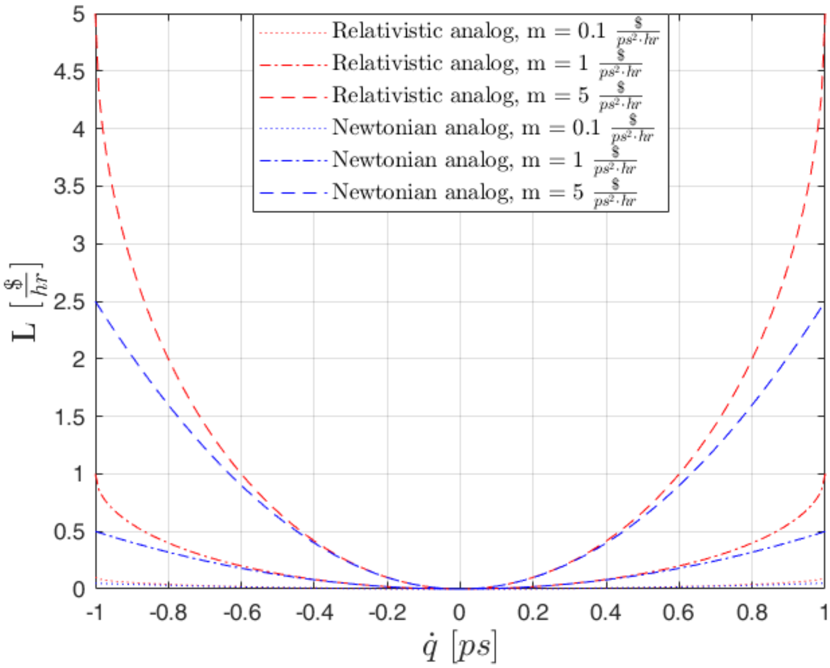


Figure 3-17: The relativistic (red) and Newtonian analog (blue) for running labor cost $\mathcal{L}$

The first observation is that the Newtonian analog shows a quadratic curve, whilst the relativistic analog yields an exponential curve. Intuitively, would it be equally tiresome to do an hour of overtime after you worked for 6 hours or an additional hour after working for 12 hours? If no, which one would be more tiresome? If you answered "no, working an additional hour after already having worked for 12 hours would be more tiresome', then for you the relativistic curve would be applicable. Because the rate of change in running labor cost of the relativistic curve is dependent on the flow of labor services, whilst in the Newtonian analog it is not.

Furthermore, it shows that the labor cost, when performing labor services at the maximum rate, is mc^2 (or, without that added constant, the running labor cost whilst not performing labor is $-mc^2$). It thus shows, as expected, that the running labor cost at the maximum flow, is a function of that maximum flow squared c^2, and the measure of how much this flow affects the wage the laborer requires, m.

The newly developed analogies in this paragraph are presented in Table 3-2.

Labor economics	Symbol	Unit
Periodic labor cost	S	[$]
Running labor cost	$\mathcal{L}$	$[\frac{\$}{hr}]$
Laborer's running benefit	$-\mathcal{L}$	$[\frac{\$}{hr}]$

Table 3-2: Labor economics analogous to the stationary action principle

3-3-2 The wage as momentum

The marginal increase in running labor cost per unit increase in the flow of labor services is defined as the wage p in $[\frac{\$}{ps \cdot hr}]$. Thus, if a laborer consistently performs more labor services, the laborer's wage goes up and vice versa. This definition is presented in equation (3-18).

$$p := \left.\frac{\partial \mathcal{L}}{\partial \dot{q}}\right|_q \tag{3-18}$$

Geometrically, this means that at each point of L, in the $(L, \dot{q})$ space, a tangent line is determined. The slope of this tangent line is in turn the wage p associated with the $\dot{q}$ value of the point the line is tangent to. This process is visualized in Figure 3-18. In which the running cost L is shown in red, in the case of $m = 1\ \frac{\$}{hr \cdot ps^2}$ and $c = 1\ ps$, and the tangent lines are visualized in light grey. [60]

When filling equation (3-18) in using equation (3-5) and equation (3-17), equation (3-19) is found when using equation (3-20) and equation (3-21).

$$\begin{aligned} p &= \frac{d(-mc^2 \operatorname{sech}\phi + mc^2)}{dc \tanh\phi} \\ &= -mc\frac{d(\operatorname{sech}\phi + 1)}{d\tanh\phi} \\ &= mc\frac{-\operatorname{sech}\phi\tanh\phi d\phi}{\operatorname{sech}\phi^2 d\phi} \\ &= mc\frac{\tanh\phi}{\operatorname{sech}\phi} \\ &= mc\frac{\sinh\phi}{\cosh\phi}\cdot\cosh\phi \\ &= mc\sinh\phi \end{aligned} \tag{3-19}$$

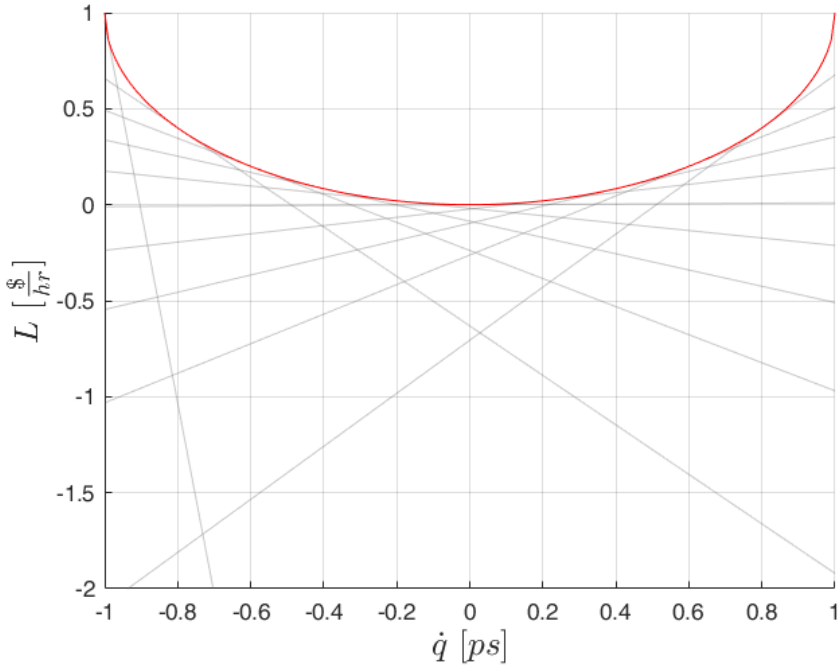


Figure 3-18: Lines (grey) with slope p, tangent to the running cost (red) with $m = 1\ \frac{\$}{hr \cdot ps^2}$ and $c = 1\ ps$

$$d\tanh\phi = \frac{d\tanh\phi d\phi}{d\phi} = \operatorname{sech}^2\phi d\phi \tag{3-20}$$

$$d(\operatorname{sech}\phi + 1) = \frac{d(\operatorname{sech}\phi + 1)d\phi}{d\phi} = -\operatorname{sech}\phi\tanh\phi d\phi \tag{3-21}$$

This result is visualized in Figure 3-19.

In the figure, it is visible that the relativistic analogy yields a nonlinear curve whereas the Newtonian analogy results in a linear line. Intuitively, consider what an employer would need to pay you to consistently perform 8 hours of labor services a day. Now consider how much that employer would need to pay you to consistently perform 16 hours a day. Answering that the reward required for the 16 hours is more than double that of the 8 hours, is in line with the relativistic curve. Furthermore, you can ask yourself, how much hourly wage would I require to consistently perform 23 of labor services a day? From this, it is easy to see that a nonlinear curve is more intuitive than a linear one.

Furthermore, Figure 3-19 shows that when performing additional labor services ($\dot{q} > 0$), the additional wage is positive ($p > 0$). On the other hand, a laborer can take additional leisure ($\dot{q} < 0$), by hiring someone else to perform their labor. Consequently, our laborer needs to pay the person they are hiring ($p < 0$).

The relativistic analogy to the wage, described by equation (3-19) has two asymptotes whereas

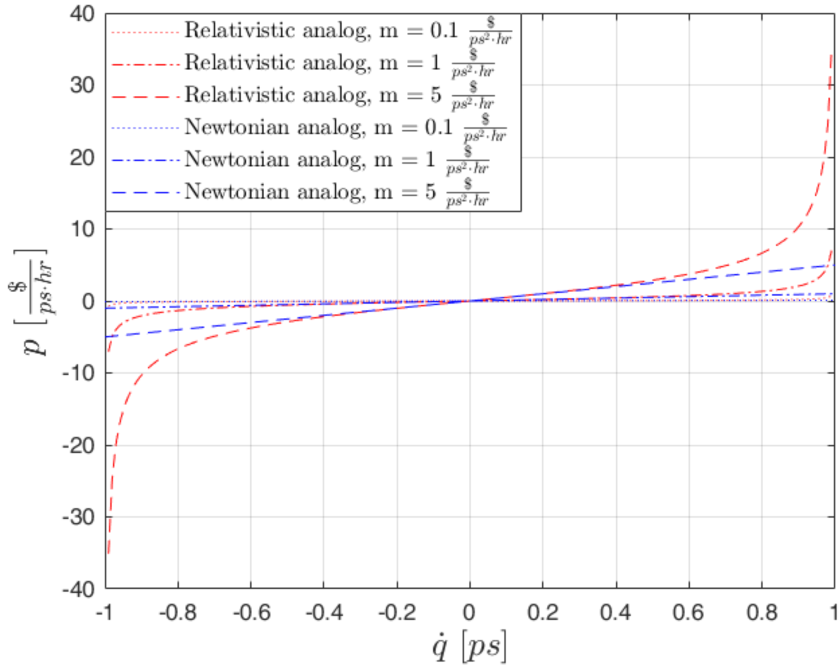


Figure 3-19: The relativistic (red) and Newtonian analog (blue) for wage p

the Newtonian analogy, described by equation (2-7), has none. The asymptotes of the relativistic analog are $\dot{q} = mc$ and $\dot{q} = -mc$. These asymptotes are in line with the expected kinematics described in Section 2-4 and Subsection 3-2-2. Therefore, the relativistic analog is more realistic.

Furthermore, we obtain a nonlinear supply curve with a constant m. As a result of this, a laborer can be described using one wage inelasticity. Whilst economists need many different wage inelasticities that vary based on the flow and wage.

Table 3-3 shows the analogy to the momentum.

Labor economics	Symbol	Unit
Wage	p	$[\frac{\$}{hr}]$

Table 3-3: Labor economics analogous to the wage

3-3-3 The surplus as Hamiltonian

Thus far, the flow of labor services $\dot{q}$, the running labor cost $\mathcal{L}$, and the wage p are developed. These are now used in the Legendre transform presented in equation (3-22).

$$(\mathcal{L} + \mathcal{H})\ dt = pdq \tag{3-22}$$

Through rewriting and filling in equation (3-22), the Hamiltonian is derived as in equation (3-23).

$$
\begin{aligned}
\mathcal{H} &= -\mathcal{L} + p\dot{q} \\
&= mc^2 \operatorname{sech}\phi - mc^2 + mc \sinh\phi \cdot c \tanh\phi \\
&= mc^2 \frac{\sinh\phi^2}{\cosh\phi} + mc^2 \frac{1}{\cosh\phi} - mc^2 \\
&= mc^2 \frac{\sinh^2\phi + 1}{\cosh\phi} - mc^2 \\
&= mc^2 \frac{\cosh\phi^2}{\cosh\phi} - mc^2 \\
&= mc^2 \cosh\phi - mc^2
\end{aligned} \quad (3\text{-}23)
$$

This Hamiltonian is visualized in Figure 3-20 and Figure 3-21. The pink dot in figure Figure 3-20 is the location at which the tangent line to point $(\dot{q}, \mathcal{L})$ crosses the y-axis. The y-value of this point is the negative of the Hamiltonian associated with that $\dot{q}$ and plotted underneath it, represented by the green dot. The length of the thick green line represents the Hamiltonian $\mathcal{H}$, whilst the thick red line represents the running cost $\mathcal{L}$.

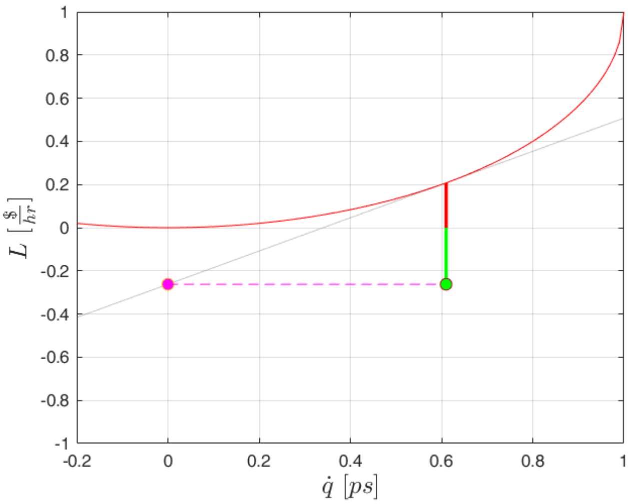


Figure 3-20: Lines (grey) with slope p, tangent to the running cost (thin red), the point where the line intersects the y-axis (magenta), $-\mathcal{H}(\dot{q})$ value (green), $\mathcal{L}(\dot{\text{II}})$ (thick red), with $m = 1\ \frac{\$}{hr \cdot ps^2}$, $c = 1\ ps$

When considering the meaning of the wage multiplied by the flow of labor services $(p\dot{q})$, the *labor value* of the labor performed is found. When subtracting the running labor cost from the labor value it results in the *laborer's surplus* in $[\frac{\$}{hr}]$.

The next step is visualizing the laborer's surplus as a function of the wage. [60] The observation that is made from Figure 3-22 is that (p, H), forms a hyperbola implying that there is a limit to the rate at which the surplus increases with an increase in the wage. It is known

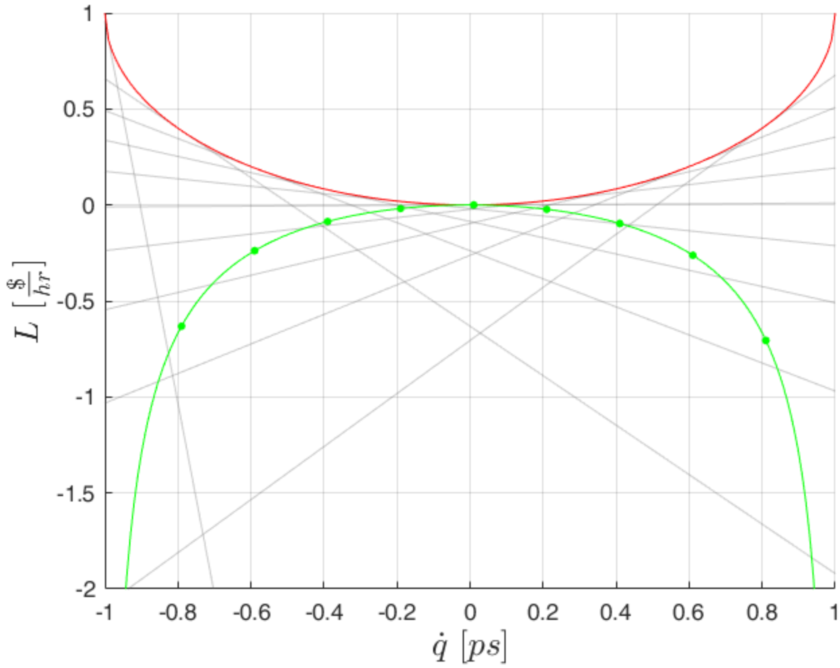


Figure 3-21: Lines (grey) with slope p, tangent to the running cost and $-\mathcal{H}$ in green, with $m = 1\ \frac{\$}{hr \cdot ps^2}$, $c = 1\ ps$ (pink).

that the Hamiltonian represents the co-energy which is the integral over the flow of labor in Figure 3-19 [37]. From here, and the second postulate, it is clear that $\dot{q}$ does not become larger than c. Therefore, the integral will not increase faster than c. Thus, the hyperbolic shape of (p, H) is in line with the expectation of the analog to the co-energy.

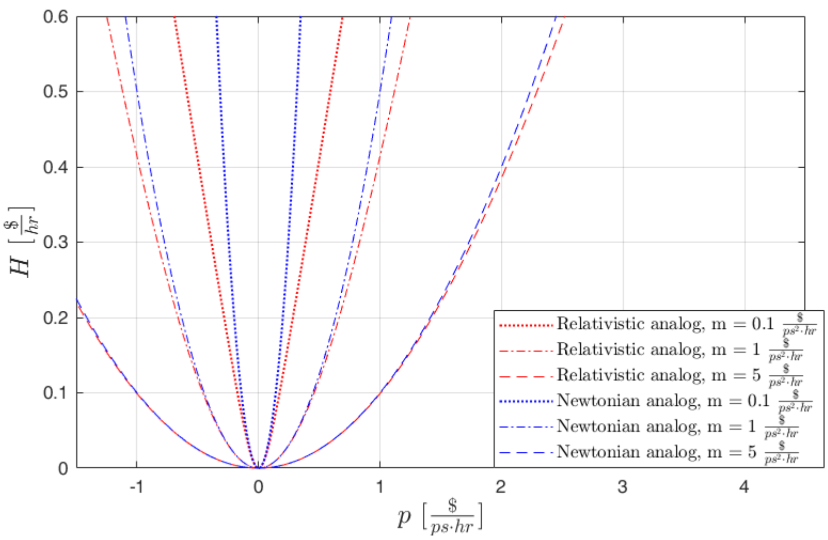


Figure 3-22: The relativistic (red) and Newtonian analogy (blue) for the laborer's surplus $\mathcal{H}$

Table 3-4 summarizes the labor economics analogies to the Legendre transform.

Labor economics	**Symbol**	**Unit**
Laborer's surplus	$\mathcal{H}$	$[\frac{\$}{hr}]$
Running labor cost	$\mathcal{L}$	$[\frac{\$}{hr}]$
Labor value	$p\dot{q}$	$[\frac{\$}{hr}]$

Table 3-4: Labor economics analog to the Legendre transform

3-4 The surplus-wage relationships

For the purpose of finding causal relationships, the analogies to forces are determined by analyzing the Legendre transform. First, dS is interpreted. Then, an equivalent of the Legendre transform is determined to come to the surplus-momentum relationship. From here, the wage vector is defined, considering not only spatial dimensions but also temporal dimensions. Finally, the derivative of the wage vector is interpreted.

3-4-1 The opportunity cost

The Legendre transform has more terms that are possible to interpret in labor economics. From equation (3-12), equation (3-24) is obtained.

$$dS = \mathcal{L}dt \tag{3-24}$$

When using this in the Legendre transform, equation (3-22) is rewritten as in equation (3-25). Here, pdq represents the wage obtained for the additional quantity of labor services performed. making it a laborers *periodic income*, expressed in [$]. The $\mathcal{H}dt$ term expresses the surplus accumulated during time span dt, therefore it is called the *accumulated surplus* which also has the unit [$].

$$\mathcal{H}dt = pdq - dS \tag{3-25}$$

Finally, the term dS is the running labor cost over time period dt. To the laborer, the running cost is the money they need to spend to be able to perform the labor and the value they could have obtained from leisuring during the time it is now performing labor. Therefore, dS is deemed the *opportunity cost* for the laborer, with the unit [$].

The results of determining what is analogous to the various terms in the Legendre transform are Table 3-5.

Labor economics	**Symbol**	**Unit**
Periodic income	pdq	[$]
Opportunity cost	$dS = \mathcal{L}dt$	[$]
Accumulated surplus	$\mathcal{H}dt$	[$]

Table 3-5: Labor economics analogous to the Legendre transform (2)

3-4-2 The surplus-wage relation from the Legendre transform

The Legendre transform as presented in equation (3-22) is expressed in dt. Therefore, it is not time-invariant. As per the first postulate, the goal is to obtain equations independent of the demand frame of reference. Therefore, equation (3-7) is used to rewrite the Legendre transform as in equation (3-26).

$$\begin{aligned}(\mathcal{L}+\mathcal{H})dt &= p\frac{dq}{dt} \\ (\mathcal{L}+\mathcal{H})\ \cosh\phi\ d\tau &= p\dot{q}\cosh\phi\ d\tau \\ (-mc^2\frac{1}{\cosh\phi}+mc^2+mc^2\cosh\phi-mc^2)\cosh\phi\ d\tau &= mc\sinh\phi\cdot c\tanh\phi\cdot\cosh\phi\ d\tau \\ mc^2(\cosh^2\phi-1)\ d\tau &= mc^2\sinh^2\phi\ d\tau\end{aligned} \tag{3-26}$$

When dividing both sides of the equation by the proper time $d\tau$, and multiplying with the proportionality factor mc^2, equation (3-27) is obtained.

$$m^2c^4(\cosh^2\phi-1)\ = (mc^2\sinh\phi)^2 \tag{3-27}$$

In the right-hand side of this equation, the wage p can be recognized, resulting in equation (3-28).

$$(mc^2\cosh\phi)^2-(mc^2)^2\ = (cp)^2 \tag{3-28}$$

Furthermore, it contains the constant used in the running cost and laborer's surplus as a result of which they start at 0. Based on this, and the knowledge about its units $\frac{\$}{hr}$, mc^2 is deemed to be the *inherent surplus* of the laborer. Lastly, using the knowledge on the structure of the energy-momentum relation [47] the first term on the right-hand side is deemed analog to E, the *total surplus* in $\frac{\$}{hr}$. Resulting in a time-invariant equivalent to the Legendre transform, presented in equation (3-29).

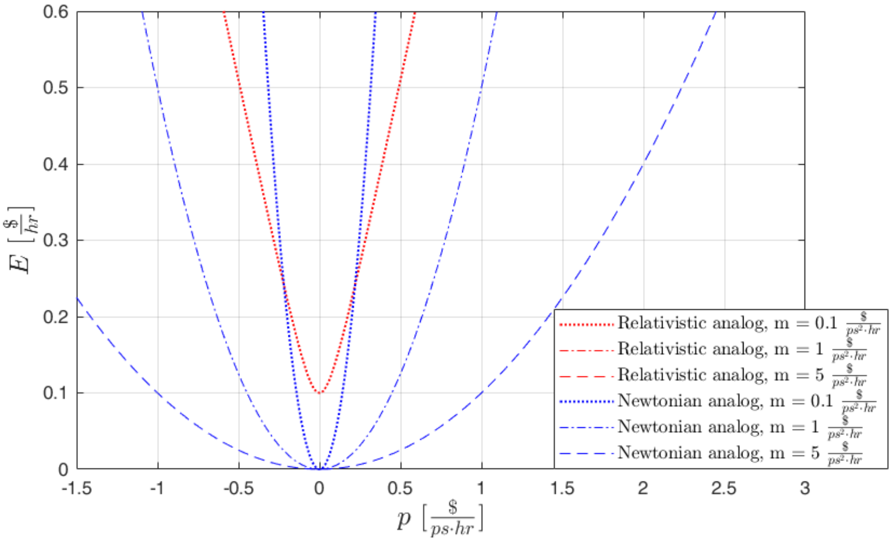
$$E^2-(mc^2)^2=(cp)^2 \tag{3-29}$$

Visualizing the surplus-wage plane results in Figure 3-23. The limit, which is represented through the robot, is shown in Figure 3-24 as the yellow line. In Figure 3-23a and Figure 3-23b, the comparison between the Newtonian analog and the Relativistic analog is shown. The total surplus E and the laborer's surplus $\mathcal{H}$ have the same shape, the only difference being that a constant, the inherent surplus, is added. The intuition of the hyperbola is thus the same as that of the laborer's surplus.

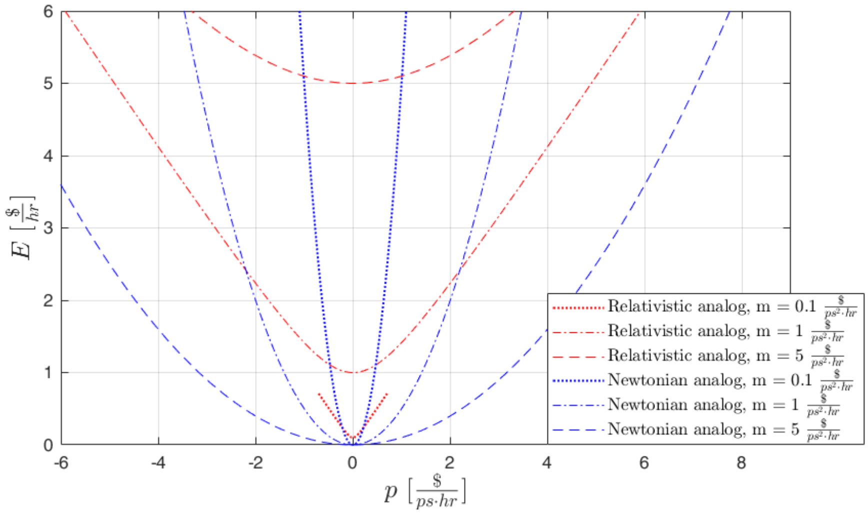
The results of finding the surplus-wage relation are shown in Table 3-6.

Labor economics	**Symbol**	**Unit**
Total surplus	E	$[\frac{\$}{hr}]$
Inherent surplus	mc^2	$[\frac{\$}{hr}]$

Table 3-6: Labor economics analogous to energy



(a) The Surplus-Wage plane, visualizing the relativistic (red) as well as Newtonian (blue) analogy, zoomed in



(b) The Surplus-Wage plane, visualizing the relativistic (red) as well as Newtonian (blue) analogy, zoomed out

Figure 3-23: Surplus-wage plane

3-4-3 The wage vector

So far, wage has only been in the context of the spatial dimensions, not yet in the time dimension. equation (3-29) is now used as the first step towards defining the *wage-vector* $\mathbf{P}$. To do this, it is first rewritten as equation (3-30).

$$-m^2c^2 = p^2 - \frac{E^2}{c^2} \tag{3-30}$$

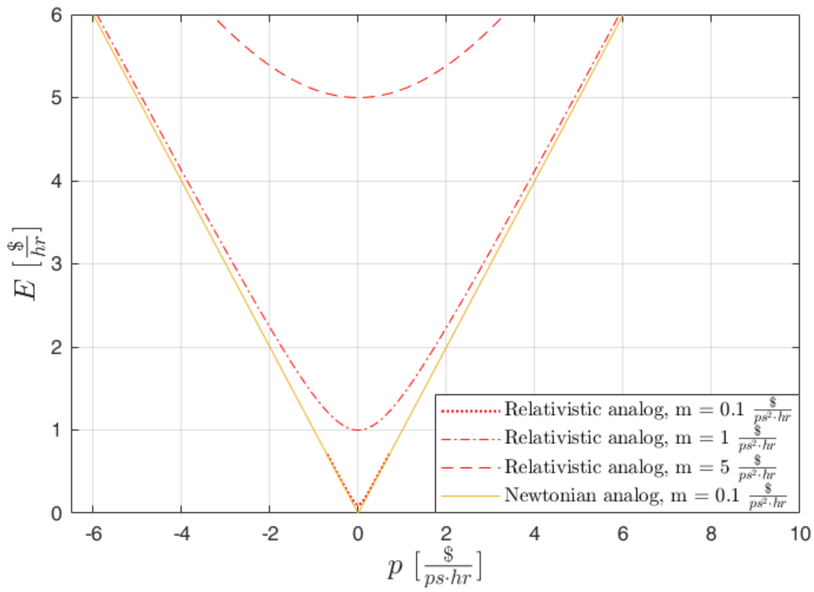


Figure 3-24: Surplus-wage plane, the relativistic (red) analogy and the theoretical maximum represented by the robot (orange)

The size of the individual vectors is presented in equation (3-31). With n denoting the number of dimensions of q. The first term on the right-hand side represents the spatial dimensions, and the second represents the time dimension. Which is equal to the invariant constants on the left hand of the equation.

$$-[1 \times 1] = [1 \times n][n \times 1] - [1 \times 1][1 \times 1] \tag{3-31}$$

Then the invariant wage-vector is defined as in equation (3-32). equation (3-32)

$$\mathbf{P} := (\frac{E}{c}, p) \tag{3-32}$$

It is conveniently chosen such that the inner product satisfies equation (3-30), as shown in equation (3-33) and equation (3-34) written in Einstein notation. Here, $\eta_{\kappa\nu}$ represents the metric tensor. Thus, the wage vector satisfies the first postulate and has units $[\frac{\$}{ps \cdot hr}]$.

$$\langle \mathbf{P}, \mathbf{P} \rangle = P^{\kappa}\eta_{\kappa\nu}P^{\nu} = \begin{pmatrix} \frac{E}{c} & p \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p \end{pmatrix} = -\left(\frac{E}{c}\right)^2 + p^2 \tag{3-33}$$

$$
\begin{aligned}
\langle \mathbf{P}, \mathbf{P} \rangle = |\mathbf{P}|^2 \\
&= \sqrt{\mathbf{P}^2}^2 \\
&= \sqrt{-\frac{E^2}{c^2} + (mc \sinh \phi)^2}^2 \\
&= -\frac{(mc^2 \cosh \phi)^2}{c^2} + m^2 c^2 \sinh \phi^2 \\
&= m^2 c^2 (-\cosh \phi^2 + \sinh \phi^2) \\
&= -(mc)^2
\end{aligned}
\tag{3-34}
$$

The wants-vector

To lay bare causal relations, the analog to the four-force vector is determined through finding the time derivative of the wage-vector. For this purpose, the time derivative of the individual components is determined first. Starting with the spatial component. Similar to how in physics, the derivative of the momentum is the force [58], the derivative of the wage is considered its driver. Therefore, it is deemed the laborer's *want for labor F*. The want for labor represents the change in marginal running labor cost per change in quantity of labor services the laborer has performed, as presented in equation (3-35), with the units $[\frac{\$}{ps \cdot hr^2}]$. [48]

$$
F := \dot{p} = \left. \frac{\partial \mathcal{L}}{\partial q} \right|_{\dot{q}} \tag{3-35}
$$

Next, the time derivative of $\frac{E}{c}$ is considered and named ρ as in equation (3-36). Whereas $\dot{p}$ is a want in the spatial dimension, ρ is the want in the time domain and is dependent on the surplus. Therefore ρ is defined as the *want for profit*, which also has the units $[\frac{\$}{ps \cdot hr^2}]$.

$$
\rho := \frac{\dot{E}}{c} \tag{3-36}
$$

ρ becomes the want for profit. For this is the time derivative, the units are $[\frac{\$}{ps \cdot hr^2}]$.

Combining the results in the *want-vector*, as presented in equation (3-37). The results are furthermore presented in Table 3-7.

$$
\dot{\mathbf{P}} = (\rho, F) \tag{3-37}
$$

Labor economics	Symbol	Unit
Wage-vector	$\mathbf{P}$	$[\frac{\$}{ps \cdot hr}]$
Want for labor	F	$[\frac{\$}{ps \cdot hr^2}]$
Want for profit	ρ	$[\frac{\$}{ps \cdot hr^2}]$
Wants-vector	$\dot{\mathbf{P}}$	$[\frac{\$}{ps \cdot hr^2}]$

Table 3-7: Labor economics analogous to the four-force

Chapter 4

The geometric description of flows of labor services

4-1 Introduction

In this chapter, the flow space is further analyzed for the case where LS spacetime is three-dimensional. The hyperboloid representing the flow space is an infinite manifold which can be projected onto finite maps using descriptive geometry. The flow space consists of points representing different flows of labor services due to the projective structure, as detailed in the previous chapter. Thus the projections provide information on time t, quantities of labor services q's, perceived time τ ($= 1$ hr in the case assessed) and the variables dependent on these, such as the flow of labor services and the real flow of labor services.

First, the flow space is visualized in Section 4-2. Then descriptive geometry is used to project the 3D hyperboloid onto a 2D poincaré disk in 4-3. Finally, the interpretation of the projection onto the poincaré disk is presented.

The MATLAB code used for the transformations is provided in E-4. Appendix E furthermore details transformations of the points on the hyperboloid to maps other than the Poincaré disk.

4-2 The hyperboloid in LS spacetime

4-2-1 Real flow of labor services

The flow space provides knowledge on the flows of labor services of equilibrium laborers, as each point setting up the flow space represents a different constant flow due to the projective structure. Using the flow space in LS spacetime, it is furthermore possible to determine the values of the variables t, q and τ. The flow of labor services as in equation (3-6) is the TC derivative of the quantity of labor services. Besides the TC derivative, also the proper time derivative is assessed. This proper time derivative of the quantity of labor services is presented in equation (4-1).

$$\dot{q}(\tau) = \frac{dq}{d\tau} \tag{4-1}$$

Which represents the flow of labor services demanded from the perspective of the laborer. $\dot{q}(\tau)$, is therefore named the *real flow of labor services.* When functioning at higher $\dot{q}(\tau)$ the wage companies need to pay to laborers becomes exponentially higher, as can be seen in equation (3-19). Because companies want to minimize the wage, functioning at higher $v(\tau)$ indicates a labor shortage. On the other hand, a lower $\dot{q}(\tau)$ indicates a plentitude, and thus high unemployment rates. The demand for labor services as perceived by the laborer thus informs economists on the strain felt on the labor market as a consequence of too few or too many laborers available.

4-2-2 Projecting flows of labor services on one hyperboloid

The first step towards visualizing the real flows in 2D is obtaining the real flow from LS spacetime. For the flow space to be a hyperboloid in 3D, LS spacetime needs to be in $\mathbb{R}^3$, thus describing $n = 2$ different types of labor. The two types considered are desk labor (q_1) and physical labor (q_2). The third dimension is time. Next, the projective space is determined, which describes timesheets at $\tau = 1\ hr$ for all possible career paths of equilibrium laborers at different flows of labor services. It is obtained by assessing the career paths of equilibrium laborers at different flows of labor services after $\tau = 1\ hr$. The obtained manifold is described as in equation (4-2) and represents the flow space. It is formulated from the perspective of a laborer initially located at $(0, 0, 0)$.

$$1 = c^2t^2 - q_1^2 - q_2^2 \tag{4-2}$$

The obtained hyperboloid can be visualized as in Figure 4-1 as a green-yellow mesh. Here, the laborer at $\tau = 0\ hr$ is visualized as a green marker. The timesheet of the same laborer, but observed at $tau = 1\ hr$, is visualized by the blue marker. This is a randomly chosen possible timesheet. The axes set up a demand frame of reference, this particular demand frame will be called "demand frame A".

From the perspective of the laborer at $\tau = 1\ hr$ (blue marker), another flow space is set up. This flow space describes the possible timesheets the laborer can obtain at $\tau = 2\ hr$ from the perspective of $\tau = 1\ hr$. This flow space is visualized by the blue-green mesh in Figure 4-2. Here, the red marker depicts, again a randomly chosen possible next timesheet of the same laborer at $\tau = 2\ hr$.

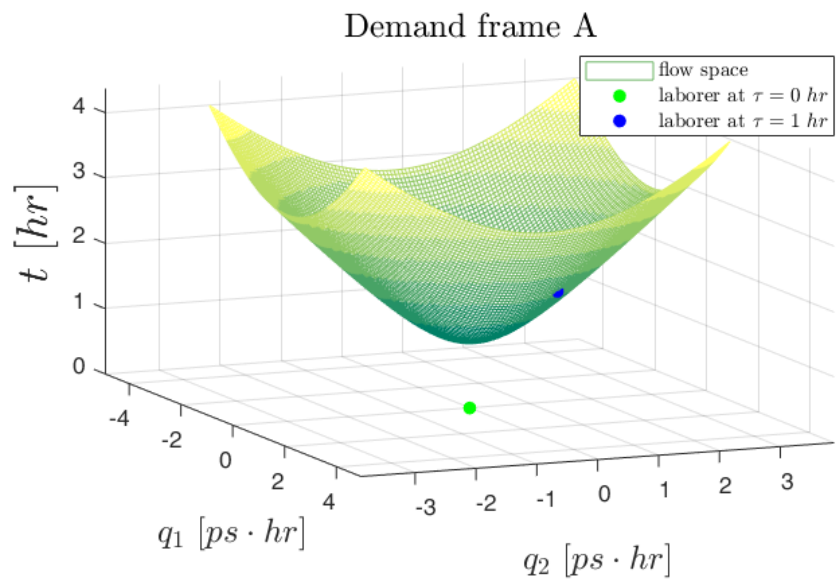


Figure 4-1: The flow space in 3D LS spacetime as viewed from demand frame A, with a laborer at the origin (green marker), and the timesheet that laborer obtains after $\tau = 1\ hr$ (blue marker)

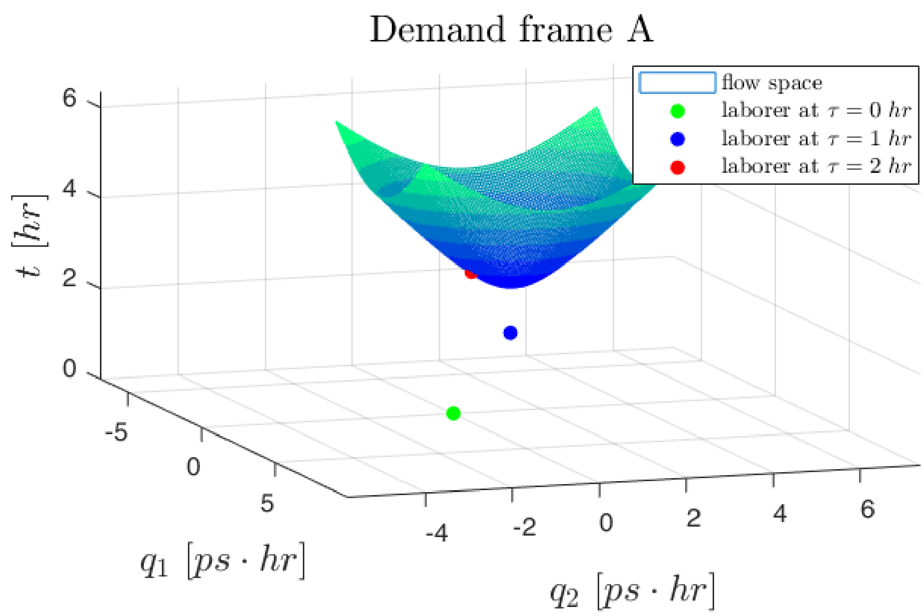


Figure 4-2: The flow space in 3D LS spacetime as viewed from demand frame A, with a laborer at the origin (green marker), the timesheet that laborer obtains after $\tau = 1\ hr$ (blue marker), and the timesheet the laborer obtains after $\tau = 2\ hr$ (red marker)

The flow space can also be viewed from the perspective of the laborer at $\tau = 1\ hr$, which will be called "demand frame B". Demand frame B has the same real flow of labor services as demand frame A, thus through simple addition to the coordinates of frame A, frame B can be found. The perspective from frame B is depicted in Figure 4-3.

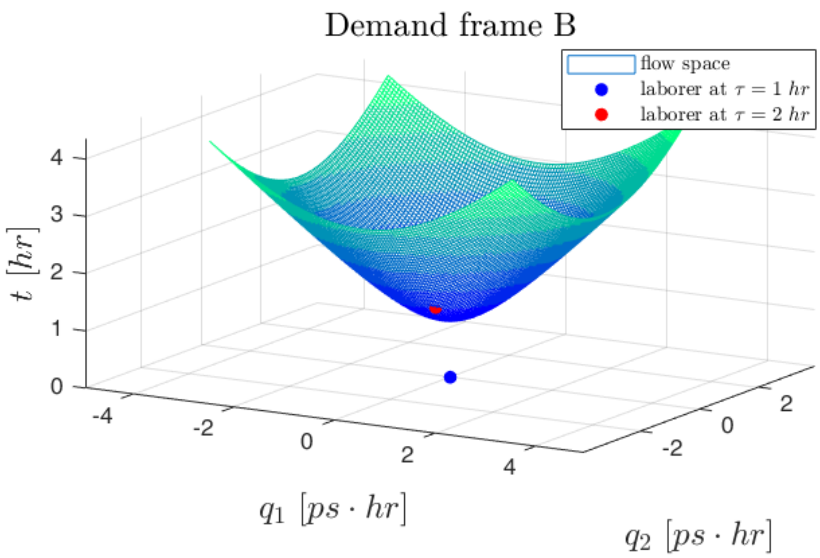


Figure 4-3: The flow space in 3D LS spacetime as viewed from demand frame B, the timesheet that laborer obtains after $\tau = 1\ hr$ (blue marker), and the timesheet the laborer obtains after $\tau = 2\ hr$ (red marker)

Because there is no difference in flow of labor services between the demand frames of reference, the flow of the laborer over time can be projected onto one hyperbola. The result is visualized in Figure 4-4. The difference between the different time sheets of the laborer is now $\Delta\tau = 1\ hr$. This perceived time step can be made infinitesimally for the flow space is a projective space. Thus continuous trajectories representing $\dot{q}(\tau)$ over perceived time can be derived, which in turn can be visualized on the manifold which in 3D forms a hyperboloid.

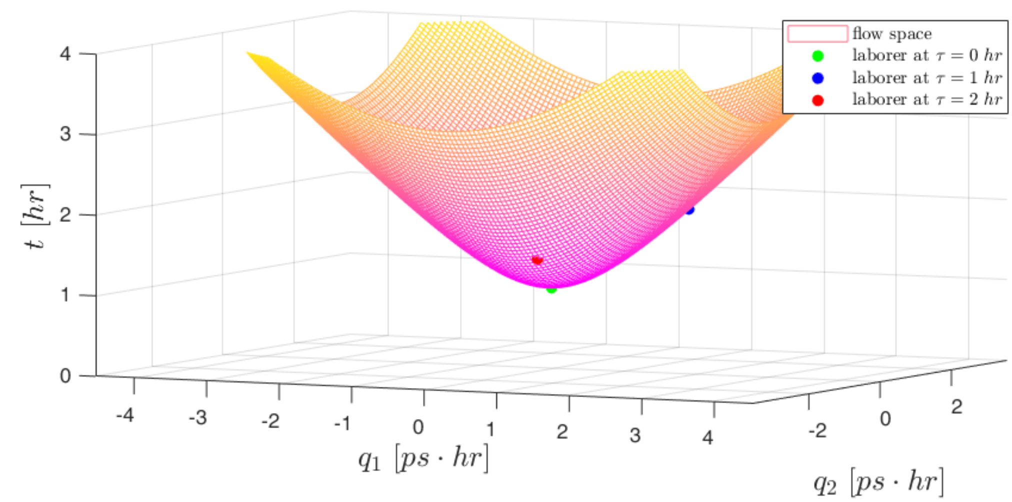


Figure 4-4: The flow space in 3D LS spacetime projecting the view from three different demand frames, with a laborer at the origin in frame A (green marker), the timesheet that laborer obtains after $\tau = 1\ hr$ (blue marker), and the timesheet the laborer obtains after $\tau = 2\ hr$ (red marker)

4-3 Real flows of labor services on the Poincaré disk

The hyperboloid representing the flow space is an infinite manifold that can be projected onto a poincaré disk, which is a unit circle. Thus making it possible to capture all $\dot{q}(\tau)$ and $\dot{q}(t)$ in a finite 2D image, making it easier to analyze. It is possible because the points of the hyperboloid represent $\dot{q}(\tau)$ for the cases that $\Delta\tau = 1\ hr$, and the hyperboloid details q and t which allows determining $\dot{q}(t)$.

The hyperboloid can thus be used to determine the real flow of labor services as well as the flow of labor services. In the following sections we will focus on the real flow of labor services, but where it says "real flow", it can be replaced with "flow" if the correction factor between these is applied, the Lorentz factor as per equation (3-7).

The projection onto the poincaré disk is obtained by projecting the point on the hyperboloid to the coordinates $(q_1, q_2, t) = (0, 0, -1)$. The values q_1 and q_2 of where this projection crosses the $q_1 q_2$-plane are the coordinates on the poincaré disk. This is visualized in Figure 4-5. The mathematical transformation from coordinates on the hyperboloid to coordinates on the poincaré disk is presented in equation (4-3) in which an underscore h represents the value associated with the hyperboloid and underscore p represents the value on the Poincaré disk.

[1]

$$
\begin{aligned}
q_{1p} &= \frac{q_{1h}}{1+t_h} \\
q_{2p} &= \frac{q_{2h}}{1+t_h} \\
t_p &= 0
\end{aligned}
\tag{4-3}
$$

Figure 4-6 shows the result of applying the transformation to the laborer presented at 3 different points in perceived time.

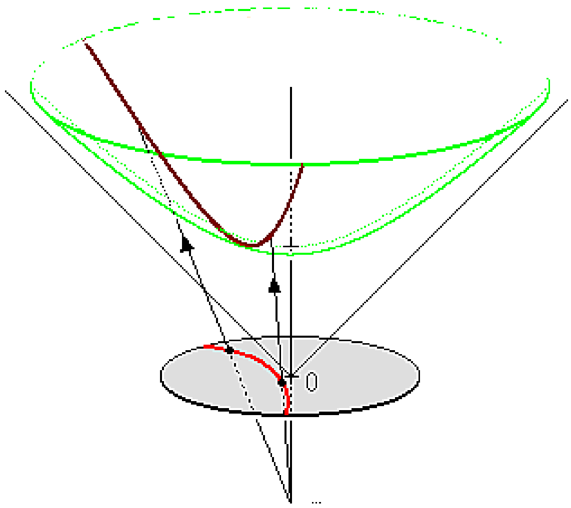


Figure 4-5: Projecting the flow space onto the poincaré disk [62]

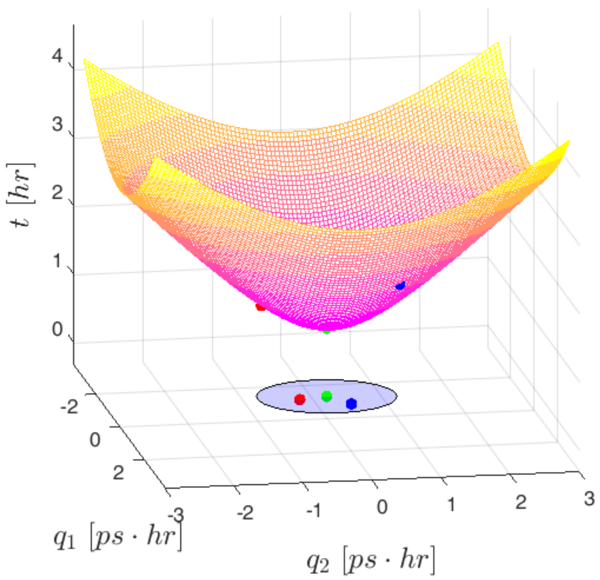


Figure 4-6: The flow at different perceived time τ, projected onto the poincaré disk

Other transformations projecting points on the hyperboloid onto surfaces, such as the Klein disk, half-plane and pseudosphere, are provided in appendix E. In this appendix, the example of the laborer at different perceived times is continued and projected onto the different surfaces. It furthermore provides initial ideas for interpretation of these different surfaces.[19], [11], [62], [2].

4-3-1 Angles representing occupational composition

The projecting of a flow space onto the poincaré disk is conformal projection. Meaning, the angles and areas on the hyperboloid are preserved on the poincaré disk. The angles represent the ratio between the relative real flows of desk labor $\dot{q}_1$ and physical labor $\dot{q}_2$ the laborer performs. Therefore it is named the *Occupational composition* θ.

For example, let us consider the angle between the q_1 and the laborer's coordinates. The poincaré disk is visualized in Figure 4-7. The coordinates of the laborer at $\tau = 1\ hr$ (blue marker) are $(0.50, 0.27)$ and thus the angle with the q_1 axis is 28°. Thus for each $1\ ps \cdot hr$ of desk labor the laborer performs, the laborer performs $0.54\ ps \cdot hr$ of physical labor.

Now consider the laborer at perceived time $\tau = 2\ hr$, which has a negative real flow of physical labor. The coordinates on the poincaré disk are $(0.12, -0.41)$, resulting in an angle of 287°. When the laborer performs $1\ ps \cdot hr$ of desk labor, it means that the laborer performs $-3.33\ ps \cdot hr$. This may sound counter-intuitive, however, the frame is relative. It thus means that the laborer has an occupational composition that has more desk labor compared to the demand frame of reference, which can represent another laborer or company average, etc.

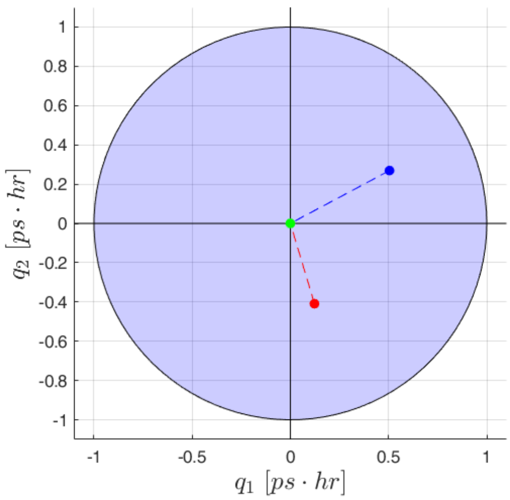


Figure 4-7: The real flow of labor services at different perceived time τ, projected onto the poincaré disk

4-3-2 Arc representing job transition

Arcs between points on the poincaré disk represent the progression from one real flow to another. The different locations on the poincaré disk represent different occupational compositions of the real flows of labor a laborer is performing. Thus, progressing to a different location on the disk represents a change in the composition of the labor a laborer is performing. Therefore, the arc visualizing this change in occupational composition is named a *job transition*.

An example of such a job transition visualized is presented in Figure 4-8. In Figure 4-8,

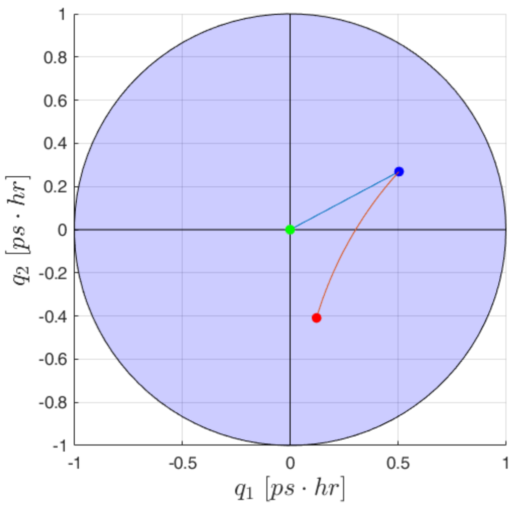


Figure 4-8: Job transition projected onto the poincaré disk

the job transition is not just any job transition, it is the optimal job transition. The optimal job transition is found when the arc between the two real flows is a geodesic, and therefore the shortest path. The shortest path does not always look like a straight line (see the red geodesic). The geodesic is determined by setting-up a plane through the two real flows on the hyperboloid, and the origin of the demand frame. The geodesic representing the optimal job transition is where this plane intersects the hyperboloid.

Chapter 5

Conclusion and Recommendations

5-1 Conclusions

This thesis presents the development of an economic engineering analogy which describes how much (more) labor a single laborer performs in exchange for (additional) wage. Furthermore, we present a method to visualize how much labor, and at what rate, this laborer performs labor services when analyzing one and two types of labor.

We assert that labor services cannot be performed faster than time. By building up LS spacetime, containing the mathematical structures describing the laborer, we show that this theoretical limit results in the flow space being described by a model of hyperbolic geometry. As opposed to the Newtonian analogy, in which the flow space is a model of Euclidean geometry. Whilst continuing to build up LS spacetime we recognize a mathematical structure similar to the one defined in the special theory of relativity. Therefore, we use the derivation of the special theory of relativity as a guide to develop our relativistic analogy for labor economics.

From LS spacetime and the flow space, we obtain timesheets (positions), career progressions (displacement), career paths (trajectories), flows (velocities) and additional hirings (accelerations). Thus providing us the kinematics of a laborer.

We furthermore find an exponential function for the running labor cost. This shows that the increase in wage demanded to perform more labor services, depends on how high the flow of labor services of the laborer already is. Contrary to the Newtonian analogy where the increase in running cost according to the capital or commodity analogy does not depend on the value of the existing flow.

Moreover, we find that the labor supply curve is exponential for a constant wage inelasticity (mass). We attribute this to the dependence on the difference in perception of time when a laborer is performing at a different flow of labor services. This is contrary to the view of economists, who ascribe this phenomenon to a change in wage (in)elasticity. Our viewpoint has the advantage that a constant wage inelasticity yields a supply curve in the shape of a hyperbolic sine function, thus an exponential labor supply curve. As a result of which, we only need to determine one wage inelasticity to obtain the labor supply curve, whereas economists need to determine all the different wage (in)elasticities for the different flows.

We also identify the wants-vector (force-vector) as the causation for changes in the flow of labor services, also known as the laborer's behavioral changes. This wants-vector provides insights into the dynamics. The wants-vector consists of the want for labor and the want for profit, which are dependent on the change in surplus and wage.

Concerning the next steps in the development of the dynamics of the laborer, I recommend clearly defining what is analogous to a damper and spring. For the damper, I suggest looking at phenomena such as taxes and travel time to and from the job. For the springs, I propose considering activities such as preliminary work.

We furthermore present a method to visualize the flow of labor services onto the Poincaré disk. On this disk, we find the laborer's job composition and (optimal) job transition. As a next step in the process of visualizing the kinematics of a laborer, I encourage the continuation of the interpretation of different maps from descriptive geometry. From these maps, I recommend starting with the half-plane and pseudosphere because of their metrics.

Ultimately, our developed model describing the dynamics of an individual laborer in the context of labor economics is a pre-existing model in physics. In physics, it describes the special theory of relativity. This is in line with economic engineering principles.

5-2 Further recommendations

5-2-1 Labor market modeling

To come to a model that performs policy analysis which includes analysis of the distribution and group difference, the theory developed needs to be applied. The model can be developed for the purpose of, for example, analyzing the unemployment rate or wage gaps. Both are considered topics relevant by today's government.

To build the macroeconomic model I suggest designing a closed-loop control system. Here, the controller represents the government, and the system represents the labor market which consists of laborers and companies. For designing the system, I propose using agent-based modeling, which allows the modeling of large numbers of individual laborers using our proposed analogy [34]. Because it is an agent-based model containing microeconomics[1] as well as the macroeconomics, the causal relationships are established both ways, as opposed to contemporary models of the CPB. [24] [15]

[1]Microeconomics studies the decisions individual laborers or firms make based on changes in wages, incentives, types of labor and available resources [70]

A suggestion for studying wage gaps would be to start with two types of jobs possible for laborers, flexible work and greedy work. The wage inelasticity of the demand curve representing greedy work is higher, for greedy work describes a laborer who needs to always be available for emergencies and regularly works overtime. The next step is to consider a couple that consists of two laborers with wage inelasticities dependent on the stage of career they are at. When, for example, a child enters the couple, it is required for the couple that at least one of them is not performing greedy work. This requires the couple to choose based on what is financially the most advantageous decision, over the full time-span of their careers. Such a model describes the core of the scenario Goldin attributes to wage gaps for which she received the Nobel prize in economic sciences in 2023. [28], [65]

5-2-2 Economic engineering

At the basis of the labor service analogy stands a laborer who cannot supply labor at a rate faster than time. The consequence of this is the difference in geometrical structure compared to the commodity and capital analog.

The labor services are not the only economic quantity whose flow cannot become faster than time. I recommend the usage of the structure of the labor service analogy presented in this thesis when using economic engineering to model services. The main differences will be that "labor" should be replaced by the name of the type of service considered, and "wage" is the type of payment the service demands. An example of other types of services is housing services, for the space is rented out for a specified duration. This duration is dependent on time and thus cannot progress faster than time.

5-2-3 The perspective of the laborer

The variables from the perspective of the laborer are not considered in economics. For economists do not ascribe the nonlinearity of the labor supply curve to the difference in perceived time but instead to a difference in wage elasticity. The difference in perceived time has implications for the way the labor market is observed, leading to different perspectives for demand frames with different flows. These flows from the perspective of the laborer elaborate on, for example, the strain on the laborers, describing the consequences on laborers of the flow they are performing, related to the unemployment rate. Because these variables are new, I advise exploring the interpretation of the different perspectives further.

A starting point would be continuing the interpretation of descriptive geometry, such as the Klein disk, the half-plane and the pseudosphere. Whereas the Klein disk preserves straight lines, thus showing whether arcs are optimal job transitions (geodesics) or not. The half-plane shows whether these job transitions consist of a change in the flow assigned to one type of job or multiple, through straight lines or half circles. Finally, the pseudosphere allows for Euclidean measurements of distance, allowing for an easily measurable arc length.

Appendix A

The initial economic engineering approach

This appendix provides a short overview of economic engineering and provides the aspects relevant to the development of the analogy in this thesis. First, the method of analogies is discussed. Furthermore, the advantages of the use of analytical mechanics in economic engineering are detailed. Lastly, the decision to start with the development of the Newtonian analogy is substantiated.

A-1 Extending the method of analogies to economic engineering

Economic engineering is the modeling of economic systems analogous to mechanical systems. Modeling economic systems analogous to mechanical systems opens up the possibility to apply the tools available in the field of systems and control engineering to economics. For example, performing transient analysis to tune system parameters influencing i.a. settling time and overshoot. Which, in the context of labor economics, could be the duration until a desired unemployment rate is achieved and the maximum unemployment rate encountered in the process. Another example would be using system identification to identify the value of constants, such as constants analogous to the spring constant or the resistance [40]. The last example is the information obtainable from the frequency domain. Economists only focus on relations that were found between variables based on averages obtained through historical data. Therefore not accounting for differences during economic cycles, such as trade or inventory cycles, provides less accurate results. [17], [49]

The method of analogies is set up within various domains in engineering, such as the linear mechanical, rotational mechanical, electrical and thermal domain. Thus far, the analogy to Newtonian mechanics has often been used [31], [41], [64], [40], [49]. This is because the Newtonian analogy has an explicit interpretation for most variables, and is often most familiar to students with a mechanical engineering background. Both factors make the process of defining the economic analogy more intuitive, forming the reasons why this thesis will also start with developing a Newtonian analog.

Prior to this thesis, the analogies to the electrical domain and thermodynamics have also been explored. Both analogies focus on combining micro- and macroeconomics [26], and an attempt to describe economic growth through thermodynamics [46]. The electrical domain analogy resulted in a hands-on method to realize a model that combines micro- and macroeconomics, namely through Agent-Based Modeling (ABM). [34]

The reader may be wondering, if there is a method in the electrical domain that can link micro- and macroeconomics, and that is so valuable, why not start by developing a new analogy there? This is precisely where the strength of the method of analogies comes in. When an analogy is developed between two domains it can thereafter easily be translated to another domain. As long as the other domain already has an analogy to one of the two domains between which the analogy is initially developed. This means that in our case, when developing the analogy for labor services between the economic engineering domain and Newtonian mechanics, it can easily be translated to the electrical domain. Because the analogy between Newtonian mechanics and the electrical domain has already been established.

A-2 Analytical mechanics in economic engineering

Besides Newtonian mechanics, analytical mechanics is also used in economic engineering. Analytical mechanics is a collection of alternative formulations of mechanics. The branches of analytical mechanics that have so far been applied are Lagrangian mechanics, Hamiltonian mechanics, which is equivalent to Lagrangian mechanics by a Legendre transform, and Hamilton-Jacobi. Newtonian mechanics is a force-momentum formulation, with the force in economic engineering analogous to a *want* and momentum to the *price*. Whereas the Lagrangian represents the co-energy, analogous to the *cost*, and the Hamiltonian represents the energy, analogous to the *surplus*.

In economic engineering Lagrangian mechanics has for example been used to derive the analogy between Mechanics and the economic processes of the mortgage market, deriving the economic forces responsible for mortgage prepayments by Krabbenborg. [40] Hamiltonian mechanics, for example, has been used to derive the economic rent by Fränkel [26]. Lastly, the Hamilton-Jacobi equation has been used by Legrand to formulate the effects of dissipation for general mechanical systems. Which allows economic engineers to model the equivalent of dissipation in economic systems. [44]

In engineering, the advantage of applying Newtonian mechanics over Lagrangian mechanics is that for simple systems, such as a single particle, it can be more intuitive. Whilst Lagrangian mechanics become more advantageous when systems become more complex, in the case of many particles. The reason for this is that Newtonian mechanics uses Cartesian coordinates whilst Lagrangian mechanics uses generalized coordinates. Whilst Cartesian coordinates do not vary, generalized coordinates can be strategically defined based on the system and its holonomic constraint forces. Thus, building the holonomic constraints directly into the equations of motion, as opposed to Newtonian mechanics in which they require to be individually applied to each particle. [45]

Particles are considered analogous to agents in economic engineering, and in the context of labor economics to *laborers*. Therefore, the consideration between approaching the system through Newtonian mechanics or Lagrangian mechanics extends into economic engineering. Making the analogy to Newtonian mechanics advantageous for a system consisting of a low number of laborers, when an intuitive interpretation is advantageous. Whilst Lagrangian becomes advantageous when considering an economic system with a large number of laborers. Therefore, the choice is made to initially approach the derivation of the theory to describe a single laborer as analogous to Newtonian mechanics, prioritizing intuition at this early stage.

Appendix B

Overview of the analogies

This appendix provides an overview of the different analogies for labor services. Starting with the commodity analogy for labor services, also known as the linear mechanical analogy. Then the capital analogy is presented, also known as the rotational analog. Finally, the relativistic analogy is presented.

B-1 Commodity analogy for labor services

The commodity analogy for labor services is presented in Table B-1.

Labor economics	**Symbol**	**Unit**	**Linear mechanical analog**	**Unit in physics**
Quantity of labor services	q	$ps \cdot hr$	Displacement	m
Flow of labor services	$\dot{q}$	ps	Velocity	$\frac{m}{s}$
Additional hiring	$\ddot{q}$	$\frac{ps}{hr}$	Acceleration	$\frac{m}{s^2}$
Time clock	t	s	Time	s
Wage	p	$\frac{\$}{ps \cdot hr}$	Momentum	$\frac{kg \cdot m}{s}$
Want for labor	F	$\frac{\$}{ps \cdot hr^2}$	Force	$\frac{kg \cdot m}{s^2}$
Wage inelasticity	m	$\frac{\$}{ps^2 \cdot hr}$	Mass	kg
Wage elasticity	ϵ	$\frac{ps^2 \cdot hr}{\$}$	Mass^{-1}	$\frac{1}{kg}$
Labor cost	T^*	$\frac{\$}{hr}$	Kinetic co-energy	$\frac{kg \cdot m^2}{s^2}$
Labor surplus	T	$\frac{\$}{hr}$	Kinetic energy	$\frac{kg \cdot m^2}{s^2}$

Table B-1: Linear labor service analogy (Commodity analogy)

B-2 Capital analogy for labor services

The capital analogy for labor services is presented in Table B-2.

Labor economics	Symbol	Unit	Linear mechanical analog	Unit in physics
Quantity of labor services	q_1	$[ps \cdot hr]$	Displacement	$[m]$
Labor force	q_0	$[ps \cdot hr]$	Displacement	$[m]$
Flow of labor services	$\dot{q}$	$[ps]$	Velocity	$[\frac{m}{s}]$
Additional hiring	$\ddot{q}$	$[\frac{ps}{hr}]$	Acceleration	$[\frac{m}{s^2}]$
Time clock	t	$[s]$	Time	$[s]$
Wage	p	$[\frac{\$}{ps \cdot hr}]$	Momentum	$[\frac{kg \cdot m}{s}]$
Want for labor	F	$[\frac{\$}{ps \cdot hr^2}]$	Force	$[\frac{kg \cdot m}{s^2}]$
Wage inelasticity	m	$[\frac{\$}{ps^2 \cdot hr}]$	Mass	$[kg]$
Wage elasticity	ϵ	$[\frac{ps^2 \cdot hr}{\$}]$	Mass^{-1}	$[\frac{1}{kg}]$
Labor cost	T^*	$[\frac{\$}{hr}]$	Kinetic co-energy	$[\frac{kg \cdot m^2}{s^2}]$
Labor surplus	T	$[\frac{\$}{hr}]$	Kinetic energy	$[\frac{kg \cdot m^2}{s^2}]$
Working hours	θ	$[-]$	Angle	$[-]$
Productivity	ω	$[\frac{1}{hr}]$	Angular velocity	$[\frac{1}{s}]$
Change in productivity	α	$[\frac{1}{hr^2}]$	Angular acceleration	$[\frac{1}{s^2}]$
Full employment cost	I	$[\$ \cdot hr]$	Inertia	$[kg \cdot m^2]$
Workforce's wage	L	$[\$]$	Angular momentum	$[\frac{kg \cdot m^2}{s}]$
Wage inflation	τ_N	$[\frac{\$}{hr}]$	Torque	$[\frac{kg \cdot m^2}{s^2}]$

Table B-2: Rotational labor service analogy (Capital analogy)

B-3 Relativistic analogy for labor services

The relativistic analogy for labor services is presented in Table B-3.

Labor economics	Symbol	Unit	Linear mechanical analog	Unit in physics
Quantity of labor services	q	$[ps \cdot hr]$	Distance	$[m]$
Flow of labor services	$\dot{q}$	$[ps]$	Velocity	$[\frac{m}{s}]$
Flow of a robot	c	$[ps]$	Light speed	$[\frac{m}{s}]$
Wage inelasticity	m	$[\frac{\$}{ps^2 \cdot hr}]$	Mass	$[kg]$
Perceived time	τ	$[s]$	proper time	$[s]$
Time clock	t	$[s]$	lab time	$[s]$
Labor cost	$\mathcal{L}$	$[\frac{\$}{hr}]$	Lagrangian	$[\frac{kg \cdot m^2}{s^2}]$
Degree of demand	ϕ	$[-]$	Rapidity	$[-]$
Wage	p	$[\frac{\$}{ps \cdot hr}]$	Momentum	$\frac{kg \cdot m}{s}$
Lorentz factor	γ	$[-]$	lorentz factor	$[-]$
Labor surplus	$\mathcal{H}$	$[\frac{\$}{hr}]$	Hamiltonian	$[\frac{kg \cdot m^2}{s^2}]$
Want for wage	F	$[\frac{\$}{ps \cdot hr^2}]$	Force	$[\frac{kg \cdot m}{s^2}]$
Wage-vector	$\mathbf{P}$	$[\frac{\$}{ps \cdot hr}]$	four-momentum vector	$[\frac{kg \cdot m}{s}]$
Want for profit	ρ	$[\frac{\$}{ps \cdot hr^2}]$	power	$[\frac{kg \cdot m}{s^2}]$
Wants-vector	$\dot{\mathbf{P}}$	$[\frac{\$}{ps \cdot hr^2}]$	Four-force vector	$[\frac{kg \cdot m}{s^2}]$

Table B-3: Relativistic labor service analogy

Appendix C

Lorentz transform

To derive the Lorentz transformation, the first step is defining two different demand frames. One frame of demand, called s, with $\{q, t\}$. This is the demand frame from which we observe LS spacetime. Therefore, from our perspective, it has no flow of labor services in the q direction. The other frame of demand s' with $\{q', t'\}$ has a flow of labor services $\dot{q}$ in the q-direction.

The goal is to obtain a transformation linear in nature. Because that is what we found for the transformation applicable to the robot. As described in 3-2-1. Furthermore, if they are not linear, the transformation equations could result in accelerations which should not be possible since accelerations can only be caused by forces. Therefore the q' coordinate is defined as in equation (C-1), with γ and b constants. The derivation works towards expressing these constants in the variables and constants assumed available, which are the flow of labor services $\dot{q}$ and the robot's flow c.

$$q' = \gamma q + bt \tag{C-1}$$

The next step is to imagine a time sheet at the origin of s'. Which is described as equation (C-2).

$$\begin{aligned} q &= \dot{q}t \\ q' &= 0 \end{aligned} \tag{C-2}$$

When filling equation (C-2) in equation (C-1) it yields equation (C-3).

$$0 = \gamma(\dot{q}t) + bt \qquad \Rightarrow b = -\gamma\dot{q} \tag{C-3}$$

Substituting this value for b into equation (C-3) yields equation (C-4).

$$q' = \gamma q - \gamma\dot{q}t \qquad \Rightarrow q' = \gamma(q - \dot{q}t) \tag{C-4}$$

The reverse transformation is defined in equation (C-5).

$$q = \gamma(q' - \dot{q}t') \tag{C-5}$$

The second part of the derivation is dependent on the second postulate stating the robot's flow is the same for all observers. In this part, a robot is considered to start performing labor at $t = 0\ s$ at the origin of s and s' that overlap at time $t = 0\ s$. The displacement of a robot is described in the different frames of demand as in equation (C-6).

$$\begin{aligned} q &= ct \\ q' &= ct' \end{aligned} \tag{C-6}$$

When substituting q in the transformation equation (C-4) it yields equation (C-7).

$$\begin{aligned} &ct' = \gamma(ct - \dot{q}t) \\ &\Rightarrow \frac{t}{t'} = \frac{c}{\gamma(c - \dot{q})} \end{aligned} \tag{C-7}$$

When substituting q' from equation (C-6) into equation (C-5) it yields equation (C-8).

$$\begin{aligned} &ct = \gamma(ct' + \dot{q}t') \\ &\Rightarrow \frac{t}{t'} = \frac{\gamma(c + \dot{q})}{c} \end{aligned} \tag{C-8}$$

When equating $\frac{t}{t'} = \frac{t}{t'}$, γ is derived as in equation (C-9).

$$\begin{aligned} &\frac{\gamma(c + \dot{q})}{c} = \frac{c}{\gamma(c - \dot{q})} \\ &c^2 = \gamma^2(c + \dot{q})(c - \dot{q}) = \gamma^2(c^2 - \dot{q}^2) \\ &\gamma^2 = \frac{c^2}{c^2 - \dot{q}^2} = \frac{1}{1 - \frac{\dot{q}^2}{c^2}} \\ &\Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} \end{aligned} \tag{C-9}$$

This γ is the Lorentz factor. Now this will be used to derive the transformation to obtain t' from t and the reverse transformation to t from t'. When filling γ in in equation (C-5) it yields equation (C-10).

$$q = \frac{q' + \dot{q}t'}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}}$$

$$\begin{aligned}
q\sqrt{1 - \frac{\dot{q}^2}{c^2}} &= \frac{q - \dot{q}t}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} + \dot{q}t' = \frac{q - \dot{q}t + \dot{q}t'\sqrt{1 - \frac{\dot{q}^2}{c^2}}}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} \\
q(1 - \frac{\dot{q}^2}{c^2}) &= q - \dot{q}t + \dot{q}t'\sqrt{1 - \frac{\dot{q}^2}{c^2}} \\
q - \frac{q\dot{q}^2}{c^2} &= q - \dot{q}t + \dot{q}t'\sqrt{1 - \frac{\dot{q}^2}{c^2}} \\
t - \frac{\dot{q}q}{c^2} &= t'\sqrt{1 - \frac{\dot{q}^2}{c^2}} \\
t' &= \frac{t - \frac{\dot{q}q}{c^2}}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}} \Rightarrow t' = \gamma(t - \frac{\dot{q}q}{c^2})
\end{aligned} \tag{C-10}$$

A similar derivation can be followed to determine the reverse transformation. Finally, this leads to the transformation equations shown in equation (C-11) for the regular transformation, and the reverse transformation is shown in equation (C-12). These are a special case of the Lorentz transformation called the Lorentz boost for it is a rotation-free Lorentz transformation.

$$\begin{aligned}
q' &= \gamma(q - \dot{q}t) \\
t' &= \gamma(t - \frac{\dot{q}q}{c^2})
\end{aligned} \tag{C-11}$$

$$\begin{aligned}
q &= \gamma(q' - \dot{q}t') \\
t &= \gamma(t' - \frac{\dot{q}q'}{c^2})
\end{aligned} \tag{C-12}$$

Appendix D

Proportionality constant of the periodic cost

In this appendix, the proportionality constant is derived. We obtain this by considering that it is desirable to be able to compare the Newtonian analogy to the relativistic analogy. Therefore, similar to physics, the regions with low flows (velocities) should yield almost the same values. By comparing these, we find that the proportionality constant is mc^2.

Let us start with defining the difference ds between two timesheets in LS Spacetime. This difference is determined using equation (D-1) when applying the metric $[+-]$.

$$ds^2 = c^2 dt^2 - dq^2 \tag{D-1}$$

Then, the length along the curve is parameterized by a parameter such as n to find equation (D-2).

$$ds = \sqrt{c^2 \dot{t}^2 - d\dot{q}^2} dn \tag{D-2}$$

With $\dot{t}$ and $\dot{q}$ defined as in equation (D-3).

$$\dot{t} = \frac{dt}{dn} \quad \text{and} \quad \dot{q} = \frac{dq}{dn} \tag{D-3}$$

The next step is determining the action, which in LS spacetime is the *period cost* of an equilibrium laborer. These period costs for the laborer are proportional to the the length-along-the-curve. The factor which allows for this is named w. The period cost can then be written as in equation (D-4).

$$S = w \int \sqrt{c^2 \dot{t}^2 - d\dot{q}^2} dn \tag{D-4}$$

The parameter n can be changed without the interpretation of the length of the line segment changing. This can be shown through introducing the parameter $\eta(n)$ and using equation

(D-5).

$$\dot{t} = \frac{dt}{dn} = \frac{dt}{d\eta}\frac{d\eta}{dn} \text{ and } \dot{q} = \frac{dq}{dn} = \frac{dq}{d\eta}\frac{d\eta}{dn} \tag{D-5}$$

Which allows us to rewrite equation (D-4) into equation (D-7). Here, the derivative dots do not refer to the derivative w.r.t. n but to η.

$$S = w\int\sqrt{c^2(\frac{dt}{d\eta})^2 - (\frac{dq}{d\eta})^2}\frac{d\eta}{dn}\frac{dn}{d\eta}d\eta \tag{D-6}$$

$$= w\int\sqrt{c^2\dot{t}^2 - \dot{q}^2}d\eta \tag{D-7}$$

The next step is attempting to identify parameter w by determining the Lagrangian for this period cost and then comparing it with the Lagrangian found in the Newtonian labor analog. Because it is compared to the Newtonian labor analog the period cost of the free laborer is determined from the inertial demand frame of the company (considered the laboratory frame in special relativity), thus not the inertial demand frame of the free laborer (considered the COM frame in special relativity). Therefore, η will be defined as time t, the period cost can then be written as in equation (D-8).

$$S = w\int\sqrt{c^2(\frac{dt}{dt})^2 - (\frac{dq}{dt})^2}dt$$

$$= w\int\sqrt{c^2 - \dot{q}^2}dt$$

$$= wic\int\sqrt{-1 + \frac{\dot{q}^2}{c^2}} \tag{D-8}$$

Furthermore, it is known that the integrant of the action is the Lagrangian. Analog to this, the integrant of the period costs is the *running cost* L for the laborer, as presented in equation (D-9) [69].

$$S = \int_{t_1}^{t_2} L \ dt \tag{D-9}$$

This results in the running cost as presented in equation (D-10).

$$L = wic\sqrt{-1 + \frac{\dot{q}^2}{c^2}} \tag{D-10}$$

The next step is comparing this running cost at low flow in the relativistic analog with the running cost of the Newtonian analog L_N. Because it is known that for a low flow of labor services, the results are comparable. This is done using a Taylor series polynomial of degree 1 around $\dot{q} = 0$, presented in equation (D-11).

$$L \approx wic(-1 + \frac{1}{2}\frac{\dot{q}^2}{c^2}) = -wic + wi\frac{1}{2}\frac{\dot{q}^2}{c} = L_N = \frac{1}{2}m\dot{q}^2 \tag{D-11}$$

It is known that w, i and c are all constants that form a term. This new, constant term, does not influence the Euler-Lagrange equations. Thus, the constants can be identified through equation (D-12).

$$\frac{wi}{c} = m \Rightarrow wi = mc \tag{D-12}$$

Filling this back into the running cost of equation (D-11) yields equation (D-13).

$$L = -mc^2\sqrt{1 - \frac{\dot{q}^2}{c^2}} + mc^2 \tag{D-13}$$

Which can be rewritten in hyperbolic functions using equation (D-14) expressing the Lorentz factor as a hyperbolic function of the degree of demand.

$$\gamma = \frac{1}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}}$$
$$= \frac{1}{\sqrt{1 - \tanh\phi^2}}$$
$$\frac{1}{\sqrt{\operatorname{sech}\phi^2}} =$$
$$\frac{1}{\operatorname{sech}\phi} =$$
$$\cosh\phi \tag{D-14}$$

Which yields equation (D-15), in the thesis referred to as equation (3-17).

$$\mathcal{L}(\phi) = -mc^2\frac{1}{\cosh\phi} + mc^2 \tag{D-15}$$

[12]

Appendix E

Descriptive geometry

Besides the Poincaré disk, descriptive geometry details more methods to visualize the 3D hyperboloid onto a 2D plane. In this appendix, we continue with the example presented in chapter 4. Where a single laborer is visualized at perceived time $\tau = 0$ hr using a green marker, at $\tau = 1$ hr using a blue marker and $\tau = 2$ hr using a red marker.

The different maps presented below have different properties. Whereas the Poincaré disk is a conformal model, the Klein disk is a model that preserves straight lines. The Klein disk is thus useful for assessing and determining optimal job transitions (geodesics). The Poincaré half-plane is conformal and shows geodesics through straight vertical lines or circles with the center at $y = 0$ axis. Because of its conformality, the half-plane preserves the occupational composition. Finally, the distances on the Pseudosphere are Euclidean distances, making it easy to measure. The different projects thus each have their own advantages. [56]

E-1 Klein disk

The projection from the hyperboloid to the Klein disk is presented in equation (E-1). Here, q_k and t_k represents the location of the coordinates after projecting it on the Klein disk. Whereas q_h and t_h represent the coordinates on the hyperboloid in LS spacetime.

$$\begin{aligned} q_{1k} &= \frac{q_{1h}}{t_h} \\ q_{2k} &= \frac{q_{2h}}{t_h} \\ t_k &= 1 \end{aligned} \tag{E-1}$$

The advantage of the Klein disk is that it is easy to determine the optimal job transition between two coordinates both containing the information on the flow and job composition. This optimal job transition is found by simply drawing a straight line between the two coordinates. [1]

Visualizing the laborer and its job transitions on the Klein disk yields Figure E-1.

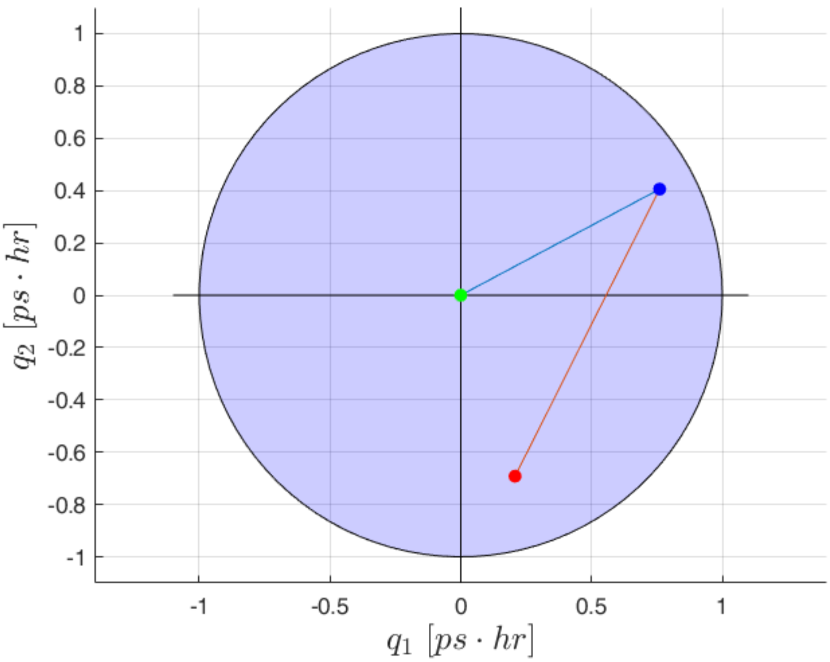


Figure E-1: Job transition projected onto the Klein disk

The distance ds_k on the Klein disk is not Euclidean but instead provided by the metric in equation (E-2) [23].

$$ds_k^2 = \frac{\left(1 - \|\mathbf{q_p}\|^2\right) \|d\mathbf{q_p}\|^2 + (\mathbf{q_p} \cdot d\mathbf{q_p})^2}{\left(1 - \|\mathbf{q_p}\|^2\right)^2} \tag{E-2}$$

E-2 Poincaré half-plane

The projection from the hyperboloid onto the Poincaré half-plane is presented in equation (E-3). [66] Here, q_{hp} represents the location of the quantity of labor services projected on the Poincaré half-plane and q_p represents these coordinates projected on the Poincaré disk.

On the half-plane, the optimal job transitions can be identified as straight lines and circles with a center on the real axis [27]. To keep the expression simple, the coordinates on the Poincaré half-plane are expressed in coordinates from the Poincaré disk. In the code presented at the bottom of this appendix, the direct transformation between the hyperboloid and the Poincaré half-plane is also presented, see line 91.

$$\begin{aligned} q_{1hp} &= \frac{2 \cdot q_{1p}}{q_{1p}^2 + (q_{2p}+1)^2} \\ q_{2hp} &= \frac{2 \cdot (q_{2p}+1)}{q_{1p}^2 + (q_{2p}+1)^2} - 1 \end{aligned} \tag{E-3}$$

Visualizing results in Figure E-2.

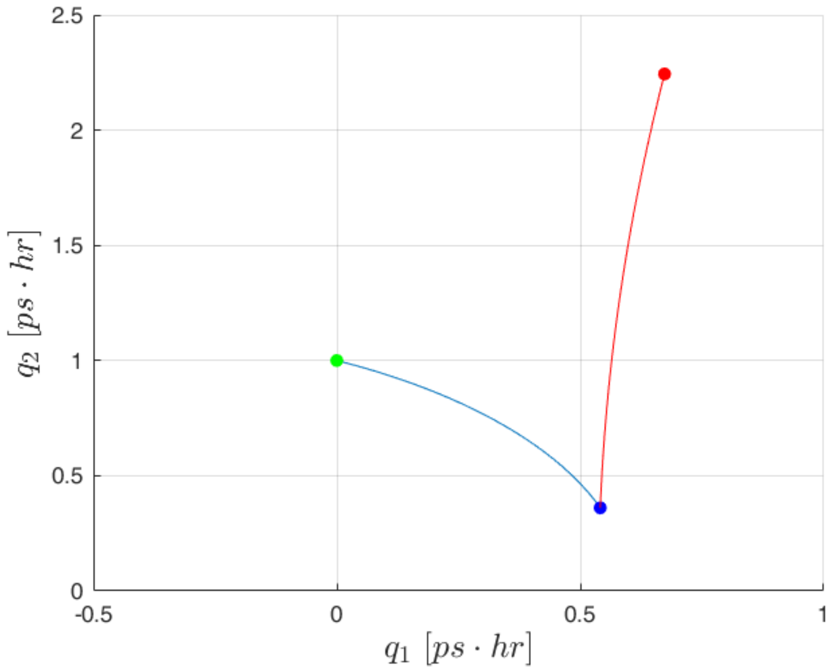


Figure E-2: Job transition projected onto the half-plane

The half-plane also does not measure Euclidean distances. The metric applicable to determine distance ds_{hp}, is presented in equation (E-4). [66] From this metric, the inspiration is obtained that this plane potentially would allow for the expression of the value of goods or commodities in a number of labor services the laborer provides. Instead of converting the labor services to money first and then to the good or commodity.

$$(ds_{hp})^2 = \frac{(dx)^2 + (dy)^2}{y^2} \tag{E-4}$$

E-3 Pseudosphere

To obtain the pseudosphere from the Poincaré half-plane equation (E-5) is used. In which q_{ps} and t_{ps} represent the locations of the coordinates from the hyperboloid projected on the pseudosphere.

$$\begin{aligned} q_{1ps} &= \text{sech}(\text{arcCosh}(q_{2hp})) \cdot \cos(q_{1ph}) \\ q_{2ps} &= \text{sech}(\text{arcCosh}(q_{2hp})) \cdot \sin(q_{1hp}) \\ t_{ps} &= \text{arcCosh}(q_{2hp}) - \tanh(\text{arcCosh}(q_{2hp})) \end{aligned} \tag{E-5}$$

The advantage of the pseudosphere is that it measures Euclidean distances. Thus allowing for a direct measuring of the different timesheets. It is important to note that not all coordinates described by the hyperboloid can be projected onto the pseudosphere. The coordinates that cannot be projected onto the half-plane are the ones that adhere to $q_{2hp} < 1$. When $q_{2hp} \geq 1$, it is possible for the coordinates to be projected onto the pseudosphere. [61]

As visible in Figure E-2, it is not possible to project all coordinates of the job transitions onto the pseudosphere, for not all coordinates satisfy $q_{2hp} \geq 1$. The coordinates that do satisfy this requirement are projected onto the pseudosphere and visualized in Figure E-3.

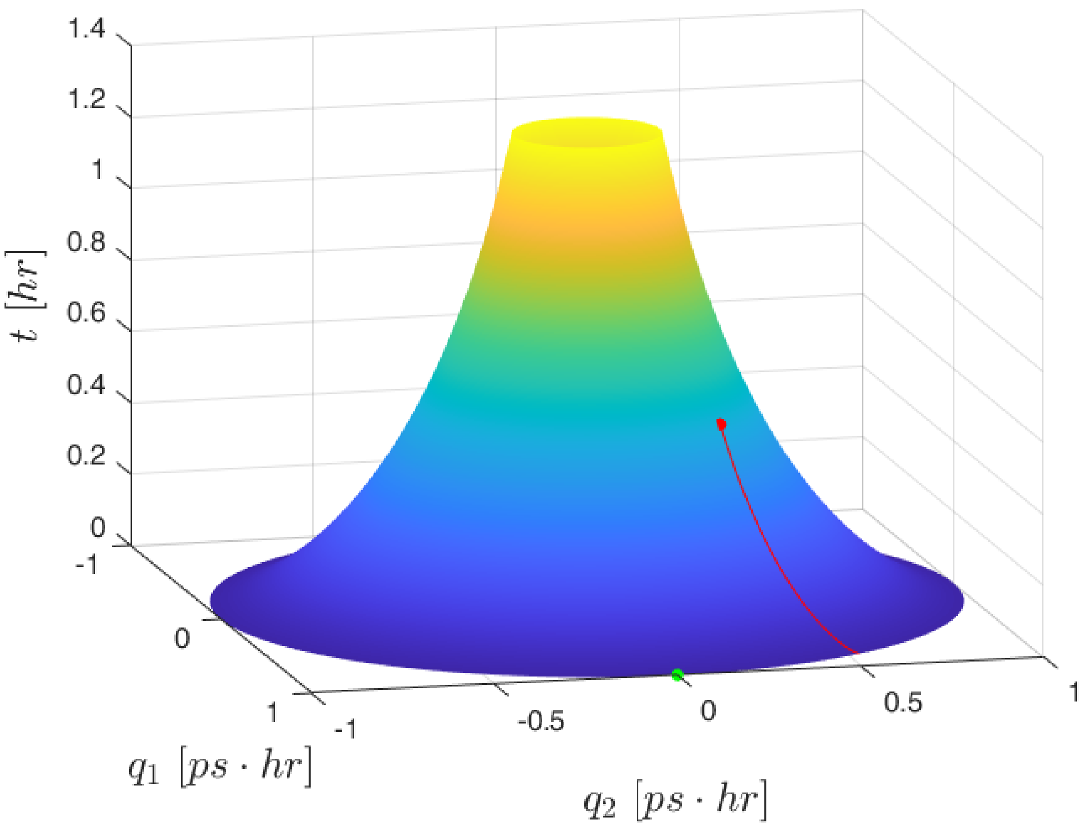


Figure E-3: Job transition projected onto the Klein disk

E-4 MATLAB code determining the projections from the hyperboloid

In the code, ϕ represents the degree of demand, which is a hyperbolic angle between the q_1q_2-plane at height $t = 0$ hr. θ represents the occupational composition as measured from q_1. In the code q_1 is referred to as qx and q_2 as qy.

```
clear all; close all;

%% Hyperbola
% Define a position on the hyperboloid through defining the real demand
    in the q_1 and q_2 direction
```

```matlab
c = 1;                  % Work of a robot [ps] (max work)
tau = 1;                % Perceived time [hr]
phi = [0.5; 0.7];       % Degree of demand [-] in [q1, q2] direction
n = 100;                % Nr. of points taken on geodesic
m = 1;                  % Wage inelasticity, [$ / (ps^2 hr)], mass

% If q_x and q_y displacements are known
qx_3 = -1.5;
qy_3 = 7;

% Calculate other info
[t_3, phi_3, the_3] = disp_2_angles(qx_3, qy_3, tau, c);

% If phi and theta are known, using the same example as before for easy
% verification
phi_4 = phi_3;
the_4 = the_3;

% Calculate other info
[t_4, qx_4, qy_4] = angles_2_disp(phi_4, the_4, tau, c);


%% Functions

function [t, phi, theta] = disp_2_angles(qx, qy, tau, c)  % Determines angles based on known distances
    for i = 1:size(qx,1)
        t(i, 1) = 1/c*sqrt(tau^2*c^2+qx(i, 1)^2+qy(i, 1)^2);
        phi(i, 1) = acosh(t(i,1)/tau);
        theta(i, 1) = acos(qx(i, 1)/qy(i, 1));
    end
end

function [t, qx, qy] = angles_2_disp(phi, theta, tau, c)  % Determines distances based on known angles
    for i = 1:size(phi, 1)
        t(i, 1) = tau*cosh(phi(i, 1));
        qy(i, 1) = c*tau*sqrt((cosh(phi(i, 1))^2 -1)/(1+cos(theta(i, 1))^2));
        qx(i, 1) = qy*cos(theta(i, 1));
    end
end

function [th, xh, yh] = klein_2_hyper(xk, yk, c)    % Maps trajectory on the Klein disk to the hyperboloid
    for i = 1:size(xk, 1)
        th(i, 1) = sqrt(c^2/(c^2-xk(i, 1)^2-yk(i, 1)^2));
        xh(i, 1) = xk(i, 1)*th(i, 1);
        yh(i, 1) = yk(i, 1)*th(i, 1);
    end
end
```

```
function [tk, xk, yk] = hyper_2_klein(th, xh, yh, c)    % Maps trajectory
    on the hyperboloid to the Klein disk
    for i = 1:size(xh, 1)
        xk(i, 1) = xh(i, 1)/th(i, 1);
        yk(i, 1) = yh(i, 1)/th(i, 1);
        tk(i, 1) = 1;
    end
end

function [tpc, xpc, ypc] = hyper_2_poincare(th, xh, yh, c)      % Maps
    trajectory on the hyperboloid to Poincare disk
    for i = 1:size(xh, 1)
        % Project this path onto the Poincare disk
        xpc(i, 1) = xh(i, 1)/(1+th(i, 1));
        ypc(i, 1) = yh(i, 1)/(1+th(i, 1));
        tpc(i, 1) = 0;
    end
end

function [xhp, yhp] = poincare_2_halfplane(xpc, ypc)    % Maps trajectory
    on Poincare disk to Poincare half plane
    for i = 1:size(xpc, 1)
        xhp(i, 1) = 2*xpc(i, 1)/(xpc(i, 1)^2+(ypc(i, 1)+1)^2);
        yhp(i, 1) = 2*(ypc(i, 1)+1)/(xpc(i, 1)^2+(ypc(i, 1)+1)^2)-1;
    end
end

function [tps, xps, yps, go_nogo] = halfplane_2_pseudo(xhp, yhp)    %
    Maps trajectory on Poincare half plane to pseudo sphere
    if any(yhp(:) < 1)
        go_nogo = {'no go, yhp too small to project onto pseudosphere'};
            % Not all points on the can be visualized on the
            pseudosphere, this signals if some of the points you are
            trying to put on the pseudosphere cannot be put on the
            pseudospher
    else
        go_nogo = {'go, yhp values can be projected onto pseudosphere'};
    end

    for i = 1:size(xhp, 1)
        xps(i, 1) = sech(acosh(yhp(i, 1)))*cos(xhp(i, 1));
        yps(i, 1) = sech(acosh(yhp(i, 1)))*sin(xhp(i, 1));
        tps(i, 1) = acosh(yhp(i, 1)) - tanh(acosh(yhp(i, 1)));
    end
end

function [xhp, yhp] = hyper_2_halfplane(th, xh, yh, c)  % Maps trajectory
    on the Poincare half plane to the hyperboloid
    for i = 1:size(xh, 1)
        xhp(i,1) = 2*xh(i, 1) / ((xh(i, 1)^2+yh(i, 1)^2) / (1+th(i, 1)) +
            2*yh(i, 1)+1+th(i, 1));
        yhp(i,1) = (2*yh(i, 1)+2*th(i, 1)+2) / ((xh(i, 1)^2+yh(i, 1)^2) /
            (1+th(i, 1))+2*yh(i, 1)+1+th(i, 1))-1;
```

```
    end
end

function [xpc, ypc] = halfplane_2_poincare(xhp, yhp, c) % Maps trajectory
    on the Poincare half plane to the Poincare disk
    for i = 1:size(xhp, 1)
        xpc(i,1) = (2*xhp(i,1))/(xhp(i,1)^2 + yhp(i,1)^2 + 2*yhp(i,1) +
            1)
        ypc(i,1) = -(xhp(i,1)^2 + yhp(i,1)^2 - 1)/(xhp(i,1)^2 + yhp(i,1)
            ^2 + 2*yhp(i,1) + 1)
    end
end
```

Appendix F

MATLAB for comparing Newtonian analogy and Relativistic analogy

This appendix contains the code used to visualize the comparison between the Newtonian analogy for labor services and the Relativistic analogy for labor services. These visualizations, and the mathematical descriptions for the variables are presented throughout chapter 3.

```
clear all; close all;

%%
c = 1;                          % Flow of a robot [ps]

m = [0.1, 1, 5];                % Wage inelasticity [$ / (ps^2 hr]
v = -c:0.01:c;                  % Flow of labor services, [ps], also can have negative values because it is all relative

for i = 1:size(v,2)
    phi(i) = atanh(v(i)/c);                                             % Rapidity, real demand
    % varieties of the wage inelasticity (m)
    for i2 = 1:(size(m, 2))
        p(i, i2) = m(1, i2)*c*sinh(phi(i));                             % Momentum, wage [$ / (ps hr)]
        L(i, i2) = -m(1,i2)*c^2 / cosh(phi(i))+m(1,i2)*c^2;             % Laborer's cost, Lagrangian, kinetic co-energy [$ / hr]
        H(i, i2) = m(1, i2)*c^2*cosh(phi(i))-m(1,i2)*c^2;               % Laborer's surplus, Hamiltonian, kinetic energy [$ / hr]
        pv(i, i2) = v(i) * m(1, i2) *c*sinh(phi(i));                    % Legendre transform, L+H should overlap with pv [$ / hr]
        gamma(i, i2) = cosh(phi(i));                                    % Lorentz factor [-]
        E(i, i2) = m(1, i2)*c^2*cosh(phi(i));                           % Total surplus  [$ / hr]
```

```
        E_rob(i, i2) = abs(p(i, i2));                             % Line representing robot  [$ / hr]
        K_e_n(i, i2) = 1/2*m(1, i2)*v(i)^2;                       % Newtonian laborer's cost, kinetic energy  [$ / hr]
        p_n(i, i2) = m(1, i2)*v(i);                               % Newtonian laborers wage, momentum  [$ / (ps hr)]
        K_ce_n(i, i2) = p_n(i, i2)^2/(2*m(1, i2));                % Newtonian laborer's surplus, kinetic co-energy [$ / hr]
        E_n(i, i2) = K_e_n(i,i2);                                 % Newtonian total surplus, energy [$ / hr]
    end
end


%% Visualizing Legendre Transform

figure();
hold on
grid on
n_a = 2;
tan_length = 5;
targetX = 0;

for n=1:10:100
    slope(n) = (L(n*n_a+1, 2)-L(n*n_a-1, 2)) / (v(n*n_a+1) - v(n*n_a-1));          % Determine slope

    % Point where the tangent line touches the Lagrangian
    X_neutral(n,1) = v(n*n_a);
    Y_neutral(n,1) = L(n*n_a,2);

    % Determine begin and end point of tangent line
    X(n, 2) = v(n*n_a)+tan_length;                   % Determine begin point of line
    Y(n, 2) = L(n*n_a, 2)+tan_length*slope(n);       % Determine begin point of line
    X(n, 1) = v(n*n_a)-tan_length;                   % Determine end point of line
    Y(n, 1) = L(n*n_a, 2)-tan_length*slope(n);       % Determine end point of line

    % Visualize the tangent lines
    plot([X(n,1), X(n,2)], [Y(n,1), Y(n,2)], 'Color', [0.3, 0.3, 0.3, 0.3])

    % Visualizing legrendre transform
    y0(n) = Y_neutral(n,1) + slope(n) * (targetX - X_neutral(n,1));
    plot(X_neutral(n,1), y0(n), 'go', 'MarkerSize', 3, 'MarkerFaceColor', 'g')
end
plot(v, L(:,2), 'r')        % Visualize Lagrangian
plot(v, -H(:,2), 'g')       % Visualize - Hamiltonian

```

```
xlabel('$\dot{q}$ $[ps]$','Interpreter','Latex', fontsize=15);
ylabel('$L$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize=15);
xlim([-1, 1]);
ylim([-2, 1]);
hold off
%title('$\dot{q}$, L', 'Interpreter','Latex');

%%
n_1 = 81;                          % The number of the tangent line that is being visualized
m_2 = (Y(n_1, 2) - Y(n_1, 1)) / (X(n_1, 2) - X(n_1, 1));    % Determine the slope of the tangent line
b = Y(n_1, 1) - m_2 * X(n_1, 1);    % Identify the value of the negative of the laborer's surpus, Hamiltonian

figure()
plot([X(n_1,1), X(n_1,2)], [Y(n_1,1), Y(n_1,2)], 'Color', [0.3, 0.3, 0.3, 0.3]) % Visualize a tangent line
hold on
plot(v, L(:,2), 'r')                                                          % Visualize Lagrangian
plot(0, b,'o', 'MarkerSize', 7, 'MarkerFaceColor', 'm')                       % Visualize the point where the tangent line crosses the y axis
plot([v(n_1*n_a),v(n_1*n_a)],[0 ,L(n_1*n_a, 2)], 'r', 'LineWidth', 2)         % Visualize line from y axis to -H
plot([v(n_1*n_a),v(n_1*n_a)],[b, 0], 'g', 'LineWidth', 2)                     % Visualize line from L to x-axis
plot([0,v(n_1*n_a)],[b,b], 'm--')                                             % Visualize line from -H to (L, \dot{q})
plot(v(n_1*n_a), b,'o', 'MarkerSize', 7, 'MarkerFaceColor', 'g')
grid on
xlim([-0.2, 1]);
ylim([-1, 1]);
xlabel('$\dot{q}$ $[ps]$','Interpreter','Latex', fontsize=15);
ylabel('$L$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize=15);
hold off

%% Visualizations Comparing Relativistic and Newtonian

%% Laborer's cost - Lagrangian
figure();
plot(v, L(:,1), 'r:');
hold on
plot(v, L(:,2), 'r-.');
plot(v, L(:,3), 'r--');
grid on
xlabel('$\dot{q}$ $[ps]$','Interpreter','Latex', fontsize=15);
ylabel('$L$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize=15);
%title('$\dot{q}$, L', 'Interpreter','Latex');
plot(v, K_e_n(:,1), 'b:');
```

```
plot(v, K_e_n(:,2), 'b-.');
plot(v, K_e_n(:,3), 'b--');
legend(sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$
    ',m(1,1)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\
    cdot hr}$',m(1,2)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{
    ps^2 \\cdot hr}$',m(1,3)), sprintf('Newtonian analog, m = %d $\\frac
    {\\$}{ps^2 \\cdot hr}$',m(1,1)), sprintf('Newtonian analog, m = %d $\\
    frac{\\$}{ps^2 \\cdot hr}$',m(1,2)), sprintf('Newtonian analog, m = %d
     $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,3)), 'Interpreter','Latex',
    fontsize=10)
hold off

%% Laborer's surplus - Hamiltonian
figure();
plot(p(:,1), H(:,1), 'r:');
hold on
plot(p(:,2), H(:,2), 'r-.');
plot(p(:,3), H(:,3), 'r--');
grid on
xlabel('$p$ $[\frac{\$}{ps \cdot hr}]$', 'Interpreter','Latex', fontsize
    = 15);
ylabel('$H$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize = 15);
%title('$p$, H', 'Interpreter', 'Latex');
plot(p_n(:,1), K_ce_n(:,1), 'b:');
plot(p_n(:,2), K_ce_n(:,2), 'b-.');
plot(p_n(:,3), K_ce_n(:,3), 'b--');
legend(sprintf('Relativistic with m = %d',m(1,1)), sprintf('Relativistic
    with m = %d',m(1,2)), sprintf('Relativistic with m = %d',m(1,3)),
    sprintf('Newtonian with m = %d',m(1,1)), sprintf('Newtonian with m = %
    d',m(1,2)), sprintf('Newtonian with m = %d',m(1,3)))
hold off

%% Only one Hamiltonian
figure();
hold on
plot(p(:,2), H(:,2), 'r');
grid on
xlabel('$p$ $[\frac{\$}{ps \cdot hr}]$', 'Interpreter','Latex', fontsize
    = 15);
ylabel('$H$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize = 15);
%title('$p$, H', 'Interpreter', 'Latex');
plot(p_n(:,2), K_ce_n(:,2), 'b-.');
legend(sprintf('Relativistic with m = %d',m(1,2)), sprintf('Newtonian
    with m = %d',m(1,2)))
hold off

%% Momentum - wage, pv-diagram
figure();
plot(v, p(:,1), 'r:');
hold on
plot(v, p(:,2), 'r-.');
plot(v, p(:,3), 'r--');
grid on
```

```
xlabel('$\dot{q}$ $[ps]$','Interpreter','Latex', fontsize = 15);
ylabel('$p$ $[\frac{\$}{ps \cdot hr}]$', 'Interpreter','Latex', fontsize = 15);
%title('$\dot{q}$, p', 'Interpreter', 'Latex', fontsize = 15);
plot(v, p_n(:,1), 'b:');
plot(v, p_n(:,2), 'b-.');
plot(v, p_n(:,3), 'b--');
legend(sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,1)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,2)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,3)), sprintf('Newtonian analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,1)), sprintf('Newtonian analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,2)), sprintf('Newtonian analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,3)), 'Interpreter','Latex', fontsize=10)
hold off

%% Momentum - wage, pv-diagram only Newtonian analog
v_n1 = -1.5*c:0.01:1.5*c;
for i = 1:size(v_n1,2)
    %phi(i) = atanh(v_n(i)/c);             % Rapidity, real demand
    % vary the wage inelasticity (m)
    for i2 = 1:(size(m, 2))
        p_n1(i, i2) = m(1, i2)*v_n1(i);                          % Momentum, wage
    end
end

figure();
hold on
grid on
xlabel('$\dot{q}$ $[ps]$','Interpreter','Latex', fontsize = 15);
ylabel('$p$ $[\frac{\$}{ps \cdot hr}]$', 'Interpreter','Latex', fontsize = 15);
%title('$\dot{q}$, p', 'Interpreter', 'Latex', fontsize = 15);
plot(v_n1, p_n1(:,1), 'b:');
plot(v_n1, p_n1(:,2), 'b-.');
plot(v_n1, p_n1(:,3), 'b--');
legend(sprintf('Newtonian analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,1)), sprintf('Newtonian analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,2)), sprintf('Newtonian analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,3)), 'Interpreter','Latex', fontsize=10)
hold off


%% Energy Momentum equation

figure();
plot(p(:,1), E(:,1), 'r:');
hold on
plot(p(:,2), E(:,2), 'r-.');
plot(p(:,3), E(:,3), 'r--');
grid on
```

```
plot(p_n(:,1), E_n(:,1), 'b:');
plot(p_n(:,2), E_n(:,2), 'b-.');
plot(p_n(:,3), E_n(:,3), 'b--');
%plot(p(:,2), E_rob(:,2), 'm');
xlabel('$p$ $[\frac{\$}{ps \cdot hr}]$', 'Interpreter','Latex', fontsize
    = 15);
ylabel('$E$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize = 15);
%title('p, E');
legend(sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$
    ',m(1,1)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\
    cdot hr}$',m(1,2)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{
    ps^2 \\cdot hr}$',m(1,3)), sprintf('Newtonian analog, m = %d $\\frac
    {\\$}{ps^2 \\cdot hr}$',m(1,1)), sprintf('Newtonian analog, m = %d $\\
    frac{\\$}{ps^2 \\cdot hr}$',m(1,2)), sprintf('Newtonian analog, m = %d
     $\\frac{\\$}{ps^2 \\cdot hr}$',m(1,3)), 'Interpreter','Latex',
    fontsize=10)
hold off

%% Energy momentum only relativistic and robot

figure();
plot(p(:,1), E(:,1), 'r:');
hold on
plot(p(:,2), E(:,2), 'r-.');
plot(p(:,3), E(:,3), 'r--');
grid on
%functioning at the max flow
plot(p(:,3), E_rob(:,3), 'color', [0.9290 0.6940 0.1250], 'LineStyle', '-
    ');
xlabel('$p$ $[\frac{\$}{ps \cdot hr}]$', 'Interpreter','Latex', fontsize
    = 15);
ylabel('$E$ $[\frac{\$}{hr}]$', 'Interpreter','Latex', fontsize = 15);
%title('p, E');
legend(sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\cdot hr}$
    ',m(1,1)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{ps^2 \\
    cdot hr}$',m(1,2)), sprintf('Relativistic analog, m = %d $\\frac{\\$}{
    ps^2 \\cdot hr}$',m(1,3)), sprintf('Robot'), 'Interpreter','Latex',
    fontsize=10)
hold off
```

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Glossary

List of Acronyms

LS	labor service (often used in the context of "LS spacetime")
PS	producer surplus
CS	consumer surplus
TC	time clock

List of Symbols

α	Change in productivity
β	Proportionality constant
γ	Lorentz factor
ω	Productivity
ϕ	Degree of demand
ρ	Want for profit
τ	Perceived time
τ_N	Wage inflation
θ	Occupational composition
θ_h	Working hours
ξ	Continuous map of labor timesheets
$-\mathcal{L}$	Laborer's running benefit
$\ddot{q}$	Additional hiring
$\dot{q}$	Flow of labor services
$\dot{q}(\tau)$	Real flow of labor services
$\dot{q}_1$	Real flow of desk labor services
$\dot{q}_2$	Real flow of physical labor services

ϵ	Wage elasticity
$\dot{\mathbf{P}}$	Wants-vector
$\mathbf{P}$	Wage-vector
$\mathcal{H}$	Laborer's surplus
$\mathcal{L}$	Running labor cost
c	Flow of labor services of a robot performing labor
ds	Length in LS spacetime
ds_{hp}	Distance on the Poincaré half-plane
ds_k	Distance on the Klein disk
E	Total surplus
F	Want for labor
H	Set of points of all different career paths after one unit interval of perceived time
I	Full employment cost
L	Workforce's wage
m	Wage inelasticity
mc^2	Inherent surplus
n	Number of different types of labor
p	Wage
$p\dot{q}$	Labor value
q	Quantity of labor services
q_0	Labor force
q_1	Quantity of desk labor services (in Chapter 4)
q_1	Quantity of labor services (in Chapter 2)
q_2	Quantity of physical labor services
q_b	Quantity of blue-collar labor services
q_h	Quantity of labor services on the hyperboloid in LS spacetime
q_k	Coordinate of the quantity of labor services projected onto the Klein disk
q_w	Quantity of white-collar labor services
q_{hp}	Coordinate of the quantity of labor services projected onto the Poincaré half-plane
q_{ps}	Coordinates of the quantity of labor services projected onto the pseudosphere
q_p	Coordinate of the quantity of labor services projected onto the Poincaré disk
R	Possible positions of the laborer after one time interval of perceived time
S	Periodic labor cost
T	Labor surplus
t	Time clock
T^*	Labor cost
t_h	Time on the hyperboloid in LS spacetime
t_k	Time value representing the height of the plane containing the Klein disk
t_{ps}	Coordinates of the time projected onto the Poincaré disk
s	Length in LS spacetime